Intro to Communication System EE140

## Assignment

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Problem 1 (8.9, otrhogonal signal sets; continuation of Exercise 8.8) Score: \_\_\_\_\_. Consider a set  $\mathcal{A} = \{a_m, 0 \leq m \leq M-1\}$  of M orthogonal vectors in  $\mathbb{R}^M$  with equal energy E.

(a) Use the union bound to show that Pr(e), using ML detection, is bounded by

$$\Pr(e) \le (M-1)Q(\sqrt{E/N_0}).$$

(b) Let  $M \to \infty$  with  $E_b = E/\log M$  held constant. Using the upperbound for Q(x) in Exercise 8.7(b), show that if  $E_b/N_0 > 2 \ln 2$ , then  $\lim_{M\to\infty} \Pr(e) = 0$ . How close is this to the ultimate Shannon limit on  $E_b/N_0$ ? What is the limit of the spectral efficiency  $\rho$ ?

**Solution:** (a) Suppose  $U = a_m$  is transmitted and V is received. Using ML detection, if  $||V - a_j|| \le ||V - a_k|| \, \forall k$ , we determine that  $\tilde{U} = a_j$ . If  $\exists j$  s.t.  $||V - a_j|| \le ||V - a_m||$ , then an error occurs. Use  $A_{jm}$  to denote the event  $||V - a_j|| < ||V - a_m||$ . The error probability is

$$\Pr(e|\boldsymbol{U}=\boldsymbol{a}_m) = \Pr(\cup_{j\neq m} A_{jm}|\boldsymbol{U}=\boldsymbol{a}_m) \le \sum_{j\neq m} \Pr(A_{jm}|\boldsymbol{U}=\boldsymbol{a}_m). \tag{1}$$

Suppose that V = U + Z, where Z is Gaussian noise with variance of  $N_0/2$  per dimension. Since  $\{a_m, 0 \le m \le M-1\}$  are orthogonal with equal energy E,

$$\Pr(A_{jm}|\boldsymbol{U}=\boldsymbol{a}_{m})=Q\left(\frac{\|\boldsymbol{a}_{j}-\boldsymbol{a}_{m}\|/2}{\sqrt{N_{0}/2}}\right)=Q\left(\sqrt{\frac{\|\boldsymbol{a}_{j}-\boldsymbol{a}_{m}\|^{2}}{2N_{0}}}\right)=Q\left(\sqrt{\frac{E}{N_{0}}}\right). \tag{2}$$

Therefore,

$$\Pr(e|\boldsymbol{U}=\boldsymbol{a}_m) \le (M-1)Q(\sqrt{E/N_0}). \tag{3}$$

(b) According to Exercise 8.7, for  $x \ge 0$ ,

$$Q(x) \le \frac{1}{2} \exp\left(-\frac{x^2}{2}\right),\tag{4}$$

so

$$0 < \Pr(e) \le \frac{M-1}{2} \exp\left(-\frac{E}{2N_0}\right) = \frac{M-1}{2} \exp\left(-\frac{E_b \log_2 M}{2N_0}\right) = \frac{M-1}{2} \left(\frac{1}{M}\right)^{\frac{E_b}{2N_0 \ln 2}}.$$
 (5)

If  $E_b/N_0 > 2 \ln 2$ , then  $\frac{E_b}{2N_0 \ln 2} > 1$ . Let  $M \to \infty$ ,

$$\lim_{M \to \infty} \Pr(e) = 0. \tag{6}$$

The limit of the spectral efficiency  $\rho$  is

$$\lim_{M \to \infty} \rho = \lim_{M \to \infty} \frac{2\log_2 M}{M} = 0. \tag{7}$$

**Problem 2 (8.11) Score:** \_\_\_\_\_\_. Section 8.3.4 discusses detection for binary complex vectors in WGN by viewing complex n-dimensional vectors as 2n-dimensional real vectors. Here you will treat the vectors directly as n-dimension complex vectors. Let  $\mathbf{Z} = (Z_1, \dots, Z_n)^T$  be a vector of complex idd Gaussian rvs with iid real and imaginary parts, each  $\mathcal{N}(0, N_0/2)$ . The input  $\mathbf{U}$  is binary antipodal, taking on values  $\mathbf{a}$  and  $-\mathbf{a}$ . The observation  $\mathbf{V}$  is  $\mathbf{U} + \mathbf{Z}$ .

(a) The probability density of Z is given by

$$f_{\mathbf{Z}}(z) = \frac{1}{(\pi N_0)^n} \exp \sum_{j=1}^n \frac{-|z_j|^2}{N_0} = \frac{1}{(\pi N_0)^n} \exp \frac{-\|z\|^2}{N_0}.$$

Explain what this probability density represents (i.e. probability per unit what?)

- (b) Give expression for  $f_{V|U}(v|a)$  and  $f_{V|U}(v|-a)$ .
- (c) Show that the log likelihood ratio for the observation v is given by

LLR(
$$v$$
) =  $\frac{-\|v - a\|^2 + \|v + a\|^2}{N_0}$ .

- (d) Explain why this implies that ML detection is minimum distance detection (defining the distance between two complex vectors as the norm of their difference).
- (e) Show that LLR(v) can also be written as  $4 \operatorname{Re} \left[ \langle v, a \rangle \right] / N_0$ .
- (f) The appearance of the real part, Re  $[\langle \boldsymbol{v}, \boldsymbol{a} \rangle]$ , in part (e) is surprising. Point out why log likelihood ratios must be real. Also explain why replacing Re  $[\langle \boldsymbol{v}, \boldsymbol{a} \rangle]$  by  $|\langle \boldsymbol{v}, \boldsymbol{a} \rangle|$  in the above expression would give a nonsensical result in the ML test.
- (g) Does the set of points  $\{v : LLR(v) = 0\}$  form a complex vector space?

**Solution:** (a) The probability density represents the probability per unit volume in 2n-dimensional real vector space.

(b)

$$f_{V|U}(v|a) = \frac{1}{(\pi N_0)^n} \exp \frac{-\|v - a\|^2}{N_0},$$
 (8)

$$f_{V|U}(v|-a) = \frac{1}{(\pi N_0)^n} \exp \frac{-\|v+a\|^2}{N_0}.$$
 (9)

(c) The log likelihood ratio for the observation v is given by

$$LLR(\mathbf{v}) = \ln \frac{f_{\mathbf{V}|\mathbf{U}}(\mathbf{v}|\mathbf{a})}{f_{\mathbf{V}|\mathbf{U}}(\mathbf{v}|-\mathbf{a})} = \ln \left[ \exp \frac{-\|\mathbf{v}-\mathbf{a}\|^2 + \|\mathbf{v}+\mathbf{a}\|^2}{N_0} \right] = \frac{-\|\mathbf{v}-\mathbf{a}\|^2 + \|\mathbf{v}+\mathbf{a}\|^2}{N_0}.$$
 (10)

(d) The ML detection rule is

$$LLR(v) = \frac{-\|v - a\|^2 + \|v + a\|^2}{N_0} \stackrel{\tilde{U} = a}{\underset{\tilde{U} = -a}{\gtrless}} 0.$$
(11)

Since  $\|\boldsymbol{v} - \boldsymbol{a}\|$  is the distance between  $\boldsymbol{v}$  and  $\boldsymbol{a}$  and  $\|\boldsymbol{v} + \boldsymbol{a}\|^2$  is the distance between  $\boldsymbol{v}$  and  $-\boldsymbol{a}$ , if the distance between  $\boldsymbol{v}$  and  $\boldsymbol{a}$  is less than that between  $\boldsymbol{v}$  and  $-\boldsymbol{a}$ , ML detection gives  $\tilde{U} = \boldsymbol{a}$ , otherwise, giving  $\tilde{U} = -\boldsymbol{a}$ . Therefore, this LLR( $\boldsymbol{v}$ ) implies that ML detection is minimum distance detection.

(e) LLR can be written as

$$LLR(\boldsymbol{v}) = \frac{2\langle \boldsymbol{v}, \boldsymbol{a} \rangle + \langle \boldsymbol{a}, \boldsymbol{v} \rangle}{N_0} = \frac{4 \operatorname{Re} \left[ \langle \boldsymbol{v}, \boldsymbol{a} \rangle \right]}{N_0}.$$
 (12)

- (f) The likelihood ratio is the ratio of two positive numbers, so its logarithm LLR(v) is also a real number. Consider the one-dimensional case with a = 1. The noise in the imaginary direction is irrelevant and only the real component of noise is relevant.
- (g) No. Consider the one-dimensional case with a = 1. The set  $\{v : LLR(v)\}$  is pure imaginary numbers, which is not closed under scalar multiplication by complex number.