

**Problem 1 (Noise in DSB-SC Receiver, 30pts) Score:** \_\_\_\_\_. A DSB-SC modulated signal is transmitted over a noisy channel. The power spectral density of the noise is shown in Figure 1. The message bandwidth is 3 kHz and the carrier frequency is 200 kHz. Assuming that the average power of the modulated signal is 12 watts, determine the input signal-to-noise ratio (predetection SNR), output signal-to-noise ratio (postdetection SNR) and the detection gain (output SNR / input SNR).

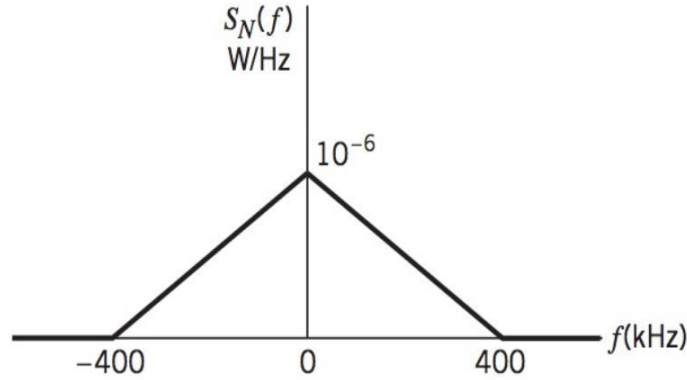


Figure 1:

**Solution:** The received signal is the sum of the DSB-SC modulated signal and the noise:

$$x_r(t) = x_c(t) + n(t) = A_c m(t) \cos(2\pi f_c t + \theta) + n(t) = A_c \cos(2\pi \cdot 200000t + \theta) + n(t). \quad (1)$$

After passing the bandpass filter, the received signal becomes

$$\begin{aligned} e_2(t) &= x_c(t) + n'(t) = A_c \cos(2\pi f_c t + \theta) + n_c \cos(2\pi f_c t + \theta) - n_s \sin(2\pi f_c t + \theta) \\ &= A_c \cos(2\pi \cdot 200000t + \theta) + n_c \cos(2\pi \cdot 200000t) - n_s \sin(2\pi \cdot 200000t). \end{aligned} \quad (2)$$

The average power of the modulated signal  $x_c(t)$  is

$$P_T = \frac{A_c^2}{2} \overline{m^2(t)} = 12 \text{ W}. \quad (3)$$

According to figure 1, the power spectral density of the noise is

$$S_N(f) = 10^{-6} \Lambda(400000f). \quad (4)$$

After passing the narrowband filter, the average power of the noise component is

$$\begin{aligned} P_{n'} &= \int_{-f_c-W}^{-f_c+W} S_N(f) df + \int_{f_c-W}^{f_c+W} S_N(f) df \\ &= 2 \int_{f_c-W}^{f_c+W} S_N(f) df \\ &= 2 \int_{200000-3000}^{200000+3000} 10^{-6} \Lambda(400000f) df \\ &= 0.006 \text{ (W)}. \end{aligned} \quad (5)$$

The input SNR is

$$\text{SNR}_T = \frac{P_T}{P_{n'}} = \frac{12}{0.006} = 2000. \quad (6)$$

After demodulation, the output is

$$\begin{aligned} y_D(t) &= \text{LP}[e_2(t) \cdot 2 \cos(2\pi f_c t + \theta)] \\ &= \text{LP}\{A_c m(t)[1 + \cos(4\pi f_c t + 2\theta)] + n_c(t)[1 + \cos(4\pi f_c t + 2\theta)] - n_s(t) \sin(2\pi f_c t + \theta)\} \\ &= A_c m(t) + n_c(t). \end{aligned}$$

The average power of the demodulated signal component is

$$A_c^2 \overline{m^2(t)} = 2P_T = 24\text{W}. \quad (7)$$

The average power of the noise component is

$$P_{n_c} = P_{n'} = 0.006\text{W}. \quad (8)$$

The output SNR is

$$\text{SNR}_D = \frac{2P_T}{P_{n_c}} = \frac{24}{0.006} = 4000. \quad (9)$$

The detection gain is

$$\frac{\text{SNR}_D}{\text{SNR}_T} = 2. \quad (10)$$

□

**Problem 2 (Noise in SSB Receiver, 25pts) Score:** \_\_\_\_\_. Derive the equation for  $y_D(t)$  for an USB-SSB system assuming that the noise is expanded about the frequency  $f_c + \frac{W}{2}$ . Derive the output SNR (postdetection SNR), detection gain and the figure of merit. Determine and plot the power spectral density of the in-phase component  $n_c(t)$  and the quadrature component  $n_s(t)$  of the narrowband noise.

**Solution:** The USB-SSB modulated signal is

$$x_c(t) = A_c[m(t) \cos(2\pi f_c t + \theta) - \hat{m}(t) \sin(2\pi f_c t + \theta)]. \quad (11)$$

The received signal is

$$x_r(t) = x_c(t) + w(t) = A_c[m(t) \cos(2\pi f_c t + \theta) - \hat{m}(t) \sin(2\pi f_c t + \theta)] + w(t). \quad (12)$$

After passing the band filter whose center frequency is  $f_c + \frac{W}{2}$  and bandwidth is  $W$ , the signal becomes

$$\begin{aligned} e_2(t) &= A_c[m(t) \cos(2\pi f_c t + \theta) - \hat{m}(t) \sin(2\pi f_c t + \theta)] + n(t) \\ &= A_c[m(t) \cos(2\pi f_c t + \theta) - \hat{m}(t) \sin(2\pi f_c t + \theta)] \\ &\quad + n_c(t) \cos\left[2\pi\left(f_c + \frac{W}{2}\right)t + \theta\right] - n_s(t) \sin\left[2\pi\left(f_c + \frac{W}{2}\right)t + \theta\right]. \end{aligned} \quad (13)$$

The average power of the modulated signal is

$$P_T = A_c^2 \overline{m^2(t)} \quad (14)$$

The average power of the filtered noise is  $N_0 \cdot W$ . Therefore, the input SNR is

$$\text{SNR}_T = \frac{P_T}{N_0 W} = \frac{A_c^2 \overline{m^2(t)}}{N_0 W}. \quad (15)$$

After passing demodulator (i.e., multiplying with a local oscillator  $2 \cos(2\pi f_c t + \theta)$  and passing a lowpass filter), the output signal is

$$y_D(t) = A_c m(t) + n_{c, \text{expanded about } f_c}(t). \quad (16)$$

The average power of the demodulated signal is

$$P_D = A_c \overline{m^2(t)}. \quad (17)$$

The average power of the noise in output signal is  $N_0 \cdot W$ . Therefore, the output SNR is

$$\text{SNR}_D = \frac{P_D}{N_0 W} = \frac{A_c^2 \overline{m^2(t)}}{N_0 W}. \quad (18)$$

The detection gain is

$$\frac{\text{SNR}_D}{\text{SNR}_T} = 1. \quad (19)$$

The channel SNR is

$$\text{SNR}_c = \frac{P_T}{N_0 W} = \frac{A_c^2 \overline{m^2(t)}}{N_0 W}. \quad (20)$$

The figure of merit is

$$\frac{\text{SNR}_D}{\text{SNR}_c} = 1. \quad (21)$$

The power spectral density of the narrowband noise is

$$S_n(f) = \frac{N_0}{2} \left[ \Pi \left( \frac{f - (f_c + \frac{W}{2})}{W} \right) + \Pi \left( \frac{f + (f_c + \frac{W}{2})}{W} \right) \right]. \quad (22)$$

Expanded about the frequency  $f_c + \frac{W}{2}$ , the power spectral density of the in-phase component  $n_c(t)$  and the quadrature component  $n_s(t)$  are both

$$S_{n_c}(f) = S_{n_s}(f) = \text{LP} \left[ S_n \left( f - f_c - \frac{W}{2} \right) + S_n \left( f + f_c + \frac{W}{2} \right) \right] = N_0 \Pi \left( \frac{f}{W} \right), \quad (23)$$

as shown in figure 2. □

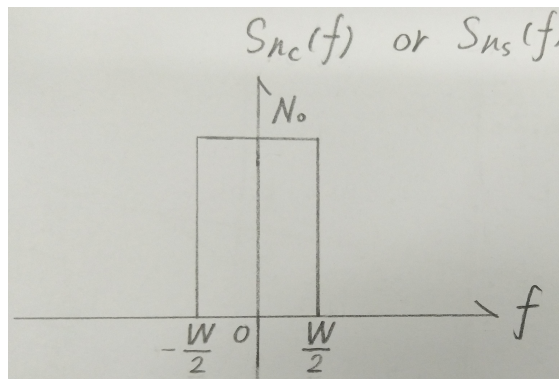


Figure 2: The power spectral density of the in-phase component  $n_c(t)$  and the quadrature component  $n_s(t)$

**Problem 3 (Noise in AM Receiver, 25pts) Score:** \_\_\_\_\_. Assume an AM system operates with a modulation index  $a = 0.4$ . The message signal is  $m(t) = 5 \cos(10\pi t)$ .

- 1) Compute the transmission efficiency.
- 2) Assume the envelope detector operates above the threshold. Compute the output SNR (postdetection SNR) in decibel relative to the input SNR.
- 3) Compute the output SNR in decibels relative to the baseband SNR ( $P_T/N_0W$ ).
- 4) Keep  $P_T$  (the average power of modulated signal) unchanged, determine the improvement (in decibels) in the output SNR if the modulation index is increased from 0.4 to 0.8. (Hint: Since the input SNR and baseband SNR are unchanged, we can calculate the improvement of output SNR based on its relationship with the input SNR and baseband SNR.)

**Solution:** 1) In AM modulation,

$$m_n(t) = \frac{m(t)}{|\min[m(t)]|} = \cos(10\pi t). \quad (24)$$

The transmission efficiency is

$$\mu = \frac{a^2 \overline{m_n^2(t)}}{a^2 \overline{m_n^2(t)} + 1} = \frac{a^2/2}{a^2/2 + 1} = \frac{2}{27}. \quad (25)$$

- 2) The AM modulated signal is

$$x_c(t) = A_c[1 + am_n(t)] \cos(2\pi f_c t + \theta). \quad (26)$$

The received signal is

$$x_c(t) = x_c(t) + w(t). \quad (27)$$

After passing the bandpass filter, the signal becomes

$$e_2(t) = A_c[1 + am_n(t)] \cos(2\pi f_c t + \theta) + n_c(t) \cos(2\pi f_c t + \theta) - n_s(t) \sin(2\pi f_c t + \theta) = r(t) \cos[2\pi f_c t + \phi(t)], \quad (28)$$

where the envelope of the signal is

$$r(t) = \sqrt{\{A_c[1 + am_n(t)] + n_c(t)\}^2 + n_s^2(t)}. \quad (29)$$

The average power of the modulated signal is

$$P_t = \overline{\{A_c[1 + am_n(t)] \cos(2\pi f_c t + \theta)\}^2} = \frac{A_c^2}{2} [1 + a^2 \overline{m_n(t)}]. \quad (30)$$

The average power of the narrowband noise is  $2N_0W$ . Therefore, the input SNR is

$$\text{SNR}_T = \frac{P_T}{2N_0W} = \frac{A_c^2 [1 + a^2 \overline{m_n(t)}]}{4N_0W}. \quad (31)$$

After the envelope detector, the output signal is the envelope

$$y_D(t) = r(t) = \sqrt{\{A_c[1 + am_n(t)] + n_c(t)\}^2 + n_s^2(t)}. \quad (32)$$

Since the envelope detector operates above the threshold, i.e.,  $\text{SNR}_T$  is large and thus  $|A_c[1 + am_n(t)] + n_c(t)| \gg |n_s(t)|$ ,

$$y_D(t) \approx A_c[1 + am_n(t)] + n_c(t). \quad (33)$$

The average of the modulated signal after removal of DC component is

$$P_D = \overline{[A_c a m_n(t)]^2} = A_c^2 a^2 \overline{m_n^2(t)}. \quad (34)$$

The average power of the noise in the output signal is  $2N_0W$ . Therefore, the output SNR is

$$\text{SNR}_D = \frac{P_D}{2N_0W} = \frac{A_c^2 a^2 \overline{m_n^2(t)}}{2N_0W}. \quad (35)$$

The output SNR in decibels relative to the input SNR is

$$10 \lg \frac{\text{SNR}_D}{\text{SNR}_T} = 10 \lg \frac{2a^2 \overline{m_n^2(t)}}{1 + a^2 \overline{m_n^2(t)}} = 10 \lg 2\mu = -8.29(\text{dB}). \quad (36)$$

- 3) The average power of the modulated signal is  $P_T = \frac{A_c^2}{2}[1 + a^2 \overline{m_n^2(t)}]$  and the average power of the noise in the message bandwidth is  $N_0W$ , so the baseband SNR is

$$\text{SNR}_c = \frac{P_T}{N_0W} = \frac{A_c^2[1 + a^2 \overline{m_n^2(t)}]}{2N_0W}. \quad (37)$$

The output SNR in decibels relative to the baseband SNR is

$$10 \lg \frac{\text{SNR}_D}{\text{SNR}_c} = 10 \lg \frac{a^2 \overline{m_n^2(t)}}{1 + a^2 \overline{m_n^2(t)}} = 10 \lg \mu = -11.30(\text{dB}). \quad (38)$$

- 4) Since the baseband SNR is unchanged, we can calculate the improvement of output SNR based on its relation with the baseband SNR:

$$\begin{aligned} 10 \lg \frac{\text{SNR}_D(a=0.8)}{\text{SNR}_D(a=0.4)} &= 10 \lg \frac{\text{SNR}_D(a=0.8)/\text{SNR}_c}{\text{SNR}_D(a=0.4)/\text{SNR}_c} = 10 \lg \frac{\text{SNR}_D(a=0.8)}{\text{SNR}_c} - 10 \lg \frac{\text{SNR}_D(a=0.4)}{\text{SNR}_c} \\ &= 10 \lg \mu(a=0.8) - 10 \lg \mu(a=0.4) = 5.15\text{dB}. \end{aligned} \quad (39)$$

The output SNR improves by 5.15 dB.

□

**Problem 4 (Noise in FM Receiver and FDM, 20pts) Score: \_\_\_\_\_.** An FDM system uses single-sideband modulation to form the baseband, and FM modulation for transmission of the baseband. Assume that there are eight channels and that all eight message signals have equal power  $P_0$  and equal bandwidth  $W$ . For each signal, only the upper sideband is transmitted. The sub-carrier waves used for the first stage of modulation are defined by  $c_k(t) = A_k \cos(2\pi k f_0 t)$ ,  $0 \leq k \leq 7$ . The width of the guardbands is  $3W$ . The received signal consists of the transmitted FM signal plus white Gaussian noise of zero mean and two-sided power spectral density  $N_0/2$ . Assume the frequency discriminator at the receiver operates above the threshold.

- 1) Sketch the power spectral density of the signal produced at the frequency discriminator output, showing both the signal and the noise components.
- 2) Find the relationship between the subcarrier amplitudes  $A_k$  such that the SSB modulated signals corresponding to different channels have equal output SNRs at the frequency discriminator output.

**Solution:** 1) Since each message signal has equal bandwidth  $W$  and the width of the guardbands is  $3W$ , we have

$$f_1 = 5W, \quad (40)$$

and the  $k$ th channel occupies the frequency range

$$5kW = kf_1 \leq f \leq W + kf_1 = (5k+1)W, \quad -(1+5k)W = -W - kf_1 \leq f \leq -kf_1 = -5kW. \quad (41)$$

Since received signal includes the white Gaussian noise of two-sided power spectral density  $N_0/2$ , the power spectral density of the noise component is

$$S_n(f) = \frac{K_D^2}{A_c^2} N_0 f^2. \quad (42)$$

The power spectral density of the signal and the noise components are shown in figure 3.

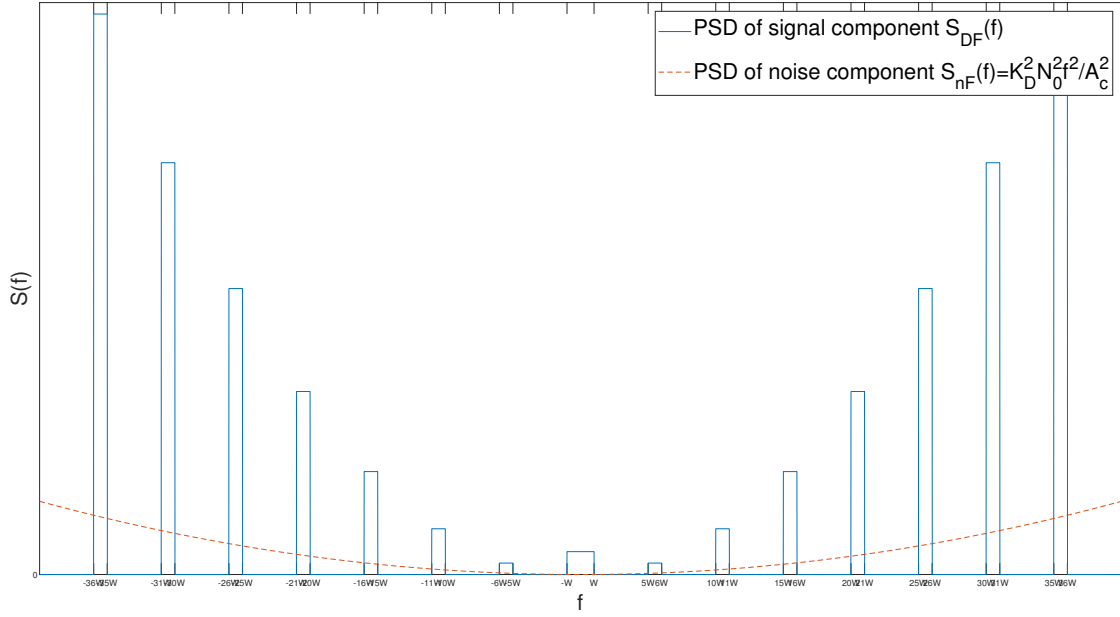


Figure 3: The power spectral density of the signal and the noise components

2) The noise power in the  $k$ th channel is

$$P_{nk} = 2 \int_{5kW}^{(5k+1)W} S_n(f) df = 2 \int_{5kW}^{(5k+1)W} \frac{K_D^2}{A_c^2} N_0 f^2 df = \frac{2}{3} \frac{K_D^2}{A_c^2} N_0 (75k^2 + 15k + 1) W^3. \quad (43)$$

The signal power of each channel is

$$P_T = \begin{cases} \frac{K_D^2 f_d^2 A_k^2}{2}, & k = 0 \\ \frac{K_D^2 f_d^2 A_k^2}{4}, & 1 \leq k \leq 7. \end{cases} \quad (44)$$

The output of the  $k$ th channel is

$$\text{SNR}_D = \frac{P_T}{P_{nk}} = \begin{cases} \frac{3}{4} \frac{f_d^2 A_c^2 W^3}{N_0} \frac{A_k^2}{75k^2 + 15k + 1} = \frac{3}{4} \frac{f_d^2 A_c^2 W^3}{N_0} A_0^2, & k = 0, \\ \frac{3}{8} \frac{f_d^2 A_c^2 W^3}{N_0} \frac{A_k^2}{75k^2 + 15k + 1}, & 1 \leq k \leq 7. \end{cases} \quad (45)$$

To make that the SSB modulated signals corresponding to different channels have equal output SNRs at the frequency discriminator output,

$$2A_0^2 = \frac{A_k^2}{75k^2 + 15k + 1} = \text{constant}, \quad \forall 1 \leq k \leq 7. \quad (46)$$

□