Chapter 10

正规算子的结构理论

10.1 线性算子的伴随

先来回顾一下算子伴随: $\mathcal{B}$ 和 $\mathcal{C}$ 分别为线性空间 V 和 W 的定序基, V^* 和 W^* 分别为 V 和 W 的对偶空间, $\mathcal{B}^*$ 和 $\mathcal{C}^*$ 分别为 $\mathcal{B}$ 和 $\mathcal{C}$ 的对偶基, 对给定的线性变换 $\tau \in \mathcal{L}(V, W)$, 有算子伴随 $\tau^\times : W^* \to V^*$, $g \mapsto \tau^\times = g \circ \tau$, 线性变换在定序基上与其算子伴随在对偶基上的表示存在关系: $[\tau]_{\mathcal{B}\mathcal{C}} = [\tau^\times]_{\mathcal{C}^*\mathcal{B}^*}$.

$$
\begin{tikzcd}
V \arrow[rr, "\tau"] \arrow[dr, "\tau^\times = g \circ \tau"'] & & W \arrow[dl, "g"] \\
& F &
\end{tikzcd}
$$

下面来定义另一种伴随: 对于有限维内积向量空间 V 和 W, $\dim V = n$, $\dim W = m$, Riesz 映射 $\mathcal{R}_V : V^* \to V$, $\mathcal{R}_W : W^* \to W$, $\because \mathcal{R}_W$ 为共轭同构, $\therefore$ 有其逆同构 $\mathcal{R}_W^{-1}$, 从而有映射 $\tau^* = \mathcal{R}_V \circ \tau^\times \circ \mathcal{R}_W$.

$$
\begin{tikzcd}
\mathcal{R}_V(\tau^\times(\mathcal{R}_W^{-1}(u))) & & & u \\
& V \arrow[r, "\tau"] & W \arrow[l, dashed, bend left, "\tau^*?"] & \\
& V^* \arrow[u, "\mathcal{R}_V"] & W^* \arrow[l, "\tau^\times"] \arrow[u, "\mathcal{R}_W"'] & \\
\tau^\times(\mathcal{R}_W^{-1}(u)) \arrow[uuu, mapsto] & & & \mathcal{R}_W^{-1}(u) \arrow[lll, mapsto] \arrow[uuu, mapsto]
\end{tikzcd}
$$

定理 10.1 (课本定理10.1): (1) τ^* 为线性变换, 即 $\tau^* \in \mathcal{L}(W,V)$.

(2) $\langle v, \tau^*(w)\rangle = \langle \tau(v), w\rangle$, 称 τ^* 为 τ 的伴随.

(3) $[\tau]_{\mathcal{BC}} = [\tau^*]^\dagger_{\mathcal{CB}}$.

证: (1) $\tau^* = \mathcal{R}_V \circ \tau^\times \circ \mathcal{R}_W^{-1} : W \to V$.
$\forall u_1, u_2 \in W$, $\tau^*(ru_1 + tu_2) = \mathcal{R}_V \circ \tau^\times \circ \mathcal{R}_W^{-1}(ru_1+tu_2) = \mathcal{R}_V \circ \tau^\times(\mathcal{R}_W^{-1}(ru_1+tu_2)) = \mathcal{R}_V \circ \tau^\times(\bar{r}\mathcal{R}_W^{-1}(u_1) + \bar{t}\mathcal{R}_W^{-1}(u_2)) = \mathcal{R}_V(\tau^\times(\bar{r}\mathcal{R}_W^{-1}(u_1) + \bar{t}\mathcal{R}_W^{-1}(u_2))) = \mathcal{R}_V(\bar{r}\tau^\times(\mathcal{R}_W^{-1}(u_1)) + \bar{t}\tau^\times(\mathcal{R}_W^{-1}(u_2))) = r\mathcal{R}_V(\tau^\times(\mathcal{R}_W^{-1}(u_1))) + t\mathcal{R}_V(\tau^\times(\mathcal{R}_W^{-1}(u_2))) = r\tau^*(u_1) + t\tau^*(u_2)$, 其中利用了引理 10.1, 得证.

(2) $\forall v \in V$, $\forall w \in W$, $\langle v, \tau^*(w)\rangle = \langle v, \mathcal{R}_V(\tau^\times \circ \mathcal{R}_W^{-1}(w))\rangle = \tau^\times \circ \mathcal{R}_W^{-1}(w)(v) = \mathcal{R}_W^{-1}(w) \circ \tau(v) = \mathcal{R}_W^{-1}(w)(\tau(v)) = \langle \tau(v), w\rangle$.

(3) 设 V 的正交归一基 $\mathcal{B} = \{b_1, \cdots, b_n\}$, W 的正交归一基 $\mathcal{C} = \{c_1, \cdots, c_n\}$.
$[\tau]_{\mathcal{BC}} = \begin{pmatrix} [\tau(b_1)]_{\mathcal{C}} & \cdots & [\tau(b_n)]_{\mathcal{C}} \end{pmatrix}$, $[\tau^*]_{\mathcal{CB}} = \begin{pmatrix} [\tau(c_1)]_{\mathcal{B}} & \cdots & [\tau(c_m)]_{\mathcal{B}} \end{pmatrix}$.

设 $[\tau(b_i)]_{\mathcal{C}} = \begin{pmatrix} \alpha_{1i} \\ \vdots \\ \alpha_{mi} \end{pmatrix}$, $\tau(b_i) = \sum_{k=1}^m \alpha_{ki}c_k$, $\langle \tau(b_i), c_j\rangle = \langle \sum_{k=1}^m \alpha_{ki}c_k, c_j\rangle = \sum_{k=1}^m \alpha_{ki}\langle c_k, c_j\rangle = \sum_{k=1}^m \alpha_{ki}\delta_{kj} = \alpha_{ji}$.

同理, 设 $[\tau(c_j)]_{\mathcal{B}} = \begin{pmatrix} \beta_{1j} \\ \vdots \\ \beta_{nj} \end{pmatrix}$, $\tau^*(c_j) = \sum_{k=1}^n \beta_{kj}b_k$, $\langle \tau^*(c_j), b_i\rangle = \langle \sum_{k=1}^n \beta_{kj}b_k, b_i\rangle = \sum_{k=1}^n \beta_{kj}\langle b_k, b_i\rangle = \sum_{k=1}^n \beta_{kj}\delta_{ki} = \beta_{ij}$.
又 $\because \langle \tau(b_i), c_j\rangle = \langle b_i, \tau^*(c_j)\rangle = \overline{\langle \tau^*(c_j), b_i\rangle}$, $\therefore \alpha_{ji}^* = \beta_{ij} \Longrightarrow [\tau]_{\mathcal{BC}} = [\tau^*]^\dagger_{\mathcal{CB}}$.

□

引理 10.1: Riesz 映射的逆 $\mathcal{R}^{-1}$ 共轭线性.

证: $\forall x_1, x_2 \in V$, $\exists f_1 = \mathcal{R}^{-1}(x_1), f_2 = \mathcal{R}^{-1}(x_2) \in V^*$, s.t. $\forall v \in V$, $f_1(v) = \langle v, x_1\rangle$, $f_2(v) = \langle v, x_2\rangle$
$\Longrightarrow \forall \bar{r}, \bar{t} \in F$, $(\bar{r}\mathcal{R}^{-1}(x_1) + \bar{t}\mathcal{R}^{-1}(x_2))(v) = (\bar{r}f_1 + \bar{t}f_2)(v) = \bar{r}f_1(v) + \bar{t}f_2(v) = \bar{r}\langle v, x_1\rangle + \bar{t}\langle v, x_2\rangle = \langle v, rx_1\rangle + \langle v, tx_2\rangle = \langle v, rx_1 + tx_2\rangle = \mathcal{R}^{-1}(rx_1 + rx_2)(v)$
$\Longrightarrow \mathcal{R}^{-1}(rx_1 + tx_2) = \bar{r}\mathcal{R}^{-1}(x_1) + \bar{t}\mathcal{R}^{-1}(x_2)$. □

$\because [\tau]_{\mathcal{BC}} = [\tau^\times]^T_{\mathcal{C}^*\mathcal{B}^*}$, $[\tau]_{\mathcal{BC}} = [\tau^*]^\dagger_{\mathcal{CB}}$, $\therefore [\tau^\times]_{\mathcal{C}^*\mathcal{B}^*} = \overline{[\tau^*]_{\mathcal{CB}}}$. 当然这也可用类似定理 10.1 (3) 的证明方法证明:

证: $[\tau^\times]_{\mathcal{C}^*\mathcal{B}^*} = \begin{pmatrix} [\tau^\times(c_1^*)]_{\mathcal{B}^*} & \cdots & [\tau^\times(c_n^*)]_{\mathcal{B}^*} \end{pmatrix}$, $[\tau^*]_{\mathcal{CB}} = \begin{pmatrix} [\tau^*(c_1)]_{\mathcal{B}^*} & \cdots & [\tau^*(c_n)]_{\mathcal{B}} \end{pmatrix}$,

设 $[\tau^\times(c_i^*)]_{\mathcal{B}^*} = \begin{pmatrix} \alpha_{1i} \\ \vdots \\ \alpha_{ni} \end{pmatrix}$, 则 $\tau^\times(c_i^*) = \sum_{k=1}^n \alpha_{ki}b_k^*$.

设 $[\tau^*(c_i)]_{\mathcal{B}} = \begin{pmatrix} \beta_{1i} \\ \vdots \\ \beta_{mi} \end{pmatrix}$, 则 $\tau^*(c_i) = \sum_{k=1}^m \beta_{ki}b_k$.

一方面, $\because \mathcal{R}_W^{-1}(c_i)(c_j) = \langle c_j, c_i\rangle = \delta_{ij}$, $\therefore \mathcal{R}_V(c_i) = c_i^*$
$\Longrightarrow \langle b_j, \tau^*(c_i)\rangle = \langle b_j, \mathcal{R}_V \circ \tau^\times \circ \mathcal{R}_W^{-1}(c_i)\rangle = \langle b_j, \mathcal{R}_V(\tau^\times(\mathcal{R}_W^{-1}(c_i)))\rangle = \langle b_j, \mathcal{R}_V(\tau^\times(c_i^*))\rangle = \tau^\times(c_i^*)(b_j) = \left(\sum_{k=1}^n \alpha_{ki}b_k^*\right)(b_j) =$

$\sum_{k=1}^n \alpha_{ki} b_k^*(b_j) = \sum_{k=1}^n \alpha_{ki}\delta_{jk} = \alpha_{ji}$;
另一方面, $\langle b_j, \tau^*(c_i)\rangle = \langle b_j, \sum_{k=1}^m \beta_{ki} b_k\rangle = \sum_{k=1}^m \overline{\beta_{ki}}\langle b_j, b_k\rangle = \sum_{k=1}^m \overline{\beta_{ki}}\delta_{jk} = \overline{\beta_{ji}}$.
故 $\alpha_{ji} = \overline{\beta_{ji}}$, 得证. □

定理 10.2 (课本定理10.2): V, W 为有限维内积向量空间, $\forall \sigma, \tau \in \mathcal{L}(V, W)$, $\forall r \in F$,

(1) $(\sigma + \tau)^* = \sigma^* + \tau^*$.

(2) $(r\tau)^* = \bar{r}\tau^*$.

(3) $\tau^{**} = \tau$.

(4) 若 $V = W$, 则 $(\tau \circ \sigma)^* = \sigma^* \circ \tau^*$.

(5) $V = W$, 若 τ 可逆, 则 $(\tau^{-1})^* = (\tau^*)^{-1}$.

(6) $V = W$, $p(x) \in \mathbb{R}[x]$, 则 $p(\tau)^* = p(\tau^*)$.

(7) S 是 V 的子空间, $\tau \in \mathcal{L}(V)$, 则 S 是 τ 的不变子空间 $\Longleftrightarrow$ $S^\perp$ 是 τ^* 的不变子空间.

证: (1) $\forall u \in W$, $\forall v \in V$, $\langle v, (\sigma+\tau)^*(u)\rangle = \langle(\sigma+\tau)(v), u\rangle = \langle\sigma(v)+\tau(v), u\rangle = \langle\sigma(v), u\rangle + \langle\tau(v), u\rangle = \langle v, \sigma^*(u)\rangle + \langle v, \tau^*(u)\rangle = \langle v, \sigma^*(u) + \tau^*(u)\rangle = \langle v, (\sigma^* + \tau^*)(u)\rangle \Longrightarrow (\sigma+\tau)^*(u) = (\sigma^*+\tau^*)(u) \Longrightarrow (\sigma+\tau)^* = \sigma^* + \tau^*$.

(2) $\forall u \in W$, $\forall v \in V$, $\langle v, (r\tau)^*(u)\rangle = \langle r\tau(v), u\rangle = r\langle\tau(v), u\rangle = r\langle v, \tau^*(u)\rangle = \langle v, \bar{r}\tau^*(u)\rangle \Longrightarrow (r\tau)^*(u) = \bar{r}\tau(u) \Longrightarrow (r\tau)^* = \bar{r}\tau^*$.

(3) $\forall u \in W$, $\forall v \in V$, $\langle u, \tau^{**}(v)\rangle = \langle u, (\tau^*)^*(v)\rangle = \langle\tau^*(u), v\rangle = \overline{\langle v, \tau^*(u)\rangle} = \overline{\langle\tau(v), u\rangle} = \langle u, \tau(v)\rangle \Longrightarrow \tau^{**}(v) = \tau(v) \Longrightarrow \tau^{**} = \tau$.

(4) $\forall v \in V$, $\forall u \in W$, $\langle u, (\tau\circ\sigma)^*(v)\rangle = \langle(\tau\circ\sigma)(u), v\rangle = \langle\tau(\sigma(u)), v\rangle = \langle\sigma(u), \tau^*(v)\rangle = \langle u, \sigma^*(\tau^*(v))\rangle = \langle u, (\sigma^* \circ \tau^*)(v)\rangle \Longrightarrow (\tau\circ\sigma)^*(v) = (\sigma^*\circ\tau^*)(v) \Longrightarrow (\tau\circ\sigma)^* = \sigma^*\circ\tau^*$.

$$\begin{tikzcd} V \arrow[r, shift left, "\sigma"] & V \arrow[l, shift left, "\sigma^*"] \arrow[r, shift left, "\tau"] & V \arrow[l, shift left, "\tau^*"] \end{tikzcd}$$

$$\begin{tikzcd} V \arrow[r, shift left, "\sigma\circ\tau"] & V \arrow[l, shift left, "(\sigma\circ\tau)^*"] \end{tikzcd}$$

(5) $(\tau^{-1})^* \circ \tau^* = (\tau \circ \tau^{-1})^* = 1_V^* = 1_V \Longrightarrow (\tau^{-1})^* = (\tau^*)^{-1}$.

$$(\because \langle u, v\rangle = \mathcal{R}^{-1}(v)(u) = (\mathcal{R}^{-1}(v)\circ 1_V)(u) = \langle u, \mathcal{R}_V(\mathcal{R}_V^{-1}(v)\circ 1_V)\rangle)$$

$$\begin{tikzcd}
\mathcal{R}_V(\mathcal{R}_V^{-1}(v)\circ 1_V) = v & & & v \\
& V \arrow[r, "1_V"] & V \arrow[l, dashed, bend left, "1_V^* = 1_V?"] & \\
& V^* \arrow[u, "\mathcal{R}_V"] & V^* \arrow[u, "\mathcal{R}_V"'] \arrow[l, "1_V^\times"'] & \\
1_V^\times(\mathcal{R}_V^{-1}(v)) = \mathcal{R}_V^{-1}(v)\circ 1_V \arrow[uuu, mapsto] & & & \mathcal{R}_V^{-1}(v) \arrow[lll, mapsto] \arrow[uuu, mapsto]
\end{tikzcd}$$

(6) $(\tau\circ\tau)^* = \tau^*\circ\tau^*$, $(\tau^k)^* = (\tau^*)^k$.
若 $r\in\mathbb{R}$, 则 $(r\tau)^* = rt^*$, $(r\tau^k)^* = r(\tau^*)^k$
$\Longrightarrow (p(\tau))^* = p(\tau^*)$.

(7) "$\Longrightarrow$": $\because S$ 是 τ 的不变子空间, $\therefore \tau(S)\subseteq S$.
$\forall v\in S^\perp$, $\forall u\in S$, $\tau(u)\in S \Longrightarrow \langle u, \tau^*(v)\rangle = \langle\tau(u), v\rangle = 0 \Longrightarrow \tau^*(v)\in S^\perp \Longrightarrow S^\perp$ 是 τ^* 的不变子空间.
"$\Longleftarrow$": $\because S^{\perp\perp} = S$, 故同理得证.

□

定理 10.3 (<u>课本定理10.3</u>): V, W 为有限维内积向量空间, $\tau\in\mathcal{L}(V,W)$, 则

(1) $\ker\tau^* = (\operatorname{Im}\tau)^\perp$, 等价地, $\operatorname{Im}\tau^* = (\ker\tau)^\perp$.

(2) $\ker\tau^*\tau = \ker\tau$, $\ker\tau\tau^* = \ker\tau^*$.

(3) $\operatorname{Im}\tau^*\tau = \operatorname{Im}\tau^*$, $\operatorname{Im}\tau\tau^* = \operatorname{Im}\tau$.

(4) $\rho_{ST}^* = \rho_{T^\perp S^\perp}$.

证: (1) $\forall w\in\operatorname{Im}\tau$, $\exists u\in V$, s.t. $w = \tau(u)$.
$v\in\ker\tau^* \iff \tau^*(v) = 0 \iff \langle w, v\rangle = \langle\tau(u), v\rangle = \langle u, \tau^*(v)\rangle = \langle u, 0\rangle = 0 \iff v\in(\operatorname{Im}\tau)^\perp$, 故 $\ker\tau^* = (\operatorname{Im}\tau)^\perp$.

$\forall w\in\ker\tau$, $\iff\tau(w) = 0\in W$.
$v\in\operatorname{Im}\tau^* \iff \exists u\in W$, s.t. $\tau^*(u) = v \iff \langle w, v\rangle = \langle w, \tau^*(u)\rangle = \langle\tau(w), u\rangle = \langle 0, u\rangle = 0 \iff v\in(\ker\tau)^\perp$, 故 $\operatorname{Im}\tau^* = (\ker\tau)^\perp$.

(2) $\forall v\in\ker\tau\tau^*$, $\tau^*\tau(v) = 0 \Longrightarrow \langle v, \tau^*\tau(v)\rangle = 0 \iff \langle\tau(v), \tau(v)\rangle = 0 \Longrightarrow \tau(v) = 0 \iff v\in\ker\tau$, 故 $\ker\tau^*\tau\subseteq\ker\tau$.

$\forall v\in\ker\tau$, $\tau(v) = 0 \Longrightarrow \tau^*\tau(v) = 0 \iff v\in\ker\tau^*\tau$, 故 $\ker\tau\subseteq\ker\tau^*\tau$.

综上, $\ker\tau^*\tau = \ker\tau$.

同理, $\ker\tau\tau^* = \ker\tau^*$.

(3) $\forall v \in \operatorname{Im} \tau^*\tau$, $\exists u \in V$, s.t. $u = \tau^*\tau(v) = \tau^*(\tau(v))$, 即 $\exists w = \tau(v) \in W$, s.t. $v = \tau^*(w)$, 故 $\operatorname{Im} \tau^*\tau \in \operatorname{Im} \tau^*$.

$\forall v \in \operatorname{Im} \tau^*$, $\exists w \in W$, s.t. $v = \tau^*(w)$.
$\because \tau$ 为共轭同构, $\therefore \exists u \in V$, s.t. $w = \tau(u) \Longrightarrow v = \tau^*\tau(u)$, 故 $\operatorname{Im} \tau^* \in \operatorname{Im} \tau^*\tau$.

综上, $\operatorname{Im} \tau^*\tau = \operatorname{Im} \tau^*$.

同理, $\operatorname{Im} \tau\tau^* = \operatorname{Im} \tau$.

(4) $\forall u, v \in V$,
$\rho_{ST} : V \to V$, $u = u_S + u_T \mapsto u_S$, $v = v_S + v_T \mapsto v_S$, 其中 $u_S \in S$, $u_T \in T$, $v_S \in S$, $v_T \in T$, $V = S \oplus T$,
$\rho_{T^\perp S^\perp} : V \to V$, $u = u_{S^\perp} + u_{T^\perp} \mapsto u_{T^\perp}$, $v = v_{S^\perp} + v_{T^\perp} \mapsto v_{T^\perp}$, 其中 $u_{S^\perp} \in S^\perp$, $u_{T^\perp} \in T^\perp$, $v_{S^\perp} \in S^\perp$, $v_{T^\perp} \in T^\perp$, $V = S^\perp \oplus T^\perp$.
$\because \langle u, \rho_{ST}^*(v)\rangle = \langle \rho_{ST}(u), v\rangle = \langle u_S, v\rangle$, $\langle u, \rho_{T^\perp S^\perp}(v)\rangle = \langle u, v_{T^\perp}\rangle$,
$\therefore \langle u, \rho_{ST}^*(v)\rangle - \langle u, \rho_{T^\perp S^\perp}(v)\rangle = \langle u_S, v\rangle - \langle u, v_{T^\perp}\rangle = \langle u_S, v\rangle - \langle u_S, v_{T^\perp}\rangle + \langle u_S, v_{T^\perp}\rangle - \langle u, v_{T^\perp}\rangle = \langle u_S, v - v_{T^\perp}\rangle - \langle u - u_S, v_{T^\perp}\rangle = \langle u_S, v_{S^\perp}\rangle - \langle u_T, v_{T^\perp}\rangle$.
又 $\because v_S \in S$, $u_{S^\perp} \in S^\perp$, $v_T \in T$, $u_{T^\perp} \in T^\perp$, $\therefore \langle u_S, u_{S^\perp}\rangle = 0$, $\langle u_T, v_{T^\perp}\rangle = 0 \Longrightarrow \langle u, \rho_{ST}^*(v)\rangle - \langle u, \rho_{T^\perp S^\perp}(v)\rangle = 0 \Longrightarrow \langle u, \rho_{ST}^*(v)\rangle = \langle u, \rho_{T^\perp S^\perp}(v)\rangle \Longrightarrow \rho_{ST}^*(v) = \rho_{T^\perp S^\perp}(v) \Longrightarrow \rho_{ST}^* = \rho_{T^\perp S^\perp}$.

□

10.2 正交(/幺正)对角化

先来回顾一下线性变换可对角化的充要条件: $\tau \in \mathcal{L}(V)$, τ 可对角化 (即 $\exists$ 一组基 $\mathcal{B}$, $[\tau]_\mathcal{B}$ 为对角阵)
$\Longleftrightarrow m_\tau(x) = (x - \lambda_1) \cdots (x - \lambda_k)$, 其中 λ_i 互不相同
$\Longleftrightarrow V = \mathcal{E}_{\lambda_1} \oplus \cdots \oplus \mathcal{E}_{\lambda_k}$
$\Longleftrightarrow \tau$ 的特征向量构成 V 的一组基
$\Longleftrightarrow$ 几何重数 (特征子空间的维数) = 代数重数 (特征多项式的根的重数)
$\Longleftrightarrow \tau = \lambda_1\rho_1 + \cdots + \lambda_k\rho_k$, 其中 λ_i 互不相同, $\rho_1 + \cdots + \rho_k = 1$ 为单位分解 (即 ρ_i 为投影, $\sum_i \rho_i = 1$ 且 $\rho_i\rho_j = \rho_j\rho_i = \delta_i j\rho_i$).

再来回顾一下向量正交: 向量 u 与 v 正交 $\Longleftrightarrow \langle u, v\rangle = 0$.
非零元构成的正交集线性无关.
若 $\dim V < \infty$, 则 V 有正交归一基.

那么, τ 是否可正交对角化? 哪一类 τ 可正交对角化?

定义 10.1 **<u>正交(/幺正)对角化</u>:** $\tau \in \mathcal{L}(V)$, 若 $\exists$ 一组正交归一基 $\mathcal{O}$, s.t. $[\tau]_\mathcal{O}$ 为对角阵, 则称 τ 可**正交(/幺正)对角化**.

定理 10.4: τ 可正交归一化 $\Longleftrightarrow \tau$ 的特征向量构成 V 的正交基.

定义 10.2 **<u>正规算子</u>:** $\dim V < \infty$, $\tau \in \mathcal{L}(V)$, 若 $\tau^*\tau = \tau\tau^*$, 则称 τ 为**正规算子**.

定理 10.5 (课本第3 版定理10.8): 对正规算子 $\tau \in \mathcal{L}(V)$,

τ^*, τ^{-1} (在 τ 可逆的前提下), $p(\tau)$ ($p(x) \in F[x]$) 正规.

(2) $\|\tau(v)\| = \|\tau^*(v)\|$, 从而 $\ker \tau = \ker \tau^*$.

(3) $\forall k \in \mathbb{Z}^+$, $\ker \tau^k = \ker \tau$.

(4) $m_\tau(x) = p_1(x) \cdots p_m(x)$, 其中 $p_i(x)$ 不可约且互不相同.

(5) $\tau(v) = \lambda v \Longrightarrow \tau^*(v) = \bar{\lambda} v$.

(6) $\lambda_i \neq \lambda_j \Longrightarrow \mathcal{E}_{\lambda_i} \perp \mathcal{E}_{\lambda_j}$.

证: (1) $(\tau^*)^*\tau^* = \tau^{**}\tau^* = \tau\tau^* = \tau^*\tau = \tau^*\tau^{**} = \tau^*(\tau^*)^* \Longrightarrow \tau^*$ 正规.

$(\tau^{-1})^*\tau^{-1} = (\tau^*)^{-1}\tau^{-1} = (\tau\tau^*)^{-1} = (\tau^*\tau)^{-1} = \tau^{-1}(\tau^*)^{-1} = \tau^{-1}(\tau^{-1})^* \Longrightarrow \tau^{-1}$ 正规.

$(\tau^i)^*\tau^i = (\tau^*)^i\tau^i = (\because \tau$ 正规, 即 τ 与 τ^* 可交换$)\tau^i(\tau^*)^i = \tau^i(\tau^i)^* \Longrightarrow \tau^i$ 正规

$\Longrightarrow (r\tau^i)^*(r\tau^i) = \bar{r}(\tau^i)^*r\tau^i = r\tau^i\bar{r}(\tau^i)^* = (r\tau^i)(r\tau^i)^* \Longrightarrow r\tau^i$ 正规

$\Longrightarrow p(\tau)p^*(\tau) = \left(\sum_i r_i\tau^i\right)\left(\sum_j r_j\tau^j\right)^* = \sum_{ij} r_i\tau^i\bar{r}_j(\tau^j)^* = \sum_{ij} \bar{r}_j(\tau^j)^*\bar{r}_i\tau^i = \left(\sum_j r_j\tau^j\right)^*\left(\sum_i r_i\tau^i\right) = p^*(\tau)p(\tau) \Longrightarrow$ $p(\tau)$ 正规.

(2) $\|\tau(v)\|^2 = \langle \tau(v), \tau(v)\rangle = \langle v, \tau^*(\tau(v))\rangle = \langle v, (\tau^* \circ \tau)(v)\rangle = \langle v, (\tau \circ \tau^*)(v)\rangle = \langle v, \tau(\tau^*(v))\rangle = \langle \tau^*(v), \tau^*(v)\rangle = \|\tau^*(v)\|^2 \Longrightarrow \|\tau(v)\|^2 = \|\tau^*(v)\|$,

故 $\ker \tau = \{v \mid \tau(v) = 0\} = \{v \mid \|\tau(v)\| = 0\} = \{v \mid \|\tau^*(v)\| = 0\} = \{v \mid \tau^*(v) = 0\} = \ker \tau^*$.

(3) $\ker \tau \subseteq \ker \tau^k$ 显然. 下面来证 $\ker \tau^k \subseteq \ker \tau$:

令 $\sigma = \tau^*\tau$, 则 $\sigma^* = (\tau^*\tau)^* = \tau^*\tau^{**} = \tau^*\tau = \sigma$.

$\forall v \in \ker \tau^k$, $\tau^k(v) = 0 \Longrightarrow \sigma^k(v) = (\tau^*\tau)^k(v) = (\because \tau$ 正规, 即 τ 与 τ^* 可交换$)(\tau^*)^k\tau^k(v) = 0$,

$0 = \langle 0, \sigma^{k-2}(v)\rangle = \langle \sigma^k(v), \sigma^{k-2}(v)\rangle = \langle \sigma \circ \sigma^{k-1}(v), \sigma^{k-2}(v)\rangle = \langle \sigma^{k-1}(v), \sigma^* \circ \sigma^{k-2}\rangle = (\because \sigma^* = \sigma)\langle \sigma^{k-1}(v), \sigma \circ \sigma^{k-2}(v)\rangle = \langle \sigma^{k-1}(v), \sigma^{k-1}(v)\rangle = \left\|\sigma^{k-1}(v)\right\| \Longrightarrow \sigma^{k-1}(v) = 0$, 以此类推得 $\sigma(v) = 0$

$\Longrightarrow 0 = \langle v, 0\rangle = \langle v, \sigma(v)\rangle = \langle v, \tau^*(\tau(v))\rangle = \langle \tau(v), \tau(v)\rangle = \|\tau(v)\|^2 \Longrightarrow \tau(v) = 0 \Longrightarrow v \in \ker \tau \Longrightarrow \ker \tau^k \subseteq \ker \tau$.

综上, 得证.

(4) $m_\tau(x) = up_1^{e_1}(x) \cdots p_m^{e_m}(x)$, 其中 p_i 不可约且互不相同, $e_i \in \mathbb{Z}^+$.

要证 $m_\tau(x) = p_1(x) \cdots p_m(x)$, 即证 $e_i = 1 \forall i$.

$\forall v \in V$, $m_\tau(\tau)(v) = p_1^{e_1}(\tau) \cdots p_m^{e_m}(\tau)(v) = p_1^{e_1}(\tau)[p_2^{e_2}(\tau) \cdots p_m^{e_m}(\tau)(v)] = 0$.

$\because \tau$ 正规, $\therefore p_1(\tau)$ 正规 $\Longrightarrow \ker p_1(\tau) = \ker p_1^{e_1}(\tau)$

$\Longrightarrow p_1(\tau)[p_2^{e_2}(\tau) \cdots p_m^{e_m}(\tau)(v)] = 0 \Longrightarrow p_1(x)p_2^{e_2}(x) \cdots p_m^{e_m}(x) \in \langle m_\tau(x)\rangle \Longrightarrow m_\tau(x) = p_1^{e_1}(x)p_2^{e_2}(x) \cdots p_m^{e_m}(x) \mid p_1^1(x)p_2^{e_2}(x) \cdots p_m^{e_m}(x) \Longrightarrow e_1 = 1$.

$\because p_i(\tau)$ 正规, 即 $p_i(\tau)$ 可交换, $\therefore$ 同理可得 $e_i = 1 \forall i$, 故得证.

(5) $\tau(v) = \lambda v \Longrightarrow (\tau - \lambda)(v) = 0 \Longrightarrow v \in \ker(\tau - \lambda)$,

$\because \tau$ 正规, $\therefore \tau - \lambda$ 正规 $\Longrightarrow \ker(\tau - \lambda) = \ker(\tau - \lambda)^*$

$\Longrightarrow v \in \ker(\tau - \lambda)^* = \ker(\tau^* - \bar{\lambda})$.

(6) $\forall 0 \neq v \in \mathcal{E}_{\lambda_i}$, $\forall 0 \neq u \in \mathcal{E}_{\lambda_j}$, 其中 $\lambda_i \neq \lambda_j$, $\lambda_i\langle v, u\rangle = \langle \lambda_i v, u\rangle = \langle \tau(v), u\rangle = \langle v, \tau^*(u)\rangle = \langle v, \overline{\lambda_j}u\rangle = \lambda_j\langle v, u\rangle \Longrightarrow (\lambda_i - \lambda_j)\langle v, u\rangle = 0$.

$\because \lambda_i - \lambda_j \neq 0$, $\therefore \langle v, u\rangle = 0$.

□

定理 10.6 正规算子的谱的结构: 复情形(课本定理10.13): $F=\mathbb{C}$, $\dim V<\infty$, $\tau\in\mathcal{L}(V)$, 则下列叙述等价:

(1) τ 正规.

(2) τ 可正交对角化, $V=\mathcal{E}_{\lambda_i}\odot\cdots\odot\mathcal{E}_{\lambda_k}$.

(3) $\tau=\lambda_1\rho_1+\cdots+\lambda_k\rho_k$, 其中 $\rho_1+\cdots+\rho_k=1$ 为单位分解, 对 $i\neq j$, $\operatorname{Im}\rho_i\perp\operatorname{Im}\rho_j$, $\operatorname{Im}\rho_i\perp\ker\rho_i$.

证: "(1) $\Longrightarrow$ (2)": $\because$ τ 正规, $\therefore$ τ 的极小多项式的不可约多项式的次数均为 1, 即 $m_\tau(x)=p_1(x)\cdots p_k(x)=(x-\lambda_1)\cdots(x-\lambda_k)$, 其中 $p_i(x)\in\mathbb{C}[x]$ 为不可约多项式, λ_i 互不相等
$\Longrightarrow V=\mathcal{E}_{\lambda_1}\oplus\cdots\oplus\mathcal{E}_{\lambda_k}$.
又 $\because$ 对 $i\neq j$, $\mathcal{E}_{\lambda_i}\perp\mathcal{E}_{\lambda_j}$, $\therefore V=\mathcal{E}_{\lambda_i}\odot\cdots\odot\mathcal{E}_{\lambda_k}$.

"(2) $\Longrightarrow$ (1)": $\because\tau$ 可正交对角化, $\therefore\exists$ 正交归一基 $\mathcal{O}$, s.t. $[\tau]_\mathcal{O}=\operatorname{diag}(\lambda_1,\cdots,\lambda_k)$, $[\tau^*]_\mathcal{O}=\operatorname{diag}(\overline{\lambda_1},\cdots,\overline{\lambda_k})$
$\Longrightarrow[\tau]_\mathcal{O}[\tau^*]_\mathcal{O}=\operatorname{diag}(|\lambda_1|^2,\cdots,|\lambda_k|^2)=[\tau^*]_\mathcal{O}[\tau]_\mathcal{O}\Longrightarrow[\tau^*\tau(v)]_\mathcal{O}=[\tau^*]_\mathcal{O}[\tau]_\mathcal{O}[v]_\mathcal{O}=[\tau]_\mathcal{O}[\tau^*]_\mathcal{O}[v]_\mathcal{O}=[\tau\tau^*(v)]_\mathcal{O}\Longrightarrow\tau\tau^*=\tau^*\tau$.

"(3) $\Longleftrightarrow$ (1)": 利用引理 10.2, $\ker\rho^*=(\operatorname{Im}\rho)^\perp=\ker\rho$, $\operatorname{Im}\rho^*=(\ker\rho)^\perp=\operatorname{Im}\rho\Longrightarrow\rho^*=\rho$
$\Longrightarrow\tau^*=\overline{\lambda_1}\rho_1+\cdots+\overline{\lambda_k}\rho_k$.
$\tau\tau^*=\left(\sum_i\lambda_i\rho_i\right)\left(\sum_j\overline{\lambda_j}\rho_j\right)=\sum_{ij}\lambda_i\overline{\lambda_j}\rho_i\rho_j=\sum_{ij}\lambda_i\overline{\lambda_j}\delta_{ij}\rho_i=\sum_i|\lambda_i|^2\rho_i=\sum_{ij}\overline{\lambda_j}\lambda_i\rho_j\rho_i=\left(\sum_j\overline{\lambda_j}\rho_j\right)\left(\sum_i\lambda_i\rho_i\right)=\tau^*\tau\Longrightarrow\tau$ 正规.

综上, 得证. □

引理 10.2: $V=S\odot T$, 正交投影 $\rho_{ST}:V\to V$, $u=u_S+u_T\to u_S$, 则 $\ker\rho\perp\operatorname{Im}\rho$.

证: $\forall v\in\ker\rho_{ST}$, $v=v_S+v_T$ 其中 $v_S\in S$, $v_T\in T$, $\rho_{ST}(v)=v_S=0\Longrightarrow v=v_T\in T$.
$\forall w_S\in\operatorname{Im}\rho_{ST}$, $\exists w\in V$, s.t. $\rho_{ST}(w)=w_S\Longrightarrow w=w_S+w_T$, 其中 $w_S\in S$, $w_T\in T$.
$\because v\in T$, $w_S\in S$, $\therefore v\perp w\Longrightarrow\ker\rho\perp\operatorname{Im}\rho$. □

由于 $\mathbb{R}[x]$ 中不可约多项式的最高次数为 2, 故实数域上的向量空间的线性算子的最小多项式的分解形式与复情形有所不同.

定理 10.7 正规算子的谱的结构: 实情形(课本定理10.14): $F=\mathbb{R}$, $\dim V<\infty$, $\tau\in\mathcal{L}(V)$ 正规 $\Longleftrightarrow V=\mathcal{E}_{\lambda_1}\odot\cdots\odot\mathcal{E}_{\lambda_k}\odot D_1\odot\cdots\odot D_l$, 其中 $\mathcal{E}_{\lambda_i}$ 为 τ 的不变子特征空间, λ_i 为 τ 的谱, D_i 为 τ_i 的二维不可约不变子空间且 D_i 中有基 $\mathcal{B}_i'$, s.t. $[\tau]_{\mathcal{B}'}=\begin{pmatrix}s_i & t_i\\ -t_i & s_i\end{pmatrix}$,

$$[\tau]_{\mathcal{B}}=\begin{pmatrix}\lambda_1 &&&&& \\ & \ddots &&&& \\ && \lambda_k &&& \\ &&& \begin{pmatrix} s_1 & t_1 \\ -t_1 & s_1\end{pmatrix} && \\ &&&& \ddots & \\ &&&&& \begin{pmatrix} s_l & t_l \\ -t_l & s_l\end{pmatrix}\end{pmatrix}_{n\times n}.$$

证: τ 的极小多项式 $m_\tau(x)=(x-\lambda_1)\cdots(x-\lambda_t)q_1(x)\cdots q_l(x)$, 其中 $q_j(x)$ 不可约, $\deg q_j(x)=2$, $\lambda_i\in\mathbb{R}$ 互不相同, $V=\mathcal{E}_{\lambda_1}\oplus\cdots\oplus\mathcal{E}_{\lambda_t}\oplus D_1\oplus\cdots\oplus D_l$, 其中 $V_{p_i}=\{v\mid(\tau-\lambda_i)v=0\}$, $\dim V_{p_i}=1$, $\operatorname{ann}(D_i)=\langle q_i(x)\rangle$, $\deg q_i(x)=2$, 无妨 $q_i(x)=x^2+b_ix+c_i$, $\because q_i$ 不可约, $\therefore \Delta=b_i^2-4c_i<0$.

取 D_i 的基 $\mathcal{B}_i\equiv\{v_i,\tau(v_i)\}$, 则 $[\tau]_{\mathcal{B}_i}=\begin{pmatrix}0 & -c_i\\ 1 & -b_i\end{pmatrix}$.

为使 τ 在 D_i 中的表示更对称, 对 $[\tau]_{\mathcal{B}_i}$ 做相似变换到基 $\mathcal{B}_i'$ 上, s.t. $[\tau]_{\mathcal{B}_i'}=\begin{pmatrix}s_i & t_i\\ -t_i & s_i\end{pmatrix}$, 其中 $s_i=-\frac{b_i}{2}$, $t_i=\frac{\sqrt{4c_i-b_i^2}}{2}$.

问题 10.1: 如何相似变换? $\mathcal{B}_i'=?$ □

解: $\because [\tau]_{\mathcal{B}_i}$ 和 $[\tau]_{\mathcal{B}_i'}$ 的特征多项式相同, 均为 $q_i(x)$, 特征值相同, 均为 $q_i(x)$ 的根 $x_i^\pm=\frac{-b_i\pm i\sqrt{4c_i-b_i^2}}{2}$, $\therefore$ 这一相似变换和 $\mathcal{B}_i'$ 必 $\exists$.

$[\tau]_{\mathcal{B}_i}$ 的特征向量为 $\frac{1}{\sqrt{s_i^2+t_i^2+1}}\begin{pmatrix}-x_i^-\\ 1\end{pmatrix}=\frac{1}{\sqrt{s_i^2+t_i^2+1}}(-x_i^-v_i+\tau(v_i))$, $\frac{1}{\sqrt{s_i^2+t_i^2+1}}\begin{pmatrix}-x_i^+\\ 1\end{pmatrix}=\frac{1}{\sqrt{s_i^2+t_i^2+1}}(-x_i^+v_i+\tau(v_i))$,

即 $[\tau]_{\mathcal{B}_i}$ 的特征分解为 $[\tau]_{\mathcal{B}_i}=Q\Lambda Q^{-1}$, 其中 $Q=\frac{1}{\sqrt{s_i^2+t_i^2+1}}\begin{pmatrix}-x_i^- & -x_i^+\\ 1 & 1\end{pmatrix}$, $\Lambda=\begin{pmatrix}x_i^+ & 0\\ 0 & x_i^-\end{pmatrix}$.

$[\tau]_{\mathcal{B}_i'}$ 的特征向量为 $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$, $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$, 即 $[\tau]_{\mathcal{B}_i'}$ 的特征分解为 $[\tau]_{\mathcal{B}_i'}=P\Lambda P^{-1}$, 其中 $P=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}$.

相似变换下, $[\tau]_{\mathcal{B}_i'}=T[\tau]_{\mathcal{B}_i}T^{-1}$, 故其中 $T=PQ^{-1}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\frac{\sqrt{s_i^2+t_i^2+1}}{x_i^+-x_i^-}\begin{pmatrix}1 & x_i^+\\ -1 & -x_i^-\end{pmatrix}=\frac{\sqrt{s_i^2+t_i^2+1}}{\sqrt{2}(x_i^+-x_i^-)}\begin{pmatrix}0 & x_i^+-x_i^-\\ 2i & -ib_i\end{pmatrix}=$ $\frac{\sqrt{c_i+1}}{\sqrt{2}\sqrt{4c_i-b_i^2}}\begin{pmatrix}0 & \sqrt{4c_i-b_i^2}\\ 2 & -b_i\end{pmatrix}$, $T^{-1}=QP^{-1}=\frac{1}{\sqrt{s_i^2+t_i^2+1}}\begin{pmatrix}-x_i^- & -x_i^+\\ 1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}=\frac{1}{\sqrt{2(s_i^2+t_i^2+1)}}\begin{pmatrix}-(x_i^++x_i^-) & -i(x_i^+ - \\ 2 & 0\end{pmatrix}$ $\frac{1}{\sqrt{2(c_i+1)}}\begin{pmatrix}b_i & \sqrt{4c_i-b_i^2}\\ 2 & 0\end{pmatrix}$.

当然, 也可调整 T 前的系数从而得 $T=\begin{pmatrix}0 & 1\\ \frac{2}{\sqrt{4c_i-b_i^2}} & -\frac{b_i}{\sqrt{4c_i-b_i^2}}\end{pmatrix}=\begin{pmatrix}0 & 1\\ \frac{1}{t_i} & \frac{s_i}{t_i}\end{pmatrix}$, $T^{-1}=\begin{pmatrix}\frac{b_i}{2} & \frac{\sqrt{4c_i-b_i^2}}{2}\\ 1 & 0\end{pmatrix}=\begin{pmatrix}-s_i & t_i\\ 1 & 0\end{pmatrix}$.

$[\tau]_{\mathcal{B}_i'}=M_{\mathcal{B}_i\mathcal{B}_i'}[\tau]_{\mathcal{B}}M_{\mathcal{B}'\mathcal{B}}$, 其中 $M_{\mathcal{B}_i\mathcal{B}_i'}=\begin{pmatrix}[v_i]_{\mathcal{B}_i'} & [\tau(v_i)]_{\mathcal{B}_i'}\end{pmatrix}=T$, $M_{\mathcal{B}_i'\mathcal{B}_i}=\begin{pmatrix}[b_1']_{\mathcal{B}_i} & [b_2']_{\mathcal{B}_i'}\end{pmatrix}=T^{-1}\Longrightarrow\mathcal{B}_i'=\{b_1'=-s_iv+\tau(v_i),b_2'=t_iv_i\}$.

$$\begin{tikzcd}
F^n \arrow[r, "\tau'_A"] & F^n \\
V \arrow[u, "\phi'_{\mathcal{B}}"] \arrow[r, "\tau"] \arrow[d, "\phi_{\mathcal{B}}"] & V \arrow[u, "\phi'_{\mathcal{B}}"] \arrow[d, "\phi_{\mathcal{B}}"] \\
F^n \arrow[r, "\tau_A"] & F^n
\end{tikzcd}$$

□

□

对 $F=\mathbb{Q}$, 由于 $\mathbb{Q}[x]$ 中的不可约多项式无次数限制, 线性算子的极小多项式可分解成任意次数不可约多项式的乘积, 此时子空间没有确定的维数, 故此时没有普适的定理.

10.3 特殊的正规算子

定义 10.3 自伴随(/厄米)算子: 满足 $\tau=\tau^*$.

定义 10.4 斜伴随(/反厄米)算子: 满足 $\tau=-\tau^*$.

定义 10.5 酉(/幺正)算子: 满足 $\tau^*=\tau^{-1}$.

定理 10.8 厄米算子的性质(课本第3 版定理10.11): $\dim V<\infty$, $\tau,\sigma\in\mathcal{L}(V)$, 则

(1) 若 τ,σ 厄米, 则 $\tau+\sigma$, τ^{-1}, $p(\tau)$ ($p(x)\in\mathbb{R}[x]$) 厄米.

(2) $F=\mathbb{C}$, 则 τ 厄米 $\Longleftrightarrow\langle\tau(v),v\rangle\in\mathbb{R}$.

(3) τ 为复算子或实对称算子, 则 $\tau=0\Longleftrightarrow\forall v\in V,\langle\tau(v),v\rangle=0$.

(4) τ 厄米, 则 $m_\tau(x)$ 仅有实根.

证: (1) $(\tau+\sigma)^*=\tau^*+\sigma^*=\tau+\sigma\Longrightarrow\tau+\sigma$ 厄米.
$(\tau^{-1})^*=(\tau^*)^{-1}=\tau^{-1}\Longrightarrow\tau^{-1}$ 厄米.
$p^*(\tau)=\left(\sum_i r_i\tau^i\right)^*=\sum_i r_i(\tau^i)^*=\sum_i r_i(\tau^*)^i=\sum_i r_i\tau^i=p(\tau)\Longrightarrow p(\tau)$ 厄米.

(2) "$\Longleftrightarrow$": τ 厄米, 即 $\tau^*=\tau\Longleftrightarrow\forall v,\ \tau(v)=\tau^*(v)\Longleftrightarrow\langle\tau(v),v\rangle=\langle v,\tau^*(v)\rangle=\langle v,\tau(v)\rangle=\overline{\langle\tau(v),v\rangle}\Longrightarrow\langle\tau(v),v\rangle\in\mathbb{R}$.

(3) "$\Longrightarrow$": 显然. 复算子的 "$\Longleftarrow$" 见定理 **??**, 下证实对称算子的 "$\Longleftarrow$".
实对称算子即实厄米算子. $\because F=\mathbb{R}$, $\because\langle u,v\rangle=\langle v,u\rangle$.
$\forall u,v\in V$, $0=\langle\tau(u+v),u+v\rangle=\cancel{\langle\tau(u),u\rangle}0+\langle\tau(u),v\rangle+\langle\tau(v),u\rangle+\cancel{\langle\tau(v),v\rangle}0=\langle\tau(u),v\rangle+\langle\tau(v),u\rangle=\langle u,\tau^*(v)\rangle+\langle\tau(v),u\rangle=\langle u,\tau(v)\rangle+\langle\tau(v),u\rangle=2\langle\tau(v),u\rangle\Longrightarrow\tau(v)=0\Longrightarrow\tau=0$.
综上, 得证.

(4) $\because \tau$ 厄米, $\therefore \tau$ 正规.
设 λ 为 τ 的特征值, 亦即 $m_\tau(x)$ 的根, 则 $\bar{\lambda}$ 为 τ^* 的特征值.
$\lambda v = \tau(v) = \tau^*(v) = \bar{\lambda} v \Longrightarrow \lambda = \bar{\lambda} \Longrightarrow \lambda \in \mathbb{R}$, 故 $m_\tau(x)$ 仅有实根.

□

定理 10.9 酉算子的性质(课本第3 版定理10.12): $\dim V < \infty$, $\sigma, \tau \in \mathcal{L}(V)$, 则

(1) σ, τ 酉 $\Longrightarrow r\tau$ $(|r| = 1)$, $\sigma \circ \tau$, τ^{-1} 酉.

(2) τ 酉 $\Longleftrightarrow \tau$ 等距同构.

(3) τ 酉 $\Longleftrightarrow \tau$ 将一组正交归一基变换为正交归一基.

(4) τ 酉, 则 τ 的特征值模长 $= 1$.

证: (1) $(r\tau)^*(r\tau) = \bar{r}\tau^* r\tau = \bar{r}r\tau^*\tau = |r|^2 \tau^{-1}\tau = 1 \Longrightarrow r\tau$ 酉.
$(\sigma \circ \tau)^*(\sigma \circ \tau) = \tau^*\sigma^*\sigma\tau = \tau^{-1}\sigma^{-1}\sigma\tau = 1 \Longrightarrow \sigma \circ \tau$ 酉.
$(\tau^{-1})^* = (\tau^*)^{-1} = (\tau^{-1})^{-1} \Longrightarrow \tau^{-1}$ 酉.

(2) "$\Longrightarrow$": $\because$ 酉算子有逆, $\therefore$ 必双射, 下证等距.
$\langle \tau(u), \tau(v) \rangle = \langle u, \tau^*(\tau(v)) \rangle = \langle u, \tau^{-1}(\tau(v)) \rangle = \langle u, v \rangle \Longrightarrow \tau$ 等距, 故 τ 等距同构.
"$\Longleftarrow$": $\because \tau$ 等距同构, $\therefore \langle u, \tau^*(\tau(v)) \rangle = \langle \tau(u), \tau(v) \rangle = \langle u, v \rangle \Longrightarrow \tau^*(\tau(v)) = v \Longrightarrow \tau^* \circ \tau = 1 \Longrightarrow \tau$ 酉.
综上, 得证.

(3) "$\Longrightarrow$": 取一组正交归一基 $\mathcal{O} = \{o_1, \cdots, o_n\}$, $\langle o_i, o_j \rangle = \delta_{ij}$.
$\because \langle \tau(o_i), \tau(o_j) \rangle = \delta_{ij}$ 且 $\dim \tau(\mathcal{O}) = \dim \mathcal{O}$, $\therefore \tau(\mathcal{O})$ 为一组正交归一基.
"$\Longleftarrow$": $\forall u, v \in V$, $u = \sum_{i=1}^n \alpha_i o_i$, $v = \sum_{j=1}^n \beta_j o_j$.
$\because \tau(\mathcal{O})$ 为正交归一基, $\therefore \langle \tau(u), \tau(v) \rangle = \langle \tau\left(\sum_{i=1}^n \alpha_i o_i\right), \tau\left(\sum_{j=1}^n \beta_i o_i\right) \rangle = \sum_{i,j=1}^n \alpha_i \overline{\beta_j} \langle \tau(o_i), \tau(o_j) \rangle = \sum_{i,j=1}^n \alpha_i \overline{\beta_j} \delta_{ij} = \sum_{i=1}^n \alpha_i \overline{\beta_i} = \sum_{i,j=1}^n \alpha_i \overline{\beta_j} \langle o_i, o_j \rangle = \langle \sum_{i=1}^n \alpha_i o_i, \sum_{j=1}^n \beta_j o_j \rangle = \langle u, v \rangle \Longrightarrow \tau$ 等距同构 $\Longrightarrow \tau$ 酉.
综上, 得证.

(4) 设 λ 为 τ 的特征值, $\tau(v) = \lambda v$, $\tau^*(v) = \bar{\lambda} v$.
$v = \tau^{-1}(\tau(v)) = \tau^*(\tau(v)) = \bar{\lambda}\lambda v = |\lambda|^2 v \Longrightarrow |\lambda| = 1$.

□

定理 10.10 正规算子的结构(课本第3 版定理10.18): (1) $F = \mathbb{C}$,

(a) τ 正规 $\Longleftrightarrow \tau$ 正交归一对角化 $\Longleftrightarrow \tau$ 有正交谱分解 $\tau = \lambda_1\rho_1 + \cdots + \lambda_k\rho_k$, 其中 λ_i 互不相等, $\rho_1 + \cdots + \rho_k = 1$ 为单位分解, 对 $i \neq j$, $\operatorname{Im}\rho_i \perp \operatorname{Im}\rho_j$, $\operatorname{Im}\rho_i \perp \ker\rho_i$.

(b) 特征值为实数的正规算子厄米.

(c) 特征值的模长 $= 1$ 的正规算子酉.

(2) $F = \mathbb{R}$,

(a) τ 正规 $\Longleftrightarrow \tau = \mathcal{E}_{\lambda_1} \odot \cdots \odot \mathcal{E}_{\lambda_k} \odot D_1 \odot \cdots \odot D_l$, 其中 D_i 为二维不可约的 τ 不变子空间, D_i 上 τ 的

矩阵表示为 $\begin{pmatrix} s_i & t_i \\ -t_i & s_i \end{pmatrix}$.

(b) 若上述正交直和式中无 D_i, 则 τ 厄米.

(c) 若在 D_i 上的 τ 的矩阵表示为 $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, 则 τ 酉, 称为 **正交算子**.

定义 10.6 正交算子: $F = \mathbb{R}$ 的酉算子.

10.4 (半)正定算子

定义 10.7 (半)正定算子: $F = \mathbb{R}$, $\dim V < \infty$, $\tau \in \mathcal{L}(V)$ 厄米, 若 $\forall v \in V$, $\langle \tau(v), v\rangle > (\geq) 0$, 则 τ **(半)正定**.

定理 10.11 (课本第3 版定理10.22): $F = \mathbb{R}$, $\dim V < \infty$, $\tau \in \mathcal{L}(V)$ 厄米, 则

(1) τ 半正定 $\Longleftrightarrow \tau$ 的特征值 ≥ 0.

(2) τ 正定 $\Longleftrightarrow \tau$ 的特征值 > 0.

证: "$\Longrightarrow$": 设 λ 为 τ 的特征值.
$\because \tau$ (半)正定, $\therefore \langle \tau(v), v\rangle > (\geq) 0$.
又 $\because \langle \tau(v), v\rangle = \langle \lambda v, v\rangle = \lambda\langle v, v\rangle \geq 0$, $\langle v, v\rangle > 0$, $\therefore \lambda > (\geq) 0$, $\therefore \lambda > (\geq) 0$.

"$\Longleftarrow$": $\because \rho$ 厄米, $\therefore \tau$ 的正交谱分解 $\tau = \lambda_1\rho_1 + \cdots + \lambda_k\rho_k$.
对 τ 的函数操作均等效于作用于其谱分解的特征值上: $\tau^2 = \sum_{ij} \lambda_i\lambda_j\rho_i\rho_j = \sum_{ij} \lambda_i\lambda_j\delta_{ij}\rho_i = \sum_i \lambda_i^2\rho_i$.
类似地, $\tau^k = \sum_i \lambda^k \rho_k$.
$r\tau^k = r\sum_i \lambda_i^k\rho_i = \sum_i r\lambda_r^k\rho_i \Longrightarrow$ 对 $f(x) \in F[x]$, $f(\tau) = \sum_i f(\lambda_i)\rho_i$,
$\forall$ 可由多项式近似的 $g(x)$, $g(\tau) = \sum_i g(\lambda_i)\rho_i$.
$\because \tau$ (半)正定, $\therefore$ 必可定义其平方根 $\sqrt{\tau} = \sqrt{\lambda_1}\rho_1 + \cdots + \sqrt{\lambda_k}\rho_k$, 此处 $\lambda_i > (\geq) 0$, 否则 $\sqrt{\tau}$ 不一定合法. □

定理 10.12 (课本第3 版定理10.23): τ 厄米, 则

(1) τ 半正定 $\Longleftrightarrow \tau$ 有平方根.

(2) τ 半正定 $\Longleftrightarrow \tau = \sigma^* \circ \sigma$, 其中 $\sigma \in \mathcal{L}(V)$ (注意这里的 σ 不唯一).

证: (1) τ 半正定, 即 $\tau = \sum_i \lambda_i\rho_i$, 其中 $\lambda_i \geq 0 \Longleftrightarrow \sqrt{\tau} = \sum_i r_i\rho_i$, 其中 $r_i = \sqrt{\lambda_i}$.

(2) "$\Longrightarrow$": 取 $\sigma = \sqrt{\tau}$ 即得证.

"$\Longleftarrow$": $\langle \tau(v), v\rangle = \langle \sigma^* \circ \sigma(v), v\rangle = \langle \sigma(v), \sigma(v)\rangle = \|\sigma(v)\|^2 \geq 0 \Longrightarrow \tau$ 半正定.

□

定理 10.13 半正定算子的复合半正定的条件(课本第3 版定理10.24): $\sigma,\tau\in\mathcal{L}(V)$ 半正定, 若 $\sigma\circ\tau=\tau\circ\sigma$, 则 $\sigma\circ\tau$ 半正定.

证: $\because\sigma\tau=\tau\sigma,\therefore\sqrt{\sigma}\sqrt{\tau}=\sqrt{\tau}\sqrt{\sigma}\Longrightarrow\sigma\tau=\sqrt{\sigma}\sqrt{\sigma}\sqrt{\tau}\sqrt{\tau}=(\sqrt{\sigma}\sqrt{\tau})(\sqrt{\sigma}\sqrt{\tau})$, 故 $\sigma\tau$ 半正定. □

10.5 算子的极分解

定理 10.14 算子的极分解(课本第3 版定理10.25): $F=\mathbb{C}$, 有限维内积向量空间 V, $\tau\in\mathcal{L}(V)$, 则 $\exists!$ 半正定算子 ρ 及酉算子 ν, s.t. $\tau=\nu\circ\rho$, 且若 τ 可逆, 则 ν 唯一.

证: 取 $\rho=\sqrt{\tau^*\tau}$, 则 $\|\rho(v)\|^2=\langle\rho(v),\rho(v)\rangle=\langle v,\rho^*\rho(v)\rangle=\langle v,\rho^2(v)\rangle=\langle v,\tau^*\tau(v)\rangle=\langle\tau(v),\tau(v)\rangle=\|\tau(v)\|^2$.
取 $\nu:\operatorname{Im}\rho\to\operatorname{Im}\tau$, $\rho(v)\mapsto\tau(v)$.
先证 ν 为映射: 若 $\rho(v)=\rho(u)$, 则 $\rho(u-v)=0\Longrightarrow\|\rho(u-v)\|^2=0\Longrightarrow\|\tau(u-v)\|=^2=\langle\tau(u-v),\tau(u-v)\rangle=0\Longrightarrow\tau(u-v)=0\Longrightarrow\tau(v)=\tau(u)$, 故 ν 为映射.
$\because\|\nu(\rho(v))\|=\|\tau(v)\|=\|\rho(v)\|,\therefore\nu$ 等距同构 $\Longrightarrow\nu$ 酉.
当 τ 不可逆时, 拓展 $\operatorname{Im}\rho$ 为 $\operatorname{Im}\tau$ 的方式不唯一, 故 ν 不唯一.

□

若 σ 酉, 则其特征值模 $|\lambda|=1$, 可记为 $\lambda_i=e^{i\theta_i}$, 其中 $\theta_i\in\mathbb{R}$,
则 $\sigma=\lambda_1\rho_1+\cdots+\lambda_k\rho_k=e^{i\theta_1}\rho_1+\cdots+e^{i\theta_k}\rho_k$, 其中 $\rho_1+\cdots+\rho_k=1$ 为正交单位分解.
令 $H=\theta_1\rho_1+\cdots+\theta_k\rho_k$, 则

(1) $H^*=H$.

(2) $\sigma=e^{iH}$.

此处 H 类似量子力学中的哈密顿量, σ 类似演化算子.