

Chapter 5

模 II: 自由与诺特模

定义 5.1 诺特(Noetherian) 模: $M \in R - \text{mod}$, S_1, \dots, S_n, \dots 是 M 的子模且 $S_1 \subseteq \dots \subseteq S_n \subseteq \dots$, 若 $\exists K \in \mathbb{Z}^+$, s.t. $S_K = S_{K+1} = \dots$, 则称 M 满足升链条件 (A.C.C.), 称满足 ACC 的模为诺特模.

定理 5.1 (课本定理5.7): (1) $M \in R - \text{mod}$ 为诺特模 $\iff M$ 的子模是有限生成的.
(2) R 是诺特环 $\iff R$ 的理想都是有限生成的.

证: (1) “ \implies ”: 设 S 是 M 的子模.

若 $S = \{0\}$, 则 $S = \langle \langle 0 \rangle \rangle$ 显然有限生成,

若 $S \neq \{0\}$, 则 $\exists 0 \neq v_1 \in S$, 令 $S_1 = \langle \langle v_1 \rangle \rangle \subseteq S$,

若 $S_1 = S$, 则 S 有限生成,

若 $S_1 \neq S$, 则 $\exists v_2 \in S - S_1$, 令 $S_2 = \langle \langle v_1, v_2 \rangle \rangle \subseteq S$, 则 $S_1 \subseteq S_2 \subseteq S$,

若 $S_2 = S$, 则 S 有限生成,

若 $S_2 \neq S$, 则 $\exists 0 \neq v_3 \in S - S_2$, 令 $S_3 = \langle \langle v_1, v_2, v_3 \rangle \rangle \subseteq S$, 则 $S_1 \subseteq S_2 \subseteq S_3 \subseteq S$,

若 $S_3 = S$, 则 S 有限生成,

若 $S_3 \neq S$, 则 $\exists 0 \neq v_4 \in S - S_3$, 令 $S_4 = \langle \langle v_1, v_2, v_3, v_4 \rangle \rangle \subseteq S$, 则 $S_1 \subseteq S_2 \subseteq S_3 \subseteq S_4 \subseteq S$,

\dots , 以此类推, 得 $S_1 \subseteq S_2 \subseteq \dots \subseteq S_n \subseteq \dots$.

$\therefore S$ 满足 ACC, $\therefore \exists K \in \mathbb{Z}^+$, s.t. $S_K = S_{K+1} = \dots = S = \langle \langle v_1, \dots, v_n \rangle \rangle$, 故 S 有限生成.

“ \impliedby ”: 取 M 的任一子模升链 $S_1 \subseteq \dots \subseteq S_n \subseteq \dots$, 则 $S = \cup_{i \in J} S_i$ 是 M 的子模.

$\therefore M$ 的子模是有限生成的, $\therefore S$ 必然是有限生成, 故设 $S = \langle \langle v_1, \dots, v_m \rangle \rangle$.

$\forall k = 1, \dots, m, v_k \in S = \cup_{i \in J} S_i \implies \exists i_k \in J$, s.t. $u_k \in S_{i_k}$.

令 $K = \max\{i_1, \dots, i_m\}$, 则由升链的性质, $u_1, \dots, u_m \in S_K$

$\implies S_K = S$, 故升链必终止于 S_K .

综上, 得证. □

例 5.1: $\therefore \mathbb{Z}$ 的任意理想均由单个元素生成, 具体地说, I 是 \mathbb{Z} 的理想, 则 $I = \langle n \rangle$, 其中 n 为 I 中的最小整数, $\therefore \mathbb{Z}$ 是诺特环. □

例 5.2: $F[x] = \{\sum_{i=0}^n a_i x^i \mid a_i \in F, n \in \mathbb{Z}\}$, I 是 $F[x]$ 的理想, 则 $I = \langle f(x) \rangle$, 其中 $\deg f(x)$ 是 I 中最小的¹, 故

¹ 多项式间的除法: 若 $\deg g(x) \geq \deg f(x)$, 则 $\exists q(x), r(x) \in F[x]$, s.t. $g(x) = q(x)f(x) + r(x)$ 且 $(r(x) = 0 \text{ 或 } 0 < \deg r(x) < \deg f(x))$

$(F[x], +, \cdot)$ 是诺特环. □

定义 5.2 主理想: 由一个元素生成的诺特环.

定理 5.2 (课本定理5.8): R 为有单位元的交换环, 则 R 是诺特环 $\iff R$ 上的有限生成模都是诺特模.

上述定理意味着有限生成的性质对诺特环是遗传的.

证: “ \Leftarrow ”: $R \in R\text{-mod}$ 且 $R = \langle\langle 1 \rangle\rangle$, 故 R 为诺特环.

“ \Rightarrow ”: 取 R 上的有限生成模 $M = \langle\langle v_1, \dots, v_n \rangle\rangle \in R\text{-mod}$, $M = \{\sum_{i=1}^n r_i v_i \mid r_i \in R\}$.

映射 $\tau: R^n \rightarrow M$, $(r_1, \dots, r_n) \mapsto \sum_{i=1}^n r_i u_i$ 满足

$$(1) \because \tau(r(r_1, \dots, r_n) + t(l_1, \dots, l_n)) = \tau(rr_1 + tl_1, \dots, rr_n + tl_n) = \sum_{i=1}^n (rr_i + tl_i)u_i = r \sum_{i=1}^n r_i u_i + t \sum_{i=1}^n l_i u_i = r\tau(r_1, \dots, r_n) + t\tau(l_1, \dots, l_n), \therefore \tau \text{ 是 } R^n \text{ 到 } M \text{ 上的模同态},$$

$$(2) \because \forall v \in M, \exists (r_1, \dots, r_n), \text{ s.t. } v = \sum_{i=1}^n r_i u_i = \tau(r_1, \dots, r_n), \therefore \tau \text{ 满射},$$

故 τ 满同态.

设 S 是 M 的任一子模, 则 $\tau^{-1}(S)$ 是 R^n 的子模, 且 $\because \tau$ 满同态, $\therefore \tau(\tau^{-1}(S)) = S$.

【思路】根据定理 5.2, 要证 M 诺特, 即证 M 的子模 S 有限生成, 于是先证 R^n 的子模有限生成, 从而 R^n 诺特, 进而利用引理 5.1 得 S 有限生成.

数学归纳法: 当 $n = 1$ 时, R 诺特 $\implies R^n$ 诺特.

假设当 $n = k$ 时, R^k 诺特, 则当 $n = k + 1$ 时, 要证 R^{k+1} 诺特, 即证 R^{k+1} 的子模有限生成.

取 I 为 R^{n+1} 子模, 取 $I_1 = \{(0, \dots, 0, a_{k+1}) \mid \exists a_1, \dots, a_k \in R, \text{ s.t. } (a_1, \dots, a_k, a_{k+1}) \in I\}$, $I_2 = \{(a_1, \dots, a_k, 0) \mid \exists a_{k+1} \in R, \text{ s.t. } (a_1, \dots, a_k, a_{k+1}) \in I\}$.

$\forall (0, \dots, 0, a_{k+1}), (0, \dots, 0, b_{k+1}) \in I_1, \exists a_1, \dots, a_k, b_1, \dots, b_k \in R, \text{ s.t. } (a_1, \dots, a_k, a_{k+1}), (b_1, \dots, b_k, b_{k+1}) \in I$.

又 $\because I$ 是子模, $\therefore \forall r, t \in R, r(a_1, \dots, a_k, a_{k+1}) + t(b_1, \dots, b_k, b_{k+1}) = (ra_1 + tb_1, \dots, ra_k + tb_k) \in I \implies r(0, \dots, 0, a_{k+1}) + t(0, \dots, 0, b_{k+1}) = (0, \dots, 0, ra_{k+1} + tb_{k+1}) \in I_1$, 故 I_1 为 R^{k+1} 的子模.

$\forall (a_1, \dots, a_k, 0), (b_1, \dots, b_k, 0) \in I_2, \exists a_{k+1}, b_{k+1}, \text{ s.t. } (a_1, \dots, a_k, a_{k+1}), (b_1, \dots, b_k, b_{k+1}) \in I$.

又 $\because I$ 是子模, $\therefore \forall r, t \in R, r(a_1, \dots, a_k, a_{k+1}) + t(b_1, \dots, b_k, b_{k+1}) = (ra_1 + tb_1, \dots, ra_k + tb_k) \in I \implies r(a_1, \dots, a_k, 0) + t(b_1, \dots, b_k, 0) = (ra_1 + tb_1, \dots, ra_k + tb_k, 0) \in I_2$, 故 I_2 为 R^{k+1} 的子模.

令 $J_1 = \{a_{k+1} \mid (0, \dots, 0, a_{k+1}) \in I_1\}$, $J_2 = \{(a_1, \dots, a_k) \mid (a_1, \dots, a_k) \in I_2\}$, 易证 J_1 是 R 的子模, J_2 是 R^k 的子模.

$\because R, R^k$ 诺特, $\therefore J_1, J_2$ 有限生成, 设 $J_1 = \langle\langle g_1, \dots, g_m \rangle\rangle$, $J_2 = \langle\langle f_1, \dots, f_n \rangle\rangle$, 其中 $g_i \in R, f_i \in R^k$.

于是 $\forall i = 1, \dots, m, (0, \dots, 0, g_i) \in I_1$, 由 I_1 的定义, $\exists g_{i_1}, \dots, g_{i_k} \in R, \text{ s.t. } \bar{g}_i \equiv (g_{i_1}, \dots, g_{i_k}, g_i) \in I$.

同理可记 $\bar{f}_i = (f_i, 0)$.

$\forall r = (r_1, \dots, r_k, r_{k+1}) \in I$, 有 $(0, \dots, 0, r_{k+1}) \in I_1$, 即 $r_{k+1} \in J_1 = \langle\langle g_1, \dots, g_m \rangle\rangle$.

于是 $r_{k+1} = \sum_{i=1}^m \alpha_i g_i, (h, 0) \equiv r - \sum_{i=1}^m \alpha_i \bar{g}_i = (r_1, \dots, r_k, 0) \in I_2 \implies h \in J_2$.

设 $h = \sum_{i=1}^n \beta_i f_i$, 则 $r = \sum_{i=1}^m \alpha_i \bar{g}_i + \sum_{i=1}^n \beta_i \bar{f}_i$, 故 I 由 $\bar{g}_1, \dots, \bar{g}_m, \bar{f}_1, \dots, \bar{f}_n$ 生成 $\implies R^{k+1}$ 诺特 $\implies R^n$ 诺特 $\forall n \implies S = \tau(\tau^{-1}(S))$ 有限生成. □

引理 5.1: $\tau: M \rightarrow N$ 满同态, 则 M 有限生成 $\implies N$ 有限生成, 即有限生成模的满同态像有限生成.

证: $\because M$ 有限生成, \therefore 设 $M = \langle v_1, \dots, v_n \rangle = \{ \sum_{i=1}^n r_i v_i \mid r_i \in R \}$.

$\because \tau$ 满同态, $\therefore N = \text{Im } \tau = \{ \tau(u) \mid u \in M \} = \{ \tau(u) \mid u = \sum_{i=1}^n r_i v_i, r_i \in R \} = \{ \tau(\sum_{i=1}^n r_i v_i) \mid r_i \in R \} = \{ \sum_{i=1}^n r_i \tau(v_i) \mid r_i \in R \} = \langle \tau(v_1), \dots, \tau(v_n) \rangle$, 故 N 有限生成. \square

定理 5.3 Hilbert 基底定理(课本定理5.9): R 是诺特环 $\implies R[x] \equiv \{ \sum_{i=0}^n a_i x^i \mid a_i \in R, n \in \mathbb{Z}^+ \}$ 诺特, 其中 $\sum_{i=0}^n a_i x^i + \sum_{j=0}^m b_j x^j = \sum_{k=0}^{\max\{n,m\}} (a_k + b_k) x^k$, $(\sum_{i=0}^n a_i x^i) \left(\sum_{j=0}^m b_j x^j \right) = \sum_{k=0}^{nm} \left(\sum_{i+j=k} a_i b_j \right) x^k$.

证: 设 I 是 $R[x]$ 的理想, 则 $I_k = \{ r_k \in R \mid \exists a_0 + a_1 x + \dots + a_{k-1} x^{k-1} + r_k x^k \in I \}$ 是 R 的理想.

又 $\because \forall f(x) \in I, xf(x) \in I, \therefore I_0 \subseteq I_1 \subseteq \dots \subseteq I_K \subseteq \dots$.

又 $\because R$ 诺特, $\therefore \exists K \in \mathbb{Z}^+$, s.t. $I_K = I_{K+1} = \dots$, 且 R 的理想均有限生成.

设 $I_0 = \langle r_{01}, r_{02}, \dots, r_{0t_0} \rangle, I_1 = \langle r_{11}, r_{12}, \dots, r_{1t_1} \rangle, \dots, I_K = \langle r_{K1}, r_{K2}, \dots, r_{Kt_K} \rangle$.

$g_{01} = r_{01} \in I, g_{02} = r_{02} \in I, \dots, g_{0t_0} = r_{0t_0} \in I,$

$g_{11} = r_{11}x + O(1) \in I, g_{12} = r_{12}x + O(1) \in I, \dots, g_{1t_1} = r_{1t_1}x + O(1) \in I,$

$\dots,$

$g_{K1} = r_{K1}x^K + O(x^{K-1}) \in I, g_{K2} = r_{K2}x^K + O(x^{K-1}) \in I, \dots, g_{Kt_K} = r_{Kt_K}x^K + O(x^{K-1}) \in I,$

则 I 由 $\{ g_{ij} \mid i = 1, \dots, K; j = 1, \dots, t_i \}$ 生成.

$\forall f(x) \in I$, 设 $f(x) = \sum_{i=0}^n a_i x^i$.

取 $a_n \in I_n$, 若 $n > K$, 则 $I_n = I_K = \langle r_{K1}, \dots, r_{Kt_K} \rangle$, 从而 $a_n = \sum_{i=1}^{t_K} \alpha_i r_{Ki}$,

$\implies f(x) = a_n x^n + O(x^{n-1}) = \sum_{i=1}^{t_K} \alpha_i r_{Ki} x^n + O(x^{n-K}) = x^{n-K} \left(\sum_{i=1}^{t_K} \alpha_i g_{Ki} \right) + O(x^{n-K}),$

$\implies f(x) - x^{n-K} \left(\sum_{i=1}^{t_K} \alpha_i g_{Ki} \right) = \beta_{n-1} x^{n-1} + O(x^{n-2}),$

重复以上操作有限次直至多项式的最高次数 $n < K$, 此时, $f(x)$ 可完全由 g_{ij} 表示 $\implies I$ 有限生成, 故由定理 5.1 得, $R[x]$ 诺特. \square

例 5.3: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ 诺特 $\implies \mathbb{Z}[x], \mathbb{Q}[x], \mathbb{R}[x], \mathbb{C}[x]$ 诺特.

$\mathbb{R}[z] = \{ \sum_{i=0}^n a_i z^i \mid a_i \in \mathbb{R}, n \in \mathbb{Z}^+ \}$.

方程组 $\begin{cases} f_1(x) = a_{1n}x^n + a_{1,n-1}x^{n-1} + \dots + a_{11}x + a_{10} = 0, \\ \dots \\ f_m(x) = a_{mn}x^n + a_{m,n-1}x^{n-1} + \dots + a_{m1}x + a_{m0} = 0, \end{cases}$ 的解为 \mathbb{R} 的子集合,

令 $h(x) = \sum_{i=1}^m \alpha_i f_i(x)$, 若 $f_i(x) = 0 \forall i$, 则 $h(x) = 0$.

方程组与解集合之间存在的一一对应的关系, 正如 $\mathbb{R}[x]$ 与 \mathbb{R} 之间的对应关系. \square