习题 IX

截止时间：2021. 5. 4（周二）

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成绩：________

第 1 题 得分: ________. 一稀薄气体处于外力场内，相应的势能为 $V(\vec{r})$. 假设 $V(\vec{r})$ 在分子相互作用力程范围内的变化很小，求出 Boltzmann 方程的近似解并用平均数密度 n，平均动量 $\vec{p}_0$ 和 $\vec{p}_0=0$ 时的平均动能表示所得的解.

解: 势能 $V(\vec{r})$ 分布下的外力场为

$$\vec{F}=-\vec{\nabla}V(\vec{r}). \tag{1}$$

故 Boltzmann 方程

$$\begin{aligned}&\frac{\partial}{\partial t}f(\vec{r},\vec{p};t)+\left(\frac{\vec{p}}{m}\cdot\vec{\nabla}_{\vec{r}}+\vec{F}\cdot\vec{\nabla}_{\vec{p}}\right)f(\vec{r},\vec{p};t)\\=&\int \mathrm{d}^3\vec{p}_2\,\mathrm{d}^3\vec{p}_1'\,\mathrm{d}^3\vec{p}_2'\,|T_{if}|^2\,\delta^4(P'-P)[f(\vec{r},\vec{p}_1';t)f(\vec{r},\vec{p}_2';t)-f(\vec{r},\vec{p};t)f(\vec{r},\vec{p}_2;t)]\end{aligned} \tag{2}$$

可化为

$$\begin{aligned}&\frac{\partial}{\partial t}f(\vec{r},\vec{p};t)+\left(\frac{\vec{p}}{m}\cdot\vec{\nabla}_{\vec{r}}-\vec{\nabla}_{\vec{p}}V(\vec{r})\cdot\vec{\nabla}_{\vec{p}}\right)f(\vec{r},\vec{p};t)\\=&\int \mathrm{d}^3\vec{p}_2\,\mathrm{d}^3\vec{p}_1'\,\mathrm{d}^3\vec{p}_2'\,|T_{if}|^2\,\delta^4(P'-P)[f(\vec{r},\vec{p}_1';t)f(\vec{r},\vec{p}_2';t)-f(\vec{r},\vec{p};t)f(\vec{r},\vec{p}_2;t)]\end{aligned} \tag{3}$$

设 Boltzmann 方程的试探解为

$$f(\vec{r},\vec{p};t)=C_1\rho(\vec{r},t)e^{-A(\vec{p}-\vec{p}_0)^2}, \tag{4}$$

其中 A 和 C 为待定常数. 坐标空间的粒子数密度可表为

$$\rho(\vec{r})=C_1\rho(\vec{r},t)\int \mathrm{d}^3\vec{p}\,e^{-A(\vec{p}-\vec{p}_0)^2}=C_1\rho(\vec{r},t)\left(\frac{\pi}{A}\right)^{3/2}, \tag{5}$$

$$\Longrightarrow C_1=\left(\frac{A}{\pi}\right)^{3/2}. \tag{6}$$

平均动量即为 $\vec{p}_0$：

$$\langle\vec{p}\rangle=\frac{\int \mathrm{d}^3\vec{p}\,\vec{p}f(\vec{r},\vec{p};t)}{\int \mathrm{d}^3\vec{p}\,f(\vec{r},\vec{p};t)}=\frac{\int \mathrm{d}^3\vec{p}\,\vec{p}e^{-A(\vec{p}-\vec{p}_0)^2}}{\int \mathrm{d}^3\vec{p}\,e^{-A(\vec{p}-\vec{p}_0)^2}}=\vec{p}_0. \tag{7}$$

$\vec{p}_0=0$ 时的平均动能可表为

$$\varepsilon=\left\langle\frac{p^2}{2m}\right\rangle=\frac{\int \mathrm{d}^3\vec{p}\,\frac{p^2}{2m}f(\vec{r},\vec{p};t)}{\int \mathrm{d}^3\vec{p}\,f(\vec{r},\vec{p};t)}=\frac{\int \mathrm{d}^3\vec{p}\,\frac{p^2}{2m}e^{-A\vec{p}^2}}{\int \mathrm{d}^3\vec{p}\,e^{-A\vec{p}^2}}=\frac{3}{4Am}, \tag{8}$$

$$\Longrightarrow A=\frac{3}{4m\varepsilon}. \tag{9}$$

在这一试探解下，

$$f(\vec{r},\vec{p}_1';t)f(\vec{r},\vec{p}_2';t)=f(\vec{r},\vec{p};t)f(\vec{r},\vec{p}_2;t), \tag{10}$$

故 Boltzmann 方程右侧等于零，从而

$$\frac{\partial}{\partial t}f(\vec{r},\vec{p};t)+\frac{\vec{p}}{m}\cdot\vec{\nabla}_{\vec{r}}f(\vec{r},\vec{p};t)-\vec{\nabla}_{\vec{r}}V(\vec{r})\cdot\vec{\nabla}_{\vec{p}}f(\vec{r},\vec{p};t)=0, \tag{11}$$

$$\Longrightarrow\frac{\partial}{\partial t}\rho(\vec{r},t)+\frac{\vec{p}}{m}\cdot\vec{\nabla}_{\vec{r}}\rho(\vec{r},t)-\vec{\nabla}_{\vec{r}}V(\vec{r})\cdot[-2A(\vec{p}-\vec{p}_0)]\,\rho(\vec{r},t)=0. \tag{12}$$

当 $\vec{p}_0=0$ 时，可取

$$\rho(\vec{r})=C_2e^{-2mAV(\vec{r})}, \tag{13}$$

从而

$$f(\vec{r},\vec{p};t)=\frac{3C_2}{4\pi m\varepsilon}\exp\left\{-\frac{3}{2\varepsilon}\left[\frac{(\vec{p}-\vec{p}_0)^2}{2m}+V(\vec{r})\right]\right\}. \tag{14}$$

其中 C 为待定系数. 对于一般情况，$\vec{p}_0\neq 0$，需作变换

$$\vec{p}\rightarrow\vec{p}-\vec{p}_0, \tag{15}$$

$$\vec{r}\rightarrow\vec{r}-\frac{1}{m}\vec{p}_0 t \tag{16}$$

以保证式 (12) 仍然成立. 此时，

$$\rho(\vec{r},t)=C_2\exp\left[-2mAV\left(\vec{r}-\frac{1}{m}\vec{p}_0 t\right)\right], \tag{17}$$

从而

$$f(\vec{r},\vec{p};t)=\frac{3C_2}{4\pi m\varepsilon}\exp\left\{-\frac{3}{2\varepsilon}\left[\frac{(\vec{p}-\vec{p}_0)^2}{2m}+V\left(\vec{r}-\frac{1}{m}\vec{p}_0 t\right)\right]\right\}. \tag{18}$$

其中 ε 为 $\vec{p}_0=0$ 时的平均动能，待定系数 C_2 满足归一化条件

$$\int \mathrm{d}^3\vec{r}\,\mathrm{d}\vec{p}\,f(\vec{r},\vec{p};t)=N. \tag{19}$$

□

第 2 题 得分: ________. 写下一个均匀且无外力作用的气体的 Boltzmann 方程并证明下列 Boltzmann H - 定理：

$$\frac{\mathrm{d}H}{\mathrm{d}t}\leq 0$$

其中

$$H\equiv\int \mathrm{d}^3\vec{p}\,f(\vec{p},t)\ln f(\vec{p},t).$$

证: 对于一个均匀且无外力作用的气体，

$$\vec{\nabla}_{\vec{r}}f=0, \tag{20}$$

$$\vec{F}=0. \tag{21}$$

故 Boltzmann 方程

$$\begin{aligned}&\frac{\partial}{\partial t}f(\vec{r},\vec{p};t)+\left(\frac{\vec{p}}{m}\cdot\vec{\nabla}_{\vec{r}}+\vec{F}\cdot\vec{\nabla}_{\vec{p}}\right)f(\vec{r},\vec{p};t)\\=&\int \mathrm{d}^3\vec{p}_2\,\mathrm{d}^3\vec{p}_1'\,\mathrm{d}^3\vec{p}_2'\,\left|T_{if}\right|^2\delta^4(P'-P)[f(\vec{r},\vec{p}_1';t)f(\vec{r},\vec{p}_2';t)-f(\vec{r},\vec{p};t)f(\vec{r},\vec{p}_2;t)]\end{aligned} \tag{22}$$

可化为

$$\frac{\partial}{\partial t}f(\vec{p},t)=\int \mathrm{d}^3\vec{p}_2\,\mathrm{d}^3\vec{p}_1'\,\mathrm{d}^3\vec{p}_2'\,\left|T_{if}\right|^2\delta^4(P'-P)[f(\vec{p}_1',t)f(\vec{p}_2',t)-f(\vec{p},t)f(\vec{p}_2,t)]. \tag{23}$$

$\frac{\mathrm{d}H}{\mathrm{d}t}$ 可表为

$$\frac{\mathrm{d}H}{\mathrm{d}t}=\int \mathrm{d}^3\vec{p}\,\frac{\mathrm{d}f(\vec{p},t)}{\mathrm{d}t}[\ln f(\vec{p},t)+1], \tag{24}$$

其中

$$\int \mathrm{d}^3\vec{p}\frac{\mathrm{d}f(\vec{p},t)}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\vec{p}\,f(\vec{p})=\frac{\partial N}{\partial t}=0, \tag{25}$$

故

$$\frac{\mathrm{d}H}{\mathrm{d}t}=\int\mathrm{d}^3\vec{p}\,\frac{\mathrm{d}f(\vec{p},t)}{\mathrm{d}t}\ln f(\vec{p},t). \tag{26}$$

将 Boltzmann 方程 (23) 代入上式得

$$\frac{\mathrm{d}H}{\mathrm{d}t}=\int\mathrm{d}^3\vec{p}\,\mathrm{d}^3\vec{p}_2\,\mathrm{d}^3\vec{p}_1'\,\mathrm{d}^3\vec{p}_2'\,\left|T_{if}\right|^2\delta(P'-P)[f(\vec{p}_1',t)f(\vec{p}_2',t)-f(\vec{p},t)f(\vec{p}_2,t)]\ln f(\vec{p},t) \tag{27}$$

根据 H 的定义，$f(\vec{p})$ 相对于 $f(\vec{p}_2)$ 并没有特殊性，由于上式中同时对 $\mathrm{d}^3\vec{p}$ 和 $\mathrm{d}^3\vec{p}_2$ 进行积分，根据对称性，交换碰撞的两个粒子的状态，即交换 $\vec{p}$ 和 $\vec{p}_2$，$\vec{p}_1$ 和 $\vec{p}_2$，上面的等式仍成立，

$$\frac{\mathrm{d}H}{\mathrm{d}t}=\int\mathrm{d}^3\vec{p}_2\,\mathrm{d}^3\vec{p}\,\mathrm{d}^3\vec{p}_2'\,\mathrm{d}^3\vec{p}_1'\,\left|T_{ij}\right|^2\delta(P'-P)[f(\vec{p}_2',t)f(\vec{p}_1',t)-f(\vec{p}_2,t)f(\vec{p},t)]\ln f(\vec{p}_2,t) \tag{28}$$

$\frac{\mathrm{d}H}{\mathrm{d}t}$ 又可表为

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t}=&\frac{1}{2}[\text{式 }(27)+\text{式 }(28)]\\=&\frac{1}{2}\int\mathrm{d}^3\vec{p}\,\mathrm{d}^3\vec{p}_2\,\mathrm{d}^3\vec{p}_1'\,\mathrm{d}^3\vec{p}_2'\,\left|T_{ij}\right|^2\delta(P'-P)[f(\vec{p}_1',t)f(\vec{p}_2',t)-f(\vec{p},t)f(\vec{p}_2',t)]\ln[f(\vec{p},t)f(\vec{p}_2,t)].\end{aligned} \tag{29}$$

根据 H 的定义，$(\vec{p},\vec{p}_2)$ 相对于 $(\vec{p}_1',\vec{p}_2')$ 并没有特殊性，由于上式中同时对 $\mathrm{d}^3\vec{p}$，$\mathrm{d}^3\vec{p}_2$，$\mathrm{d}^3\vec{p}_1'$ 和 $\mathrm{d}^3\vec{p}_2'$ 进行积分，根据对称性，交换 $(\vec{p},\vec{p}_2)$ 和 $(\vec{p}_1,\vec{p}_2)$，上面的等式仍然成立，

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t}=&\frac{1}{2}\int\mathrm{d}^3\vec{p}_1'\,\mathrm{d}^3\vec{p}_2'\,\mathrm{d}^3\vec{p}\,\mathrm{d}^3\vec{p}_2\,\left|T_{ji}\right|^2\delta(P-P')[f(\vec{p},t)f(\vec{p}_2,t)-f(\vec{p}_1',t)f(\vec{p}_2',t)]\ln[f(\vec{p}_1',t)f(\vec{p}_2',t)]\\=&\frac{1}{2}\int\mathrm{d}^3\vec{p}_1'\,\mathrm{d}^3\vec{p}_2'\,\mathrm{d}^3\vec{p}\,\mathrm{d}^3\vec{p}_2\,\left|T_{ij}\right|^2\delta(P'-P)[f(\vec{p},t)f(\vec{p}_2,t)-f(\vec{p}_1',t)f(\vec{p}_2',t)]\ln[f(\vec{p}_1',t)f(\vec{p}_2',t)]\end{aligned} \tag{30}$$

$\frac{\mathrm{d}H}{\mathrm{d}t}$ 又可表为

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t}=&\frac{1}{2}[\text{式 }(29)+\text{式 }(30)]\\=&\frac{1}{4}\int\mathrm{d}^3\vec{p}\,\mathrm{d}^3\vec{p}_2\,\mathrm{d}^3\vec{p}_1'\,\mathrm{d}^3\vec{p}_2'\,\left|T_{ij}\right|\delta(P'-P)[f(\vec{p}_1',t)f(\vec{p}_2',t)-f(\vec{p},t)f(\vec{p}_2,t)]\ln\frac{f(\vec{p},t)f(\vec{p}_2,t)}{f(\vec{p}_1',t)f(\vec{p}_2',t)}.\end{aligned} \tag{31}$$

当 $f(\vec{p}_1',t)f(\vec{p}_2',t)\geq f(\vec{p},t)f(\vec{p}_2,t)$，$[f(\vec{p}_1',t)f(\vec{p}_2',t)-f(\vec{p},t)f(\vec{p}_2,t)]\geq 0$，$\ln\frac{f(\vec{p},t)f(\vec{p}_2,t)}{f(\vec{p}_1',t)f(\vec{p}_2',t)}\leq 0$；当 $f(\vec{p}_1',t)f(\vec{p}_2',t)<f(\vec{p},t)f(\vec{p}_2,t)$，$[f(\vec{p}_1',t)f(\vec{p}_2',t)-f(\vec{p},t)f(\vec{p}_2,t)]<0$，$\ln\frac{f(\vec{p},t)f(\vec{p}_2,t)}{f(\vec{p}_1',t)f(\vec{p}_2',t)}>0$，故

$$\frac{\mathrm{d}H}{\mathrm{d}t}=\frac{1}{4}\int\mathrm{d}^3\vec{p}\,\mathrm{d}^3\vec{p}_2\,\mathrm{d}^3\vec{p}_1'\,\mathrm{d}^3\vec{p}_2'\,\left|T_{ij}\right|\delta(P'-P)[f(\vec{p}_1',t)f(\vec{p}_2',t)-f(\vec{p},t)f(\vec{p}_2,t)]\ln\frac{f(\vec{p},t)f(\vec{p}_2,t)}{f(\vec{p}_1',t)f(\vec{p}_2',t)}\leq 0. \tag{32}$$

□