

Chapter 3

同构定理

3.1 商空间

定义 3.1 商空间: F 为域, V 为 F 上的向量空间, S 为 V 的子空间, 则称 $\frac{V}{S} \equiv \{[v] \mid v \in V\}$ 为 F 的商空间, 其中 $[v] \equiv \{u \in V \mid u - v \in S\} = S + v$.

$\frac{V}{S}$ 为 F 上的向量空间.

证: $[u] + [v] = \{a \in V \mid a - u \in S\} + \{b \in V \mid b - v \in S\} = \{(a + b) \in V \mid a - u \in S, b - v \in S\}$.

$[u + v] = \{w \in V \mid w - (u + v) \in S\}$.

$\forall (a + b) \in [u] + [v], (a - u) + (b - v) = (a + b) - (u + v) \in S \implies (a + b) \in [u + v] \implies [u] + [v] \subseteq [u + v]$.

$\forall w \in [u + v], \exists c, d \in S, \text{ s.t. } w = c + d + (u + v) = (c + u) + (d + v), \text{ 其中 } (c + u) \in [u], (d + v) \in [v] \implies w \in [u] + [v] \implies [u + v] \subseteq [u] + [v]$.

故 $[u] + [v] = [u + v]$.

假设 $u \sim u', v \sim v'$, 即 $[u] = [u'], [v] = [v']$.

$\because [u] = [u'], \therefore u + S = u' + S \implies \exists s_1, s'_1 \in S, \text{ s.t. } u + s_1 = u' + s'_1 \iff u' = u + s_1 - s'_1$,

$\because [v] = [v'], \therefore v + S = v' + S \implies \exists s_2, s'_2 \in S, \text{ s.t. } v + s_2 = v' + s'_2 \iff v' = v + s_2 - s'_2$

$\implies u' + v' = u + s_1 - s'_1 + v + s_2 - s'_2$, 其中 $\because s_1, s'_1, s_2, s'_2 \in S, \therefore s_1 - s'_1 \in S, s_2 - s'_2 \in S$.

又 $\because V$ 是交换群, $\therefore u' + v' = u + v + (s_1 - s'_1 + s_2 - s'_2)$

$\implies (u' + v') + S = (u + v + (s_1 - s'_1 + s_2 - s'_2)) + S \implies [u' + v'] = [u + v]$,

即 $[u] + [v] = [u + v]$ 与代表元选取无关, 故 $[u] + [v] = [u + v]$ 为运算.

$r[u] = r\{v \in V \mid v - u \in S\} = \{rv \mid v \in V, v - u \in S\} = \{rv \in V \mid rv - ru \in S\} = [ru]$.

假设 $u \sim u'$, 即 $[u] = [u']$.

$\because [u] = [u'], \therefore u + S = u' + S \implies \exists s, s' \in S, \text{ s.t. } u + s = u' + s' \iff u' = u + s - s'$

$\implies ru' = r(u + s - s') = ru + r(s - s')$, 其中 $s - s' \in S \implies (ru') + S = (ru + r(s - s')) + S = (ru) + S \implies r[u'] = [ru]$,

即 $r[u] = [ru]$ 与代表元选取无关, 故 $r[u] = [ru]$ 为运算.

$(\frac{V}{S}, +)$ 满足

(1) **结合律:** $([v] + [u]) + [w] = [u + v] + [w] = [u + v + w] = [u + (v + w)] = [u] + [v + w] = [u] + ([v] + [w]),$

(2) **有单位元** $[0]$: $[0] + [u] = [0 + u] = [u] = [u + 0] = [u] + [0],$

(3) **有逆元:** $\forall v \in V, \exists -v, \text{ s.t. } [a] + [-a] = [a + (-a)] = [0] = [(-a) + a] = [-a] + [a],$

且 $[u] + [v] = [u + v] = [v + u] = [v] + [u]$, 即 $(\frac{V}{S}, +)$ 交换, 故 $(\frac{V}{S}, +)$ 为交换群. (总之就是因为 $\frac{V}{S}$ 中的元素 $[v]$ 保持了 V 中的元素 v 的各种运算性质, 所以 $(V, +)$ 是交换群就可以推出 $\frac{V}{S}$ 也是交换群.)

$\frac{V}{S}$ 满足

- (1) $r([u + v]) = r([u] + [v]) = r[u] + r[v]$,
- (2) $(r + t)[u] = [(r + t)u] = [ru + tu] = [ru] + [tu] = r[u] + t[u]$,
- (3) $(r \cdot t)[u] = [(r \cdot t)u] = [r(tu)] = r[tu] = r(t[u])$,
- (4) 有单位元 1: $[1][u] = [1u] = [u]$,

故 $\frac{V}{S}$ 为 F 上的向量空间. □

定理 3.1 (课本定理3.2): (1) $\Pi_S : V \rightarrow \frac{V}{S}, v \mapsto [v]$ 是线性变换.

(2) Π_S 是满线性变换, 即 $\text{Im } \Pi_S = \frac{V}{S}$.

(3) $\ker \Pi_S = S$.

证: (1) 显然 Π_S 是唯一的, 故 Π_S 是映射.

如前所证, V 和 $\frac{V}{S}$ 均为 F 上的向量空间.

$\because [u] + [v] = [u + v], r[u] = [ru], \therefore r[u] + t[v] = [ru] + [tv] = [ru + tv]$, 故 Π_S 为线性变换.

(2) $\forall [v] \in \frac{V}{S}, \exists v \in V, \text{ s.t. } \Pi_S(v) = [v]$, 故 Π_S 是满线性变换.

(3) $\ker \Pi_S = \{v \in S \mid \Pi_S(v) = [0]\}$.

$v \in \ker \Pi_S, \Pi_S(v) = [v] = v + S = [0] = 0 + S = S \iff v \in S$, 故 $\ker \Pi_S \subseteq S$. □

定理 3.2 (课本定理3.3): (1) S, T 为 V 的子空间且 $S \subseteq T$, 则 $\frac{T}{S}$ 是 $\frac{V}{S}$ 的子空间.

(2) 取 X 为 $\frac{V}{S}$ 的子空间, 则 $\exists V$ 的子空间 T , s.t. $\emptyset \neq S \subseteq T, \frac{T}{S} = X$.

证: (1) $\frac{T}{S} = \{[u] \mid u \in T\}, \frac{V}{S} = \{[v] \mid v \in V\}$.

$\forall [u] \in \frac{T}{S}, u \in T, \because T$ 是 V 的子空间, $\therefore u \in V \implies [u] \in \frac{V}{S}$, 故 $\frac{T}{S} \subseteq \frac{V}{S}$.

$\forall [u_1], [u_2] \in \frac{T}{S}, \forall r, t \in F, r[u_1] + t[u_2] = [ru_1 + tu_2]$.

$\because u_1, u_2 \in T, \therefore ru_1 + tu_2 \in T \implies [ru_1 + tu_2] \in \frac{T}{S}$, 故 $\frac{T}{S}$ 为向量空间.

综上, 得证.

(2) 取 $T = \cup_{[v] \in X} [v]$. 显然 $T \subseteq V$.

$\forall u, v \in T$, 根据 T 的定义, $[u], [v] \in X$.

$\because X$ 为子空间, $\therefore r[u] + t[v] = [ru + tv] \in X \subseteq T = \cup_{[v] \in X} [v] \implies ru + tv \in T$.

故 T 为 V 的子空间.

$\because S = [0] \in X, \therefore S \subseteq T = \cup_{[v] \in X} [v]$.

$\frac{T}{S} = \{[v] = S + v \mid v \in T\}$.

$\forall [u] \in \frac{T}{S}, u \in T = \cup_{[v] \in X} [v] \implies [u] \in X \implies \frac{T}{S} \subseteq X$.

$$\forall [u] \in X, u \in T = \cap_{[v] \in X} [v] \implies [u] \in \frac{T}{S} \implies X \subseteq \frac{T}{S}.$$

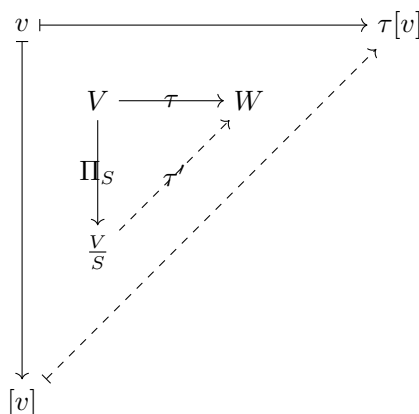
$$\text{故 } \frac{T}{S} = X.$$

综上, 得证.

□

3.2 第一同构定理

定理 3.3 第一同态基本定理(课本定理3.4): S 是 V 的子空间, $\tau \in \mathcal{L}(V, W)$,



若 $S \subseteq \ker \tau$, 即 $\ker \Pi_S \subseteq \ker \tau$, 则 $\exists! \tau' : \frac{V}{S} \rightarrow W$, s.t. $\tau = \tau' \circ \Pi_S$, 即 $\forall v \in V, \tau(v) = \tau'([v])$, 此时上图可交换, $\ker \tau' = \frac{\ker \tau}{S}$, $\text{Im } \tau' = \text{Im } \tau$.

^a该定理回答了 τ' 的存在性 (即 τ' 在什么条件下存在) 的问题. 之所以称“基本”, 是因为若将该定理中的向量空间换成其他代数结构, 定理仍然成立.

证: τ' 的唯一性要求, 若 $[u] = [v]$, 则 $\tau'([u]) = \tau'([v])$,

即若 $u \sim v$, 则 $\tau(u) = \tau(v)$,

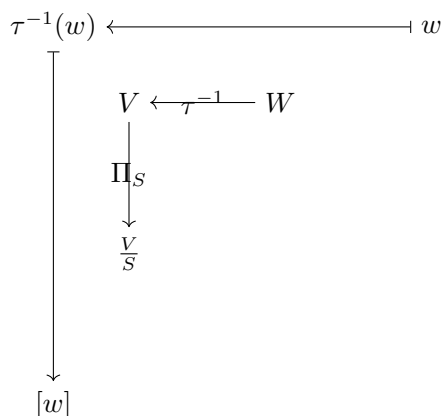
即若 $u - v \in S$, 则 $\tau(u - v) = 0$,

即 $S \subseteq \ker \tau$.

此时, $\ker \tau' = \{[v] \in \frac{V}{S} \mid \tau'([v]) = 0\} = \{[v] \in \frac{V}{S} \mid \tau(v) = 0\} = \{[v] \in \frac{V}{S} \mid v \in \ker \tau\} = \frac{\ker \tau}{S}$,

$\text{Im } \tau' = \{\tau'([v]) \mid [v] \in \frac{V}{S}\} = \{\tau'([v]) \mid v \in V\} = \{\tau(v) \mid v \in V\} = \text{Im } \tau$ ($\because \Pi_S$ 满射, $\therefore \forall [v] \in \frac{V}{S}, \exists v \in V$). □

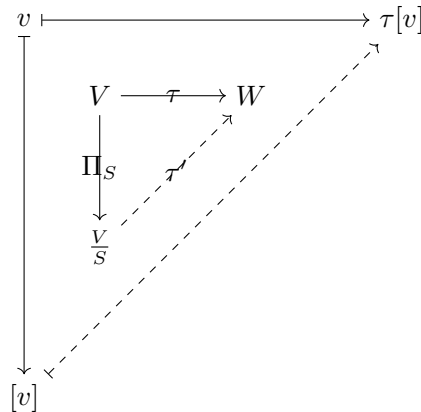
那么, 若 τ 双射, 即 $\exists \tau^{-1} \in \mathcal{L}(W, V)$, 且 $\ker \tau = S$, 如何?



此时, $\ker \tau' = \frac{\ker \tau}{S} = \{[v] \mid v \in \ker \tau\} = \{[v] \mid v \in S\} = \{[0]\} \implies \tau'$ 单射.

由上面关于第一同态基本定理的延伸讨论我们得到:

定理 3.4 第一同构定理(课本定理3.5): 若 $\ker \tau = S$, 则 τ' 单射, $\frac{V}{\ker \tau} = \frac{V}{S} \approx \text{Im } \tau$.



证: $V = \ker \tau \oplus (\ker \tau)^c$, 其中 $(\ker \tau)^c \approx \text{Im } \tau \implies \frac{V}{\ker \tau} \approx (\ker \tau)^c \approx \text{Im } \tau$. □

更一般地, 若 $V = S \oplus T$, 则 $\frac{V}{S} \approx T, \frac{V}{T} \approx S$.

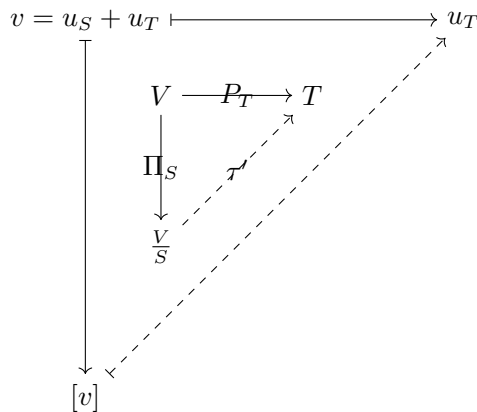
证: $\forall v \in V, v = u_S + u_T$, 其中 $u_S \in S, u_T \in T$.

令投影映射 $P_T: V \rightarrow T, v = u_S + u_T \mapsto u_T$.

$\ker P_T = \{v \in V \mid P_T(v) = 0\} = S = [0] = \ker \Pi_S$.

由第一同构定理 (定理 3.4), $\exists! \tau'$ 单射, s.t. $P_T = \tau' \circ \Pi_S$.

又 $\because \text{Im } P_T = T$, 即 P_T 满射, $\therefore \tau'$ 满射 $\implies \tau'$ 同构 $\implies \frac{V}{S} \approx T$.



同理可证 $\frac{V}{T} \approx S$. □

3.3 线性泛函

定义 3.2 对偶(空间)和线性泛函: $V^* = \mathcal{L}(V, F)$ 为 F 上的向量空间, 称 V^* 为 V 的对偶(空间).
若 $f \in V^*$, 则称 f 为线性泛函.

- (1) $\ker V^*$ 为 F 上的向量空间.
- (2) $\because \dim F = 1, \text{Im } f \subseteq F, \therefore \dim \text{Im } f \leq 1, \dim \ker f \geq \dim V - 1$.
- (3) \because 必有零映射 $0 \in V^*, 0: V \rightarrow F, v \mapsto 0, \therefore V^*$ 非空.

(4) 若 $\dim \operatorname{Im} f = 0$, 则 $\operatorname{Im} f = \{0\}$, f 为零映射.

(5) 若 $\dim \operatorname{Im} f = 1$, 则 $\operatorname{Im} f = \langle r \rangle$, 其中 $0 \neq r \in F \implies \operatorname{Im} f = F$.

由反证法易证, 若 $v \in f^{-1}(r) = \{v \in V \mid f(v) = r\}$, 其中 $r \neq 0$, 则 $v \neq 0$, 且必有 $f(\langle v \rangle^c) = \{0\}$.

证明 (5) 的末句:

证: 假设 $\exists u \in \langle v \rangle^c$, s.t. $f(u) \neq 0$.

$$f\left(\frac{ru}{f(u)}\right) = r \implies \frac{ru}{f(u)} \in f^{-1}(r).$$

又 $\because u \in \langle v \rangle^c$, $\therefore \dim f^{-1}(r) \geq 2$, 这与 $f^{-1}(r) \subseteq (\ker f)^c$, $\dim(\ker f)^c = \dim \operatorname{Im} f \leq 1$ 矛盾,

故假设错误, $\forall u \in \langle v \rangle^c$, $f(u) = 0 \implies f(\langle v \rangle^c) = \{0\}$. □

定理 3.5 (课本定理3.11): (1) $\forall 0 \neq v \in V$, $\exists 0 \neq f \in V^*$, s.t. $f(v) \neq 0$.

(2) $v = 0 \iff \forall f \in V^*$, $f(v) = 0$.

(3) $f \in V^*$, 若 $f(x) \neq 0$, 则 $V = \ker f \oplus \langle x \rangle$, 即 $\operatorname{Im} f \approx \langle x \rangle$.

(4) $0 \neq f, g \in V^*$, $\ker f = \ker g \iff \exists 0 \neq \lambda \in F$, s.t. $f = \lambda g$.

证: (1) $v \neq 0$, 则 $V = \langle v \rangle \oplus \langle v \rangle^c$, 其中 $\langle v \rangle = \{rv \mid r \in F\}$.

令 $f: V \rightarrow F$, $rv + w \mapsto r$, 其中 $rv \in \langle v \rangle$, $w \in \langle v \rangle^c$, 故 $f(v) = 1$, $f \in V^*$.

下证 f 为线性变换: $\forall u_1, u_2 \in V$, $u_1 = r_1v + w_1$, $u_2 = r_2v + w_2$, 其中 $w_1, w_2 \in \langle v \rangle^c$,

$$f(ru_1 + tu_2) = f(r(r_1v + w_1) + t(r_2v + w_2)) = f((rr_1v + rw_1) + (tr_2v + tw_2)) = f((rr_1 + tr_2)v + (rw_1 + tw_2)) = rr_1 + tr_2 = rf(r_1v + w_1) + tf(r_2v + w_2) = rf(u_1) + tf(u_2).$$

故得证.

注意此处 f 的构造并非唯一: 构造 $f: V \rightarrow F$, $rv + u \mapsto rt$, 其中 $u \in \langle v \rangle^c$, 同理可得证.

(2) “ \implies ”: 若 $v = 0$, 则 $\forall u \in V$, $f(v) + f(u) = f(v + u) = f(u) \implies f(v) = 0$.

“ \impliedby ”: 假设 $v \neq 0$, 则由 (1), $\exists f \in V^*$, s.t. $f(v) \neq 0$, 矛盾, 故假设错误, $v = 0$.

(3) $f(x) \neq 0 \implies \operatorname{Im} f \neq \{0\} \implies \dim \operatorname{Im} f \neq 0 \implies \dim(\ker f)^c = \dim \operatorname{Im} f = 1$

$\implies \exists v \in V$, s.t. $\ker f^c = \langle v \rangle \implies V = \ker f \oplus \ker f^c = \langle v \rangle^c \oplus \langle v \rangle$ 且 $\operatorname{Im} f \approx \ker f^c = \langle v \rangle$. $\because f(x) \neq 0$, $\therefore x = rv + w$, 其中 $rv \in \langle v \rangle$, $w \in \langle v \rangle^c \implies \langle x \rangle \approx \langle v \rangle \implies V = \ker f \oplus \langle x \rangle$ 且 $\operatorname{Im} f \approx \langle x \rangle$.

(4) “ \implies ”: 令 $K = \ker f = \ker g$.

$\forall x \notin K$, 由 (3) 有, $V = K \oplus \langle x \rangle$.

取 $\lambda = \frac{f(x)}{g(x)}$, $\forall v \in V$, $x = rx + w$, 其中 $rx \in \langle x \rangle$, $w \in K$

$$\implies f(v) = f(rx + w) = rf(x) = r \frac{f(x)}{g(x)} g(x) = r \lambda g(x) = \lambda g(rx) = \lambda g(rx + w) = \lambda g(v) \implies f = \lambda g.$$

“ \impliedby ”: 若 $\exists 0 \neq \lambda \in F$, s.t. $f = \lambda g$, 则显然 $\ker f = \ker g$. □

3.4 对偶基

定义 3.3 对偶基: $\mathcal{B} = \{b_1, \dots, b_n\}$ 为 V 的基, 则 $\forall i, \exists b_i^* \in V$, s.t. $b_i^*(b_i) = 1$, $b_i^*(b_j) = 0 \forall i \neq j$, 即 $b_i^*(b_j) = \delta_{ij}$,

从而可构造出 $\mathcal{B}^* = \{b_1^*, \dots, b_n^*\} \subseteq V^*$, 称为 \mathcal{B} 的对偶基.

定理 3.6 (课本定理3.12): (1) $\mathcal{B}^* = \{b_1^*, \dots, b_n^*\}$ 线性无关.

(2) $\dim V < \infty$, 则 \mathcal{B}^* 是 V^* 的基.

证: (1) $\sum_{i=1}^m r_i b_i^* = 0 \implies \forall v \in V, \sum_{i=1}^m r_i b_i^*(v) = (\sum_{i=1}^m r_i b_i^*)(v) = 0(v) = 0$.

取 $v = b_j$, 则 $\sum_{i=1}^m r_i b_i^*(b_j) = \sum_{i=1}^m r_i \delta_{ij} = r_j = 0$.

对各个 b_j 如法炮制, 从而可得 $r_j = 0 \forall i$, 故得证.

(2) $\forall f \in V^*, \forall v \in V, \because \mathcal{B}$ 是 V 的基, $\therefore v = \sum_{i=1}^n r_i b_i$

$\implies b_j^*(v) = b_j^*(\sum_{i=1}^n r_i b_i) = \sum_{i=1}^n r_i b_j^*(b_i) = \sum_{i=1}^n r_i \delta_{ij} = r_j$

回代得 $v = \sum_{i=1}^n b_i^*(v) b_i$

$\implies f(v) = f(\sum_{i=1}^n b_i^*(v) b_i) = \sum_{i=1}^n b_i^*(v) f(b_i) = \sum_{i=1}^n f(b_i) b_i^*(v) = (\sum_{i=1}^n f(b_i) b_i^*)(v)$, 此处 $b_i^*(v), f(b_i) \in F$,

因此可交换位置, 我们可视 $\{b_i^*(v)\}$ 为基, $f(b_i)$ 为 $f(v)$ 在这组基上的展开系数

$\implies f = \sum_{i=1}^n f(b_i) b_i^*$, 即 f 可展开为 $\{\mathcal{B}^*\}$ 的线性表示, 结合 (1) 得证. □

仿照上面的方法, $\forall v \in V$, 我们都可构造 $v^* \in V^*$, s.t. $v^*(v) = 1, \forall u_2 \in \langle v \rangle^c, v^*(u_2) = 0$, 从而有映射 $V \rightarrow V^*, v \mapsto v^*, 0 \mapsto 0$ (零映射).

V^* 本身也是向量空间.

定义 3.4 二重对偶(空间): $V^{**} = \mathcal{L}(V^*, F)$ 称为二重对偶(空间), 其中的元素为 $v^{**} : V^* \rightarrow F, f \mapsto v^{**}(f) = f(v)$.

$V \rightarrow V^* \rightarrow V^{**}, v \mapsto v^* \mapsto v^{**}, b_i \mapsto b_i^* \mapsto b_i^{**}$, 满足 $b_i^*(b_j) = \delta_{ij}, b_i^{**}(b_j^*) = b_j^*(b_i)$, 两个映射复合得 $\tau : V \rightarrow V^{**}, v \mapsto v^{**}$.

(1) τ 是映射.

(2) τ 是线性变换.

(3) $\ker \tau = \{v \in V \mid \tau(v) = 0\} = \{0\} \iff \tau$ 单射.

证: (1) 若 $u = v \in V$, 则 $\forall f \in V^*, u^{**}(f) = f(u) = f(v) = v^{**}(f)$, 即得证.

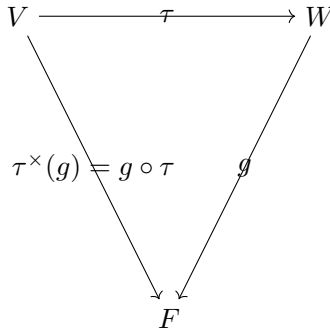
(2) $\forall f \in V^*, (\tau(ru + tv))(f) = (ru + tv)^{**}(f) = f(ru + tv) = rf(u) + tf(v) = ru^{**}(f) + tv^{**}(f) = r\tau(u)(f) + t\tau(v)(f) = (r\tau(u) + t\tau(v))(f) \implies \tau(ru + tv) = r\tau(u) + t\tau(v)$, 结合 (1) 得证.

(3) $\tau(v) = 0 \implies \forall f \in V^*, v^{**}(f) = 0 \implies f(v) = 0 \implies$ (定理 3.5 (1)) $v = 0$, 即得证. □

引理 3.1 (课本引理3.13): 若 $\dim V < \infty$, 则 $\dim V^* = \dim V^{**}$, V^{**} 与 V 同构, 一个线性空间的二重对偶就回到自身, 故实际上套娃式的 V^{****} 是没有意义的, 此处我们就写成 $V^{**} = V$.

3.5 伴随算子

定义 3.5 伴随算子: 由线性变换 $\tau: V \rightarrow W$ 可引出伴随算子 $\tau^\times: W^* \rightarrow V^*$, $g \mapsto \tau^\times(g) = g \circ \tau$.



(1) τ^\times 是映射.

(2) τ^\times 是线性变换.

证: (1) 若 $f = g \in W^*$, 则 $\tau^\times(f) = f \circ \tau = g \circ \tau = \tau^\times(g)$, 故得证.

(2) $\tau^\times(rg_1 + tg_2) = (rg_1 + tg_2) \circ \tau = rg_1 \circ \tau + tg_2 \circ \tau = r\tau^\times(g_1) + t\tau^\times(g_2)$, 结合 (1) 得证.

□

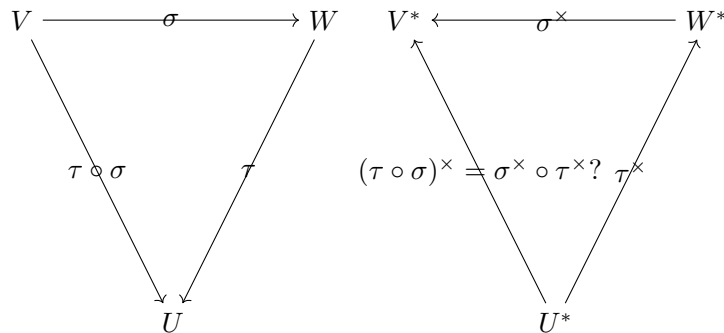
定理 3.7 (课本定理3.18): (1) $\forall \tau, \sigma \in \mathcal{L}(V, W), \forall a, b \in F, (a\tau + b\sigma)^\times = a\tau^\times + b\sigma^\times$.

(2) $\sigma \in \mathcal{L}(V, W), \tau \in \mathcal{L}(W, U)$, 则 $(\tau \circ \sigma)^\times = \sigma^\times \circ \tau^\times$.

(3) $\tau \in \mathcal{L}(V)$ 可逆 $\implies (\tau^{-1})^\times = (\tau^\times)^{-1}$.

证: (1) $\forall f \in W^*, (a\tau + b\sigma)^\times(f) = f \circ (a\tau + b\sigma) = af \circ \tau + bf \circ \sigma = a\tau^\times(f) + b\sigma^\times(f) = (a\tau^\times + b\sigma^\times)(f)$, 即得证.

(2) $\forall f \in U^*, (\tau \circ \sigma)^\times(f) = f \circ (\tau \circ \sigma) = (f \circ \tau) \circ \sigma = \sigma^\times(f \circ \tau) = \sigma^\times(\tau^\times(f)) = (\sigma^\times \circ \tau^\times)(f)$, 即得证.



(3) $1^\times = (\tau \circ \tau^{-1})^\times = (\tau^{-1})^\times \circ \tau^\times \implies (\tau^{-1})^\times = (\tau^\times)^{-1}$.

□

定理 3.8 (课本定理3.18): $\dim V < \infty, \dim W < \infty, \tau \in \mathcal{L}(V, W), \tau^\times \in \mathcal{L}(W^*, V^*), \tau^{\times \times} \in \mathcal{L}(V^{**}, W^{**}) = \mathcal{L}(V, W)$, 则 $\tau^{\times \times} = \tau$.

定理 3.9 (课本定理3.22): $\tau \in \mathcal{L}(V, W)$, 其中 $\dim V < \infty$, $\dim W < \infty$, \mathcal{B} 和 \mathcal{C} 分别是 V 和 W 的定序基, \mathcal{B}^* 和 \mathcal{C}^* 分别是 \mathcal{B} 和 \mathcal{C} 的对偶基, 则 $[\tau^\times]_{\mathcal{C}^* \mathcal{B}^*} = ([\tau]_{\mathcal{B} \mathcal{C}})^T$.

证: 设 $\dim V = n$, $\dim W = m$, V 的定序基 $\mathcal{B} = \{b_1, \dots, b_n\}$, W 的定序基 $\mathcal{C} = \{c_1, \dots, c_m\}$, $\tau \in \mathcal{L}(V, W)$ 的矩阵表示为 $[\tau]_{\mathcal{B} \mathcal{C}} = [\alpha_{ij}]_{m \times n}$, $\tau^\times \in \mathcal{L}(W^*, V^*)$ 的矩阵表示为 $[\tau^\times]_{\mathcal{C}^* \mathcal{B}^*} = [\beta_{ij}]_{n \times m}$.

$$[\tau]_{\mathcal{B} \mathcal{C}} = \begin{pmatrix} [\tau(b_1)]_{\mathcal{C}} & \cdots & [\tau(b_n)]_{\mathcal{C}} \end{pmatrix}, \text{ 其中 } [\tau(b_i)]_{\mathcal{C}} = \begin{pmatrix} \alpha_{1i} \\ \vdots \\ \alpha_{mi} \end{pmatrix}, \text{ 即 } \tau(b_i) = \sum_{k=1}^m \alpha_{ki} c_k.$$

$$[\tau^\times]_{\mathcal{C}^* \mathcal{B}^*} = \begin{pmatrix} [\tau^\times(c_1^*)]_{\mathcal{B}^*} & \cdots & [\tau^\times(c_m^*)]_{\mathcal{B}^*} \end{pmatrix}, \text{ 其中 } [\tau^\times(c_i^*)]_{\mathcal{B}^*} = \begin{pmatrix} \beta_{1i} \\ \vdots \\ \beta_{ni} \end{pmatrix}, \text{ 即 } \tau^\times(c_i^*) = \sum_{l=1}^n \beta_{li} b_l^*.$$

又由 τ^\times 的定义, $\tau^\times(c_i^*) = c_i^* \circ \tau$, 故将该复合函数作用于 b_j 上有 $c_i^*(\tau(b_j)) = (c_i^* \circ \tau)(b_j) = (\tau^\times(c_i^*))(b_j) = (\sum_{l=1}^n \beta_{li} b_l^*)(b_j) = \sum_{l=1}^n \beta_{li} b_l^*(b_j) = \sum_{l=1}^n \beta_{li} \delta_{lj} = \beta_{ji} \implies \beta_{ji} = c_i^*(\tau(b_j))$,

代入上面的 $\tau(b_j)$ 的展开式得 $\beta_{ji} = c_i^*(\sum_{k=1}^m \alpha_{kj} c_k) = \sum_{k=1}^m \alpha_{kj} c_i^*(c_k) = \sum_{k=1}^m \alpha_{kj} \delta_{ik} = \alpha_{ij}$, 故得证. \square