Assignment 12

Due time : Nil

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Grade : ________

Problem 1 (8.9, otrhogonal signal sets; continuation of Exercise 8.8) Score: ________. Consider a set $\mathcal{A} = \{\boldsymbol{a}_m, 0 \leq m \leq M-1\}$ of M orthogonal vectors in $\mathbb{R}^M$ with equal energy E.

(a) Use the union bound to show that $\Pr(e)$, using ML detection, is bounded by

$$\Pr(e) \leq (M-1)Q(\sqrt{E/N_0}).$$

(b) Let $M \to \infty$ with $E_b = E/\log M$ held constant. Using the upperbound for $Q(x)$ in Exercise 8.7(b), show that if $E_b/N_0 > 2\ln 2$, then $\lim_{M\to\infty}\Pr(e) = 0$. How close is this to the ultimate Shannon limit on E_b/N_0? What is the limit of the spectral efficiency ρ?

Solution: (a) Suppose $\boldsymbol{U} = \boldsymbol{a}_m$ is transmitted and $\boldsymbol{V}$ is received. Using ML detection, if $\|\boldsymbol{V} - \boldsymbol{a}_j\| \leq \|\boldsymbol{V} - \boldsymbol{a}_k\|\ \forall k$, we determine that $\tilde{\boldsymbol{U}} = a_j$. If $\exists j$ s.t. $\|\boldsymbol{V} - \boldsymbol{a}_j\| \leq \|\boldsymbol{V} - \boldsymbol{a}_m\|$, then an error occurs. Use A_{jm} to denote the event $\|\boldsymbol{V} - \boldsymbol{a}_j\| < \|\boldsymbol{V} - \boldsymbol{a}_m\|$. The error probability is

$$\Pr(e|\boldsymbol{U} = \boldsymbol{a}_m) = \Pr(\cup_{j\neq m}A_{jm}|\boldsymbol{U} = \boldsymbol{a}_m) \leq \sum_{j\neq m}\Pr(A_{jm}|\boldsymbol{U} = \boldsymbol{a}_m). \tag{1}$$

Suppose that $\boldsymbol{V} = \boldsymbol{U} + \boldsymbol{Z}$, where $\boldsymbol{Z}$ is Gaussian noise with variance of $N_0/2$ per dimension. Since $\{\boldsymbol{a}_m, 0 \leq m \leq M-1\}$ are orthogonal with equal energy E,

$$\Pr(A_{jm}|\boldsymbol{U} = \boldsymbol{a}_m) = Q\left(\frac{\|\boldsymbol{a}_j - \boldsymbol{a}_m\|/2}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{\|\boldsymbol{a}_j - \boldsymbol{a}_m\|^2}{2N_0}}\right) = Q\left(\sqrt{\frac{E}{N_0}}\right). \tag{2}$$

Therefore,

$$\Pr(e|\boldsymbol{U} = \boldsymbol{a}_m) \leq (M-1)Q(\sqrt{E/N_0}). \tag{3}$$

(b) According to Exercise 8.7, for $x \geq 0$,

$$Q(x) \leq \frac{1}{2}\exp\left(-\frac{x^2}{2}\right), \tag{4}$$

so

$$0 < \Pr(e) \leq \frac{M-1}{2}\exp\left(-\frac{E}{2N_0}\right) = \frac{M-1}{2}\exp\left(-\frac{E_b\log_2 M}{2N_0}\right) = \frac{M-1}{2}\left(\frac{1}{M}\right)^{\frac{E_b}{2N_0\ln 2}}. \tag{5}$$

If $E_b/N_0 > 2\ln 2$, then $\frac{E_b}{2N_0\ln 2} > 1$. Let $M \to \infty$,

$$\lim_{M\to\infty}\Pr(e) = 0. \tag{6}$$

The limit of the spectral efficiency ρ is

$$\lim_{M\to\infty}\rho = \lim_{M\to\infty}\frac{2\log_2 M}{M} = 0. \tag{7}$$

□

Problem 2 (8.11) Score: ________. Section 8.3.4 discusses detection for binary complex vectors in WGN by viewing complex n-dimensional vectors as $2n$-dimensional real vectors. Here you will treat the vectors directly as n-dimension complex vectors. Let $\boldsymbol{Z} = (Z_1, \cdots, Z_n)^T$ be a vector of complex idd Gaussian rvs with iid real and imaginary parts, each $\mathcal{N}(0, N_0/2)$. The input $\boldsymbol{U}$ is binary antipodal, taking on values $\boldsymbol{a}$ and $-\boldsymbol{a}$. The observation $\boldsymbol{V}$ is $\boldsymbol{U} + \boldsymbol{Z}$.

(a) The probability density of $\boldsymbol{Z}$ is given by

$$f_{\boldsymbol{Z}}(z) = \frac{1}{(\pi N_0)^n} \exp \sum_{j=1}^{n} \frac{-\left|z_j\right|^2}{N_0} = \frac{1}{(\pi N_0)^n} \exp \frac{-\left\|z\right\|^2}{N_0}.$$

Explain what this probability density represents (i.e. probability per unit what?)

(b) Give expression for $f_{\boldsymbol{V}|\boldsymbol{U}}(\boldsymbol{v}|\boldsymbol{a})$ and $f_{\boldsymbol{V}|\boldsymbol{U}}(\boldsymbol{v}|-\boldsymbol{a})$.

(c) Show that the log likelihood ratio for the observation $\boldsymbol{v}$ is given by

$$\mathrm{LLR}(\boldsymbol{v}) = \frac{-\left\|\boldsymbol{v}-\boldsymbol{a}\right\|^2 + \left\|\boldsymbol{v}+\boldsymbol{a}\right\|^2}{N_0}.$$

(d) Explain why this implies that ML detection is minimum distance detection (defining the distance between two complex vectors as the norm of their difference).

(e) Show that $\mathrm{LLR}(\boldsymbol{v})$ can also be written as $4\operatorname{Re}[\langle \boldsymbol{v}, \boldsymbol{a}\rangle]/N_0$.

(f) The appearance of the real part, $\operatorname{Re}[\langle \boldsymbol{v}, \boldsymbol{a}\rangle]$, in part (e) is surprising. Point out why log likelihood ratios must be real. Also explain why replacing $\operatorname{Re}[\langle \boldsymbol{v}, \boldsymbol{a}\rangle]$ by $|\langle \boldsymbol{v}, \boldsymbol{a}\rangle|$ in the above expression would give a nonsensical result in the ML test.

(g) Does the set of points $\{\boldsymbol{v} : \mathrm{LLR}(\boldsymbol{v}) = 0\}$ form a complex vector space?

Solution: (a) The probability density represents the probability per unit volume in $2n$-dimensional real vector space.

(b)

$$f_{\boldsymbol{V}|\boldsymbol{U}}(\boldsymbol{v}|\boldsymbol{a}) = \frac{1}{(\pi N_0)^n} \exp \frac{-\left\|\boldsymbol{v}-\boldsymbol{a}\right\|^2}{N_0}, \tag{8}$$

$$f_{\boldsymbol{V}|\boldsymbol{U}}(\boldsymbol{v}|-\boldsymbol{a}) = \frac{1}{(\pi N_0)^n} \exp \frac{-\left\|\boldsymbol{v}+\boldsymbol{a}\right\|^2}{N_0}. \tag{9}$$

(c) The log likelihood ratio for the observation $\boldsymbol{v}$ is given by

$$\mathrm{LLR}(\boldsymbol{v}) = \ln \frac{f_{\boldsymbol{V}|\boldsymbol{U}}(\boldsymbol{v}|\boldsymbol{a})}{f_{\boldsymbol{V}|\boldsymbol{U}}(\boldsymbol{v}|-\boldsymbol{a})} = \ln\left[\exp \frac{-\left\|\boldsymbol{v}-\boldsymbol{a}\right\|^2 + \left\|\boldsymbol{v}+\boldsymbol{a}\right\|^2}{N_0}\right] = \frac{-\left\|\boldsymbol{v}-\boldsymbol{a}\right\|^2 + \left\|\boldsymbol{v}+\boldsymbol{a}\right\|^2}{N_0}. \tag{10}$$

(d) The ML detection rule is

$$\mathrm{LLR}(\boldsymbol{v}) = \frac{-\left\|\boldsymbol{v}-\boldsymbol{a}\right\|^2 + \left\|\boldsymbol{v}+\boldsymbol{a}\right\|^2}{N_0} \underset{\tilde{U}=-\boldsymbol{a}}{\overset{\tilde{U}=\boldsymbol{a}}{\gtrless}} 0. \tag{11}$$

Since $\left\|\boldsymbol{v}-\boldsymbol{a}\right\|$ is the distance between $\boldsymbol{v}$ and $\boldsymbol{a}$ and $\left\|\boldsymbol{v}+\boldsymbol{a}\right\|^2$ is the distance between $\boldsymbol{v}$ and $-\boldsymbol{a}$, if the distance between $\boldsymbol{v}$ and $\boldsymbol{a}$ is less than that between $\boldsymbol{v}$ and $-\boldsymbol{a}$, ML detection gives $\tilde{U} = \boldsymbol{a}$, otherwise, giving $\tilde{U} = -\boldsymbol{a}$. Therefore, this $\mathrm{LLR}(\boldsymbol{v})$ implies that ML detection is minimum distance detection.

(e) LLR can be written as

$$\mathrm{LLR}(\boldsymbol{v}) = \frac{2\langle \boldsymbol{v}, \boldsymbol{a}\rangle + \langle \boldsymbol{a}, \boldsymbol{v}\rangle}{N_0} = \frac{4\operatorname{Re}[\langle \boldsymbol{v}, \boldsymbol{a}\rangle]}{N_0}. \tag{12}$$

(f) The likelihood ratio is the ratio of two positive numbers, so its logarithm $\text{LLR}(\boldsymbol{v})$ is also a real number. Consider the one-dimensional case with $\boldsymbol{a} = 1$. The noise in the imaginary direction is irrelevant and only the real component of noise is relevant.

(g) No. Consider the one-dimensional case with $\boldsymbol{a} = 1$. The set $\{\boldsymbol{v} : \text{LLR}(\boldsymbol{v})\}$ is pure imaginary numbers, which is not closed under scalar multiplication by complex number.

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