2.24

$$V(x) = -\nu_0\delta(x)$$

薛定谔方程为：

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}[E + \nu_0\delta(x)]\psi = 0$$

两边做积分极限：

$$\lim_{\varepsilon\to 0+}\left(\int_{-\varepsilon}^{\varepsilon}\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}[E + \nu_0\delta(x)]\psi dx\right) = \psi'(0+) - \psi'(0-) + \frac{2m\nu_0}{\hbar^2}\psi(0) = 0$$

在$x \neq 0$的区域

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}E\psi = 0$$

为满足$|x| \to \infty$的边界条件

$$\psi \propto e^{-\frac{\sqrt{-2mE}}{\hbar}|x|}$$

偶宇态下：

$$\psi = Ce^{-\frac{\sqrt{-2mE}}{\hbar}|x|}$$

由：$\psi'(0+) - \psi'(0-) + \frac{2m\nu_0}{\hbar^2}\psi(0) = 0$

$$-2\frac{\sqrt{-2mE}}{\hbar} + \frac{2m\nu_0}{\hbar^2} = 0 \to E = -\frac{m\nu_0^2}{2\hbar^2}$$

由归一化条件积的：$C = \frac{\sqrt{m\nu_0}}{\hbar}, \psi = \frac{\sqrt{m\nu_0}}{\hbar}e^{-\frac{m\nu_0}{\hbar^2}|x|}$

奇宇态下：

$$\psi = \begin{cases} Ae^{-\frac{\sqrt{-2mE}}{\hbar}|x|} \\ -Ae^{-\frac{\sqrt{-2mE}}{\hbar}|x|} \end{cases}$$

由ψ在 0 处连续，得：$A = 0$

故无奇宇态束缚态，只有一个偶宇态束缚态，$\psi = \frac{\sqrt{m\nu_0}}{\hbar}e^{-\frac{m\nu_0}{\hbar^2}|x|}$，$E = -\frac{m\nu_0^2}{2\hbar^2}$，无其他激发束缚态

2.26

（a）

$$V(x) = \lambda x$$

薛定谔方程为：

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}[E - \lambda x]\psi = 0$$

令$y = \sqrt[3]{\frac{m\lambda}{\hbar^2}}x, \mathcal{E} = \sqrt[3]{\frac{m}{\lambda^2\hbar^2}}E$

$$\frac{d^2\psi}{dy^2} + 2[\mathcal{E} - y]\psi = 0$$

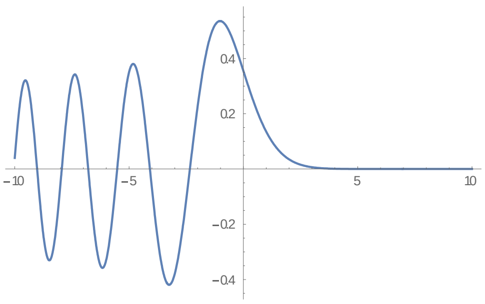
做变换：$z = 2^{\frac{1}{3}}(y - \mathcal{E})$

$$\frac{d^2\psi}{dz^2}-z\psi=0$$

其解为艾里函数：$\psi=Ai(z)$，$z=0$对应于经典转折点：$E=\lambda x$

其能谱是连续的

$$\psi \propto \begin{cases} \sqrt{2^{\frac{1}{3}}\left(\sqrt[3]{\frac{m\lambda}{\hbar^2}}x-\sqrt[3]{\frac{m}{\lambda^2\hbar^2}}E\right)}K_{1/3}(\frac{2}{3}\left(2^{\frac{1}{3}}\left(\sqrt[3]{\frac{m\lambda}{\hbar^2}}x-\sqrt[3]{\frac{m}{\lambda^2\hbar^2}}E\right)\right)^{\frac{3}{2}})(E<\lambda x) \\ \frac{\pi}{3}\sqrt{2^{\frac{1}{3}}\left(\sqrt[3]{\frac{m}{\lambda^2\hbar^2}}E-\sqrt[3]{\frac{m\lambda}{\hbar^2}}x\right)}\times \\ [J_{1/3}\left(\frac{2}{3}\left(2^{\frac{1}{3}}\left(\sqrt[3]{\frac{m}{\lambda^2\hbar^2}}E-\sqrt[3]{\frac{m\lambda}{\hbar^2}}x\right)\right)^{\frac{3}{2}}\right)+J_{-1/3}(\frac{2}{3}\left(2^{\frac{1}{3}}\left(\sqrt[3]{\frac{m}{\lambda^2\hbar^2}}E-\sqrt[3]{\frac{m\lambda}{\hbar^2}}x\right)\right)^{\frac{3}{2}})](E>\lambda x) \end{cases}$$



波函数示意图

其中已取 E=0 为例

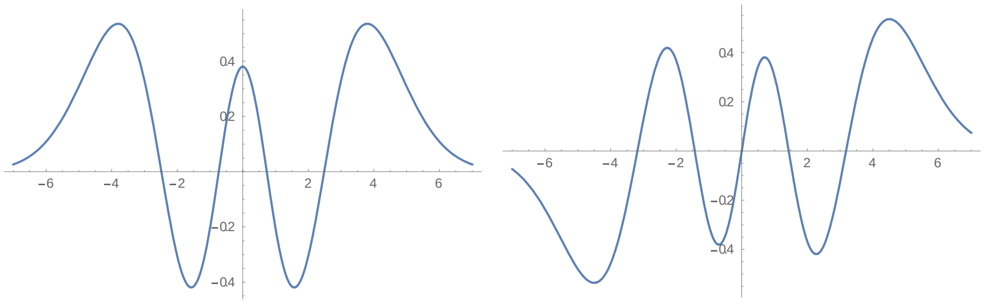
（b）

若

$$V(x)=\lambda|x|$$

$$\psi=\begin{cases} Ai(2^{\frac{1}{3}}\left(\sqrt[3]{\frac{m\lambda}{\hbar^2}}x-\sqrt[3]{\frac{m}{\lambda^2\hbar^2}}E\right))(x>0) \\ Ai(2^{\frac{1}{3}}\left(-\sqrt[3]{\frac{m\lambda}{\hbar^2}}x-\sqrt[3]{\frac{m}{\lambda^2\hbar^2}}E\right))(x<0) \end{cases}$$

此时要求$x=0$处波函数及其一阶导连续，则此时能谱是分立的



波函数示意图如上所示，分别对应$n=5,6$能级

2.31

$$K(x'',t;x',t_0)=\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}dp'\exp\,[-i\frac{(t-t_0)}{2m\hbar}(p'^2-\frac{2m(x''-x')}{(t-t_0)}p')]$$

$$=\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}dp'\exp\,[-i\frac{(t-t_0)}{2m\hbar}((p'-\frac{m(x''-x')}{(t-t_0)})^2-\frac{m^2(x''-x')^2}{(t-t_0)^2})]=$$

$$=\frac{1}{2\pi\hbar}\sqrt{\frac{2m\hbar}{i(t-t_0)}}\exp\left[\frac{im(x''-x')^2}{2\hbar(t-t_0)}\right]\int_{-\infty}^{+\infty}dt\exp[-t^2]$$

$$=\frac{1}{2\pi\hbar}\sqrt{\frac{2\pi m\hbar}{i(t-t_0)}}\exp\left[\frac{im(x''-x')^2}{2\hbar(t-t_0)}\right]=\sqrt{\frac{m}{2\pi i\hbar(t-t_0)}}\exp\left[\frac{im(x''-x')^2}{2\hbar(t-t_0)}\right]$$

三维：

$$K(\overrightarrow{x''},t;\overrightarrow{x'},t_0)=\frac{1}{(2\pi\hbar)^3}\int_{-\infty}^{+\infty}d^3p'\exp\,[-i\frac{(t-t_0)}{2m\hbar}(\overrightarrow{p'}^2-\frac{2m(\overrightarrow{x''}-\overrightarrow{x'})}{(t-t_0)}\overrightarrow{p'})]$$

$$=\frac{1}{(2\pi\hbar)^3}\prod_{i=1}^{3}\int_{-\infty}^{+\infty}dp'_i\exp\,[-i\frac{(t-t_0)}{2m\hbar}({p'_i}^2-\frac{2m(\overrightarrow{x''}-\overrightarrow{x'})_i}{(t-t_0)}p'_i)]$$

$$=\frac{1}{(2\pi\hbar)^3}\prod_{i=1}^{3}\sqrt{\frac{2\pi m\hbar}{i(t-t_0)}}\exp\left[\frac{im(\overrightarrow{x''}-\overrightarrow{x'})_i^2}{2\hbar(t-t_0)}\right]$$

$$=(\frac{m}{2\pi i\hbar(t-t_0)})^{\frac{3}{2}}\exp\left[\frac{im(\overrightarrow{x''}-\overrightarrow{x'})^2}{2\hbar(t-t_0)}\right]$$

2.34

（a）

$$S(n,n-1)=\int_{t_{n-1}}^{t_n}\frac{1}{2}m(\frac{dx}{dt})^2-\frac{1}{2}m\omega^2x^2dt=\frac{m}{2}(\frac{(x_n-x_{n-1})^2}{(t_n-t_{n-1})}-\omega^2x_n^2(t_n-t_{n-1}))$$

此处已取t_n-t_{n-1},x_n-x_{n-1}均为小量

（b）

$$\langle x_n,t_n|x_{n-1},t_{n-1}\rangle=\sqrt{\frac{m}{2\pi i\hbar\Delta t}}\exp\left[\frac{iS(n,n-1)}{\hbar}\right]$$

$$=\sqrt{\frac{m}{2\pi i\hbar\Delta t}}\exp\left[\frac{im}{2\hbar}(\frac{(x_n-x_{n-1})^2}{(t_n-t_{n-1})}-\omega^2x_n^2(t_n-t_{n-1}))\right]$$

对于简谐振子：

$$K(x_n,t_n;x_{n-1},t_{n-1})=\sqrt{\frac{m\omega}{2\pi i\hbar sin\omega\Delta t}}\exp\left[\frac{im\omega}{2\hbar sin\omega\Delta t}\times((x_n^2+x_{n-1}^2)cos\omega\Delta t-2x_nx_{n-1})\right]$$

$$=\sqrt{\frac{m}{2\pi i\hbar\Delta t}}\exp\left[\frac{im}{2\hbar\Delta t}\times((x_n^2+x_{n-1}^2)(1-\frac{\Delta t^2}{2})-2x_nx_{n-1})\right]$$

$$=\sqrt{\frac{m}{2\pi i\hbar\Delta t}}\exp\left[\frac{im}{2\hbar}\times(\frac{(x_n-x_{n-1})^2}{\Delta t}-\frac{\Delta t}{2}(2x_n^2))\right]$$

$$=\sqrt{\frac{m}{2\pi i\hbar\Delta t}}\exp\left[\frac{im}{2\hbar}(\frac{(x_n-x_{n-1})^2}{(t_n-t_{n-1})}-\omega^2x_n^2(t_n-t_{n-1}))\right]=\langle x_n,t_n|x_{n-1},t_{n-1}\rangle$$

证毕

2.37

（a）

$$[\Pi_i,\Pi_j]=\left[p_i-\frac{e}{c}A_i,p_j-\frac{e}{c}A_j\right]=-\frac{e}{c}([p_i,A_j]+[p_j,A_i])=\frac{i\hbar e}{c}\left(\frac{\partial A_j}{\partial x_i}-\frac{\partial A_i}{\partial x_j}\right)=\frac{i\hbar e}{c}\varepsilon_{ijk}B_k$$

$$m\frac{d^2\vec{x}}{dt^2}=\frac{d\Pi}{dt}=\frac{[\Pi,H]}{i\hbar}=\frac{[\Pi,\Pi^2]}{2im\hbar}+e\frac{[\vec{p},\phi]}{i\hbar}$$

$$=-e\nabla\phi+\frac{1}{2im\hbar}([\Pi_1,\Pi_2^2+\Pi_3^2]\widehat{e_1}+[\Pi_2,\Pi_3^2+\Pi_1^2]\widehat{e_2}+[\Pi_3,\Pi_1^2+\Pi_2^2]\widehat{e_3})$$

$$=e\vec{E}$$

$$+\frac{e}{2mc}[(\Pi_2B_3+B_3\Pi_2-\Pi_3B_2-B_2\Pi_3)\widehat{e_1}+(\Pi_3B_1+B_1\Pi_3-\Pi_1B_3-B_3\Pi_1)\widehat{e_2}$$

$$+(\Pi_1B_2+B_2\Pi_1-\Pi_2B_1-B_1\Pi_2)\widehat{e_3}]=e\vec{E}+\frac{e}{2mc}\left[\Pi\times\vec{B}+\vec{B}\times\Pi\right]$$

$$=e\vec{E}+\frac{e}{2c}\left[\frac{d\vec{x}}{dt}\times\vec{B}+\vec{B}\times\frac{d\vec{x}}{dt}\right]$$

证毕

（b）

$$\frac{\partial\rho}{\partial t}+\nabla'\vec{j}=\frac{\partial\psi^*\psi}{\partial t}+\nabla'\left(\frac{\hbar}{2im}(\psi^*\nabla'\psi-\psi\nabla'\psi^*)-\frac{e}{mc}\vec{A}\psi^*\psi\right)$$

$$=\psi^*\frac{\partial\psi}{\partial t}+\psi\frac{\partial\psi^*}{\partial t}+\frac{\hbar}{2im}\psi^*\nabla'^2\psi-\frac{\hbar}{2im}\psi\nabla'^2\psi^*-\frac{e}{mc}\psi^*\psi\nabla'\vec{A}-\frac{e}{mc}\vec{A}\psi^*\nabla'\psi$$

$$-\frac{e}{mc}\vec{A}\psi\nabla'\psi^*$$

$$=\psi^*\left(\frac{\partial\psi}{\partial t}+\frac{\hbar}{2im}\nabla'^2\psi-\frac{e}{mc}\vec{A}\nabla'\psi\right)+\psi\left(\frac{\partial\psi^*}{\partial t}-\frac{\hbar}{2im}\nabla'^2\psi^*-\frac{e}{mc}\vec{A}\nabla'\psi^*\right)$$

$$=\psi^*\left(\frac{1}{i\hbar}(i\hbar\frac{\partial\psi}{\partial t}+\frac{\hbar^2}{2m}\nabla'^2\psi-\frac{i\hbar e}{mc}\vec{A}\nabla'\psi-\frac{i\hbar e}{mc}\psi\nabla'\vec{A}+e\phi\psi))\right)$$

$$+\psi\left(-\frac{1}{i\hbar}\left(-i\hbar\frac{\partial\psi^*}{\partial t}+\frac{\hbar^2}{2m}\nabla'^2\psi^*+\frac{i\hbar e}{i\hbar mc}\vec{A}\nabla'\psi^*+\frac{i\hbar e}{mc}\psi^*\nabla'\vec{A}+e\phi\psi^*\right)\right)=0$$

2.39

(a)

由：$[\Pi_i,\Pi_j]=\frac{i\hbar e}{c}\varepsilon_{ijk}B_k$

$$[\Pi_x,\Pi_y]=\frac{i\hbar eB}{c}$$

(b)

$$H=\frac{\Pi^2}{2m}=\frac{\Pi_x^2}{2m}+\frac{\Pi_y^2}{2m}+\frac{\Pi_z^2}{2m}$$

由：$A_z=0\rightarrow\ \Pi_z=p_z$

$$\left[\Pi_x,\frac{c}{eB}\Pi_y\right]=i\hbar$$

Π_x类比p，$\frac{c}{eB}\Pi_y$类比x

$$\mathrm{H}=\frac{\Pi^2}{2m}=\frac{\Pi_x^2}{2m}+\frac{1}{2}m(\frac{eB}{mc})^2(\frac{c}{eB}\Pi_y)^2+\frac{\mathrm{p}_z^2}{2m}$$

故:

$$\mathrm{H}|n\rangle=(\left(n+\frac{1}{2}\right)\hbar\frac{|eB|}{mc}+\frac{\hbar^2k^2}{2m})|n\rangle$$

$$E_n=\left(n+\frac{1}{2}\right)\hbar\frac{|eB|}{mc}+\frac{\hbar^2k^2}{2m}$$

2.40

在没有磁场的区域:

$$H=\frac{p^2}{2m}$$

在有磁场的区域:

$$H=\frac{p^2}{2m}+g_n\frac{|e|\hbar}{2mc}B$$

$$\Delta\phi=\frac{\Delta H}{\hbar}t=g_n\frac{|e|}{2mc}\Delta Bt=2\pi$$

穿越磁场的时间

$$t=\frac{ml}{p}=\frac{\lambda\!\!\!^{-} ml}{\hbar}$$

$$\Delta\phi=g_n\frac{|e|}{2mc}\Delta B\frac{\lambda\!\!\!^{-} ml}{\hbar}=2\pi$$

$$\Delta B=\frac{4\pi c\hbar}{g_n|e|\lambda\!\!\!^{-} l}$$

补充题:

设$t=0$时刻，高斯波包波函数为:

$$\psi(x)=\frac{1}{\pi^{1/4}\sqrt{d}}\exp\left[ikx-\frac{x^2}{2d^2}\right]$$

$$\psi(x,t)=\int K(x,t;x',0)\psi(x')dx'$$

$$=\frac{1}{\pi^{1/4}\sqrt{d}}\sqrt{\frac{m}{2\pi i\hbar t}}\int\exp\left[\frac{im(x-x')^2}{2\hbar t}\right]\exp\left[ikx'-\frac{{x'}^2}{2d^2}\right]dx'$$

$$=\frac{1}{\pi^{1/4}\sqrt{d}}\sqrt{\frac{m}{2\pi i\hbar t}}\exp\left[\frac{imx^2}{2\hbar t}\right]\int\exp\left[ikx'-\frac{{x'}^2}{2d^2}-\frac{imx}{\hbar t}x'+\frac{im{x'}^2}{2\hbar t}\right]dx'$$

$$=\frac{1}{\sqrt{\left(\frac{im}{2\hbar t}-\frac{1}{2d^2}\right)}}\frac{\sqrt{\pi}}{\pi^{1/4}\sqrt{d}}\sqrt{\frac{m}{2\pi i\hbar t}}\exp\left[\frac{imx^2}{2\hbar t}-\frac{1}{2}\frac{\left(ik-\frac{imx}{\hbar t}\right)^2}{\left(\frac{im}{\hbar t}-\frac{1}{d^2}\right)}\right]$$

$$=\frac{1}{\sqrt{1+\frac{i\hbar t}{md^2}}}\frac{1}{\pi^{1/4}\sqrt{d}}\exp\left[\frac{-(x-\frac{\hbar k}{m}t)^2}{2d^2\left(1+\frac{i\hbar t}{md^2}\right)}\right]\exp\left[ik(x-\frac{\hbar k}{m}t)\right]$$

与$t=0$时刻对比，以读出其$\Delta x=\frac{d}{\sqrt{2}}\sqrt{1+\frac{\hbar^2t^2}{m^2d^4}}$

$$\varphi(p,t)=\frac{1}{\sqrt{2\pi\hbar}}\frac{1}{\sqrt{1+\frac{i\hbar t}{md^2}}}\frac{1}{\pi^{1/4}\sqrt{d}}\int dx\exp\left[-i\frac{p}{\hbar}x\right]\exp\left[\frac{-(x-\frac{\hbar k}{m}t)^2}{2d^2\left(1+\frac{i\hbar t}{md^2}\right)}\right]\exp\left[ik(x-\frac{\hbar k}{m}t)\right]$$

$$=\frac{1}{\sqrt{2\pi\hbar}}\frac{1}{\sqrt{1+\frac{i\hbar t}{md^2}}}\frac{1}{\pi^{1/4}\sqrt{d}}\exp\left[i\frac{p}{\hbar}\frac{\hbar k}{m}t\right]\int \mathrm{dx}\exp\left[-i\frac{p}{\hbar}\left(x\right.\right.$$

$$\left.\left.-\frac{\hbar k}{m}t\right)\right]\exp\left[\frac{-\left(x-\frac{\hbar k}{m}t\right)^2}{2d^2\left(1+\frac{i\hbar t}{md^2}\right)}\right]\exp\left[ik\left(x-\frac{\hbar k}{m}t\right)\right]$$

$$=\frac{1}{\sqrt{\hbar}}\frac{\sqrt{d}}{\pi^{1/4}}\exp\left[i\frac{p}{\hbar}\frac{\hbar k}{m}t\right]\exp\left[\frac{-(p-\hbar k)^2}{2\hbar^2}d^2\left(1+\frac{i\hbar t}{md^2}\right)\right]$$

$$\Delta p=\frac{\hbar}{\sqrt{2}d}$$

$$\Delta x\Delta p=\frac{\hbar}{2}\sqrt{1+\frac{\hbar^2t^2}{m^2d^4}}$$