

Chapter 6

主理想整环上的模

定义 6.1 主理想整环(PID): 每个理想均由一个元素生成的整环.

例 6.1: $\mathbb{Z}, \mathbb{C}[x]$ 为 PID. □

PID 必诺特.

\mathbb{R} 为整环, $a, b, r, s \in R$,

(1)

定义 6.2 整除: r 整除 $s \iff s = xr, x \in R$, 记作 $r \mid s$.

(2)

定义 6.3 单位: R 中的可逆元.

例 6.2: \mathbb{Z} 中的 1 和 -1 互逆, 故 1 和 -1 均为单位, 实际上, $\mathbb{Z}^* \equiv \mathbb{Z} - \{0\}$ 中的元素均为单位. □

(3)

定义 6.4 素元: $0 \neq q \in R$, 若 $p \mid ab \implies p \mid a$ 或 $p \mid b$, 则称 p 为素元.

(4)

定义 6.5 不可约元: $0 \neq r \in R$, 若 $r = ab \implies a$ 或 b 为单位, 则称 r 为不可约元.

(5)

定义 6.6 互素: r 与 b 互素 $\implies a$ 与 b 无非单位公因子.

注意:

- 单元必素, 必不可约.

证: 设 $0 \neq r \in R$ 为单位, 则必 $\exists a$ 的逆 a^{-1} .

若 $r \mid ab$, 则 $(ar^{-1})r = a, (br^{-1})r = b \implies r$ 为素元.

若 $r = ab$, 则 $r^{-1}r = r^{-1}(ab) = (r^{-1}a)b = 1, r^{-1}a$ 为 b 的逆元, 即 b 可逆 $\implies r$ 为不可约元. □

- 对于整环来说, 素元不可约, 反之未必.

证: 设 p 为素元, 若 $p = ab$, 则 $1p = p = ab \implies p \mid ab$.

$\because p$ 为素元, $\therefore p \mid a$ 或 $p = b$.

无妨 $p \mid a$, 则 $a = px$, 其中 $x \in R$

$\implies p = ab = pxb \implies p(1 - xb) = 0$,

$\because p \neq 0$ 且 R 为整环 (R 无零因子), $\therefore 1 - xb = 0 \implies xb = 1 \implies b$ 为单位, 故 p 为不可约元. □

例 6.3: (不可约元非素的例子) $R = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$ 为整环.

$9 = 3^2 = (2 + \sqrt{-5})(2 - \sqrt{-5})$,

3 不可约 (证略), $3 \mid (2 + \sqrt{-5})(2 - \sqrt{-5})$, 但 $3 \nmid (2 + \sqrt{-5})$, $3 \nmid (2 - \sqrt{-5}) \implies 3$ 非素. □

- 对于非整环来说, 素元未必不可约.

例 6.4: $(\mathbb{Z}_6, +, \cdot)$ 非整环, $[2]$ 为素元, 但 $[2] = [2][4]$, $[2]$ 和 $[4]$ 均非单位 $\implies [2]$ 可约. □

定理 6.1 (课本定理0.29): R 为 PID, $a, b \in R$,

a 与 b 互素 $\iff \exists r, t \in R$, s.t. $ra + tb = 1$.

证: “ \implies ”: R 为 PID, 令 $I = \langle a, b \rangle$,

$\because R$ 是主理想, $\therefore I$ 可由一个元素生成, 设 $I = \langle c \rangle$, 其中 $c \in R$,

又 $\because a \in I, b \in I, \therefore c \mid a, c \mid b \implies c$ 为 a 和 b 的公因子,

$\because a, b$ 互素, $\therefore c$ 为单位, 即 $\exists c^{-1} \in R$, s.t. $1 = c^{-1}c \in I$,

$\therefore 1 \in I, \therefore 1 = ra + tb$.

“ \impliedby ”: 取 c 为 a 和 b 的公因子,

$\because 1 = ra + tb, \therefore c \mid 1 \implies c$ 可逆, 即 c 为单位. □

有算法可以在给定 a, b 下找到 s, t , 此处不赘述.

定理 6.2 (课本定理0.29): R 是 PID, $\forall 0 \neq r \in R, r = up_1 \cdots p_n$ 且该分解式唯一, 其中 u 为单位, p_i 是 R 中的不可约元, $n \in \mathbb{Z}^+$.

证: 若 r 不可约, 则直接得证.

若 r 可约, 则设 $r = r_1 r_2$, r_1 和 r_2 至少有一个非单位,

无妨 r_1 不是单位, 则 r_1 不可约.

若 r_2 不可约, 则得证,

若 r_2 可约, 则 $\langle r \rangle \subseteq \langle r_2 \rangle$,

对 r_2 继续如上分解, 可得 $\langle r \rangle \subseteq \langle r_2 \rangle \subseteq \cdots$,

又 $\because R$ 为 PID, $\therefore R$ 诺特, 即 $\exists K \in \mathbb{Z}^+$, s.t. $\langle r_K \rangle = \langle r_{K+1} \rangle = \cdots$,

故重复如上分解操作, 最终可将 r 表为有限个不可约元的乘积. □

定义 6.7 挠元(Torsion): $M \in R - \text{mod}$, $v \in M$, 若 $\exists 0 \neq r \in R$, s.t. $rv = 0$, 则称 v 为 M 的挠元.

定义 6.8 挠模: 所有元素均为挠元的模.

定义 6.9 无挠: 若一模无非零挠元, 则称该模无挠.

与线性无关类似, 若 $0 \neq v \in M, r \in R, rv = 0$, 且 M 无挠, 则 $r = 0$.

定义 6.10 挠子模: $M_{\text{tor}} = \{v \in M \mid v \text{ 为挠元}\}.$

$\because 0$ 为 M 的挠元, $0 \in M_{\text{tor}}, \therefore M_{\text{tor}} \neq \emptyset.$

M_{tor} 为 M 的子模.

证: $\forall u, v \in M_{\text{tor}}, \exists 0 \neq r_1, r_2 \in R, \text{ s.t. } r_1 u = 0, r_2 v = 0,$

$\forall s, t \in R, (r_1 r_2)(su + tv) = r_2 s(r_1 u) + r_1 t(r_2 v) = r_2 s \cdot 0 + r_1 t \cdot 0 = 0 + 0 = 0$ 且 $r_1 r_2 \neq 0 \implies (su + tv) \in M_{\text{tor}},$ 故得证. \square

$\frac{M}{M_{\text{tor}}}$ 无挠.

证: 假设 $[0] \neq [v] \in \frac{M}{M_{\text{tor}}}$ 为挠元, 则 $\exists 0 \neq r \in R, r[v] = [rv] = [0] = M_{\text{tor}} \implies rv \in M_{\text{tor}} \implies v = r^{-1}(rv) \in M_{\text{tor}} \implies [v] = M_{\text{tor}} = [0],$ 与假设矛盾, 故假设错误, 得证. \square

定义 6.11 零化子: $v \in M \in R - \text{mod}, v$ 的零化子 $\text{ann}(v) \equiv \{r \in R \mid rv = 0\} \subseteq R.$

N 是 M 的子模, 则 $\text{ann}(N) = \{r \in R \mid rN \equiv \{rv \mid v \in N\} = \{0\}\} \subseteq R.$

$\text{ann}(v)$ 是 R 的理想.

证: $\forall s, t \in \text{ann}(v), sv = tv = 0 \implies sv - tv = (s - t)v = 0 \implies s - t \in \text{ann}(v),$

$\forall r \in R, (rs)v = r(sv) = r \cdot 0 = 0 \implies rs \in \text{ann}(v).$

综上, 得证. \square

同理, $\text{ann}(N)$ 也是 R 的理想

定义 6.12 阶: 若 R 为 PID, 则 $\text{ann}(v), \text{ann}(N)$ 均为主理想, 其生成元分别称为 v 和 N 的阶.

定理 6.3 (课本定理6.5): R 为 PID, $M \in R - \text{mod}$ 自由, 则 M 的子模均自由.

证: (不严谨的证明, 仅针对 M 有限生成 (的特殊情况) 且自由. 设 $M = \langle \langle v_1, \dots, v_n \rangle \rangle = \{\sum_{i=1}^n r_i v_i \mid r_i \in R\},$ 其中 $\{v_1, \dots, v_n\}$ 线性无关.

$\forall v \in M, v = \sum_{i=1}^n r_i v_i$ 展开唯一, 定序后, $M \longleftrightarrow R^n, v \longleftrightarrow \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$ 模同构.

设 S 是 R^n 的子模, 取 R 的理想 $I_k = \{r_k \in R \mid \exists a_1, \dots, a_{k-1} \in R, \text{ s.t. } (a_1, \dots, a_{k-1}, r_k, 0, \dots, 0) \in S\}.$

$\because R$ 为 PID, $\therefore I_k$ 由一个元素生成, 设 $I_k = \langle r_k \rangle$, 其中 $r_k \neq 0, k = 1, \dots, n.$

取 $u_k = (a_1^k, \dots, a_{k-1}^k, r_k, 0, \dots, 0) \in S, S = \langle u_1, \dots, u_n \rangle$ 生成 (下证) 且显然 $\{u_1, \dots, u_n\}$ 线性无关.

取 $(b_1, \dots, b_n) \in S,$ 若 $b_n \neq 0,$ 则 $b_n \in I_n = \langle r_n \rangle \implies \exists x_n \in R, \text{ s.t. } b_n = x_n r_n \implies (b_1, \dots, b_n) - x_n b_n = (\dots, 0),$ 重复如上操作, 最终可将 (b_1, \dots, b_n) 用 $\{u_1, \dots, u_n\}$ 表示.

故得证. \square

定理 6.4 (课本定理6.6): R 为 PID, $M \in R - \text{mod}$ 有限生成,
 M 自由 $\iff M$ 无挠.

证: “ \implies ”: 设 $M = \langle \langle v_1, \dots, v_n \rangle \rangle$ 且 $\{v_1, \dots, v_n\}$ 线性无关.

$$\forall v \in V, v = \sum_{i=1}^n r_i v_i,$$

$$\text{若 } rv = 0, \text{ 则 } r(\sum_{i=1}^n r_i v_i) = \sum_{i=1}^n (rr_i) v_i = 0,$$

$$\because \{v_1, \dots, v_n\} \text{ 线性无关, } \therefore rr_1 = \dots = rr_n = 0,$$

$$\because R \text{ 为整环 (无零因子), } \therefore \text{若 } r \neq 0, \text{ 则 } r_1 = \dots = r_n = 0 \implies v = 0, \text{ 故 } M \text{ 无挠.}$$

$$\text{“}\Leftarrow\text{”}: \text{取 } M = \langle \langle u_1, \dots, u_m \rangle \rangle,$$

无妨设 u_1, \dots, u_k 是其中最大的线性无关组, 即 $\forall i = k+1, \dots, m, \{u_1, \dots, u_k, u_i\}$ 线性相关

$$\implies \exists \text{不全为零的 } a_{i1}, \dots, a_{ik}, a_i, \text{ s.t. } a_{i1}u_1 + \dots + a_{ik}u_k + a_i u_i = 0,$$

$$\text{显然 } a_i \neq 0 \text{ (否则 } a_{i1}u_1 + \dots + a_{ik}u_k = 0 \implies a_{i1} = \dots = a_{ik} = 0, \text{ 矛盾)} \implies a_i u_i = -(a_{i1}u_1 + \dots + a_{ik}u_k).$$

$$\text{令 } a = a_{k+1} \cdots a_m, \text{ 则 } a \neq 0,$$

$$aM = \langle \langle au_1, \dots, au_k, au_{k+1}, \dots, au_m \rangle \rangle \subseteq \langle \langle u_1, \dots, u_k \rangle \rangle,$$

$$\because \{u_1, \dots, u_k\} \text{ 线性无关, } \therefore \langle \langle u_1, \dots, u_k \rangle \rangle \text{ 是自由模,}$$

$$\because R \text{ 为 PID, 自由具有遗传性, } \therefore aM \text{ 自由. 构造映射 } \tau: M \rightarrow aM, v \mapsto av.$$

(1) τ 线性.

$$(2) \because M \text{ 无挠且 } a \neq 0, \therefore \ker \tau = \{v \in M \mid av = 0\} = \{0\}.$$

(3) τ 满射.

故 τ 同构 $\implies M$ 也自由.

综上, 得证. □

$$\because M \text{ 自由, } \therefore M = \langle \langle v_1, \dots, v_n \rangle \rangle,$$

$$\text{又 } \because \{v_1, \dots, v_n\} \text{ 线性无关, } \therefore \text{对 } i \neq j, \langle \langle v_i \rangle \rangle \cap \langle \langle v_j \rangle \rangle = \{0\} \implies M = \langle \langle v_1 \rangle \rangle \oplus \dots \oplus \langle \langle v_n \rangle \rangle.$$

定理 6.5 (课本定理6.8): R 是 PID, $M \in R - \text{mod}$ 有限生成, 则 $M = M_{\text{free}} \oplus M_{\text{tor}}$.

证: M_{tor} 为挠子模且 $\frac{M}{M_{\text{tor}}}$ 无挠.

$$\because \Pi: M \rightarrow \frac{M}{M_{\text{tor}}}, u \mapsto [u] \text{ 满同态且 } M \text{ 有限生成, 由引理 6.1 得 } \frac{M}{M_{\text{tor}}} \text{ 有限生成.}$$

$$\text{又 } \because \frac{M}{M_{\text{tor}}} \text{ 无挠, } \therefore \frac{M}{M_{\text{tor}}} \text{ 自由.}$$

$$\text{取 } \frac{M}{M_{\text{tor}}} = \langle \langle [u_1], \dots, [u_t] \rangle \rangle, \text{ 其中 } \{u_1, \dots, u_t\}, \text{ 线性无关 (下证),}$$

$$\text{证: 若 } \sum_{i=1}^t r_i u_i = 0, \text{ 则 } \Pi\left(\sum_{i=1}^t r_i u_i\right) = \sum_{i=1}^t r_i \Pi(u_i) = \sum_{i=1}^t r_i [u_i] = 0,$$

$$\text{又 } \because \{[u_1], \dots, [u_t]\} \text{ 线性无关, } \therefore r_1 = \dots = r_t = 0 \implies \{u_1, \dots, u_t\} \text{ 线性无关.} \quad \square$$

故 $\langle \langle u_1, \dots, u_t \rangle \rangle$ 为自由模, 记为 M_{free} .

确定了 M_{free} 和 M_{tor} 后, 下面来证 $M = M_{\text{free}} \oplus M_{\text{tor}}$:

$$\forall v \in M, \Pi(v) = [v] \in \frac{M}{M_{\text{tor}}} = \langle \langle [u_1], \dots, [u_t] \rangle \rangle \implies \Pi(v) = [v] = \sum_{i=1}^t l_i [u_i].$$

$$\text{令 } u = \sum_{i=1}^t l_i u_i \in M_{\text{free}}, \text{ 则 } \tau(u) = \tau\left(\sum_{i=1}^t l_i u_i\right) = \sum_{i=1}^t l_i \Pi(u_i) = \sum_{i=1}^t l_i [u_i] = \Pi(v).$$

$$\Pi(v - u) = \Pi(v) - \Pi(u) = 0 \implies v - u \in \ker \Pi = M_{\text{tor}},$$

$$\text{于是 } v = u + (v - u), \text{ 其中 } u \in M_{\text{free}}, v - u \in M_{\text{tor}} \implies M = M_{\text{free}} + M_{\text{tor}}.$$

取 $w \in M_{\text{free}} \cap M_{\text{tor}}$, 则 $w \in M_{\text{free}} \iff w = \sum_{i=1}^t \alpha_i u_i$,
 且 $w \in M_{\text{tor}} \iff \Pi(w) = 0$
 $\implies 0 = \Pi(w) = \Pi\left(\sum_{i=1}^t \alpha_i u_i\right) = \sum_{i=1}^t \alpha_i \Pi(u_i) \implies \alpha_1 = \cdots = \alpha_t = 0 \implies w = 0 \implies M_{\text{free}} \cap M_{\text{tor}} = \{0\}$.
 综上, 得证. □

引理 6.1: $\tau: M \rightarrow N$ 满同态, 若 M 有限生成, 则 N 有限生成.

证: $\because \tau: M \rightarrow N$ 满同态, $\therefore \forall w \in N, \exists u \in M, \text{ s.t. } w = \tau(u)$,
 又 $\because M$ 有限生成, 设 $M = \langle \langle v_1, \dots, v_k \rangle \rangle$, $\therefore u = \sum_{i=1}^k r_i u_i \implies \tau(u) = \tau\left(\sum_{i=1}^k r_i u_i\right) = \sum_{i=1}^k r_i \tau(u_i)$,
 故 $N = \langle \langle \tau(u_1), \dots, \tau(u_k) \rangle \rangle$, 即 N 有限生成. □

至此, $M_{\text{free}} = \langle \langle u_1, \dots, u_t \rangle \rangle = \langle \langle u_1 \rangle \rangle \oplus \cdots \oplus \langle \langle u_t \rangle \rangle$ 已拆解到位. 那么能否以及如何继续拆解 M_{tor} 呢?

定理 6.6 (课本定理6.10): R 为 PID, $M \in R - \text{mod}$ 为挠模且 $\text{ann}(M) = \langle \langle \mu \rangle \rangle$, 其中 $\mu = up_1 e_1 \cdots p_m^{e_m}$, p_i 均不可约且互不相等, $e_i \in \mathbb{Z}^+ \forall i$,
 则 $M = M_{p_1} \oplus \cdots \oplus M_{p_m}$, 其中 $M_{p_i} = \{v \in M \mid p_i^{e_i} v = 0\}$ 是阶为 $p_i^{e_i}$ (即 $\text{ann}(M_{p_i}) = \langle p_i^{e_i} \rangle$) 的准素子模.

证: 不失一般性, 设 $\mu = pq$, p 与 q 互素, 要证 $M = M_p \oplus M_q$, 其中 $M_p = \{v \mid pv = 0\}$, $M_q = \{v \mid qv = 0\}$.
 $\because p$ 与 q 互素, $\therefore \exists r, t \in R, \text{ s.t. } rp + tq = 1$.

$\forall v \in M, v = 1v = (rp + tq)v = (rp)v + (tq)v$,
 $q(rp)v = (qrp)v = (rpq)v = r(pq)v = r\mu v$,
 又 $\because \langle \langle \mu \rangle \rangle$ 为零化子, $\therefore q(rp)v = r\mu v = 0 \implies rp v \in M_q$,
 同理, $tqv \in M_p$, 故 $M = M_p + M_q$.

若 $v \in M_p \cap M_q$, 则 $v \in M_p \iff pv = 0$,
 且 $v \in M_q \iff qv = 0$
 $\implies v = 1v = (rp + tq)v = rp v + tqv = r0 + t0 = 0 + 0 = 0 \implies M_p = M_q = \{0\}$.
 $\because M_p = \{v \mid pv = 0\}$, $\therefore \text{ann}(M_p) = \langle p \rangle$, 易推广得 $M_{p_i} = \langle p_i^{e_i} \rangle$.
 综上, 得证. □

然后准子模能否进一步分解呢?

定理 6.7 (课本定理6.11): R 为 PID, $M \in R - \text{mod}$ 有限生成, $\text{ann}(M) = \langle p^e \rangle$, 其中 p 不可约, $e \in \mathbb{Z}^+$,
 则 $M = \langle \langle v_1 \rangle \rangle \oplus \cdots \oplus \langle \langle v_n \rangle \rangle$, 其中 $\text{ann}(v_i) = \langle p^{e_i} \rangle$, 且 $e = e_1 \geq \cdots \geq e_n$.

证: (存在性证明) 不失一般性, 只需证 M 由两个生成元时, 定理成立, 即可由数学归纳法推广到一般情况.

设 $M = \langle \langle u_1, u_2 \rangle \rangle$ 且 $u_1, u_2 \neq 0$, $\text{ann}(M) = \{r \in R \mid rM = \{0\}\} = \langle p^e \rangle$.

$\because u_1 \in M, \therefore p^e u_1 = 0 \implies p^e \in \text{ann}(u_1)$,

同理, $p^e \in \text{ann}(u_2)$.

若 $\text{ann}(u_1) = \langle b_1 \rangle$, 则 $\because p$ 不可约, $\therefore b_1 \mid p^e \implies b_1 = p^{l_1}, l_1 \leq e$,

同理, 若 $\text{ann}(u_2) = \langle b_2 \rangle$, 则 $b_2 = p^{l_2}, l_2 \leq e$.

假设 $l_1 < e, l_2 < e$, 令 $l = \max\{l_1, l_2\}$, 则 $p^e \nmid p^l$ 且 $p^l \in \text{ann}(M)$, 与 $\text{ann}(M) = \langle p^e \rangle$ 矛盾, 故假设错误, l_1, l_2 中至少有一个 $= e$.

不妨设 $l_1 = e$ 即 $\text{ann}(u_1) = \langle p^e \rangle$.

$M = \langle \langle u_1, u_2 \rangle \rangle \implies M = \langle \langle u_1 \rangle \rangle + \langle \langle u_2 \rangle \rangle$,

若 $\langle \langle u_1 \rangle \rangle \cap \langle \langle u_2 \rangle \rangle = \{0\}$, 则 $M = \langle \langle u_1 \rangle \rangle \oplus \langle \langle u_2 \rangle \rangle$, 得证.

若 $\langle\langle u_1 \rangle\rangle \cap \langle\langle u_2 \rangle\rangle \neq \{0\}$, 则 $\exists 0 \neq r \in R$, s.t. $ru_2 \in \langle\langle u_1 \rangle\rangle$.

取 R 的理想 $J = \{r \in R \mid ru_2 \in \langle\langle u_1 \rangle\rangle\}$.

$\because R$ 为 PID, $\therefore J$ 由一个元素生成, 设 $J = \langle\langle t \rangle\rangle$.

$\because p^e u_2 = 0 \implies p^e \in J$, $\therefore p^e \in J \implies t \mid p^e$,

又 $\because p$ 不可约, $\therefore t = p^{e_2}$ 且 $e_2 \leq e$,

又 $\because J = \{r \in R \mid ru_2 \in \langle\langle u_1 \rangle\rangle\} = \langle\langle t \rangle\rangle$, $\therefore p^{e_2} u_2 \in \langle\langle u_1 \rangle\rangle$, 即 $\exists \alpha \in R$, s.t. $p^{e_2} u_2 - \alpha u_1 = 0$

$\implies p^{e-e_2}(p^{e_2} u_2 - \alpha u_1) = 0 \implies p^e u_2 - p^{e-e_2} \alpha u_1 = 0$,

又 $\because p^e u_2 = 0$, $\therefore p^{e-e_2} \alpha u_1 = 0 \implies p^{e-e_2} \alpha \in \text{ann}(u_1)$,

又 $\because \text{ann}(u_1) = \langle\langle p^e \rangle\rangle$, $\therefore p^e \mid p^{e-e_2} \alpha \implies p^{e_2} \mid \alpha \implies \exists \beta \in R$, s.t. $\alpha = \beta p^{e_2}$,

回代到 $p^{e_2} u_2 - \alpha u_1 = 0$ 得 $p^{e_2} u_2 - p^{e_2} \beta u_1 = 0 \implies p^{e_2}(u_2 - \beta u_1) = 0$.

令 $w = u_2 - \beta u_1$, 则 $M = \langle\langle u_1, w \rangle\rangle$, 且 $\langle\langle u_1 \rangle\rangle \cap \langle\langle w \rangle\rangle = \{0\}$ (下证),

证: 设 $v \in \langle\langle u_1 \rangle\rangle \cap \langle\langle w \rangle\rangle$, 则 $v \in \langle\langle u_1 \rangle\rangle$,

且 $v \in \langle\langle w \rangle\rangle \implies \exists r \in R$, $v = rw$

$\implies v = rw = ru_2 - r\beta u_1 \in \langle\langle u_1 \rangle\rangle$,

$\because r\beta u_1 \in \langle\langle u_1 \rangle\rangle$, $\therefore ru_2 \in \langle\langle u_1 \rangle\rangle$, (由 J 的定义) 即 $r = p^{e_2} r_1$,

回代得 $v = rw = p^{e_2} r_1 u_2 - p^{e_2} r\beta u_1 = p^{e_2} r_1 u_2 - p^{e_2} r_1 \beta u_1 = p^{e_2} r_1 u_2 - r_1 (\beta p^{e_2}) u_1 = r_2 (p^{e_2} u_2 - \alpha u_1) = r_2 0 = 0 \implies \langle\langle u_1 \rangle\rangle \cap \langle\langle w \rangle\rangle = \{0\}$. □

故 $M = \langle\langle u_1 \rangle\rangle \oplus \langle\langle w \rangle\rangle$, 其中 u_1 的阶为 p^{e_1} , w 的阶为 p^{e_2} , $e_2 \leq e_1 = e$. □

总结定理 6.5, 6.6 和 6.7, 可得:

定理 6.8 (课本定理6.12): R 为 PID, $M \in R\text{-mod}$ 有限生成且为挠模, $\text{ann}(M) = \langle\langle \mu \rangle\rangle$, 其中 $\mu = up_1^{e_1} \cdots p_m^{e_m}$, 则 $M = M_{p_1} \oplus \cdots \oplus M_{p_m} = \left(\bigoplus_{i=1}^{t_1} \langle\langle v_{1i} \rangle\rangle \right) \oplus \cdots \oplus \left(\bigoplus_{i=1}^{t_m} \langle\langle v_{mi} \rangle\rangle \right)$, 其中 $\text{ann}(v_{ij}) = p_i^{e_{ij}}$, $e_i = e_{i1} \geq \cdots \geq e_{it_i}$.