Assignment 9

Due time : 10:15, Dec 4, 2020 (Friday)

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Grade : ________

Problem 1 (3.1) Score: ________. Let U be an analog rv uniformly distributed between -1 and 1.

(a) Find the 3-bit ($M = 8$) quantizer that minimizes the MSE.

(b) Argue that your quantizer satisfies the necessary condition for optimality.

(c) Show that the quantizer is unique in the sense that no other 3-bit quantizer satisfies the necessary condition for optimality.

Solution: (a) The 3-bit quantizer that minimizes the MSE for the uniformly distributed analog rv U should be a uniform quantizer with 8 equally-spaced quantization intervals bounded by endpoints

$$b_0 = -1, \quad b_1 = -\frac{3}{4}, \quad b_2 = -\frac{1}{2}, \quad b_3 = -\frac{1}{4}, \quad b_4 = 0, \quad b_5 = \frac{1}{4}, \quad b_6 = \frac{1}{2}, \quad b_7 = \frac{3}{4}, \quad b_8 = 1 \tag{1}$$

and 8 equally-spaced representation points

$$a_1 = -\frac{7}{8}, \quad a_2 = -\frac{5}{8}, \quad a_3 = -\frac{3}{8}, \quad a_4 = -\frac{1}{8}, \quad a_5 = \frac{1}{8}, \quad a_6 = \frac{3}{8}, \quad a_7 = \frac{5}{8}, \quad a_8 = \frac{7}{8}. \tag{2}$$

(b) The above uniform quantizer satisfies the Lloyd-Max necessary conditions:

(i) For the given representation points $\{a_j\}$, the interval endpoints (excepts the first and the last ones) are the midpoints of the neighboring representation points:

$$b_j = \frac{a_j + a_{j+1}}{2}, \quad \forall j = 1, 2, \cdots, 7. \tag{3}$$

(ii) For the given quantization intervals $\{(b_j, b_{j+1})\}$, the representation points are the expectation of the analog rv in the corresponding quantization intervals

$$a_j = E[U|U \in \mathcal{R}_j] = \frac{\int_{\mathcal{R}_j} f_U(u) u \, \mathrm{d}u}{\int_{\mathcal{R}_j} f_U(u) \, \mathrm{d}u} = \frac{\int_{u_{j-1}}^{u_j} \frac{1}{2} u \, \mathrm{d}u}{\int_{u_{j-1}}^{u_j} \frac{1}{2} \, \mathrm{d}u} = \frac{\frac{1}{2}(b_j^2 - b_{j-1}^2)}{\frac{1}{2}(b_j - b_{j-1})} = \frac{1}{2}(b_j + b_{j+1}),$$
$$\forall j = 1, 2 \cdots, 8. \tag{4}$$

(c) Plugging equation (4) into equation (3), we get

$$b_j = \frac{\frac{b_{j-1}+b_j}{2} + \frac{b_j+b_{j+1}}{2}}{2} = \frac{1}{4} b_{j-1} + \frac{1}{2} b_j + \frac{1}{4} b_{j+1} \Longrightarrow b_{j+1} - b_j = b_j - b_{j-1}, \quad \forall j = 1, 2, \cdots, 7, \tag{5}$$

which means that for any 3-bit quantizers satisfying the necessary condition, it must have equally-spaced quantization intervals bounded by endpoints as equation (1) shows. Besides, according to equation (4), for any 3-bit quantizers satisfying the necessary condition, it must have equally-spaced representation points as equation (2) shows.

Therefore, there is only one 3-bit quantizer that minimizes the MSE — the one with 8 equally-spaced quantization intervals and 8 equally-spaced representation points. It is unique.

□

Problem 2 (3.3) Score: ________. Consider a binary scalar quantizer that partitions the set of reals $\mathbb{R}$ into two subsets $(-\infty, b]$ and (b, ∞) and then presents $(-\infty, b]$ by $a_1 \in \mathbb{R}$ and (b, ∞) by $a_2 \in \mathbb{R}$. This quantizer is used on each letter U_n of a sequence $\cdots, U_{-1}, U_0, U_1, \cdots$ of iid random variables, each having the probability density $f(u)$. Assume throughout this exercise that $f(u)$ is symmetric, i.e. that $f(u) = f(-u)$ for all $u \geq 0$.

(a) Given the representation levels a_1 and $a_2 > a_1$, how should b be chosen to minimize the mean-squared distortion in the quantization? Assume that $f(u) > 0$ for $a_1 \leq u \leq a_2$ and explain why this assumption is relevant.

(b) Given $b \geq 0$, find the values of a_1 and a_2 that minimize the mean-squared distortion. Given both answer in terms of the two functions $Q(x) = \int_x^\infty f(u)\,\mathrm{d}u$ and $y(x) = \int_x^\infty uf(u)\,\mathrm{d}u$.

(c) Show that for $b = 0$, the minimizing values of a_1 and a_2 satisfy $a_1 = -a_2$.

(d) Show that the choice of b, a_1, and a_2 in part (c) satisfies the Lloyd-Max conditions for minimum mean-squared distortion.

(e) Consider the particular symmetric density

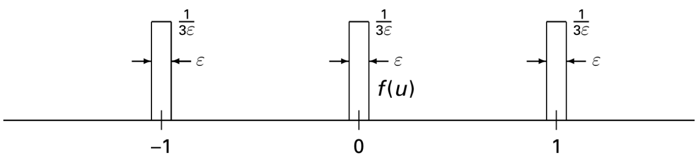

Figure 1:

Find all sets of triples $\{b, a_1, a_2\}$ that satisfy the Lloyd-Max conditions and evaluate the MSE for each. You are welcome in your calculation to replace each region of nonzero probability density above with an impulse, i.e. $f(u) = (1/3)[\delta(-1) + \delta(0) + \delta(1)]$, but you should use Figure 1 to resolve the ambiguity about regions that occurs when b is $-1, 0$ or $+1$.

(f) Given the MSE for each of your solutions above (in the limit of $\varepsilon \to 0$). Which of your solutions minimizes the MSE?

Solution: (a) The mean-squared distortion is

$$\mathrm{MSE} = E[|U - V|^2] = \int_{-\infty}^{b} f(u)(u - a_1)^2\,\mathrm{d}u + \int_{b}^{+\infty} f(u)(u - a_2)^2\,\mathrm{d}u. \tag{6}$$

Given a_1 and a_2, to minimize the mean-squared distortion, we require that

$$\frac{\partial \mathrm{MSE}}{\partial b} = f(b)(b - a_1)^2 - f(b)(b - a_2)^2 = 2(a_1 - a_2)f(b)\left(\frac{a_1 + a_2}{2} - b\right) = 0. \tag{7}$$

Here, $a_1 - a_2 \neq 0$. Under the assumption that $f(u) > 0$, we need that

$$b = \frac{a_1 + a_2}{2}, \tag{8}$$

which means that we quantize u to the nearest representation point.

The above assumption is relevant, since if $\exists u_0 \in [a_1, a_2]$ such that $f(u_0) = 0$, then $b = u_0$ is also a solution to equation (7). However, this solution, $b = u_0$, may not necessarily guarantee that we can quantize u to the nearest representation point and thus can not minimize the mean-squared distortion. Besides, consider such a case that $f(u) = 0 \forall a_1 \leq u \leq a_2$. In this case, we can choose any $b \in [a_1, a_2]$ to minimize the mean-squared distortion, not necessarily $b = \frac{a_1 + a_2}{2}$.

(b) Given b, to minimize the mean-squared distortion, we require that

$$\frac{\partial \text{MSE}}{\partial a_1} = -2\int_{-\infty}^{b} f(u)(u-a_1)\,\mathrm{d}u = 0, \tag{9}$$

$$\Longrightarrow a_1 = \frac{\int_{-\infty}^{b} f(u)u\,\mathrm{d}u}{\int_{-\infty}^{b} f(u)\,\mathrm{d}u} = \frac{-\int_{-b}^{+\infty} f(u)u\,\mathrm{d}u}{1-\int_{b}^{\infty} f(u)\,\mathrm{d}u} = -\frac{y(-b)}{1-Q(b)} = -\frac{y(b)}{1-Q(b)}. \tag{10}$$

and that

$$\frac{\partial \text{MSE}}{\partial a_2} = -2\int_{b}^{+\infty} f(u)(u-a_2)\,\mathrm{d}u = 0, \tag{11}$$

$$\Longrightarrow a_2 = \frac{\int_{b}^{+\infty} f(u)u\,\mathrm{d}u}{\int_{b}^{+\infty} f(u)\,\mathrm{d}u} = \frac{y(b)}{Q(b)}. \tag{12}$$

(c) Since $f(u) = f(-u)\forall u \geq 0$, we have

$$\begin{aligned} Q(0) = &\int_{0}^{+\infty} f(u)\,\mathrm{d}u = \frac{1}{2}\int_{0}^{+\infty} f(u)\,\mathrm{d}u + \frac{1}{2}\int_{0}^{+\infty} f(u)\,\mathrm{d}u = \frac{1}{2}\int_{0}^{+\infty} f(u)\,\mathrm{d}u + \frac{1}{2}\int_{-\infty}^{0} f(u)\,\mathrm{d}u \\ =&\frac{1}{2}\int_{-\infty}^{+\infty} f(u)\,\mathrm{d}u = \frac{1}{2}. \end{aligned} \tag{13}$$

For $b = 0$,

$$a_1 = -\frac{y(0)}{1-Q(0)} = -2y(0), \tag{14}$$

and

$$a_2 = \frac{y(0)}{Q(0)} = 2y(0), \tag{15}$$

so

$$a_1 = -a_2. \tag{16}$$

(d) The choice of b, a_1, and a_2 in part (c) satisfies the Lloyd-Max conditions for minimum mean-squared distortion:

(i) The quantization interval endpoint b is the midpoint of the representation points a_1 and a_2:

$$b = 0 = \frac{a_1 + a_2}{2}. \tag{17}$$

(ii) The representation points a_1 and a_2 are the expectation of the random variable U in their corresponding quantization intervals:

$$a_1 = \frac{\int_{-\infty}^{0} f(u)u\,\mathrm{d}u}{\int_{-\infty}^{0} f(u)\,\mathrm{d}u} = E[U|U \leq 0]. \tag{18}$$

and

$$a_2 = \frac{\int_{0}^{+\infty} f(u)u\,\mathrm{d}u}{\int_{0}^{+\infty} f(u)\,\mathrm{d}u} = E[U|U > 0]. \tag{19}$$

(e) For this particular symmetric density,

$$y(x)=\begin{cases}0, & x\le -1-\varepsilon/2,\\ \frac{1}{3}+\frac{1}{6\varepsilon}[(-1+\varepsilon/2)^2-x^2], & -1-\varepsilon/2<x\le -1+\varepsilon/2,\\ \frac{1}{3}, & -1+\varepsilon/2<x\le -\varepsilon/2,\\ \frac{1}{3}+\frac{1}{6\varepsilon}[(\varepsilon/2)^2-x^2], & -\varepsilon/2<x\le \varepsilon/2,\\ \frac{1}{3}, & \varepsilon/2<x\le 1-\varepsilon/2,\\ \frac{1}{6\varepsilon}[(1+\varepsilon/2)^2-x^2], & 1-\varepsilon/2<x\le 1+\varepsilon/2,\\ 0, & x>1+\varepsilon/2,\end{cases} \tag{20}$$

and

$$Q(x)=\begin{cases}1, & x\le -1-\varepsilon/2,\\ 1-\frac{1}{3\varepsilon}[x-(-1-\varepsilon/2)], & -1-\varepsilon/2<x\le -1+\varepsilon/2,\\ \frac{2}{3}, & -1+\varepsilon/2<x\le -\varepsilon/2,\\ \frac{2}{3}-\frac{1}{3\varepsilon}[x-(-\varepsilon/2)], & -\varepsilon/2<x\le \varepsilon/2,\\ \frac{1}{3}, & \varepsilon/2<x\le 1-\varepsilon/2,\\ \frac{1}{3}-\frac{1}{3\varepsilon}[x-(1-\varepsilon/2)], & 1-\varepsilon/2<x\le 1+\varepsilon/2,\\ 0, & x>1+\varepsilon/2.\end{cases}, \tag{21}$$

so

$$-\frac{y(x)}{1-Q(x)}=\begin{cases}\frac{1}{2}[x-(1+\varepsilon/2)], & -1-\varepsilon/2<x\le -1+\varepsilon/2,\\ -1, & -1+\varepsilon/2<x\le -\varepsilon/2,\\ \frac{x^2-(\varepsilon/2)^2-2\varepsilon}{2x+3\varepsilon}, & -\varepsilon/2<x\le \varepsilon/2,\\ -\frac{1}{2}, & \varepsilon/2<x\le 1-\varepsilon/2,\\ \frac{x^2-(1+\varepsilon/2)^2}{2x+5\varepsilon-2}, & 1-\varepsilon/2<x1+\varepsilon/2,\\ 0, & x>1+\varepsilon/2,\end{cases} \tag{22}$$

and

$$\frac{y(x)}{Q(x)}=\begin{cases}0, & x\le -1-\varepsilon/2,\\ \frac{-x^2+(1+\varepsilon/2)^2}{-2x+3\varepsilon}, & -1-\varepsilon/2<x\le -1+\varepsilon/2,\\ \frac{1}{2}, & -1+\varepsilon/2<x\le -\varepsilon/2,\\ \frac{x^2-(\varepsilon/2)^2-2\varepsilon}{2x-3\varepsilon}, & -\varepsilon/2<x\le \varepsilon/2,\\ 1, & \varepsilon/2<x\le 1-\varepsilon/2,\\ \frac{x^2-(1+\varepsilon/2)^2}{2x-\varepsilon-2}, & 1-\varepsilon/2<x\le 1+\varepsilon/2.\end{cases} \tag{23}$$

Sets of triple $\{b, a_1, a_2\}$ satisfying the Lloyd-Max conditions are (assuming $\epsilon\to 0$):

(i) $b=0, a_1=-\frac{2}{3}, a_2=\frac{2}{3}$;

(ii) $b=-\frac{1}{4}, a_1=-1, a_2=\frac{1}{2}$;

(iii) $b=\frac{1}{4}, a_1=-\frac{1}{2}, a_2=1$.

(f) (i) For $b=0, a_1=-\frac{2}{3}, a_2=\frac{2}{3}$,

$$\mathrm{MSE}=\lim_{b\to 0}\int_{-\infty}^{0} f(u)(u+\frac{2}{3})^2\,\mathrm{d}u+\int_{0}^{+\infty} f(u)(u-\frac{2}{3})^2\,\mathrm{d}u=\frac{2}{9}. \tag{24}$$

(ii) For $b = -\frac{1}{4}, a_1 = -1, a_2 = \frac{1}{2}$,

$$\text{MSE} = \lim_{b\to 0} \int_{-\infty}^{-\frac{1}{4}} f(u)(u+1)^2 \, \mathrm{d}u + \int_{-\frac{1}{4}}^{+\infty} f(u)(u-\frac{1}{2})^2 \, \mathrm{d}u = \frac{1}{6}. \tag{25}$$

(iii) For $b = \frac{1}{4}, a_1 = -\frac{1}{2}, a_2 = 1$,

$$\text{MSE} = \lim_{b\to 0} \int_{-\infty}^{\frac{1}{4}} f(u)(u+\frac{1}{2})^2 \, \mathrm{d}u + \int_{\frac{1}{4}}^{+\infty} f(u)(u-1)^2 \, \mathrm{d}u = \frac{1}{6}. \tag{26}$$

The solutions (ii) ($b = -\frac{1}{4}, a_1 = -1, a_2 = \frac{1}{2}$) and (iii) ($b = \frac{1}{4}, a_1 = -\frac{1}{2}, a_2 = 1$) minimize the MSE.

□

Problem 3 (3.4) Score: ________. Section 3.4 partly analyzed a minimum-MSE quantizer for a pdf in which $f_U(u) = f_1$ over an interval of size L_1, $f_U(u) = f_2$ over an interval of size L_2, and $f_U(u) = 0$ elsewhere. Let M be the total number of representation points to be used, with M_1 in the first interval and $M_2 = M - M_1$ in the second. Assume (from symmetry) that the quantization intervals are of equal size $\Delta_1 = L_1/M_1$ in interval 1 and of equal size $\Delta_2 = L_2/M_2$ in interval 2. Assume that M is very large, so that we can approximately minimize the MSE over M_1, M_2 without an integer constraint on M_1, M_2 (that is, assume that M_1, M_2 can be arbitrary real numbers).

(a) Show that the MSE is minimized if $\Delta_1 f_1^{1/3} = \Delta_2 f_2^{1/3}$, i.e. the quantization interval sizes are inversely proportional to the cube root of the density. [Hint. Use a Lagrange multiplier to perform the minimization. That is, to minimize a function $\text{MSE}(\Delta_1, \Delta_2)$ subject to a constraint $M = f(\Delta_1, \Delta_2)$, first minimize $\text{MSE}(\Delta_1, \Delta_2) + \lambda f(\Delta_1, \Delta_2)$ without the constraint, and, second, choose λ so that the solution meets the constraint.]

(b) Show that the minimum MSE under the above assumption is given by

$$\text{MSE} = \frac{(L_1 f_1^{1/3} + L_2 f_2^{1/3})^3}{12M^2}.$$

(c) Assume that the Lloyd-Max algorithm is started with $0 < M_1 < M$ representation points in the first interval and $M_2 = M - M_1$ points in the second interval. Explain where the Lloyd-Max algorithm converges for this starting point. Assume from here on that the distance between the two intervals is very large.

(d) Redo part (c) under the assumption that the Lloyd-Max algorithm is started with $0 < M_1 < M - 2$ representation points in the first interval, one point between the two intervals, and the remaining points in the second interval.

(e) Express the exact minimum MSE as a minimum over $M - 1$ possibilities, with one term for each choice of $0 < M_1 < M$. (Assume there are no representation points between the two intervals.)

(f) Now consider an arbitrary choice of Δ_1 and Δ_2 (with no constraint on M). Show that the entropy of the set of quantization points is given by

$$H(V) = -f_1 L_1 \log(f_1 \Delta_1) - f_2 L_2 \log(f_2 \Delta_2).$$

(g) Show that if the MSE is minimized subject to a constraint on the entropy (ignoring the integer constraint on quantization level), then $\Delta_1 = \Delta_2$.

Solution: (a) The MSE of the quantization is

$$\begin{aligned}\text{MSE} =&E[(U-V)^2] = \sum_{i=1,2}\sum_{j=1}^{M_i}\int_{a_j^{(i)}-\frac{1}{2}\Delta_i}^{a_j^{(i)}+\frac{1}{2}\Delta_i} f_i\left[u-a_j^{(i)}\right]^2 \mathrm{d}u\\ =&\sum_{j=1}^{M_1}\int_{a_j-\frac{1}{2}\Delta_1}^{a_j+\frac{1}{2}\Delta_1} f_1\left[u-a_j^{(1)}\right]^2 \mathrm{d}u + \sum_{j=1}^{M_2}\int_{a_j^{(2)}-\frac{1}{2}\Delta_2}^{a_j^{(2)}+\frac{1}{2}\Delta_2} f_2\left[u-a_j^{(2)}\right]^2 \mathrm{d}u\\ =&M_1f_1\frac{\Delta_1^3}{12}+M_2f_2\frac{\Delta_2^3}{12}\\ =&f_1L_1\frac{\Delta_1^2}{12}+f_2L_2\frac{\Delta_2^2}{12},\end{aligned} \tag{27}$$

where $a_j^{(i)}$ is the jth representation point in the ith interval. The constraint is

$$M=f(\Delta_1,\Delta_2)=\frac{L_1}{\Delta_1}+\frac{L_2}{\Delta_2}. \tag{28}$$

To minimize $\text{MSE}+\lambda f(\Delta_1,\Delta_2)$, we have

$$\frac{\partial}{\partial\Delta_1}[\text{MSE}+\lambda f(\Delta_1,\Delta_2)] =\frac{1}{6}f_1L_1\Delta_1-\frac{\lambda L_1}{\Delta_1^2}=0, \tag{29}$$

$$\frac{\partial}{\partial\Delta_2}[\text{MSE}+\lambda f(\Delta_1,\Delta_2)] =\frac{1}{6}f_1L_1\Delta_1-\frac{\lambda L_2}{\Delta_2^2}=0. \tag{30}$$

$$\Longrightarrow \lambda=\frac{1}{6}f_1\Delta_1^3=\frac{1}{6}f_1\Delta_2^3, \tag{31}$$

$$\Longrightarrow \Delta_1 f_1^{1/3}=\Delta_2 f_2^{1/3}. \tag{32}$$

Therefore, MSE is minimized if $\Delta_1 f_1^{1/3}=\Delta_2 f_2^{1/3}$.

(b) From equation (31) together with the constraint (28), we get

$$\lambda =\frac{(L_1f_1^{1/3}+L_2f_2^{1/3})^3}{6M^3}, \tag{33}$$

$$\Delta_1 =\frac{L_1f_1^{1/3}+L_2f_2^{1/3}}{Mf_1^{1/3}}, \tag{34}$$

$$\Delta_2 =\frac{L_1f_2^{1/3}+L_2f_2^{1/3}}{Mf_2^{1/3}}. \tag{35}$$

so the minimized MSE under the above assumption is

$$\text{MSE}=\frac{f_1L_1^{1/3}(L_1f_1^{1/3}+L_2f_2^{1/3})^2}{12M^2}+\frac{f_2L_2^{1/3}(L_1f_1^{1/3}+L_2f_2^{1/3})^2}{12M^2}=\frac{(L_1f_1^{1/3}+f_2f_2^{1/3})^3}{12M^2}. \tag{36}$$

(c) The Lloyd-Max algorithm converges at such a scheme:

$$\text{representation points}: \overbrace{u_1+\frac{L_1}{2M_1},\quad u_1+3\frac{L_1}{2M_1},\quad \cdots,\quad u_1+(2M_1-3)\frac{L_1}{2M_1},\quad u_1+(2M_1-1)\frac{L_2}{2M_1}}^{M_1\text{ uniformly-spaced representation points in the first interval}},$$
$$\overbrace{u_2+\frac{L_2}{2M_2},\quad u_2+3\frac{L_2}{2M_2},\quad \cdots,\quad u_2+(2M_2-3)\frac{L_2}{2M_2},\quad u_2+(2M_2-1)\frac{L_2}{2M_2}}^{M_2\text{ uniformly-spaced representation points in the second interval}},$$

quantization intervals :

$$\overbrace{\left[u_1, u_1+\frac{L_1}{M_1}\right], \left(u_1+\frac{L_1}{M}, u_1+2\frac{L_1}{M_1}\right], \cdots, \left(u_1+(M_1-2)\frac{L_1}{M_1}, u_1+(M_1-1)\frac{L_1}{M_1}\right]}^{M_1-1 \text{ uniformly distributed quantization intervals in the first interval}}, \overbrace{\left(u_1+(M_1-1)\frac{L_1}{M_1}, \frac{u_1+(M_1-\frac{1}{2})\frac{L_1}{M_1}+u_2+\frac{1}{2}\frac{L_2}{M_2}}{2}\right]}^{M_1\text{th quantization interval}}$$

$$\overbrace{\left(\frac{u_1+(M_1-\frac{1}{2})\frac{L_1}{M_1}+u_2+\frac{1}{2}\frac{L_2}{M_2}}{2}, u_2+\frac{L_2}{M_2}\right]}^{M_1+1\text{th quantization interval}}, \overbrace{\left(u_2+\frac{L_2}{M_2}, u_2+2\frac{L_2}{M_2}\right], \cdots, \left(u_2+(M_2-2)\frac{L_2}{M_2}, u_2+(M_2-1)\frac{L_2}{M_2}\right], \left(u_2+(M_2-1)\frac{L_2}{M_2}, u_2+L_2\right]}^{M_2-1 \text{ uniformly distributed quantization intervals in the second interval}}.$$

where $[u_1, u_1+L_1]$ is the first interval and $[u_2+L_2]$ is the second interval.

Explanation: if the Lloyd-Max algorithm starts with $0<M_1<M$ representation points in the first interval and $M_2=M-M_1$ points in the second interval, the second step of Lloyd-Max algorithm (choose the boundary points of the quantization intervals as the midpoints of the neighboring representation points) will remain the first M_1 boundary points of the quantization intervals in the first interval, last M_2 boundary points of quantization intervals in the second interval and the M_1+1th boundary point of the interval in the gap between the two intervals. The third step (choose the representation points as the expectation of U in their corresponding quantization intervals) will remain the first M_1 representation points in the first interval and the last M_2 representation points in the second interval. Repeating the Lloyd-Max algorithm, eventually, we will reach the quantizer scheme described above, since it satisfies the Lloyd-Max conditions.

(d) The Lloyd-Max algorithm will end in such a scheme:

$$\text{representation points}: \overbrace{u_1+\frac{L_1}{2M_1}, \quad u_1+3\frac{L_1}{2M_1}, \quad \cdots, \quad u_1+(2M_1-3)\frac{L_1}{2M_1}, \quad u_1+(2M_1-1)\frac{L_1}{2M_1}}^{M_1 \text{ uniformly-spaced representation points in the first interval}}, \overbrace{u_0}^{\text{initial position of the } M_1+1\text{th representation points}},$$

$$\overbrace{u_2+\frac{L_2}{2(M-M_1-1)}, \quad u_2+3\frac{L_2}{2(M-M_1-1)}, \quad \cdots, \quad u_2+[2(M-M_1)-5]\frac{L_2}{2(M-M_1-1)}, \quad u_2+[2(M-M_1)-3]\frac{L_2}{2(M-M_1-1)}}^{M-M_1-1 \text{ uniformly-spaced representation points in the second interval}},$$

$$\text{quantization intervals}:$$

$$\overbrace{\left[u_1, u_1+\frac{L_1}{M_1}\right], \left(u_1+\frac{L_1}{M}, u_1+2\frac{L_1}{M_1}\right], \cdots, \left(u_1+(M_1-2)\frac{L_1}{M_1}, u_1+(M_1-1)\frac{L_1}{M_1}\right]}^{M_1-1 \text{ uniformly distributed quantization intervals in the first interval}}, \overbrace{\left(u_1+(M_1-1)\frac{L_1}{M_1}, \frac{u_1+(M_1-\frac{1}{2})\frac{L_1}{M_1}+u_0}{2}\right]}^{M_1\text{th quantization interval}},$$

$$\overbrace{\left(\frac{u_1+(M_1-\frac{1}{2})\frac{L_1}{M_1}+u_0}{2}, \frac{u_0+u_2+\frac{L_2}{2(M-M_1-1)}}{2}\right]}^{(M_1+1)\text{th quantization interval}}, \overbrace{\left(\frac{u_0+u_2+\frac{L_2}{2(M-M_1-1)}}{2}, u_2+\frac{L_2}{M-M_1-1}\right]}^{(M_1+2)\text{th quantization interval}},$$

$$\overbrace{\left(u_2+\frac{L_2}{M-M_1-1}, u_2+2\frac{L_2}{M-M_1-1}\right], \left(u_2+2\frac{L_2}{M-M_1-1}, u_2+3\frac{L_2}{M-M_1-1}\right], \cdots, \left(u_2+(M-M_1-2)\frac{L_2}{M-M_1-1}, u_2+L_2\right]}^{M-M_1-1 \text{ uniformly distributed quantization quantization intervals in the second interval}}.$$

where the coordinate of the M_1+1th representation u_0 remains its initial value, since the Lloyd algorithm is not well-defined within the interval where the pdf is 0. Other representation points and quantization intervals converges similarly as (c).

(e) Now M_1 and M_2 are integers, so equation (27) can be expressed as

$$\text{MSE} = \frac{f_1 L_1^3}{12M_1^2} + \frac{f_2 L_2^3}{12M_2^2}. \tag{37}$$

(f) The entropy of the set of quantization points is

$$H(V) = -\sum_{i=1,2} M_i f_i \Delta_i \log_2(f_i\Delta_i) = -f_1 L_1 \log_2(f_1\Delta_1) - f_2 L_2 \log_2(f_2\Delta_2). \tag{38}$$

(g) To minimize MSE under a constraint on the entropy, we have

$$\frac{\partial}{\partial\Delta_1}[\text{MSE}+\lambda H(V)] = \frac{1}{6}f_1 L_1 \Delta_1 - \frac{\lambda f_1 L_1}{\Delta_1} = 0, \tag{39}$$

$$\frac{\partial}{\partial \Delta_2}[\text{MSE} + \lambda H(V)] = \frac{1}{6} f_2 L_2 \Delta_2 - \frac{\lambda f_2 L_2}{\Delta_2} = 0, \tag{40}$$

$$\Longrightarrow \Delta_1 = \Delta_2 = \sqrt{6\lambda}. \tag{41}$$

□

Problem 4 (3.5) Score: _________. (a) Assume that a continuous-valued rv Z has probability density that is 0 except over the interval $[-A, +A]$. Show that the differential entropy $h(Z)$ is upperbounded $1 + \log_2 A$.

(b) Show that $h(Z) = 1 + \log_2 A$ if and only if Z is uniformly distributed between $-A$ and $+A$.

Solution: (a) The differential entropy of the continuous-valued rv Z is

$$h(Z) = -\int_{-\infty}^{+\infty} f_Z(z) \log_2 f_Z(z)\, \mathrm{d}z = -\int_{-A}^{+A} f_Z(z) \log_2 f_Z(z)\, \mathrm{d}z. \tag{42}$$

Method I: Variational method. Regard $h(Z)$ as a functional of $f_Z(z)$:

$$h(Z) = h[f_Z(z)]. \tag{43}$$

Suppose that $f_{Z0}(z)$ is the pdf of Z that maximizes $h(Z)$, and $f_{Z1}(z)$ an arbitrary function defined on $[-A, +A]$ satisfying

(i) $\int_{-A}^{+A} f_{Z1}(z)\, \mathrm{d}z = 0$;

(ii) the first order derivative of $f_{Z1}(z)$, $f'_{Z1}(z)$ exists.

Since $f_{Z0}(z)$ is the pdf that maximizes $h(Z)$, the derivative of $h[f_{Z0}(z) + \epsilon f_{Z1}(z)]$ about ϵ is 0 at $\epsilon = 0$ for arbitrary $h_{Z1}(z)$ satisfying the above conditions:

$$\begin{aligned}
\frac{\partial}{\partial \epsilon} h[f_{Z0}(z) + \epsilon f_{Z1}(z)] &= -\frac{\partial}{\partial \epsilon} \int_{-A}^{+A} [f_{Z0}(z) + \epsilon f_{Z1}(z)] \log_2 [f_{Z0}(z) + \epsilon f_{Z1}(z)]\, \mathrm{d}z \bigg|_{\epsilon=0} \\
&= -\int_{-A}^{+A} \left\{ f_{Z1}(z) \log_2 [f_{Z0}(z) + \epsilon f_{Z1}(z)] + [f_{Z0}(z) + \epsilon f_{Z1}(z)] \frac{f_{Z1}(z)}{[f_{Z0}(z) + \epsilon f_{Z1}(z)] \ln 2} \right\} \bigg|_{\epsilon=0} \mathrm{d}z \\
&= -\int_{-A}^{+A} f_{Z1}(z) \left\{ \log_2 [f_{Z0}(z)] + \frac{1}{\ln 2} \right\} \mathrm{d}z \\
&= -\int_{-A}^{+A} f_{Z1}(z) \log_2 [f_{Z0}(z)]\, \mathrm{d}z = 0, \quad \forall f_{Z1}(z) \text{ satisfying conditions (i) and (ii).}
\end{aligned} \tag{44}$$

To make the above equation hold for all $f_{Z1}(z)$ satisfying conditions (i) and (ii) and considering the condition (i) that $\int_{-A}^{+A} f_{Z1}(z)\, \mathrm{d}z = 0$, $f_{Z0}(z)$ must be a constant over the range of $[-A, +A]$. Therefore, the pdf that maximizes $h(Z)$ is

$$f_{Z0}(z) = \frac{1}{2a}, \quad \text{for } x \in [-A, +A], \tag{45}$$

and the upperbound of $h(Z)$ is

$$h_0(Z) = -\int_{-A}^{+A} \frac{1}{2A} \log_2 \frac{1}{2A}\, \mathrm{d}z = 1 + \log_2 A. \tag{46}$$

Method II: The pdf of uniform distribution over $[-A, +A]$ is

$$f_{Z0}(z) = \frac{1}{2A}, \quad \text{for } z \in [-A, +A], \tag{47}$$

whose corresponding differential entropy is

$$h_0(Z) = -\int_{-A}^{+A} \frac{1}{2A} \log_2 \frac{1}{2A} \, \mathrm{d}z = 1 + \log_2 A. \tag{48}$$

For an arbitrary pdf $f_Z(z)$,

$$\begin{aligned}
h(Z) - h_0(Z) =& -\int_{-A}^{+A} f_Z(z) \log_2[f_Z(z)] \, \mathrm{d}z - (1 + \log_2 A) \\
=& -\int_{-A}^{+A} f_Z(z) \log_2[f_Z(z)] \, \mathrm{d}z + \int_{-A}^{+A} f_Z(z) \log_2 \frac{1}{2A} \, \mathrm{d}z \\
=& \int_{-A}^{+A} f_Z(z) \log_2 \frac{f_{Z0}(z)}{f_Z(z)} \, \mathrm{d}z \\
=& \frac{1}{\ln 2} \int_{-A}^{+A} f_Z(z) \ln \frac{f_{Z0}(z)}{f_Z(z)} \, \mathrm{d}z \\
\leq& \frac{1}{\ln 2} \int_{-A}^{+A} f_Z(z) \left[\frac{f_{Z0}(z)}{f_Z(z)} - 1 \right] \mathrm{d}z \\
=& \frac{1}{\ln 2} \int_{-A}^{+A} [f_{Z0}(z) - f_Z(z)] \, \mathrm{d}z = 0.
\end{aligned} \tag{49}$$

Therefore, the differential entropy $h(Z)$ is upperbounded $1 + \log_2 A$.

(b) **Method I: Variational Method**: As mentioned in (a), to make the equation (44) hold for all $f_{Z1}(z)$ satisfying conditions (i) and (ii), $f_{Z0}(z)$ must be a constant over the range of $[-A, +A]$. Therefore, $h(Z) = 1 + \log_2 A$ if and only if Z is uniformly distributed between $-A$ and $+A$.

Method II: The two sides of the inequality in (49) reach equal only when $\frac{f_{Z0}(z)}{f_Z(z)} = 1$. Therefore, $h(Z) = 1 + \log_2 A$ if and only if Z is uniformly distributed between $-A$ and $+A$.

□