Intro to Communication System
EE140
Fall, 2020

Assignment 1

Due time : 10:15, Sept 18, 2020 (Friday)

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Grade : ________

Problem 1 (15 pts) Score: ________. Classify each of the following signals as an energy signal or as a power signal by calculating E (energy) or P (power). Note: the parameters involved are positive constants.

a) $x(t) = e^{-\alpha|t|}\cos\pi t$,

b) $x(t) = \Pi(t-3)\cos 3\pi t$, $\left(\Pi(t) = \begin{cases} 1, & |t| \leq 0.5 \\ 0, & \text{otherwise} \end{cases}\right)$

c) $x(t) = |t|$,

d) $x(t) = \sum_{n=-\infty}^{\infty} \Lambda(\frac{t}{2} - n)$, $\left(\Lambda(t) = \begin{cases} 1 - |t|, & |t| \\ 0, & \text{otherwise} \end{cases}\right)$

e) $x(t) = e^{j2\pi 3t}u(t)$, $\left(u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}\right)$

Solution: a) The energy of the signal:

$$\begin{aligned}
E =& \int_{-\infty}^{+\infty} |x(t)|^2\, dt = \int_{-\infty}^{+\infty} e^{-2\alpha|t|}\cos^2\pi t\, dt \\
=& 2\int_0^{+\infty} e^{-2\alpha t}\cos^2\pi t\, dt \\
=& \int_0^{+\infty} e^{-2\alpha t}(\cos 2\pi t + 1)\, dt \\
=& \int_0^{+\infty} e^{-2\alpha t}\,\mathrm{Re}\left[e^{i2\pi t}\right] dt - \frac{1}{2\alpha}e^{-2\alpha t}\bigg|_0^{+\infty} \\
=& \mathrm{Re}\left[\int_0^{+\infty} e^{(-2\alpha + i2\pi)t}\, dt\right] + \frac{1}{2\alpha} \\
=& \mathrm{Re}\left[\frac{1}{2\alpha - i2\pi}\right] + \frac{1}{2\alpha} \\
=& \frac{\alpha}{2(\alpha^2 + \pi^2)} + \frac{1}{2\alpha} < +\infty.
\end{aligned} \tag{1}$$

Therefore, the signal is **an energy signal**.

b) The energy of the signal:

$$\begin{aligned}
E =& \int_{-\infty}^{+\infty} |x(t)|^2\, dt = \int_{-\infty}^{+\infty} |\Pi(t-3)\cos 3\pi t|^2\, dt \\
=& \int_{5/2}^{7/2} \cos^2 3\pi t\, dt \\
=& \frac{1}{2}\int_{5/2}^{7/2} (\cos 6\pi t + 1)\, dt \\
=& \frac{1}{2} < +\infty.
\end{aligned}$$

Therefore, the signal is **an energy signal**.

c) The average power of the signal is

$$P = \lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt = \lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{T/2} |t|^2\, dt = \lim_{T\to+\infty} \frac{2}{T}\int_0^{T/2} t^2\, dt = \lim_{T\to+\infty} \frac{T^2}{12} = +\infty. \tag{2}$$

Therefore, the signal is **neither an energy signal nor a power signal**.

d) The signal function can be written as

$$x(t) = 1. \tag{3}$$

The average power of the signal:

$$P = \lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2\,dt = \lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}dt = 1, \tag{4}$$

$$\Longrightarrow 0 < P < +\infty. \tag{5}$$

Therefore, the signal is **a power signal**.

e) The average power of the signal:

$$\begin{aligned}P =& \lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2\,dt = \lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}\left|e^{j6\pi t}u(t)\right|^2\,dt\\ =& \lim_{T\to+\infty}\frac{1}{T}\int_{0}^{T/2}\left|e^{j6\pi t}\right|^2\,dt\\ =& \lim_{T\to+\infty}\frac{1}{T}\int_{0}^{T/2}dt\\ =& \frac{1}{2}.\end{aligned} \tag{6}$$

$$\Longrightarrow 0 < P < +\infty. \tag{7}$$

Therefore, the signal is **a power signal**.

□

Problem 2 (20 pts) Score: ________. Calculate **the Fourier transform and the energy** of the following signals.

a) $x_1(t) = 5\,\mathrm{sinc}\,(2t)e^{j2\pi 3t}$

b) $x_2(t) = \mathrm{sinc}^2(t-1)$

c) $x_a(t) = x_1(t) + x_2(-t)$

d) $x_b(t) = x_1(-t) + x_2(t)$

e) $x_c(t) = 2x_1(t)\cos 6\pi t + x_2(t)e^{j6\pi t}$

Solution: a) We first look for the Fourier transform of $\frac{1}{\pi t}$ (see reference at [1]). Consider such a function:

$$f_\alpha(t) = \begin{cases} e^{-2\pi\alpha t}, & t > 0\\ 0, & t = 0\\ -e^{2\pi\alpha t}, & t < 0\end{cases} \tag{8}$$

where $\alpha > 0$. The Fourier transform of the above function is

$$\mathscr{F}[f_\alpha(t)] = \int_{-\infty}^{+\infty} f(t)e^{-j2\pi f\tau}\,d\tau = -\int_{-\infty}^{0} e^{2\pi(\alpha-jf)\tau}\,d\tau + \int_{0}^{+\infty} e^{-2\pi(\alpha+if)\tau}\,d\tau$$

[1] https://math.stackexchange.com/questions/1033870/does-the-fourier-transform-exist-for-ft-1-t

$$= -\frac{1}{2\pi(\alpha - jf)} + \frac{1}{2\pi(\alpha + if)} = -\frac{2jf}{2\pi(\alpha^2 + f^2)}. \tag{9}$$

Since the sign function is the limit of $f_\alpha(t)$ when $a \to 0$:

$$\operatorname{sgn}(t) = \begin{cases} 1, & t > 0, \\ 0, & t = 0, \\ -1, & t < 0, \end{cases} = \lim_{\alpha \to 0} f_\alpha(t), \tag{10}$$

by taking the limit $\alpha \to 0$, we get the Fourier transform of the sign function:

$$\mathscr{F}[\operatorname{sgn}(t)] = \lim_{\alpha \to 0} [f_\alpha(t)] = \frac{1}{j\pi f}. \tag{11}$$

Using the duality property, the Fourier transform of $\frac{1}{\pi t}$ is

$$\mathscr{F}[\frac{1}{\pi t}] = -j \operatorname{sgn}(f). \tag{12}$$

Then, using the multiplication property, the Fourier transform of the sinc function is

$$\begin{aligned} \mathscr{F}[\operatorname{sinc}(t)] =& \mathscr{F}\left[\frac{\sin(\pi t)}{\pi t}\right] = \mathscr{F}[\sin(\pi t)] * \mathscr{F}\left[\frac{1}{\pi t}\right] \\ =& \frac{1}{2j}\left[\delta\left(f - \frac{1}{2}\right) - \delta\left(f + \frac{1}{2}\right)\right] * [-j \operatorname{sgn}(f)] \\ =& -\frac{1}{2}\int_{-\infty}^{+\infty}\left[\delta\left(\nu - \frac{1}{2}\right) - \delta\left(\nu + \frac{1}{2}\right)\right] \operatorname{sgn}(f - \nu)\, d\nu \\ =& -\frac{1}{2}\left[\operatorname{sgn}\left(f - \frac{1}{2}\right) - \operatorname{sgn}\left(f + \frac{1}{2}\right)\right] \\ =& \Pi(f). \end{aligned} \tag{13}$$

Finally, using the scaling shifting property, we have

$$\mathscr{F}[\operatorname{sinc}(2t)] = \frac{1}{2}\Pi\left(\frac{f}{2}\right). \tag{14}$$

And using the frequency shifting property, we get the Fourier transform of $x_1(t)$:

$$\mathscr{F}[x_1(t)] = \mathscr{F}[5 \operatorname{sinc}(2t) e^{j2\pi 3t}] = \frac{5}{2}\Pi\left(\frac{f-3}{2}\right). \tag{15}$$

The energy of $x_1(t)$ is

$$E = \int_{-\infty}^{+\infty} |\mathscr{F}[x_1(t)]|^2 \, df = \int_{-\infty}^{+\infty} \left|\frac{5}{2}\Pi\left(\frac{f-3}{2}\right)\right|^2 df = \frac{25}{2}. \tag{16}$$

b) Using the time shifting property, we have

$$\mathscr{F}[\operatorname{sinc}(t-1)] = \Pi(f)e^{-j2\pi f}. \tag{17}$$

Using the multiplication property, we get the Fourier transform of $x_2(t)$:

$$\begin{aligned} \mathscr{F}[x_2(t)] =& \mathscr{F}[\operatorname{sinc}^2(t-1)] = \mathscr{F}[\operatorname{sinc}(t-1)] * \mathscr{F}[\operatorname{sinc}(t-1)] \\ =& \int_{-\infty}^{+\infty} \Pi(\nu)e^{-j2\pi\nu}\Pi(f-\nu)e^{-j2\pi(f-\nu)}\, d\nu \end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{+\infty} \Pi(\nu)\Pi(f-\nu)e^{-j2\pi f}\, d\nu \\
&= \Lambda(f)e^{-j2\pi f}.
\end{aligned} \tag{18}
$$

The energy of $x_2(t)$ is

$$
\begin{aligned}
E &= \int_{-\infty}^{+\infty} \left|\mathscr{F}[x_2(t)]\right|^2\, df = \int_{-\infty}^{+\infty} \left|\Lambda(f)e^{-j2\pi f}\right|^2\, df \\
&= \int_{-\infty}^{+\infty} \left|\Lambda(f)\right|^2\, df \\
&= 2\int_0^1 (1-f)^2\, df \\
&= \frac{2}{3}.
\end{aligned}
$$

c) Using the superposition and the scaling properties, the Fourier transform of $x_a(t)$ is

$$
\mathscr{F}[x_a(t)] = \mathscr{F}[x_1(t)+x_2(-t)] = \mathscr{F}[x_1(t)] + \mathscr{F}[x_2(-t)] = \frac{5}{2}\Pi\left(\frac{f-3}{2}\right) + \Lambda(f)e^{-j2\pi f}. \tag{19}
$$

The energy of $x_a(t)$ is

$$
\begin{aligned}
E &= \int_{-\infty}^{+\infty} \left|\mathscr{F}[x_a(t)]\right|^2\, df = \int_{-\infty}^{+\infty} \left|\frac{5}{2}\Pi\left(\frac{f-3}{2}\right) + \Lambda(f)e^{-j2\pi f}\right|^2\, df \\
&= \int_2^4 \left|\frac{5}{2}\Pi\left(\frac{f-3}{2}\right)\right|^2\, df + \int_{-1}^1 \left|\Lambda(f)\right|^2\, df \\
&= \int_2^4 \frac{25}{4}\, df + 2\int_0^1 (1-f)^2\, df \\
&= \frac{25}{2} + \frac{2}{3} = \frac{79}{6}.
\end{aligned} \tag{20}
$$

d) Using the superposition and the scaling properties, the Fourier transform of $x_b(t)$ is

$$
\mathscr{F}[x_b(t)] = \mathscr{F}[x_1(-t)+x_2(t)] = \mathscr{F}[x_1(-t)] + \mathscr{F}[x_2(t)] = \frac{5}{2}\Pi\left(\frac{-f-3}{2}\right) + \Lambda(f)e^{-j2\pi f}. \tag{21}
$$

~~Since the Fourier transform of $x_b(t)$ is the same as $x_a(t)$.~~ The energy of $x_b(t)$ is ~~also the same as that of $x_a(t)$~~:

$$
E = \frac{79}{6}. \tag{22}
$$

e) The Fourier transform of the first term is

$$
\begin{aligned}
\mathscr{F}[2x_1(t)\cos 6\pi t] &= \mathscr{F}\left[x_1(t)(e^{j2\pi 3t} + e^{-j2\pi 3t})\right] = \mathscr{F}\left[x_1(t)e^{j2\pi 3t}\right] + \mathscr{F}\left[x_1(t)e^{-j2\pi 3t}\right] \\
&= \frac{5}{2}\left[\Pi\left(\frac{f}{2}-3\right) + \Pi\left(\frac{f}{2}\right)\right].
\end{aligned} \tag{23}
$$

The Fourier transform of the second term is

$$
\mathscr{F}\left[x_2(t)e^{j6\pi t}\right] = \mathscr{F}\left[x_2(t)e^{j2\pi 3t}\right] = \Lambda(f-3)e^{-j2\pi(f-3)}. \tag{24}
$$

Therefore, the Fourier transform of $x_c(t)$ is

$$
\mathscr{F}[x_c(t)] = \mathscr{F}[2x_1(t)\cos 6\pi t + x_2(t)e^{j6\pi t}] = \mathscr{F}[2x_1(t)\cos 6\pi t] + \mathscr{F}\left[x_2(t)e^{j6\pi t}\right]
$$

$$=\frac{5}{2}\left[\Pi\left(\frac{f}{2}-3\right)+\Pi\left(\frac{f}{2}\right)\right]+\Lambda(f-3)e^{-j2\pi(f-3)}. \tag{25}$$

The energy of $x_c(t)$ is

$$\begin{aligned}E=&\int_{-\infty}^{+\infty}|\mathscr{F}[x_c(t)]|^2\,df=\int_{-\infty}^{+\infty}\left|\frac{5}{2}\left[\Pi\left(\frac{f}{2}-3\right)+\Pi\left(\frac{f}{2}\right)\right]+\Lambda(f-3)e^{-j2\pi(f-3)}\right|^2\,df\\=&\int_5^7\left|\frac{5}{2}\Pi\left(\frac{f}{2}-3\right)\right|^2\,df+\int_{-1}^1\left|\frac{5}{2}\Pi\left(\frac{f}{2}\right)\right|^2\,df+\int_2^4\left|\Lambda(f-3)e^{-j2\pi(f-3)}\right|^2\,df\\=&\int_5^7\frac{25}{4}\,df+\int_{-1}^1\frac{25}{4}\,df+2\int_3^4(4-f)^2\,df\\=&\frac{25}{2}+\frac{25}{2}+\frac{2}{3}=\frac{77}{3}.\end{aligned} \tag{26}$$

□

Problem 3 (10 pts) Score: _________. Calculate the convolution of the following signal.

a) $y(t)=e^{-|t|}*\Pi(t-2)$

b) $y(t)=\operatorname{sgn}(t)*\Lambda(t-2)$

Solution: a) The convolution can be written as

$$\begin{aligned}y(t)=&e^{-|t|}*\Pi(t-2)=\int_{-\infty}^{+\infty}e^{-|\tau|}\Pi(t-\tau-2)\,d\tau\\=&\int_{t-\frac{5}{2}}^{t-\frac{3}{2}}e^{-|\tau|}\,d\tau.\end{aligned} \tag{27}$$

If $t\geq\frac{5}{2}$,

$$y(t)=\int_{t-\frac{5}{2}}^{t-\frac{3}{2}}e^{-\tau}\,d\tau=e^{\frac{5}{2}-t}-e^{\frac{3}{2}-t}. \tag{28}$$

If $\frac{3}{2}\leq t<\frac{5}{2}$,

$$y(t)=\int_{t-\frac{5}{2}}^{0}e^{\tau}\,d\tau+\int_0^{t-\frac{3}{2}}e^{-\tau}\,d\tau=2-e^{t-\frac{5}{2}}-e^{t-\frac{3}{2}}. \tag{29}$$

If $t<\frac{3}{2}$,

$$y(t)=\int_{t-\frac{5}{2}}^{t-\frac{3}{2}}e^{\tau}\,d\tau=e^{t-\frac{3}{2}}-e^{t-\frac{5}{2}}. \tag{30}$$

Therefore, in general,

$$y(t)=\begin{cases}e^{\frac{5}{2}-t}-e^{\frac{3}{2}-t}, & t\geq\frac{5}{2},\\ 2-e^{t-\frac{5}{2}}-e^{t-\frac{3}{2}}, & \frac{3}{2}\leq t<\frac{3}{2},\\ e^{t-\frac{3}{2}}-e^{t-\frac{5}{2}}, & t<\frac{3}{2}.\end{cases} \tag{31}$$

b) The convolution can be written as

$$y(t)=\operatorname{sgn}(t)*\Lambda(t-2)=\int_{-\infty}^{+\infty}\operatorname{sgn}(\tau)\Lambda(t-\tau-2)\,d\tau$$

$$
\begin{aligned}
&\cancel{=\int_{t-\frac{5}{2}}^{t-\frac{3}{2}} \operatorname{sgn}(\tau)\, d\tau.}\\
&=-\int_{-\infty}^{0} \Lambda(t-\tau-2)\, d\tau + \int_{0}^{+\infty} \Lambda(t-\tau-2)\, d\tau.
\end{aligned} \tag{32}
$$

If $t \geq 3$,

$$
\begin{aligned}
y(t) =& \int_{0}^{+\infty} \Lambda(t-\tau-2)\, d\tau\\
=& \int_{t-3}^{t-2} (t-\tau-1)\, d\tau + \int_{t-2}^{t-1} (3-t+\tau)\, d\tau\\
=&1.
\end{aligned} \tag{33}
$$

If $2 \leq t < 3$,

$$
\begin{aligned}
y(t) =& -\int_{-\infty}^{0} \Lambda(t-\tau-2)\, d\tau + \int_{0}^{+\infty} \Lambda(t-\tau-2)\, d\tau\\
=& -\int_{t-3}^{0} (t-\tau-1)\, d\tau + \int_{0}^{t-2} (t-\tau-1)\, d\tau + \int_{t-2}^{t-1} (3-t-\tau)\, d\tau\\
=& -t^2 + 6t - 8.
\end{aligned} \tag{34}
$$

If $1 \leq t < 2$,

$$
\begin{aligned}
y(t) =& -\int_{t-3}^{0} \Lambda(t-\tau-2)\, d\tau + \int_{0}^{+\infty} \Lambda(t-\tau-2)\, d\tau\\
=& -\int_{t-3}^{t-2} (t-\tau-1) - \int_{t-2}^{0} (3-t-\tau)\, d\tau + \int_{0}^{t-1} (3-t-\tau)\, d\tau\\
=&t^2 - 2t.
\end{aligned} \tag{35}
$$

If $t < 1$,

$$
\begin{aligned}
y(t) =& -\int_{-\infty}^{0} \Lambda(t-\tau-2)\, d\tau\\
=& -\int_{t-3}^{t-2} (t-\tau-1)\, d\tau - \int_{t-2}^{t-1} (3-t-\tau)\, d\tau\\
=& -1.
\end{aligned} \tag{36}
$$

Therefore, in general,

$$
y(t) = \begin{cases} 1, & t \geq 3,\\ -t^2+6t-8, & 2 \leq t < 3,\\ t^2-2t, & 1 \leq t < 2,\\ -1, & t < 3. \end{cases} \tag{37}
$$

□

Problem 4 (20 pts) Score: __________. Calculate the Fourier transform of the following periodic signal.

a) $\sum_{n=-\infty}^{\infty} \Lambda(\frac{t}{2} - 2n)$

b) $\left[\sum_{n=-\infty}^{\infty} \delta(t-2n)\right] * \left[\Pi(\frac{t}{2})\cos(2\pi t)\right]$

Solution: a) The Fourier series of the signal is

$$\sum_{n=-\infty}^{\infty} \Lambda\left(\frac{t}{2}-2n\right) = \sum_{n=-\infty}^{\infty} c_n e^{jn2\pi\frac{t}{4}}, \tag{38}$$

where

$$\begin{aligned}
c_n =& \frac{1}{4}\int_{-2}^{2} \sum_{n=-\infty}^{\infty} \Lambda\left(\frac{\tau}{2}-2n\right) e^{-jn2\pi\frac{\tau}{4}}\, d\tau \\
=& \frac{1}{2}\int_{0}^{2} \left(1-\frac{\tau}{2}\right)\cos\left(n2\pi\frac{\tau}{4}\right) d\tau \\
=& \frac{\sin(n\pi)}{n\pi} - \frac{1}{4}\frac{1}{n2\pi\frac{1}{4}}\int_0^2 \tau\, d\left[\sin\left(n2\pi\frac{\tau}{4}\right)\right] \\
=& \frac{\sin(n\pi)}{n\pi} - \frac{\sin(n\pi)}{n\pi} + \frac{1}{4}\frac{1}{n2\pi\frac{1}{4}}\int_0^2 \sin\left(n2\pi\frac{\tau}{4}\right) d\tau \\
=& \frac{1-\cos(n\pi)}{(n\pi)^2}.
\end{aligned} \tag{39}$$

Note that $c_n = 2$ when $n = 0$. The fourier transform of the signal is

$$\begin{aligned}
\mathscr{F}\left[\sum_{n=-\infty}^{\infty} \Lambda\left(\frac{t}{2}-2n\right)\right] & \cancel{= \mathscr{F}\left[\sum_{n=-\infty}^{\infty} \frac{1-\cos(n\pi)}{(n\pi)^2} e^{jn2\pi\frac{t}{4}}\right]} \\
& = \mathscr{F}\left[\sum_{n=-\infty}^{\infty} c_n e^{jn2\pi\frac{t}{4}}\right] \\
& \cancel{= \sum_{n=-\infty}^{\infty} \frac{1-\cos(n\pi)}{(n\pi)^2} \mathscr{F}\left[e^{-j2\pi(f-\frac{n}{4})t}\right]} \\
& = \sum_{n=-\infty}^{\infty} c_n \mathscr{F}\left[e^{-j2\pi(f-\frac{n}{4})t}\right] \\
& \cancel{= \sum_{n=-\infty}^{\infty} \frac{1-\cos(n\pi)}{(n\pi)^2} \delta(f-\frac{n}{4}).} \\
& = \sum_{n=-\infty}^{\infty} c_n \delta\left(f-\frac{n}{4}\right).
\end{aligned} \tag{40}$$

b) The Fourier series of $\sum_{n=-\infty}^{\infty} \delta(t-2n)$ is

$$\sum_{n=-\infty}^{\infty} \delta(t-2n) = \sum_{n=-\infty}^{\infty} c_n e^{jn2\pi\frac{t}{2}}, \tag{41}$$

where

$$c_n = \frac{1}{2}\int_{-1}^{1} \sum_{n=-\infty}^{\infty} \delta(t-2n) e^{-jn2\pi\frac{\tau}{2}}\, d\tau = \frac{1}{2}. \tag{42}$$

The Fourier transform of $\sum_{n=-\infty}^{\infty} \delta(t-2n)$ is

$$\mathscr{F}\left[\sum_{n=-\infty}^{\infty} \delta(t-2n)\right] = \mathscr{F}\left[\sum_{n=-\infty}^{\infty} \frac{1}{2} e^{jn2\pi\frac{t}{2}}\right] = \sum_{n=-\infty}^{\infty} \frac{1}{2}\mathscr{F}\left[e^{jn2\pi\frac{t}{2}}\right] = \frac{1}{2}\sum_{n=-\infty}^{\infty} \delta\left(f-\frac{n}{2}\right). \tag{43}$$

The Fourier transform of $\Pi\left(\frac{t}{2}\right)\cos(2\pi t)$ is

$$\mathscr{F}\left[\Pi\left(\frac{t}{2}\right)\cos(2\pi t)\right] = \int_{-\infty}^{+\infty} \Pi\left(\frac{\tau}{2}\right)\cos(2\pi\tau) e^{-j2\pi f\tau}\, d\tau$$

$$
\begin{aligned}
&=2\int_0^1 \cos(2\pi\tau)\cos(2\pi f\tau)\,d\tau\\
&=\int_0^1 \{\cos[2\pi(f+1)\tau]+\cos[2\pi(f-1)\tau]\}\,d\tau\\
&=\frac{\sin[2\pi(f+1)]}{2\pi(f+1)}+\frac{\sin[2\pi(f-1)]}{2\pi(f-1)}.
\end{aligned}
$$

Using the time convolution property, we get the Fourier transform of the signal

$$
\begin{aligned}
\mathscr{F}\left\{\left[\sum_{n=-\infty}^{\infty}\delta(t-2n)\right]*\left[\Pi\left(\frac{t}{2}\right)\cos(2\pi t)\right]\right\}=&\mathscr{F}\left[\sum_{n=-\infty}^{\infty}\delta(t-2n)\right]\mathscr{F}\left[\Pi\left(\frac{t}{2}\right)\cos(2\pi t)\right]\\
=&\frac{1}{2}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{2}\right)\left\{\frac{\sin[2\pi(f+1)]}{2\pi(f+1)}+\frac{\sin[2\pi(f-1)]}{2\pi(f-1)}\right\}. \quad (44)
\end{aligned}
$$

□

Problem 5 (15 pts) Score: ________. Determine the range of permissible cutoff frequencies for the ideal lowpass filter used to reconstruct the following signal

$$x(t)=4\cos^2(200\pi t)\cos(1000\pi t)$$

Which is sampled at 2000 samples per second. Sketch $X(f)$ and $X_\delta(f)$ (spectrum after the sampling). Find the minimum allowable sampling frequency.

Solution: The signal can be written as

$$
\begin{aligned}
x(t)=&4\cos^2(200\pi t)\cos(1000\pi t)=2[1+\cos(400\pi t)]\cos(1000\pi t)\\
=&2\cos(1000\pi t)+\cos(1400\pi t)+\cos(600\pi t). \quad (45)
\end{aligned}
$$

The spectrum of the signal is

$$
\begin{aligned}
X(f)=&\mathscr{F}[x(t)]=\mathscr{F}[2\cos(1000\pi t)+\cos(1400\pi t)+\cos(600\pi t)]\\
=&\delta(f-500)+\delta(f+500)+\frac{1}{2}\delta(f-700)+\frac{1}{2}\delta(f+700)+\frac{1}{2}\delta(f-300)+\frac{1}{2}\delta(f+300), \quad (46)
\end{aligned}
$$

as shown in figure 1(a). Sampling at 2000 samples per second means that the sampling frequency is $f_s=2000$ Hz. The spectrum after the sampling is

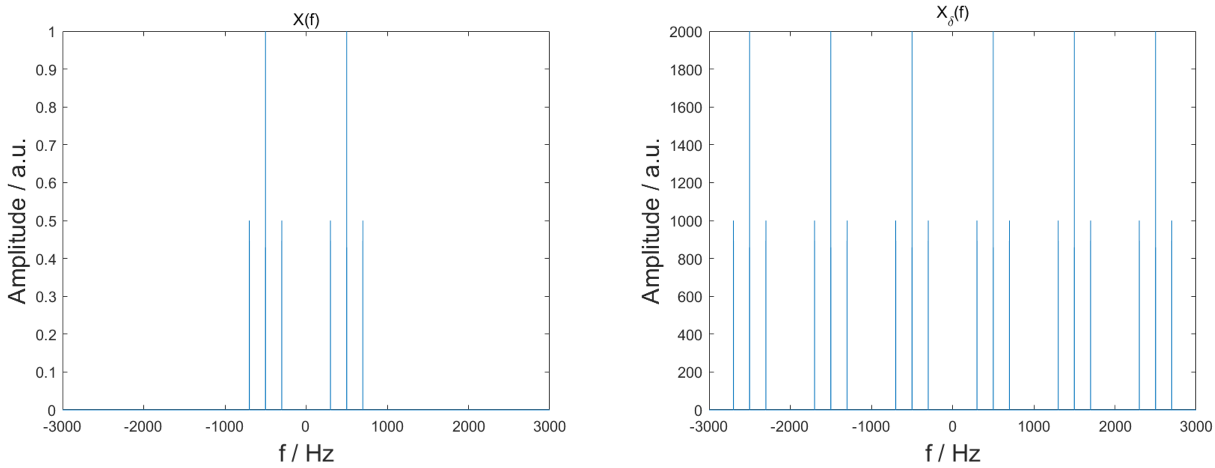
$$
\begin{aligned}
X_\delta(f)=&2000\sum_{n=-\infty}^{\infty}X(f-2000n)\\
=&2000\sum_{n=-\infty}^{\infty}\left[\delta(f-2000n-500)+\delta(f-2000n+500)+\frac{1}{2}\delta(f-2000n-700)+\frac{1}{2}\delta(f-2000n+700)\right.\\
&\left.+\frac{1}{2}\delta(f-2000n-300)+\frac{1}{2}\delta(f-2000n+300)\right], \quad (47)
\end{aligned}
$$

as shown in figure 1(b). According to the sampling theorem, the range of permissible cutoff frequencies for the ideal lowpass filter to reconstruct the signal is 700Hz $< f_c <$ 1300Hz and the minimum allowable sampling frequency is 1400Hz. □

Problem 6 Score: ________. 1) Express the spectrum $Y(f)$ of

$$y(t)=x(t)\cos(400\pi t)+\hat{x}(t)\sin(400\pi t)$$

using the spectrum $X(f)$ of $x(t)$, where $\hat{x}(t)$ is the Hilbert transform of $x(t)$. (5 pts)



(a) Original spetrum of the signal: $X(f)$

(b) Spectrum after the sampling: $X_\delta(f)$

Figure 1:

2) if $x(t) = \operatorname{sinc}(t)$, sketch $Y(f)$. (5 pts)

Solution: 1) The spectrum of $y(t)$ is

$$
\begin{aligned}
Y(f) =& \mathscr{F}[y(t)] = \mathscr{F}[x(t)\cos(400\pi t) + \hat{x}(t)\sin(400\pi t)] \\
=& \mathscr{F}\left[x(t)\frac{e^{j400\pi t} + e^{-j400\pi t}}{2}\right] + \mathscr{F}\left\{\mathscr{F}^{-1}[-j\operatorname{sgn}(f)X(f)]\frac{e^{j400\pi t} - e^{-j400\pi t}}{2j}\right\} \\
=& \frac{1}{2}X(f-200) + \frac{1}{2}X(f+200) - \frac{1}{2}\operatorname{sgn}(f-200)X(f-200) + \frac{1}{2}\operatorname{sgn}(f+200)X(f+200) \quad (48)
\end{aligned}
$$

2) If $x(t) = \operatorname{sinc}(t)$, then

$$X(f) = \Pi(f) \quad (49)$$

and

$$Y(f) = \frac{1}{2}\Pi(f-200) + \frac{1}{2}\Pi(f+200) - \frac{1}{2}\operatorname{sgn}(f-200)\Pi(f-200) + \frac{1}{2}\operatorname{sgn}(f+200)\Pi(f+200), \quad (50)$$

as shown in figure 2.

□

Problem 7 Score: ________. Consider $x(t) = 2\cos(60\pi t)$, the reference frequency $f_0 = 40$ Hz. Calculate the following signals.

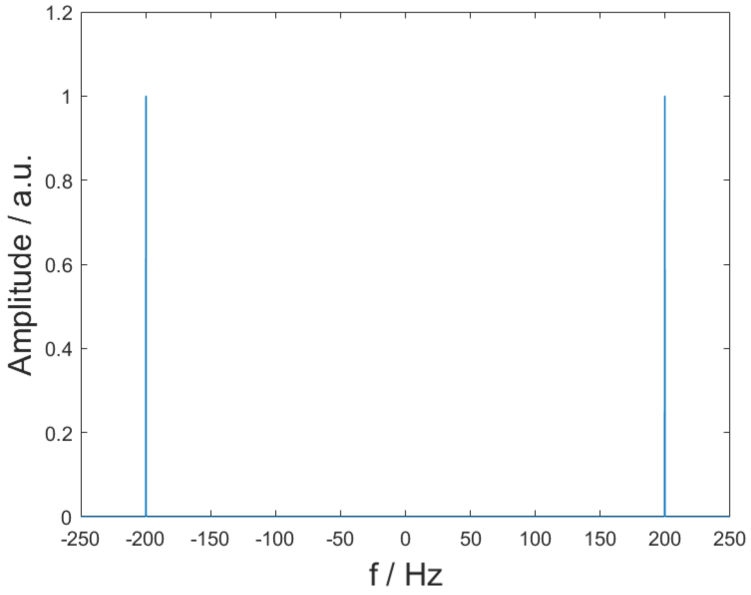
a) The Hilbert transform of $x(t)$, i.e., $\hat{x}(t)$.

b) The analytic signal $x_p(t)$.

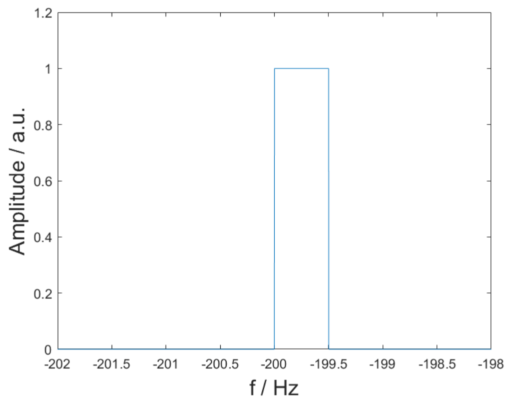
c) The complex envelope of $x(t)$, i.e., $\tilde{x}(t)$.

d) The inphase and quadrature component of $x(t)$, i.e., $x_R(t)$ and $x_I(t)$.
（Please refer to Lecture 2, Slide 36 or Page 88 of reference textbook, we will learn this in the next class. I am sorry for the lagging.）

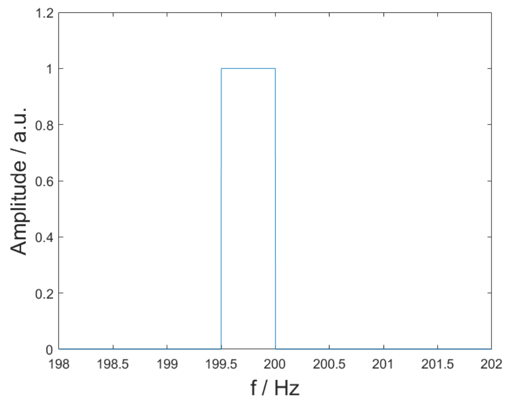
e) Determine and plot the spectrum of the following signals:



(a) Overall view of $Y(f)$



(b) $Y(f)$ near -200



(c) $Y(f)$ near $+200$

Figure 2: Spectrum of $y(t)$: $Y(f)$

i. $x_1(t) = \frac{2}{3}x(t) + \frac{1}{3}j\hat{x}(t)$

ii. $x_2(t) = \left[\frac{1}{5}x(t) + \frac{4}{5}j\hat{x}(t)\right]e^{j2\pi f_0 t}$

Solution: a) The Fourier spectrum of $x(t)$ is

$$X(f) = \mathscr{F}[x(t)] = \delta(f-30) + \delta(f+30).$$

The spectrum of $x(t)$ after Hilbert transform is

$$\hat{X}(f) = -j\,\mathrm{sgn}\,(f)X(f) = \delta(f-30)e^{-j\pi/2} + \delta(f+30)e^{j\pi/2}. \tag{51}$$

The Hilbert transform of $x(t)$ is

$$\begin{aligned}\hat{x}(t) =& \mathscr{F}^{-1}[\hat{X}(f)] = \mathscr{F}[\delta(f-30)e^{-j\pi/2} + \delta(f+30)e^{j\pi/2}]\\ =& e^{j60\pi t}e^{-j\pi/2} + e^{-j60\pi t}e^{j\pi/2}\\ =& 2\sin(60\pi t).\end{aligned} \tag{52}$$

b) The analytic signal of $x(t)$ is

$$x_p(t) = x(t) + j\hat{x}(t) = 2\cos(60\pi t) + j2\sin(60\pi t) = 2e^{j60\pi t}. \tag{53}$$

c) The complex envelope of $x(t)$ is

$$\tilde{x}(t) = x_p(t)e^{-j2\pi 40t} = 2e^{-j20\pi t}. \tag{54}$$

d) The inphase component of $x(t)$ is

$$x_R(t) = \operatorname{Re}[\tilde{x}(t)] = 2\cos(20\pi t). \tag{55}$$

The quadrature component of $x(t)$ is

$$x_I(t) = \operatorname{Im}[\tilde{x}(t)] = -2\sin(20\pi t). \tag{56}$$

e) i. The spectrum of $x_1(t)$ is

$$\begin{aligned}X_1(f) =&\mathscr{F}[x_1(t)] = \mathscr{F}\left[\frac{2}{3}x(t) + \frac{1}{3}j\hat{x}(t)\right]\\=&\frac{2}{3}X(f) + \frac{1}{3}j\hat{X}(f)\\=&\frac{2}{3}[\delta(f-30)+\delta(f+30)] + \frac{1}{3}j\left[\delta(f-30)e^{-j\pi/2} + \delta(f+30)e^{j\pi/2}\right]\\=&\delta(f-30) + \frac{1}{3}\delta(f+30).\end{aligned} \tag{57}$$

Its amplitude spectrum is

$$|X_1(f)| = \delta(f-30) + \frac{1}{3}\delta(f+30), \tag{58}$$

and angular spectrum is

$$\theta_1(f) = 0. \tag{59}$$

as shown in figure 3.

ii. The Fourier transform of $\left[\frac{1}{5}x(t) + \frac{4}{5}j\hat{x}(t)\right]$ is

$$\begin{aligned}\mathscr{F}\left[\frac{1}{5}x(t) + \frac{4}{5}j\hat{x}(t)\right] =&\frac{1}{5}X(f) + \frac{4}{5}j\hat{X}(f)\\=&\frac{1}{5}[\delta(f-30)+\delta(f+30)] + \frac{4}{5}j\left[\delta(f-30)e^{-j\pi/2} + \delta(f+30)e^{j\pi/2}\right]\\=&\delta(f-30) - \frac{3}{5}\delta(f+30).\end{aligned} \tag{60}$$

Using the frequency shifting property, we get the spectrum of $x_2(t)$:

$$X_1(f) = \mathscr{F}[x_2(t)] = \mathscr{F}\left\{\left[\frac{1}{5}x(t) + \frac{4}{5}j\hat{x}(t)\right]e^{j2\pi f_0 t}\right\} = \delta(f-f_0-30) - \frac{3}{5}\delta(f-f_0+30). \tag{61}$$

Its amplitude spectrum is

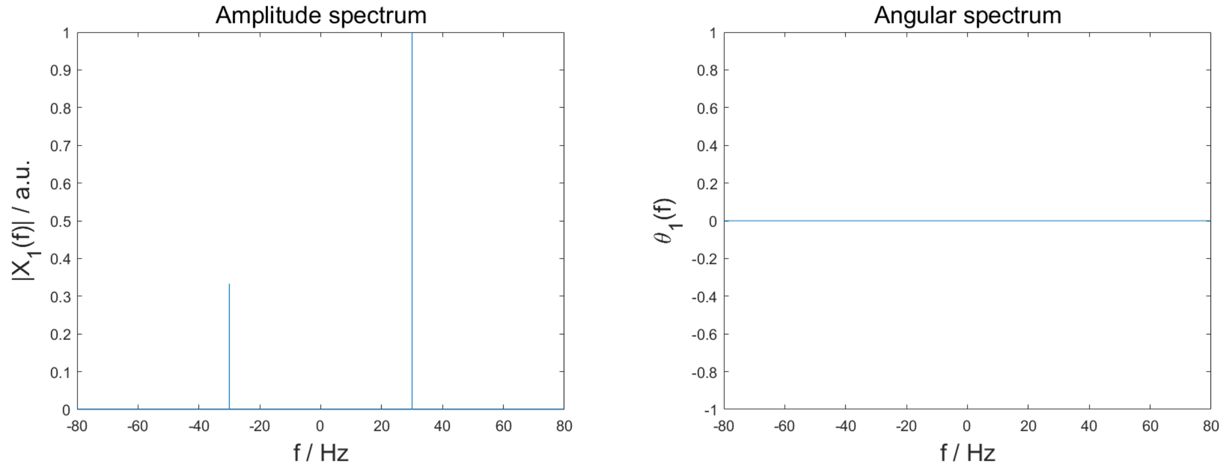
$$|X_1(f)| = \delta(f-f_0-30) - \frac{3}{5}\delta(f-f_0+30) = \delta(f-70) - \frac{3}{5}\delta(f-10), \tag{62}$$

and angular spectrum is

$$\theta_2(f) = 0, \tag{63}$$

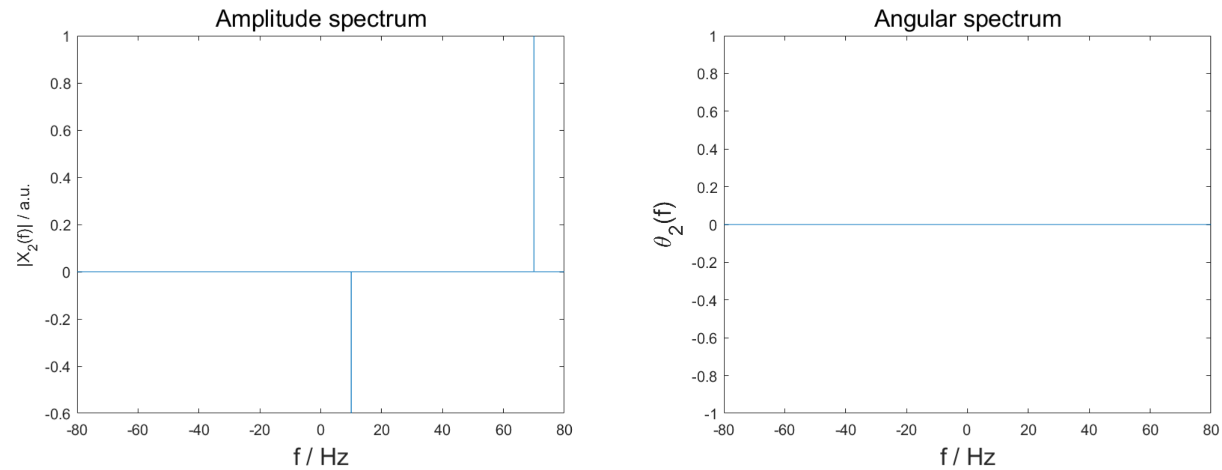
as shown in figure 4.

□



(a) $|X_1(f)|$ (b) $|\theta_1(f)|$

Figure 3: Spectrum of $x_1(t)$



(a) $|X_2(f)|$ (b) $|\theta_2(f)|$

Figure 4: Spectrum of $x_2(t)$