3.1

From $\det(\sigma_y - \lambda I) = \lambda^2 - 1 = 0$ we obtain $\lambda_\pm = \pm 1$, and from $\sigma_y \mathbf{x}_\pm = \lambda \pm \mathbf{x}_\pm$ we obtain

$$\mathbf{x}_\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

The probability of measuring result $s_y = \hbar/2$ for state $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is

$$\begin{aligned} P = & \left| \mathbf{x}_+^\dagger \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right|^2 \\ = & \frac{1}{2} |\alpha - i\beta|^2 = \frac{1}{2} (\alpha - i\beta)(\alpha^\dagger + i\beta^\dagger) \\ = & \frac{1}{2} (|\alpha|^2 + |\beta|^2) + \mathrm{Im}(\alpha^\dagger \beta) = \frac{1}{2} + \mathrm{Im}(\alpha^\dagger \beta) \end{aligned}$$

3.2

$$H = -\frac{2\mu}{\hbar} \mathbf{S} \cdot \mathbf{B} = -\mu \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$$

Eigenvalues are obtained from equation

$$\det \begin{pmatrix} B_z - \lambda & B_x - iB_y \\ B_x + iB_y & -B_z - \lambda \end{pmatrix} = \lambda^2 - B_z^2 - (B_x^2 + B_y^2) = \lambda^2 - B^2$$

The solution is $\lambda_\pm = \pm B$ and corresponding energy eigenvalues are $E_\pm = \pm \mu B$, where we have interchanged the subscript $\pm$ in accordance with positive and negative energies.

3.5

$$S_z(S_z + \hbar)(S_z - \hbar) = \hbar^3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = 0$$

$$\begin{aligned} S_x(S_x + \hbar)(S_x - \hbar) = & \frac{\hbar^3}{8} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 2 & \sqrt{2} & 0 \\ \sqrt{2} & 2 & \sqrt{2} \\ 0 & \sqrt{2} & 2 \end{pmatrix} \begin{pmatrix} -2 & \sqrt{2} & 0 \\ \sqrt{2} & -2 & \sqrt{2} \\ 0 & \sqrt{2} & -2 \end{pmatrix} \\ = & \frac{\hbar^3}{8} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{pmatrix} = 0 \end{aligned}$$

In fact for a spin 1 particle S_z and S_x both can only take eigenvalues $0, \pm\hbar$, so $S_z(S_z + \hbar)(S_z - \hbar)$ acting on eigenkets for S_z will always yield 0, and similarly for $S_x(S_x + \hbar)(S_x - \hbar)$.

3.9

A finite rotation of angle θ around axis (n_x, n_y, n_z) is

$$\exp\left(\frac{-i\sigma\cdot\hat{\mathbf{n}}\theta}{2}\right) = \begin{pmatrix} \cos\frac{\theta}{2} - in_z\sin\frac{\theta}{2} & (-in_x - n_y)\sin\frac{\theta}{2} \\ (-in_x + n_y)\sin\frac{\theta}{2} & \cos\frac{\theta}{2} + in_z\sin\frac{\theta}{2} \end{pmatrix}$$

Comparing matrix element we see that

$$\begin{aligned} \cos\frac{\theta}{2} &= \cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2} \\ n_z\sin\frac{\theta}{2} &= \sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2} \\ n_x\sin\frac{\theta}{2} &= -\sin\frac{\alpha-\gamma}{2}\sin\frac{\beta}{2} \\ n_y\sin\frac{\theta}{2} &= \cos\frac{\alpha-\gamma}{2}\sin\frac{\beta}{2} \end{aligned}$$

Thus $\theta = 2\mathrm{acos}\left(\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\right)$.

3.14

$$\begin{aligned} [G_i, G_j]_{mn} &= (G_iG_j - G_jG_i)_{mn} = \sum_l (G_i)_{ml}(G_j)_{ln} - (G_j)_{ml}(G_i)_{ln} \\ &= -\hbar^2\sum_l \varepsilon_{iml}\varepsilon_{jln} - \varepsilon_{jml}\varepsilon_{iln} = -\hbar^2(\delta_{in}\delta_{jm} - \delta_{ij}\delta_{nm} - \delta_{jn}\delta_{im} + \delta_{ji}\delta_{nm}) \\ &= \hbar^2(\delta_{jn}\delta_{im} - \delta_{in}\delta_{jm}) = i\hbar\varepsilon_{ijk}(-i\hbar\varepsilon_{kmn}) = i\hbar\varepsilon_{ijk}(G_k)_{mn} \end{aligned}$$

Thus G_i satisfy the angular-momentum commutation relations. The transformation matrix that satisfy $U^\dagger G_i U = J_i$ is found by considering diagonalization of G_3, which reads

$$G_3 = -i\hbar\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Solving the eigenvalue equation we find eigenvalues and corresponding eigenvectors are

$$\lambda_1 = \hbar \qquad \mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$$

$$\lambda_2 = 0 \qquad \mathbf{u}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_3 = -\hbar \qquad \mathbf{u}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ i \\ 0 \end{pmatrix}$$

So that

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -1 \\ i & 0 & i \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

and

$$J_3 = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\{G_i\}$ and $\{J_i\}$ satisfy the same Lie algebra and thus are two representation of the same rotation group. Thus the transformation from $\{G_i\}$ to $\{J_i\}$ corresponds to a rotation in the group space, which is a compound of infinitesimal rotations

$$\mathbf{G} \to \mathbf{G} + \hat{\mathbf{n}}\delta\phi \times \mathbf{G}$$

3.15

(a)

$$\begin{aligned} \mathbf{J}^2 &= J_z^2 + J_x^2 + J_y^2 = J_z^2 + J_x^2 + J_y^2 + iJ_yJ_x - iJ_xJ_y + i[J_x, J_y] \\ &= J_z^2 + (J_x + iJ_y)(J_x - iJ_y) - \hbar J_z = J_z^2 + J_+J_- - \hbar J_z \end{aligned}$$

(b)

$$\begin{aligned} \langle j, m|J_-^\dagger J_-|j, m\rangle &= \langle j, m|\mathbf{J}^2 - J_z^2 + \hbar J_z|j, m\rangle = \hbar^2\langle j, m|j(j+1) - m^2 + m|j, m\rangle \\ &= \hbar^2[j(j+1) - m^2 + m] \end{aligned}$$

Since $J_-|j, m\rangle = c_-|j, m-1\rangle$, we have $|c_-|^2 = \hbar^2[j(j+1) - m^2 + m]$, and it can be chosen so that

$$c_- = \sqrt{j(j+1) - m^2 + m}\hbar = \sqrt{(j+m)(j-m+1)}\hbar$$