Intro to Communication System
EE140
Fall, 2020

Assignment 11

Due time : 10:15, Dec 18, 2020 (Friday)

Name : 陈 稼 霖
Student ID : 45875852
Grade : ________

Problem 1 (8.1, Binary minimum cost detection) Score: ________. (a) Consider a binary hypothesis testing problem with a priori probabilities p_0, p_1 and likelihoods $f_{V|U}(v|i)$, $i = 0, 1$. Let C_{ij} be the cost of deciding on hypothesis j when i is correct. Conditional on a observation $V = v$, find the expected cost (over $U = 0, 1$) of making the decision $\tilde{U} = j$ for $j = 0, 1$. Show that the decision of minimum expected cost is given by

$$\tilde{U}_{\text{mincost}} = \arg\min_j \left[C_{0j} p_{U|V}(0|v) + C_{1j} p_{U|V}(1|v)\right]$$

(b) Show that the min cost decision above can be expressed as the following threshold test:

$$\Lambda(v) = \frac{f_{V|U}(v|0)}{f_{V|U}(v,1)} \underset{\tilde{U}=1}{\overset{\tilde{U}=0}{\gtrless}} \frac{p_1(C_{10} - C_{11})}{p_0(C_{01} - C_{00})} = \eta.$$

(c) Interpret the result above as saying that the only difference between a MAP test and a minimum cost test is an adjustment of the threshold to take account of the cost. i.e., a large cost of an error of one type equivalent to having a large a priori probability for that hypothesis.

Solution: (a) The expected cost of making the decision $\tilde{U} = 0$ is

$$E[\text{cost of making decision } \tilde{U} = 0] = C_{00} p_{U|V}(0|v) + C_{10} p_{U|V}(1|v). \tag{1}$$

The expected cost of making the decision $\tilde{U} = 1$ is

$$E[\text{cost of making decision } \tilde{U} = 1] = C_{01} p_{U|V}(0|v) + C_{11} p_{U|V}(1|v) \tag{2}$$

In general, the expected cost of making decision $\tilde{U} = j$ is

$$\begin{aligned} E[\text{cost of making decision } \tilde{U} = j] &= \begin{cases} C_{00} p_{U|V}(0|v) + C_{10} p_{U|V}(1|v), & j = 0; \\ C_{01} p_{U|V}(0|v) + C_{11} p_{U|V}(1|v), & j = 1, \end{cases} \\ &= C_{0j} p_{U|V}(0|v) + C_{1j} p_{U|V}(1|v). \end{aligned} \tag{3}$$

To give minimum expected cost, the decision should be

$$\begin{aligned} \tilde{U}_{\text{mincost}} &= \arg\min_j \{E[\text{cost of making decision } \tilde{U} = j] \\ &= \arg\min_j [C_{0j} p_{U|V}(0|v) + C_{1j} p_{U|V}(1|v)]. \end{aligned} \tag{4}$$

(b) The min cost decision can be expressed as

$$C_{00} p_{U|V}(0|v) + C_{10} p_{U|V}(1|v) \underset{\tilde{U}=1}{\overset{\tilde{U}=0}{\gtrless}} C_{01} p_{U|V}(0|v) + C_{11} p_{U|V}(1|v), \tag{5}$$

$$\Longrightarrow (C_{00} - C_{01}) p_{U|V}(0|v) \underset{\tilde{U}=1}{\overset{\tilde{U}=0}{\gtrless}} (C_{11} - C_{10}) p_{U|V}(1|v). \tag{6}$$

where according to Bayes' theorem

$$p_{U|V}(0|v) = \frac{p_0 f_{V|U}(v|0)}{P_V(v)}, \tag{7}$$

$$p_{U|V}(1|v) = \frac{p_1 f_{V|U}(v|1)}{P_V(v)}. \tag{8}$$

Therefore, the min cost decision can be expressed as

$$(C_{00}-C_{01})\frac{p_0 f_{V|U}(v|0)}{P_V(v)} \underset{\tilde{U}=1}{\overset{\tilde{U}=0}{\gtrless}} (C_{11}-C_{10})\frac{p_1 f_{V|U}(v|1)}{P_V(v)}, \tag{9}$$

$$\Longrightarrow \Lambda(v)=\frac{f_{V|U}(v|0)}{f_{V|U}(v|1)} \underset{\tilde{U}=1}{\overset{\tilde{U}=0}{\gtrless}} \frac{p_1(C_{10}-C_{11})}{p_0(C_{01}-C_{00})}=\eta. \tag{10}$$

(c) The MAP rule is

$$\hat{U}(v)=\arg\max_j [p_{U|V}(j|v)], \tag{11}$$

$$\Longrightarrow p_{U|V}(0|v) \underset{\tilde{U}=1}{\overset{\tilde{U}=0}{\gtrless}} p_{U|V}(1|v), \tag{12}$$

$$\Longrightarrow \frac{p_0 f_{V|U}(v|0)}{P_V(v)} \underset{\tilde{U}=1}{\overset{\tilde{U}=0}{\gtrless}} \frac{p_1 f_{V|U}(v|1)}{P_V(v)}, \tag{13}$$

$$\Longrightarrow \Lambda(v)=\frac{f_{V|U}(v|0)}{f_{V|U}(v|1)} \underset{\tilde{U}=1}{\overset{\tilde{U}=0}{\gtrless}} \frac{p_1}{p_0}, \tag{14}$$

which can be regarded as a special case of the min cost decision rule that $C_{10}-C_{11}=C_{01}-C_{00}$. Therefore, the only difference between a MAP test and a minimum cost test is an adjustment of the threshold to take account of the cost, i.e., a large cost of an error of one type equivalent to having a large a priori probability for that hypothesis.

□

Problem 2 Score: ________. Consider the following two equiprobable hypothesis:

$$\begin{aligned} U=0 \quad &: \quad V_1=a\cos\Theta+Z_1, \quad V_2=a\sin\Theta+Z_2, \\ U=1 \quad &: \quad V_1=-a\cos\Theta+Z_1, \quad V_2=-a\sin\Theta+Z_2. \end{aligned}$$

Z_1 and Z_2 are iid $\mathcal{N}(0,\sigma^2)$, and Θ takes on the values $\{-\pi/4, 0, \pi/4\}$ each with probability 1/3.
Find the ML decision rule when V_1 and V_2 are observed.
Hint: Sketch the possible values of V_1 and V_2 for $\boldsymbol{Z}=0$ given each hypothesis. Then, without doing any calculations try to come up with a good intuitive decision rule. Then try to verify that it is optimal.

Solution: If $\boldsymbol{Z}=0$, then possible value of (V_1,V_2) are shown as the points in figure 1.

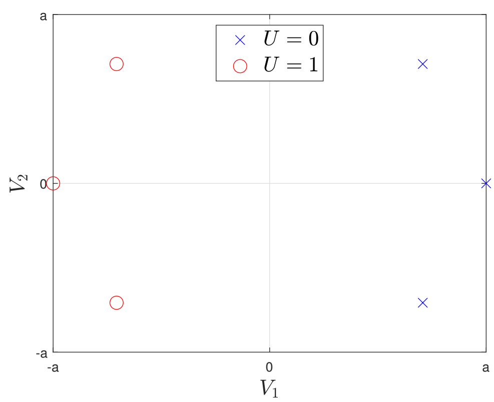

Figure 1: Possible value of (V_1,V_2) if $\boldsymbol{Z}=0$.

According to figure 1, guess that the ML decision rule is

$$V_1 \mathop{\gtrless}_{\hat{U}=1}^{\hat{U}=0} \cancel{1}0. \tag{15}$$

Now we prove that the above rule is optimal: The joint pdf of (V_1, V_2) conditional on $U=0$ is

$$\begin{aligned}
f_{(V_1,V_2)|U}((v_1,v_2)|U=0) =& \frac{1}{3}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{\left(v_1-\frac{a}{\sqrt{2}}\right)^2}{2\sigma^2}\right]\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{\left(v_2+\frac{a}{\sqrt{2}}\right)^2}{2\sigma^2}\right]\\
&+\frac{1}{3}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(v_1-a)^2}{2\sigma^2}\right]\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{v_2^2}{2\sigma^2}\right]\\
&+\frac{1}{3}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{\left(v_1-\frac{a}{\sqrt{2}}\right)^2}{2\sigma^2}\right]\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{\left(v_2-\frac{a}{\sqrt{2}}\right)^2}{2\sigma^2}\right]\\
=&\frac{1}{6\pi\sigma^2}\left\{\exp\left[-\frac{\left(v_1-\frac{a}{\sqrt{2}}\right)^2+\left(v_2+\frac{a}{\sqrt{2}}\right)^2}{2\sigma^2}\right]+\exp\left[-\frac{(v_1-a)^2+v_2^2}{2\sigma^2}\right]+\exp\left[-\frac{\left(v_1-\frac{a}{\sqrt{2}}\right)^2+\left(v_2-\frac{a}{\sqrt{2}}\right)^2}{2\sigma^2}\right]\right\}.
\end{aligned} \tag{16}$$

Similarly, the joint pdf of (V_1, V_2) conditional on $U=1$ is

$$f_{(V_1,V_2)|U}((v_1,v_2)|U=1) = \frac{1}{6\pi\sigma^2}\left\{\exp\left[-\frac{\left(v_1+\frac{a}{\sqrt{2}}\right)^2+\left(v_2+\frac{a}{\sqrt{2}}\right)^2}{2\sigma^2}\right]+\exp\left[-\frac{(v_1+a)^2+v_2^2}{2\sigma^2}\right]+\exp\left[-\frac{\left(v_1+\frac{a}{\sqrt{2}}\right)^2+\left(v_2-\frac{a}{\sqrt{2}}\right)^2}{2\sigma^2}\right]\right\}. \tag{17}$$

ML decision rule is

$$\frac{f_{(V_1,V_2)|U}((v_1,v_2)|0)}{f_{(V_1,V_2)|U}((v_1,v_2)|1)} \mathop{\gtrless}_{\hat{U}=1}^{\hat{U}=0} 1, \tag{18}$$

$$\Longrightarrow \frac{\exp\left[-\frac{\left(v_1-\frac{a}{\sqrt{2}}\right)^2+\left(v_2+\frac{a}{\sqrt{2}}\right)^2}{2\sigma^2}\right]+\exp\left[-\frac{(v_1-a)^2+v_2^2}{2\sigma^2}\right]+\exp\left[-\frac{\left(v_1-\frac{a}{\sqrt{2}}\right)^2+\left(v_2-\frac{a}{\sqrt{2}}\right)^2}{2\sigma^2}\right]}{\exp\left[-\frac{\left(v_1+\frac{a}{\sqrt{2}}\right)^2+\left(v_2+\frac{a}{\sqrt{2}}\right)^2}{2\sigma^2}\right]+\exp\left[-\frac{(v_1+a)^2+v_2^2}{2\sigma^2}\right]+\exp\left[-\frac{\left(v_1+\frac{a}{\sqrt{2}}\right)^2+\left(v_2-\frac{a}{\sqrt{2}}\right)^2}{2\sigma^2}\right]} \mathop{\gtrless}_{\hat{U}=1}^{\hat{U}=0} 1, \tag{19}$$

$$\Longrightarrow \frac{\exp\left[\frac{\sqrt{2}av_1-\sqrt{2}av_2}{2\sigma^2}\right]+\exp\left[\frac{2av_1}{2\sigma^2}\right]+\exp\left[\frac{\sqrt{2}av_1+\sqrt{2}av_2}{2\sigma^2}\right]}{\exp\left[\frac{-\sqrt{2}av_1-\sqrt{2}av_2}{2\sigma^2}\right]+\exp\left[\frac{-2av_1}{2\sigma^2}\right]+\exp\left[\frac{-\sqrt{2}av_1+\sqrt{2}av_2}{2\sigma^2}\right]} \mathop{\gtrless}_{\hat{U}=1}^{\hat{U}=0} 1, \tag{20}$$

$$\Longrightarrow \exp\left[\frac{2\sqrt{2}av_1}{2\sigma^2}\right]\frac{\exp\left[\frac{-\sqrt{2}av_2}{2\sigma^2}\right]+\exp\left[\frac{(2-\sqrt{2})av_1}{2\sigma^2}\right]+\exp\left[\frac{\sqrt{2}av_2}{2\sigma^2}\right]}{\exp\left[\frac{-\sqrt{2}av_2}{2\sigma^2}\right]+\exp\left[\frac{(-2+\sqrt{2})av_1}{2\sigma^2}\right]+\exp\left[\frac{\sqrt{2}av_2}{2\sigma^2}\right]} \mathop{\gtrless}_{\hat{U}=1}^{\hat{U}=0} 1, \tag{21}$$

$$\Rightarrow v_1 \mathop{\gtrless}_{\hat{U}=1}^{\hat{U}=0} \cancel{1}0. \tag{22}$$

$\square$