

第 1 题 (课本习题 2.24) 得分: _____. 考虑一维粒子被一个 δ 函数势

$$V(x) = -\nu_0 \delta(x), \quad (\nu_0 \text{ 为正实数})$$

束缚于一个固定的中心位置处, 求波函数和基态束缚能. 有激发的束缚态吗?

解: 该粒子的哈密顿量为

$$H = \frac{p^2}{2m} + V(x) = \begin{cases} \frac{p^2}{2m}, & x \neq 0, \\ \frac{p^2}{2m} - \nu_0 \delta(x), & x = 0. \end{cases} \quad (1)$$

在 $x \neq 0$ 处的薛定谔方程为

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = E \psi. \quad (2)$$

考虑到束缚态波函数在无穷远处的函数值为 0, $x < 0$ 处和 $x > 0$ 处波函数的通解分别为

$$\psi(x < 0) = A e^{kx}, \quad (3)$$

$$\psi(x > 0) = B e^{-kx}, \quad (4)$$

其中 $k = \frac{\sqrt{2mE}}{\hbar}$. 利用连续性条件

$$\psi(x = 0^-) = A = \psi(x = 0^+) = B, \quad (5)$$

$$\psi'(x = 0^+) - \psi'(x = 0^-) = -Ak - Bk = \int_{0^-}^{0^+} \frac{\partial^2 \psi}{\partial x'^2} dx' = \frac{2m}{\hbar^2} \int_{0^-}^{0^+} [-\nu_0 \delta(x) - E] \psi(0) dx' = -\frac{2m\nu_0}{\hbar^2} A, \quad (6)$$

解得

$$A = B, \quad k = \frac{m\nu_0}{\hbar^2}. \quad (7)$$

考虑到波函数满足归一化条件, 得波函数

$$\psi(x) = \frac{\sqrt{m\nu_0}}{\hbar} e^{-m\nu_0|x|/\hbar^2}, \quad (8)$$

基态束缚能为

$$E = \frac{\hbar^2 k^2}{2m} = \frac{m\nu_0^2}{2\hbar^2}. \quad (9)$$

该系统仅有一个束缚基态, 没有激发的束缚态. □

第 2 题 (课本习题 2.26) 得分: _____. 一个一维粒子 ($-\infty < x < \infty$) 受到一个可从

$$V = \lambda x, \quad (\lambda > 0)$$

导出的恒力的作用.

(a) 其能谱是连续的还是分立的? 写出由 E 所确定的能量本征函数的近似表达式. 然后粗略地画出其示意图.

(b) 简略地讨论, 如果用

$$V = \lambda |x|.$$

代替 V , 什么地方需要改动?

解: (a) 在 $x \rightarrow -\infty$ 处必有 $E > V$, 粒子非束缚, 故其能谱必为连续的.

能量本征函数的近似表达式为

$$\psi(x) \sim \begin{cases} \frac{1}{[E-\lambda x]^{1/4}} e^{\pm \frac{i}{\hbar} \int_{-\infty}^x \sqrt{2m(E-\lambda x')} dx'}, & \text{for } x < \frac{E}{\lambda}, \\ \frac{1}{[\lambda x-E]^{1/4}} e^{-\frac{1}{\hbar} \int_{E/\lambda}^x \sqrt{2m(\lambda x'-E)} dx'}, & \text{for } x > \frac{E}{\lambda}. \end{cases} \quad (10)$$

如图 1 所示.

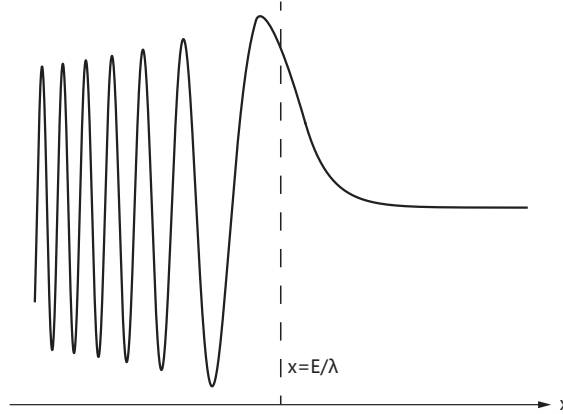


图 1: 能量本征函数的近似图像. 注意在 $x < E/\lambda$ 处波函数为波动形式, 且随着 $x \rightarrow -\infty$, 波动频率加快, 波动幅度变小; 在 $x > E/\lambda$ 处波函数为衰减形式.

(2) 若 $V = \lambda|x|$, 则粒子处于束缚态, 能谱为一系列分离的本征能量构成. 能量本征函数的近似表达式为

$$\psi(x) \sim \begin{cases} \frac{1}{[-\lambda x-E]^{1/4}} e^{\frac{1}{\hbar} \int_{-\infty}^x \sqrt{2m(-\lambda x'-E)} dx'}, & \text{for } x < -\frac{E}{\lambda}, \\ \frac{1}{[E-\lambda|x|]^{1/4}} \cos \left[\frac{1}{\hbar} \int_{-E/\lambda}^x \sqrt{2m(E-\lambda|x'|)} dx' \right], & \text{for } -\frac{E}{\lambda} < x < \frac{E}{\lambda}, \\ \frac{1}{[\lambda x-E]^{1/4}} e^{-\frac{1}{\hbar} \int_{E/\lambda}^x \sqrt{2m(\lambda x'-E)} dx'}, & \text{for } x > \frac{E}{\lambda}, \end{cases} \quad (11)$$

或

$$\psi(x) \sim \begin{cases} \frac{1}{[-\lambda x-E]^{1/4}} e^{\frac{1}{\hbar} \int_{-\infty}^x \sqrt{2m(-\lambda x'-E)} dx'}, & \text{for } x < -\frac{E}{\lambda}, \\ \frac{1}{[E-\lambda|x|]^{1/4}} \sin \left[\frac{1}{\hbar} \int_{-E/\lambda}^x \sqrt{2m(E-\lambda|x'|)} dx' \right], & \text{for } -\frac{E}{\lambda} < x < \frac{E}{\lambda}, \\ \frac{1}{[\lambda x-E]^{1/4}} e^{-\frac{1}{\hbar} \int_{E/\lambda}^x \sqrt{2m(\lambda x'-E)} dx'}, & \text{for } x > \frac{E}{\lambda}, \end{cases} \quad (12)$$

这些本征函数均为偶函数或奇函数. 能量本征值满足

$$\int_{-E/\lambda}^{E/\lambda} \sqrt{2m(E-\lambda|x'|)} dx' = (n + \frac{1}{2})\pi\hbar, \quad n = 0, 1, 2, 3, \dots \quad (13)$$

解得能量本征值为

$$E_n = \frac{[3(n + \frac{1}{4})\pi\hbar\lambda]^{2/3}}{2m^{1/3}}, \quad n = 0, 1, 2, 3, \dots \quad (14)$$

□

第 3 题 (课本习题 2.31) 得分: _____. 导出 (2.6.16) 式, 并求得 (2.6.16) 式的三维推广.

证: 一维自由粒子的哈密顿量为

$$H = \frac{p^2}{2m}, \quad (15)$$

显然其与动量算符 p 对易; $\{|p'\rangle\}$ 为哈密顿量 H 和动量算符 p 的共同本征态:

$$p|p'\rangle = p'|p'\rangle, \quad H|p'\rangle = \left(\frac{p'^2}{2m}\right)|p'\rangle. \quad (16)$$

由传播子的定义, 一维自由粒子的传播子为

$$\begin{aligned} K(x'', t; x', t_0) &= \int_{-\infty}^{\infty} dp' \langle x''|p'\rangle \langle p'|x'\rangle \exp\left[\frac{-iE_{p'}(t-t_0)}{\hbar}\right] \\ &\quad (\text{利用 } \langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ip'x'}{\hbar}\right)) \\ &= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp' \exp\left[\frac{ip'(x''-x')}{\hbar} - \frac{ip'^2(t-t_0)}{2m\hbar}\right] \\ &= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp' \exp\left\{-\frac{i(t-t_0)}{2m\hbar} \left[p' - \frac{m(x''-x')}{t-t_0}\right]^2 + \frac{im(x''-x')^2}{2\hbar(t-t_0)}\right\}, \\ &\quad (\text{令 } \xi' = \sqrt{\frac{i(t-t_0)}{2m\hbar}} \left[p' - \frac{m(x''-x')}{t-t_0}\right]) \\ &= \frac{1}{2\pi\hbar} \sqrt{\frac{2m\hbar}{i(t-t_0)}} \int_{-\infty}^{\infty} d\xi \exp(-\xi^2) \exp\left[\frac{im(x''-x')^2}{2\hbar(t-t_0)}\right], \\ &\quad (\text{利用高斯积分 } \int_{-\infty}^{\infty} e^{-\xi^2} d\xi = \sqrt{\pi}) \\ &= \sqrt{\frac{m}{2\pi i\hbar(t-t_0)}} \exp\left[\frac{im(x''-x')^2}{2\hbar(t-t_0)}\right], \end{aligned} \quad (17)$$

此即 (2.6.16) 式.

下面将上式推广至三维情况. 三维自由粒子的哈密顿量为

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m}, \quad (18)$$

显然与动量算符 p_x, p_y, p_z 对易; $\{|p'_x, p'_y, p'_z\rangle\}$ 为哈密顿量 H 和动量算符 p_x, p_y, p_z 的共同本征态:

$$p_x|p'_x, p'_y, p'_z\rangle = p'_x|p'_x, p'_y, p'_z\rangle, \quad p_y|p'_x, p'_y, p'_z\rangle = p'_y|p'_x, p'_y, p'_z\rangle, \quad p_z|p'_x, p'_y, p'_z\rangle = p'_z|p'_x, p'_y, p'_z\rangle, \quad (19)$$

$$H|p'_x, p'_y, p'_z\rangle = \frac{p_x'^2 + p_y'^2 + p_z'^2}{2m}|p'_x, p'_y, p'_z\rangle. \quad (20)$$

由传播子的定义, 三维自由粒子的传播子为

$$\begin{aligned} K(\mathbf{x}'', t; \mathbf{x}', t_0) &= \int_{-\infty}^{\infty} dp'_x \int_{-\infty}^{\infty} dp'_y \int_{-\infty}^{\infty} dp'_z \langle \mathbf{x}''|p'_x, p'_y, p'_z\rangle \langle p'_x, p'_y, p'_z|\mathbf{x}'\rangle \exp\left[-\frac{iE_{p'_x, p'_y, p'_z}(t-t_0)}{\hbar}\right] \\ &\quad (\text{利用 } \langle \mathbf{x}'|\mathbf{p}'\rangle = \frac{1}{(2\pi\hbar)^{3/2}} \exp\left(\frac{i\mathbf{p}' \cdot \mathbf{x}'}{\hbar}\right)) \\ &= \frac{1}{(2\pi\hbar)^3} \int_{-\infty}^{\infty} dp'_x \int_{-\infty}^{\infty} dp'_y \int_{-\infty}^{\infty} dp'_z \times \\ &\quad \exp\left\{\frac{i[p'_x(x''-x') + p'_y(y''-y') + p'_z(z''-z')]}{\hbar} - \frac{i(p_x'^2 + p_y'^2 + p_z'^2)(t-t_0)}{2m\hbar}\right\} \\ &= \frac{1}{(2\pi\hbar)^3} \int_{-\infty}^{\infty} dp'_x \exp\left\{-\frac{i(t-t_0)}{2m\hbar} \left[p'_x - \frac{m(x''-x')}{t-t_0}\right]^2 + \frac{im(x''-x')^2}{2\hbar(t-t_0)}\right\} \times \end{aligned}$$

$$\begin{aligned}
& \int_{-\infty}^{\infty} dp'_y \exp \left\{ -\frac{i(t-t_0)}{2m\hbar} [p'_y - \frac{m(y''-y')}{t-t_0}]^2 + \frac{im(y''-y')}{2\hbar(t-t_0)} \right\} \times \\
& \int_{-\infty}^{\infty} dp'_z \exp \left\{ -\frac{i(t-t_0)}{2m\hbar} [p'_z - \frac{m(z''-z')}{t-t_0}]^2 + \frac{im(z''-z')}{2\hbar(t-t_0)} \right\} \\
& = \left(\frac{m}{2\pi i\hbar(t-t_0)} \right)^{3/2} \exp \left\{ \frac{im[(x''-x')^2 + (y''-y')^2 + (z''-z')^2]}{2\hbar(t-t_0)} \right\}.
\end{aligned} \tag{21}$$

□

第 4 题 (补充题) 得分: _____. 求一维自由粒子高斯波包坐标与动量测不准关系随时间的变化.

解: $t = 0$ 时刻一维自由粒子高斯波包的波函数为

$$\psi(x', 0) = e^{ip_0 x' / \hbar} \frac{\exp\left(-\frac{x'^2}{2d_0^2}\right)}{(\pi d_0^2)^{1/4}}. \tag{22}$$

一维自由粒子的传播子为

$$K(x'', t; x', t_0) = \sqrt{\frac{m}{2\pi i\hbar(t-t_0)}} \exp \left[\frac{im(x''-x')^2}{2\hbar(t-t_0)} \right]. \tag{23}$$

t 时刻该波包的波函数演化为

$$\begin{aligned}
\psi(x, t) &= \int K(x, t; x', 0) \psi(x', 0) dx' \\
&= \int_{-\infty}^{\infty} \sqrt{\frac{m}{2\pi i\hbar t}} \exp \left[\frac{im(x-x')^2}{2\hbar t} \right] e^{ip_0 x' / \hbar} \frac{\exp\left(-\frac{x'^2}{2d_0^2}\right)}{(\pi d_0^2)^{1/4}} dx' \\
&= \sqrt{\frac{m}{2\pi i\hbar t}} \frac{1}{(\pi d_0^2)^{1/4}} \int_{-\infty}^{\infty} \exp \left\{ \left(\frac{im}{2\hbar t} - \frac{1}{2d_0^2} \right) x'^2 - \left(\frac{imx}{\hbar t} - \frac{ip_0}{\hbar} \right) x' + \frac{imx^2}{2\hbar t} \right\} dx' \\
&= \sqrt{\frac{m}{2\pi i\hbar t}} \frac{1}{(\pi d_0^2)^{1/4}} \int_{-\infty}^{\infty} \exp \left\{ -\left(\frac{im}{2\hbar t} + \frac{1}{2d_0^2} \right) \left[x'^2 - 2 \frac{x - \frac{p_0 t}{m}}{1 + i \frac{\hbar t}{m d_0^2}} + \frac{x^2}{1 + i \frac{\hbar t}{m d_0^2}} \right] \right\} dx' \\
&= \sqrt{\frac{m}{2\pi i\hbar t}} \frac{1}{(\pi d_0^2)^{1/4}} \int_{-\infty}^{\infty} \exp \left\{ -\left(\frac{im}{2\hbar t} + \frac{1}{2d_0^2} \right) \left(x' - \frac{x - \frac{p_0 t}{m}}{1 + i \frac{\hbar t}{m d_0^2}} \right)^2 + \frac{-\frac{im}{2\hbar t} \left(x - \frac{p_0 t}{m} \right)^2 + \left(\frac{im}{2\hbar t} - \frac{1}{2d_0^2} \right) x^2}{1 + \frac{i\hbar t}{m d_0^2}} \right\} dx' \\
&\quad \left(\text{令 } \xi = \left(\frac{im}{2\hbar t} + \frac{1}{2d_0^2} \right)^{1/2} \left(x' - \frac{x - \frac{p_0 t}{m}}{1 + i \frac{\hbar t}{m d_0^2}} \right) \right) \\
&= \sqrt{\frac{m}{2\pi i\hbar t}} \frac{1}{(\pi d_0^2)^{1/4}} \left(-\frac{im}{2\hbar t} + \frac{1}{2d_0^2} \right)^{-1/2} \int_{-\infty}^{\infty} \exp \left\{ -\xi^2 + \frac{-\frac{im}{2\hbar t} \left(x - \frac{p_0 t}{m} \right)^2 + \left(\frac{im}{2\hbar t} - \frac{1}{2d_0^2} \right) x^2}{1 + \frac{i\hbar t}{m d_0^2}} \right\} d\xi \\
&\quad \left(\text{利用高斯积分 } \int_{-\infty}^{\infty} e^{-\xi^2} = \sqrt{\pi} \right) \\
&= \left[\pi^{1/2} \left(d_0 + \frac{i\hbar t}{m d_0} \right) \right]^{-1/2} \exp \left[\frac{-\frac{im}{2\hbar t} \left(x - \frac{p_0 t}{m} \right)^2 + \left(\frac{im}{2\hbar t} - \frac{1}{2d_0^2} \right) x^2}{1 + \frac{i\hbar t}{m d_0^2}} \right] \\
&= \left[\pi^{1/2} \left(d_0 + \frac{i\hbar t}{m d_0} \right) \right]^{-1/2} \exp \left[\frac{-\left(x - \frac{p_0 t}{m} \right)^2 \left(1 - \frac{i\hbar t}{m d_0^2} \right)}{2d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^2} \right)} \right] \exp \left[\frac{ip_0}{\hbar} \left(x - \frac{p_0 t}{m} \right) \right].
\end{aligned} \tag{24}$$

此时位置坐标的期望值为

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi^*(x', t) x \psi(x', t) dx' = \frac{p_0 t}{m}. \tag{25}$$

位置坐标的平方的期望值为

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \psi^*(x', t) x^2 \psi(x', t) dx'$$

$$\begin{aligned}
&= \pi^{-1/2} \left(d_0^2 + \frac{\hbar^2 t^2}{m^2 d_0^2} \right)^{-1/2} \int_{-\infty}^{\infty} \exp \left[\frac{-(x' - \frac{p_0 t}{m})^2}{d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} \right] x'^2 dx' \\
&= \pi^{-1/2} \left(d_0^2 + \frac{\hbar^2 t^2}{m^2 d_0^2} \right)^{-1/2} \int_{-\infty}^{\infty} \exp \left[\frac{-(x' - \frac{p_0 t}{m})^2}{d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} \right] \left[\left(x' - \frac{p_0 t}{m} \right)^2 + 2 \frac{p_0 t}{m} \left(x' - \frac{p_0 t}{m} \right) + \left(\frac{p_0 t}{m} \right)^2 \right] dx' \\
&= \pi^{-1/2} \left(d_0^2 + \frac{\hbar^2 t^2}{m^2 d_0^2} \right)^{-1/2} \left\{ \int_{-\infty}^{\infty} \exp \left[\frac{-(x' - \frac{p_0 t}{m})^2}{d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} \right] \left(x' - \frac{p_0 t}{m} \right)^2 dx' \right. \\
&\quad + 2 \frac{p_0 t}{m} \int_{-\infty}^{\infty} \exp \left[\frac{-(x' - \frac{p_0 t}{m})^2}{d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} \right] \left(x' - \frac{p_0 t}{m} \right) dx' \\
&\quad \left. + \left(\frac{p_0 t}{m} \right)^2 \int_{-\infty}^{\infty} \exp \left[\frac{-(x' - \frac{p_0 t}{m})^2}{d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} \right] dx' \right\} \\
&\quad (\text{利用 } \int_{-\infty}^{\infty} e^{-\alpha \xi^2} d\xi = \frac{\pi^{1/2}}{\alpha^{1/2}}, \quad \int_{-\infty}^{\infty} e^{-\alpha \xi^2} \xi d\xi = 0, \quad \text{及} \quad \int_{-\infty}^{\infty} e^{-\alpha \xi^2} \xi^2 d\xi = \frac{\pi^{1/2}}{2\alpha^{3/2}}) \\
&= \pi^{-1/2} \left(d_0^2 + \frac{\hbar^2 t^2}{m^2 d_0^2} \right)^{-1/2} \left\{ \frac{1}{2} \pi^{1/2} d_0^3 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)^{3/2} + \left(\frac{p_0 t}{m} \right)^2 \pi^{1/2} d_0 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)^{1/2} \right\} \\
&= \frac{1}{2} d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right) + \left(\frac{p_0 t}{m} \right)^2. \tag{26}
\end{aligned}$$

位置坐标的不确定度为

$$\langle (\Delta x)^2 \rangle^{1/2} = [\langle x^2 \rangle - \langle x \rangle^2]^{1/2} = \frac{1}{\sqrt{2}} d_0 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)^{1/2}. \tag{27}$$

动量的期望值为

$$\begin{aligned}
\langle p \rangle &= \int_{-\infty}^{\infty} \psi^*(x', t) \left(-i\hbar \frac{\partial}{\partial x'} \right) \psi(x', t) dx' \\
&= \pi^{-1/2} \left(d_0^2 + \frac{\hbar^2 t^2}{m^2 d_0^2} \right)^{1/2} \int_{-\infty}^{\infty} \exp \left[\frac{-(x - \frac{p_0 t}{m})^2}{2d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} \right] \exp \left[\frac{ip_0}{\hbar} \left(x - \frac{p_0 t}{2m} \right) \right] \times \\
&\quad \left(-i\hbar \frac{\partial}{\partial x} \right) \exp \left[\frac{-(x - \frac{p_0 t}{m})^2}{2d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} \right] \exp \left[-\frac{ip_0}{\hbar} \left(x - \frac{p_0 t}{2m} \right) \right] dx' \\
&= \pi^{-1/2} \left(d_0^2 + \frac{\hbar^2 t^2}{m^2 d_0^2} \right)^{-1/2} \int_{-\infty}^{\infty} \exp \left[\frac{-(x - \frac{p_0 t}{m})^2}{d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} \right] \left[\frac{i\hbar \left(x - \frac{p_0 t}{m} \right) \left(1 + \frac{i\hbar t}{md_0^2} \right)}{d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} - p_0 \right] dx' \\
&= \pi^{-1/2} \left(d_0^2 + \frac{\hbar^2 t^2}{m^2 d_0^2} \right)^{-1/2} \left\{ \frac{i\hbar \left(1 + \frac{i\hbar t}{md_0^2} \right)}{d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} \int_{-\infty}^{\infty} \exp \left[\frac{-(x - \frac{p_0 t}{m})^2}{d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} \right] \left(x - \frac{p_0 t}{m} \right) dx' \right. \\
&\quad \left. - p_0 \int_{-\infty}^{\infty} \exp \left[\frac{-(x - \frac{p_0 t}{m})^2}{d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} \right] dx' \right\} \\
&\quad (\text{利用 } \int_{-\infty}^{\infty} e^{-\alpha \xi^2} d\xi = \frac{\pi^{1/2}}{\alpha^{1/2}} \quad \text{及} \quad \int_{-\infty}^{\infty} e^{-\alpha \xi^2} \xi d\xi = 0) \\
&= p_0. \tag{28}
\end{aligned}$$

动量的平方的期望值为

$$\begin{aligned}
\langle p^2 \rangle &= \int_{-\infty}^{\infty} \psi^*(x', t) \left(-\hbar^2 \frac{\partial^2}{\partial x'^2} \right) \psi(x', t) dx' \\
&= \pi^{-1/2} \left(d_0^2 + \frac{\hbar^2 t^2}{m^2 d_0^2} \right)^{-1/2} \int_{-\infty}^{\infty} \exp \left[\frac{-(x' - \frac{p_0 t}{m})^2 \left(1 - \frac{i\hbar t}{m d_0^2} \right)}{2 d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} \right] \exp \left[\frac{i p_0}{\hbar} \left(x' - \frac{p_0 t}{2m} \right) \right] \times \\
&\quad \left(-\hbar^2 \frac{\partial^2}{\partial x'^2} \right) \exp \left[\frac{-(x' - \frac{p_0 t}{m})^2 \left(1 + \frac{i\hbar t}{m d_0^2} \right)}{2 d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} \right] \exp \left[-\frac{i p_0}{\hbar} \left(x' - \frac{p_0 t}{2m} \right) \right] dx' \\
&= \pi^{-1/2} \left(d_0^2 + \frac{\hbar^2 t^2}{m^2 d_0^2} \right)^{-1/2} (-\hbar^2) \int_{-\infty}^{\infty} \exp \left[\frac{-(x' - \frac{p_0 t}{m})^2}{d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} \right] \times \\
&\quad \left\{ -\frac{1 + \frac{i\hbar t}{m d_0^2}}{d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} + \left[\frac{(x' - \frac{p_0 t}{m}) \left(1 + \frac{i\hbar t}{m d_0^2} \right)}{d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} + \frac{i p_0}{\hbar} \right]^2 \right\} \\
&= \pi^{-1/2} \left(d_0^2 + \frac{\hbar^2 t^2}{m^2 d_0^2} \right)^{-1/2} (-\hbar^2) \int_{-\infty}^{\infty} \exp \left[\frac{-(x' - \frac{p_0 t}{m})^2}{d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} \right] \times \\
&\quad \left\{ \frac{\left(1 + \frac{i\hbar t}{m d_0^2} \right)^2}{d_0^4 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)^2} \left(x' - \frac{p_0 t}{m} \right)^2 + \frac{2i p_0 \left(1 + \frac{i\hbar t}{m d_0^2} \right)}{\hbar d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} \left(x' - \frac{p_0 t}{m} \right) - \frac{p_0^2}{\hbar^2} - \frac{1 + \frac{i\hbar t}{m d_0^2}}{d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} \right\} dx' \\
&= \pi^{-1/2} \left(d_0^2 + \frac{\hbar^2 t^2}{m^2 d_0^2} \right)^{-1/2} (-\hbar^2) \left\{ \frac{\left(1 + \frac{i\hbar t}{m d_0^2} \right)^2}{d_0^4 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)^2} \int_{-\infty}^{\infty} \exp \left[\frac{-(x' - \frac{p_0 t}{m})^2}{d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} \right] \left(x' - \frac{p_0 t}{m} \right)^2 dx' \right. \\
&\quad + \frac{2i p_0 \left(1 + \frac{i\hbar t}{m d_0^2} \right)}{\hbar d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} \int_{-\infty}^{\infty} \exp \left[\frac{-(x' - \frac{p_0 t}{m})^2}{d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} \right] \left(x' - \frac{p_0 t}{m} \right) dx' \\
&\quad \left. - \left[\frac{p_0^2}{\hbar^2} + \frac{1 + \frac{i\hbar t}{m d_0^2}}{d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} \right] \int_{-\infty}^{\infty} \exp \left[\frac{-(x' - \frac{p_0 t}{m})^2}{d_0^2 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)} \right] dx' \right\} \\
&\quad (\text{利用 } \int_{-\infty}^{\infty} e^{-\alpha \xi^2} d\xi = \frac{\pi^{1/2}}{\alpha^{1/2}}, \quad \int_{-\infty}^{\infty} e^{-\alpha \xi^2} \xi d\xi = 0, \quad \text{及} \quad \int_{-\infty}^{\infty} e^{-\alpha \xi^2} \xi^2 d\xi = \frac{\pi^{1/2}}{2\alpha^{3/2}}) \\
&= \pi^{-1/2} \left(d_0^2 + \frac{\hbar^2 t^2}{m^2 d_0^2} \right)^{-1/2} (-\hbar^2) \left[-\frac{\pi^{1/2} \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)^{1/2}}{2 d_0} - \frac{\pi^{1/2} p_0^2 d_0 \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)^{1/2}}{\hbar^2} \right] \\
&= \frac{\hbar^2}{2 d_0^2} + p_0^2.
\end{aligned} \tag{29}$$

动量的不确定度为

$$\langle (\Delta p)^2 \rangle^{1/2} = [\langle p^2 \rangle - \langle p \rangle^2]^{1/2} = \frac{\hbar}{\sqrt{2} d_0}. \tag{30}$$

此时位置坐标和动量满足不确定性关系:

$$\langle (\Delta x)^2 \rangle^{1/2} \langle (\Delta p)^2 \rangle^{1/2} = \frac{\hbar}{2} \left(1 + \frac{\hbar^2 t^2}{m^2 d_0^4} \right)^{1/2} \geq \frac{\hbar}{2}. \tag{31}$$

□

第 5 题 (课本习题 2.34) 得分: _____. (a) 写出一个简谐振子对于一个有限时间间隔的经典作用量.

(b) 对于一个简谐振子, 利用费曼方法构造出微小的 $t_n - t_{n-1} = \Delta t$ 情况下的 $\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle$. 只保留到 $(\Delta t)^2$ 量级的项, 证明它与由 (2.6.26) 式给出的传播子在 $t - t_0 \rightarrow 0$ 时的极限完全一致.

解: (a) 简谐振子的经典拉格朗日量为

$$L(x, \dot{x}) = \frac{m\dot{x}^2}{2} - \frac{m\omega^2 x^2}{2}. \quad (32)$$

对于一个有限时间间隔 (t_{n-1}, t_n) 的经典作用量为

$$\begin{aligned} S(n, n-1) &= \int_{t_{n-1}}^{t_n} dt L(x, \dot{x}) \\ &\approx (t_n - t_{n-1}) \left[\frac{m}{2} \left(\frac{x_n - x_{n-1}}{t_n - t_{n-1}} \right)^2 - \frac{m\omega^2}{2} \left(\frac{x_n + x_{n-1}}{2} \right)^2 \right] \\ &\approx \frac{m(\Delta x)^2}{2\Delta t} - \frac{m\omega^2 x_n^2 \Delta t^2}{2}, \end{aligned} \quad (33)$$

其中时间间隔 $\Delta t = t_n - t_{n-1}$, $\Delta x = x_n - x_{n-1} = x(t_n) - x(t_{n-1})$.

(b) 微小的 $t_n - t_{n-1} \equiv \Delta t$ 情况下,

$$\begin{aligned} \langle x_n, t_n | x_{n-1}, t_{n-1} \rangle &= \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \exp \left[\frac{i S(n, n-1)}{\hbar} \right] \\ &= \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \exp \left\{ \frac{i}{\hbar} \left[\frac{m(\Delta x)^2}{2\Delta t} - \frac{m\omega^2 x_n^2 \Delta t^2}{2} \right] \right\}. \end{aligned} \quad (34)$$

由课本 (2.6.26) 式和 (2.6.18),

$$\begin{aligned} \langle x'', t | x', t_0 \rangle &= K(x'', t; x', t_0) \\ &= \sqrt{\frac{m\omega}{2\pi i \hbar \sin[\omega(t - t_0)]}} \exp \left\{ \left[\frac{im\omega}{2\hbar \sin[\omega(t - t_0)]} \right] \times \right. \\ &\quad \left. \{ (x''^2 + x'^2) \cos[\omega(t - t_0)] - 2x''x' \} \right\}, \end{aligned} \quad (35)$$

在 $t - t_0 \rightarrow 0$ 的极限下, $\sin[\omega(t - t_0)] \approx \omega(t - t_0)$, $\cos[\omega(t - t_0)] \approx 1 - \frac{\omega^2(t - t_0)^2}{2}$, 值保留 $(\Delta t)^2$ 量级的项, 上式可简化为

$$\begin{aligned} \langle x'', t | x', t_0 \rangle &\approx \sqrt{\frac{m}{2\pi i \hbar (t - t_0)}} \exp \left[\frac{im}{2\hbar(t - t_0)} \left\{ [x'^2 + (x' + (x'' - x'))^2] \left[1 - \frac{\omega^2(t - t_0)^2}{2} \right] - 2[x' + (x'' - x')]x' \right\} \right] \\ &\approx \sqrt{\frac{m}{2\pi i \hbar (t - t_0)}} \exp \left[\frac{im}{\hbar} \left(\frac{(x'' - x')^2}{2(t - t_0)} - \frac{\omega^2 x'^2 (t - t_0)}{2} \right) \right]. \end{aligned} \quad (36)$$

可见, 利用高斯方法构造的微小的 $t_n - t_{n-1} = \Delta t$ 情况下的 $\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle$ 与传播子在 $t - t_0 \rightarrow 0$ 时的极限具有一致的形式. □

第 6 题 (课本习题 2.37) 得分: _____. (a) 证明 (2.7.25) 式和 (2.7.27) 式的正确性.

(b) 证明具有由 (2.7.31) 式给定的 j 的连续性方程 (2.7.30) 的正确性.

证: (a) 课本 (2.7.25) 式:

$$[\Pi_i, \Pi_j] = [p_i - \frac{eA_i}{c}, p_j - \frac{eA_j}{c}]$$

$$\begin{aligned}
&= -\frac{e}{c}([p_i, A_j] + [A_i, p_j]) \\
&= i\hbar \frac{e}{c} \left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} \right) \\
&= \frac{i\hbar e}{c} \varepsilon_{ijk} (\nabla \times \mathbf{A})_k \\
&= \frac{i\hbar e}{c} \varepsilon_{ijk} B_k,
\end{aligned} \tag{37}$$

其中 $\varepsilon_{ijk} = \begin{cases} 1, & i, j, k \text{ 为偶排列,} \\ -1, & i, j, k \text{ 为奇排列,} \\ 0, & i, j, k \text{ 中有重复.} \end{cases}$

课本式 (2.7.27):

$$\begin{aligned}
m \frac{d^2 \mathbf{x}}{dt^2} &= \frac{d\mathbf{\Pi}}{dt} = \frac{1}{i\hbar} [\mathbf{\Pi}, H] \\
&= \frac{1}{i\hbar} [\mathbf{\Pi}, \frac{\mathbf{\Pi}^2}{2m} + e\phi] \\
&= \frac{1}{2i\hbar m} [\Pi_x \hat{x} + \Pi_y \hat{y} + \Pi_z \hat{z}, \mathbf{\Pi}^2] + \frac{e}{i\hbar} [\Pi_x \hat{x} + \Pi_y \hat{y} + \Pi_z \hat{z}, \phi] \\
&= \frac{1}{2i\hbar m} ([\Pi_x, \mathbf{\Pi}^2] \hat{x} + [\Pi_y, \mathbf{\Pi}^2] \hat{y} + [\Pi_z, \mathbf{\Pi}^2] \hat{z}) + \frac{e}{i\hbar} ([\Pi_x, \phi] \hat{x} + [\Pi_y, \phi] \hat{y} + [\Pi_z, \phi] \hat{z}),
\end{aligned} \tag{38}$$

其中

$$[\Pi_i, \Pi_j^2] = \Pi_j [\Pi_i, \Pi_j] + [\Pi_i, \Pi_j] \Pi_j = \frac{i\hbar e}{c} \varepsilon_{ijk} (\Pi_j B_k + B_k \Pi_j), \tag{39}$$

$$\begin{aligned}
\Rightarrow [\Pi_x, \mathbf{\Pi}^2] &= [\Pi_x, \Pi_x^2 + \Pi_y^2 + \Pi_z^2] \\
&= [\Pi_x, \Pi_x^2] + [\Pi_x, \Pi_y^2] + [\Pi_x, \Pi_z^2] \\
&= \frac{i\hbar e}{c} [(\Pi_y B_z + B_z \Pi_y) - (\Pi_z B_y + B_y \Pi_z)] \\
&= \frac{i\hbar e}{c} [(\mathbf{\Pi} \times \mathbf{B})_x - (\mathbf{B} \times \mathbf{\Pi})_x],
\end{aligned} \tag{40}$$

以及

$$[\Pi_i, \phi] = [p_i - \frac{e\mathbf{A}(\mathbf{x})}{c}, \phi(\mathbf{x})] = [p_i, \phi] = -i\hbar \frac{\partial \phi}{\partial x_i}, \tag{41}$$

从而

$$\begin{aligned}
m \frac{d^2 \mathbf{x}}{dt^2} &= \frac{1}{2i\hbar m} \frac{i\hbar e}{c} [(\mathbf{\Pi} \times \mathbf{B})_x \hat{x} - (\mathbf{B} \times \mathbf{\Pi})_x \hat{x} + (\mathbf{\Pi} \times \mathbf{B})_y \hat{y} - (\mathbf{B} \times \mathbf{\Pi})_y \hat{y} + (\mathbf{\Pi} \times \mathbf{B})_z \hat{z} - (\mathbf{B} \times \mathbf{\Pi})_z \hat{z}] \\
&\quad + \frac{e}{i\hbar} (-i\hbar) \left(\frac{\partial \phi}{\partial x} \hat{x} + \frac{\partial \phi}{\partial y} \hat{y} + \frac{\partial \phi}{\partial z} \hat{z} \right) \\
&= \frac{e}{2mc} (\mathbf{\Pi} \times \mathbf{B} - \mathbf{B} \times \mathbf{\Pi}) - e \nabla \phi \\
&= e \left[\mathbf{E} + \frac{1}{2c} \left(\frac{d\mathbf{x}}{dt} \times \mathbf{B} - \mathbf{B} \times \frac{d\mathbf{x}}{dt} \right) \right].
\end{aligned} \tag{42}$$

(b) 电磁场中粒子的薛定谔方程为

$$i\hbar \frac{\partial}{\partial t} \langle \mathbf{x}' | \alpha, t_0; t \rangle = \langle \mathbf{x}' | H | \alpha, t_0; t \rangle$$

$$\begin{aligned}
&= \langle \mathbf{x}' | \left\{ \frac{1}{2m} \left[p - \frac{e\mathbf{A}(\mathbf{x})}{c} \right]^2 + e\phi(\mathbf{x}) \right\} | \alpha, t_0; t \rangle \\
&= \frac{1}{2m} \left[-i\hbar\nabla' - \frac{e\mathbf{A}(\mathbf{x}')}{c} \right] \cdot \left[-i\hbar\nabla' - \frac{e\mathbf{A}(\mathbf{x}')}{c} \right] \langle \mathbf{x}' | \alpha, t_0; t \rangle + e\phi(\mathbf{x}') \langle \mathbf{x}' | \alpha, t_0; t \rangle \\
&= \frac{1}{2m} \left\{ -\hbar^2\nabla'^2 + \frac{i\hbar e}{c} \nabla' \cdot \mathbf{A}(\mathbf{x}') + \frac{i\hbar e\mathbf{A}(\mathbf{x}')}{c} \cdot \nabla' + \frac{e^2\mathbf{A}^2(\mathbf{x}')}{c} + e\phi(\mathbf{x}') \right\} \langle \mathbf{x}' | \alpha, t_0; t \rangle, \quad (43)
\end{aligned}$$

即

$$i\hbar \frac{\partial}{\partial t} \psi = \frac{1}{2m} \left\{ -\hbar^2\nabla'^2 + \frac{i\hbar e}{c} \nabla' \cdot \mathbf{A}(\mathbf{x}') + \frac{i\hbar e\mathbf{A}(\mathbf{x}')}{c} \cdot \nabla' + \frac{e^2\mathbf{A}^2(\mathbf{x}')}{c} + e\phi(\mathbf{x}') \right\} \psi, \quad (44)$$

其中 $\psi = \langle \mathbf{x}' | \alpha, t_0; t \rangle$, 注意大括号中第二项的 ∇' 是同时作用在 $\mathbf{A}'(\mathbf{x}')\psi$ 上. 上式的厄米共轭为

$$-i\hbar \frac{\partial}{\partial t} \psi^* = \frac{1}{2m} \left\{ -\hbar^2\nabla'^2 - \frac{i\hbar e}{c} \nabla' \cdot \mathbf{A}(\mathbf{x}') - \frac{i\hbar e\mathbf{A}(\mathbf{x}')}{c} \cdot \nabla' + \frac{e^2\mathbf{A}^2(\mathbf{x}')}{c} + e\phi(\mathbf{x}') \right\} \psi^*. \quad (45)$$

— 式 (45) $\times \psi + \psi^* \times$ 式 (44) 得

$$\begin{aligned}
i\hbar \frac{\partial \psi^*}{\partial t} \psi + i\hbar \psi^* \frac{\partial \psi}{\partial t} &= \frac{\hbar^2}{2m} [(\nabla'^2 \psi^*)\psi - \psi^* \nabla'^2 \psi] \\
&\quad + \frac{i\hbar e}{2mc} \{ (\nabla' \cdot [\mathbf{A}(\mathbf{x}')\psi^*])\psi + \mathbf{A}(\mathbf{x}') \cdot (\nabla' \psi^*)\psi + \psi^* \nabla' \cdot [\mathbf{A}'(\mathbf{x}')\psi] + \psi^* \mathbf{A}(\mathbf{x}') \cdot \nabla' \psi \}, \quad (46)
\end{aligned}$$

其中,

$$\text{式 (46) 左边} = i\hbar \frac{\partial(\psi^* \psi)}{\partial t} = i\hbar \frac{\partial \psi}{\partial t}, \quad (47)$$

此处 $\rho = \psi^* \psi$,

$$\begin{aligned}
\frac{\hbar^2}{2m} [(\nabla'^2 \psi^*)\psi - \psi^* \nabla'^2 \psi] &= \frac{\hbar^2}{2m} [(\nabla'^2 \psi^*)\psi + \nabla' \psi^* \cdot \nabla' \psi - \nabla' \psi^* \cdot \nabla' \psi - \psi^* \nabla'^2 \psi] \\
&= \frac{\hbar^2}{2m} \nabla' \cdot [(\nabla' \psi^*)\psi - \psi^* \nabla' \psi] \\
&= \frac{\hbar^2}{2m} \nabla' \cdot [-2i \text{Im}(\psi^* \nabla' \psi)] \\
&= -\frac{i\hbar^2}{m} \nabla' \cdot \text{Im}(\psi^* \nabla' \psi), \quad (48)
\end{aligned}$$

$$\frac{i\hbar e}{2mc} \{ (\nabla' \cdot [\mathbf{A}(\mathbf{x}')\psi^*])\psi + \mathbf{A}(\mathbf{x}') \cdot (\nabla' \psi^*)\psi + \psi^* \nabla' \cdot [\mathbf{A}'(\mathbf{x}')\psi] + \psi^* \mathbf{A}(\mathbf{x}') \cdot \nabla' \psi \} = \frac{i\hbar e}{mc} \nabla' \cdot [\mathbf{A}(\mathbf{x}')\psi^* \psi]. \quad (49)$$

将以上三式代回式 (46) 得

$$i\hbar \frac{\partial \rho}{\partial t} = -\frac{i\hbar^2}{m} \nabla' \cdot \text{Im}(\psi^* \nabla \psi) + \frac{i\hbar e}{mc} \nabla' \cdot [\mathbf{A}(\mathbf{x}')\psi^* \psi], \quad (50)$$

$$\implies \frac{\partial \rho}{\partial t} + \nabla' \cdot \mathbf{j} = 0, \quad (51)$$

此即课本式 (2.7.30), 其中概率流 \mathbf{j} 为课本式 (2.7.31) 式所给定的形式:

$$\mathbf{j} = \left(\frac{\hbar}{m} \right) \text{Im}(\psi^* \nabla' \psi) - \left(\frac{e}{mc} \right) \mathbf{A} |\psi|^2. \quad (52)$$

□

第 7 题 (课本习题 2.39) 得分: _____. 一个电子在一个均匀的、沿 z 方向的磁场 ($\mathbf{B} = B\hat{z}$) 中运动.

(a) 求

$$[\Pi_x, \Pi_y],$$

其中

$$\Pi_x \equiv p_x - \frac{eA_x}{c}, \quad \Pi_y \equiv p_y - \left(\frac{eA_y}{c}\right).$$

(b) 通过将哈密顿量及 (a) 中得到的对易关系与一维谐振子问题中相应的结果比较, 展示我们怎样能够立即写出能量本征值

$$E_{k,n} = \frac{\hbar^2 k^2}{2m} + \left(\frac{|eB|\hbar}{mc}\right) \left(n + \frac{1}{2}\right).$$

解: (a) 由上一题 (a) 中的结论,

$$[\Pi_x, \Pi_y] = \frac{i\hbar e}{c} B_z = \frac{i\hbar e}{c} B. \quad (53)$$

(b) 由于磁场 $\mathbf{B} = B\hat{z} = \nabla \times \mathbf{A}$ 沿 z 方向, 故 \mathbf{A} 仅有沿 y 和 x 方向的分量. 该电子的哈密顿量为

$$H = \frac{\Pi_x^2}{2m} + \frac{\Pi_y^2}{2m} + \frac{p_z^2}{2m}. \quad (54)$$

回顾一维谐振子问题: 哈密顿量为

$$H_{1D \text{ HO}} = \frac{p^2}{2m} + \frac{m\omega x^2}{2}, \quad (55)$$

且有对易关系

$$[x, p] = i\hbar. \quad (56)$$

受此启发, 我们可以定义 $Y = \frac{c\Pi_x}{eB}$, 从而该电子的哈密顿量为

$$H = \frac{\left(\frac{eBY}{c}\right)^2}{2m} + \frac{\Pi_y^2}{2m} + \frac{p_z^2}{2m} = \frac{m\left(\frac{eB}{mc}\right)^2 Y^2}{2} + \frac{\Pi_y^2}{2m} + \frac{p_z^2}{2m}, \quad (57)$$

且有对易关系

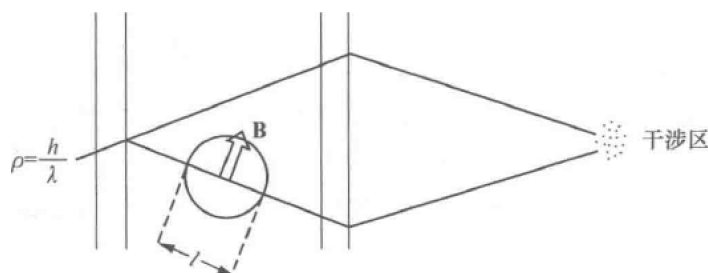
$$[Y, \Pi_y] = i\hbar. \quad (58)$$

可见, 该电子的哈密顿量相当于一个本征频率 $\omega = \frac{|eB|\hbar}{mc}$ 的一维谐振子的哈密顿量和沿 z 方向运动的自由粒子的哈密顿量之和, 从而其能量本征值为

$$E_{k,n} = \frac{\hbar^2 k^2}{2m} + \frac{|eB|\hbar}{mc} \left(n + \frac{1}{2}\right). \quad (59)$$

□

第 8 题 (课本习题 2.40) 得分: _____. 考虑中子干涉仪



证明在计数率中产生两个相继极大值的磁场差由下式给出

$$\Delta B = \frac{4\pi\hbar c}{|e|g_n\bar{\lambda}l},$$

其中 $g_n = (-1.91)$ 是中子磁矩. 单位为 $-e\hbar/2m_n c$. (假如你在 1967 年解出了这个问题的话, 你将会在 Physical Review Letters 上发表你的解!)

证: 中子磁矩在磁场中的势能为

$$V = -g_n \left(-\frac{e\hbar}{2m_n c} \right) B = \frac{g_n e\hbar}{2m_n c} B. \quad (60)$$

由上下两臂到达干涉区的中子束的相位差为

$$\Delta\phi = -\frac{V}{\hbar} \frac{l}{p/m_n} = -\frac{g_n e}{2m_n c} B \frac{l}{(\hbar/\bar{\lambda})/m_n}. \quad (61)$$

计数中产生两个相继极大值的情形的相位差为 2π :

$$\Delta\phi_2 - \Delta\phi_1 = -\frac{g_n e}{2m_n c} B_2 \frac{l}{(\hbar/\bar{\lambda})/m_n} + \frac{g_n e}{2m_n c} B_1 \frac{l}{(\hbar/\bar{\lambda})/m_n} = 2\pi, \quad (62)$$

对应的磁场差为

$$\Delta B = |B_2 - B_1| = \frac{4\pi\hbar c}{|eg_n|\bar{\lambda}l}. \quad (63)$$

□