Assignment 3

Due time : 10:15, Oct 9, 2020 (Friday)

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Grade : ________

Problem 1 (Autocorrelation and cross-correlation function, 30 pts) Score: ________. The random process are given by $X(t) = n(t) + A\cos(2\pi f_0 t + \theta)$, $Y(t) = n(t) + A\sin(2\pi f_0 t + \theta)$, where A and f_0 are positive constants and θ is a random variable uniformly distributed in the interval $[-\pi, \pi)$. The first term $n(t)$ represents a stationary random noise process with autocorrelation function $R_n(\tau) = B\Lambda(\tau) + C$, where B and C are positive constants. We further assume the random process $n(t)$ and $A\cos(2\pi f_0 t+\theta)$ are uncorrelated, $n(t)$ and $A\sin(2\pi f_0 t+\theta)$ are also uncorrelated.

1) Find the autocorrelation functions of $X(t)$ and $Y(t)$, respectively.
2) Find the cross-correlation function of $X(t)$ and $Y(t)$.
3) Find the power spectral densities of $X(t)$ and $Y(t)$, respectively.
4) Find the cross power spectral density of $X(t)$ and $Y(t)$.
5) Find the total power of $X(t)$ and $Y(t)$, respectively.
6) Find the DC powers of $X(t)$ and $Y(t)$, respectively.

(Hint: $\Lambda(\tau) = \begin{cases} 1 - |\tau|, & |\tau| \leq 1 \\ 0, & \text{otherwise} \end{cases}$, the DC power of $X(t)$ is $\overline{X(t)}^2 = E^2[X(t)]$)

Solution: 1) The autocorrelation function of $X(t)$ is

$$\begin{aligned}
R_X(t, t+\tau) =& E[X(t)X(t+\tau)] = E\{[n(t) + A\cos(2\pi f_0 t + \theta)][n(t+\tau) + A\cos(2\pi f_0(t+\tau) + \theta)]\} \\
=& E[n(t)n(t+\tau)] + E[n(t)A\cos(2\pi f_t(t+\tau) + \theta)] \\
&+ E[A\cos(2\pi f_0 t + \theta)n(t+\tau)] + E[A\cos(2\pi f_0 t + \theta)A\cos(2\pi f_0(t+\tau) + \theta)] \\
&(\because n(t) \text{ and } A\cos(2\pi f_0 t + \theta) \text{ are uncorrelated}) \\
=& R_X(\tau) + E[n(t)]E[A\cos(2\pi f_0(t+\tau) + \theta)] \\
&+ E[A\cos(2\pi f_0 t + \theta)]E[n(t+\tau)] + A^2 E[\cos(2\pi f_0 t + \theta)\cos(2\pi f_0(t+\tau) + \theta)] \\
=& B\Lambda(\tau) + C + \frac{A^2}{2}\int_{-\pi}^{\pi}[\cos(4\pi f_0 t + 2\pi f_0\tau + 2\theta) + \cos(2\pi f_0\tau)]\frac{1}{2\pi}\,d\theta \\
=& B\Lambda(\tau) + C + \frac{A^2}{2}\cos(2\pi f_0\tau).
\end{aligned} \tag{1}$$

Similarly, the autocorrelation function of $Y(t)$ is

$$\begin{aligned}
R_Y(t, t+\tau) =& E[Y(t)Y(t+\tau)] = E\{[n(t) + A\sin(2\pi f_0 t + \theta)][n(t+\tau) + A\sin(2\pi f_0(t+\tau) + \theta)]\} \\
=& E[X(t)X(t+\tau)] + E[n(t)A\sin(2\pi f_0(t+\tau) + \theta)] \\
&+ E[A\sin(2\pi f_0 t + \theta)n(t+\tau)] + E[A\sin(2\pi f_0 t + \theta)A\sin(2\pi f_0(t+\tau) + \theta)] \\
&(\because n(t) \text{ and } A\sin(2\pi f_0 t + \theta) \text{ are uncorrelated}) \\
=& R_n(\tau) + E[n(t)]E[A\sin(2\pi f_0 t + \theta)] \\
&+ E[A\sin(2\pi f_0 t + \theta)]E[n(t+\tau)] + A^2 E[\sin(2\pi f_0 t + \theta)\sin(2\pi f_0(t+\tau) + \theta)] \\
=& B\Lambda(\tau) + C + \frac{A^2}{2}\int_{-\pi}^{\pi}[\cos(2\pi f_0\tau) - \cos(4\pi f_0 t + 2\pi f_0\tau + 2\theta)]\frac{1}{2\pi}\,d\theta \\
=& B\Lambda(\tau) + C + \frac{A^2}{2}\cos(2\pi f_0\tau).
\end{aligned} \tag{2}$$

2) The cross-correlation function of $X(t)$ and $Y(t)$ is

$$\begin{aligned}
R_{X,Y}(t,\tau) =& E[X(t)Y(t+\tau)] = E\{[n(t)+A\cos(2\pi f_0 t+\theta)][n(t+\tau)+A\sin(2\pi f_0(t+\tau)+\theta)]\}\\
=& E[n(t)n(t+\tau)] + E[n(t)A\sin(2\pi f_0(t+\tau)+\theta)]\\
&+E[A\cos(2\pi f_0 t+\theta)n(t+\tau)] + E[A\cos(2\pi f_0 t+\theta)A\sin(2\pi f_0(t+\tau)+\theta)]\\
&(\because n(t) \text{ and } A\cos(2\pi f_0 t+\theta) \text{ are uncorrelated, } n(t) \text{ and } A\sin(2\pi f_0 t+\theta) \text{ are uncorrelated})\\
=& R_n(\tau) + E[n(t)]E[A\sin(2\pi f_0(t+\tau)+\theta)]\\
&+ E[A\cos(2\pi f_0 t+\theta)]E[n(t+\tau)] + A^2 E[\cos(2\pi f_0 t+\theta)\sin(2\pi f_0(t+\tau)+\theta)]\\
=& B\Lambda(\tau) + C + \frac{A^2}{2}\int_{-\pi}^{\pi}[\sin(4\pi f_0 t + 2\pi f_0\tau + 2\theta) + \sin(2\pi f_0 \tau)]\frac{1}{2\pi}\,d\theta\\
=& B\Lambda(\tau) + C + \frac{A^2}{2}\sin(2\pi f_0\tau).
\end{aligned} \tag{3}$$

3) The first-order statistics of $X(t)$

$$\begin{aligned}
E[X(t)] =& E[n(t)+A\cos(2\pi f_0 t+\theta)] = E[n(t)] + AE[\cos(2\pi f_0 t+\theta)] = \cancel{0+0=0,}\\
=& E[n(t)],
\end{aligned} \tag{4}$$

$$E\{[X(t)-E[X(t)]]^2\} = E[X^2(t)] - E^2[X(t)] = E[X^2(t)] = R_X(t,t) = B + C + \frac{A^2}{2}, \tag{5}$$

are not dependent on t, and as obtained in 1), the second-order statistics of $X(t)$ only depends on the gap, so $X(t)$ is wide-sense stationary. According to Wiener-Khinchine, the power spectral density of $X(t)$ is

$$\begin{aligned}
S_X(t) =& \mathscr{F}[R_X(\tau)] = \mathscr{F}\left[B\Lambda(\tau) + C + \frac{A^2}{2}\cos(2\pi f_0\tau)\right]\\
=& B\,\text{sinc}^2 f + C\delta(f) + \frac{A^2}{4}\left[\delta(f-f_0) + \delta(f+f_0)\right].
\end{aligned} \tag{6}$$

Similarly, the first-order statistics of $Y(t)$

$$E[Y(t)] = E[n(t)+A\sin(2\pi f_0 t+\theta)] = E[n(t)] + AE[\sin(2\pi f_0 t+\theta)] = 0+0=0, \tag{7}$$

$$E\{[Y(t)-E[Y(t)]]^2\} = E[Y^2(t)] - E^2[Y(t)] = E[Y^2(t)] = R_Y(t,t) = B + C + \frac{A^2}{2}, \tag{8}$$

are not dependent on t, and as obtained in 2), the second-order statistics of $Y(t)$ only depends on the gap, so $Y(t)$ is wide-sense stationary. According to Wiener-Khinchine theorem, the power spectral density of $Y(t)$ is

$$\begin{aligned}
S_Y(t) =& \mathscr{F}[R_Y(\tau)] = \mathscr{F}\left[B\Lambda(\tau) + C + \frac{A^2}{2}\cos(2\pi f_0\tau)\right]\\
=& B\,\text{sinc}^2 f + C\delta(f) + \frac{A^2}{4}\left[\delta(f-f_0) + \delta(f+f_0)\right].
\end{aligned} \tag{9}$$

4) As we obtained in 3), both $X(t)$ and $Y(t)$ are wide-sense stationary. The cross power of $X(t)$ and $Y(t)$ is

$$\begin{aligned}
S_{XY}(f) =& \mathscr{F}[R_{XY}(\tau)] = \mathscr{F}\left[B\Lambda(\tau) + C + \frac{A^2}{2}\sin(2\pi f_0\tau)\right]\\
=& B\,\text{sinc}^2 f + C\delta(f) + \frac{A^2}{4j}\left[\delta(f-f_0) - \delta(f+f_0)\right].
\end{aligned} \tag{10}$$

5) The total power of $X(t)$ is

$$P_X = E[X^2(t)] = R_X(\tau=0) = B + C + \frac{A^2}{2}. \tag{11}$$

Similarly, the total power of $Y(t)$ is

$$P_Y = E[Y^2(t)] = R_Y(\tau = 0) = B + C + \frac{A^2}{2}. \tag{12}$$

6) As obtained in 3), the mean of $X(t)$ is $E[X(t)] = 0$, so the DC power of $X(t)$ is

$$\overline{X(t)}^2 = E^2[X(t)] = \cancel{0} E^2[n(t)] = \lim_{\tau\to\infty} R_X(\tau) = C. \tag{13}$$

Similarly, as obtained in 3), the mean of $Y(t)$ is $E[Y(t)] = 0$, so the DC power of $Y(t)$ is

$$\overline{Y(t)}^2 = E^2[Y(t)] = \cancel{0} E^2[n(t)] = \lim_{\tau\to\infty} R_Y(\tau) = C. \tag{14}$$

□

Problem 2 (Gaussian random process transmission through a linear system, 30 pts) Score: ________.
The input to a lowpass filter with impulse response $h(t) = \exp(-10t)u(t)$ is white, Gaussian noise with two-sided power spectral density of 2 W/Hz. Obtain the following:

1) The mean of the output.

2) The power spectral density of the output.

3) The autocorrelation function of the output.

4) The probability density function of the output at an arbitrary time t_1.

5) The joint probability density function of the output at times t_1 and $t_1 + 2$.

6) Find the noise equivalent bandwidth of the filter.

(Hint: $\mathscr{F}[\exp(-\alpha t)u(t), \alpha > 0] = \frac{1}{\alpha + j2\pi f}$, $\mathscr{F}[\exp(-\alpha|t|), \alpha > 0] = \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$)

Solution:

1) The output is

$$Y(t) = n_w(t) * h(t) = \int_{-\infty}^{+\infty} h(\tau) n_w(t-\tau)\, d\tau. \tag{15}$$

The mean of the output is

$$\begin{aligned} E[Y(t)] =& E\left[\int_{-\infty}^{+\infty} h(\tau) n_w(t-\tau)\, d\tau\right] \\ =& \int_{-\infty}^{+\infty} h(\tau) E[n_w(t-\tau)]\, d\tau \\ & (\because n_w(t) \text{ is a white, Gaussian noise}) \\ =& \int_{-\infty}^{+\infty} h(\tau) \cdot 0\, d\tau \\ =& 0. \end{aligned} \tag{16}$$

2) The spectral of the response of the lowpass filter is

$$H(f) = \mathscr{F}[h(t)] = \mathscr{F}[\exp(-10t)u(t)] = \frac{1}{10 + i2\pi f}. \tag{17}$$

The power spectral density of the output is

$$S_{n_w}(f) = |H(f)|^2 S_{n_w}(f) = \frac{1}{100 + 4\pi^2 f^2} \frac{N_0}{2} = \frac{1}{50 + 2\pi^2 f^2} \qquad \text{(W/Hz)}. \tag{18}$$

3) The input is white, Gaussian noise, and thus, stationary, so the autocorrelation function of the output is

$$\begin{aligned}R_Y(\tau) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(\tau_1)h(\tau_2)R_{n_w}(t-\tau_1+\tau_2)\,d\tau_1\,d\tau_2\\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(\tau_1)h(\tau_2)\frac{N_0}{2}\delta(\tau-\tau_1+\tau_2)\,d\tau_1\,d\tau_2\\ &= \frac{N_0}{2}\int_{-\infty}^{+\infty} h(\tau+\tau_2)h(\tau_2)\,d\tau_2\\ &= \frac{N_0}{2}\int_{-\infty}^{+\infty} \exp[-10(\tau+\tau_2)]u(\tau+\tau_2)\exp(-10\tau_2)u(\tau_2)\,d\tau_2\\ &= \frac{N_0}{2}\exp(-10\tau)\int_{\max\{0,-\tau\}}^{+\infty}\exp(-20\tau_2)\,d\tau_2\\ &= \frac{N_0}{2}\exp(-10\tau)\frac{1}{20}\exp[-20\max\{0,-\tau\}]\\ &= \frac{1}{10}\exp(-10\,|\tau|) \qquad (\mathrm{W}^2/\mathrm{Hz}^2).\end{aligned} \tag{19}$$

4) Since the input is a Gaussian noise and the lowpass filter is a linear system, the output is a Gaussian random process. As obtained in 1), the mean of the output is 0. The variance of the output is

$$\sigma_Y^2 = E[Y^2(t)] = R_Y(0) = \frac{1}{10}. \tag{20}$$

The probability density function of the output at an arbitrary time t_1 is

$$f_Y = \frac{1}{\sqrt{2\pi\sigma_Y^2}}\exp\left\{-\frac{1}{2\sigma_Y^2}(y-E[Y(t)])^2\right\} = \sqrt{\frac{5}{\pi}}\exp(-5y^2). \tag{21}$$

5) The autocovariance of $Y(t)$ and $Y(t+2)$ is

$$\sigma_{Y(t),Y(t+2)} = E\{[Y(t)-E[Y(t)]][Y(t+2)-E[Y(t+2)]]\} = E[Y(t)Y(t+2)] = R_Y(2) = \frac{1}{10}\exp(-20). \tag{22}$$

The correlation coefficient of $Y(t)$ and $Y(t+2)$ is

$$\rho_{Y(t),Y(t+2)} = \frac{\sigma_{Y(t),Y(t+2)}}{\sigma_{Y(t)}\sigma_{Y(t+2)}} = \frac{\frac{1}{10}\exp(-20)}{\frac{1}{10}\times\frac{1}{10}} = 10\exp(-20). \tag{23}$$

The joint probability density function of the output at times t_1 and t_1+2 is

$$\begin{aligned}f_{Y(t),Y(t+\tau)}(y_1,y_2) = &\frac{1}{2\pi\sigma_{Y(t)}\sigma_{Y(t+2)}\sqrt{1-\rho^2_{Y(t),Y(t+2)}}}\times\\ &\exp\left\{-\frac{1}{2(1-\rho^2_{Y(t),Y(t+2)})}\left[\frac{(y_1-E[Y(t)])^2}{\sigma^2_{Y(t)}} - \frac{2\rho_{Y(t),Y(t+2)}(y_1-E[Y(t)])(y_2-E[Y(t+2)])}{\sigma_{Y(t)}\sigma_{Y(t+2)}} + \frac{(y_2-E[Y(t+2)])^2}{\sigma^2_{Y(t+2)}}\right]\right\}\\ = &\frac{50}{\pi\sqrt{1-100\exp(-40)}}\exp\left\{-\frac{100\exp(40)}{2[\exp(40)-100]}[y_1^2-20\exp(-20)y_1y_2+y_2^2]\right\}.\end{aligned} \tag{24}$$

6) The noise equivalent bandwidth of the filter is

$$B_N = \frac{\int_0^{+\infty}|H(f)|^2\,df}{H_0^2} = \frac{\int_0^{+\infty}\frac{1}{100+4\pi^2f^2}\,df}{1/100} = \frac{5}{\pi}\arctan\left(\frac{\pi}{5}x\right)\Big|_0^{+\infty} = \frac{5}{2}. \tag{25}$$

□

Problem 3 (Narrowband noise, 40pts) Score: ________. Noise $n(t)$ has the power spectral density shown in the figure 1. We write $n(t) = n_c(t)\cos(2\pi f_0 t+\theta) - n_s(t)\sin(2\pi f_0 t+\theta)$, find and plot $S_{n_c}(f)$, $S_{n_s}(f)$ and $S_{n_cn_s}(f)$ fot the following case.

1) $f_0 = f_1$

2) $f_0 = f_2$

3) $f_0 = (f_1 + f_2)/2$

4) For which of these cases are $n_c(t)$ and $n_s(t)$ uncorrelated.

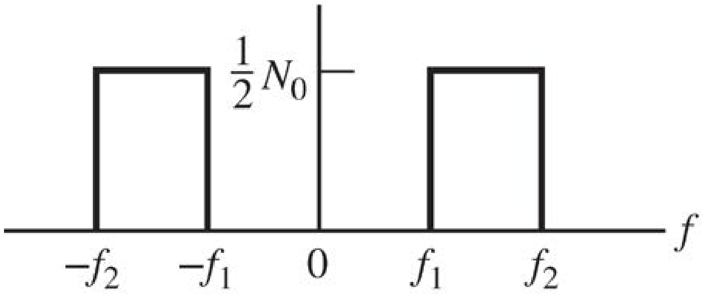


Figure 1: $S_n(f)$

Solution: The power spectral density of $n(t)$ can be written as

$$S_n(f) = \frac{N_0}{2}\left[\Pi\left(\frac{f - \frac{f_1+f_2}{2}}{f_2 - f_1}\right) + \Pi\left(\frac{f + \frac{f_1+f_2}{2}}{f_2 - f_1}\right)\right]. \tag{26}$$

1) For $f_0 = f_1$,

$$\begin{aligned} S_{n_c}(f) = S_{n_s}(f) =& \mathrm{LP}[S_n(f - f_0) + S_n(f + f_0)] = \frac{N_0}{2}\left[\Pi\left(\frac{f + \frac{f_2-f_1}{2}}{f_2 - f_1}\right) + \Pi\left(\frac{f - \frac{f_2-f_1}{2}}{f_2 - f_1}\right)\right] \\ =& \frac{N_0}{2}\Pi\left(\frac{f}{2(f_2 - f_1)}\right), \end{aligned} \tag{27}$$

as shown in figure 2(a).

$$S_{n_c n_s}(f) = j\mathrm{LP}[S_n(f - f_0) - S_n(f + f_0)] = j\frac{N_0}{2}\left[\Pi\left(\frac{f + \frac{f_2-f_1}{2}}{f_2 - f_1}\right) - \Pi\left(\frac{f - \frac{f_2-f_1}{2}}{f_2 - f_1}\right)\right], \tag{28}$$

as shown in figure 2(b).

2) For $f_0 = f_2$,

$$\begin{aligned} S_{n_c}(f) = S_{n_s}(f) =& \mathrm{LP}[S_n(f - f_0) + S_n(f + f_0)] = \frac{N_0}{2}\left[\Pi\left(\frac{f - \frac{f_2-f_1}{2}}{f_2 - f_1}\right) + \Pi\left(\frac{f + \frac{f_2-f_1}{2}}{f_2 - f_1}\right)\right] \\ =& \frac{N_0}{2}\Pi\left(\frac{f}{2(f_2 - f_1)}\right), \end{aligned} \tag{29}$$

as shown in figure 3(a).

$$S_{n_c n_s}(f) = j\mathrm{LP}[S_n(f - f_0) - S_n(f + f_0)] = j\frac{N_0}{2}\left[\Pi\left(\frac{f - \frac{f_2-f_1}{2}}{f_2 - f_1}\right) - \Pi\left(\frac{f + \frac{f_2-f_1}{2}}{f_2 - f_1}\right)\right], \tag{30}$$

as shown in figure 3(b).

3) For $f_0 = \frac{f_1+f_2}{2}$,

$$S_{n_c}(f) = S_{n_s}(f) = \mathrm{LP}\left[S_n(f-f_0) + S_n(f+f_0)\right] = N_0 \Pi\left(\frac{f}{f_2 - f_1}\right), \tag{31}$$

as shown in figure 4(a).

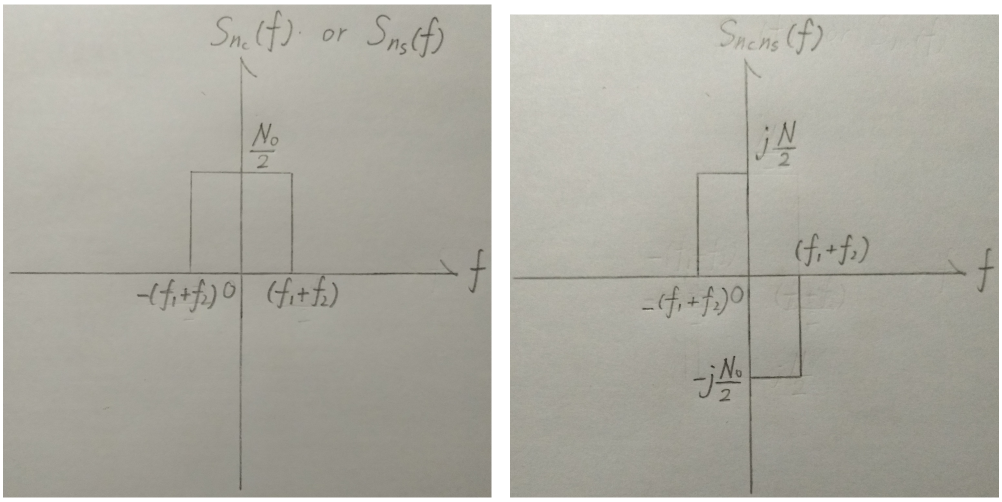
$$S_{n_c n_s}(f) = j\mathrm{LP}[S_n(f-f_0) - S_n(f+f_0)] = 0, \tag{32}$$

as shown in figure 4(b).

4) ~~For cases **1), 2) and 3)**, $n_c(t)$ and $n_s(t)$ are uncorrelated. This is because, for all the cases above, the cross power spectral density is pure imaginary, so that the correlation function of $n_s(t)$ and $n_c(t)$, $R_{n_s n_c}(\tau) = \mathscr{F}^{-1}[S_{n_s n_c}(f)]$, is an odd function and thus $R_{n_s n_c}(0) = 0$. Therefore, for all the cases above, $n_c(t)$ and $n_s(t)$ are uncorrelated.~~ Only in case 3), $n_c(t)$ and $n_s(t)$ are uncorrelated, since only in case 3), $R_{n_s n_c}(t) = \mathscr{F}^{-1}[S_{n_s n_c}(f)] = 0 \quad \forall \tau$. In case 1) and 2), $R_{n_s n_c}(\tau) = \mathscr{F}^{-1}[S_{n_s n_c}(f)] = \pm\frac{N_0}{2}(f_2 - f_1)\,\mathrm{sinc}\,[(f_2-f_1)\tau]e^{-j2\pi\frac{f_2-f_1}{2}\tau} \mp \frac{N_0}{2}(f_2-f_1)\,\mathrm{sinc}\,[(f_2 - f_1)\tau]e^{+j2\pi\frac{f_2-f_1}{2}\tau} = \pm N_0\pi(f_2-f_1)^2\tau\,\mathrm{sinc}^2[(f_2-f_1)\tau] = 0$ only for $\tau = \frac{m}{f_2-f_1}$, $m = 0, \pm1, \pm2, \cdots$

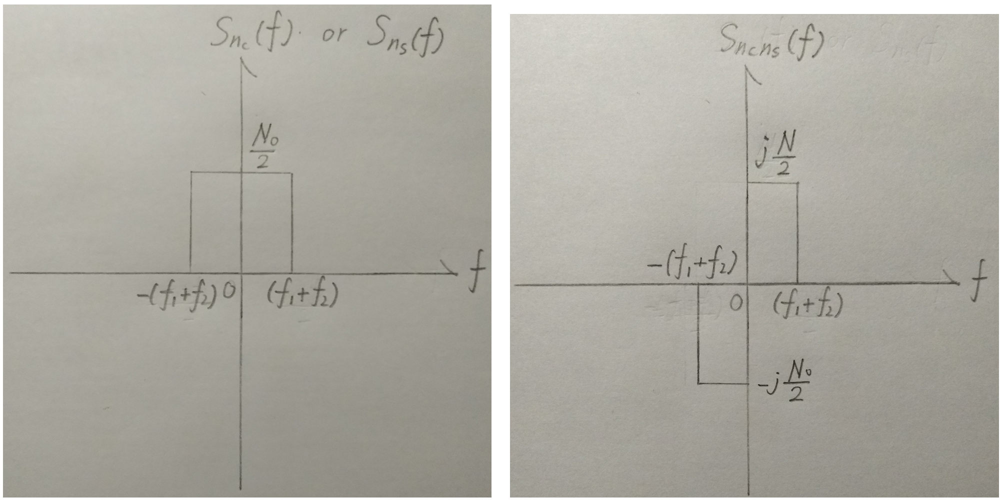
$\square$

There are some mistakes in these figures, please refer to the scan version.



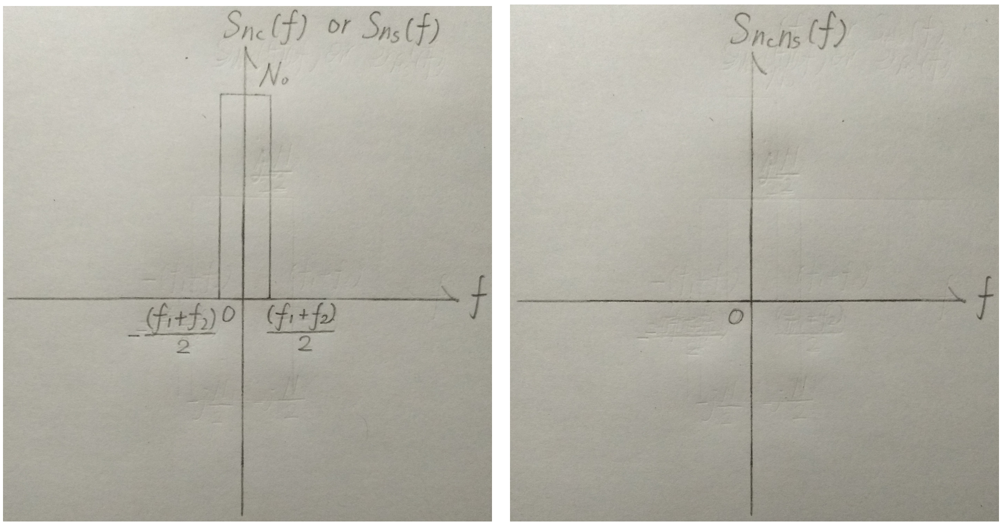
(a) $S_{n_c}(f)$ or $S_{n_s}(f)$ (b) $S_{n_cn_s}(f)$

Figure 2: Case 1)



(a) $S_{n_c}(f)$ or $S_{n_s}(f)$ (b) $S_{n_cn_s}(f)$

Figure 3: Case 2)



(a) $S_{n_c}(f)$ or $S_{n_s}(f)$ (b) $S_{n_cn_s}(f)$

Figure 4: Case 3)