Intro to Communication System
EE140
Fall, 2020

Assignment 1

Due time : 10:15, Sept 18, 2020 (Friday)

Name : 陈 栋 霖
Student ID : 45875852
Grade : ________

Problem 1 (15 pts) Score: ________. Classify each of the following signals as an energy signal or as a power signal by calculating E (energy) or P (power). Note: the parameters involved are positive constants.

a) $x(t) = e^{-\alpha|t|}\cos\pi t$,

b) $x(t) = \Pi(t-3)\cos 3\pi t$, $\left(\Pi(t) = \begin{cases} 1, & |t| \le 0.5 \\ 0, & \text{otherwise} \end{cases}\right)$

c) $x(t) = |t|$,

d) $x(t) = \sum_{n=-\infty}^{\infty} \Lambda(\frac{t}{2} - n)$, $\left(\Lambda(t) = \begin{cases} 1-|t|, & |t| \\ 0, & \text{otherwise} \end{cases}\right)$

e) $x(t) = e^{j2\pi 3t}u(t)$, $\left(u(t) = \begin{cases} 1, & t \ge 0 \\ 0, & \text{otherwise} \end{cases}\right)$

求 $\int AB\,dt$

		D		F
A	$\cos 2\pi t$	$-2\pi\sin 2\pi t$		$-4\pi^2\cos 2\pi t$
B	$e^{-2\alpha t}$	$-\frac{1}{2\alpha}e^{-2\alpha t}$		$\frac{1}{4\alpha^2}e^{-2\alpha t}$
		C		E

$\int AB\,dt = AC - DE + \int FE\,dt$.

Solution: a) The energy of the signal:

$$\begin{aligned} E &= \int_{-\infty}^{+\infty} |x(t)|^2\,dt = \int_{-\infty}^{+\infty} e^{-2\alpha|t|}\cos^2\pi t\,dt \\ &= 2\int_0^{+\infty} e^{-2\alpha t}\cos^2\pi t\,dt \\ &= \int_0^{+\infty} e^{-2\alpha t}(\cos 2\pi t + 1)\,dt \\ &= \int_0^{+\infty} e^{-2\alpha t}\,\mathrm{Re}\left[e^{i2\pi t}\right]dt - \frac{1}{2\alpha}e^{-2\alpha t}\Big|_0^{+\infty} \\ &= \mathrm{Re}\left[\int_0^{+\infty} e^{(-2\alpha+i2\pi)t}\,dt\right] + \frac{1}{2\alpha} \\ &= \mathrm{Re}\left[\frac{1}{2\alpha - i2\pi}\right] + \frac{1}{2\alpha} \\ &= \frac{\alpha}{2(\alpha^2+\pi^2)} + \frac{1}{2\alpha} < +\infty. \end{aligned} \tag{1}$$

-1

Therefore, the signal is **an energy signal**.

b) The energy of the signal:

$$\begin{aligned} E &= \int_{-\infty}^{+\infty} |x(t)|^2\,dt = \int_{-\infty}^{+\infty} |\Pi(t-3)\cos 3\pi t|^2\,dt \\ &= \int_{5/2}^{7/2} \cos^2 3\pi t\,dt \\ &= \frac{1}{2}\int_{5/2}^{7/2} (\cos 6\pi t + 1)\,dt \\ &= \frac{1}{2} < +\infty. \end{aligned}$$

Therefore, the signal is **an energy signal**.

c) The average power of the signal is

$$P = \lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\,dt = \lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{T/2} |t|^2\,dt = \lim_{T\to+\infty} \frac{2}{T}\int_0^{T/2} t^2\,dt = \lim_{T\to+\infty} \frac{T^2}{12} = +\infty. \tag{2}$$

Therefore, the signal is **neither an energy signal nor a power signal**.

d) The signal function can be written as

$$x(t) = 1. \tag{3}$$

The average power of the signal:

$$P = \lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2\,dt = \lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}dt = 1, \tag{4}$$

$$\Longrightarrow 0 < P < +\infty. \tag{5}$$

Therefore, the signal is **a power signal.**

e) The average power of the signal:

$$\begin{aligned}P = &\lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2\,dt = \lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}\left|e^{j6\pi t}u(t)\right|^2\,dt\\ =&\lim_{T\to+\infty}\frac{1}{T}\int_{0}^{T/2}\left|e^{j6\pi t}\right|^2\,dt\\ =&\lim_{T\to+\infty}\frac{1}{T}\int_{0}^{T/2}dt\\ =&\frac{1}{2}.\end{aligned} \tag{6}$$

$$\Longrightarrow 0 < P < +\infty. \tag{7}$$

Therefore, the signal is **a power signal.**

□

Problem 2 (20 pts) Score: ________. Calculate **the Fourier transform and the energy** of the following signals.

a) $x_1(t) = 5\,\mathrm{sinc}\,(2t)e^{j2\pi 3t}$

b) $x_2(t) = \mathrm{sinc}^2(t-1)$

c) $x_a(t) = x_1(t) + x_2(-t)$

d) $x_b(t) = x_1(-t) + x_2(t)$

e) $x_c(t) = 2x_1(t)\cos 6\pi t + x_2(t)e^{j6\pi t}$

Solution: a) We first look for the Fourier transform of $\frac{1}{\pi t}$ (see reference at [1]). Consider such a function:

$$f_\alpha(t) = \begin{cases} e^{-2\pi\alpha t}, & t > 0\\ 0, & t = 0\\ -e^{2\pi\alpha t}, & t < 0\end{cases} \tag{8}$$

where $\alpha > 0$. The Fourier transform of the above function is

$$\mathscr{F}[f_\alpha(t)] = \int_{-\infty}^{+\infty} f(t)e^{-j2\pi f\tau}\,d\tau = -\int_{-\infty}^{0} e^{2\pi(\alpha-jf)\tau}\,d\tau + \int_{0}^{+\infty} e^{-2\pi(\alpha+jf)\tau}\,d\tau$$

[1] `https://math.stackexchange.com/questions/1033870/does-the-fourier-transform-exist-for-ft-1-t`

$$= -\frac{1}{2\pi(\alpha - jf)} + \frac{1}{2\pi(\alpha + if)} = -\frac{2jf}{2\pi(\alpha^2 + f^2)}. \tag{9}$$

Since the sign function is the limit of $f_\alpha(t)$ when $a \to 0$:

$$\operatorname{sgn}(t) = \begin{cases} 1, & t > 0. \\ 0, & t = 0. \\ -1, & t < 0. \end{cases} = \lim_{a \to 0} f_\alpha(t). \tag{10}$$

by taking the limit $\alpha \to 0$, we get the Fourier transform of the sign function:

$$\mathscr{F}[\operatorname{sgn}(t)] = \lim_{a \to 0} [f_\alpha(t)] = \frac{1}{j\pi f}. \tag{11}$$

Using the duality property, the Fourier transform of $\frac{1}{\pi t}$ is

$$\mathscr{F}[\frac{1}{\pi t}] = -j \operatorname{sgn}(f). \tag{12}$$

Then, using the multiplication property, the Fourier transform of the sinc function is

$$\begin{aligned}
\mathscr{F}[\operatorname{sinc}(t)] =& \mathscr{F}\left[\frac{\sin(\pi t)}{\pi t}\right] = \mathscr{F}[\sin(\pi t)] * \mathscr{F}\left[\frac{1}{\pi t}\right] \\
=& \frac{1}{2j}\left[\delta\left(f - \frac{1}{2}\right) - \delta\left(f + \frac{1}{2}\right)\right] * [-j \operatorname{sgn}(f)] \\
=& -\frac{1}{2}\int_{-\infty}^{+\infty}\left[\delta\left(\nu - \frac{1}{2}\right) - \delta\left(\nu + \frac{1}{2}\right)\right] \operatorname{sgn}(f - \nu)\, d\nu \\
=& -\frac{1}{2}\left[\operatorname{sgn}\left(f - \frac{1}{2}\right) - \operatorname{sgn}\left(f + \frac{1}{2}\right)\right] \\
=& \Pi(f).
\end{aligned} \tag{13}$$

Finally, using the scaling shifting property, we have

$$\mathscr{F}[\operatorname{sinc}(2t)] = \frac{1}{2}\Pi\left(\frac{f}{2}\right). \tag{14}$$

And using the frequency shifting property, we get the Fourier transform of $x_1(t)$:

$$\mathscr{F}[x_1(t)] = \mathscr{F}[5\operatorname{sinc}(2t)e^{j2\pi 3t}] = \frac{5}{2}\Pi\left(\frac{f-3}{2}\right) \tag{15}$$

The energy of $x_1(t)$ is

$$E = \int_{-\infty}^{+\infty} |\mathscr{F}[x_1(t)]|^2\, df = \int_{-\infty}^{+\infty}\left|\frac{5}{2}\Pi\left(\frac{f-3}{2}\right)\right|^2 df = \frac{25}{2} \tag{16}$$

b) Using the time shifting property, we have

$$\mathscr{F}[\operatorname{sinc}(t-1)] = \Pi(f)e^{-j2\pi f}. \tag{17}$$

Using the multiplication property, we get the Fourier transform of $x_2(t)$:

$$\begin{aligned}
\mathscr{F}[x_2(t)] =& \mathscr{F}[\operatorname{sinc}^2(t-1)] = \mathscr{F}[\operatorname{sinc}(t-1)] * \mathscr{F}[\operatorname{sinc}(t-1)] \\
=& \int_{-\infty}^{+\infty} \Pi(\nu)e^{-j2\pi\nu}\Pi(f-\nu)e^{-j2\pi(f-\nu)}\, d\nu
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{+\infty} \Pi(\nu)\Pi(f-\nu)e^{-j2\pi f} d\nu \\
&= \Lambda(f)e^{-j2\pi f}.
\end{aligned} \tag{18}
$$

The energy of $x_2(t)$ is

$$
\begin{aligned}
E = &\int_{-\infty}^{+\infty} |\mathscr{F}[x_2(t)]|^2 \, df = \int_{-\infty}^{+\infty} \left|\Lambda(f)e^{-j2\pi f}\right|^2 df \\
= &\int_{-\infty}^{+\infty} |\Lambda(f)|^2 \, df \\
= &2\int_0^1 (1-f)^2 \, df \\
= &\frac{2}{3}.
\end{aligned}
$$

c) Using the superposition and the scaling properties, the Fourier transform of $x_a(t)$ is

$$
\mathscr{F}[x_a(t)] = \mathscr{F}[x_1(t) + x_2(-t)] = \mathscr{F}[x_1(t)] + \mathscr{F}[x_2(-t)] = \frac{5}{2}\Pi\left(\frac{f-3}{2}\right) + \Lambda(f)e^{-j2\pi f}. \tag{19}
$$

The energy of $x_a(t)$ is

$$
\begin{aligned}
E = &\int_{-\infty}^{+\infty} |\mathscr{F}[x_a(t)]|^2 \, df = \int_{-\infty}^{+\infty} \left|\frac{5}{2}\Pi\left(\frac{f-3}{2}\right) + \Lambda(f)e^{-j2\pi f}\right|^2 df \\
= &\int_2^4 \left|\frac{5}{2}\Pi\left(\frac{f-3}{2}\right)\right|^2 df + \int_{-1}^1 |\Lambda(f)|^2 \, df \\
= &\int_2^4 \frac{25}{4} \, df + 2\int_0^1 (1-f)^2 \, df \\
= &\frac{25}{2} + \frac{2}{3} = \frac{79}{6}.
\end{aligned} \tag{20}
$$

d) Using the superposition and the scaling properties, the Fourier transform of $x_b(t)$ is

$$
\mathscr{F}[x_b(t)] = \mathscr{F}[x_1(-t) + x_2(t)] = \mathscr{F}[x_1(-t)] + \mathscr{F}[x_2(t)] = \frac{5}{2}\Pi\left(\frac{f-3}{2}\right) + \Lambda(f)e^{-j2\pi f}. \tag{21}
$$

Since the Fourier transform of $x_b(t)$ is the same as $x_a(t)$. The energy of $x_b(t)$ is also the same as that of $x_a(t)$:

$$
E = \frac{79}{6}. \tag{22}
$$

e) The Fourier transform of the first term is

$$
\begin{aligned}
\mathscr{F}[2x_1(t)\cos 6\pi t] = &\mathscr{F}\left[x_1(t)(e^{j2\pi 3t} + e^{-j2\pi 3t})\right] = \mathscr{F}\left[x_1(t)e^{j2\pi 3t}\right] + \mathscr{F}\left[x_1(t)e^{-j2\pi 3t}\right] \\
= &\frac{5}{2}\left[\Pi\left(\frac{f}{2}-3\right) + \Pi\left(\frac{f}{2}\right)\right].
\end{aligned} \tag{23}
$$

The Fourier transform of the second term is

$$
\mathscr{F}\left[x_2(t)e^{j6\pi t}\right] = \mathscr{F}\left[x_2(t)e^{j2\pi 3t}\right] = \Lambda(f-3)e^{-j2\pi(f-3)}. \tag{24}
$$

Therefore, the Fourier transform of $x_c(t)$ is

$$
\mathscr{F}[x_c(t)] = \mathscr{F}\left[2x_1(t)\cos 6\pi t + x_2(t)e^{j6\pi t}\right] = \mathscr{F}\left[2x_1(t)\cos 6\pi t\right] + \mathscr{F}\left[x_2(t)e^{j6\pi t}\right]
$$

$$=\frac{5}{2}\left[\Pi\left(\frac{f}{2}-3\right)+\Pi\left(\frac{f}{2}\right)\right]+\Lambda(f-3)e^{-j2\pi(f-3)}. \tag{25}$$

The energy of $x_c(t)$ is

$$\begin{aligned}E=&\int_{-\infty}^{+\infty}|\mathscr{F}[x_c(t)]|^2\,df=\int_{-\infty}^{+\infty}\left|\frac{5}{2}\left[\Pi\left(\frac{f}{2}-3\right)+\Pi\left(\frac{f}{2}\right)\right]+\Lambda(f-3)e^{-j2\pi(f-3)}\right|^2 df\\ =&\int_5^7\left|\frac{5}{2}\Pi\left(\frac{f}{2}-3\right)\right|^2 df+\int_{-1}^{1}\left|\frac{5}{2}\Pi\left(\frac{f}{2}\right)\right|^2 df+\int_2^4|\Lambda(f-3)e^{-j2\pi(f-3)}|^2\,df\\ =&\int_5^7\frac{25}{4}\,df+\int_{-1}^{1}\frac{25}{4}\,df+2\int_3^4(4-f)^2\,df\\ =&\frac{25}{2}+\frac{25}{2}+\frac{2}{3}=\frac{77}{3}.\end{aligned} \tag{26}$$

□

Problem 3 (10 pts) Score: ________. Calculate the convolution of the following signal.

a) $y(t)=e^{-|t|}*\Pi(t-2)$

b) $y(t)=\operatorname{sgn}(t)*\Lambda(t-2)$

Solution: a) The convolution can be written as

$$\begin{aligned}y(t)=&e^{-|t|}*\Pi(t-2)=\int_{-\infty}^{+\infty}e^{-|\tau|}\Pi(t-\tau-2)\,d\tau\\ =&\int_{t-\frac{5}{2}}^{t-\frac{3}{2}}e^{-|\tau|}\,d\tau.\end{aligned} \tag{27}$$

If $t\geq\frac{5}{2}$,

$$y(t)=\int_{t-\frac{5}{2}}^{t-\frac{3}{2}}e^{-\tau}\,d\tau=e^{\frac{5}{2}-t}-e^{\frac{3}{2}-t}. \tag{28}$$

If $\frac{3}{2}\leq t<\frac{5}{2}$,

$$y(t)=\int_{t-\frac{5}{2}}^{0}e^{\tau}\,d\tau+\int_0^{t-\frac{3}{2}}e^{-\tau}\,d\tau=2-e^{t-\frac{5}{2}}-e^{t-\frac{3}{2}}. \tag{29}$$

If $t<\frac{3}{2}$,

$$y(t)=\int_{t-\frac{5}{2}}^{t-\frac{3}{2}}e^{\tau}\,d\tau=e^{t-\frac{3}{2}}-e^{t-\frac{5}{2}}. \tag{30}$$

Therefore, in general,

$$y(t)=\begin{cases}e^{\frac{5}{2}-t}-e^{\frac{3}{2}-t}, & t\geq\frac{5}{2},\\ 2-e^{t-\frac{5}{2}}-e^{t-\frac{3}{2}}, & \frac{3}{2}\leq t<\frac{5}{2},\\ e^{t-\frac{3}{2}}-e^{t-\frac{5}{2}}, & t<\frac{3}{2}.\end{cases} \tag{31}$$

b) The convolution can be written as

$$y(t)=\operatorname{sgn}(t)*\Lambda(t-2)=\int_{-\infty}^{+\infty}\operatorname{sgn}(\tau)\Lambda(t-\tau-2)\,d\tau$$

$$= \int_{t-\frac{5}{2}}^{t-\frac{3}{2}} \operatorname{sgn}(\tau)\, d\tau. \tag{32}$$

If $t \geq \frac{5}{2}$,

$$y(t) = \int_{t-\frac{5}{2}}^{t-\frac{3}{2}} d\tau = 1. \tag{33}$$

If $\frac{3}{2} < t \leq \frac{5}{2}$,

$$y(t) = \int_{t-\frac{5}{2}}^{0} (-1)\, d\tau + \int_{0}^{t-\frac{3}{2}} d\tau = 2t - 4. \tag{34}$$

If $t < \frac{3}{2}$,

$$y(t) = \int_{t-\frac{5}{2}}^{t-\frac{3}{2}} (-1)\, d\tau = -1. \tag{35}$$

Therefore, in general,

$$y(t) = \begin{cases} -1 & t \leq 1 \\ t^2 - 2t & 1 < t \leq 2 \\ -t^2 + 6t - 8 & 2 < t \leq 3 \\ 1 & t > 3 \end{cases}$$

$$y(t) = \begin{cases} 1, & t \geq \frac{5}{2}, \\ 2t - 4, & \frac{3}{2} \leq t < \frac{5}{2}, \\ -1, & t < \frac{3}{2}. \end{cases} \tag{36}$$

-5

□

Problem 4 (20 pts) Score: ________. Calculate the Fourier transform of the following periodic signal.

a) $\sum_{n=-\infty}^{\infty} \Lambda(\frac{t}{2} - 2n)$

b) $\left[\sum_{n=-\infty}^{\infty} \delta(t - 2n)\right] * \left[\Pi(\frac{t}{2}) \cos(2\pi t)\right]$

对周期信号$f(t)$，$F(f) = \sum_{n=-\infty}^{\infty} F_n \delta(f - nf_1)$

① $A\Pi(\frac{t}{T}) \xrightarrow{FT} \underbrace{AT}_{面积} \operatorname{sinc}(\underbrace{T}_{门宽}f)$

② $\Lambda(\frac{t}{T}) \xrightarrow{FT} \underbrace{T}_{面积} \operatorname{sinc}^2(\underbrace{\frac{T}{2}}_{门宽一半}f)$

其中F_1是$f(t)$的一个周期的FT，$F_n = \frac{1}{T} F_1(f)$

$f_1 = \frac{1}{T}$，T是$f(t)$的周期

Solution: a) The Fourier series of the signal is

$$\sum_{n=-\infty}^{\infty} \Lambda\left(\frac{t}{2} - 2n\right) = \sum_{n=-\infty}^{\infty} c_n e^{jn2\pi\frac{t}{4}}, \tag{37}$$

where

$$\begin{aligned} c_n =& \frac{1}{4} \int_{-2}^{2} \sum_{n=-\infty}^{\infty} \Lambda\left(\frac{\tau}{2} - 2n\right) e^{-jn2\pi\frac{\tau}{4}}\, d\tau \\ =& \frac{1}{2} \int_{0}^{2} \left(1 - \frac{\tau}{2}\right) \cos\left(n2\pi\frac{\tau}{4}\right) d\tau \\ =& \frac{\sin(n\pi)}{n\pi} - \frac{1}{4}\frac{1}{n2\pi\frac{1}{4}} \int_{0}^{2} \tau\, d\left[\sin\left(n2\pi\frac{\tau}{4}\right)\right] \\ =& \frac{\sin(n\pi)}{n\pi} - \frac{\sin(n\pi)}{n\pi} + \frac{1}{4}\frac{1}{n2\pi\frac{1}{4}} \int_{0}^{2} \sin\left(n2\pi\frac{\tau}{4}\right) d\tau \\ =& \frac{1 - \cos(n\pi)}{(n\pi)^2}. \end{aligned} \tag{38}$$

Note that $c_n = 2$ when $n = 0$. The fourier transform of the signal is

$$\mathscr{F}\left[\sum_{n=-\infty}^{\infty} \Lambda\left(\frac{t}{2} - 2n\right)\right] = \mathscr{F}\left[\sum_{n=-\infty}^{\infty} \frac{1 - \cos(n\pi)}{(n\pi)^2} e^{jn2\pi\frac{t}{4}}\right]$$

$F_n = \frac{1}{4} F_1(f) = \frac{1}{4} 2 \operatorname{sinc}^2(2nf_1).$

$$
\begin{aligned}
&= \sum_{n=-\infty}^{\infty} \frac{1-\cos(n\pi)}{(n\pi)^2} \mathscr{F}\left[e^{-j2\pi(f-\frac{n}{4})t}\right] \\
&= \sum_{n=-\infty}^{\infty} \frac{1-\cos(n\pi)}{(n\pi)^2} \delta(f-\frac{n}{4}). \qquad (39)
\end{aligned}
$$

$\frac{1}{2}\sum\delta(f-\frac{n}{4})\mathrm{sinc}^2(\frac{n}{2})$ −2

b) The Fourier series of $\sum_{n=-\infty}^{\infty}\delta(t-2n)$ is

$$\sum_{n=-\infty}^{\infty}\delta(t-2n) = \sum_{n=-\infty}^{\infty} c_n e^{jn2\pi\frac{t}{2}}, \tag{40}$$

where

$$c_n = \frac{1}{2}\int_{-1}^{1}\sum_{n=-\infty}^{\infty}\delta(t-2n)e^{-jn2\pi\frac{\tau}{2}}\,d\tau = \frac{1}{2}. \tag{41}$$

The Fourier transform of $\sum_{n=-\infty}^{\infty}\delta(t-2n)$ is

$$\mathscr{F}\left[\sum_{n=-\infty}^{\infty}\delta(t-2n)\right] = \mathscr{F}\left[\sum_{n=-\infty}^{\infty}\frac{1}{2}e^{jn2\pi\frac{t}{2}}\right] = \sum_{n=-\infty}^{\infty}\frac{1}{2}\mathscr{F}\left[e^{jn2\pi\frac{t}{2}}\right] = \frac{1}{2}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{2}\right). \tag{42}$$

The Fourier transform of $\Pi\left(\frac{t}{2}\right)\cos(2\pi t)$ is

$$
\begin{aligned}
\mathscr{F}\left[\Pi\left(\frac{t}{2}\right)\cos(2\pi t)\right] &= \int_{-\infty}^{+\infty}\Pi\left(\frac{\tau}{2}\right)\cos(2\pi\tau)e^{-j2\pi f\tau}\,d\tau \\
=&2\int_0^1\cos(2\pi\tau)\cos(2\pi f\tau)\,d\tau \\
=&\int_0^1\{\cos[2\pi(f+1)\tau]+\cos[2\pi(f-1)\tau]\}\,d\tau \\
=&\frac{\sin[2\pi(f+1)]}{2\pi(f+1)}+\frac{\sin[2\pi(f-1)]}{2\pi(f-1)}.
\end{aligned}
$$

Using the time convolution property, we get the Fourier transform of the signal

$$
\begin{aligned}
\mathscr{F}\left\{\left[\sum_{n=-\infty}^{\infty}\delta(t-2n)\right]*\left[\Pi\left(\frac{t}{2}\right)\cos(2\pi t)\right]\right\} =&\mathscr{F}\left[\sum_{n=-\infty}^{\infty}\delta(t-2n)\right]\mathscr{F}\left[\Pi\left(\frac{t}{2}\right)\cos(2\pi t)\right] \\
=&\frac{1}{2}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{2}\right)\left\{\frac{\sin[2\pi(f+1)]}{2\pi(f+1)}+\frac{\sin[2\pi(f-1)]}{2\pi(f-1)}\right\}. \qquad (43)
\end{aligned}
$$

□

Problem 5 (15 pts) Score: 15. **Determine** the range of permissible cutoff frequencies for the ideal lowpass filter used to reconstruct the following signal

$$x(t) = 4\cos^2(200\pi t)\cos(1000\pi t)$$

Which is sampled at 2000 samples per second. Sketch $X(f)$ and $X_\delta(f)$ (spectrum after the sampling). Find the minimum allowable sampling frequency.

Solution: The signal can be written as

$$
\begin{aligned}
x(t) =&4\cos^2(200\pi t)\cos(1000\pi t) = 2[1+\cos(400\pi t)]\cos(1000\pi t) \\
=&2\cos(1000\pi t)+\cos(1400\pi t)+\cos(600\pi t). \qquad (44)
\end{aligned}
$$

采样可以当是时域上乘一梳函数，从而频域上卷一梳函数

The spectrum of the signal is

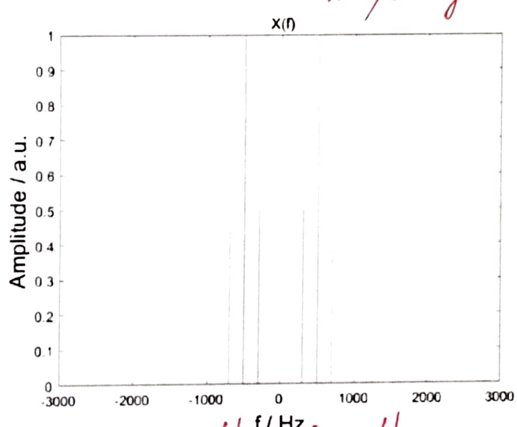
$$
\begin{aligned}
X(f) =& \mathscr{F}[x(t)] = \mathscr{F}[2\cos(1000\pi t) + \cos(1400\pi t) + \cos(600\pi t)] \\
=& \delta(f-500) + \delta(f+500) + \frac{1}{2}\delta(f-700) + \frac{1}{2}\delta(f+700) + \frac{1}{2}\delta(f-300) + \frac{1}{2}\delta(f+300).
\end{aligned} \tag{45}
$$

as shown in figure 1(a). Sampling at 2000 samples per second means that the sampling frequency is $f_s = 2000$ Hz. The spectrum after the sampling is

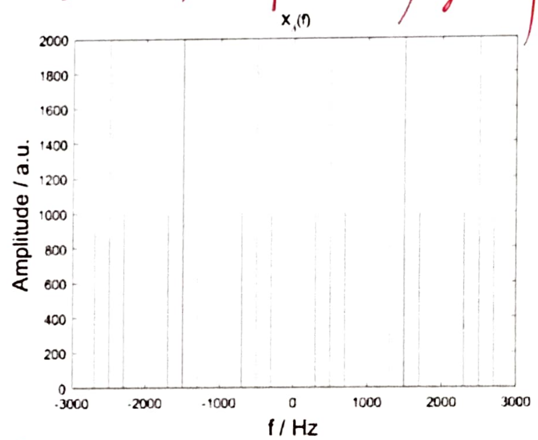
$$
\begin{aligned}
X_\delta(f) =& 2000\sum_{n=-\infty}^{\infty} X(f-2000n) \\
=& 2000\sum_{n=-\infty}^{\infty}\left[\delta(f-2000n-500) + \delta(f-2000n+500) + \frac{1}{2}\delta(f-2000n-700) + \frac{1}{2}\delta(f-2000n+700)\right. \\
& \left. + \frac{1}{2}\delta(f-2000n-300) + \frac{1}{2}\delta(f-2000n+300)\right].
\end{aligned} \tag{46}
$$

as shown in figure 1(b). According to the sampling theorem, the range of permissible cutoff frequencies for the ideal

Maybe you should sketch the spectrum by yourself!



(a) Original spetrum of the signal: $X(f)$



(b) Spectrum after the sampling: $X_\delta(f)$

Figure 1:

(带宽)
$B = f_H - f_L = 700 - 300\,\text{Hz} = 400\,\text{Hz}.$
$f_s \geq 2f_H = 1400$
$k = \lfloor \frac{f_H}{B} \rfloor$ $f_{s\min} = \frac{2f_H}{k}$

lowpass filter to reconstruct the signal is 700Hz $< f_c <$ 1300Hz and the minimum allowable sampling frequency is 1400Hz. □

Problem 6 Score: 10. 1) Express the spectrum $Y(f)$ of

$$y(t) = x(t)\cos(400\pi t) + \hat{x}(t)\sin(400\pi t)$$

using the spectrum $X(f)$ of $x(t)$, where $\hat{x}(t)$ is the Hilbert transform of $x(t)$. (5 pts)

2) if $x(t) = \text{sinc}\,(t)$, sketch $Y(f)$. (5 pts)

$x(t)\sin(2\pi f_0 t) \xrightarrow{FT} \frac{1}{2j}[X(f-f_0) - X(f+f_0)]$

Solution: 1) The spectrum of $y(t)$ is

$$
\begin{aligned}
Y(f) =& \mathscr{F}[y(t)] = \mathscr{F}[x(t)\cos(400\pi t) + \hat{x}(t)\sin(400\pi t)] \\
=& \mathscr{F}\left[x(t)\frac{e^{j400\pi t} + e^{-j400\pi t}}{2}\right] + \mathscr{F}\left\{\mathscr{F}^{-1}[-j\,\text{sgn}\,(f)X(f)]\frac{e^{j400\pi t} - e^{-j400\pi t}}{2j}\right\} \\
=& \frac{1}{2}X(f-200) + \frac{1}{2}X(f+200) - \frac{1}{2}\text{sgn}\,(f-200)X(f-200) + \frac{1}{2}\text{sgn}\,(f+200)X(f+200)
\end{aligned} \tag{47}
$$

$\hat{x}(t) = x(t) * \frac{1}{\pi t} \xrightarrow{FT} \hat{X}(f) = X(f)[-j\,\text{sgn}(f)].$

$x(t)\cos(2\pi f_0 t) \xrightarrow{FT} \frac{1}{2}[X(f-f_0) + X(f+f_0)]$

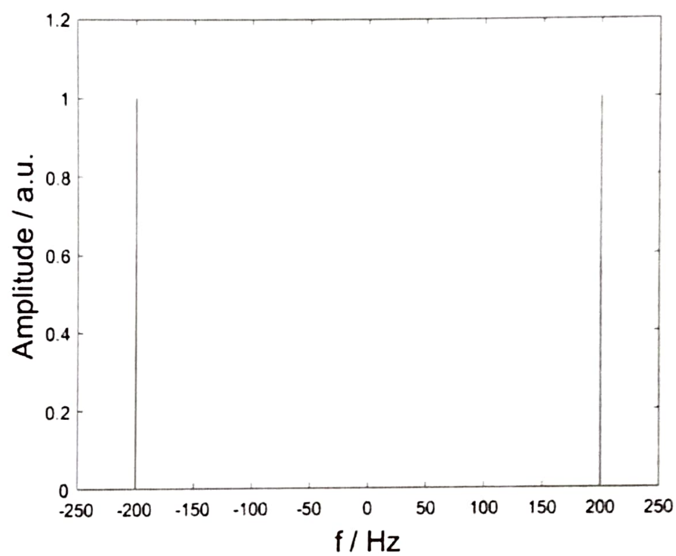
2) If $x(t) = \text{sinc}(t)$, then

$$X(f) = \Pi(f) \tag{48}$$

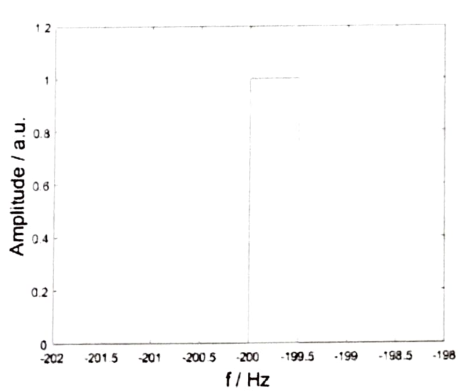
and

$$Y(f) = \frac{1}{2}\Pi(f-200) + \frac{1}{2}\Pi(f+200) - \frac{1}{2}\,\text{sgn}(f-200)\Pi(f-200) + \frac{1}{2}\,\text{sgn}(f+200)\Pi(f+200). \tag{49}$$

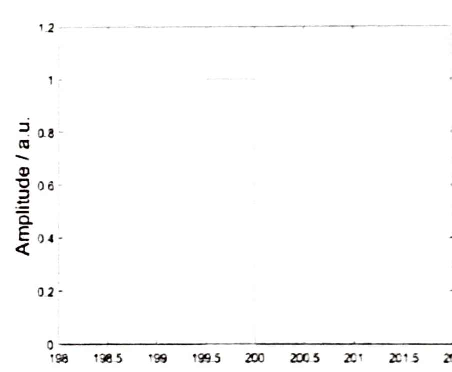
as shown in figure 2. $= \Pi(f-200)\,u(200-f) + \Pi(f+200)\,u(f+200)$



(a) Overall view of $Y(f)$



(b) $Y(f)$ near -200



(c) $Y(f)$ near $+200$

Figure 2: Spectrum of $y(t)$: $Y(f)$

□

Problem 7 Score: 9.5. Consider $x(t) = 2\cos(60\pi t)$, the reference frequency $f_0 = 40$ Hz. Calculate the following signals.

a) The Hilbert transform of $x(t)$, i.e., $\hat{x}(t)$.

b) The analytic signal $x_p(t)$.

c) The complex envelope of $x(t)$, i.e., $\tilde{x}(t)$.

d) The inphase and quadrature component of $x(t)$, i.e., $x_R(t)$ and $x_I(t)$.
(Please refer to Lecture 2, Slide 36 or Page 88 of reference textbook, we will learn this in the next class. I am sorry for the lagging.)

e) Determine and plot the spectrum of the following signals:

i. $x_1(t) = \frac{2}{3}x(t) + \frac{1}{3}j\hat{x}(t)$

ii. $x_2(t) = \left[\frac{1}{5}x(t) + \frac{4}{5}j\hat{x}(t)\right] e^{j2\pi f_0 t}$

Solution: a) The Fourier spectrum of $x(t)$ is

$$X(f) = \mathscr{F}[x(t)] = \delta(f-30) + \delta(f+30).$$

The spectrum of $x(t)$ after Hilbert transform is

$$\hat{X}(f) = -j\,\mathrm{sgn}\,(f)X(f) = \delta(f-30)e^{-j\pi/2} + \delta(f+30)e^{j\pi/2}. \tag{50}$$

The Hilbert transform of $x(t)$ is

$$\begin{aligned}\hat{x}(t) =& \mathscr{F}^{-1}[\hat{X}(f)] = \mathscr{F}[\delta(f-30)e^{-j\pi/2} + \delta(f+30)e^{j\pi/2}]\\ =& e^{j60\pi t}e^{-j\pi/2} + e^{-j60\pi t}e^{j\pi/2}\\ =& 2\sin(60\pi t).\end{aligned} \tag{51}$$

$A\cos(2\pi f_0 t+\varphi) \xrightarrow{HT} A\sin(2\pi f_0 t+\varphi)$

$A\sin(2\pi f_0 t+\varphi) \xrightarrow{HT} -A\sin(2\pi f_0 t+\varphi)$

b) The analytic signal of $x(t)$ is

$$x_p(t) = x(t) + j\hat{x}(t) = 2\cos(60\pi t) + j2\sin(60\pi t) = 2e^{j60\pi t}. \tag{52}$$

c) The complex envelope of $x(t)$ is

$$\tilde{x}(t) = x_p(t)e^{-j2\pi 40t} = 2e^{-j20\pi t}. \tag{53}$$

d) The inphase component of $x(t)$ is

$$x_R(t) = \mathrm{Re}\,[\tilde{x}(t)] = 2\cos(20\pi t). \tag{54}$$

The quadrature component of $x(t)$ is

$$x_I(t) = \mathrm{Im}\,[\tilde{x}(t)] = 2\sin(20\pi t). \tag{55}$$

?

e) i. The spectrum of $x_1(t)$ is

$$\begin{aligned}X_1(f) =& \mathscr{F}[x_1(t)] = \mathscr{F}\left[\frac{2}{3}x(t) + \frac{1}{3}j\hat{x}(t)\right]\\ =& \frac{2}{3}X(f) + \frac{1}{3}j\hat{X}(f)\\ =& \frac{2}{3}[\delta(f-30)+\delta(f+30)] + \frac{1}{3}j\left[\delta(f-30)e^{-j\pi/2} + \delta(f+30)e^{j\pi/2}\right]\\ =& \delta(f-30) + \frac{1}{3}\delta(f+30).\end{aligned} \tag{56}$$

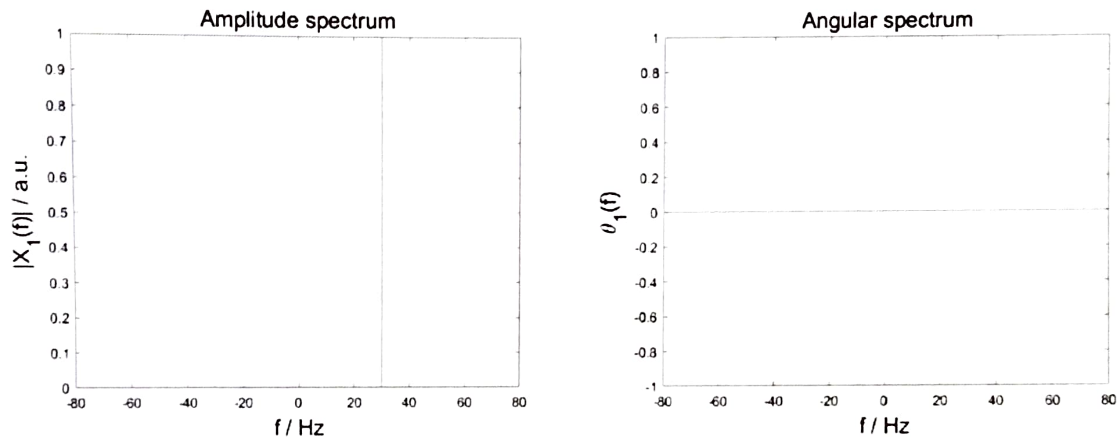
Its amplitude spectrum is

$$|X_1(f)| = \delta(f-30) + \frac{1}{3}\delta(f+30), \tag{57}$$

and angular spectrum is

$$\theta_1(f) = 0. \tag{58}$$

as shown in figure 3.



(a) $|X_1(f)|$ (b) $|\theta_1(f)|$

Figure 3: Spectrum of $x_1(t)$

ii. The Fourier transform of $\left[\frac{1}{5}x(t) + \frac{4}{5}j\hat{x}(t)\right]$ is

$$\begin{aligned}\mathscr{F}\left[\frac{1}{5}x(t) + \frac{4}{5}j\hat{x}(t)\right] =& \frac{1}{5}X(f) + \frac{4}{5}j\hat{X}(f)\\ =& \frac{1}{5}[\delta(f-30)+\delta(f+30)] + \frac{4}{5}j\left[\delta(f-30)e^{-j\pi/2} + \delta(f+30)e^{j\pi/2}\right]\\ =& \delta(f-30) - \frac{3}{5}\delta(f+30).\end{aligned} \tag{59}$$

Using the frequency shifting property, we get the spectrum of $x_2(t)$:

$$X_1(f) = \mathscr{F}[x_2(t)] = \mathscr{F}\left\{\left[\frac{1}{5}x(t) + \frac{4}{5}j\hat{x}(t)\right]e^{j2\pi f_0 t}\right\} = \delta(f-f_0-30) - \frac{3}{5}\delta(f-f_0+30). \tag{60}$$

Its amplitude spectrum is

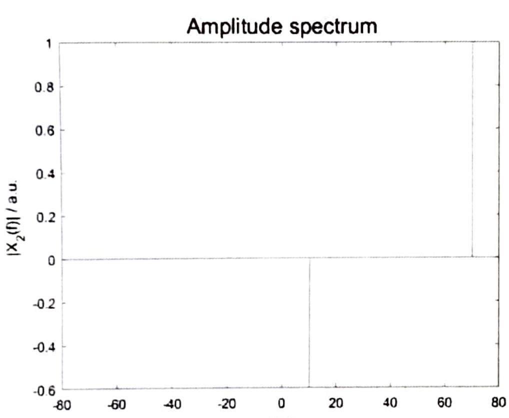
$$|X_1(f)| = \delta(f-f_0-30) - \frac{3}{5}\delta(f-f_0+30) = \delta(f-70) - \frac{3}{5}\delta(f-10), \tag{61}$$

and angular spectrum is

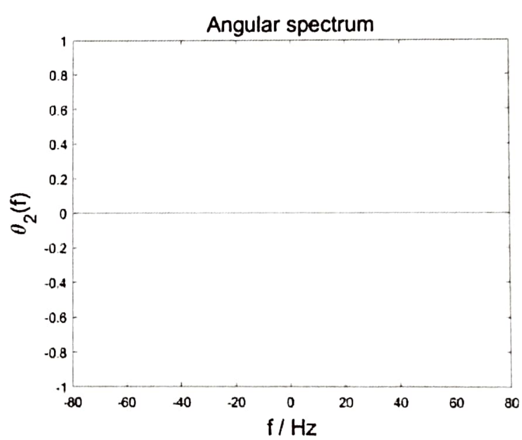
$$\theta_2(f) = 0, \tag{62}$$

as shown in figure 4.

□



(a) $|X_2(f)|$



(b) $|\theta_2(f)|$

Figure 4: Spectrum of $x_2(t)$