Homework 2 (Due on Sep 25)

1. (Conditional Probability) (10pts)

   Given a binary communication channel where A=input and B=output, let P(A)=0.3, P(B|A)=0.8, and $P(\bar{B}|\bar{A}) = 0.6$. Find P(A|B) and $P(\bar{A}|\bar{B})$.

2. (Property of PDF) (20pts)

   The joint pdf of random variables X and Y is $f(x,y) = Axye^{-2(x+y)}, x \geq 0, y \geq 0$. Find the following.

   1) Find the constant A.
   2) Find the marginal pdfs of X and Y, $f_X(x)$ and $f_Y(y)$.
   3) Are X and Y statistically independent? Justify your answer.
   4) Find the probability densify function of Z=X+Y.

3. (Gaussian random variables)(20pts)

   Assume two random variables X and Y are jointly Gaussian with mean $m_x = 1, m_y = 2,$ and variances $\sigma_x^2 = 1, \sigma_Y^2 = 4$.

   1) Write down the expressions for their marginal pdfs.
   2) Assume the correlation coefficient is $\rho_{XY} = 0.5$, a new random variable is defined as Z=X+Y, find the expression for the pdf of Z.
   3) Express the following probabilities using the Q function.
      a) $P(|X| \leq 5)$
      b) $P(-2 < Y \leq 12)$

4. (Random Process: probability) (20pts)

   A fair die is thrown. Depending on the number of spots on the up face, the following random processes are generated.

   1) Sketch several examples of sample functions of each case. (A>0 is a constant).

      a) $X(t,\zeta) = \begin{cases} 2A, 1 \ or \ 2 \ spots \ up \\ 0, \ 3 \ or \ 4 \ spots \ up \\ -2A, 5 \ or \ 6 \ spots \ up \end{cases}$

      b) $X(t,\zeta) = \begin{cases} 2A, 1 \ or \ 2 \ spots \ up \\ At, \ 3 \ or \ 4 \ spots \ up \\ -At, 5 \ or \ 6 \ spots \ up \end{cases}$

   2) What are the following probabilities for each case (case a and case b in 1)) ?
      i. $P(\mathrm{X}(4) \geq A)$ (X(4) means X(t=4, ζ), which is a random variable.)
      ii. $P(\mathrm{X}(2) \leq 0)$ (X(2) means X(t=2, ζ), which is a random variable.)

5. (Random Process: mean, variance, autocorrelation) (30pts)

   Let the sample function of a random process be given by $X(t) = Acos2\pi f_0 t,$ where f_0 is fixed and A has the pdf

   $$f_A(a) = \frac{1}{\sqrt{2\pi\sigma_a^2}} e^{-\frac{|a-\mu_a|^2}{2\sigma_a^2}}.$$

   The mean and variance of A are given by $\mu_a = 0, \ \sigma_a^2 = 4$.

   1) Find the mean and variance of random process X(t) at time t_0.
      (Hint: ensemble-mean; Note that $cos2\pi f_0 t_0$ is just a constant.)
   2) Find the autocorrelation function of X(t).

3) Is X(t) stationary?
4) Find the time average mean and autocorrelation function of X(t).

   (Hint1: the time average mean can be calculated as $< X(t) >= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)dt$, the time average autocorrelation function can be calculated as $< X(t)X(t+\tau) > = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$.

   Hint2: For periodic signal, $< X(t) >= \frac{1}{T}\int_{0}^{T} x(t)dt,\ and\ < X(t)X(t+\tau) >= \frac{1}{T}\int_{0}^{T} x(t)x(t+\tau)dt.$)
5) Is the X(t) ergodic?