Assignment 9

Due time : 10:15, Dec 4, 2020 (Friday)

Name : 陈 稼 霖
Student ID : 45875852
Grade : ________

Problem 1 (3.1) Score: ________. Let U be an analog rv uniformly distributed between -1 and 1.

(a) Find the 3-bit ($M = 8$) quantizer that minimizes the MSE.

(b) Argue that your quantizer satisfies the necessary condition for optimality.

(c) Show that the quantizer is unique in the sense that no other 3-bit quantizer satisfies the necessary condition for optimality.

Solution: (a)

(b)

(c)

□

Problem 2 (3.3) Score: ________. Consider a binary scalar quantizer that partitions the set of reals $\mathbb{R}$ into two subsets $(-\infty, b]$ and (b, ∞) and then presents $(-\infty, b]$ by $a_1 \in \mathbb{R}$ and (b, ∞) by $a_2 \in \mathbb{R}$. This quantizer is used on each letter U_n of a sequence $\cdots, U_1, U_0, U_1, \cdots$ of iid random variables, each having the probability density $f(u)$. Assume throughout this exercise that $f(u)$ is symmetric, i.e. that $f(u) = f(-u)$ for all $u \geq 0$.

(a) Given the representation levels a_1 and $a_2 > a_1$, how should b be chosen to minimize the mean-squared distortion in the quantization? Assume that $f(u) > 0$ for $a_1 \leq u \leq a_2$ and explain why this assumption is relevant.

(b) Given $b \geq 0$, find the values of a_1 and a_2 that minimize the mean-squared distortion. Given both answer in terms of the two functions $Q(x) = \int_x^\infty f(u)\,\mathrm{d}u$ and $y(x) = \int_x^\infty u f(u)\,\mathrm{d}u$.

(c) Show that for $b = 0$, the minimizing values of a_1 and a_2 satisfy $a_1 = -a_2$.

(d) Show that the choice of b, a_1, and a_2 in part (c) satisfies the Lloyd-Max conditions for minimum mean-squared distortion.

(e) Consider the particular symmetric density

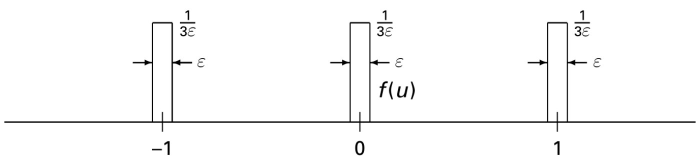

Figure 1:

Find all sets of triples $\{b, a_1, a_2\}$ that satisfy the Lloyd-Max conditions and evaluate the MSE for each. You are welcome in your calculation to replace each region of nonzero probability density above with an impulse, i.e. $f(u) = (1/3)[\delta(-1) + \delta(0) + \delta(1)]$, but you should use Figure 1 to resolve the ambiguity about regions that occurs when b is $-1, 0$ or $+1$.

(f) Given the MSE for each of your solutions above (in the limit of $\varepsilon \to 0$). Which of your solutions minimizes the MSE?

Solution: (a)

(b)

(c)

(d)

(e)

(f)

□

Problem 3 (3.4) Score: _________. Section 3.4 partly analyzed a minimum-MSE quantizer for a pdf in which $f_U(u) = f_1$ over an interval of size L_1, $f_U(u) = f_2$ over an interval of size L_2, and $f_U(u) = 0$ elsewhere. Let M be the total number of representation points to be used, with M_1 in the first interval and $M_2 = M - M_1$ in the second. Assume (from symmetry) that the quantization intervals are of equal size $\Delta_1 = L_1/M_1$ in interval 1 and of equal size $\Delta_2 = L_2/M_2$ in interval 2. Assume that M is very large, so that we can approximately minimize the MSE over M_1, M_2 without an integer constraint on M_1, M_2 (that is, assume that M_1, M_2 can be arbitrary real numbers).

(a) Show that the MSE in minimized if $\Delta_1 f_1^{1/3} = \Delta_2 f_2^{1/3}$, i.e. the quantization interval sizes are inversely proportional to the cube root of the density. [Hint. Use a Lagrange multiplier to perform the minimization. That is, to minimize a function $\mathrm{MSE}(\Delta_1, \Delta_2)$ subject to a constraint $M = f(\Delta_1, \Delta_2)$, first minimize $\mathrm{MSE}(\Delta_1, \Delta_2) + \lambda f(\Delta_1, \Delta_2)$ without the constraint, and, second, choose λ so that the solution meets the constraint.]

(b) Show that the minimum MSE under the above assumption is given by

$$\mathrm{MSE} = \frac{(L_1 f_1^{1/3} + L_2 f_2^{1/3})^3}{12M^2}.$$

(c) Assume that the Lloyd-Max algorithm is started with $0 < M_1 < M$ representation points in the first interval and $M_2 = M - M_1$ points in the second interval. Explain where the Lloyd-Max algorithm converges for this starting point. Assume from here on that the distance between the two intervals is very large.

(d) Redo part (c) under the assumption that the Lloyd-Max algorithm is started with $0 < M_1 < M - 2$ representation points in the first interval, one point between the two intervals, and the remaining points in the second interval.

(e) Express the exact minimum MSE as a minimum over $M - 1$ possibilities, with one term for each choice of $0 < M_1 < M$. (Assume there are no representation points between the two intervals.)

(f) Now consider an arbitrary choice of Δ_1 and Δ_2 (with no constraint on M). Show that the entropy of the set of quantization points is given by

$$H(V) = -f_1 L_1 \log(f_1 \Delta_1) - f_2 L_2 \log(f_2 \Delta_2).$$

(g) Show that if the MSE is minimized subject to a constraint on the entropy (ignoring the integer constraint on quantization level), then $\Delta_1 = \Delta_2$.

Solution: (a)

(b)

(c)

(d)

(e)

(f)

(g)

□

Problem 4 (3.5) Score: ________. (a) Assume that a continuous-valued rv Z has probability density that is 0 except over the interval $[-A, +A]$. Show that the differential entropy $h(Z)$ is upperbounded $1 + \log_2 A$.

(b) Show that $h(Z) = 1 + \log_2 A$ if and only if Z is uniformly distributed between $-A$ and $+A$.

Solution: (a)

(b)

□