Chapter 3

同构定理

3.1 商空间

> **定义 3.1 商空间:** F 为域, V 为 F 上的向量空间, S 为 V 的子空间, 则称 $\frac{V}{S} \equiv \{[v] \mid v \in V\}$ 为 F 的**商空间**, 其中 $[v] \equiv \{u \in V \mid u - v \in S\} = S + v$.

$\frac{V}{S}$ 为 F 上的向量空间.

证: $[u] + [v] = \{a \in V \mid a - u \in S\} + \{b \in V \mid b - v \in S\} = \{(a+b) \in V \mid a - u \in S, b - v \in S\}$.
$[u+v] = \{w \in V \mid w - (u+v) \in S\}$.
$\forall (a+b) \in [u] + [v]$, $(a-u) + (b-v) = (a+b) - (u+v) \in S \implies (a+b) \in [u+v] \implies [u]+[v] \subseteq [u+v]$.
$\forall w \in [u+v]$, $\exists c, d \in S$, s.t. $w = c + d + (u+v) = (c+u) + (d+v)$, 其中 $(c+u) \in [u]$, $(d+v) \in [v] \implies w \in [u]+[v] \implies [u+v] \subseteq [u+v]$.
故 $[u]+[v] = [u+v]$.

假设 $u \sim u'$, $v \sim v'$, 即 $[u] = [u']$, $[v] = [v']$.
$\because [u] = [u']$, $\therefore u + S = u' + S \implies \exists s_1, s_1' \in S$, s.t. $u + s_1 = u' + s_1' \iff u' = u + s_1 - s_1'$,
$\because [v] = [v']$, $\therefore v + S = v' + S \implies \exists s_2, s_2' \in S$, s.t. $v + s_2 = v' + s_2' \iff v' = v + s_2 - s_2'$
$\implies u' + v' = u + s_1 - s_1' + v + s_1 - s_1'$, 其中 $\because s_1, s_1', s_2, s_2' \in S$, $\therefore s_1 - s_1' \in S$, $s_2 - s_2' \in S$.
又 $\because V$ 是交换群, $\therefore u' + v' = u + v + (s_1 - s_1' + s_2 - s_2')$
$\implies (u'+v') + S = (u + v + (s_1 - s_1' + s_2 - s_2')) + S \implies [u'+v'] = [u+v]$,
即 $[u]+[v] = [u+v]$ 与代表元选取无关, 故 $[u]+[v] = [u+v]$ 为运算.

$r[u] = r\{v \in V \mid v - u \in S\} = \{rv \mid v \in V, v - u \in S\} = \{rv \in V \mid rv - ru \in S\} = [ru]$.

假设 $u \sim u'$, 即 $[u] = [u']$.
$\because [u] = [u']$, $\therefore u + S = u' + S \implies \exists s, s' \in S$, s.t. $u + s = u' + s' \iff u' = u + s - s'$
$\implies ru' = r(u + s - s') = ru + r(s - s')$, 其中 $s - s' \in S \implies (ru') + S = (ru + r(s-s')) + S = (ru) + S \implies r[u'] = [ru'] = [ru]$,
即 $r[u] = [ru]$ 与代表元选取无关, 故 $r[u] = [ru]$ 为运算.

$(\frac{V}{S}, +)$ 满足

(1) **结合律**: $([v]+[u]) + [w] = [u+v] + [w] = [u+v+w] = [u + (v+w)] = [u] + [v+w] = [u] + ([v]+[w])$,

(2) **有单位元** $[0]$: $[0] + [u] = [0+u] = [u] = [u+0] = [u] + [0]$,

(3) **有逆元**: $\forall v \in V$, $\exists -v$, s.t. $[a] + [-a] = [a + (-a)] = [0] = [(-a) + a] = [-a] + [a]$,

且 $[u]+[v]=[u+v]=[v+u]=[v]+[u]$, 即 $\left(\frac{V}{S},+\right)$ 交换, 故 $\left(\frac{V}{S},+\right)$ 为交换群. (总之就是因为 $\frac{V}{S}$ 中的元素 $[v]$ 保持了 V 中的元素 v 的各种运算性质, 所以 $(V,+)$ 是交换群就可以推出 $\frac{V}{S}$ 也是交换群.)

$\frac{V}{S}$ 满足

(1) $r([u+v])=r([u]+[v])=r[u]+r[v]$,

(2) $(r+t)[u]=[(r+t)u]=[ru+tu]=[ru]+[tu]=r[u]+t[u]$,

(3) $(r\cdot t)[u]=[(r\cdot t)u]=[r(tu)]=r[tu]=r(t[u])$,

(4) **有单位元** 1: $[1][u]=[1u]=[u]$,

故 $\frac{V}{S}$ 为 F 上的向量空间. □

定理 3.1 (课本定理3.2): (1) $\Pi_S: V\to\frac{V}{S}$, $v\mapsto[v]$ 是线性变换.

(2) Π_S 是满线性变换, 即 $\operatorname{Im}\Pi_S=\frac{V}{S}$.

(3) $\ker\Pi_S=S$.

证: (1) 显然 Π_S 是唯一的, 故 Π_S 是映射.
如前所证, V 和 $\frac{V}{S}$ 均为 F 上的向量空间.
$\because [u]+[v]=[u+v]$, $r[u]=[ru]$, $\therefore r[u]+t[v]=[ru]+[tv]=[ru+tv]$, 故 Π_S 为线性变换.

(2) $\forall[v]\in\frac{V}{S}$, $\exists v\in V$, s.t. $\Pi_S(v)=[v]$, 故 Π_S 是满线性变换.

(3) $\ker\Pi_S=\{v\in S\mid\Pi_S(v)=[0]\}$.
$v\in\ker\Pi_S$, $\Pi_S(v)=[v]=v+S=[0]=0+S=S\iff v\in S$, 故 $\ker\Pi_S\subseteq S$.

□

定理 3.2 (课本定理3.3): (1) S,T 为 V 的子空间且 $S\subseteq T$, 则 $\frac{T}{S}$ 是 $\frac{V}{S}$ 的子空间.

(2) 取 X 为 $\frac{V}{S}$ 的子空间, 则 $\exists V$ 的子空间 T, s.t. $\emptyset\neq S\subseteq T$, $\frac{T}{S}=X$.

证: (1) $\frac{T}{S}=\{[u]\mid u\in T\}$, $\frac{V}{S}=\{[v]\mid v\in V\}$.
$\forall[u]\in\frac{T}{S}$, $u\in T$, $\because T$ 是 V 的子空间, $\therefore u\in V\Longrightarrow[u]\in\frac{V}{S}$, 故 $\frac{T}{S}\subseteq\frac{V}{S}$.
$\forall[u_1],[u_2]\in\frac{T}{S}$, $\forall r,t\in F$, $r[u_1]+t[u_2]=[ru_1+tu_2]$.
$\because u_1,u_2\in T$, $\therefore ru_1+tu_2\in T\Longrightarrow[ru_1+tu_2]\in\frac{T}{S}$, 故 $\frac{T}{S}$ 为向量空间.
综上, 得证.

(2) 取 $T=\cup_{[v]\in X}[v]$. 显然 $T\subseteq V$.
$\forall u,v\in T$, 根据 T 的定义, $[u],[v]\in X$.
$\because X$ 为子空间, $\therefore r[u]+t[v]=[ru+tv]\in X\subseteq T=\cup_{v\in X}[v]\Longrightarrow ru+tv\in T$.
故 T 为 V 的子空间.
$\because S=[0]\in X$, $\therefore S\subseteq T=\cup_{[v]\in X}[v]$.
$\frac{T}{S}=\{[v]=S+v\mid v\in T\}$.
$\forall[u]\in\frac{T}{S}$, $u\in T=\cup_{[v]\in X}[v]\Longrightarrow[u]\in X\Longrightarrow\frac{T}{S}\subseteq X$.

$\forall [u] \in X$, $u \in T = \cap_{[v]\in X}[v] \Longrightarrow [u] \in \frac{T}{S} \Longrightarrow X \subseteq \frac{T}{S}$.
故 $\frac{T}{S} = X$.

综上, 得证.

□

3.2 第一同构定理

定理 3.3 第一同态基本定理(课本定理3.4): [a] S 是 V 的子空间, $\tau \in \mathcal{L}(V, W)$,

$$\begin{tikzcd}
v \arrow[rrr, mapsto] \arrow[ddd] & & & \tau[v] \\
& V \arrow[r, "\tau"] \arrow[d, "\Pi_S"'] & W & \\
& \frac{V}{S} \arrow[ur, dashed, "\tau'"'] & & \\
{[v]} \arrow[uuurrr, dashed, mapsto] & & &
\end{tikzcd}$$

若 $S \subseteq \ker\tau$, 即 $\ker\Pi_S \subseteq \ker\tau$, 则 $\exists!\tau' : \frac{V}{S} \to W$, s.t. $\tau = \tau' \circ \Pi_S$, 即 $\forall v \in V$, $\tau(v) = \tau'([v])$, 此时上图可交换, $\ker\tau' = \frac{\ker\tau}{S}$, $\operatorname{Im}\tau' = \operatorname{Im}\tau$.

[a] 该定理回答了 τ' 的存在性 (即 τ' 在什么条件下存在) 的问题. 之所以称"基本", 是因为若将该定理中的向量空间换成其他代数结构, 定理仍然成立.

证: τ' 的唯一性要求, 若 $[u] = [v]$, 则 $\tau'([u]) = \tau'([v])$,
即若 $u \sim v$, 则 $\tau(u) = \tau(v)$,
即若 $u - v \in S$, 则 $\tau(u - v) = 0$,
即 $S \subseteq \ker\tau$.

此时, $\ker\tau' = \{[v] \in \frac{V}{S} \mid \tau'([v]) = 0\} = \{[v] \in \frac{V}{S} \mid \tau(v) = 0\} = \{[v] \in \frac{V}{S} \mid v \in \ker\tau\} = \frac{\ker\tau}{S}$,
$\operatorname{Im}\tau' = \{\tau'([v]) \mid [v] \in \frac{V}{S}\} = \{\tau'([v]) \mid v \in V\} = \{\tau(v) \mid v \in V\} = \operatorname{Im}\tau$ ($\because \Pi_S$ 满射, $\therefore \forall [v] \in \frac{V}{S}$, $\exists v \in V$). □

那么, 若 τ 双射, 即 $\exists \tau^{-1} \in \mathcal{L}(W, V)$, 且 $\ker\tau = S$, 如何?

$$\begin{tikzcd}
\tau^{-1}(w) \arrow[ddd] & & & w \arrow[lll, mapsto] \\
& V \arrow[d, "\Pi_S"'] & W \arrow[l, "\tau^{-1}"'] & \\
& \frac{V}{S} & & \\
{[w]} & & &
\end{tikzcd}$$

此时, $\ker\tau' = \frac{\ker\tau}{S} = \{[v] \mid v \in \ker\tau\} = \{[v] \mid v \in S\} = \{[0]\} \Longrightarrow \tau'$ 单射.

由上面关于第一同态基本定理的延伸讨论我们得到:

定理 3.4 第一同构定理(课本定理3.5): 若 $\ker\tau = S$, 则 τ' 单射, $\frac{V}{\ker\tau} = \frac{V}{S} \approx \operatorname{Im}\tau$.

$$\begin{tikzcd}
v \arrow[rrr, mapsto] \arrow[ddd] & & & \tau[v] \\
& V \arrow[r, "\tau"] \arrow[d, "\Pi_S"'] & W & \\
& \frac{V}{S} \arrow[ur, dashed, "\tau'"] & & \\
{[v]} \arrow[uuurrr, dashed, mapsto] & & &
\end{tikzcd}$$

证: $V = \ker\tau \oplus (\ker\tau)^c$, 其中 $(\ker\tau)^c \approx \operatorname{Im}\tau \Longrightarrow \frac{V}{\ker\tau} \approx (\ker\tau)^c \approx \operatorname{Im}\tau$. □

更一般地, 若 $V = S \oplus T$, 则 $\frac{V}{S} \approx T$, $\frac{V}{T} \approx S$.

证: $\forall v \in V$, $v = u_S + u_T$, 其中 $u_S \in S$, $u_T \in T$.
令投影映射 $P_T : V \to T$, $v = u_S + u_T \mapsto u_T$.
$\ker P_T = \{v \in V \mid P_T(v) = 0\} = S = [0] = \ker\Pi_S$.
由第一同构定理 (定理 3.4), $\exists!\tau'$ 单射, s.t. $P_T = \tau' \circ \Pi_S$.
又 $\because \operatorname{Im} P_T = T$, 即 P_T 满射, $\therefore \tau'$ 满射 $\Longrightarrow \tau'$ 同构 $\Longrightarrow \frac{V}{S} \approx T$.

$$\begin{tikzcd}
v = u_S + u_T \arrow[rrr, mapsto] \arrow[ddd] & & & u_T \\
& V \arrow[r, "P_T"] \arrow[d, "\Pi_S"'] & T & \\
& \frac{V}{S} \arrow[ur, dashed, "\tau'"] & & \\
{[v]} \arrow[uuurrr, dashed, mapsto] & & &
\end{tikzcd}$$

同理可证 $\frac{V}{T} \approx S$. □

3.3 线性泛函

定义 3.2 对偶(空间)和线性泛函: $V^* = \mathcal{L}(V, F)$ 为 F 上的向量空间, 称 V^* 为 V 的**对偶(空间)**.
若 $f \in V^*$, 则称 f 为**线性泛函**.

(1) $\ker V^*$ 为 F 上的向量空间.

(2) $\because \dim F = 1$, $\operatorname{Im} f \subseteq F$, $\therefore \dim\operatorname{Im} f \le 1$, $\dim\ker f \ge \dim V - 1$.

(3) $\because$ 必有零映射 $0 \in V^*$, $0 : V \to F$, $v \mapsto 0$, $\therefore V^*$ 非空.

(4) 若 $\dim \operatorname{Im} f = 0$, 则 $\operatorname{Im} f = \{0\}$, f 为零映射.

(5) 若 $\dim \operatorname{Im} f = 1$, 则 $\operatorname{Im} f = \langle r \rangle$, 其中 $0 \neq r \in F \Longrightarrow \operatorname{Im} f = F$.
由反证法易证, 若 $v \in f^{-1}(r) = \{v \in V \mid f(v) = r\}$, 其中 $r \neq 0$, 则 $v \neq 0$, 且必有 $f(\langle v \rangle^c) = \{0\}$.

证明 (5) 的末句:

证: 假设 $\exists u \in \langle v \rangle^c$, s.t. $f(u) \neq 0$.
$f\left(\frac{ru}{f(u)}\right) = r \Longrightarrow \frac{ru}{f(u)} \in f^{-1}(r)$.
又 $\because u \in \langle v \rangle^c$, $\therefore \dim f^{-1}(r) \geq 2$, 这与 $f^{-1}(r) \subseteq (\ker f)^c$, $\dim(\ker f)^c = \dim \operatorname{Im} f \leq 1$ 矛盾,
故假设错误, $\forall u \in \langle v \rangle^c$, $f(u) = 0 \Longrightarrow f(\langle v \rangle^c) = \{0\}$. □

定理 3.5 (课本定理3.11): (1) $\forall 0 \neq v \in V$, $\exists 0 \neq f \in V^*$, s.t. $f(v) \neq 0$.

(2) $v = 0 \Longleftrightarrow \forall f \in V^*$, $f(v) = 0$.

(3) $f \in V^*$, 若 $f(x) \neq 0$, 则 $V = \ker f \oplus \langle x \rangle$, 即 $\operatorname{Im} f \approx \langle x \rangle$.

(4) $0 \neq f, g \in V^*$, $\ker f = \ker g \Longleftrightarrow \exists 0 \neq \lambda \in F$, s.t. $f = \lambda g$.

证: (1) $v \neq 0$, 则 $V = \langle v \rangle \oplus \langle v \rangle^c$, 其中 $\langle v \rangle = \{rv \mid r \in F\}$.
令 $f: V \to F$, $rv + w \mapsto r$, 其中 $rv \in \langle v \rangle$, $w \in \langle v \rangle^c$, 故 $f(v) = 1$, $f \in V^*$.
下证 f 为线性变换: $\forall u_1, u_2 \in V$, $u_1 = r_1 v + w_2$, $u_2 = r_2 v + w_2$, 其中 $w_1, w_2 \in \langle v \rangle^c$,
$f(ru_1 + tu_2) = f(r(r_1 v + w_1) + t(r_2 v + w_2)) = f((rr_1 v + rw_1) + (tr_2 v + tw_2)) = f((rr_1 + tr_2)v + (rw_1 + tw_2)) = rr_1 + tr_2 = rf(r_1 v + w_1) + tf(r_2 v + w_2) = rf(u_1) + tf(u_2)$.
故得证.

注意此处 f 的构造并非唯一: 构造 $f: V \to F$, $rv + u \mapsto rt$, 其中 $u \in \langle v \rangle^c$, 同理可得证.

(2) "$\Longrightarrow$": 若 $v = 0$, 则 $\forall u \in V$, $f(v) + f(u) = f(v + u) = f(u) \Longrightarrow f(v) = 0$.

"$\Longleftarrow$": 假设 $v \neq 0$, 则由 (1), $\exists f \in V^*$, s.t. $f(v) \neq 0$, 矛盾, 故假设错误, $v = 0$.

(3) $f(x) \neq 0 \Longrightarrow \operatorname{Im} f \neq \{0\} \Longrightarrow \dim \operatorname{Im} f \neq 0 \Longrightarrow \dim(\ker f)^c = \dim \operatorname{Im} f = 1$
$\Longrightarrow \exists v \in V$, s.t. $\ker f^c = \langle v \rangle \Longrightarrow V = \ker f \oplus \ker f^c = \langle v \rangle^c \oplus \langle v \rangle$ 且 $\operatorname{Im} f \approx \ker f^c = \langle v \rangle$. $\because f(x) \neq 0$, $\therefore x = rv + w$, 其中 $rv \in \langle v \rangle$, $w \in \langle v \rangle^c \Longrightarrow \langle x \rangle \approx \langle v \rangle \Longrightarrow V = \ker f \oplus \langle x \rangle$ 且 $\operatorname{Im} f \approx \langle x \rangle$.

(4) "$\Longrightarrow$": 令 $K = \ker f = \ker g$.
$\forall x \notin K$, 由 (3) 有, $V = K \oplus \langle x \rangle$.
取 $\lambda = \frac{f(x)}{g(x)}$, $\forall v \in V$, $x = rx + w$, 其中 $rx \in \langle x \rangle$, $w \in K$
$\Longrightarrow f(v) = f(rx + w) = rf(x) = r\frac{f(x)}{g(x)}g(x) = r\lambda g(x) = \lambda g(rx) = \lambda g(rx + w) = \lambda g(v) \Longrightarrow f = \lambda g$.

"$\Longleftarrow$": 若 $\exists 0 \neq \lambda \in F$, s.t. $f = \lambda g$, 则显然 $\ker f = \ker g$.

□

3.4 对偶基

定义 3.3 对偶基: $\mathcal{B} = \{b_1, \cdots, b_n\}$ 为 V 的基, 则 $\forall i$, $\exists b_i^* \in V$, s.t. $b_i^*(b_i) = 1$, $b_i^*(b_j) = 0 \forall i \neq j$, 即 $b_i^*(b_j) = \delta_{ij}$,

从而可构造出 $\mathcal{B}^* = \{b_1^*, \cdots, b_n^*\} \subseteq V^*$, 称为 $\mathcal{B}$ 的**对偶基**.

定理 3.6 (课本定理3.12): (1) $\mathcal{B}^* = \{b_1^*, \cdots, b_n^*\}$ 线性无关.

(2) $\dim V < \infty$, 则 $\mathcal{B}^*$ 是 V^* 的基.

证: (1) $\sum_{i=1}^m r_i b_i^* = 0 \Longrightarrow \forall v \in V$, $\sum_{i=1}^m r_i b_i^*(v) = \left(\sum_{i=1}^m r_i b_i^*\right)(v) = 0(v) = 0$.
取 $v = b_j$, 则 $\sum_{i=1}^m r_i b_i^*(b_j) = \sum_{i=1}^m r_i \delta_{ij} = r_j = 0$.
对各个 b_j 如法炮制, 从而可得 $r_j = 0 \forall i$, 故得证.

(2) $\forall f \in V^*, \forall v \in V$, $\because \mathcal{B}$ 是 V 的基, $\therefore v = \sum_{i=1}^n r_i b_i$
$\Longrightarrow b_j^*(v) = b_j^*\left(\sum_{i=1}^n r_i b_i\right) = \sum_{i=1}^n r_i b_j^*(b_i) = \sum_{i=1}^n r_i \delta_{ij} = r_j$
回代得 $v = \sum_{i=1}^n b_i^*(v) b_i$
$\Longrightarrow f(v) = f\left(\sum_{i=1}^n b_i^*(v) b_i\right) = \sum_{i=1}^n b_i^*(v) f(b_i) = \sum_{i=1}^n f(b_i) b_i^*(v) = \left(\sum_{i=1}^n f(b_i) b_i^*\right)(v)$, 此处 $b_i^*(v), f(b_i) \in F$, 因此可交换位置, 我们可视 $\{b_i^*(v)\}$ 为基, $f(b_i)$ 为 $f(v)$ 在这组基上的展开系数
$\Longrightarrow f = \sum_{i=1}^n f(b_i) b_i^*$, 即 f 可展开为 $\{\mathcal{B}^*\}$ 的线性表示, 结合 (1) 得证.

□

仿照上面的方法, $\forall v \in V$, 我们都可构造 $v^* \in V^*$, s.t. $v^*(v) = 1$, $\forall u_2 \in \langle v \rangle^c$, $v^*(u_2) = 0$, 从而有映射 $V \to V^*$, $v \mapsto v^*$, $0 \mapsto 0$ (零映射).

V^* 本身也是向量空间.

定义 3.4 二重对偶(空间): $V^{**} = \mathcal{L}(V^*, F)$ 称为**二重对偶(空间)**, 其中的元素为 $v^{**}: V^* \to F$, $f \to v^* * = f(v)$.

$V \to V^* \to V^{**}$, $v \mapsto v^* \mapsto v^{**}$, $b_i \mapsto b_i^* \mapsto b_i^{**}$, 满足 $b_i^*(b_j) = \delta_{ij}$, $b_i^{**}(b_j^*) = b_j^*(b_i)$, 两个映射复合得 $\tau: V \to V^{**}$, $v \mapsto v^{**}$.

(1) τ 是映射.

(2) τ 是线性变换.

(3) $\ker \tau = \{v \in V \mid \tau(v) = 0\} = \{0\} \Longleftrightarrow \tau$ 单射.

证: (1) 若 $u = v \in V$, 则 $\forall f \in V^*$, $u^{**}(f) = f(u) = f(v) = v^{**}(f)$, 即得证.

(2) $\forall f \in V^*$, $(\tau(ru + tv))(f) = (ru + tv)^{**}(f) = f(ru + tv) = rf(u) + tf(v) = ru^{**}(f) + tv^{**}(f) = r\tau(u)(f) + t\tau(v)(f) = (r\tau(u) + t\tau(v))(f) \Longrightarrow \tau(ru + tv) = r\tau(u) + t\tau(v)$, 结合 (1) 得证.

(3) $\tau(v) = 0 \Longrightarrow \forall f \in V^*$, $v^{**}(f) = 0 \Longrightarrow f(v) = 0 \Longrightarrow$(定理 3.5 (1)) $v = 0$, 即得证.

□

引理 3.1 (课本引理3.13): 若 $\dim V < \infty$, 则 $\dim V^* = \dim V^{**}$, V^{**} 与 V 同构, 一个线性空间的二重对偶就回到自身, 故实际上套娃式的 $V^{*\cdots *}$ 是没有意义的, 此处我们就写成 $V^{**} = V$.

3.5 伴随算子

定义 3.5 **伴随算子:** 由线性变换 $\tau : V \to W$ 可引出**伴随算子** $\tau^\times : W^* \to V^*$, $g \mapsto \tau^\times(g) = g \circ \tau$.

$$\begin{tikzcd}
V \arrow[rr, "\tau"] \arrow[dr, "\tau^\times(g) = g\circ\tau"'] & & W \arrow[dl, "g"] \\
& F &
\end{tikzcd}$$

(1) $\tau^\times$ 是映射.

(2) $\tau^\times$ 是线性变换.

证: (1) 若 $f = g \in W^*$, 则 $\tau^\times(f) = f \circ \tau = g \circ \tau = \tau^\times(g)$, 故得证.

(2) $\tau^\times(rg_1 + tg_2) = (rg_1 + tg_2) \circ \tau = rg_1 \circ \tau + tg_2 \circ \tau = r\tau^\times(g_1) + t\tau^\times(g_2)$, 结合 (1) 得证.

□

定理 3.7 (课本定理3.18): (1) $\forall \tau, \sigma \in \mathcal{L}(V, W)$, $\forall a, b \in F$, $(a\tau + b\sigma)^\times = a\tau^\times + b\sigma^\times$.

(2) $\sigma \in \mathcal{L}(V, W)$, $\tau \in \mathcal{L}(W, U)$, 则 $(\tau \circ \sigma)^\times = \sigma^\times \circ \tau^\times$.

(3) $\tau \in \mathcal{L}(V)$ 可逆 $\Longrightarrow (\tau^{-1})^\times = (\tau^\times)^{-1}$.

证: (1) $\forall f \in W^*$, $(a\tau + b\sigma)^*(f) = f \circ (a\tau + b\sigma) = af \circ \tau + bf \circ \sigma = a\tau^\times(f) + b\tau^\times(f) = (a\tau^\times + b\sigma^\times)(f)$, 即得证.

(2) $\forall f \in U^*$, $(\tau \circ \sigma)^*(f) = f \circ (\tau \circ \sigma) = (f \circ \tau) \circ \sigma = \sigma^\times(f \circ \tau) = \sigma^\times(\tau^\times(f)) = (\sigma^\times \circ \tau^\times)(f)$, 即得证.

$$\begin{tikzcd}
V \arrow[rr, "\sigma"] \arrow[dr, "\tau\circ\sigma"'] & & W \arrow[dl, "\tau"] & V^* & & W^* \arrow[ll, "\sigma^\times"'] \\
& U & & & U^* \arrow[ul, "(\tau\circ\sigma)^\times = \sigma^\times\circ\tau^\times?"] \arrow[ur, "\tau^\times"'] &
\end{tikzcd}$$

(3) $1^\times = (\tau \circ \tau^{-1})^\times = (\tau^{-1})^\times \circ \tau^\times \Longrightarrow (\tau^{-1})^\times = (\tau^\times)^{-1}$.

□

定理 3.8 (课本定理3.18): $\dim V < \infty$, $\dim W < \infty$, $\tau \in \mathcal{L}(V, W)$, $\tau^\times \in \mathcal{L}(W^*, V^*)$, $\tau^{\times\times} \in \mathcal{L}(V^{**}, W^{**}) = \mathcal{L}(V, W)$, 则 $\tau^{\times\times} = \tau$.

定理 3.9 (课本定理3.22): $\tau \in \mathcal{L}(V,W)$, 其中 $\dim V < \infty$, $\dim W < \infty$, $\mathcal{B}$ 和 $\mathcal{C}$ 分别是 V 和 W 的定序基, $\mathcal{B}^*$ 和 $\mathcal{C}^*$ 分别是 $\mathcal{B}$ 和 $\mathcal{C}$ 的对偶基, 则 $[\tau^\times]_{\mathcal{C}^*\mathcal{B}^*} = ([\tau]_{\mathcal{BC}})^T$.

证: 设 $\dim V = n$, $\dim W = m$, V 的定序基 $\mathcal{B} = \{b_1, \cdots, b_n\}$, W 的定序基 $\mathcal{C} = \{c_1, \cdots, c_m\}$, $\tau \in \mathcal{L}(V,W)$ 的矩阵表示为 $[\tau]_{\mathcal{BC}} = [\alpha_{ij}]_{m\times n}$, $\tau^\times \in \mathcal{L}(W^*, V^*)$ 的矩阵表示为 $[\tau^\times]_{\mathcal{C}^*\mathcal{B}^*} = [\beta_{ij}]_{n\times m}$.

$[\tau]_{\mathcal{BC}} = \begin{pmatrix} [\tau(b_1)]_{\mathcal{C}} & \cdots & [\tau(b_n)]_{\mathcal{C}} \end{pmatrix}$, 其中 $[\tau(b_i)]_{\mathcal{C}} = \begin{pmatrix} \alpha_{1i} \\ \vdots \\ \alpha_{mi} \end{pmatrix}$, 即 $\tau(b_i) = \sum_{k=1}^m \alpha_{ki} c_k$.

$[\tau^\times]_{\mathcal{C}^*\mathcal{B}^*} = \begin{pmatrix} [\tau^\times(c_1^*)]_{\mathcal{B}^*} & \cdots & [\tau^\times(c_m^*)]_{\mathcal{B}^*} \end{pmatrix}$, 其中 $[\tau^*(c_i^*)]_{\mathcal{B}^*} = \begin{pmatrix} \beta_{1i} \\ \vdots \\ \beta_{ni} \end{pmatrix}$, 即 $\tau^\times(c_i^*) = \sum_{l=1}^n \beta_{li} b_l^*$.

又由 $\tau^\times$ 的定义, $\tau^\times(c_i^*) = c_i^* \circ \tau$, 故将该复合函数作用于 b_j 上有 $c_i^*(\tau(b_j)) = (c_i^* \circ \tau)(b_j) = (\tau^\times(c_i^*))(b_j) = \left(\sum_{l=1}^n \beta_{li} b_l^*\right)(b_j) = \sum_{l=1}^n \beta_{li} b_l^*(b_j) = \sum_{l=1}^n \beta_{li}\delta_{lj} = \beta_{ji} \Longrightarrow \beta_{ji} = c_i^*(\tau(b_j))$,
代入上面的 $\tau(b_j)$ 的展开式得 $\beta_{ji} = c_i^*\left(\sum_{k=1}^m \alpha_{kj} c_k\right) = \sum_{k=1}^m \alpha_{kj} c_i^*(c_k) = \sum_{k=1}^m \alpha_{kj}\delta_{ik} = \alpha_{ij}$, 故得证. $\square$