Assignment 8

Due time : 10:15, Nov 27, 2020 (Friday)

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Grade : _____

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Problem 1 (2.11) Score: _____. Proof of the Kraft inequality for uniquely decodable codes.

(a) Assume a uniquely decodable code has lengths $l_1, \dots l_M$. In order to show that $\sum_j 2^{-l_j} \le 1$, demonstrate the following identity for each integer $n \ge 1$:

$$\left[\sum_{i=1}^{M} 2^{-l_i}\right]^n = \sum_{i_1=1}^{M} \sum_{j_2=1}^{M} \cdots \sum_{i_n=1}^{M} 2^{-(l_{j_1} + l_{j_2} + \cdots + l_{j_n})}.$$

- (b) Show that there is one term on the right for each concatenation of n codewords (i.e. for the encoding of one n-tuple x^n) where $l_{j_1} + l_{j_2} + \cdots + l_{j_n}$ is the aggregate length of that concatenation.
- (c) Let A_i be the number of concatenations which have overall length i and show that

$$\left[\sum_{j=1}^{M} 2^{-l_j}\right]^n = \sum_{i=1}^{nl_{max}} A_i 2^{-i}.$$

(d) Using the unique decodability, upperbound each A_i and show that

$$\left[\sum_{j=1}^{M} 2^{-l_j}\right]^n \le n l_{\max}.$$

(e) By taking the nth root and letting $n \to \infty$, demonstrate the Kraft inequality.

Proof: (a)

$$\left[\sum_{j=1}^{M} M 2^{-l_j}\right]^n = \left[\sum_{j_1=1}^{M} 2^{-l_{j_1}}\right] \left[\sum_{j_2=1}^{M} 2^{-l_{j_2}}\right] \cdots \left[\sum_{j_n=1}^{M} 2^{-l_{j_n}}\right] = \sum_{j_1=1}^{M} \sum_{j_2=1}^{M} \cdots \sum_{j_n=1}^{M} 2^{-(l_{j_1}+l_{j_2}+\cdots+l_{j_n})}, \quad \forall n \ge 1. \quad (1)$$

- (b) For each concatenation of n codewords x^n , the length of the kth codeword is l_{j_k} and there is n codewords in total $(1 \le k \le n)$, so the aggregate length of the concatenation is $l_{j_1} + l_{j_2} + \cdots + l_{j_n}$.
- (c) Using the conclusion we obtained in (b), we have

$$\left[\sum_{j=1}^{M} 2^{-l_j}\right]^n = \sum_{\boldsymbol{x}^n} e^{-(\text{length of } \boldsymbol{x}^n)} = \sum_{i=1}^{nl_{\text{max}}} A_i 2^{-i}.$$
 (2)

(d) Because of unique decodability (i.e. each concatenation should be different),

$$A_i \le 2^i, \quad \forall i.$$
 (3)

Thus,

$$\left[\sum_{j=1}^{M} 2^{-l_j}\right]^n = \sum_{i=1}^{nl_{\text{max}}} A_i 2^{-i} \le \sum_{i=1}^{nl_{\text{max}}} 2^i 2^{-i} = \sum_{i=1}^{nl_{\text{max}}} 1 = nl_{\text{max}}.$$
 (4)

(e) Taking the nth root of equation (4), we have

$$\sum_{j=1}^{M} 2^{-l_j} \le [nl_{\text{max}}]^{1/n}.$$
 (5)

Letting $n \to \infty$, we have

$$\sum_{i=1}^{M} 2^{-l_j} \le \lim_{n \to \infty} [nl_{\text{max}}]^{1/n} = 1, \tag{6}$$

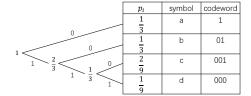
which is the Kraft inequality.

Problem 2 (2.12) Score: _____. A source with an alphabet size of $M = |\mathcal{X}| = 4$ has a symbol probabilities $\{1/3, 1/3, 2/9, 1/9\}$.

- (a) Use the Huffman algorithm to find an optimal prefix-free code for this source.
- (b) Use the Huffman algorithm to find another optimal prefix-free code with a different set of lengths.
- (c) Find another prefix-free code that is optimal but cannot result from using the Huffman algorithm.

Solution: Suppose the four symbols corresponding to the probabilities $\{1/3, 1/3, 2/9, 1/9\}$ are a,b,c,d.

- (a) An optimal prefix-free code for this source is shown in figure 1(a).
- (b) Another optimal prefix-free code for this source is shown in figure 1(b).



	p_i	symbol	codeword
2 0	$\frac{1}{3}$	а	00
$0 \frac{2}{3} 1$	$\frac{1}{3}$	b	01
1 0	2 9	С	10
1 $\frac{1}{3}$ $\frac{1}{1}$	1 9	d	11

(a) An optimal prefix-free code derived from the Huffman algo- (b) Another optimal prefix-free code derived from the Huffman rithm.

Figure 1: Two optimal prefix-free code scheme.

(c) Another prefix-free code that is optimal but cannot result from using the Huffman algorithm is shown in table 1.

Table 1: Another prefix-free code that is optimal but cannot result from using the Huffman algorithm

symbol	codeword		
a	00		
b	11		
С	10		
d	01		

Problem 3 (2.14) Score: _____. Consider a source with M equiprobable symbols.

- (a) Let $k = \lceil \log M \rceil$. Show that, for a Huffman code, the only possible codeword lengths are k and k-1.
- (b) As a function of M, find how many codewords have length $k = \lceil \log M \rceil$. What is the expected codeword length \bar{L} in bits per source code?
- (c) Define $y = M/2^k$. Express $\bar{L} \log M$ as a function of y. Find the maximum value of this function over $1/2 < y \le 1$. This illustrates that the entropy bound, $\bar{L} = H[X] + 1$, is rather loose in this equiprobable case.

- **Solution:** (a) For a Huffman code, if M is a power of 2, say $M=2^k$, then the Huffman tree should be a full binary tree and the length is $k=\log_2 M$ for all codewords. If M is not a power of 2, say $M=2^{k-1}+k_0$ where $0 < k_0 \le 2^{k-1}$, the Huffman tree should be a complete but not full binary tree. In this case, some codewords are at the bottom layer of the Huffman tree whose lengths are all $k = \lceil \log_2 M \rceil$, other codewords are at the bottom but one layer whose lengths are all $k 1 = \lceil \log_2 M \rceil 1$.
 - (b) Suppose the number of codewords with length k is x, then the number of codewords with length k-1 is M-x. The number of node at the bottom but one layer of the Huffman tree should be

$$\frac{x}{2} + (M - x) = 2^{k-1},\tag{7}$$

so the number of codeword with length k is

$$x = 2M - 2^k. (8)$$

The expected code length per source code is

$$\bar{L} = \frac{k(2M - 2^k) + (k - 1)(2^k - M)}{M} = k + 1 - \frac{2^k}{M}.$$
(9)

(c) Using $k = \log_2 \frac{M}{y}$, we have

$$\bar{L} - \log_2 M = -\log_2 y + 1 - \frac{1}{y}.\tag{10}$$

The derivative of the above function is

$$\frac{\mathrm{d}}{\mathrm{d}y} = -\frac{1}{y\ln 2} + \frac{1}{y^2} \begin{cases} > 0, & \frac{1}{2} \le y < \ln 2, \\ < 0, & \ln 2 < y < 1, \end{cases}$$
 (11)

so the maximum value of the above function is

$$\left[\bar{L} - \log_2 M\right]_{\text{max}} = \left[\bar{L} - \log_2 M\right]_{y=\ln 2} = -\log_2(\ln 2) + 1 - \frac{1}{\ln 2} = 0.086,$$
 (12)

which means that

$$\bar{L} \le \log_2 M + 0.086 \le H(X) + 0.086.$$
 (13)

Therefore, the entropy bound L = H[X] + 1, is rather loose in this equiprobable case.

Problem 4 (2.21) Score: _____. A discrete memoryless source emits iid random symbols X_1, X_2, \cdots Each random symbol X has the symbols $\{a, b, c\}$ with probabilities $\{0.5, 0.4, 0.1\}$, respectively.

- (a) Find the expected length \bar{L}_{\min} of the best variable-length prefix-free code for X.
- (b) Find the expected length $\bar{L}_{\text{min},2}$, normalized to bits per symbol, of the best variable-length prefix-free code for X^2 .
- (c) Is it true that for any DMS, $\bar{L}_{\min} \geq \bar{L}_{\min,2}$? Explain your answer.

Solution: (a) The best variable prefix-free code for X is shown in figure 2(a), whose expected length is

$$\bar{L}_{\min} = 0.5 \times 1 + 0.4 \times 2 + 0.1 \times 2 = 1.5$$
 (14)

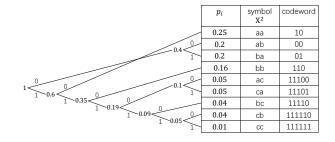
(b) The best variable prefix-free code for X^2 is shown in figure 2(b), whose expected length normalized to bits per symbol is

$$\bar{L}_{\min,2} = \frac{1}{2} (0.25 \times 2 + 0.2 \times 2 + 0.2 \times 2 + 0.16 \times 3 + 0.05 \times 5 + 0.05 \times 5 + 0.04 \times 5 + 0.04 \times 6 + 0.01 \times 6)$$

$$= 1.39 \quad . \tag{15}$$

(c) It is true that for any DMS, $\bar{L}_{\min} \geq \bar{L}_{\min,2}$. One method for source coding of X^2 is to use the concatenation of two codewords of X as the codeword of X^2 , whose expected length per symbol equals L_{\min} . Because this method is not necessarily the best coding method, L_{\min} can not be less than $\bar{L}_{\min,2}$, which is the minimal expected length per symbol of X^2 .

	p_i	symbol X	codeword
0	0.5	а	0
1 < 05	0.4	b	10
1 0.5	0.1	С	11



- (a) The best variable-length prefix-free code for X
- (b) The best variable-length prefix-free code for X^2

Figure 2: The best variable-length prefix-free code schemes.

Problem 5 (2.33) Score: _____. Perform an LZ77 parsing of the string $\underline{00011101}0010101100$. Assume a window of length W=8; the initial window is underlined above. You should parse the rest of the string using the Lempel-Ziv algorithm.

Solution: The rest of the string can be parsed as

$$\underbrace{00011101}^{u=7,n=3}\underbrace{001}^{u=2,n=4}\underbrace{0101}^{u=8,n=3}\underbrace{100}$$

whose corresponding encoded sequence is

011 111 00100 010 011 000.

Problem 6 (4.35 Aliasing) Score: _____. The following exercise is designed to illustrate the sampling of an approximately baseband waveform. To avoid messy computation, we look at a waveform baseband-limited to 3/2 which is sampled at rate 1 (i.e. sampled at only 1/3 the rate that it should be sampled at). In particular, let u(t) = sinc(3t).

- (a) Sketch $\hat{u}(f)$. Sketch the function $\hat{v}_m(f) = \text{rect}(f m)$ for each integer m such that $v_m(f) \neq 0$. Note that $\hat{u}(f) = \sum_m \hat{v}_m(f)$.
- (b) Sketch the inverse transforms $v_m(t)$ (real and imaginary parts if complex).

- (c) Verify directly from the equations that $u(t) = \sum v_m(t)$. [Hint. This is easier if you express the sine part of the sinc function as a sum of complex exponentials.]
- (d) Verify the sinc-weighted sinusoid expansion, (4.73). (There are only three nonzero terms in the expansion.)
- (e) For the approximation $s(t) = u(0) \operatorname{sinc}(t)$, find the energy in the difference between u(t) and s(t) and interpret the terms.

Solution: (a) The fourier transforms of u(t)

$$\hat{u}(f) = \mathscr{F}[u(t)] = \frac{1}{3} \operatorname{rect}\left(\frac{f}{3}\right),$$
 (16)

as shown in figure 3.

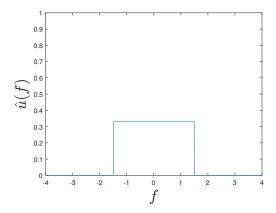


Figure 3: $\hat{u}(f)$

The segment functions are

$$\hat{v}_0(f) = \frac{1}{3} \operatorname{rect}(f),$$
 (17)

$$\hat{v}_0(f) = \frac{1}{3} \operatorname{rect}(f),$$

$$\hat{v}_1(f) = \frac{1}{3} \operatorname{rect}(f - 1),$$
(18)

$$\hat{v}_{-1}(f) = \frac{1}{3} \operatorname{rect}(f+1), \tag{19}$$

as shown in figure 4.

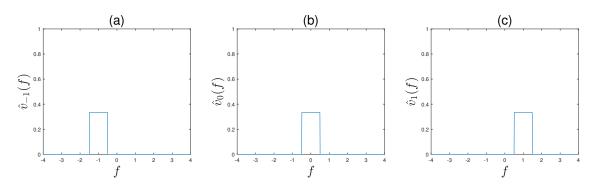


Figure 4: $\hat{v}_m(f)$ for $m = 0, \pm 1$.

(b) The inverse transform of $\hat{v}_0(f)$ is

$$v_0(t) = \mathcal{F}^{-1}[\hat{v}_0(t)] = \frac{1}{3}\operatorname{sinc}(t).$$
 (20)

as shown in figure 5.

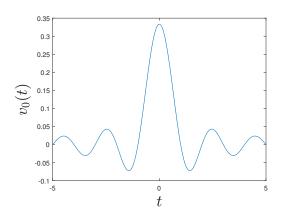


Figure 5: $v_0(t)$.

The inverse transform of $\hat{v}_1(f)$ is

$$v_1(t) = \mathcal{F}^{-1}[\hat{v}_1(t)] = \frac{1}{3}\operatorname{sinc}(t)e^{2\pi it},$$
 (21)

whose real part is

$$\operatorname{Re}\left[v_{1}(t)\right] = \frac{1}{3}\operatorname{sinc}\left(t\right)\cos(2\pi t),\tag{22}$$

and imaginary part is

$$Im [v_1(t)] = \frac{1}{3} \operatorname{sinc}(t) \sin(2\pi t),$$
 (23)

as shown in figure 6.

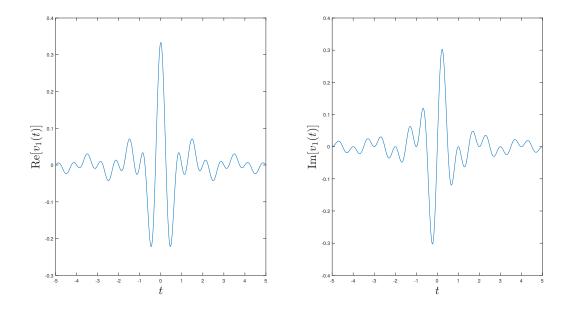


Figure 6: Real part and imaginary part of $v_1(t)$.

The inverse transform of $\hat{v}_{-1}(f)$ is

$$v_{-1}(t) = \mathcal{F}^{-1}[\hat{v}_{-1}(t)] = \frac{1}{3}\operatorname{sinc}(t)e^{-2\pi it},$$
(24)

whose real part is

$$\operatorname{Re}[v_{-1}(t)] = \frac{1}{3}\operatorname{sinc}(t)\cos(2\pi t),$$
 (25)

and imaginary part is

$$\operatorname{Im} [v_{-1}(t)] = -\frac{1}{3}\operatorname{sinc}(t)\sin(2\pi t), \tag{26}$$

as shown in figure 7.

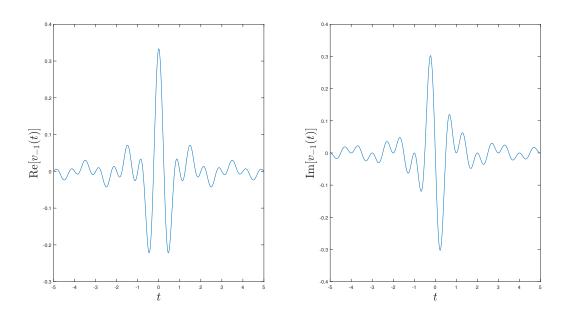


Figure 7: Real part and imaginary part of $v_{-1}(t)$.

(c)
$$\sum_{m} v_{m}(t) = v_{0}(t) + v_{1}(t) + v_{-1}(t) = \frac{1}{3} \frac{e^{\pi it} - e^{-\pi it}}{2\pi it} + \frac{1}{3} \frac{e^{\pi it} - e^{-\pi it}}{2\pi it} e^{2\pi it} + \frac{1}{3} \frac{e^{\pi it} - e^{-\pi it}}{2\pi it} e^{-2\pi it}$$

$$= \frac{1}{3} \frac{e^{3\pi it} - e^{-3\pi it}}{2\pi it} = \frac{\sin(3\pi t)}{3\pi t} = \sin(3t) = u(t) \tag{27}$$

(d) Sampling period T = 1,

$$\sum_{m,k} v_m(kT) \operatorname{sinc}\left(\frac{t}{T} - k\right) e^{2\pi i m t/T} = \sum_{m=0,\pm 1} v_m(0) \operatorname{sinc}(t) e^{2\pi i m t} = \frac{1}{3} \sin(t) + \frac{1}{3} \sin(t) e^{2\pi i t} + \frac{1}{3} \sin(t) e^{-2\pi i t}$$

$$= \frac{1}{3} \frac{e^{\pi i t} - e^{-\pi i t}}{2\pi i t} + \frac{1}{3} \frac{e^{\pi i t} - e^{-\pi i t}}{2\pi i t} e^{2\pi i t} + \frac{1}{3} \frac{e^{\pi i t} - e^{-\pi i t}}{2\pi i t} e^{-2\pi i t}$$

$$= \frac{1}{3} \frac{e^{3\pi i t} - e^{-3\pi i t}}{2\pi i t} = \frac{\sin(3\pi t)}{3\pi t} = \operatorname{sinc}(3t) = u(t).$$

(e) The approximation function

$$s(t) = u(0)\operatorname{sinc}(t) = \operatorname{sinc}(t). \tag{28}$$

Using Parseval's Theorem, the energy between u(t) and s(t) is

$$\int_{-\infty}^{+\infty} |u(t) - s(t)|^2 dt = \int_{-\infty}^{+\infty} |\hat{u}(f) - \hat{s}(f)| = \int_{-\infty}^{+\infty} \left| \frac{1}{3} \operatorname{rect} \left(\frac{f}{3} \right) - \operatorname{rect} (f) \right|^2 df = \frac{2}{3}.$$

which shows that s(t) is not a very good approximation of u(t), i.e., s(t) and u(t) are not \mathcal{L}_2 equivalent.