Assignment 6

Due time : 10:15, Oct 30, 2020 (Friday)

Name : 陈 稼 霖
Student ID : 45875852
Grade : ________

Problem 1 (Noise in DSB-SC Receiver, 30pts) Score: ________. A DSB-SC modulated signal in transmitted over a noisy channel. The power spectral density of the noise is shown in Figure 1. The message bandwidth is 3 kHz and the carrier frequency is 200 kHz. Assuming that the average power of the modulated signal is 12 watts, determine the input signal-to-noise ratio (predetection SNR), output signal-to-noise ratio (postdetection SNR) and the detection gain (output SNR / input SNR).

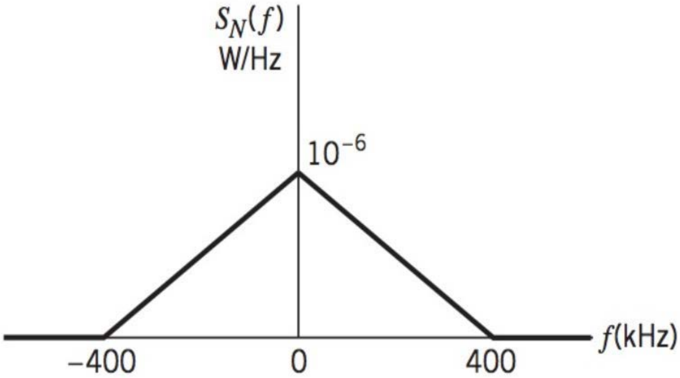

Figure 1:

Solution: The received signal is the sum of the DSB-SC modulated signal and the noise:

$$x_r(t) = x_c(t) + n(t) = A_c m(t)\cos(2\pi f_c t + \theta) + n(t) = A_c\cos(2\pi \cdot 200000t + \theta) + n(t). \tag{1}$$

After passing the bandpass filter, the received signal becomes

$$\begin{aligned} e_2(t) =& x_c(t) + n'(t) = A_c\cos(2\pi f_c t + \theta) + n_c\cos(2\pi f_c t + \theta) - n_s\sin(2\pi f_c t + \theta) \\ =& A_c\cos(2\pi \cdot 200000t + \theta) + n_c\cos(2\pi \cdot 200000t) - n_s\sin(2\pi \cdot 200000t). \end{aligned} \tag{2}$$

The average power of the modulated signal $x_c(t)$ is

$$P_T = \frac{A_c^2}{2}\overline{m^2(t)} = 12\text{W}. \tag{3}$$

According to figure 1, the power spectral density of the noise is

$$S_N(f) = 10^{-6}\Lambda(400000f). \tag{4}$$

After passing the narrowband filter, the average power of the noise component is

$$\begin{aligned} P_{n'} =& \int_{-f_c-W}^{-f_c+W} S_N(f)\,\mathrm{d}f + \int_{f_c-W}^{f_c+W} S_N(f)\,\mathrm{d}f \\ =& 2\int_{f_c-W}^{f_c+W} S_N(f)\,\mathrm{d}f \\ =& 2\int_{200000-3000}^{200000+3000} 10^{-6}\Lambda(400000f)\,\mathrm{d}f \\ =& 0.006(\text{W}). \end{aligned} \tag{5}$$

The input SNR is

$$\text{SNR}_T = \frac{P_T}{P_{n'}} = \frac{12}{0.006} = 2000. \tag{6}$$

After demodulation, the output is

$$\begin{aligned} y_D(t) =& \text{LP}[e_2(t) \cdot 2\cos(2\pi f_c t + \theta)] \\ =& \text{LP}\{A_c m(t)[1 + \cos(4\pi f_c t + 2\theta)] + n_c(t)[1 + \cos(4\pi f_c t + 2\theta)] - n_s(t)\sin(2\pi f_c t + 2\theta)\} \\ =& A_c m(t) + n_c(t). \end{aligned}$$

The average power of the demodulated signal component is

$$A_c^2 \overline{m^2(t)} = 2P_T = 24\text{W}. \tag{7}$$

The average power of the noise component is

$$P_{n_c} = P_{n'} = 0.006\text{W}. \tag{8}$$

The output SNR is

$$\text{SNR}_D = \frac{2P_T}{P_{n_c}} = \frac{24}{0.006} = 4000. \tag{9}$$

The detection gain is

$$\frac{\text{SNR}_D}{\text{SNR}_T} = 2. \tag{10}$$

□

Problem 2 (Noise in SSB Receiver, 25pts) Score: _________. Derive the equation for $y_D(t)$ for an USB-SSB system assuming that the noise is expanded about the frequency $f_c + \frac{W}{2}$. Derive the output SNR (postdetection SNR), detection gain and the figure of merit. Determine and plot the power spectral density of the in-phase component $n_c(t)$ and the quadrature component $n_s(t)$ of the narrowband noise.

Solution: The USB-SSB modulated signal is

$$x_c(t) = A_c[m(t)\cos(2\pi f_c t + \theta) - \hat{m}(t)\sin(2\pi f_c t + \theta)]. \tag{11}$$

The received signal is

$$x_r(t) = x_c(t) + w(t) = A_c[m(t)\cos(2\pi f_c t + \theta) - \hat{m}(t)\sin(2\pi f_c t + \theta)] + w(t). \tag{12}$$

After passing the band filter whose center frequency is $f_c + \frac{W}{2}$ and bandwidth is W, the signal becomes

$$\begin{aligned} e_2(t) =& A_c[m(t)\cos(2\pi f_c t + \theta) - \hat{m}(t)\sin(2\pi f_c t + \theta)] + n(t) \\ =& A_c[m(t)\cos(2\pi f_c t + \theta) - \hat{m}(t)\sin(2\pi f_c t + \theta)] \\ &+ n_c(t)\cos\left[2\pi\left(f_c + \frac{W}{2}\right)t + \theta\right] - n_s(t)\sin\left[2\pi\left(f_c + \frac{W}{2}\right)t + \theta\right]. \end{aligned} \tag{13}$$

The average power of the modulated signal is

$$P_T = A_c^2 \overline{m^2(t)} \tag{14}$$

The average power of the filtered noise is $N_0 \cdot W$. Therefore, the input SNR is

$$\text{SNR}_T = \frac{P_T}{N_0 W} = \frac{A_c^2 \overline{m^2(t)}}{N_0 W}. \tag{15}$$

After passing demodulator (i.e., multiplying with a local oscillator $2\cos(2\pi f_c t + \theta)$ and passing a lowpass filter), the output signal is

$$y_D(t) = A_c m(t) + n_{c,\text{expanded about } f_c}(t). \tag{16}$$

The average power of the demodulated signal is

$$P_D = A_c\overline{m^2(t)}. \tag{17}$$

The average power of the noise in output signal is $N_0 \cdot W$. Therefore, the output SNR is

$$\text{SNR}_D = \frac{P_D}{N_0 W} = \frac{A_c^2\overline{m^2(t)}}{N_0 W}. \tag{18}$$

The detection gain is

$$\frac{\text{SNR}_D}{\text{SNR}_T} = 1. \tag{19}$$

The channel SNR is

$$\text{SNR}_c = \frac{P_T}{N_0 W} = \frac{A_c^2\overline{m^2(t)}}{N_0 W}. \tag{20}$$

The figure of merit is

$$\frac{\text{SNR}_D}{\text{SNR}_c} = 1. \tag{21}$$

The power spectral density of the narrowband noise is

$$S_n(f) = \frac{N_0}{2}\left[\Pi\left(\frac{f - (f_c + \frac{W}{2})}{W}\right) + \Pi\left(\frac{f + (f_c + \frac{W}{2})}{W}\right)\right]. \tag{22}$$

Expanded about the frequency $f_c + \frac{W}{2}$, the power spectral density of the in-phase component $n_c(t)$ and the quadrature component $n_s(t)$ are both

$$S_{n_c}(f) = S_{n_s}(f) = \text{LP}\left[S_n\left(f - f_c - \frac{W}{2}\right) + S_n\left(f + f_c + \frac{W}{2}\right)\right] = N_0\Pi\left(\frac{f}{W}\right), \tag{23}$$

as shown in figure 2. □

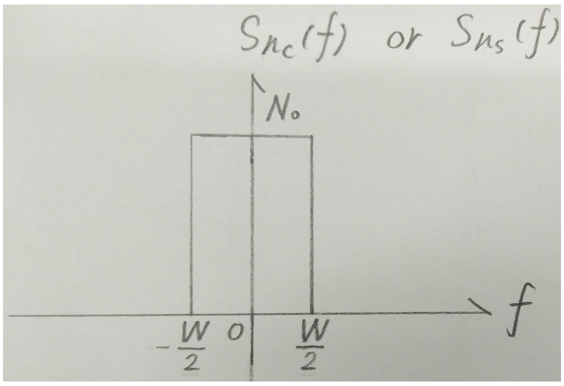

Figure 2: The power spectral density of the in-phase component $n_c(t)$ and the quadrature component $n_s(t)$

Problem 3 (Noise in AM Receiver, 25pts) Score: ________. Assume an AM system operates with a modulation index $a = 0.4$. The message signal is $m(t) = 5\cos(10\pi t)$.

1) Compute the transmission efficiency.

2) Assume the envelope detector operates above the threshold. Compute the output SNR (postdetection SNR) in decibel relative to the input SNR.

3) Compute the output SNR in decibels relative to the baseband SNR (P_T/N_0W).

4) Keep P_T (the average power of modulated signal) unchanged, determine the improvement (in decibels) in the output SNR if the modulation index is increased from 0.4 to 0.8. (Hint: Since the input SNR and baseband SNR are unchanged, we can calculate the improvement of output SNR based on its relationship with the input SNR and baseband SNR.)

Solution: 1) In AM modulation,

$$m_n(t) = \frac{m(t)}{|\min[m(t)]|} = \cos(10\pi t). \tag{24}$$

The transmission efficiency is

$$\mu = \frac{a^2\overline{m_n^2(t)}}{a^2\overline{m_n^2(t)}+1} = \frac{a^2/2}{a^2/2+1} = \frac{2}{27} = 7.41\%. \tag{25}$$

2) The AM modulated signal is

$$x_c(t) = A_c[1+am_n(t)]\cos(2\pi f_c t+\theta). \tag{26}$$

The received signal is

$$x_c(t) = x_c(t) + w(t). \tag{27}$$

After passing the bandpass filter, the signal becomes

$$e_2(t) = A_c[1+am_n(t)]\cos(2\pi f_c t+\theta) + n_c(t)\cos(2\pi f_c t+\theta) - n_s(t)\sin(2\pi f_c t+\theta) = r(t)\cos[2\pi f_c t+\phi(t)], \tag{28}$$

where the envelope of the signal is

$$r(t) = \sqrt{\{A_c[1+am_n(t)]+n_c(t)\}^2 + n_s^2(t)}. \tag{29}$$

The average power of the modulated signal is

$$P_t = \overline{\{A_c[1+am_n(t)]\cos(2\pi f_c t+\theta)\}^2} = \frac{A_c^2}{2}\left[1+a^2\overline{m_n(t)}\right]. \tag{30}$$

The average power of the narrowband noise is $2N_0W$. Therefore, the input SNR is

$$\text{SNR}_T = \frac{P_T}{2N_0W} = \frac{A_c^2\left[1+a^2\overline{m_n(t)}\right]}{4N_0W}. \tag{31}$$

After the envelope detector, the output signal is the envelope

$$y_D(t) = r(t) = \sqrt{\{A_c[1+am_n(t)]+n_c(t)\}^2 + n_s^2(t)}. \tag{32}$$

Since the envelope detector operates above the threshold, i.e., SNR_T is large and thus $|A_c[1+am_n(t)]+n_c(t)| \gg |n_s(t)|$,

$$y_D(t) \approx A_c[1+am_n(t)] + n_c(t). \tag{33}$$

The average of the modulated signal after removal of DC component is

$$P_D = \overline{[A_c a m_n(t)]^2} = A_c^2 a^2 \overline{m_n^2(t)}. \tag{34}$$

The average power of the noise in the output signal is $2N_0W$. Therefore, the output SNR is

$$\mathrm{SNR}_D = \frac{P_D}{2N_0W} = \frac{A_c^2 a^2 \overline{m_n^2(t)}}{2N_0W}. \tag{35}$$

The output SNR in decibels relative to the input SNR is

$$10\lg\frac{\mathrm{SNR}_D}{\mathrm{SNR}_T} = 10\lg\frac{2a^2\overline{m_n^2(t)}}{1+a^2\overline{m_n^2(t)}} = 10\lg 2\mu = -8.29(\mathrm{dB}). \tag{36}$$

3) The average power of the modulated signal is $P_T = \frac{A_c^2}{2}[1+a^2\overline{m_n^2(t)}]$ and the average power of the noise in the message bandwidth is N_0W, so the baseband SNR is

$$\mathrm{SNR}_c = \frac{P_T}{N_0W} = \frac{A_c^2[1+a^2\overline{m_n^2(t)}]}{2N_0W}. \tag{37}$$

The output SNR in decibels relative to the baseband SNR is

$$10\lg\frac{\mathrm{SNR}_D}{\mathrm{SNR}_c} = 10\lg\frac{a^2\overline{m_n^2(t)}}{1+a^2\overline{m_n^2(t)}} = 10\lg\mu = -11.30(\mathrm{dB}). \tag{38}$$

4) Since the baseband SNR is unchanged, we can calculate the improvement of output SNR based on its relation with the baseband SNR:

$$\begin{aligned}10\lg\frac{\mathrm{SNR}_D(a=0.8)}{\mathrm{SNR}_D(a=0.4)} =& 10\lg\frac{\mathrm{SNR}_D(a=0.8)/\mathrm{SNR}_c}{\mathrm{SNR}_D(a=0.4)/\mathrm{SNR}_c} = 10\lg\frac{\mathrm{SNR}_D(a=0.8)}{\mathrm{SNR}_c} - 10\lg\frac{\mathrm{SNR}_D(a=0.4)}{\mathrm{SNR}_c}\\ =& 10\lg\mu(a=0.8) - 10\lg(a=0.4) = 5.15\mathrm{dB}.\end{aligned} \tag{39}$$

The output SNR improves by 5.15 dB.

□

Problem 4 (Noise in FM Receiver and FDM, 20pts) Score: _________. An FDM system uses single-sideband modulation to from the baseband, and FM modulation for transmission of the baseband. Assume that there are eight channels and that all eight message signal have equal power P_0 and equal bandwidth W. For each signal, only the upper sideband is transmitted. The sub-carrier waves used for the first stage of modulation are defined by $c_k(t) = A_k\cos(2\pi k f_0 t), 0 \leq k \leq 7$. The width of the guardbands is $3W$. The received signal consists of the transmitted FM signal plus white Gaussian noise of zero mean and two-sided power spectral density $N_0/2$. Assume the frequency discriminator at the receiver operates above the threshold.

1) Sketch the power spectral density of the signal produced at the frequency discriminator output, showing both the signal and the noise components.

2) Find the relationship between the subcarrier amplitudes A_k such that the SSB modulated signals corresponding to different channels have equal output SNRs at the frequency discriminator output.

Solution: 1) Since each message signal have equal bandwidth W and the width of the guardbands is $3W$, we have

$$f_1 = \cancel{5}4W, \tag{40}$$

and the kth channel occupies the frequency range

$$\cancel{5kW = kf_1 \leq f \leq W + kf_1 = (5k+1)W, \quad -(1+5k)W = -W - kf_1 \leq f \leq -kf_1 = -5kW.} \tag{41}$$

$$4kW = kf_1 \leq f \leq W + kf_1 = (4k+1)W, \quad -(1+4k)W = -W - kf_1 \leq f \leq -kf_1 = -4kW. \tag{42}$$

Since received signal includes the white Gaussian noise of two-sided power spectral density $N_0/2$, the power spectral density of the noise component is

$$S_n(f) = \frac{K_D^2}{A_c^2} N_0 f^2. \tag{43}$$

The power spectral density of the signal and the noise components are shown in figure 3.

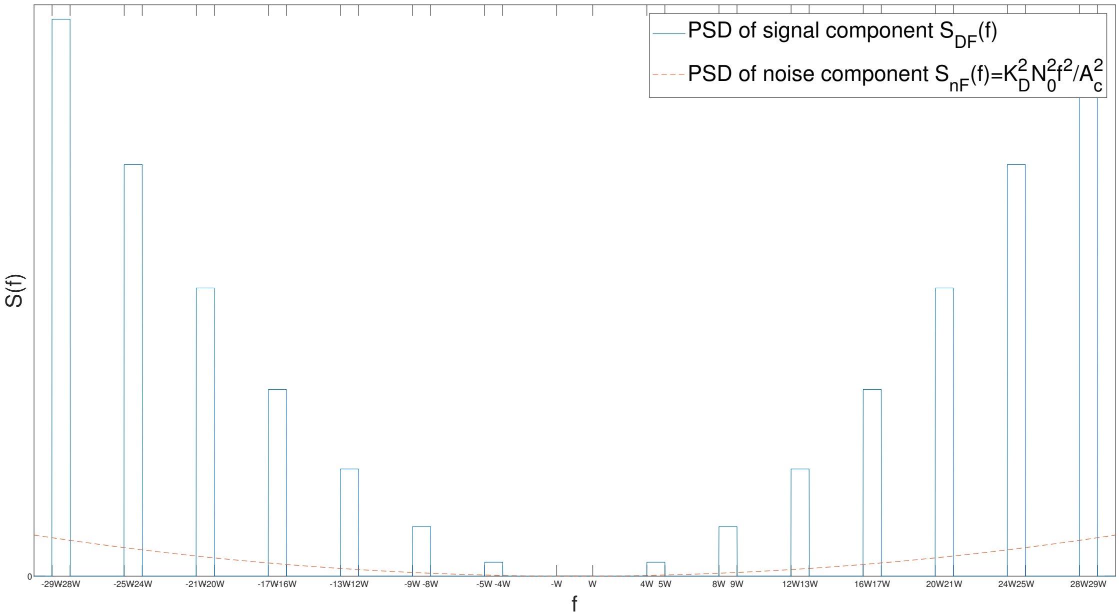

Figure 3: The power spectral density of the signal and the noise components

2) The noise power in the kth channel is

$$\cancel{P_{nk} = 2\int_{5kW}^{(5k+1)W} S_n(f)\,\mathrm{d}f = 2\int_{5kW}^{(5k+1)W} \frac{K_D^2}{A_c^2} N_0 f^2\,\mathrm{d}f = \frac{2}{3}\frac{K_D^2}{A_c^2} N_0 (75k^2 + 15k + 1)W^3.} \tag{44}$$

$$P_{nk} = 2\int_{4kW}^{(4k+1)W} S_n(f)\,\mathrm{d}f = 2\int_{4kW}^{(4k+1)W} \frac{K_D^2}{A_c^2} N_0 f^2\,\mathrm{d}f = \frac{2}{3}\frac{K_D^2}{A_c^2} N_0 (75k^2 + 15k + 1)W^3. \tag{45}$$

The signal power of each channel is

$$\cancel{P_T = \begin{cases} \frac{K_D^2 f_d^2 A_k^2}{2}, & k = 0 \\ \frac{K_D^2 f_d^2 A_k^2}{4}, & 1 \leq k \leq 7. \end{cases}} \quad P_T = \frac{K_D^2 f_d^2 A_k^2}{4}. \tag{46}$$

The output of the kth channel is

$$\mathrm{SNR}_D = \frac{P_T}{P_{nk}} = \cancel{\begin{cases} \frac{3}{4}\frac{f_d^2 A_c^2 W^3}{N_0}\frac{A_k^2}{75k^2+15k+1} = \frac{3}{4}\frac{f_d^2 A_c^2 W^3}{N_0} A_0^2, & k = 0, \\ \frac{3}{8}\frac{f_d^2 A_c^2 W^3}{N_0}\frac{A_k^2}{75k^2+15k+1}, & 1 \leq k \leq 7. \end{cases}} \quad \frac{3}{8}\frac{f_d^2 A_c^2 W^3}{N_0}\frac{A_k^2}{48k^2+12k+1}. \tag{47}$$

To make that the SSB modulated signals corresponding to different channels have equal output SNRs at the frequency dicriminator output,

$$2A_0^2 = \tfrac{A_k^2}{75k^2+15k+1}\frac{A_k^2}{48k^2+12k+1} = \text{constant}, \quad \forall \cancel{1}0 \leq k \leq 7. \tag{48}$$

$\square$