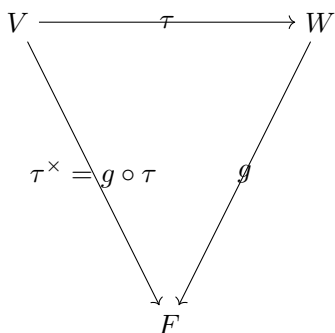


Chapter 10

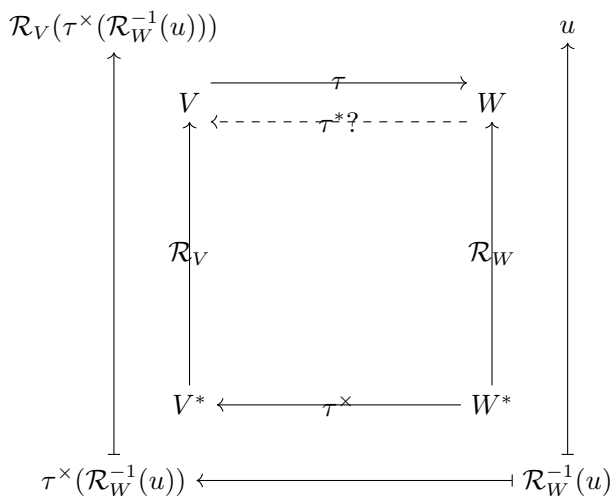
正规算子的结构理论

10.1 线性算子的伴随

先来回顾一下算子伴随: \mathcal{B} 和 \mathcal{C} 分别为线性空间 V 和 W 的定序基, V^* 和 W^* 分别为 V 和 W 的对偶空间, \mathcal{B}^* 和 \mathcal{C}^* 分别为 \mathcal{B} 和 \mathcal{C} 的对偶基, 对给定的线性变换 $\tau \in \mathcal{L}(V, W)$, 有算子伴随 $\tau^\times : W^* \rightarrow V^*$, $g \mapsto \tau^\times = g \circ \tau$, 线性变换在定序基上与其算子伴随在对偶基上的表示存在关系: $[\tau]_{\mathcal{B}\mathcal{C}} = [\tau^\times]_{\mathcal{C}^*\mathcal{B}^*}$.



下面来定义另一种伴随: 对于有限维内积向量空间 V 和 W , $\dim V = n$, $\dim W = m$, Riesz 映射 $\mathcal{R}_V : V^* \rightarrow V$, $\mathcal{R}_W : W^* \rightarrow W$, $\therefore \mathcal{R}_W$ 为共轭同构, \therefore 有其逆同构 \mathcal{R}_W^{-1} , 从而有映射 $\tau^* = \mathcal{R}_V \circ \tau^\times \circ \mathcal{R}_W$.



定理 10.1 (课本定理10.1): (1) τ^* 为线性变换, 即 $\tau^* \in \mathcal{L}(W, V)$.

(2) $\langle v, \tau^*(w) \rangle = \langle \tau(v), w \rangle$, 称 τ^* 为 τ 的伴随.

(3) $[\tau]_{\mathcal{B}\mathcal{C}} = [\tau^*]_{\mathcal{C}\mathcal{B}}^\dagger$.

证: (1) $\tau^* = \mathcal{R}_V \circ \tau^\times \circ \mathcal{R}_W^{-1} : W \rightarrow V$.

$\forall u_1, u_2 \in W, \tau^*(ru_1 + tu_2) = \mathcal{R}_V \circ \tau^\times \circ \mathcal{R}_W^{-1}(ru_1 + tu_2) = \mathcal{R}_V \circ \tau^\times(\mathcal{R}_W^{-1}(ru_1 + tu_2)) = \mathcal{R}_V \circ \tau^\times(\bar{r}\mathcal{R}_W^{-1}(u_1) + \bar{t}\mathcal{R}_W^{-1}(u_2)) = \mathcal{R}_V(\tau^\times(\bar{r}\mathcal{R}_W^{-1}(u_1) + \bar{t}\mathcal{R}_W^{-1}(u_2))) = \mathcal{R}_V(\bar{r}\tau^\times(\mathcal{R}_W^{-1}(u_1)) + \bar{t}\tau^\times(\mathcal{R}_W^{-1}(u_2))) = r\mathcal{R}_V(\tau^\times(\mathcal{R}_W^{-1}(u_1))) + t\mathcal{R}_V(\tau^\times(\mathcal{R}_W^{-1}(u_2))) = r\tau^*(u_1) + t\tau^*(u_2)$, 其中利用了引理 10.1, 得证.

(2) $\forall v \in V, \forall w \in W, \langle v, \tau^*(w) \rangle = \langle v, \mathcal{R}_V(\tau^\times \circ \mathcal{R}_W^{-1}(w)) \rangle = \tau^\times \circ \mathcal{R}_W^{-1}(w)(v) = \mathcal{R}_W^{-1}(w) \circ \tau(v) = \mathcal{R}_W^{-1}(w)(\tau(v)) = \langle \tau(v), w \rangle$.

(3) 设 V 的正交归一基 $\mathcal{B} = \{b_1, \dots, b_n\}$, W 的正交归一基 $\mathcal{C} = \{c_1, \dots, c_m\}$.

$$[\tau]_{\mathcal{B}\mathcal{C}} = \begin{pmatrix} [\tau(b_1)]_{\mathcal{C}} & \cdots & [\tau(b_n)]_{\mathcal{C}} \end{pmatrix}, [\tau^*]_{\mathcal{C}\mathcal{B}} = \begin{pmatrix} [\tau^*(c_1)]_{\mathcal{B}} & \cdots & [\tau^*(c_m)]_{\mathcal{B}} \end{pmatrix}.$$

设 $[\tau(b_i)]_{\mathcal{C}} = \begin{pmatrix} \alpha_{1i} \\ \vdots \\ \alpha_{mi} \end{pmatrix}$, $\tau(b_i) = \sum_{k=1}^m \alpha_{ki} c_k$, $\langle \tau(b_i), c_j \rangle = \langle \sum_{k=1}^m \alpha_{ki} c_k, c_j \rangle = \sum_{k=1}^m \alpha_{ki} \langle c_k, c_j \rangle = \sum_{k=1}^m \alpha_{ki} \delta_{kj} = \alpha_{ji}$.

同理, 设 $[\tau^*(c_j)]_{\mathcal{B}} = \begin{pmatrix} \beta_{1j} \\ \vdots \\ \beta_{nj} \end{pmatrix}$, $\tau^*(c_j) = \sum_{k=1}^n \beta_{kj} b_k$, $\langle \tau^*(c_j), b_i \rangle = \langle \sum_{k=1}^n \beta_{kj} b_k, b_i \rangle = \sum_{k=1}^n \beta_{kj} \langle b_k, b_i \rangle = \sum_{k=1}^n \beta_{kj} \delta_{ki} = \beta_{ij}$.

又 $\because \langle \tau(b_i), c_j \rangle = \langle b_i, \tau^*(c_j) \rangle = \overline{\langle \tau^*(c_j), b_i \rangle}$, $\therefore \alpha_{ji} = \beta_{ij} \implies [\tau]_{\mathcal{B}\mathcal{C}} = [\tau^*]_{\mathcal{C}\mathcal{B}}^\dagger$.

□

引理 10.1: Riesz 映射的逆 \mathcal{R}^{-1} 共轭线性.

证: $\forall x_1, x_2 \in V, \exists f_1 = \mathcal{R}^{-1}(x_1), f_2 = \mathcal{R}^{-1}(x_2) \in V^*$, s.t. $\forall v \in V, f_1(v) = \langle v, x_1 \rangle, f_2(v) = \langle v, x_2 \rangle$
 $\implies \forall \bar{r}, \bar{t} \in F, (\bar{r}\mathcal{R}^{-1}(x_1) + \bar{t}\mathcal{R}^{-1}(x_2))(v) = (\bar{r}f_1 + \bar{t}f_2)(v) = \bar{r}f_1(v) + \bar{t}f_2(v) = \bar{r}\langle v, x_1 \rangle + \bar{t}\langle v, x_2 \rangle = \langle v, \bar{r}x_1 + \bar{t}x_2 \rangle = \langle v, \mathcal{R}^{-1}(\bar{r}x_1 + \bar{t}x_2) \rangle = \mathcal{R}^{-1}(\bar{r}x_1 + \bar{t}x_2)(v)$
 $\implies \mathcal{R}^{-1}(\bar{r}x_1 + \bar{t}x_2) = \bar{r}\mathcal{R}^{-1}(x_1) + \bar{t}\mathcal{R}^{-1}(x_2)$.

□

$\because [\tau]_{\mathcal{B}\mathcal{C}} = [\tau^\times]_{\mathcal{C}^*\mathcal{B}^*}^T, [\tau]_{\mathcal{B}\mathcal{C}} = [\tau^*]_{\mathcal{C}\mathcal{B}}^\dagger, \therefore [\tau^\times]_{\mathcal{C}^*\mathcal{B}^*} = \overline{[\tau^*]_{\mathcal{C}\mathcal{B}}}$. 当然这也可用类似定理 10.1 (3) 的证明方法证明:

证: $[\tau^\times]_{\mathcal{C}^*\mathcal{B}^*} = \begin{pmatrix} [\tau^\times(c_1^*)]_{\mathcal{B}^*} & \cdots & [\tau^\times(c_n^*)]_{\mathcal{B}^*} \end{pmatrix}, [\tau^*]_{\mathcal{C}\mathcal{B}} = \begin{pmatrix} [\tau^*(c_1)]_{\mathcal{B}} & \cdots & [\tau^*(c_n)]_{\mathcal{B}} \end{pmatrix},$

设 $[\tau^\times(c_i^*)]_{\mathcal{B}^*} = \begin{pmatrix} \alpha_{1i} \\ \vdots \\ \alpha_{ni} \end{pmatrix}$, 则 $\tau^\times(c_i^*) = \sum_{k=1}^n \alpha_{ki} b_k^*$.

设 $[\tau^*(c_i)]_{\mathcal{B}} = \begin{pmatrix} \beta_{1i} \\ \vdots \\ \beta_{mi} \end{pmatrix}$, 则 $\tau^*(c_i) = \sum_{k=1}^m \beta_{ki} b_k$.

一方面, $\because \mathcal{R}_W^{-1}(c_i)(c_j) = \langle c_j, c_i \rangle = \delta_{ij}, \therefore \mathcal{R}_V(c_i) = c_i^*$

$\implies \langle b_j, \tau^*(c_i) \rangle = \langle b_j, \mathcal{R}_V \circ \tau^\times \circ \mathcal{R}_W^{-1}(c_i) \rangle = \langle b_j, \mathcal{R}_V(\tau^\times(\mathcal{R}_W^{-1}(c_i))) \rangle = \langle b_j, \mathcal{R}_V(\tau^\times(c_i^*)) \rangle = \tau^\times(c_i^*)(b_j) = (\sum_{k=1}^n \alpha_{ki} b_k^*)(b_j) =$

$$\sum_{k=1}^n \alpha_{ki} b_k^*(b_j) = \sum_{k=1}^n \alpha_{ki} \delta_{jk} = \alpha_{ji};$$

$$\text{另一方面, } \langle b_j, \tau^*(c_i) \rangle = \langle b_j, \sum_{k=1}^m \beta_{ki} b_k \rangle = \sum_{k=1}^m \overline{\beta_{ki}} \langle b_j, b_k \rangle = \sum_{k=1}^m \overline{\beta_{ki}} \delta_{jk} = \overline{\beta_{ji}}.$$

故 $\alpha_{ji} = \overline{\beta_{ji}}$, 得证. □

定理 10.2 (课本定理10.2): V, W 为有限维内积向量空间, $\forall \sigma, \tau \in \mathcal{L}(V, W), \forall r \in F$,

$$(1) (\sigma + \tau)^* = \sigma^* + \tau^*.$$

$$(2) (r\tau)^* = \bar{r}\tau^*.$$

$$(3) \tau^{**} = \tau.$$

$$(4) \text{ 若 } V = W, \text{ 则 } (\tau \circ \sigma)^* = \sigma^* \circ \tau^*.$$

$$(5) V = W, \text{ 若 } \tau \text{ 可逆, 则 } (\tau^{-1})^* = (\tau^*)^{-1}.$$

$$(6) V = W, p(x) \in \mathbb{R}[x], \text{ 则 } p(\tau)^* = p(\tau^*).$$

$$(7) S \text{ 是 } V \text{ 的子空间, } \tau \in \mathcal{L}(V), \text{ 则 } S \text{ 是 } \tau \text{ 的不变子空间} \iff S^\perp \text{ 是 } \tau^* \text{ 的不变子空间}.$$

证: (1) $\forall u \in W, \forall v \in V, \langle v, (\sigma + \tau)^*(u) \rangle = \langle (\sigma + \tau)(v), u \rangle = \langle \sigma(v) + \tau(v), u \rangle = \langle \sigma(v), u \rangle + \langle \tau(v), u \rangle = \langle v, \sigma^*(u) \rangle + \langle v, \tau^*(u) \rangle = \langle v, \sigma^*(u) + \tau^*(u) \rangle = \langle v, (\sigma^* + \tau^*)(u) \rangle \implies (\sigma + \tau)^*(u) = (\sigma^* + \tau^*)(u) \implies (\sigma + \tau)^* = \sigma^* + \tau^*.$

$$(2) \forall u \in W, \forall v \in V, \langle v, (r\tau)^*(u) \rangle = \langle r\tau(v), u \rangle = r \langle \tau(v), u \rangle = r \langle v, \tau^*(u) \rangle = \langle v, \bar{r}\tau^*(u) \rangle \implies (r\tau)^*(u) = \bar{r}\tau^*(u) \implies (r\tau)^* = \bar{r}\tau^*.$$

$$(3) \forall u \in W, \forall v \in V, \langle u, \tau^{**}(v) \rangle = \langle u, (\tau^*)^*(v) \rangle = \langle \tau^*(u), v \rangle = \overline{\langle v, \tau^*(u) \rangle} = \overline{\langle \tau(v), u \rangle} = \langle u, \tau(v) \rangle \implies \tau^{**}(v) = \tau(v) \implies \tau^{**} = \tau.$$

$$(4) \forall v \in V, \forall u \in W, \langle u, (\tau \circ \sigma)^*(v) \rangle = \langle (\tau \circ \sigma)(u), v \rangle = \langle \tau(\sigma(u)), v \rangle = \langle \sigma(u), \tau^*(v) \rangle = \langle u, \sigma^*(\tau^*(v)) \rangle = \langle u, (\sigma^* \circ \tau^*)(v) \rangle \implies (\tau \circ \sigma)^*(v) = (\sigma^* \circ \tau^*)(v) \implies (\tau \circ \sigma)^* = \sigma^* \circ \tau^*.$$

$$\begin{array}{ccc} V & \xrightarrow{\sigma} & V \\ \xleftarrow{\sigma^*} & & \xleftarrow{\tau^*} \\ & & V \end{array}$$

$$\begin{array}{ccc} V & \xrightarrow{\sigma \circ \tau} & V \\ \xleftarrow{(\sigma \circ \tau)^*} & & \end{array}$$

$$(5) (\tau^{-1})^* \circ \tau^* = (\tau \circ \tau^{-1})^* = 1_V^* = 1_V \implies (\tau^{-1})^* = (\tau^*)^{-1}.$$

$$(\because \langle u, v \rangle = \mathcal{R}^{-1}(v)(u) = (\mathcal{R}^{-1}(v) \circ 1_V)(u) = \langle u, \mathcal{R}_V(\mathcal{R}_V^{-1}(v) \circ 1_V) \rangle)$$

$$\begin{array}{ccc}
 & \swarrow & \\
 \mathcal{R}_V(\mathcal{R}_V^{-1}(v) \circ 1_V) = v & & v \\
 \uparrow & \xrightarrow{1_V} & \uparrow \\
 V & \xleftarrow{1_V^* = 1_V^?} & V \\
 \uparrow \mathcal{R}_V & & \uparrow \mathcal{R}_V \\
 V^* & \xleftarrow{1_V^\times} & V^* \\
 \uparrow & & \uparrow \\
 1_V^\times(\mathcal{R}_V^{-1}(v)) = \mathcal{R}_V^{-1}(v) \circ 1_V & \xleftarrow{\quad} & \mathcal{R}_V^{-1}(v)
 \end{array}$$

$$(6) \quad (\tau \circ \tau)^* = \tau^* \circ \tau^*, \quad (\tau^k)^* = (\tau^*)^k.$$

$$\begin{aligned}
 &\text{若 } r \in \mathbb{R}, \text{ 则 } (r\tau)^* = r\tau^*, \quad (r\tau^k)^* = r(\tau^*)^k \\
 &\implies (p(\tau))^* = p(\tau^*).
 \end{aligned}$$

$$(7) \quad “\implies”: \because S \text{ 是 } \tau \text{ 的不变子空间, } \therefore \tau(S) \subseteq S.$$

$$\forall v \in S^\perp, \forall u \in S, \tau(u) \in S \implies \langle u, \tau^*(v) \rangle = \langle \tau(u), v \rangle = 0 \implies \tau^*(v) \in S^\perp \implies S^\perp \text{ 是 } \tau^* \text{ 的不变子空间.}$$

$$“\impliedby”: \because S^{\perp\perp} = S, \text{ 故同理得证.}$$

□

定理 10.3 (课本定理10.3): V, W 为有限维内积向量空间, $\tau \in \mathcal{L}(V, W)$, 则

$$(1) \quad \ker \tau^* = (\operatorname{Im} \tau)^\perp, \text{ 等价地, } \operatorname{Im} \tau^* = (\ker \tau)^\perp.$$

$$(2) \quad \ker \tau^* \tau = \ker \tau, \quad \ker \tau \tau^* = \ker \tau^*.$$

$$(3) \quad \operatorname{Im} \tau^* \tau = \operatorname{Im} \tau^*, \quad \operatorname{Im} \tau \tau^* = \operatorname{Im} \tau.$$

$$(4) \quad \rho_{ST}^* = \rho_{T^\perp S^\perp}.$$

证: (1) $\forall w \in \operatorname{Im} \tau, \exists u \in V, \text{ s.t. } w = \tau(u).$

$$v \in \ker \tau^* \iff \tau^*(v) = 0 \iff \langle w, v \rangle = \langle \tau(u), v \rangle = \langle u, \tau^*(v) \rangle = \langle u, 0 \rangle = 0 \iff v \in (\operatorname{Im} \tau)^\perp, \text{ 故 } \ker \tau^* = (\operatorname{Im} \tau)^\perp.$$

$$\forall w \in \ker \tau, \iff \tau(w) = 0 \in W.$$

$$v \in \operatorname{Im} \tau^* \iff \exists u \in W, \text{ s.t. } \tau^*(u) = v \iff \langle w, v \rangle = \langle w, \tau^*(u) \rangle = \langle \tau(w), u \rangle = \langle 0, u \rangle = 0 \iff v \in (\ker \tau)^\perp, \text{ 故 } \operatorname{Im} \tau^* = (\ker \tau)^\perp.$$

$$(2) \quad \forall v \in \ker \tau \tau^*, \tau^* \tau(v) = 0 \implies \langle v, \tau^* \tau(v) \rangle = 0 \iff \langle \tau(v), \tau(v) \rangle = 0 \implies \tau(v) = 0 \iff v \in \ker \tau, \text{ 故 } \ker \tau^* \tau \subseteq \ker \tau.$$

$$\forall v \in \ker \tau, \tau(v) = 0 \implies \tau^* \tau(v) = 0 \iff v \in \ker \tau^* \tau, \text{ 故 } \ker \tau \subseteq \ker \tau^* \tau.$$

$$\text{综上, } \ker \tau^* \tau = \ker \tau.$$

$$\text{同理, } \ker \tau \tau^* = \ker \tau^*.$$

(3) $\forall v \in \text{Im } \tau^* \tau, \exists u \in V, \text{ s.t. } u = \tau^* \tau(v) = \tau^*(\tau(v))$, 即 $\exists w = \tau(v) \in W, \text{ s.t. } v = \tau^*(w)$, 故 $\text{Im } \tau^* \tau \in \text{Im } \tau^*$.

$\forall v \in \text{Im } \tau^*, \exists w \in W, \text{ s.t. } v = \tau^*(w)$.

$\therefore \tau$ 为共轭同构, $\therefore \exists u \in V, \text{ s.t. } w = \tau(u) \implies v = \tau^* \tau(u)$, 故 $\text{Im } \tau^* \in \text{Im } \tau^* \tau$.

综上, $\text{Im } \tau^* \tau = \text{Im } \tau^*$.

同理, $\text{Im } \tau \tau^* = \text{Im } \tau$.

(4) $\forall u, v \in V$,

$\rho_{ST} : V \rightarrow V, u = u_S + u_T \mapsto u_S, v = v_S + v_T \mapsto v_S$, 其中 $u_S \in S, u_T \in T, v_S \in S, v_T \in T, V = S \oplus T$,

$\rho_{T^\perp S^\perp} : V \rightarrow V, u = u_{S^\perp} + u_{T^\perp} \mapsto u_{T^\perp}, v = v_{S^\perp} + v_{T^\perp} \mapsto v_{T^\perp}$, 其中 $u_{S^\perp} \in S^\perp, u_{T^\perp} \in T^\perp, v_{S^\perp} \in S^\perp, v_{T^\perp} \in T^\perp, V = S^\perp \oplus T^\perp$.

$\therefore \langle u, \rho_{ST}^*(v) \rangle = \langle \rho_{ST}(u), v \rangle = \langle u_S, v \rangle, \langle u, \rho_{T^\perp S^\perp}(v) \rangle = \langle u, v_{T^\perp} \rangle$,

$\therefore \langle u, \rho_{ST}^*(v) \rangle - \langle u, \rho_{T^\perp S^\perp}(v) \rangle = \langle u_S, v \rangle - \langle u, v_{T^\perp} \rangle = \langle u_S, v \rangle - \langle u_S, v_{T^\perp} \rangle + \langle u_S, v_{T^\perp} \rangle - \langle u, v_{T^\perp} \rangle = \langle u_S, v - v_{T^\perp} \rangle - \langle u - u_S, v_{T^\perp} \rangle = \langle u_S, v_{S^\perp} \rangle - \langle u_T, v_{T^\perp} \rangle$.

又 $\therefore v_S \in S, u_{S^\perp} \in S^\perp, v_T \in T, u_{T^\perp} \in T^\perp, \therefore \langle u_S, u_{S^\perp} \rangle = 0, \langle u_T, v_{T^\perp} \rangle = 0 \implies \langle u, \rho_{ST}^*(v) \rangle - \langle u, \rho_{T^\perp S^\perp}(v) \rangle = 0 \implies \langle u, \rho_{ST}^*(v) \rangle = \langle u, \rho_{T^\perp S^\perp}(v) \rangle \implies \rho_{ST}^*(v) = \rho_{T^\perp S^\perp}(v) \implies \rho_{ST}^* = \rho_{T^\perp S^\perp}$.

□

10.2 正交(/么正)对角化

先来回顾一下线性变换可对角化的充要条件: $\tau \in \mathcal{L}(V)$, τ 可对角化 (即 \exists 一组基 \mathcal{B} , $[\tau]_{\mathcal{B}}$ 为对角阵)

$\iff m_\tau(x) = (x - \lambda_1) \cdots (x - \lambda_k)$, 其中 λ_i 互不相同

$\iff V = \mathcal{E}_{\lambda_1} \oplus \cdots \oplus \mathcal{E}_{\lambda_k}$

$\iff \tau$ 的特征向量构成 V 的一组基

\iff 几何重数 (特征子空间的维数) = 代数重数 (特征多项式的根的重数)

$\iff \tau = \lambda_1 \rho_1 + \cdots + \lambda_k \rho_k$, 其中 λ_i 互不相同, $\rho_1 + \cdots + \rho_k = 1$ 为单位分解 (即 ρ_i 为投影, $\sum_i \rho_i = 1$ 且 $\rho_i \rho_j = \rho_j \rho_i = \delta_{ij} \rho_i$).

再来回顾一下向量正交: 向量 u 与 v 正交 $\iff \langle u, v \rangle = 0$.

非零元构成的正交集线性无关.

若 $\dim V < \infty$, 则 V 有正交归一基.

那么, τ 是否可正交对角化? 哪一类 τ 可正交对角化?

定义 10.1 正交(/么正)对角化: $\tau \in \mathcal{L}(V)$, 若 \exists 一组正交归一基 \mathcal{O} , s.t. $[\tau]_{\mathcal{O}}$ 为对角阵, 则称 τ 可正交(/么正)对角化.

定理 10.4: τ 可正交归一化 $\iff \tau$ 的特征向量构成 V 的正交基.

定义 10.2 正规算子: $\dim V < \infty, \tau \in \mathcal{L}(V)$, 若 $\tau^* \tau = \tau \tau^*$, 则称 τ 为正规算子.

定理 10.5 (课本第3 版定理10.8): 对正规算子 $\tau \in \mathcal{L}(V)$,
 τ^*, τ^{-1} (在 τ 可逆的前提下), $p(\tau)$ ($p(x) \in F[x]$) 正规.

(2) $\|\tau(v)\| = \|\tau^*(v)\|$, 从而 $\ker \tau = \ker \tau^*$.

(3) $\forall k \in \mathbb{Z}^+$, $\ker \tau^k = \ker \tau$.

(4) $m_\tau(x) = p_1(x) \cdots p_m(x)$, 其中 $p_i(x)$ 不可约且互不相同.

(5) $\tau(v) = \lambda v \implies \tau^*(v) = \bar{\lambda}v$.

(6) $\lambda_i \neq \lambda_j \implies \mathcal{E}_{\lambda_i} \perp \mathcal{E}_{\lambda_j}$.

证: (1) $(\tau^*)^* \tau^* = \tau^{**} \tau^* = \tau \tau^* = \tau^* \tau = \tau^* \tau^{**} = \tau^* (\tau^*)^* \implies \tau^*$ 正规.

$(\tau^{-1})^* \tau^{-1} = (\tau^*)^{-1} \tau^{-1} = (\tau \tau^*)^{-1} = (\tau^* \tau)^{-1} = \tau^{-1} (\tau^*)^{-1} = \tau^{-1} (\tau^{-1})^* \implies \tau^{-1}$ 正规.

$(\tau^i)^* \tau^i = (\tau^*)^i \tau^i = (\because \tau \text{ 正规, 即 } \tau \text{ 与 } \tau^* \text{ 可交换}) \tau^i (\tau^*)^i = \tau^i (\tau^i)^* \implies \tau^i$ 正规

$\implies (r\tau^i)^* (r\tau^i) = \bar{r}(\tau^i)^* r \tau^i = r \tau^i \bar{r} (\tau^i)^* = (r\tau^i) (r\tau^i)^* \implies r\tau^i$ 正规

$\implies p(\tau) p^*(\tau) = (\sum_i r_i \tau^i) (\sum_j \bar{r}_j \tau^j)^* = \sum_{ij} r_i \tau^i \bar{r}_j (\tau^j)^* = \sum_{ij} \bar{r}_j (\tau^j)^* r_i \tau^i = (\sum_j \bar{r}_j \tau^j)^* (\sum_i r_i \tau^i) = p^*(\tau) p(\tau) \implies p(\tau)$ 正规.

(2) $\|\tau(v)\|^2 = \langle \tau(v), \tau(v) \rangle = \langle v, \tau^*(\tau(v)) \rangle = \langle v, (\tau^* \circ \tau)(v) \rangle = \langle v, (\tau \circ \tau^*)(v) \rangle = \langle v, \tau(\tau^*(v)) \rangle = \langle \tau^*(v), \tau^*(v) \rangle = \|\tau^*(v)\|^2 \implies \|\tau(v)\|^2 = \|\tau^*(v)\|^2$,

故 $\ker \tau = \{v \mid \tau(v) = 0\} = \{v \mid \|\tau(v)\| = 0\} = \{v \mid \|\tau^*(v)\| = 0\} = \{v \mid \tau^*(v) = 0\} = \ker \tau^*$.

(3) $\ker \tau \subseteq \ker \tau^k$ 显然. 下面来证 $\ker \tau^k \subseteq \ker \tau$:

令 $\sigma = \tau^* \tau$, 则 $\sigma^* = (\tau^* \tau)^* = \tau^{**} \tau^{**} = \tau^* \tau = \sigma$.

$\forall v \in \ker \tau^k, \tau^k(v) = 0 \implies \sigma^k(v) = (\tau^* \tau)^k(v) = (\because \tau \text{ 正规, 即 } \tau \text{ 与 } \tau^* \text{ 可交换}) (\tau^*)^k \tau^k(v) = 0$,

$0 = \langle 0, \sigma^{k-2}(v) \rangle = \langle \sigma^k(v), \sigma^{k-2}(v) \rangle = \langle \sigma \circ \sigma^{k-1}(v), \sigma^{k-2}(v) \rangle = \langle \sigma^{k-1}(v), \sigma^* \circ \sigma^{k-2}(v) \rangle = (\because \sigma^* = \sigma) \langle \sigma^{k-1}(v), \sigma \circ \sigma^{k-2}(v) \rangle = \langle \sigma^{k-1}(v), \sigma^{k-1}(v) \rangle = \|\sigma^{k-1}(v)\|^2 \implies \sigma^{k-1}(v) = 0$, 以此类推得 $\sigma(v) = 0$

$\implies 0 = \langle v, 0 \rangle = \langle v, \sigma(v) \rangle = \langle v, \tau^*(\tau(v)) \rangle = \langle \tau(v), \tau(v) \rangle = \|\tau(v)\|^2 \implies \tau(v) = 0 \implies v \in \ker \tau \implies \ker \tau^k \subseteq \ker \tau$.

综上, 得证.

(4) $m_\tau(x) = up_1^{e_1}(x) \cdots p_m^{e_m}(x)$, 其中 p_i 不可约且互不相同, $e_i \in \mathbb{Z}^+$.

要证 $m_\tau(x) = p_1(x) \cdots p_m(x)$, 即证 $e_i = 1 \forall i$.

$\forall v \in V, m_\tau(\tau)(v) = p_1^{e_1}(\tau) \cdots p_m^{e_m}(\tau)(v) = p_1^{e_1}(\tau)[p_2^{e_2}(\tau) \cdots p_m^{e_m}(\tau)(v)] = 0$.

$\because \tau$ 正规, $\therefore p_1(\tau)$ 正规 $\implies \ker p_1(\tau) = \ker p_1^{e_1}(\tau)$

$\implies p_1(\tau)[p_2^{e_2}(\tau) \cdots p_m^{e_m}(\tau)(v)] = 0 \implies p_1(x)p_2^{e_2}(x) \cdots p_m^{e_m}(x) \in \langle m_\tau(x) \rangle \implies m_\tau(x) = p_1^{e_1}(x)p_2^{e_2}(x) \cdots p_m^{e_m}(x) \mid p_1^1(x)p_2^{e_2}(x) \cdots p_m^{e_m}(x) \implies e_1 = 1$.

$\because p_i(\tau)$ 正规, 即 $p_i(\tau)$ 可交换, \therefore 同理可得 $e_i = 1 \forall i$, 故得证.

(5) $\tau(v) = \lambda v \implies (\tau - \lambda)(v) = 0 \implies v \in \ker(\tau - \lambda)$,

$\because \tau$ 正规, $\therefore \tau - \lambda$ 正规 $\implies \ker(\tau - \lambda) = \ker(\tau - \lambda)^*$

$\implies v \in \ker(\tau - \lambda)^* = \ker(\tau^* - \bar{\lambda})$.

(6) $\forall 0 \neq v \in \mathcal{E}_{\lambda_i}, \forall 0 \neq u \in \mathcal{E}_{\lambda_j}$, 其中 $\lambda_i \neq \lambda_j, \lambda_i \langle v, u \rangle = \langle \lambda_i v, u \rangle = \langle \tau(v), u \rangle = \langle v, \tau^*(u) \rangle = \langle v, \bar{\lambda}_j u \rangle = \lambda_j \langle v, u \rangle \implies (\lambda_i - \lambda_j) \langle v, u \rangle = 0$.

$\because \lambda_i - \lambda_j \neq 0, \therefore \langle v, u \rangle = 0$.

□

定理 10.6 正规算子的谱的结构: 复情形(课本定理10.13): $F = \mathbb{C}$, $\dim V < \infty$, $\tau \in \mathcal{L}(V)$, 则下列叙述等价:

- (1) τ 正规.
- (2) τ 可正交对角化, $V = \mathcal{E}_{\lambda_1} \odot \cdots \odot \mathcal{E}_{\lambda_k}$.
- (3) $\tau = \lambda_1 \rho_1 + \cdots + \lambda_k \rho_k$, 其中 $\rho_1 + \cdots + \rho_k = 1$ 为单位分解, 对 $i \neq j$, $\text{Im } \rho_i \perp \text{Im } \rho_j$, $\text{Im } \rho_i \perp \ker \rho_i$.

证: “(1) \implies (2)” : $\because \tau$ 正规, $\therefore \tau$ 的极小多项式的不可约多项式的次数均为 1, 即 $m_\tau(x) = p_1(x) \cdots p_k(x) = (x - \lambda_1) \cdots (x - \lambda_k)$, 其中 $p_i(x) \in \mathbb{C}[x]$ 为不可约多项式, λ_i 互不相等

$$\implies V = \mathcal{E}_{\lambda_1} \oplus \cdots \oplus \mathcal{E}_{\lambda_k}.$$

又 \because 对 $i \neq j$, $\mathcal{E}_{\lambda_i} \perp \mathcal{E}_{\lambda_j}$, $\therefore V = \mathcal{E}_{\lambda_1} \odot \cdots \odot \mathcal{E}_{\lambda_k}$.

“(2) \implies (1)” : $\because \tau$ 可正交对角化, $\therefore \exists$ 正交归一基 \mathcal{O} , s.t. $[\tau]_{\mathcal{O}} = \text{diag}(\lambda_1, \cdots, \lambda_k)$, $[\tau^*]_{\mathcal{O}} = \text{diag}(\overline{\lambda_1}, \cdots, \overline{\lambda_k})$
 $\implies [\tau]_{\mathcal{O}}[\tau^*]_{\mathcal{O}} = \text{diag}(|\lambda_1|^2, \cdots, |\lambda_k|^2) = [\tau^*]_{\mathcal{O}}[\tau]_{\mathcal{O}} \implies [\tau^* \tau(v)]_{\mathcal{O}} = [\tau^*]_{\mathcal{O}}[\tau]_{\mathcal{O}}[v]_{\mathcal{O}} = [\tau]_{\mathcal{O}}[\tau^*]_{\mathcal{O}}[v]_{\mathcal{O}} = [\tau \tau^*(v)]_{\mathcal{O}} \implies \tau \tau^* = \tau^* \tau$.

“(3) \iff (1)” : 利用引理 10.2, $\ker \rho^* = (\text{Im } \rho)^\perp = \ker \rho$, $\text{Im } \rho^* = (\ker \rho)^\perp = \text{Im } \rho \implies \rho^* = \rho$
 $\implies \tau^* = \overline{\lambda_1} \rho_1 + \cdots + \overline{\lambda_k} \rho_k$.
 $\tau \tau^* = (\sum_i \lambda_i \rho_i) \left(\sum_j \overline{\lambda_j} \rho_j \right) = \sum_{ij} \lambda_i \overline{\lambda_j} \rho_i \rho_j = \sum_{ij} \lambda_i \overline{\lambda_j} \delta_{ij} \rho_i = \sum_i |\lambda_i|^2 \rho_i = \sum_{ij} \overline{\lambda_j} \lambda_i \rho_j \rho_i = \left(\sum_j \overline{\lambda_j} \rho_j \right) \left(\sum_i \lambda_i \rho_i \right) = \tau^* \tau \implies \tau$ 正规.

综上, 得证. □

引理 10.2: $V = S \odot T$, 正交投影 $\rho_{ST} : V \rightarrow V$, $u = u_S + u_T \rightarrow u_S$, 则 $\ker \rho \perp \text{Im } \rho$.

证: $\forall v \in \ker \rho_{ST}$, $v = v_S + v_T$ 其中 $v_S \in S$, $v_T \in T$, $\rho_{ST}(v) = v_S = 0 \implies v = v_T \in T$.

$\forall w_S \in \text{Im } \rho_{ST}$, $\exists w \in V$, s.t. $\rho_{ST}(w) = w_S \implies w = w_S + w_T$, 其中 $w_S \in S$, $w_T \in T$.

$\therefore v \in T$, $w_S \in S$, $\therefore v \perp w \implies \ker \rho \perp \text{Im } \rho$. □

由于 $\mathbb{R}[x]$ 中不可约多项式的最高次数为 2, 故实数域上的向量空间的线性算子的最小多项式的分解形式与复情形有所不同.

定理 10.7 正规算子的谱的结构: 实情形(课本定理10.14): $F = \mathbb{R}$, $\dim V < \infty$, $\tau \in \mathcal{L}(V)$ 正规 $\iff V = \mathcal{E}_{\lambda_1} \odot \cdots \odot \mathcal{E}_{\lambda_k} \odot D_1 \odot \cdots \odot D_l$, 其中 \mathcal{E}_{λ_i} 为 τ 的不变子特征空间, λ_i 为 τ 的谱, D_i 为 τ_i 的二维不可约不变子空间且 D_i 中有基 \mathcal{B}'_i , s.t. $[\tau]_{\mathcal{B}'_i} = \begin{pmatrix} s_i & t_i \\ -t_i & s_i \end{pmatrix}$,

$$[\tau]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_k & & \\ & & & \begin{pmatrix} s_1 & t_1 \\ -t_1 & s_1 \end{pmatrix} & \\ & & & & \ddots \\ & & & & & \begin{pmatrix} s_l & t_l \\ -t_l & s_l \end{pmatrix} \end{pmatrix}_{n \times n}.$$

证: τ 的极小多项式 $m_\tau(x) = (x - \lambda_1) \cdots (x - \lambda_t) q_1(x) \cdots q_l(x)$, 其中 $q_j(x)$ 不可约, $\deg q_j(x) = 2$, $\lambda_i \in \mathbb{R}$ 互不相同, $V = \mathcal{E}_{\lambda_1} \oplus \cdots \oplus \mathcal{E}_{\lambda_t} \oplus D_1 \oplus \cdots \oplus D_l$, 其中 $V_{p_i} = \{v \mid (\tau - \lambda_i)v = 0\}$, $\dim V_{p_i} = 1$, $\text{ann}(D_i) = \langle q_i(x) \rangle$, $\deg q_i(x) = 2$, 无妨 $q_i(x) = x^2 + b_i x + c_i$, $\therefore q_i$ 不可约, $\therefore \Delta = b_i^2 - 4c_i < 0$.

取 D_i 的基 $\mathcal{B}_i \equiv \{v_i, \tau(v_i)\}$, 则 $[\tau]_{\mathcal{B}_i} = \begin{pmatrix} 0 & -c_i \\ 1 & -b_i \end{pmatrix}$.

为使 τ 在 D_i 中的表示更对称, 对 $[\tau]_{\mathcal{B}_i}$ 做相似变换到基 \mathcal{B}'_i 上, s.t. $[\tau]_{\mathcal{B}'_i} = \begin{pmatrix} s_i & t_i \\ -t_i & s_i \end{pmatrix}$, 其中 $s_i = -\frac{b_i}{2}$, $t_i = \frac{\sqrt{4c_i - b_i^2}}{2}$.

问题 10.1: 如何相似变换? $\mathcal{B}'_i = ?$ □

解: $\because [\tau]_{\mathcal{B}_i}$ 和 $[\tau]_{\mathcal{B}'_i}$ 的特征多项式相同, 均为 $q_i(x)$, 特征值相同, 均为 $q_i(x)$ 的根 $x_i^\pm = \frac{-b_i \pm i\sqrt{4c_i - b_i^2}}{2}$, \therefore 这一相似变换和 \mathcal{B}'_i 必 \exists .

$[\tau]_{\mathcal{B}_i}$ 的特征向量为 $\frac{1}{\sqrt{s_i^2 + t_i^2 + 1}} \begin{pmatrix} -x_i^- \\ 1 \end{pmatrix} = \frac{1}{\sqrt{s_i^2 + t_i^2 + 1}} (-x_i^- v_i + \tau(v_i))$, $\frac{1}{\sqrt{s_i^2 + t_i^2 + 1}} \begin{pmatrix} -x_i^+ \\ 1 \end{pmatrix} = \frac{1}{\sqrt{s_i^2 + t_i^2 + 1}} (-x_i^+ v_i + \tau(v_i))$,

即 $[\tau]_{\mathcal{B}_i}$ 的特征分解为 $[\tau]_{\mathcal{B}_i} = Q \Lambda Q^{-1}$, 其中 $Q = \frac{1}{\sqrt{s_i^2 + t_i^2 + 1}} \begin{pmatrix} -x_i^- & -x_i^+ \\ 1 & 1 \end{pmatrix}$, $\Lambda = \begin{pmatrix} x_i^+ & 0 \\ 0 & x_i^- \end{pmatrix}$.

$[\tau]_{\mathcal{B}'_i}$ 的特征向量为 $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$, $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$, 即 $[\tau]_{\mathcal{B}'_i}$ 的特征分解为 $[\tau]_{\mathcal{B}'_i} = P \Lambda P^{-1}$, 其中 $P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$.

相似变换下, $[\tau]_{\mathcal{B}'_i} = T [\tau]_{\mathcal{B}_i} T^{-1}$, 故其中 $T = P Q^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \frac{\sqrt{s_i^2 + t_i^2 + 1}}{x_i^+ - x_i^-} \begin{pmatrix} 1 & x_i^+ \\ -1 & -x_i^- \end{pmatrix} = \frac{\sqrt{s_i^2 + t_i^2 + 1}}{\sqrt{2}(x_i^+ - x_i^-)} \begin{pmatrix} 0 & x_i^+ - x_i^- \\ 2i & -ib_i \end{pmatrix} = \frac{\sqrt{c_i + 1}}{\sqrt{2}\sqrt{4c_i - b_i^2}} \begin{pmatrix} 0 & \sqrt{4c_i - b_i^2} \\ 2 & -b_i \end{pmatrix}$, $T^{-1} = Q P^{-1} = \frac{1}{\sqrt{s_i^2 + t_i^2 + 1}} \begin{pmatrix} -x_i^- & -x_i^+ \\ 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \frac{1}{\sqrt{2}(s_i^2 + t_i^2 + 1)} \begin{pmatrix} -(x_i^+ + x_i^-) & -i(x_i^+ - x_i^-) \\ 2 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}(c_i + 1)} \begin{pmatrix} b_i & \sqrt{4c_i - b_i^2} \\ 2 & 0 \end{pmatrix}$.

当然, 也可调整 T 前的系数从而得 $T = \begin{pmatrix} 0 & 1 \\ \frac{2}{\sqrt{4c_i - b_i^2}} & -\frac{b_i}{\sqrt{4c_i - b_i^2}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{t_i} & \frac{s_i}{t_i} \end{pmatrix}$, $T^{-1} = \begin{pmatrix} \frac{b_i}{2} & \frac{\sqrt{4c_i - b_i^2}}{2} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -s_i & t_i \\ 1 & 0 \end{pmatrix}$.

$[\tau]_{\mathcal{B}'_i} = M_{\mathcal{B}_i \mathcal{B}'_i} [\tau]_{\mathcal{B}} M_{\mathcal{B}' \mathcal{B}}$, 其中 $M_{\mathcal{B}_i \mathcal{B}'_i} = \begin{pmatrix} [v_i]_{\mathcal{B}'_i} & [\tau(v_i)]_{\mathcal{B}'_i} \end{pmatrix} = T$, $M_{\mathcal{B}' \mathcal{B}_i} = \begin{pmatrix} [b'_1]_{\mathcal{B}_i} & [b'_2]_{\mathcal{B}_i} \end{pmatrix} = T^{-1} \implies \mathcal{B}'_i = \{b'_1 = -s_i v + \tau(v_i), b'_2 = t_i v_i\}$.

$$\begin{array}{ccc}
 F^n & \xrightarrow{\tau'_A} & F^n \\
 \uparrow \phi'_B & & \uparrow \phi'_B \\
 V & \xrightarrow{\tau} & V \\
 \downarrow \phi_B & & \downarrow \phi_B \\
 F^n & \xrightarrow{\tau_A} & F^n
 \end{array}$$

□

□

对 $F = \mathbb{Q}$, 由于 $\mathbb{Q}[x]$ 中的不可约多项式无次数限制, 线性算子的极小多项式可分解成任意次数不可约多项式的乘积, 此时子空间没有确定的维数, 故此时没有普适的定理.

10.3 特殊的正规算子

定义 10.3 自伴随(/厄米)算子: 满足 $\tau = \tau^*$.

定义 10.4 斜伴随(/反厄米)算子: 满足 $\tau = -\tau^*$.

定义 10.5 酉(/么正)算子: 满足 $\tau^* = \tau^{-1}$.

定理 10.8 厄米算子的性质(课本第3版定理10.11): $\dim V < \infty$, $\tau, \sigma \in \mathcal{L}(V)$, 则

- (1) 若 τ, σ 厄米, 则 $\tau + \sigma$, τ^{-1} , $p(\tau)$ ($p(x) \in \mathbb{R}[x]$) 厄米.
- (2) $F = \mathbb{C}$, 则 τ 厄米 $\iff \langle \tau(v), v \rangle \in \mathbb{R}$.
- (3) τ 为复算子或实对称算子, 则 $\tau = 0 \iff \forall v \in V, \langle \tau(v), v \rangle = 0$.
- (4) τ 厄米, 则 $m_\tau(x)$ 仅有实根.

证: (1) $(\tau + \sigma)^* = \tau^* + \sigma^* = \tau + \sigma \implies \tau + \sigma$ 厄米.

$(\tau^{-1})^* = (\tau^*)^{-1} = \tau^{-1} \implies \tau^{-1}$ 厄米.

$p^*(\tau) = (\sum_i r_i \tau^i)^* = \sum_i r_i (\tau^i)^* = \sum_i r_i (\tau^*)^i = \sum_i r_i \tau^i = p(\tau) \implies p(\tau)$ 厄米.

(2) “ \iff ”: τ 厄米, 即 $\tau^* = \tau \iff \forall v, \tau(v) = \tau^*(v) \iff \langle \tau(v), v \rangle = \langle v, \tau^*(v) \rangle = \langle v, \tau(v) \rangle = \overline{\langle \tau(v), v \rangle} \implies \langle \tau(v), v \rangle \in \mathbb{R}$.

(3) “ \implies ”: 显然. 复算子的 “ \iff ” 见定理 ??, 下证实对称算子的 “ \iff ”.

实对称算子即实厄米算子. $\because F = \mathbb{R}, \because \langle u, v \rangle = \langle v, u \rangle$.

$\forall u, v \in V, 0 = \langle \tau(u+v), u+v \rangle = \langle \tau(u), u \rangle + \langle \tau(u), v \rangle + \langle \tau(v), u \rangle + \langle \tau(v), v \rangle = \langle \tau(u), v \rangle + \langle \tau(v), u \rangle = \langle u, \tau^*(v) \rangle + \langle \tau(v), u \rangle = \langle u, \tau(v) \rangle + \langle \tau(v), u \rangle = 2\langle \tau(v), u \rangle \implies \tau(v) = 0 \implies \tau = 0$.

综上, 得证.

(4) $\because \tau$ 厄米, $\therefore \tau$ 正规.

设 λ 为 τ 的特征值, 亦即 $m_\tau(x)$ 的根, 则 $\bar{\lambda}$ 为 τ^* 的特征值.

$\lambda v = \tau(v) = \tau^*(v) = \bar{\lambda}v \implies \lambda = \bar{\lambda} \implies \lambda \in \mathbb{R}$, 故 $m_\tau(x)$ 仅有实根.

□

定理 10.9 酉算子的性质(课本第3 版定理10.12): $\dim V < \infty$, $\sigma, \tau \in \mathcal{L}(V)$, 则

(1) σ, τ 酉 $\implies r\tau$ ($|r| = 1$), $\sigma \circ \tau$, τ^{-1} 酉.

(2) τ 酉 $\iff \tau$ 等距同构.

(3) τ 酉 $\iff \tau$ 将一组正交归一基变换为正交归一基.

(4) τ 酉, 则 τ 的特征值模长 = 1.

证: (1) $(r\tau)^*(r\tau) = \bar{r}\tau^*r\tau = \bar{r}r\tau^*\tau = |r|^2\tau^{-1}\tau = 1 \implies r\tau$ 酉.

$(\sigma \circ \tau)^*(\sigma \circ \tau) = \tau^*\sigma^*\sigma\tau = \tau^{-1}\sigma^{-1}\sigma\tau = 1 \implies \sigma \circ \tau$ 酉.

$(\tau^{-1})^* = (\tau^*)^{-1} = (\tau^{-1})^{-1} \implies \tau^{-1}$ 酉.

(2) “ \implies ”: \because 酉算子有逆, \therefore 必双射, 下证等距.

$\langle \tau(u), \tau(v) \rangle = \langle u, \tau^*(\tau(v)) \rangle = \langle u, \tau^{-1}(\tau(v)) \rangle = \langle u, v \rangle \implies \tau$ 等距, 故 τ 等距同构.

“ \impliedby ”: $\because \tau$ 等距同构, $\therefore \langle u, \tau^*(\tau(v)) \rangle = \langle \tau(u), \tau(v) \rangle = \langle u, v \rangle \implies \tau^*(\tau(v)) = v \implies \tau^* \circ \tau = 1 \implies \tau$ 酉.

综上, 得证.

(3) “ \implies ”: 取一组正交归一基 $\mathcal{O} = \{o_1, \dots, o_n\}$, $\langle o_i, o_j \rangle = \delta_{ij}$.

$\because \langle \tau(o_i), \tau(o_j) \rangle = \delta_{ij}$ 且 $\dim \tau(\mathcal{O}) = \dim \mathcal{O}$, $\therefore \tau(\mathcal{O})$ 为一组正交归一基.

“ \impliedby ”: $\forall u, v \in V$, $u = \sum_{i=1}^n \alpha_i o_i$, $v = \sum_{j=1}^n \beta_j o_j$.

$\because \tau(\mathcal{O})$ 为正交归一基, $\therefore \langle \tau(u), \tau(v) \rangle = \langle \tau(\sum_{i=1}^n \alpha_i o_i), \tau(\sum_{j=1}^n \beta_j o_j) \rangle = \sum_{i,j=1}^n \alpha_i \bar{\beta}_j \langle \tau(o_i), \tau(o_j) \rangle = \sum_{i,j=1}^n \alpha_i \bar{\beta}_j \delta_{ij} = \sum_{i=1}^n \alpha_i \bar{\beta}_i = \sum_{i,j=1}^n \alpha_i \bar{\beta}_j \langle o_i, o_j \rangle = \langle \sum_{i=1}^n \alpha_i o_i, \sum_{j=1}^n \beta_j o_j \rangle = \langle u, v \rangle \implies \tau$ 等距同构 $\implies \tau$ 酉.

综上, 得证.

(4) 设 λ 为 τ 的特征值, $\tau(v) = \lambda v$, $\tau^*(v) = \bar{\lambda}v$.

$v = \tau^{-1}(\tau(v)) = \tau^*(\tau(v)) = \bar{\lambda}\lambda v = |\lambda|^2 v \implies |\lambda| = 1$.

□

定理 10.10 正规算子的结构(课本第3 版定理10.18): (1) $F = \mathbb{C}$,

(a) τ 正规 $\iff \tau$ 正交归一对角化 $\iff \tau$ 有正交谱分解 $\tau = \lambda_1 \rho_1 + \dots + \lambda_k \rho_k$, 其中 λ_i 互不相等, $\rho_1 + \dots + \rho_k = 1$ 为单位分解, 对 $i \neq j$, $\text{Im } \rho_i \perp \text{Im } \rho_j$, $\text{Im } \rho_i \perp \ker \rho_i$.

(b) 特征值为实数的正规算子厄米.

(c) 特征值的模长 = 1 的正规算子酉.

(2) $F = \mathbb{R}$,

(a) τ 正规 $\iff V = \mathcal{E}_{\lambda_1} \oplus \dots \oplus \mathcal{E}_{\lambda_k} \oplus D_1 \oplus \dots \oplus D_l$, 其中 D_i 为二维不可约的 τ 不变子空间, D_i 上 τ

的矩阵表示为 $\begin{pmatrix} s_i & t_i \\ -t_i & s_i \end{pmatrix}$.

(b) 若上述正交直和式中无 D_i , 则 τ 厄米.

(c) 若在 D_i 上的 τ 的矩阵表示为 $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$, 则 τ 酉, 称为 **正交算子**.

定义 10.6 正交算子: $F = \mathbb{R}$ 的酉算子.

10.4 (半)正定算子

定义 10.7 (半)正定算子: $F = \mathbb{R}$, $\dim V < \infty$, $\tau \in \mathcal{L}(V)$ 厄米, 若 $\forall v \in V$, $\langle \tau(v), v \rangle > (\geq) 0$, 则 τ (半)正定.

定理 10.11 (课本第3版定理10.22): $F = \mathbb{R}$, $\dim V < \infty$, $\tau \in \mathcal{L}(V)$ 厄米, 则

(1) τ 半正定 $\iff \tau$ 的特征值 ≥ 0 .

(2) τ 正定 $\iff \tau$ 的特征值 > 0 .

证: “ \implies ”: 设 λ 为 τ 的特征值.

$\because \tau$ (半)正定, $\therefore \langle \tau(v), v \rangle > (\geq) 0$.

又 $\because \langle \tau(v), v \rangle = \langle \lambda v, v \rangle = \lambda \langle v, v \rangle \geq 0$, $\langle v, v \rangle > 0$, $\therefore \lambda > (\geq) 0$, $\therefore \lambda > (\geq) 0$.

“ \impliedby ”: $\because \rho$ 厄米, $\therefore \tau$ 的正交谱分解 $\tau = \lambda_1 \rho_1 + \cdots + \lambda_k \rho_k$.

对 τ 的函数操作均等效于作用于其谱分解的特征值上: $\tau^2 = \sum_{ij} \lambda_i \lambda_j \rho_i \rho_j = \sum_{ij} \lambda_i \lambda_j \delta_{ij} \rho_i = \sum_i \lambda_i^2 \rho_i$.

类似地, $\tau^k = \sum_i \lambda_i^k \rho_i$.

$r \tau^k = r \sum_i \lambda_i^k \rho_i = \sum_i r \lambda_i^k \rho_i \implies$ 对 $f(x) \in F[x]$, $f(\tau) = \sum_i f(\lambda_i) \rho_i$,

\forall 可由多项式近似的 $g(x)$, $g(\tau) = \sum_i g(\lambda_i) \rho_i$.

$\because \tau$ (半)正定, \therefore 必可定义其平方根 $\sqrt{\tau} = \sqrt{\lambda_1} \rho_1 + \cdots + \sqrt{\lambda_k} \rho_k$, 此处 $\lambda_i > (\geq) 0$, 否则 $\sqrt{\tau}$ 不一定合法. □

定理 10.12 (课本第3版定理10.23): τ 厄米, 则

(1) τ 半正定 $\iff \tau$ 有平方根.

(2) τ 半正定 $\iff \tau = \sigma^* \circ \sigma$, 其中 $\sigma \in \mathcal{L}(V)$ (注意这里的 σ 不唯一).

证: (1) τ 半正定, 即 $\tau = \sum_i \lambda_i \rho_i$, 其中 $\lambda_i \geq 0 \iff \sqrt{\tau} = \sum_i r_i \rho_i$, 其中 $r_i = \sqrt{\lambda_i}$.

(2) “ \implies ”: 取 $\sigma = \sqrt{\tau}$ 即得证.

“ \impliedby ”: $\langle \tau(v), v \rangle = \langle \sigma^* \circ \sigma(v), v \rangle = \langle \sigma(v), \sigma(v) \rangle = \|\sigma(v)\|^2 \geq 0 \implies \tau$ 半正定. □

定理 10.13 半正定算子的复合半正定的条件(课本第3 版定理10.24): $\sigma, \tau \in \mathcal{L}(V)$ 半正定, 若 $\sigma \circ \tau = \tau \circ \sigma$, 则 $\sigma \circ \tau$ 半正定.

证: $\because \sigma\tau = \tau\sigma, \therefore \sqrt{\sigma}\sqrt{\tau} = \sqrt{\tau}\sqrt{\sigma} \implies \sigma\tau = \sqrt{\sigma}\sqrt{\sigma}\sqrt{\tau}\sqrt{\tau} = (\sqrt{\sigma}\sqrt{\tau})(\sqrt{\sigma}\sqrt{\tau}),$ 故 $\sigma\tau$ 半正定. □

10.5 算子的极分解

定理 10.14 算子的极分解(课本第3 版定理10.25): $F = \mathbb{C}$, 有限维内积向量空间 V , $\tau \in \mathcal{L}(V)$, 则 $\exists!$ 半正定算子 ρ 及酉算子 ν , s.t. $\tau = \nu \circ \rho$, 且若 τ 可逆, 则 ν 唯一.

证: 取 $\rho = \sqrt{\tau^* \tau}$, 则 $\|\rho(v)\|^2 = \langle \rho(v), \rho(v) \rangle = \langle v, \rho^* \rho(v) \rangle = \langle v, \rho^2(v) \rangle = \langle v, \tau^* \tau(v) \rangle = \langle \tau(v), \tau(v) \rangle = \|\tau(v)\|^2.$

取 $\nu: \text{Im } \rho \rightarrow \text{Im } \tau, \rho(v) \mapsto \tau(v).$

先证 ν 为映射: 若 $\rho(v) = \rho(u)$, 则 $\rho(u-v) = 0 \implies \|\rho(u-v)\|^2 = 0 \implies \|\tau(u-v)\|^2 = \langle \tau(u-v), \tau(u-v) \rangle = 0 \implies \tau(u-v) = 0 \implies \tau(v) = \tau(u)$, 故 ν 为映射.

$\because \|\nu(\rho(v))\| = \|\tau(v)\| = \|\rho(v)\|, \therefore \nu$ 等距同构 $\implies \nu$ 酉.

当 τ 不可逆时, 拓展 $\text{Im } \rho$ 为 $\text{Im } \tau$ 的方式不唯一, 故 ν 不唯一. □

若 σ 酉, 则其特征值模 $|\lambda| = 1$, 可记为 $\lambda_i = e^{i\theta_i}$, 其中 $\theta_i \in \mathbb{R}$,

则 $\sigma = \lambda_1 \rho_1 + \cdots + \lambda_k \rho_k = e^{i\theta_1} \rho_1 + \cdots + e^{i\theta_k} \rho_k$, 其中 $\rho_1 + \cdots + \rho_k = 1$ 为正交单位分解.

令 $H = \theta_1 \rho_1 + \cdots + \theta_k \rho_k$, 则

$$(1) H^* = H.$$

$$(2) \sigma = e^{iH}.$$

此处 H 类似量子力学中的哈密顿量, σ 类似演化算子.