7.7

对于算符N的本征态$N|n\rangle = n|n\rangle$, 有

$$
\begin{aligned}
Na|n\rangle = a^\dagger aa|n\rangle = a|n\rangle - aa^\dagger a|n\rangle = (1-n)a|n\rangle \\
Na^\dagger|n\rangle = a^\dagger aa^\dagger|n\rangle = a^\dagger|n\rangle - a^\dagger a^\dagger a|n\rangle = (1-n)a^\dagger|n\rangle
\end{aligned}
$$

又由

$$
\langle n|N|n\rangle = \langle n|a^\dagger a|n\rangle = n
$$

推出$a|n\rangle = \sqrt{n}|1-n\rangle$，由

$$
\langle n|N^2|n\rangle = n\langle 1-n|aa^\dagger|1-n\rangle = n^2
$$

推出$a^\dagger|n\rangle = \sqrt{1-n}|1-n\rangle$。现选择一个满足$a|\psi\rangle = 0$的态，记为$|0\rangle$。将$a^\dagger$作用于其上，得到$|1\rangle$，从上述论证可见这两个态是$N$本征值为0与1的本征态。将$a^\dagger$再作用于$|1\rangle$上得到0，因而$N$的本征值仅有0与1。

补充2

对于T，假设其本征态为$|t_i\rangle$，则其可写作

$$
T = \sum_i t_i N_i = \sum_i t_i a_i^\dagger a_i
$$

若二次量子化采用的场算符$b_i^\dagger$对应的态为$|l_i\rangle$，则$a_i^\dagger$可以写作

$$
a_i^\dagger = \sum_j b_j^\dagger \langle l_j|t_i\rangle
$$

同样

$$
a_i = \sum_j \langle t_i|l_j\rangle b_j
$$

因而

$$
\begin{aligned}
T &= \sum_i t_i \sum_{j,k} b_j^\dagger \langle l_j|t_i\rangle\langle t_i|l_k\rangle b_k = \sum_{j,k} b_j^\dagger b_k \sum_i t_i \langle l_j|t_i\rangle\langle t_i|l_k\rangle \\
&= \sum_{j,k} b_j^\dagger b_k \langle l_j| \left[T \sum_i |t_i\rangle\langle t_i|\right] |l_k\rangle = \sum_{j,k} b_j^\dagger b_k \langle l_j|T|l_k\rangle
\end{aligned}
$$

对于V，假若与其描述的相互作用相关的本征态为$|k_i\rangle$，即两个粒子之间的相互作用完全由它们的量子数k^1、k^2决定，那么有

$$
V = \frac{1}{2}\sum_{i\neq j} v_{ij} N_i N_j + \frac{1}{2}\sum_i v_{ii} N_i(N_i - 1) = \frac{1}{2}\sum_{i,j} v_{ij}(N_i N_j - \delta_{ij} N_i)
$$

这里$N_i = a_i^\dagger a_i$是与态$|k_i\rangle$相关的粒子数算符。而

$$
N_i N_j - \delta_{ij} N_i = a_i^\dagger a_i a_j^\dagger a_j - a_i^\dagger \delta_{ij} a_i = a_i^\dagger(\delta_{ij} \pm a_j^\dagger a_i)a_j - a_i^\dagger \delta_{ij} a_i = a_i^\dagger a_j^\dagger a_j a_i
$$

因而

$$V=\frac{1}{2}\sum_{i,j}v_{ij}a_i^\dagger a_j^\dagger a_j a_i$$

仿照上述单粒子算符，将该算符变换至$b_i^\dagger$、$|l_i\rangle$对应的基展开，得到

$$V=\frac{1}{2}\sum_{i,j,k,m}\langle ij|V|km\rangle b_i^\dagger b_j^\dagger b_m b_k$$

因而在任意正交完备基$|k_i\rangle$中，H可按照这组基对应的产生、消灭算符$a^\dagger$与a展开为

$$H=\sum_{ij}a_i^\dagger a_j\langle i|T|j\rangle+\frac{1}{2}\sum_{mnpq}\langle mn|V|pq\rangle a_m^\dagger a_n^\dagger a_q a_p$$

其中$a^\dagger$，a满足

$$[a_i,a_j]=[a_i^\dagger,a_j^\dagger]=0\quad[a_i,a_j^\dagger]=\delta_{ij}$$

（对于玻色子）或

$$\{a_i,a_j\}=\{a_i^\dagger,a_j^\dagger\}=0\quad\{a_i,a_j^\dagger\}=\delta_{ij}$$

（对于费米子）。

补充3

由于没有相互作用，哈密顿量为

$$H=\sum_{ijmn}a_{im}^\dagger a_{jn}\langle p_i m|\left[\frac{p^2}{2m}+V\right]|p_j n\rangle$$

这里$m,n=1,2$表示自旋。其中

$$\begin{aligned}\langle p_i m|\left[\frac{p^2}{2m}+V\right]|p_j n\rangle&=\frac{p_i^2}{2m}\delta_{ij}\delta_{mn}+\frac{1}{L}\int\mathrm{d}xV_0e^{-ip_ix}\delta(x)e^{ip_jx}\delta_{mn}\\&=\frac{p_i^2}{2m}\delta_{ij}\delta_{mn}+\frac{V_0}{L}\delta_{mn}\end{aligned}$$

由于粒子是费米子，有

$$\{a_i,a_j\}=\{a_i^\dagger,a_j^\dagger\}=0\quad\{a_i,a_j^\dagger\}=\delta_{ij}$$

若将外场视作微扰，则基态即为能量最低的N个平面波模式占有数为1、其余模式占有数为0的态，费米能满足

$$\frac{N}{2}=\frac{L}{2\pi\hbar}\int_{|p|\le p_F}\mathrm{d}p=\frac{p_FL}{\pi\hbar}$$

得$p_F=\frac{\pi N\hbar}{2L}$。零阶能量为

$$E^{(0)}=2\int_{-p_F}^{p_F}\mathrm{d}p\rho(p)\frac{p^2}{2m}=\frac{L}{2\pi\hbar}\frac{2p_F^3}{3m}=\frac{\pi^2N^3\hbar^2}{24L^2m}$$

一阶微扰能量为

$$E^{(1)}=\sum_{ijmn}\langle F|a_{im}^\dagger a_{jn}\frac{V_0}{L}\delta_{mn}|F\rangle=2\frac{V_0}{L}\sum_{ij}\langle F|a_i^\dagger a_j|F\rangle$$

显然j必须对应已经占据的态，且$i=j$，才能使此项不为0。因而有

$$E^{(1)} = 2\frac{V_0}{L}\sum_i \langle F|a_i^\dagger a_i|F\rangle = 2\frac{V_0}{L}\frac{N}{2}$$

因而总能量为

$$E^{(0)} + E^{(1)} = \frac{\pi^2 N^3 \hbar^2}{24L^2 m} + \frac{NV_0}{L}$$

8.10

狄拉克方程及其共轭转置为

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi &= (-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\Psi \\ -i\frac{\partial}{\partial t}\Psi^\dagger &= \Psi^\dagger(i\vec{\alpha}\cdot\overleftarrow{\nabla} + \beta m) \end{aligned}$$

则

$$\begin{aligned} i\frac{\partial\rho}{\partial t} &= i\Psi^\dagger\frac{\partial}{\partial t}\Psi + i\frac{\partial}{\partial t}\Psi^\dagger\Psi \\ &= \Psi^\dagger(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\Psi - \Psi^\dagger(i\vec{\alpha}\cdot\overleftarrow{\nabla} + \beta m)\Psi \\ &= -i\Psi^\dagger(\vec{\alpha}\cdot\overleftarrow{\nabla} + \vec{\alpha}\cdot\vec{\nabla})\Psi \\ &= -i\vec{\nabla}\cdot\Psi^\dagger\vec{\alpha}\Psi \end{aligned}$$

8.11

考虑第一、第三分量，令

$$\begin{pmatrix} m & p \\ p & -m \end{pmatrix}\begin{pmatrix} u_1 \\ u_3 \end{pmatrix} = E\begin{pmatrix} u_1 \\ u_3 \end{pmatrix}$$

得到本征值与本征态

$$\begin{aligned} E_+ &= \sqrt{m^2+p^2} \qquad & u_+ &= \begin{pmatrix} 1 \\ \frac{p}{m+\sqrt{m^2+p^2}} \end{pmatrix} \\ E_- &= -\sqrt{m^2+p^2} \qquad & u_- &= \begin{pmatrix} \frac{-p}{m+\sqrt{m^2+p^2}} \\ 1 \end{pmatrix} \end{aligned}$$

对于第二、第四分量，在上式中令$p \to -p$得到

$$\begin{aligned} E_+ &= \sqrt{m^2+p^2} \qquad & u_+ &= \begin{pmatrix} 1 \\ \frac{-p}{m+\sqrt{m^2+p^2}} \end{pmatrix} \\ E_- &= -\sqrt{m^2+p^2} \qquad & u_- &= \begin{pmatrix} \frac{p}{m+\sqrt{m^2+p^2}} \\ 1 \end{pmatrix} \end{aligned}$$

因而共有四个独立的本征态，分别为

$$E_+ = \sqrt{m^2+p^2} \qquad u_+^R = \begin{pmatrix} 1 \\ 0 \\ \frac{p}{m+\sqrt{m^2+p^2}} \\ 0 \end{pmatrix} \qquad u_+^L = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-p}{m+\sqrt{m^2+p^2}} \end{pmatrix}$$

$$E_- = -\sqrt{m^2+p^2} \qquad u_-^R = \begin{pmatrix} \frac{-p}{m+\sqrt{m^2+p^2}} \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad u_-^L = \begin{pmatrix} 0 \\ \frac{p}{m+\sqrt{m^2+p^2}} \\ 0 \\ 1 \end{pmatrix}$$

补充2

由$(E - c\vec{\alpha}\cdot\vec{p} - \beta mc^2 - V)\Psi = 0$，$\Psi = \begin{pmatrix}\chi \\ \phi\end{pmatrix}$，知

$$\phi = \frac{c\vec{\sigma}\cdot\vec{p}}{E - V + mc^2}\chi$$

因而

$$|\phi| \sim \frac{c|p|}{\left(V + \frac{p^2}{2m} + mc^2\right) - V + mc^2}|\chi|$$

对于核电荷数$Z \ll 200$的类氢原子，有

$$\frac{|\phi|}{|\chi|} \sim \frac{\sqrt{Z^2 \times 13.6\text{eV} \times 2 \times 0.511\text{MeV}}}{Z^2 \times 13.6\text{eV} + 2 \times 0.511\text{MeV}} \sim 0.0036Z$$

补充3

由于$2M_iM_i = 2I$，即$M_i^2 = I$，有$(\det M_i)^2 = \det M_i^2 = 1$，因而$M_i$的所有特征值平方之积$\Pi_j \lambda_{j;i}^2 = 1$。又由于$M_i$为厄米矩阵，其本征值均为实数，因而$\lambda_{j;i} = \pm 1$。由于

$$0 = \text{tr}(M_i\{M_i, M_j\}) = \text{tr}(M_iM_iM_j + M_iM_jM_i) = \text{tr}(2IM_i) = 2\text{tr}M_i$$

有$\text{tr}M_i = 0$。由于$\text{tr}M_i = \sum_j \lambda_{j;i}$，考虑到$\lambda_{j;i} = \pm 1$，$M_i$的特征值中必须有等量的1与$-1$。因而$M_i$的维数为偶数。