习题 IV

截止时间：2021. 3. 30（周二）

姓名：陈 稼 霖
学号：45875852
成绩：________

第 1 题 得分：________. 计算下列系统内电子气体的 Fermi 能，Fermi 温度和 Fermi 速度.

1) 室温下的金属钠：密度为 0.97g/cm^3，每个原子贡献一个传导电子，假设它们的能量和动量的关系为 $E = \frac{p^2}{2m}$，（即忽略晶格场对电子运动的影响）.

2) 天狼星的伴星（白矮星）：其质量约为太阳质量的 0.98 倍，半径约为太阳半径的 0.0084 倍. 假设星体全部由氦组成.

解： 1) 电子属于 Fermion，按照自由 Fermi 气体模型，传导电子的数密度可用 Fermi 波矢表示，

$$\rho = \frac{\omega K_F^3}{6\pi^2}, \tag{1}$$

其中金属钠中的传导电子数密度 $\rho = \frac{N}{V} = \frac{N_A}{M_{\text{Na}}/\rho_m} = \frac{6.02\times 10^{23}\ \text{mol}^{-1}}{(22.99\ \text{g}\cdot\text{mol}^{-1})/(0.97\times 10^6\ \text{g}\cdot\text{m}^{-3})} = 2.540\times 10^{28}\text{m}^{-3}$，传导电子的各能级的简并度 $\omega = 2$，故得 Fermi 波矢为

$$K_F = 9.09\times 10^9\ \text{m}^{-1}. \tag{2}$$

Fermi 速度为

$$v_F = \frac{\hbar K_F}{m} = 6.62\times 10^6\ \text{m}\cdot\text{s}^{-1}, \tag{3}$$

其中电子的质量 $m = 9.11\times 10^{-31}$ kg. Fermi 能量为

$$\varepsilon_F = \frac{\hbar^2 K_F^2}{2m} = 5.05\times 10^{-19}\ \text{J} = 3.16\ \text{eV}. \tag{4}$$

Fermi 温度为

$$T_F = \frac{\varepsilon_F}{k} = 3.66\times 10^4\ \text{K}. \tag{5}$$

2) 太阳的质量为 $m_\odot = 1.989\times 10^{30}$ kg，太阳的半径为 $r_\odot = 6.6934\times 10^8$ m，该白矮星的质量密度为 $\rho_m = \frac{m}{V} = \frac{0.98 m_\odot}{\frac{4}{3}\pi(0.0084 r_\odot)^3} = 2.62\times 10^9\ \text{kg}\cdot\text{m}^{-3}$. 假设白矮星中电子全部从原子内挤出，每个氦原子挤出 $\chi = 2$ 个电子，白矮星的电子数密度为 $\rho = \frac{\chi N_A}{M_{\text{He}}/\rho_m} = \frac{2\times 6.02\times 10^{23}\ \text{mol}^{-1}}{(4.00\times 10^{-3}\ \text{kg}\cdot\text{mol}^{-1})/(2.62\times 10^9\ \text{kg}\cdot\text{m}^{-3})} = 7.89\times 10^{35}\ \text{m}^{-3}$. 对于氦原子Fermi 波矢为

$$K_F = \left(\frac{6\pi^2\rho}{\omega}\right)^{1/3} = 2.86\times 10^{12}\ \text{m}^{-1}. \tag{6}$$

Fermi 能量为

$$\varepsilon_F = c\sqrt{p_F^2 + m^2c^2} - mc^2 = c\sqrt{\hbar^2 k_F^2 + m^2c^2} - mc^2 = 4.01\times 10^{-14}\ \text{J} = 2.51\times 10^5\ \text{eV}. \tag{7}$$

Fermi 速度为

$$v_F = \frac{p_F c^2}{\varepsilon_F^2 + mc^2} = \frac{c^2\hbar K_F}{\varepsilon_F + mc^2} = 2.22\times 10^8\ \text{m}\cdot\text{s}^{-1}. \tag{8}$$

Fermi 温度为

$$T_F = \frac{\varepsilon_F}{k} = 2.91\times 10^9\ \text{K}. \tag{9}$$

□

第 2 题 得分: _________. 证明非相对论简并电子气体的热力学函数是

$$G = N\mu = N\varepsilon_F\left[1 - \frac{1}{12}\pi^2\left(\frac{kT}{\varepsilon_F}\right)^2 - \frac{1}{80}\pi^4\left(\frac{kT}{\varepsilon_F}\right)^4 + \cdots\right]$$
$$E = \frac{3}{5}N\varepsilon_F\left[1 + \frac{5}{12}\pi^2\left(\frac{kT}{\varepsilon_F}\right)^2 - \frac{1}{16}\pi^4\left(\frac{kT}{\varepsilon_F}\right)^4 + \cdots\right]$$
$$C_V = \frac{1}{2}N\pi^2\frac{k^2T}{\varepsilon_F}\left[1 - \frac{3}{10}\pi^2\left(\frac{kT}{\varepsilon_F}\right)^2 + \cdots\right]$$
$$S = \frac{1}{2}N\pi^2\frac{k^2T}{\varepsilon_F}\left[1 - \frac{1}{10}\left(\frac{kT}{\varepsilon_F}\right)^2 + \cdots\right]$$

其中 $N = \frac{V}{3\pi^2}\left(\frac{2m}{\hbar}\varepsilon_F\right)^{\frac{2}{3}}$ 为电子总数，$\cdots$ 代表 $\frac{kT}{\varepsilon_F}$ 的高阶项.

证:

Sommerfeld 方法：

$$\int_0^\infty \mathrm{d}\varepsilon\frac{f(\varepsilon)}{e^{\beta(\varepsilon-\mu)}+1} \approx 2\sum_{n=0}^\infty F^{(2n)}c_n(kT)^{2n}, \tag{10}$$

其中 $F(\varepsilon)$ 为 $f(\varepsilon)$ 的原函数，

$$c_n = \frac{1}{(2n)!}\int_0^\infty \frac{\xi^{2n}e^\xi}{(e^\xi+1)^2}. \tag{11}$$

$c_0 = \frac{1}{2}$，$c_1 = \frac{\pi^2}{12}$，$c_2 = \frac{7\pi^2}{240}$.

系统中的电子数为

$$N = V\int_0^\infty \mathrm{d}\varepsilon\frac{D(\varepsilon)}{e^{\beta(\varepsilon-\mu)}+1}, \tag{12}$$

其中对于非相对论自由电子，态密度

$$D(\varepsilon) = \frac{\sqrt{2}m^{3/2}\varepsilon^{1/2}}{\pi^2\hbar^3}, \tag{13}$$

从而对于上面这一积分式，

$$f(\varepsilon) = D(\varepsilon), \tag{14}$$
$$\Longrightarrow F(\varepsilon) = \frac{2\sqrt{2}m^{3/2}\varepsilon^{3/2}}{3\pi^2\hbar^3}, \tag{15}$$
$$F^{(2)}(\varepsilon) = D'(\varepsilon) = \frac{\sqrt{2}m^{3/2}\varepsilon^{-1/2}}{2\pi^2\hbar^3}, \tag{16}$$
$$F^{(4)}(\varepsilon) = \frac{3\sqrt{2}m^{3/2}\varepsilon^{-5/2}}{8\pi^2\hbar^3}. \tag{17}$$

从而利用 Sommerfeld 方法，得

$$N = \frac{V}{3\pi^2}\left(\frac{2m\mu}{\hbar^2}\right)^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2 + \frac{7\pi^4}{640}\left(\frac{kT}{\mu}\right)^4 + \cdots\right], \tag{18}$$
$$\Longrightarrow \rho = \frac{N}{V} = \frac{1}{3\pi^2}\left(\frac{2m\mu}{\hbar^2}\right)^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2 + \frac{7\pi^4}{640}\left(\frac{kT}{\mu}\right)^4 + \cdots\right]. \tag{19}$$

由于

$$\rho = \frac{\omega K_F^3}{6\pi^2} \overset{\omega=2}{=} \frac{K_F^3}{3\pi^2} = \frac{(2m\varepsilon_F/\hbar^2)^{3/2}}{3\pi^2}, \tag{20}$$

故有

$$\varepsilon_F^{3/2}=\mu^{3/2}\left[1+\frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2+\frac{7\pi^4}{640}\left(\frac{kT}{\mu}\right)^4+\cdots\right],\tag{21}$$

$$\begin{aligned}\Longrightarrow\varepsilon_F=&\mu\left[1+\frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2+\frac{7\pi^4}{640}\left(\frac{kT}{\mu}\right)^4+\cdots\right]^{2/3}\\=&\mu\left\{1+\frac{2}{3}\left[\frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2+\frac{7\pi^4}{640}\left(\frac{kT}{\mu}\right)^4+\cdots\right]+\frac{1}{2!}\frac{-1}{3}\frac{2}{3}\left[\frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2+\frac{7\pi^4}{640}\left(\frac{kT}{\mu}\right)^4+\cdots\right]^2+\cdots\right\}\\=&\mu\left[1+\frac{\pi^2}{12}\left(\frac{kT}{\mu}\right)^2+\frac{\pi^4}{180}\left(\frac{kT}{\mu}\right)^4+\cdots\right],\end{aligned}\tag{22}$$

$$\Longrightarrow\mu=\varepsilon_F\left[1+\frac{\pi^2}{12}\left(\frac{kT}{\mu}\right)^2+\frac{\pi^4}{180}\left(\frac{kT}{\mu}\right)^4+\cdots\right]^{-1}.$$

近似到 0 阶得

$$\mu=\varepsilon_F.\tag{23}$$

将上式回代，然后近似到 2 阶得

$$\mu=\varepsilon_F-\frac{\pi^2}{12\varepsilon_F}(kT)^2.\tag{24}$$

将上式回代，然后近似到 4 阶得

$$\mu=\varepsilon_F-\frac{\pi^2}{12\varepsilon_F\left[1-\frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2\right]}(kT)^2-\frac{\pi^4}{180\varepsilon_F^4}(kT)^4\approx\varepsilon_F-\frac{\pi^2}{12\varepsilon_F}(kT)^2-\frac{1}{80\varepsilon_F^3}(kT)^4.\tag{25}$$

故 Gibbs 势为

$$G=N\mu=N\varepsilon\left[1-\frac{1}{12}\pi^2\left(\frac{kT}{\varepsilon_F}\right)^2-\frac{1}{80}\pi^4\left(\frac{kT}{\varepsilon_F}\right)^4+\cdots\right].\tag{26}$$

系统内能为

$$E=V\int_0^\infty d\varepsilon\frac{D(\varepsilon)}{e^{-\beta(\varepsilon-\mu)}+1}\varepsilon.\tag{27}$$

从而对上面这一积分式，

$$f(\varepsilon)=\varepsilon D(\varepsilon)=\frac{\sqrt{2}m^{3/2}\varepsilon^{3/2}}{\pi^2\hbar^3},\tag{28}$$

$$\Longrightarrow F(\varepsilon)=\frac{2\sqrt{2}m^{3/2}\varepsilon^{5/2}}{5\pi^2\hbar^3},\tag{29}$$

$$F^{(2)}(\varepsilon)=\frac{3\sqrt{2}m^{3/2}\varepsilon^{1/2}}{2\pi^2\hbar^3},\tag{30}$$

$$F^{(4)}(\varepsilon)=-\frac{3\sqrt{2}m^{3/2}\varepsilon^{-3/2}}{8\pi^2\hbar^3}.\tag{31}$$

从而利用 Sommerfeld 方法，得

$$E=V\left(\frac{2m\mu}{\hbar^2}\right)^{3/2}\frac{1}{5\pi^2}\mu\left[1+\frac{5\pi^2}{8}\left(\frac{kT}{\mu}\right)^2-\frac{7\pi^4}{384}\left(\frac{kT}{\mu}\right)^4+\cdots\right].\tag{32}$$

根据式 (18)，

$$
\begin{aligned}
V\left(\frac{2m\mu}{\hbar^2}\right)^{3/2} =&3\pi^2N\left[1+\frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2+\frac{7\pi^2}{640}\left(\frac{kT}{\mu}\right)^4+\cdots\right]^{-1}\\
=&3\pi^2N\left\{1-\left[\frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2+\frac{7\pi^2}{640}\left(\frac{kT}{\mu}\right)^4\right]+\frac{1}{2!}(-1)(-2)\left[\frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2+\frac{7\pi^2}{640}\left(\frac{kT}{\mu}\right)^4+\cdots\right]^2+\cdots\right\}\\
=&3\pi^2N\left[1-\frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2+\frac{3\pi^2}{640}\left(\frac{kT}{\mu}\right)^4+\cdots\right].
\end{aligned}
\tag{33}
$$

将上式代入式 (32)，得

$$
\begin{aligned}
E=&\frac{3}{5}N\mu\left[1+\frac{5\pi^2}{8}\left(\frac{kT}{\mu}\right)^2-\frac{7\pi^4}{384}\left(\frac{kT}{\mu}\right)^4+\cdots\right]\left[1-\frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2+\frac{3\pi^2}{640}\left(\frac{kT}{\mu}\right)^4+\cdots\right]\\
=&\frac{3}{5}N\mu\left[1+\frac{\pi^2}{2}\left(\frac{kT}{\mu}\right)^2-\frac{19\pi^4}{384}\left(\frac{kT}{\mu}\right)^4+\cdots\right]\\
=&\frac{3}{5}N\left[\mu+\frac{\pi^2}{2\mu}(kT)^2-\frac{11\pi^4}{120\mu^3}(kT)^4+\cdots\right].
\end{aligned}
\tag{34}
$$

将式 (25) 代入上式，并近似至 4 阶，得

$$
\begin{aligned}
E=&\frac{3}{5}N\left\{\varepsilon_F\left[1-\frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2-\frac{\pi^4}{80}\left(\frac{kT}{\varepsilon_F}\right)^4\right]+\frac{\pi^2}{2\varepsilon_F\left[1-\frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2-\cdots\right]}(kT)^2-\frac{11\pi^4}{120\varepsilon_F^3[1-\cdots]^3}(kT)^4\right\}\\
=&\frac{3}{5}N\left[\varepsilon_F+\frac{5\pi^2}{12\varepsilon_F}(kT)^2-\frac{\pi^4}{16\varepsilon_F^3}(kT)^4+\cdots\right]\\
=&\frac{3}{5}N\varepsilon_F\left[1+\frac{5\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2-\frac{\pi^4}{16}\left(\frac{kT}{\varepsilon_F}\right)^4+\cdots\right].
\end{aligned}
\tag{35}
$$

系统的定容热容量为

$$
C_V=\left(\frac{\partial E}{\partial T}\right)_V=\frac{1}{2}N\pi^2\frac{k^2T}{\varepsilon_F}\left[1-\frac{3}{10}\left(\frac{kT}{\varepsilon_F}\right)^2+\cdots\right]
\tag{36}
$$

对于非相对论电子气体，

$$
PV=\frac{2}{3}E.
\tag{37}
$$

系统的自由能为

$$
F=G-PV=G-\frac{2}{3}E=N\varepsilon_F\left[\frac{3}{5}-\frac{\pi^2}{4}\left(\frac{kT}{\varepsilon_F}\right)^2+\frac{\pi^4}{80}\left(\frac{kT}{\varepsilon_F}\right)^4+\cdots\right].
\tag{38}
$$

系统的熵为

$$
S=-\frac{\partial G}{\partial T}=\frac{\pi^2}{2}N\frac{k^2T}{\varepsilon_F}\left[1-\frac{1}{10}\left(\frac{kT}{\varepsilon_F}\right)^2+\cdots\right].
\tag{39}
$$

□