Chapter 2

Computations of Area of Balls

I will first introduce the set-up of the computations in section 1, and the materials there follow chapter 7 of Korobenko et al. (2021). Section 2 will first introduce some new assumptions which I will follow. Then, under these assumptions, computations of the area of the metric ball when the radius approaches zero will be presented. The arguments in section 2 are original. Section 3 presents some simulations of the model deduced in section 2 using Mathematica.

2.1 Descriptions of the Metric and the Geodesics

2.1.1 Description of the metric

Consider the degenerate elliptic partial differential equation (PDE)

$$\operatorname{div}(A(x)\nabla u(x,y)) = 0 \quad \text{where} \quad A(x) = \begin{pmatrix} 1 & 0 \\ 0 & f(x)^2 \end{pmatrix} \tag{*}$$

and assume that $f \in C^2(\Omega)$ is even, $f(0) = 0$ and $f'(x) > 0$ for all $x \in \mathbb{R}_+$.[1]

Consider the Carathéodory metric associated with the PDE above. Referring to Proposition 1.2.2, say that a curve γ is *admissible* if $\langle \dot{\gamma}, \xi \rangle^2 \leq \xi^T A(x) \xi$ for every $\xi \in \mathbb{R}^n$. Observe that when $x \neq 0$, A is invertible and symmetric, so there is the following proposition.

Proposition 2.1.1. The Cathéodory distance $\mathrm{d}t$ is given by

$$\mathrm{d}t^2 = \mathrm{d}x^2 + \frac{1}{f(x)^2}\,\mathrm{d}y^2.$$

Proof. I will show the claim that $(v \cdot \xi)^2 \leq \xi^T A \xi$ for all $\xi \in \mathbb{R}^n$ if and only if $v^T A^{-1} v \leq 1$.

[1] In general, say the PDEs are degenerate elliptic if the matrix A is positive semi-definite. This thesis is concerned with the infinitely degenerate PDEs, i.e., assuming f is smooth at the origin, i.e., f cannot be represented by power series centered at the origin.

Suppose $(v \cdot \xi)^2 \leq \xi^T A \xi$ for all $\xi \in \mathbb{R}^n$, then

$$(v^T A^{-1} v)^2 = (v \cdot A^{-1} v)^2 \leq (A^{-1} v)^T A (A^{-1} v) = v^T A^{-T} v = v^T A^{-1} v.$$

Now I will show the converse direction. Because A is positive semi-definite and symmetric, the matrix $\sqrt{A}$ exists and is symmetric. Suppose $v^T A^{-1} v \leq 1$, then following the Cauchy-Schwarz inequality,

$$(v \cdot \xi)^2 = \langle \sqrt{A} A^{-1} v, \sqrt{A}^T \xi \rangle \leq |\sqrt{A} A^{-1} v|^2 |\sqrt{A}^T \xi|^2 = (v^T A^{-1} v)(\xi^T A \xi) \leq (\xi^T A \xi).$$

Thus, writing $\gamma(t) = (x(t), y(t))$, one obtain

$$1 \geq (\frac{\partial x}{\partial t})^2 + \frac{1}{f^2}(\frac{\partial y}{\partial t})^2 \quad \Longleftrightarrow \quad \mathrm{d}t^2 \geq \mathrm{d}x^2 + \frac{1}{f^2}\,\mathrm{d}y^2.$$

Recall that the Carathéodory distance $\rho(x, y)$ is $\inf\{T : \gamma[0, T] \to \mathbb{R}^n, \gamma \text{ connects } x, y\}$. Consider x, y to be infinitesimally close. For any admissible curve connecting x, y, one can always reparametrize the curve until the identity above is an equality. Then, indeed, abusing the notation to let t denote the Carathéodory distance, there is

$$\mathrm{d}t^2 = \mathrm{d}x^2 + \frac{1}{f^2}\,\mathrm{d}y^2.$$

□

Since the distance is defined by the Carathéodory metric, one can define Carathéodory metric balls. In this thesis, I will only consider balls centered at the origin. Due to the symmetry of the Carathéodory metric around the origin, the Carathéodory metric ball will be symmetrically consisting of its portion in the first quadrant and the reflections of that portion into other quadrants. Therefore I will only consider the metric ball in the first quadrant in the computations that follow.

Definition 2.1.2. Call $B_r := \{(x, y) \in \mathbb{R}^2 : \rho((0,0),(x,y)) \leq r \text{ and } x, y > 0\}$ the *metric ball of radius* r. This is a shorthand for the first quadrant of the Carathéodory metric ball with radius r centered at the origin. See figure 2.1

Denote the area of B_r by $Area(B_r)$.

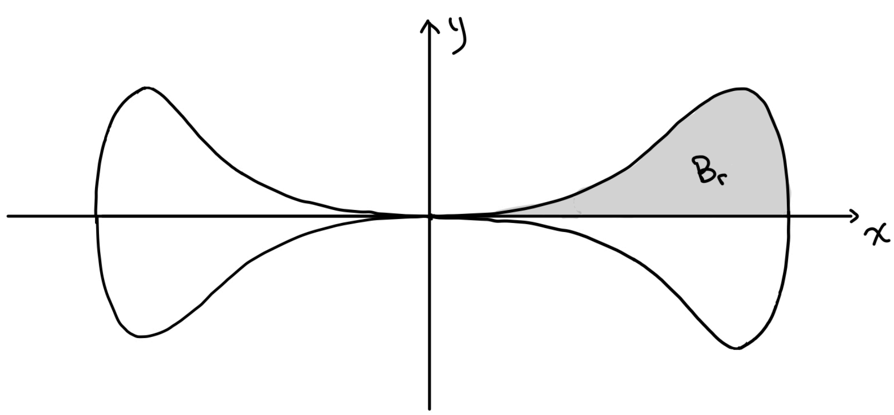

Figure 2.1: The metric ball of radius r and B_r.

2.1.2 Description of the geodesics

I am interested in calculating the asymptote of $Area(B_r)$ when $r \to 0$. One of the main tools that could be utilized is the description of the geodesics in the Carathéodory metric.

Geodesics are analogous to lines in the Euclidean metric. Lines minimize the distance between two points in the Euclidean metric, and geodesics in the Carathéodory metric minimize the distance between any two points in the Carathéodory metric. Formally, if γ are admissible curves from a to b, then γ_0 is called the geodesic if and only if for all γ,

$$\int_{\gamma_0} |\,\mathrm{d}t| \leq \int_{\gamma} |\,\mathrm{d}t|.$$

A general strategy to obtain the descriptions of geodesics is to apply the Euler-Lagrange equations. In our setting, by Euler Lagrange γ is a geodesic that minimizes

$$\int_{\gamma} |\,\mathrm{d}t| = \int_{\gamma} \sqrt{1 + \frac{1}{f(x)^2} \cdot \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.$$

if and only if it is admissible and for every points (x, y) on γ,

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{f(x)^2 \sqrt{1 + \frac{1}{f(x)^2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}} = 0.$$

This makes it possible to define a constant λ invariant along each geodesic by simply defining

$$\lambda = \frac{f(x)^2 \sqrt{1 + \frac{1}{f(x)^2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}}{\frac{\mathrm{d}y}{\mathrm{d}x}}. \tag{$\triangle$}$$

Rearrange the expression to obtain the slope of geodesic in terms of λ and coordinate x,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm \frac{f(x)^2}{\sqrt{\lambda^2 - f(x)^2}}. \tag{$\diamond$}$$

Definition 2.1.3. In ($\triangle$), call λ the *label* of the geodesic it denotes. Call the point $x > 0$ such that $f(x) = \lambda$ as the *turning point* of the geodesic λ, if the point exists. See figure 2.2.

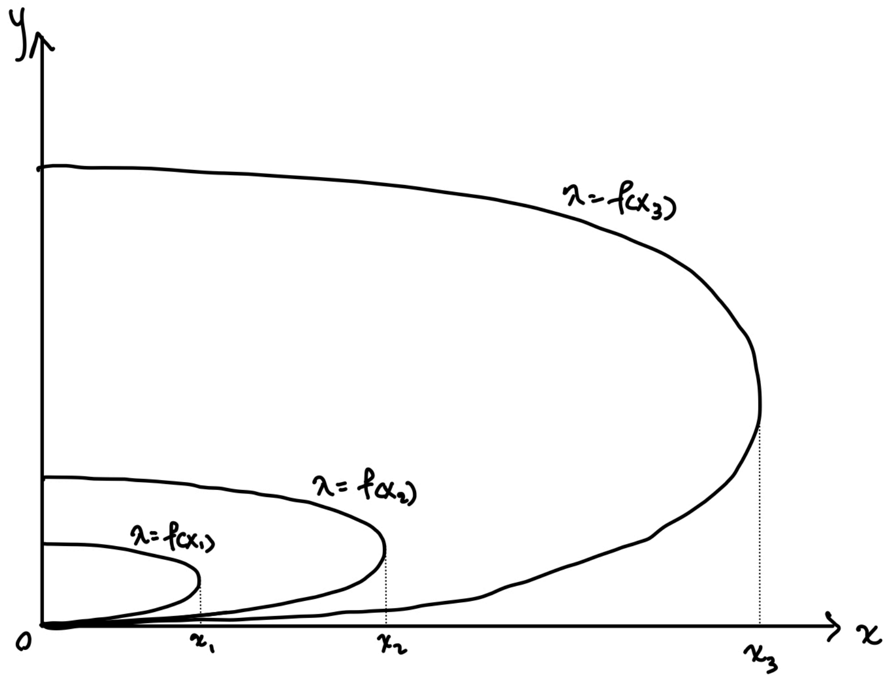

Figure 2.2: Geodesic tiling of $\mathbb{R}^2$.

Corollary 2.1.4. Some consequences are:

1. Labels $\lambda > 0$ are in bijection with the geodesics through the origin in the first quadrant.
2. Labels $\lambda > 0$ are in bijection with turning points $x > 0$.

Sketch of the proof. For (1), the only thing that should be checked is that the construction of the curves corresponding to $\lambda > 0$ is admissible. But since I am assuming the metric in Proposition 2.1.1, the only non-admissible curves possible are ones with non-zero slopes at the origin. The geodesics corresponding to all $\lambda \geq 0$ have zero slope at the origin. (2) follows the assumption that f is strictly increasing. □

Proposition 2.1.5. Along geodesic λ,

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\lambda}{\sqrt{\lambda^2 - f(x)^2}}.$$

Proof. Following

$$\mathrm{d}t^2 = \mathrm{d}x^2 + \frac{1}{f(x)^2}\,\mathrm{d}y^2,$$

one computes

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \sqrt{1 + \frac{1}{f(x)^2}\cdot\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{1 + \frac{f(x)^4}{f(x)^2(\lambda^2 - f(x)^2)}} = \frac{\lambda}{\sqrt{\lambda^2 - f(x)^2}}.$$

□

Definition 2.1.6. In the following sections, I will use the following functions extensively.

Define $X : \mathbb{R}_+ \to \mathbb{R}_+$ to be the identity function.

Define $Y : \mathbb{R}_+ \to \mathbb{R}_+$ to denote the $y-$coordinate of the turning point with parameter x, i.e.,

$$Y(x) = \int_0^x \frac{f(u)^2}{\sqrt{f(x)^2 - f(u)^2}}\,\mathrm{d}u = \int_0^x \frac{\mathrm{d}y}{\mathrm{d}u}(f(x))(u)\,\mathrm{d}u,$$

where $\frac{\mathrm{d}y}{\mathrm{d}u}(f(x))(u)$ denote that $\frac{\mathrm{d}y}{\mathrm{d}u}$ is defined along the geodesic $\lambda = f(x)$, and the argument is the variable u.

Define $R : \mathbb{R}_+ \to \mathbb{R}_+$ to denote the distance travelled along the geodesic $f(x)$ up to the turning point, i.e.,

$$R(x) = \int_0^x \frac{f(x)}{\sqrt{f(x)^2 - f(u)^2}}\,\mathrm{d}u = \int_0^x \frac{\mathrm{d}t}{\mathrm{d}u}(f(x))(u)\,\mathrm{d}u,$$

where again the parameter $f(x)$ denotes that $\frac{\mathrm{d}t}{\mathrm{d}u}$ is defined along the geodesic $f(x)$.

Proposition 2.1.7. Here are some qualitative descriptions of the behavior of the geodesics.

1. The geodesic $\lambda = \infty$ is horizontal.

2. Suppose the geodesic λ turns back at its turning point x characterized by $\lambda = f(x)$, then its path is symmetric about the line $y = Y(x)$.

Proof. For 2, the $y-$coordinate of the geodesic λ at x is given by[2]

$$y_\lambda(x) = \int_0^x \frac{f(u)^2}{\sqrt{\lambda^2 - f(u)^2}}\,\mathrm{d}u,$$

and after the geodesic turns back, the negative slope is assumed (when x increases). □

An important geometric identity first recorded in Lemma 48 of Korobenko et al. (2021) provides associations between different intrinsic quantities of the geodesics.

Proposition 2.1.8. Along any segments of geodesic λ, let $\mathrm{d}y$ denote the difference in height and $\mathrm{d}(t-x)$ represent the difference in distance r minus difference in coordinate x. Then

$$\mathrm{d}y \geq \lambda\,\mathrm{d}(t-x) \geq \frac{\mathrm{d}y}{2}.$$

Proof. Because $1 - \sqrt{1-x} = \frac{x}{1+\sqrt{1-x}}$, for $x < 1$

$$x \geq 1 - \sqrt{1-x} \geq \frac{x}{2}.$$

[2]This is true because I am only considering geodesics that go into the first quadrant.

Using the identity $1-\sqrt{1-x}\approx x$, it follows that locally

$$\frac{\mathrm{d}(t-x)}{\mathrm{d}x}=\frac{\mathrm{d}t}{\mathrm{d}x}-1=\frac{\lambda-\sqrt{\lambda^2-f(x)^2}}{\sqrt{\lambda^2-f(x)^2}}\approx\frac{1}{\lambda}\cdot\frac{f(x)^2}{\sqrt{\lambda^2-f(x)^2}}=\frac{1}{\lambda}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Then globally along any geodesic λ the same identity follows as well. □

2.2 Computations

I will assume the following assumption for the later computations.

Assumption 2.2.1. Let $F(x)=-\ln(f(x))$. Since f is strictly increasing and smooth at the origin, F strictly decreases from infinity. Then, I will assume that for $x>0$,

$$-F'(x) \text{ is decreasing}$$

and

$$\lim_{x\to 0}\frac{-F'(x)}{-F'(x\pm\frac{k}{-F'(x)})}=1 \text{ for all constants } k>0.$$

Corollary 2.2.2. Fix $k>1$ to be a constant, then

$$kf(\alpha(x))=f(x) \quad\Rightarrow\quad \lim_{x\to 0}\frac{F'(\alpha(x))}{F'(x)}=1 \tag{1}$$

where $\alpha(x)>0$ is a unique value dependent on x and k such that the first condition holds.

Proof. Compute that, for $\alpha(x)=x-\Delta x$,

$$\ln k=\ln\frac{f(x))}{f(\alpha(x))}=F(\alpha(x))-F(x)=\int_{\alpha(x)}^{x}-F'(u)\,\mathrm{d}u=\int_{x-\Delta x}^{x}-F'(u)\,\mathrm{d}u.$$

Since it is assumed that $-F'(x)$ is decreasing,

$$\ln k=\int_{x-\Delta x}^{x}-F'(u)\,\mathrm{d}u>-F'(x)\cdot\Delta x,$$

so $\alpha(x)=x-\Delta x>x-\frac{\ln k}{-F'(x)}$. Now, by assumption,

$$\lim_{x\to 0}\frac{F'(\alpha(x))}{F'(x)}=1.$$

□

By almost the exact same logic, one obtains another corollary.

Corollary 2.2.3. Suppose

$$\lim_{x\to 0}(x-\alpha(x))\cdot -F'(x) = \ln k$$

then

$$\lim_{x\to 0}\frac{f(x)}{f(\alpha(x))} = k.$$

Chapter 7 of Korobenko et al. (2021) studied functions that can be compared to $e^{-\frac{1}{x^p}}$ near the origin. The example below shows they also satisfy the assumption here.

Example 2.2.4. Let $f = e^{-\frac{1}{x^p}}$ for $p > 0$, and let $k > 0$ be a constant. Then $F(x) = \frac{1}{x^p}$ and $-F'(x) = \frac{p}{x^{p+1}}$. Thus $F'(x)$ is decreasing for $x > 0$, and

$$\lim_{x\to 0}\frac{-F'(x)}{-F'(x\pm\frac{k}{-F'(x)})} = \lim_{x\to 0}\frac{(x\pm k/p\cdot x^{p+1})^{p+1}}{x^{p+1}} = \lim_{x\to 0}(1\pm k/p\cdot x^p)^{p+1}\to 1.$$

In fact, the assumption should hold for any iterations of

$$F = e^{e^{\cdots e^{\frac{1}{x^p}}}},$$

where $p > 0$. Below is a Mathematica plot of the convergence when $F = e^{e^{\frac{1}{x}}}$, and $k = 1000$.

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F[x_] := E^E^(1/x)
Plot[-F'[x] / -F'[x - 1000/(-F'[x])], {x, 0, .6}]
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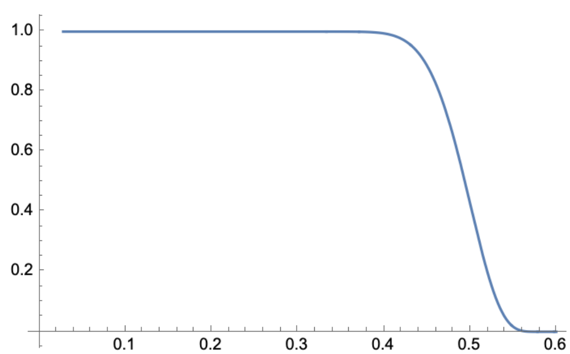

A daring conjecture is that the assumption holds for all strictly increasing f smooth at the origin.[3]

In the contents that follows I will adopt the following notion. Say $a \to b$ when $r \to 0$ if and only if $\lim_{r\to 0}\frac{a}{b} = 1$. The clause $r \to 0$ could be implicit if it is obvious.

[3] The assumption, however, is not true if one does not assume f does not have power series at the origin. One can easily check that the assumption does not hold for $f(x) = x$.

Proposition 2.2.5. For $x > 0$, and functions R, X as defined in Definition 2.1.6,

$$(R - X)(x) \cdot -F'(x) \leq \ln 2 \quad \text{and} \quad (R - X)(x) \to \frac{\ln 2}{-F'(x)} \quad \text{when} \quad x \to 0.$$

Also, $R - X$ is increasing, so R is increasing.

Proof. Recall that the unique geodesic turning around at x is labelled $f(x)$, so compute

$$\begin{aligned}
(R - X)(x) &= \int_0^x \frac{f(x) - \sqrt{f(x)^2 - f(u)^2}}{\sqrt{f(x)^2 - f(u)^2}} \,\mathrm{d}u \\
&= \int_0^x \frac{1 - \sqrt{1 - f(u)^2/f(x)^2}}{\sqrt{1 - f(u)^2/f(x)^2}} \,\mathrm{d}u \\
&= \int_0^x \frac{f(u)^2/f(x)^2}{(1 + \sqrt{1 - f(u)^2/f(x)^2})\sqrt{1 - f(u)^2/f(x)^2}} \,\mathrm{d}u \\
&= \int_0^x \frac{f(u)^2/f(x)^2}{(f(u)^2/f(x)^2)'} \cdot -\frac{\mathrm{d}}{\mathrm{d}u} 2\ln(1 + \sqrt{1 - f(u)^2/f(x)^2}) \,\mathrm{d}u \\
&= \int_0^x \frac{1}{-F'(u)} \cdot -\frac{\mathrm{d}}{\mathrm{d}u} \ln(1 + \sqrt{1 - f(u)^2/f(x)^2}) \,\mathrm{d}u.
\end{aligned}$$

An integration by parts gives

$$\begin{aligned}
(R - X)(x) = {} & \frac{1}{-F'(u)} \cdot - \ln(1 + \sqrt{1 - f(u)^2/f(x)^2})\Big|_0^x \\
& + \int_0^x \left(\frac{1}{-F'(u)}\right)' \cdot \ln(1 + \sqrt{1 - f(u)^2/f(x)^2}) \,\mathrm{d}u \\
= {} & \int_0^x \left(\frac{1}{-F'(u)}\right)' \cdot \ln(1 + \sqrt{1 - f(u)^2/f(x)^2}) \,\mathrm{d}u.
\end{aligned}$$

Observe that $R - X$ is increasing.

To show $\lim_{x\to 0}(R - X)(x) \cdot -F'(x) = \ln 2$, it is sufficient to show that for any $\varepsilon > 0$, when $x \to 0$ it holds

$$(R - X)(x) \leq \frac{\ln 2}{-F'(x)} \leq (1 + \varepsilon)(R - X)(x).$$

Since $\ln(1 + \sqrt{1 - f(u)^2/f(x)^2}) \leq \ln 2$, immediately

$$(R - X)(x) \leq \frac{\ln 2}{-F'(x)}.$$

To show the other inequality, use the contrapositive of Corollary 2.2.2. Fix $\varepsilon > 0$, and $\alpha(x)$ dependent on x and ε such that $F'(x) = (1 + \varepsilon)F'(\alpha(x))$. It is sufficient to show that

$$(R - X)(x) > (1 - \varepsilon)\frac{\ln 2}{-F'(x)},$$

or, equivalently, show that

$$\frac{\ln 2}{-F'(x)} - (R-X)(x) < C\varepsilon \frac{\ln 2}{-F'(x)},$$

where $C > 0$ is some constant.

Notice that

$$\frac{\ln 2}{-F'(x)} - (R-X)(x) = \int_0^x \left(\frac{1}{-F'(u)}\right)' \cdot (\ln 2 - \ln(1+\sqrt{1-f(u)^2/f(x)^2}))\,\mathrm{d}u$$

Split the integral into two parts: evaluated from 0 to $\alpha(x)$ and from $\alpha(x)$ to x. Compute that for the first part the difference is

$$\begin{aligned}
&\int_0^{\alpha(x)} \left(\frac{1}{-F'(u)}\right)' \cdot (\ln 2 - \ln(1+\sqrt{1-f(u)^2/f(x)^2})\,\mathrm{d}u. \\
&\le \int_0^{\alpha(x)} \left(\frac{1}{-F'(u)}\right)' \cdot (\ln 2 - \ln(1+\sqrt{1-f(\alpha)^2/f(x)^2})\,\mathrm{d}u \\
&= (\ln 2 - \ln(1+\sqrt{1-f(\alpha)^2/f(x)^2})F'(\alpha(x)).
\end{aligned}$$

But by construction of α, one can use Assumption 2.2.1 to deduce that $\lim_{x\to 0} \frac{f(\alpha(x))}{f(x)} \to 0$. Therefore the difference could be less than $\varepsilon \cdot \frac{1}{-F'(x)}$.

The difference in the second part is

$$\int_{\alpha(x)}^x \left(\frac{1}{-F'(u)}\right)' \cdot (\ln 2 - \ln(1+\sqrt{1-f(u)^2/f(x)^2})\,\mathrm{d}u) < \ln 2 \left(\frac{1}{-F'(x)} - \frac{1}{-F'(\alpha(x))}\right).$$

Here one use the construction to finish the proof:

$$\frac{-F'(x)}{-F'(\alpha(x))} = 1+\varepsilon \;\Rightarrow\; 1 - \frac{-F'(x)}{-F'(\alpha(x))} = \varepsilon \;\Rightarrow\; \frac{1}{-F'(x)} - \frac{1}{-F'(\alpha(x))} = \varepsilon \frac{1}{-F'(x)}.$$

□

Remark. In the following passages, I will use the notation $a \to b$ when $c \to 0$ to mean

$$\lim_{c\to 0} \frac{a}{b} = 1.$$

For example, I have just shown when $x \to 0$,

$$(R-X)(x) \to \frac{\ln 2}{-F'(x)}.$$

Also notice that in fact both sides of $\to$ go to zero. One trick that I will repeatedly use later is that by multiplying appropriate terms on both sides, for example, $-F'(r)$ in this case, both sides will converge to a non-zero value. So when $x \to 0$,

$$-F'(x) \cdot (R-X)(x) \to \ln 2.$$

Proposition 2.2.6. When $x \to 0$.

$$Y(x) \to \frac{f(x)}{-F'(x)}.$$

Proof. Compute that

$$\begin{aligned} Y(x) = \int_0^x \frac{f(u)^2}{\sqrt{f(x)^2 - f(u)^2}}\, \mathrm{d}u &= f(x) \int_0^x \frac{f(u)^2/f(x)^2}{\sqrt{1 - f(u)^2/f(x)^2}}\, \mathrm{d}u \\ &= f(x) \int_0^x \frac{1}{-2F'(u)} \cdot \frac{(f(u)^2/f(x)^2)'}{\sqrt{1 - f(u)^2/f(x)^2}}\, \mathrm{d}u \\ &= f(x) \int_0^x \left(\frac{1}{-F'(u)}\right)' \cdot \sqrt{1 - f(u)^2/f(x)^2}\, \mathrm{d}u. \end{aligned}$$

The error estimation follows the same logic as the proof of Proposition 2.2.5. □

Corollary 2.2.7. Consider $B(r)$, since $R-X$ is increasing - so R is strictly increasing - there is a unique geodesic $f(r^*)$ such that its intersection with the boundary of the metric ball is its turning point. It holds when $x \to 0$,

$$r - r^* \to \frac{\ln 2}{-F'(r)}$$

and

$$\frac{f(r^*)}{f(r)} \to \frac{1}{2}.$$

Proof. Fix $r > 0$. The trick is to notice that if $f(r^*)$ is the geodesic that crosses the boundary of B_r at the point with $x-$coordinate r^*, then by definition of the metric ball the distance the geodesic travelled up to the turning point is r, that is $R(r^*) = r$. Thus,

$$r - r^* = (R - X)(r^*).$$

Then by Proposition 2.2.5,

$$\lim_{r\to 0}(r - r^*) \cdot -F'(r^*) = \ln 2.$$

By Assumption 2.2.1, this implies

$$\lim_{r\to 0} \frac{-F'(r^*)}{-F'(r)} = 1$$

so indeed

$$\lim_{r\to 0}(r - r^*) \cdot -F'(r) = \ln 2.$$

Then, by Corollary 2.2.3,

$$\lim_{r\to 0} \frac{f(r^*)}{f(r)} = \frac{1}{2}.$$

□

Definition 2.2.8. Split the metric ball B_r vertically. Call the area left of r^* as *Area1* of the metric ball, and the area right of r^* as *Area2*. [4]

Here's a proposition listed as Proposition 47 (1) in Korobenko et al. (2021).

Proposition 2.2.9. Let $\varphi(x)$ denote the y-coordinate of the boundary of B_r at x. Then $\varphi(x)$ is well defined. Further, $\varphi(x)$ is decreasing when $x > r^*$.

Proof. Assume $x > 0$. As a direct corollary to $R(x)$ is increasing, concluded in Proposition 2.2.5, $\varphi(x)$ is well-defined in Area 2.

To check well-definedness in Area 1, for $\lambda_1 < \lambda_2$, check that $\frac{\mathrm{d}t}{\mathrm{d}x}(\lambda_1) > \frac{\mathrm{d}t}{\mathrm{d}x}(\lambda_2)$ at every x to the left of the turning point of λ_1. Then, if φ is not well defined, then there are two situations.

1. Geodesic λ_1 has turned back but λ_2 did not. This is impossible since $\frac{\mathrm{d}t}{\mathrm{d}x}(\lambda_1) > \frac{\mathrm{d}t}{\mathrm{d}x}(\lambda_2)$, the distance covered by λ_2 is strictly smaller.
2. Both geodesics have turned back. This is impossible. Since $\frac{\mathrm{d}t}{\mathrm{d}x}(\lambda_1) > \frac{\mathrm{d}t}{\mathrm{d}x}(\lambda_2)$, the distance not covered by λ_1 until the geodesic crosses the y-axis is strictly bigger than the distance not covered by λ_2. By symmetry of $\frac{\mathrm{d}t}{\mathrm{d}x}$ before and after turning back, this would imply that $2R_{\lambda_1} > 2R_{\lambda_2}$, a contradiction to Proposition 2.2.5.

At position $x \geq r^*$, let $\lambda(x)$ label the unique geodesic that reach distance r at x without turning back, i.e.,

$$r = \int_0^x \frac{\mathrm{d}t}{\mathrm{d}u}(\lambda(x))\,\mathrm{d}u = \int_0^x \frac{\lambda(x)}{\sqrt{\lambda(x)^2 - f(u)^2}}\,\mathrm{d}u.$$

Then by construction

$$\varphi(x) = \int_0^x \frac{\mathrm{d}y}{\mathrm{d}u}(\lambda(x))\,\mathrm{d}u = \int_0^x \frac{f(u)^2}{\sqrt{\lambda(x)^2 - f(u)^2}}\,\mathrm{d}u.$$

Differentiate both expressions above with respect to x to obtain

$$0 = \frac{\lambda(x)}{\sqrt{\lambda(x)^2 - f(x)^2}} - \left(\int_0^x \frac{f(u)^2}{\left(\lambda(x)^2 - f(u)^2\right)^{3/2}}\,\mathrm{d}u\right)\lambda'(x)$$

and

$$\varphi'(x) = \frac{f(x)^2}{\sqrt{\lambda(x)^2 - f(x)^2}} - \left(\int_0^x \frac{f(u)^2}{\left(\lambda(x)^2 - f(u)^2\right)^{3/2}}\,\mathrm{d}u\right)\lambda(x)\lambda'(x)$$

Combining equalities yields

$$\varphi'(x) = \frac{f(x)^2}{\sqrt{\lambda(x)^2 - f(x)^2}} - \lambda(x)\frac{\lambda(x)}{\sqrt{\lambda(x)^2 - f(x)^2}} = -\sqrt{\lambda(x)^2 - f(x)^2}.$$

□

[4] However, recall that B_r is only the first quadrant of the actual metric ball with radius r. By symmetry, the area of the actual metric ball is four times $Area(B_r)$.

Proposition 2.2.10. Let $x < r^*$. If the geodesic λ crosses the boundary of B_r at x, I will label $\lambda = f(x + \Delta x)$ for $x + \Delta x < r^*$ be the turning point of the geodesic. Then, for every $\varepsilon > 0$, the following inequality holds for r is sufficiently small:

$$-F'(x+\Delta x)(x+\Delta x)+\ln 2 \geq -F'(x+\Delta x)\cdot\frac{r+x}{2} \geq -F'(x+\Delta x)(x+\Delta x)+\frac{1}{2}\ln 2-\varepsilon.$$

See figure 2.3 for reference.[5]

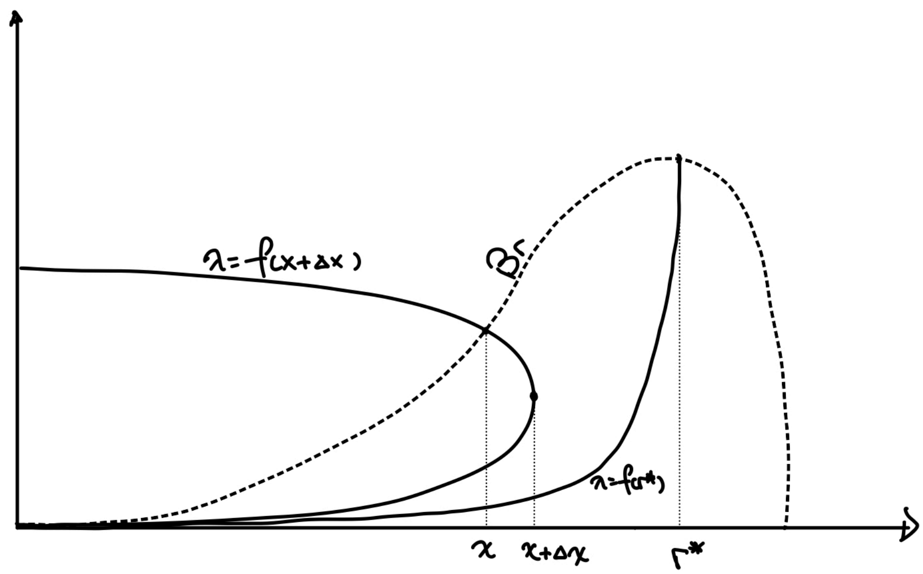

Figure 2.3: Demo 1.

Proof. Along the geodesic $f(x + \Delta x)$, define the function

$$(T_{f(x+\Delta x)} - X)(x) := \int_x^0 -\frac{\mathrm{d}t}{\mathrm{d}u}(f(x+\Delta x))\,\mathrm{d}u - x = \int_0^x \frac{\mathrm{d}t}{\mathrm{d}u}(f(x+\Delta x))\,\mathrm{d}u - x.$$

Then $(T_{f(x+\Delta x)} - X)(x)$ measures the magnitude of distance minus the $x-$coordinate traversed along the segment of the geodesic $f(x + \Delta x)$ from its intersection with the boundary of B_r to its intersection with the $y-$axis.[6] One can estimate that

$$0 \leq (T_{f(x+\Delta x)} - X)(x) \leq (R - X)(x + \Delta x).$$

[5]For small r, this proposition gives a fairly well estimate for the turning point of the unique geodesic that reaches distance r when intersecting the $y-$axis. Since $x >> \frac{1}{-F'(x)}$, the turning point would be roughly $r/2$.

[6]The main difference between T and R is that when specifying T, I have to give two arguments: the geodesic along which the measure is done, and the endpoint of the measure (recall that the measure starts from zero). However, when specifying R, I only give one argument, and that is the turning point of the geodesic. It is assumed that the measure is obtained from 0 up to the turning point.

Notice that by construction

$$r + x = 2R(x + \Delta x) - (T_{f(x+\Delta x)} - X)(x).$$

Thus, the following inequalities are obtained:

$$2R(x + \Delta x) \geq r + x \geq 2R(x + \Delta x) - (R - X)(x + \Delta x).$$

The inequalities above is equivalent to

$$2(R-X)(x+\Delta x) \geq r+x-2(x+\Delta x) \geq 2(R-X)(x+\Delta x)-(R-X)(x+\Delta x) = (R-X)(x+\Delta x).$$

When r is sufficiently small, and thus x is also sufficiently small, using Proposition 2.2.5 this implies

$$-F'(x+\Delta x)(x+\Delta x)+\ln 2 \geq -F'(x+\Delta x)\cdot\frac{r+x}{2} \geq -F'(x+\Delta x)(x+\Delta x)+\frac{1}{2}\ln 2-\varepsilon$$

for arbitrarily small $\varepsilon > 0$. So the statement in the proposition holds. $\square$

Corollary 2.2.11. For variable $\omega \geq 0$, define $x(\omega) = r^* - \frac{\omega}{-F'(r)}$. Then

$$\lim_{r\to 0}\frac{\Delta x(\omega)}{r^* - x(\omega)} \leq \frac{1}{2}.$$

Proof. Consider the inequality

$$-F'(x + \Delta x)\cdot\frac{r+x}{2} \geq -F'(x + \Delta x)(x + \Delta x) + \frac{1}{2}\ln 2$$

is equivalent to

$$-F'(x + \Delta x)\cdot(r - x) - \ln 2 \geq -2F'(x + \Delta x)\Delta x.$$

Notice that by Assumption 2.2.1 and Corollary 2.2.7,

$$\lim_{r\to 0} -F'(x + \Delta x)\cdot(r - x) = \lim_{r\to 0}\frac{-F'(x + \Delta x)}{-F'(r)}\cdot\lim_{r\to 0} -F'(r)\cdot(r - x) = \omega + \ln 2 > 0$$

so

$$\lim_{r\to 0} -F'(x + \Delta x)\cdot(r - x) - \ln 2 = \omega > 0.$$

Then, substitute this into the inequality above, cancel $-F'(x + \Delta x)$ on both sides and obtain

$$\lim_{r\to 0}\frac{\Delta x(\omega)}{r^* - x(\omega)} \leq \frac{1}{2}.$$

$\square$

Corollary 2.2.12.

$$\frac{F'(r)^2}{f(r)}Area1(B_r) \leq 8\ln 2.$$

Proof. By Assumption 2.2.1, for every $\varepsilon > 0$ there is $r > 0$ such that whenever $x < r$,

$$\frac{-F'(x - \frac{\ln 2}{-F'(x)})}{-F'(x)} > 1 - \varepsilon.$$

Repeating the idea in the proof of Corollary 2.2.2, the previous proposition allows the estimation [7]

$$F(x+\Delta x) - F\left(\frac{r+x}{2}\right) = \int_{x+\Delta x}^{\frac{r+x}{2}} -F'(u)\,\mathrm{d}u \geq \frac{\ln 2}{-2F'(x+\Delta x)} \cdot -F'\left(\frac{r+x}{2}\right) > (1-\varepsilon)\ln 2.$$

Thus

$$\frac{f(x+\Delta x)}{f\left(\frac{r+x}{2}\right)} = e^{-\left(F(x+\Delta x) - F\left(\frac{r+x}{2}\right)\right)} < c$$

where $c = 2^{-(1-\varepsilon)}$.

Using $Y(x)$ defined in Definition 2.1.6, an upper bound of $Area1(B_r)$ is

$$Area1(B_r) \leq \int_0^{r^*} 2Y(x+\Delta x)\,\mathrm{d}x$$

By Proposition 2.1.8 for the first inequality and the monotonicity of $R - X$ for the second, one can estimate

$$Y(x+\Delta x) \leq 2f(x+\Delta x)(R-X)(x+\Delta x) \leq 2cf\left(\frac{r+x}{2}\right)(R-X)\left(\frac{r+x}{2}\right).$$

So by a change of variable for the second equality, Proposition 2.2.5 for the third inequality, and recalling definition of $f(x) = e^{-F(x)}$ and the monotonicity of $\frac{1}{-F'(x)}$ for the fourth inequality

$$\begin{aligned}
Area1(B_r) &\leq 4c\int_0^{r^*} f\left(\frac{r+x}{2}\right)(R-X)\left(\frac{r+x}{2}\right)\mathrm{d}x \\
&= 8c\int_{\frac{r}{2}}^{\frac{r+r^*}{2}} f(u)(R-X)(u)\,\mathrm{d}u \\
&\leq 8c \cdot \ln 2 \int_0^r \frac{f(u)}{-F'(u)}\,\mathrm{d}u \\
&\leq c_1 \cdot \frac{1}{F'(r)^2}\int_0^r f'(u)\,\mathrm{d}u \\
&= c_1 \cdot \frac{f(r)}{F'(r)^2}
\end{aligned}$$

where $c_1 = \frac{8\ln 2}{2^{(1-\varepsilon)}} \leq 8\ln 2$ for all r.

□

[7] Note that by the previous proposition, $\frac{r+x}{2} > x + \Delta x$, so $F(x+\Delta x) > F(\frac{r+x}{2})$. Also recall $-F'(u)$ is positive and decreasing by Assumption 2.2.1.

Proposition 2.2.13. For variable $\omega \geq 0$ and for very small r, define $x(\omega) = r^* - \frac{\omega}{-F'(r)}$. Presume this denote the $x-$coordinate of the intersection point of a unique geodesic and the boundary of B_r, then denote the turning point of this geodesic be $(x + \Delta x)(\omega)$, i.e, the geodesic is labeled by $f((x + \Delta x)(\omega))$. Then,

$$-F'(r) \cdot \Delta x(\omega) \to \frac{1}{2}\omega - \frac{1}{2}\ln(2 - e^{-w}).$$

Proof. Assume $\omega \neq 0$. The geodesic $f((x + \Delta x)(\omega))$ has traveled distance $(R - X)((x + \Delta x)(\omega)) + (x + \Delta x)(\omega)$ up to the turning point, and distance r up to $x(\omega)$.[8] Then one can obtain a description of $\Delta x(\omega)$ be

$$r - (R - X)((x + \Delta x)(\omega)) - (x + \Delta x)(\omega) = \int_{x+\Delta x}^{x} -\frac{\mathrm{d}t}{\mathrm{d}u}(f((x + \Delta x)(\omega)))\,\mathrm{d}u.$$

Recall $x(\omega) = r^* - \frac{\omega}{-F'(r)}$, so by Assumption 2.2.1,[9] and recall $r - r^* = (R - X)(r)$ from Corollary 2.2.7. Multiply both side of the fraction by $-F'(r)$, the LHS (left hand side) can be approximated by

$$\begin{aligned}
&\lim_{r\to 0} \frac{-F'(r)(r - (R - X)((x + \Delta x)(\omega)) - (x + \Delta x)(\omega))}{-F'(r)(r^* - (x + \Delta x)(\omega))}\\
=&\lim_{r\to 0} \frac{-F'(r)(r^* - x(\omega)) - (-F'(r))\Delta x(\omega) + (-F'(r))((r - r^*) - (R - X)((x + \Delta x)(\omega))}{-F'(r)(r^* - x(\omega)) - (-F'(r))\Delta x(\omega)}\\
=&1 - \lim_{r\to 0} \frac{-F'(r)((R - X)(r) - (R - X)((x + \Delta x)(\omega)))}{\omega - (-F'(r))\Delta x(\omega)}.
\end{aligned}$$

The denominator is positive, since, by Corollary 2.2.11, $\lim_{r\to 0} \omega - (-F'(r))\Delta x(\omega) \geq \frac{1}{2}\omega > 0$. The numerator vanishes by Assumption 2.2.1 and Proposition 2.2.5. Therefore one obtain

$$\lim_{r\to 0} \frac{-F'(r)(r - (R - X)((x + \Delta x)(\omega)) - (x + \Delta x)(\omega))}{-F'(r)(r^* - (x + \Delta x)(\omega))} = 1.$$

Rearranging the previous two identities and times $-F'(r)$ on both the denominator and the numerator so they don't converge to zero, obtain

$$1 = \lim_{r\to 0} \frac{-F'(r)\int_x^{(x+\Delta x)(\omega)} \frac{1}{\sqrt{1-f(u)^2/f(x)^2}}\,\mathrm{d}u}{-F'(r) \cdot (r^* - (x + \Delta x)(\omega))} = \lim_{r\to 0} \frac{-F'(r)\int_x^{(x+\Delta x)(\omega)} \frac{1}{\sqrt{1-f(u)^2/f(x)^2}}\,\mathrm{d}u}{\omega - (-F'(r)) \cdot \Delta x(w)}.$$

Equivalently, by positivity, the following equality holds non-trivially:

$$\lim_{r\to 0} \omega - (-F'(r))\Delta x(w) = \lim_{r\to 0} -F'(r) \cdot \int_x^{(x+\Delta x)(\omega)} \frac{1}{\sqrt{1 - f(u)^2/f(x)^2}}\,\mathrm{d}u.$$

[8] That is, including entire the segment before the turning point, and the segment after the turning point up to $x(\omega)$

[9] It's important to note that when taking the limit $r \to 0$, w remains constant.

Now, again by Assumption 2.2.1, $\lim_{r\to 0}\frac{-F'(u)}{-F'(r)} = 1$ for all u such that $x(\omega) \le u \le (x+\Delta x)(\omega)$, so one can approximate the RHS via a change of variable

$$\begin{aligned}
&\lim_{r\to 0} -F'(r)\cdot \int_x^{(x+\Delta x)(\omega)} \frac{1}{\sqrt{1-f(u)^2/f(x)^2}}\,\mathrm{d}u \\
=&\lim_{r\to 0} -F'(r)\cdot \int_x^{(x+\Delta x)(\omega)} \frac{1}{-F'(u)}\cdot\frac{1}{\frac{f(u)}{f(x)}\sqrt{1-(\frac{f(u)}{f(x)})^2}}\,\mathrm{d}\frac{f(u)}{f(x)} \\
=&\lim_{r\to 0} \int_x^{(x+\Delta x)(\omega)} \frac{1}{\frac{f(u)}{f(x)}\sqrt{1-(\frac{f(u)}{f(x)})^2}}\,\mathrm{d}\frac{f(u)}{f(x)} \\
=&\lim_{r\to 0} \operatorname{arctanh}\sqrt{1-\left(\frac{f(x(\omega))}{f((x+\Delta x)(\omega))}\right)^2}.
\end{aligned}$$

Since both sides doesn't converge to zero, I can further apply tanh on both sides and rearrange to obtain

$$\lim_{r\to 0}\frac{f(x(\omega))}{f((x+\Delta(x))(\omega))} = \lim_{r\to 0}\sqrt{1-\tanh^2(\omega-(-F'(r))\Delta x(w))} = \lim_{r\to 0}\operatorname{sech}(\omega-(-F'(r))\Delta x(w)).$$

By Corollary 2.2.3 the following identity holds:

$$\lim_{r\to 0}\frac{f(x(\omega))}{f((x+\Delta x)(\omega))} = \lim_{r\to 0} e^{-(-F'(r))\Delta x(w)}.$$

Then, using positivity of $\lim_{r\to 0}\operatorname{sech}(\omega-(-F'(r))\Delta x(w))$, one can interchange a few more limits and ends up with the solution

$$\lim_{r\to 0} -F'(r)\cdot \Delta x(w) = \frac{1}{2}\ln\frac{e^{\omega}}{2-e^{-\omega}} = \frac{1}{2}\omega - \frac{1}{2}\ln(2-e^{-w}).$$

□

Corollary 2.2.14. Here are some corollaries (see figure 2.4 for reference):

1. $$f((x+\Delta x)(\omega)) \to \frac{f(r)}{2\sqrt{2e^{\omega}-1}}.$$

2. With abuse of notation, let $Y(\omega) = Y((x+\Delta x)(\omega))$, where the latter is defined in Definition 2.1.6 and computed in Proposition 2.2.6, then as $r \to 0$,
$$Y(\omega) \to \frac{f(r)}{2\sqrt{2e^{\omega}-1}\cdot -F'(r)}$$

3. Let Δh denote the height that the geodesic $f((x+\Delta x)(\omega))$ continued to gain after turning back and until reaching the boundary of B_r at $x(\omega)$. Then
$$\Delta h(\omega) \to \frac{f(r)}{2\sqrt{2e^{\omega}-1}\cdot -F'(r)}\tanh(\frac{\omega}{2}+\ln(2-e^{-w})).$$

Proof. (1). By Corollary 2.2.3 and Proposition 2.2.13, compute

$$\lim_{r\to 0}\frac{f((x+\Delta x)(\omega))}{f(x(\omega))}=\lim_{r\to 0}e^{-F'(r)\cdot\Delta x(\omega)}=\sqrt{\frac{e^{\omega}}{2-e^{-\omega}}}=\frac{e^{\frac{\omega}{2}}}{\sqrt{2-e^{-\omega}}}.$$

By Corollary 2.2.3 and $x(\omega)=r^*-\frac{\omega}{-F'(r)}$, compute

$$\lim_{r\to 0}\frac{f(x(\omega))}{f(r^*)}=e^{-\omega}.$$

Now, by Corollary 2.2.7,

$$\lim_{r\to 0}\frac{f((x+\Delta x)(\omega))}{f(r)}=\lim_{r\to 0}\frac{f((x+\Delta x)(\omega))}{f(x(\omega))}\cdot\lim_{r\to 0}\frac{f(x(\omega))}{f(r^*)}\cdot\lim_{r\to 0}\frac{f(r^*)}{f(r)}=\frac{1}{2\sqrt{2e^{\omega}-1}}.$$

(2) follows Proposition 2.2.6 straightforwardly.

For (3), since Δh is the height that the geodesic $f(x)$ continued to gain after turning back and until reaching the boundary of B_r,

$$\Delta h(\omega)=f(x)\int_x^{x+\Delta x}\frac{\frac{f(u)^2}{f(x)^2}}{\sqrt{1-\frac{f(u)^2}{f(x)^2}}}\,\mathrm{d}u,$$

So compute

$$\begin{aligned}\lim_{r\to 0}\Delta h(\omega)\cdot\frac{-F'(x)}{f(x)}&=\lim_{r\to 0}\int_x^{x+\Delta x}\frac{\frac{f(u)}{f(x)}}{\sqrt{1-\frac{f(u)^2}{f(x)^2}}}\,\mathrm{d}\frac{f(u)}{f(x)}\\&=\lim_{r\to 0}\sqrt{1-\frac{f(x(\omega))^2}{f((x+\Delta x)(\omega))^2}}\\&=\lim_{r\to 0}\tanh(\omega-(-F'(r))\cdot\Delta x(\omega))\\&=\tanh(\frac{\omega}{2}+\frac{1}{2}\ln(2-e^{-w})).\end{aligned}$$

Use positivity and Assumption 2.2.1 to swap $-F'(x)$ for $-F'(r)$ on the LHS, and use positivity and Proposition 2.2.13 to obtain the last equality. $\square$

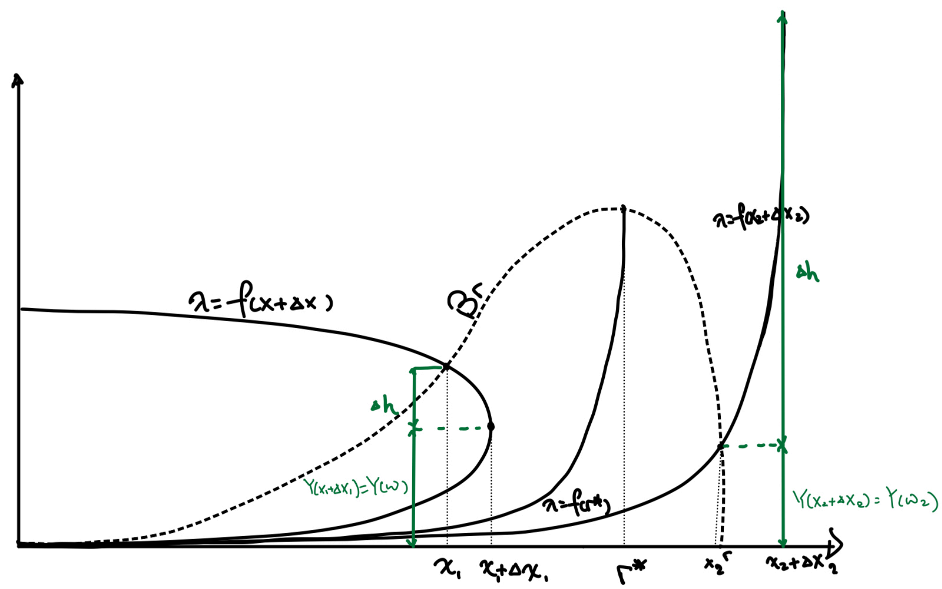

Figure 2.4: Demo 2.

Corollary 2.2.15. For $\omega \in (-\infty, \ln 2)$, define $x(\omega) = r^* + \frac{\omega}{-F'(r)}$. Then as $r \to 0$

$$-F'(r) \cdot \Delta x(\omega) \to -\frac{1}{2}\omega - \frac{1}{2}\ln(2 - e^{\omega}) \tag{1}$$

and, as $r \to 0$, the boundary of B_r is

$$\varphi(\omega) = (Y + \Delta h)(\omega) \to \frac{f(r)}{-F'(r) \cdot 2\sqrt{2e^{-\omega} - 1}}\left(1 - \tanh\left(\frac{\omega}{2} - \frac{1}{2}\ln(2 - e^{w})\right)\right). \tag{2}$$

Proof. Because I have already shown Proposition 2.2.13 and Corollary 2.2.14, it is sufficient to consider the case when $\omega > 0$. By analogy, there is identity

$$(R - X)((x + \Delta x)(\omega)) + (x + \Delta x)(\omega) - r = \int_{x+\Delta x}^{x} -\frac{\mathrm{d}t}{\mathrm{d}u}(f((x + \Delta x)(\omega)))\,\mathrm{d}u$$

where the LHS can be approximated by [10]

$$\lim_{r \to 0} \frac{(R - X)((x + \Delta x)(\omega)) + (x + \Delta x)(\omega) - r}{(x + \Delta x)(\omega) - r^*} = 1.$$

Then, as in Proposition 2.1.13, solve for

$$\lim_{r \to 0} \omega + (-F'(r))\Delta x(\omega) = \lim_{r \to 0} \operatorname{arctanh} \sqrt{1 - \left(\frac{f(x(\omega))}{f((x + \Delta x)(\omega))}\right)^2}$$

[10]Notice that I do not need an analogue for Corollary 2.2.11 for the positive ω when showing positivity by method similar to the one used in the proof of Proposition 2.2.13.

to obtain (1). Compute

$$\frac{f((x+\Delta x)(\omega))}{f(r)} = \frac{f((x+\Delta x)(\omega))}{f(x(\omega))} \cdot \frac{f(x(\omega))}{f(r^*)} \cdot \frac{f(r^*)}{f(r)} \to \frac{1}{2\sqrt{2e^{-\omega}-1}},$$

and notice that when $\omega > 0$, Δh is negative, and use that tanh is odd to obtain (2). □

See section 2.3 for what the ball looks like.

Corollary 2.2.16. Define the variable s by

$$s = -F'(r) \cdot ((x+\Delta x)(\omega) - r^*)$$

for $\omega \in (-\infty, \ln 2)$, and denote $f(s)$ be the label of the corresponding geodesic. With abuse of notation, also let $\varphi(s)$ be the height of which the geodesic $f(s)$ reach distance r. Then as $r \to 0$,

$$\varphi(s) \to \frac{1}{2}\operatorname{sech}(s) \cdot \frac{f(r)}{-F'(r)}.$$

Proof. First, compute

$$\begin{aligned}\lim_{r\to 0} s &= \lim_{r\to 0} -F'(r) \cdot ((x+\Delta x)(\omega) - r^*) \\ &= -F'(r)\left(\frac{\omega}{-F'(r)} - \frac{\frac{1}{2}\omega + \frac{1}{2}\ln(2-e^w)}{-F'(r)}\right) \\ &= \frac{\omega}{2} - \frac{1}{2}\ln(2-e^{\omega})\end{aligned}$$

Then, for $\omega \in (0, \ln 2)$, the second equality follows the identity (they are the same function when expanded out)

$$\begin{aligned}\lim_{r\to 0}\frac{-F'(r)}{f(r)}\varphi(s) &= \frac{1}{\sqrt{2e^{-\omega}-1}}\left(1+\tanh\left(\omega + \frac{1}{2}\ln\frac{e^{-2\omega}}{2e^{-\omega}-1}\right)\right) \\ &= \operatorname{sech}\left(\frac{\omega}{2} - \frac{1}{2}\ln(2-e^w)\right) \\ &= \operatorname{sech}(\lim_{r\to 0} s) \\ &= \lim_{r\to 0}\operatorname{sech}(s).\end{aligned}$$

□

Theorem 2.2.17. *As $r \to 0$,*

$$Area(B_r) \to \frac{\pi}{2} \cdot \frac{f(r)}{F'(r)^2}.$$

Proof. For each k, assign radius of the metric ball considered to be $r = r_k$. Assume that when $k \to \infty$, balls with smaller and smaller radius is measured, i.e., $r = r_k \to 0$. Let $a_k = r_k^* - \frac{k}{-F'(r_k)}$ and $b_k = r_k^* + \frac{\ln 2 - 1/(2k)}{-F'(r_k)}$. For each k in $\{k_i\} \to \infty$, associate a lower bound of $Area(B_{r_k})$ to be the area of the metric ball between a_k and b_k, that is,

$$LB_k = \int_{a_k}^{b_k} (Y(\omega) + \Delta h(\omega))\,\mathrm{d}x.$$

By a change of variable, note that $-F'(r_k)\,\mathrm{d}x = \mathrm{d}\omega$, obtain

$$-F'(r_k) \cdot LB_k = \int_{-k}^{k} (Y(\omega) + \Delta h(\omega))\,\mathrm{d}w.$$

Equivalently,

$$\frac{F'(r_k)^2}{f(r_k)} \cdot LB_k = \frac{-F'(r_k)}{f(r_k)} \left(\int_{-k}^{0} (Y(\omega) + \Delta h(\omega))\,\mathrm{d}w + \int_{0}^{k} (Y(\omega) + \Delta h(\omega))\,\mathrm{d}w \right).$$

Then LB_k measures the area of $Area1(B_{r_k})$ to the right of a_k plus the area of $Area2(B_{r_k})$ to the left of b_k, and omit the area to the left of a_k and to the right of b_k. Presume that I can apply dominated convergence theorem to the RHS to obtain, then, via Corollary 2.2.16,

$$\begin{aligned}
\lim_{k\to\infty} \frac{F'(r_k)^2}{f(r_k)} \cdot LB_k &= \frac{-F'(r_k)}{f(r_k)} \int_0^\infty \lim_{k\to\infty} (Y(\omega) + \Delta h(\omega))\,\mathrm{d}w \\
&= \int_{-\infty}^{\ln 2} \frac{1}{2\sqrt{2e^{-\omega} - 1}} (1 - \tanh(\frac{\omega}{2} - \frac{1}{2}\ln(2 - e^{w})))\,\mathrm{d}\omega \\
&= \frac{\pi}{2}.
\end{aligned}$$

I will now justify how the dominated convergence theorem could be applied. But the work is already done: an upper bound of $Area1(B_{r_k}) \cdot F'(r_k)^2/f(r_k)$ is shown in Corollary 2.2.12, and a simple rectangular upper bound (see figure 2.5) for $Area2(B_{r_k}) \cdot F'(r_k)^2/f(r_k)$ would follow Proposition 2.2.9 and Corollary 2.2.7.

It remains to show that this lower bound will accurately approximate the actual area of the ball when $k \to \infty$. I will show this separately for $Area1(B_{r_k})$ and $Area2(B_{r_k})$, so the error term concerned are $omitted\_Area1(B_{r_k}) \cdot F'(r_k)^2/f(r_k)$ and $omitted\_Area2(B_{r_k}) \cdot F'(r_k)^2/f(r_k)$. For $Area1(B_{r_k})$, for each k, only the area to the left of a_k is omitted. Then, using the same argument as in the proof of Corollary 2.2.12, one can estimate that the omitted area is less than $8\ln 2 \cdot f\left(\frac{r_k + a_k}{2}\right) / -F'(r_k)^2$. Compute using Proposition 2.2.10 and Corollary 2.2.3,

$$\lim_{k\to\infty} \frac{f\left(\frac{r_k+a_k}{2}\right)}{f(r_k)} = \lim_{k\to\infty} \frac{f\left(\frac{r_k - r_k*}{2} - \frac{k}{-2F'(r_k)}\right)}{f(r_k)} = \sqrt{\ln 2} \cdot e^{-k} = 0,$$

so

$$\lim_{k\to\infty} 8\ln 2 \cdot \frac{f\left(\frac{r_k+a_k}{2}\right)}{-F'(r_k)^2} \cdot \frac{-F'(r_k)^2}{f(r_k)} = 0.$$

For $Area2(B_{r_k})$, the undercounted area is to the right of b_k under the boundary of the metric ball. I will again use the rectangular upper bound (see figure 2.5) to bound the omitted area to the right of b_k. By construction, the width of the rectangle is $r_k - b_k$, and the height of the rectangle is less than $Y(r_k^*)$ because he boundary of B_{r_k}, is decreasing in $Area2$. Thus, by construction of b_k

$$\begin{aligned}\lim_{k\to\infty} omitted\_Area2(B_{r_k}) \cdot F'(r_k)^2/f(r_k) \leq& \lim_{k\to\infty}\left((-F'(r)\cdot(r_k - b_k))\cdot(\frac{-F'(r)}{f(r)}\cdot Y(r_k^*))\right)\\ =&\frac{1}{2}\lim_{k\to\infty}(-F'(r)\cdot(r_k - b_k))\\ =&0.\end{aligned}$$

I have shown the result holds when $k \to \infty$ so $r = r \to 0$. This is sufficient to show that the results hold when $r \to 0$. The proof is complete. □

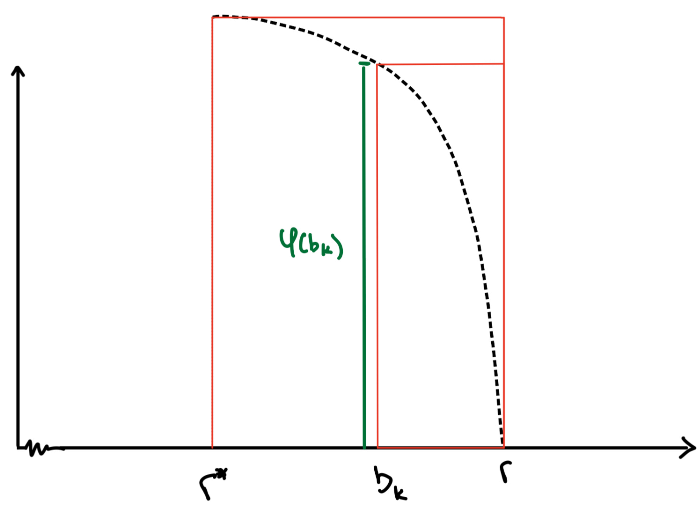

Figure 2.5: The bigger rectangle bounds Area 2, the smaller rectangle bounds the error term.

One should, however, recall that B_r is only the first quadrant of the actual metric ball with radius r. By symmetry, the area of the actual metric ball is four times $Area(B_r)$, i.e., when $r \to 0$, the area is approximately $2\pi \cdot \frac{f(r)}{F'(r)^2}$.

2.3 Gallery

Below is a picture of the "prototype ball" generated using the formula derived in Corollary 2.2.15. The variable of the horizontal axis is ω and the unit length of the vertical axis is $f(r)/F'(r)^2$. The boundary of the ball intersects the horizontal axis at $-\infty$ and $\ln 2$.

```
Plot[1/(2 Sqrt[2 E^(-w) - 1]) (1 - Tanh[w/2 - 1/2 Log[2 - e^w]]), {w, -10, Log[2]}]
```

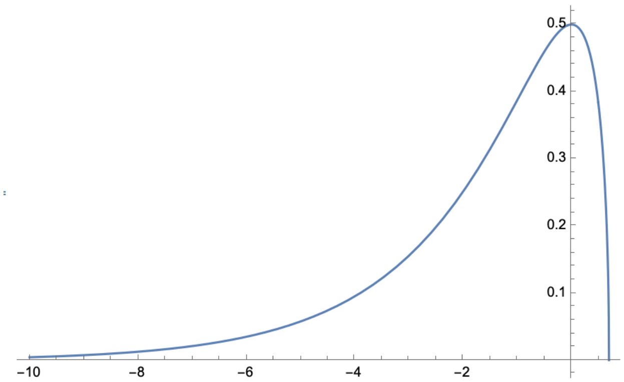

Figure 2.6: Prototype ball.

Using Mathematica, I also generated a series of pictures of how the prototype ball fits with actual balls with radius $r > 0$. I choose $f(x) = e^{-\frac{1}{x^2}}$, and $r^* = 1, .5, .25$ respectively.

Note: I adopted some changes to the original model when making these pictures.

1. In the last section I have defined $x = r^* + \frac{\omega}{-F'(r)}$. For a better fitting model I used instead $x = r^* + \frac{\omega}{-F'(r^*)}$. Because $-F'(r^*) > -F'(r)$, this gives more precision to the model. One can also recall that when $r \to 0$, Assumption 2.2.1 assumes they are asymptotic.

2. Recall that the prototype ball has left bounds at $-\infty$ and right bounds at $\ln 2$ while these conditions can never be achieved when fitting the model to actual balls with positive radius. So I rather adopted smaller upper and lower bounds for ω for each picture and truncated the model to fit the scale of the actual metric ball.

3. The picture is scaled so the unit length in the horizontal direction is r and the unit length in the vertical direction is $\frac{f(r)}{-F'(r^*)}$ (for the actual metric ball). *Blue* line is the actual metric ball and *orange* line is the model. The boundary of the actual metric ball should always reach and stop at 1 on the horizontal axis, but this is not always true because of the loss of precision in numerical computations.

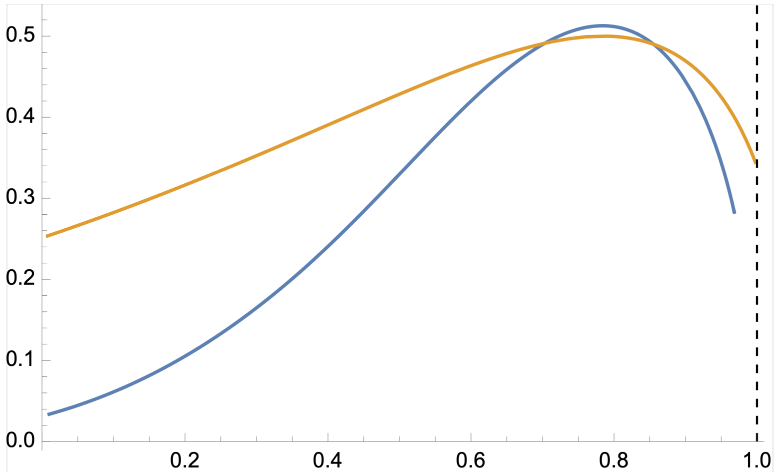

Figure 2.7: $r^* = 1$.

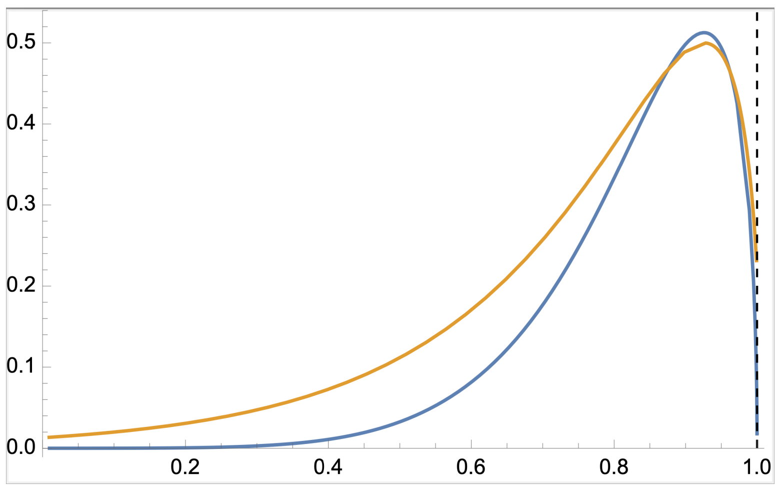

Figure 2.8: $r^* = .5$.

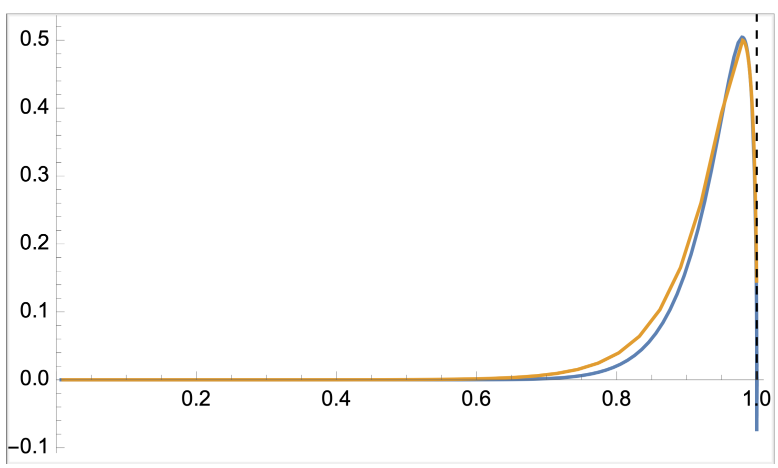

Figure 2.9: $r^* = .25$.

I used NIntegrate and NDSolve in Mathematica by Wolfram Research to obtain the boundary of the actual metric ball. The precisions of the algorithms will be lost if one further decrease r^* with the choice of $f(x) = e^{-\frac{1}{x^2}}$.

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