习题 XIV

截止时间：2021. 6. 8（周二）

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成绩：________

第 1 题 得分: ________. 设 $|z\rangle = c_0 e^{za^\dagger}|0\rangle$ 为 Boson 的相干态，证明

i) 归一化系数 $|c_0| = e^{-\frac{1}{2}|z|^2}$.

ii) 粒子数平均值为 $\langle z|a^\dagger a|z\rangle = |z|^2$.

iii) 求粒子数在 $|z\rangle$ 中分布的涨落.

证: i) Boson 的相干态可写为

$$|z\rangle = e^{za^\dagger}|0\rangle = c_0\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle. \tag{1}$$

由归一化条件

$$\langle z|z\rangle = |c_0|^2\sum_{n=0}^{\infty}\frac{z^{*n}}{\sqrt{n!}}\sum_{m=0}^{\infty}\frac{z^m}{\sqrt{m!}}\langle n|m\rangle = |c_0|^2\sum_{n=0}^{\infty}\frac{z^{*n}}{\sqrt{n!}}\sum_{m=0}^{\infty}\frac{z^m}{\sqrt{m!}}\delta_{nm} = |c_0|^2\sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!} = |c_0|^2 e^{|z|^2} = 1, \tag{2}$$

得归一化系数

$$|c_0| = e^{-\frac{1}{2}|z|^2}. \tag{3}$$

ii) 粒子数平均值为

$$\begin{aligned}\langle n\rangle =&\langle z|a^\dagger a|z\rangle = |c_0|^2\sum_{n=0}^{\infty}\frac{z^{*n}}{\sqrt{n!}}\sum_{m=0}^{\infty}\frac{z^m}{\sqrt{m!}}\langle n|a^\dagger a|m\rangle = |c_0|^2\sum_{n=0}^{\infty}\frac{z^{*n}}{\sqrt{n!}}\sum_{m=0}^{\infty}\frac{z^m}{\sqrt{m!}}m\delta_{nm} = |c_0|^2\sum_{n=1}^{\infty}\frac{|z|^{2n}}{(n-1)!}\\ =&|z|^2|c_0|^2\sum_{n=1}^{\infty}\frac{|z|^{2(n-1)}}{(n-1)!} = |z|^2|c_0|^2\sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!} = |z|^2|c_0|^2 e^{|z|^2} = |z|^2.\end{aligned} \tag{4}$$

iii) 粒子数平方的平均值为

$$\langle n^2\rangle =\langle z|a^\dagger a a^\dagger a|z\rangle = |c_0|^2\sum_{n=0}^{\infty}\frac{z^{*n}}{\sqrt{n!}}\sum_{m=0}^{\infty}\frac{z^m}{\sqrt{m!}}\langle n|a^\dagger a a^\dagger a|m\rangle = |c_0|^2\sum_{n=0}^{\infty}\frac{z^{*n}}{\sqrt{n!}}\sum_{m=0}^{\infty}\frac{z^m}{\sqrt{m!}}m^2\delta_{nm} \tag{5}$$

$$=|c_0|^2\sum_{n=0}^{\infty}\frac{n^2|z|^{2n}}{n!} = |c_0|^2 e^{|z|^2}|z|^2(|z|^2+1) = |z|^2(|z|^2+1). \tag{6}$$

粒子数的涨落为

$$\frac{\Delta n}{\langle n\rangle} = \frac{\sqrt{\langle n^2\rangle - \langle n\rangle^2}}{\langle n\rangle} = \frac{1}{|z|}. \tag{7}$$

□

第 2 题 得分: ________. 一个电子系统由下列 Hamiltonian 描述

$$\mathcal{H} = -\frac{\hbar^2}{2m}\sum_s\int \mathrm{d}^3\,\vec{r}\,\psi_s^\dagger(\vec{r})\nabla^2\psi_s(\vec{r}) + \frac{1}{2}\sum_{s_1,s_2}\int \mathrm{d}^3\vec{r}_1\,\mathrm{d}^3\vec{r}_2\,\psi_{s_1}^\dagger(\vec{r}_1)\psi_{s_2}^\dagger(\vec{r}_2)u(|\vec{r}_1-\vec{r}_2|)\psi_{s_2}(\vec{r}_2)\psi_{s_1}(\vec{r}_1) \tag{8}$$

其中 s 代表自旋的两个分量. 试写出此 Hamiltonian 用动量-自旋态的湮灭产生算符 $a_{\vec{p},s}$, $a_{\vec{p},s}^\dagger$ 和 $u(|\vec{r}_1-\vec{r}_2|)$ 的 Fourier 变换 $u_{\vec{q}} = \int \mathrm{d}^3\vec{r}\,e^{-\frac{i}{\hbar}\vec{q}\cdot\vec{r}}u(r)$ 表示的形式.

证: 动能

$$K \equiv -\frac{\hbar^2}{2m}\sum_s\int \mathrm{d}^3\vec{r}\,\psi_s^\dagger(\vec{r})\nabla^2\psi_s(\vec{r}), \tag{9}$$

其中场算符

$$\psi_s(\vec{r}) = \frac{1}{\sqrt{V}}\sum_{\vec{p}} a_{\vec{p},s} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}, \tag{10}$$

$$\psi_s^\dagger(\vec{r}) = \frac{1}{\sqrt{V}}\sum_{\vec{p}} a_{\vec{p},s}^\dagger e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{r}}, \tag{11}$$

而动量算符

$$\nabla = \frac{i}{\hbar}\hat{\vec{p}}. \tag{12}$$

代入动能（式 (8)）中可得

$$K = \sum_s \sum_{\vec{p},\vec{p}'} \frac{p^2}{2m} a_{\vec{p}',s}^\dagger a_{\vec{p},s} \frac{1}{V}\int \mathrm{d}\vec{r}\, e^{\frac{i}{\hbar}(\vec{p}-\vec{p}')\cdot\vec{r}} = \sum_s \sum_{\vec{p},\vec{p}'} \frac{p^2}{2m} a_{\vec{p}',s}^\dagger a_{\vec{p},s}\delta(\vec{p}-\vec{p}') = \sum_s\sum_{\vec{p}} \frac{p^2}{2m} a_{\vec{p},s}^\dagger a_{\vec{p},s}. \tag{13}$$

相互作用能

$$\Omega \equiv \frac{1}{2}\sum_{s_1,s_2}\int \mathrm{d}^3\vec{r}_1\,\mathrm{d}^3\vec{r}_2\,\psi_{s_1}^\dagger(\vec{r}_1)\psi_{s_2}(\vec{r}_2)u(|\vec{r}_1-\vec{r}_2|)\psi_{s_2}(\vec{r}_2)\psi_{s_1}(\vec{r}_1), \tag{14}$$

其中相互作用势的 Fourier 变换

$$u(|\vec{r}_1-\vec{r}_2|) = \frac{1}{V}\int \mathrm{d}^3\,\vec{q} e^{\frac{i}{\hbar}\vec{q}\cdot\vec{r}} u_{\vec{q}}. \tag{15}$$

将上式和场算符（式 (10), (11)）代入相互作用能（式 (14)）中可得

$$\begin{aligned}
K =& \frac{1}{2}\sum_{s_1,s_2}\sum_{\vec{p}_1,\vec{p}_1',\vec{p}_2,\vec{p}_2'} a_{\vec{p}_1',s_1}^\dagger a_{\vec{p}_2',s_2}^\dagger a_{\vec{p}_2,s_2} a_{\vec{p}_1,s_1}\frac{1}{V^2}\int \mathrm{d}^3\vec{r}_1\, e^{\frac{i}{\hbar}(\vec{p}_1-\vec{p}_1')\cdot\vec{r}_1}\int \mathrm{d}^3\vec{r}_2\, e^{\frac{i}{\hbar}(\vec{p}_2-\vec{p}_2')\cdot\vec{r}_2}\int \mathrm{d}^3\vec{q}\, e^{-\frac{i}{\hbar}\vec{q}\cdot\vec{r}} u_{\vec{q}} \\
=& \frac{1}{2}\sum_{s_1,s_2}\sum_{\vec{p}_1,\vec{p}_1',\vec{p}_2,\vec{p}_2'} a_{\vec{p}_1',s_1}^\dagger a_{\vec{p}_2',s_2}^\dagger a_{\vec{p}_2,s_2} a_{\vec{p}_1,s_1}\delta(\vec{p}_1-\vec{p}_1')\delta(\vec{p}_2-\vec{p}_2')\frac{1}{V}\int \mathrm{d}^3\vec{q}\, e^{-\frac{i}{\hbar}\vec{q}\cdot\vec{r}} u_{\vec{q}} \\
=& \frac{1}{2}\sum_{s_1,s_2}\sum_{\vec{p}_1,\vec{p}_2} a_{\vec{p}_1,s_1}^\dagger a_{\vec{p}_2}^\dagger a_{\vec{p}_2,s_2} a_{\vec{p}_1,s_1}\frac{1}{V}\int \mathrm{d}^3\vec{q}\, e^{-\frac{i}{\hbar}\vec{q}\cdot\vec{r}} u_{\vec{q}}.
\end{aligned} \tag{16}$$

因此，该电子系统哈密顿量可写为

$$\mathcal{H} = \sum_s\sum_{\vec{p}}\frac{p^2}{2m}a_{\vec{p}}^\dagger a_{\vec{p}} + \frac{1}{2}\sum_{s_1,s_2}\sum_{\vec{p}_1,\vec{p}_2} a_{\vec{p}_1,s_1}^\dagger a_{\vec{p}_2}^\dagger a_{\vec{p}_2,s_2} a_{\vec{p}_1,s_1}\frac{1}{V}\int \mathrm{d}^3\vec{q}\, e^{-\frac{i}{\hbar}\vec{q}\cdot\vec{r}} u_{\vec{q}}. \tag{17}$$

$\square$