习题 IV. 2 李政道书 191 页第5题中 μ 按 kT/ε_F 的展开：

粒子数密度:

$$\rho = \frac{N}{V} = \int_{-\infty}^{\infty} d\xi F(\mu + kT\xi)\frac{e^{\xi}}{(e^{\xi}+1)^2}$$

其中 $$F(\varepsilon) = \frac{2\sqrt{2}m^{\frac{3}{2}}\varepsilon^{\frac{3}{2}}}{3\pi^2\hbar^3}$$

对于 $kT \ll \mu$ 有

$$\rho \cong 2\sum_{n=0}^{\infty} F^{(2n)}(\mu)\, c_n (kT)^{2n}$$

$$c_0 = \frac{1}{2} \qquad c_1 = \frac{\pi^2}{12} \qquad c_2 = \frac{7\pi^4}{720}$$

$$\rho \cong \frac{2\sqrt{2}m^{\frac{3}{2}}\mu^{\frac{3}{2}}}{3\pi^2\hbar^3}\left[1 + \frac{\pi^2}{12}\left(\frac{kT}{\mu}\right)^2 + \frac{7\pi^4}{640}\left(\frac{kT}{\mu}\right)^4\right]$$

$$\varepsilon_F^{3/2} = \mu^{3/2}\left[1 + \frac{\pi^2}{12}\left(\frac{kT}{\mu}\right)^2 + \frac{7\pi^4}{640}\left(\frac{kT}{\mu}\right)^4\right]$$

$$\varepsilon_F = \mu\left[1+\frac{\pi^2}{12}\left(\frac{kT}{\mu}\right)^2+\frac{7\pi^4}{640}\left(\frac{kT}{\mu}\right)^4\right]^{2/3} = \mu\left[1+\frac{\pi^2}{12}\left(\frac{kT}{\mu}\right)^2+\frac{\pi^4}{180}\left(\frac{kT}{\mu}\right)^4\right]$$

$$\mu = \varepsilon_F - \frac{\pi^2}{12\mu}(kT)^2 - \frac{\pi^4}{180\mu^3}(kT)^4$$

零阶： $\mu = \varepsilon_F$

一阶： $\mu = \varepsilon_F - \frac{\pi^2}{12\varepsilon_F}(kT)^2$

二阶： $$\mu = \varepsilon_F - \frac{\pi^2}{12\left(\varepsilon_F - \frac{\pi^2}{12\varepsilon_F}(kT)^2\right)}(kT)^2 - \frac{\pi^4}{180\varepsilon_F^3}(kT)^4$$

$$= \varepsilon_F\left[1-\frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2-\frac{\pi^4}{80}\left(\frac{kT}{\varepsilon_F}\right)^4\right]$$

附录：用 Lagrange 公式求解 Fermi 能与化学势关系的反演:

Lagrange 展开公式:

$$z = a + t\phi(z)$$

$$f(z) = f(a) + \sum_{n=1}^{\infty} \frac{t^n}{n!} \frac{d^{n-1}}{da^{n-1}} \{f'(a)[\phi(a)]^n\}$$

令 $f(z) = z$，$a = 0$

$$z = \sum_{n=1}^{\infty} \frac{t^n}{n!} \frac{d^{n-1}}{da^{n-1}} \{f'(a)[\phi(a)]^n\} \bigg|_{a=0}$$

$$\varepsilon_F^{\frac{3}{2}} = \mu^{\frac{3}{2}}\left[1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2 + \frac{7\pi^4}{640}\left(\frac{kT}{\mu}\right)^4 + \cdots\right]$$

$$\varepsilon_F = \mu\left[1 + \frac{1}{12}\left(\frac{\pi kT}{\mu}\right)^2 + \left(\frac{7}{960} - \frac{1}{576}\right)\left(\frac{\pi kT}{\mu}\right)^4 + \cdots\right]$$

$$z = \frac{\pi kT}{\mu} \qquad t = \frac{\pi kT}{\varepsilon_F}$$

$$z = t\left[1 + \frac{1}{12}z^2 + \left(\frac{7}{960} - \frac{7}{576}\right)z^4 + \cdots\right] = t\phi(z)$$

$$z = \frac{t}{1!}\phi(0) + \frac{t^3}{3!}\frac{d^2}{dz^2}\phi^3(z)\bigg|_{z=0} + \frac{t^5}{5!}\frac{d^4}{dz^4}\phi^5(z)\bigg|_{z=0} + \cdots$$

$$= t\left(1 + \frac{1}{12}t^2 + \frac{7}{360}t^4 + \cdots\right)$$

$$\mu = \varepsilon_F\left(1 + \frac{1}{12}t^2 + \frac{7}{360}t^4 + \cdots\right)^{-1} = \varepsilon_F\left(1 - \frac{1}{12}t^2 - \frac{1}{80}t^4 + \cdots\right)$$

习题 V. 2 考虑两维自旋为零的自由 boson系统

1）推导单位面积的态密度公式；

2）推导粒子数密度（面密度）用温度和易逸度表达的公式；

3）证明此系统无凝聚现象。

解: 1）$\vec{K}=\left(K_x,K_y\right)$

$$K_x=\frac{2l\pi}{L_x}\qquad K_y=\frac{2n\pi}{L_y}\qquad l,n=0,\pm1,\pm2,\ldots$$

$$\varepsilon_{\vec{K}}=\frac{\hbar^2}{2m}\left(K_x^2+K_y^2\right)=\frac{\hbar^2K^2}{2m}$$

$$\sum_{\vec{K}}g\left(\varepsilon_{\vec{K}}\right)=L_xL_y\int\frac{d^2\vec{K}}{(2\pi)^2}g\left(\varepsilon_{\vec{K}}\right)=\frac{L_xL_y}{2\pi}\int_0^\infty dKKg\left(\varepsilon_{\vec{K}}\right)=L_xL_y\int_0^\infty d\varepsilon D(\varepsilon)\,g(\varepsilon)$$

单位面积态密度：

$$D(\varepsilon)=\frac{m}{2\pi\hbar^2}$$

2）粒子数密度：

$$\rho = \int_0^\infty d\varepsilon \frac{D(\varepsilon)}{e^{\beta(\varepsilon-\mu)}-1} = \frac{m}{2\pi\hbar^2}\int_0^\infty d\varepsilon \frac{ze^{-\beta\varepsilon}}{1-ze^{-\beta\varepsilon}} = \frac{1}{\lambda^2}\int_0^\infty dx \frac{ze^{-x}}{1-ze^{-x}}$$

$$= \frac{1}{\lambda^2}\ln\frac{1}{1-z} \qquad\qquad z = e^{\beta\mu} \qquad\qquad \lambda = \sqrt{\frac{2\pi}{mkT}}\hbar$$

3）$0 < z < 1 \quad \Longrightarrow \quad 0 < \rho < \infty$

对于任意温度和密度，存在

$$z = 1 - e^{-\rho\lambda^2}$$

故 2）的结果适用于任意温度和密度 $\Longrightarrow$ 无凝聚。

基态占据数:

$$n_0 = \frac{z}{1-z} = e^{\rho\lambda^2} - 1 \qquad \text{非宏观量。}$$

习题 V. 3 李政道书 192 页第6 题

解：由状态方程 $(\omega = 2j+1)$

$$\frac{P}{kT} = \rho\left(1 \mp \frac{1}{2^{5/2}}\frac{\rho\lambda^3}{\omega} + \cdots\right)$$

得 $$PV = NkT\left(1 + \frac{1}{2^{5/2}}\frac{\rho\lambda^3}{2j+1} + \cdots\right)$$

熵密度 $$s = \frac{S}{V} = \left(\frac{\partial P}{\partial T}\right)_\mu = \left(\frac{\partial P}{\partial T}\right)_z + \left(\frac{\partial P}{\partial \ln z}\right)_T \left(\frac{\partial \ln z}{\partial T}\right)_\mu$$

$$\left(\frac{\partial P}{\partial T}\right)_z = \frac{5}{2T}P \qquad \left(\frac{\partial P}{\partial \ln z}\right)_T = z\left(\frac{\partial P}{\partial z}\right)_T = kT\rho \qquad \left(\frac{\partial \ln z}{\partial T}\right)_\mu = -\frac{\mu}{kT^2}$$

$$\mu = kT\ln z = kT\ln\left[\frac{\rho\lambda^3}{2j+1} + \frac{1}{2^{3/2}}\left(\frac{\rho\lambda^3}{2j+1}\right)^2 + \cdots\right] \cong kT\left[\ln\frac{\rho\lambda^3}{2j+1} + \frac{\rho\lambda^3}{2^{3/2}(2j+1)}\right]$$

$$s \cong k\rho\left[\frac{5}{2}\left(1 + \frac{\rho\lambda^3}{2^{5/2}(2j+1)}\right)\right] - \ln\frac{\rho\lambda^3}{2j+1} - k\rho\frac{\rho\lambda^3}{2^{3/2}(2j+1)}$$

$$= k\rho\left[\ln\frac{(2j+1)e^{5/2}}{\rho\lambda^3} + \frac{\rho\lambda^3}{2^{7/2}(2j+1)}\right]$$

$$S = sV = Nk\left[\ln\frac{(2j+1)e^{5/2}}{\rho\lambda^3} + \frac{\rho\lambda^3}{2^{7/2}(2j+1)}\right]$$