习题 V

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成绩：________

第 1 题 得分: ________. 证明体积为 V，温度为 T 的辐射场有以下关系：

$$E = V\frac{\pi^2(kT)^4}{15(\hbar c)^3}$$
$$F = -\frac{1}{3}E$$
$$S = \frac{4}{3}\frac{E}{T}$$
$$P = \frac{1}{3}\frac{E}{V}$$

证: 辐射场即光子气体，光子是玻色子，达到平衡后遵从玻色分布. 由于容器壁不断发射和吸收光子，光子数不守恒，在导出其分布时，不存在对总粒子数的约束条件，故仅有一个拉格朗日乘子，自由能 $\mu = 0$. 辐射场的巨配分函数为

$$\mathcal{Q} = \prod_\alpha \sum_{n_\alpha=1}^{\infty} e^{-\beta n_\alpha \varepsilon_\alpha} = \prod_\alpha \frac{1}{1-e^{-\beta\varepsilon_\alpha}}. \tag{1}$$

光子的能量可表为

$$\varepsilon = cp = c\hbar K. \tag{2}$$

考虑到光子的自旋量子数为 1，自旋在动量方向的投影可取 $\pm\hbar$ 两个值，相当于左右圆偏振，对光子所有状态的求和可表为

$$\begin{aligned} V\int_0^\infty \mathrm{d}\varepsilon\, D(\varepsilon)(\cdots) =& 2V\int \frac{\mathrm{d}^3\vec{K}}{(2\pi)^3}(\cdots) = \frac{2V}{(2\pi)^3}\int_0^{2\pi}\mathrm{d}\varphi\int_0^\pi \sin\theta\,\mathrm{d}\theta\int_0^\infty \mathrm{d}K\,K^2(\cdots) \\ =& \frac{V}{\pi^2}\int_0^\infty \mathrm{d}K\,K^2(\cdots) = \frac{V}{\pi^2\hbar^3c^3}\int_0^\infty \mathrm{d}\varepsilon\,\varepsilon^2(\cdots) \end{aligned} \tag{3}$$

从而得到态密度

$$D(\varepsilon) = \frac{\varepsilon^2}{\pi^2\hbar^3c^3}. \tag{4}$$

内能为

$$\begin{aligned} E =& -\left(\frac{\partial}{\partial\beta}\ln\mathcal{Q}\right) = \sum_\alpha \frac{\varepsilon_\alpha}{e^{\beta\varepsilon_\alpha}-1} = V\int_0^\infty \mathrm{d}\varepsilon\frac{D(\varepsilon)\varepsilon}{e^{\beta\varepsilon}-1} = \frac{V}{\pi^2\hbar^3c^3}\int_0^\infty \mathrm{d}\varepsilon\frac{\varepsilon^3}{e^{\beta\varepsilon}-1} \\ & (\text{令 } x = \beta\varepsilon = \frac{\varepsilon}{kT}) \\ =& \frac{V(kT)^4}{\pi^2\hbar^3c^3}\int_0^\infty \mathrm{d}x\frac{x^3}{e^x-1} = \frac{V(kT)^4}{\pi^2\hbar^3c^3}\times\frac{\pi^4}{15} = V\frac{\pi^2(kT)^4}{15(\hbar c)^3}. \end{aligned} \tag{5}$$

压强为

$$\begin{aligned} P =& \frac{kT\ln Q}{V} = \frac{kT}{V}\sum_\alpha \ln\left(\frac{1}{1-e^{-\beta\varepsilon_\alpha}}\right) = -kT\int_0^\infty \mathrm{d}\varepsilon\,D(\varepsilon)\ln(1-e^{-\beta\varepsilon}) = -\frac{kT}{\pi^2\hbar^3c^3}\int_0^\infty \mathrm{d}\varepsilon\,\varepsilon^2\ln(1-e^{-\beta\varepsilon}) \\ =& -\frac{kT}{\pi^2\hbar^3c^3}\int_0^\infty \ln(1-e^{-\beta\varepsilon})\,\mathrm{d}\left(\frac{\varepsilon^3}{3}\right) = -\frac{kT}{\pi^2\hbar^3c^3}\frac{1}{3}\left\{\left[\varepsilon^3\ln(1-e^{-\beta\varepsilon})\right]\Big|_0^\infty - \int_0^\infty \varepsilon^3\,\mathrm{d}\ln(1-e^{-\beta\varepsilon})\right\} \\ =& \frac{1}{\pi^2\hbar^3c^3}\frac{1}{3}\int_0^\infty \mathrm{d}\varepsilon\frac{\varepsilon^3}{e^{\beta\varepsilon}-1} = \frac{1}{3}\frac{E}{V}. \end{aligned}$$

熵为

$$S = k(\ln\mathcal{Q} + \beta U) = \frac{PV}{T} + \frac{E}{T} = \frac{4}{3}\frac{E}{T}. \tag{6}$$

Helmholtz 自由能为

$$F = U - TS = -\frac{1}{3}E. \tag{7}$$

□

第 2 题 得分: ________. 考虑两维自旋为零的自由 Boson 系统

1) 推导单位面积的态密度公式;
2) 推导粒子数密度（面密度）用温度和易逸度表达的公式;
3) 证明此系统无凝聚现象.

解: 1) 两维自旋为零的自由 Boson 系统中的单个粒子的能量可表为

$$\varepsilon = \frac{\hbar^2 K^2}{2m}. \tag{8}$$

对所有状态的求和可表为

$$A\int\frac{\mathrm{d}^2\vec{K}}{(2\pi)^2} = \frac{A}{(2\pi)^2}\int_0^{2\pi}\mathrm{d}\theta\int_0^\infty K\,\mathrm{d}K = \frac{A}{2\pi}\int_0^\infty\frac{\sqrt{2m\varepsilon}}{\hbar}\mathrm{d}\left(\frac{\sqrt{2m\varepsilon}}{\hbar}\right) = \frac{Am}{2\pi\hbar^2}\int_0^\infty\mathrm{d}\varepsilon = A\int_0^\infty D(\varepsilon)\,\mathrm{d}\varepsilon, \tag{9}$$

其中 A 为系统的面积，从而得到态密度

$$D(\varepsilon) = \frac{m}{2\pi\hbar^2}. \tag{10}$$

2) 粒子数面密度为

$$\begin{aligned}\rho =& \int_0^\infty\frac{D(\varepsilon)}{e^{\beta(\varepsilon-\mu)}-1}\,\mathrm{d}\varepsilon = \frac{m}{2\pi\hbar^2}\int_0^\infty\mathrm{d}\varepsilon\frac{ze^{-\beta\varepsilon}}{1-ze^{-\beta\varepsilon}}\\ &(\text{令 } x=\beta\varepsilon=\frac{\varepsilon}{kT})\\ =&\frac{mkT}{2\pi\hbar^2}\int_0^\infty\frac{ze^{-x}}{1-ze^{-x}} = \frac{mkT}{2\pi\hbar^2}\ln(1-ze^{-x})\big|_0^\infty = \frac{mkT}{2\pi\hbar^2}\ln\frac{1}{1-z} = \frac{1}{\lambda^2}\ln\frac{1}{1-z}.\end{aligned}$$

其中 $\lambda = \sqrt{\frac{2\pi}{mkT}}\hbar$.

3) 对于玻色子，$\mu\leq 0 \Longrightarrow 0\leq z=e^{\beta\mu}\leq 1 \Longrightarrow$对任意温度和粒子数密度，均存在

$$z = 1 - e^{-\rho z^2}, \tag{11}$$

即，以上模型对任意温度和粒子数密度均成立，而不存在温度的下限或粒子数密度的上限，故该系统无凝聚现象.

□

第 3 题 得分: ________. 证明在高温或低密度区域（$\rho\lambda^3 \ll 1$），自旋为 j 的非相对论自由量子气体的状态方程和熵由下列两式给出:

$$PV = NkT\left[1 \pm \frac{\rho\lambda^3}{2^{5/2}(2j+1)} + \cdots\right]$$
$$S = Nk\ln\frac{(2j+1)e^{\frac{5}{2}}}{\rho\lambda^3} \pm Nk\frac{\rho\lambda^3}{2^{\frac{7}{2}}(2j+1)} + \cdots$$

其中上边的符号对应于 Fermions，下边的符号对应于 Bosons，λ 为热波长，$\cdots$ 代表 $\rho\lambda^3$ 的更高阶项.

证: 自旋为 j 的非相对论自由量子气体压强与体积之积为

$$\frac{P}{kT}=\frac{2j+1}{\lambda^3}\left(z\mp\frac{z^2}{2^{5/2}}+\cdots\right). \tag{12}$$

其中上面的符号对应于费米子，下面的符号对应于玻色子，下同. 粒子数密度为

$$\rho=\frac{N}{V}=\frac{2j+1}{\lambda^3}\left(z\mp\frac{z^2}{2^{3/2}}+\cdots\right). \tag{13}$$

故

$$z=\frac{\rho\lambda^3}{2j+1}\pm\frac{1}{2^{3/2}}\left(\frac{\rho\lambda^3}{2j+1}\right)^2+\cdots \tag{14}$$

将上式代入压强与体积之积的表达式可得

$$PV=NkT\left[1\pm\frac{\rho\lambda^3}{2^{5/2}(2j+1)}+\cdots\right]. \tag{15}$$

系统的吉布斯势可表为

$$G=N\mu=U+PV-TS, \tag{16}$$

其中系统的粒子数 N 和总能量（内能） U 是确定的，故单位体积的熵密度

$$s=\frac{S}{V}=\left(\frac{\partial P}{T}\right)_\mu. \tag{17}$$

从前面式 (12) 中可见，P 为 T 和 z 的函数（注意 $\lambda=\sqrt{\frac{2\pi}{mkT}}\hbar$，故实际上 $P\propto\frac{5}{2}T$），而 $z=e^{\mu/kT}$ 又为 T 的函数，故

$$s=\left(\frac{\partial P}{\partial T}\right)_{\mu,z}+\left(\frac{\partial P}{\partial z}\right)_{\mu,T}\left(\frac{\partial z}{\partial T}\right)_\mu, \tag{18}$$

其中

$$\left(\frac{\partial P}{\partial T}\right)_{\mu,z}=\frac{5}{2}\frac{P}{T}, \tag{19}$$

$$\left(\frac{\partial P}{\partial z}\right)_{\mu,T}=\frac{kT}{V}\left(\frac{\partial\ln\mathcal{Q}}{\partial z}\right)=\frac{kT}{zV}\left(z\frac{\partial}{\partial z}\ln\mathcal{Q}\right)=\frac{kT\rho}{z}, \tag{20}$$

$$\left(\frac{\partial z}{\partial T}\right)_\mu=\left(\frac{\partial e^{\mu/kT}}{\partial T}\right)_\mu=-\frac{\mu}{kT^2}e^{\mu/kT}=-\frac{\mu}{kT^2}z, \tag{21}$$

$$\begin{aligned}\mu=&kT\ln z=kT\ln\left[\frac{\rho\lambda^3}{2j+1}\pm\frac{1}{2^{3/2}}\left(\frac{\rho\lambda^3}{2j+1}\right)^2+\cdots\right]=kT\left[\ln\frac{\rho\lambda^3}{2j+1}+\ln\left(1\pm\frac{\rho\lambda^3}{2^{3/2}(2j+1)}\right)\right]\\ \approx&\,kT\left[\ln\frac{\rho\lambda^3}{2j+1}\pm\frac{\rho\lambda^3}{2^{3/2}(2j+1)}\right]\end{aligned} \tag{22}$$

从而

$$\begin{aligned}s=&\frac{5}{2}\frac{P}{T}-\frac{kT\rho}{z}\frac{\mu}{kT^2}z=\frac{5}{2}\frac{Nk}{V}\left(1\pm\frac{1}{2^{5/2}}\frac{\rho\lambda^3}{2j+1}+\cdots\right)-\frac{\rho\mu}{T}\\ =&\frac{5}{2}\frac{Nk}{V}\left(1\pm\frac{1}{2^{5/2}}\frac{\rho\lambda^3}{2j+1}+\cdots\right)-\frac{Nk}{V}\left[\ln\frac{\rho\lambda^3}{2j+1}\pm\frac{\rho\lambda^3}{2^{3/2}(2j+1)}\right]\\ =&k\rho\left[\ln\frac{(2j+1)e^{5/2}}{\rho\lambda^2}\pm\frac{\rho\lambda^3}{2^{7/2}(2j+1)}+\cdots\right].\end{aligned} \tag{23}$$

系统的熵为

$$S=sV=Nk\ln\frac{(2j+1)e^{5/2}}{\rho\lambda^3}\pm Nk\frac{\rho\lambda^3}{2^{7/2}(2j+1)}+\cdots. \tag{24}$$

□