2.9

① 类比 S_z & S_x, $H = |+\rangle\delta\langle+| - |-\rangle\delta\langle-|$

其中 $|+\rangle = \frac{1}{\sqrt{2}}(|a'\rangle + |a''\rangle)$, $|-\rangle = \frac{1}{\sqrt{2}}(|a'\rangle - |a''\rangle)$

eigenvalue: $+\delta$, $-\delta$

② $|a'\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$

$|\psi(t)\rangle = \exp\{\frac{-iHt}{\hbar}\}|a'\rangle = \frac{1}{\sqrt{2}}(e^{-i\delta t/\hbar}|+\rangle + e^{i\delta t/\hbar}|-\rangle)$

$= \frac{1}{2}[e^{-i\delta t/\hbar}(|a'\rangle + |a''\rangle) + e^{i\delta t/\hbar}(|a'\rangle - |a''\rangle)]$

$= \cos(\delta t/\hbar)|a'\rangle - i\sin(\delta t/\hbar)|a''\rangle$

③ $P = |\langle a''|\psi(t)\rangle|^2 = |-i\sin(\delta t/\hbar)|^2 = \sin^2(\delta t/\hbar)$

④ $|a'\rangle \to |S_z;+\rangle$, $|a''\rangle \to |S_z;-\rangle$, $H \to \delta S_x$

与此问题几乎等价

2.10

① 同 2.9: eigenvalue $\Delta \Rightarrow$ ket $|+\rangle = \frac{1}{\sqrt{2}}(|L\rangle + |R\rangle)$

eigenvalue $-\Delta \Rightarrow$ ket $|-\rangle = \frac{1}{\sqrt{2}}(|L\rangle - |R\rangle)$

② $|\psi(t)\rangle = \exp\{\frac{-iHt}{\hbar}\}|\alpha\rangle = e^{-i\Delta t/\hbar}|+\rangle\langle+|\alpha\rangle + e^{i\Delta t/\hbar}|-\rangle\langle-|\alpha\rangle$

$= \frac{1}{2}[e^{-i\Delta t/\hbar}(|L\rangle + |R\rangle)(\langle L|\alpha\rangle + \langle R|\alpha\rangle) +$

$e^{i\Delta t/\hbar}(|L\rangle - |R\rangle)(\langle L|\alpha\rangle - \langle R|\alpha\rangle)]$

$= \cos(\Delta t/\hbar)(|L\rangle\langle L| + |R\rangle\langle R|)|\alpha\rangle - i\sin(\Delta t/\hbar)(|L\rangle\langle R| + |R\rangle\langle L|)|\alpha\rangle$

③ $|\psi(t)\rangle = \cos(\Delta t/\hbar)|R\rangle - i\sin(\Delta t/\hbar)|L\rangle$

$P = |\langle L|\psi(t)\rangle|^2 = \sin^2(\Delta t/\hbar)$

④ $\phi_R = \langle R|\psi(t)\rangle$, $\phi_L = \langle L|\psi(t)\rangle$, 则

$i\hbar\dot{\phi}_R = \langle R|H|\psi(t)\rangle = \Delta\langle L|\psi(t)\rangle = \Delta\phi_L$, $i\hbar\dot{\phi}_L = \Delta\phi_R$

故 $\ddot{\phi}_R = -\frac{\Delta^2}{\hbar^2}\phi_R$, $\phi_R = C_0\cos\frac{\Delta}{\hbar}t + C_1\sin\frac{\Delta}{\hbar}t$, 同理 $\phi_L = C_2\cos\frac{\Delta}{\hbar}t + C_3\sin\frac{\Delta}{\hbar}t$

$|\psi(t)\rangle = \phi_R|R\rangle + \phi_L|L\rangle$, 与②中形式相同

⑤ 同 2.2，不再重复证明

2.2 违反了厄密性：$U(t_0+dt, t_0) = 1 - i\frac{Ht}{\hbar}$，$U^\dagger U = 1$ 要求 H 厄密

设 $H = a|1\rangle\langle 2|$，$H^2 = 0$

$U(t) = \exp\{-\frac{i}{\hbar}Ht\} = 1 - \frac{i}{\hbar}Ht = 1 - \frac{i}{\hbar}at|1\rangle\langle 2|$

$U^\dagger(t)U(t) = (1 + \frac{i}{\hbar}a^* t|2\rangle\langle 1|)\cdot(1 - \frac{i}{\hbar}at|1\rangle\langle 2|)$

$= 1 + \frac{|a|^2t^2}{\hbar^2}|2\rangle\langle 2| \neq 1$，不 unitary $\Rightarrow$ 不满足概率守恒

例如：

尝试作用于态 $|2\rangle$：$|\psi(t)\rangle = U(t)|2\rangle = |2\rangle - \frac{i}{\hbar}at|1\rangle$

$\langle\psi(t)|\psi(t)\rangle = 1 + \frac{|a|^2t^2}{\hbar^2} \geq 1$，故不满足概率守恒

2.12

设 $A = \exp\{\frac{-ipa}{\hbar}\}$ 则 $[x, A] = i\hbar\cdot\frac{-ia}{\hbar}A = aA$，$[p, A] = 0$

由 2.3.50，$x(t) = x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t$

故 $\langle x(t)\rangle = \langle 0|A^\dagger \exp\{\frac{iHt}{\hbar}\}x(t)\exp\{\frac{-iHt}{\hbar}\}A|0\rangle$

$= \langle 0|A^\dagger x(0)\cos\omega t A + A^\dagger\frac{p(0)}{m\omega}\sin\omega t A|0\rangle$

$= \langle 0|a\cos\omega t + x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t|0\rangle$

$= a\cos\omega t$

2.3.50 的一个简单证明：

$\frac{dx}{dt} = \frac{1}{i\hbar}[x, H] = \frac{p}{m}$　　$\frac{dp}{dt} = \frac{1}{i\hbar}[p, H] = -m\omega^2 x$

$\Rightarrow \ddot{x} = -\omega^2 x \Rightarrow x = A\cos\omega t + B\sin\omega t$.

显然 $t=0$ 时，$x = x(0)$，$\dot{x} = \frac{p(0)}{m}$ 故 $x = x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t$

2.14

① $x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$，$p = -i\sqrt{\frac{m\omega\hbar}{2}}(a - a^\dagger)$

$\langle m|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\sqrt{n}\langle m|n-1\rangle + \sqrt{n+1}\langle m|n+1\rangle)$

$\langle m|p|n\rangle = i\sqrt{\frac{m\omega\hbar}{2}}(\sqrt{n}\langle m|n-1\rangle + \sqrt{n+1}\langle m|n+1\rangle)$

$xp + px = -\frac{i\hbar}{2}(a^2 + a^\dagger a - aa^\dagger - a^{\dagger 2}) - \frac{i\hbar}{2}(a^2 - a^\dagger a + aa^\dagger - a^{\dagger 2})$

$= i\hbar(a^{\dagger 2} - a^2)$

$$\langle m|\{x,p\}|n\rangle = i\hbar\left(\sqrt{(n+1)(n+2)}\langle m|n+2\rangle - \sqrt{n(n-1)}\langle m|n-2\rangle\right)$$

$$x^2 = \frac{\hbar}{2m\omega}(a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2})$$

$$\langle m|x^2|n\rangle = \frac{\hbar}{2m\omega}\left(\sqrt{n(n-1)}\langle m|n-2\rangle + (2n+1)\langle m|n\rangle + \sqrt{(n+1)(n+2)}\langle m|n+2\rangle\right)$$

$$p^2 = -\frac{m\omega\hbar}{2}(a^2 - aa^\dagger - a^\dagger a + a^{\dagger 2})$$

$$\langle m|p^2|n\rangle = -\frac{m\omega\hbar}{2}\left(\sqrt{n(n-1)}\langle m|n-2\rangle - (2n+1)\langle m|n\rangle + \sqrt{(n+1)(n+2)}\langle m|n+2\rangle\right)$$

② $\langle n|\frac{p^2}{m}|n\rangle = \frac{1}{4}\omega\hbar(2n+1)$

$\langle n|m\omega^2x^2|n\rangle = \frac{1}{4}\omega\hbar(2n+1) = \langle n|\vec{x}\cdot\nabla V|n\rangle$

位力定理得证

2.22

① 此边界要求 $\langle x|\alpha\rangle = \psi_\alpha = 0$，当 $x \leq 0$

$x>0$ 时 ψ_α 有与一般简谐势阱类似的解

先求解一般简谐势波函数

$a|0\rangle = 0 \Rightarrow \left(x + \frac{ip}{m\omega}\right)\psi_0(x) = 0$

$x\psi_0(x) + \frac{\hbar}{m\omega}\partial_x\psi_0(x) = 0 \Rightarrow \psi_0(x) = C_0 \cdot e^{-\frac{m\omega}{2\hbar}x^2}$

C_0为归一化系数

$\psi_1(x) = \frac{1}{\sqrt{0+1}}a^\dagger\psi_0(x) = C_1\left(x - \frac{ip}{m\omega}\right)\phi_0(x)$

$= C_1 \cdot xe^{-\frac{m\omega}{2\hbar}x^2}$

后续项不逐一列出，可证明 $\begin{cases}\psi_{2k}(0) \neq 0 \\ \psi_{2k+1}(0) = 0\end{cases}$ $k \in N$

因此选奇数波函数作为 $x>0$ 的解

$\phi_{2k+1}(x) = \begin{cases}\sqrt{2}\psi_{2k+1}(x), & x>0 \\ 0, & x \leq 0\end{cases}$

ϕ_1为基态，基态能量 $\frac{3}{2}\hbar\omega$

② 对基态有 $\int_0^\infty C_1^2 x^2 e^{-\frac{m\omega}{\hbar}x^2}dx = 1 \Rightarrow$

故 $\langle x^2\rangle = \int_0^\infty C_1^2 x^4 \ e^{-\frac{m\omega}{\hbar}x^2}dx$

$= -C_1^2\frac{\hbar}{2m\omega}x^3 e^{-\frac{m\omega}{\hbar}x^2}\Big|_0^\infty + \int_0^\infty C_1^2\frac{3\hbar}{2m\omega}x^2 e^{-\frac{m\omega}{\hbar}x^2}dx$

$= \frac{3\hbar}{2m\omega}$

2.23

由 $\phi_n(0)=\phi(L)=0$ 边界条件有

$$\phi_n = \begin{cases}\sqrt{\frac{2}{L}}\sin\frac{n\pi}{L}x & ,0\le x\le L\\ 0, & \text{other}\end{cases} \qquad E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

$\psi(0) = \delta(x-L/2)$, $\psi(0) = \sum_n c_n\phi_n(x)$

$\langle\phi_n(x)|\psi(0)\rangle = \sqrt{\frac{2}{L}}\sin\frac{n\pi}{2} = C_n$

$\psi(t) = \sum_n \frac{2}{L}\sin\frac{n\pi}{2} e^{-i\frac{n^2\pi^2\hbar t}{2mL^2}}\sin\frac{n\pi}{L}x$, $0\le x\le L$