

Problem 1 (Conditional probability, 10 pts) Score: _____. Given a binary communication channel where A =input and B =output, let $P(A) = 0.3$, $P(B|A) = 0.8$, and $P(\bar{B}|\bar{A}) = 0.6$. Find $P(A|B)$ and $P(\bar{A}|\bar{B})$.

Solution: Using the law of total probability, the probability of getting output B is

$$\begin{aligned} P(B) &= P(B|A)P(A) + P(B|\bar{A})P(\bar{A}) = P(B|A)P(A) + [1 - P(\bar{B}|\bar{A})][1 - P(A)] \\ &= 0.8 \times 0.3 + (1 - 0.6) \times (1 - 0.3) = 0.52. \end{aligned} \quad (1)$$

Using Bayes' theorem, we have

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{0.8 \times 0.3}{0.52} = \frac{6}{13}, \quad (2)$$

and

$$P(\bar{A}|\bar{B}) = \frac{P(\bar{B}|\bar{A})P(\bar{A})}{P(\bar{B})} = \frac{P(\bar{B}|\bar{A})[1 - P(A)]}{P(\bar{B})} = \frac{0.6 \times (1 - 0.3)}{1 - 0.52} = \frac{7}{8}. \quad (3)$$

□

Problem 2 (Property of PDF, 20pts) Score: _____. The joint pdf of random variables X and Y is $f(x, y) = Axye^{-2(x+y)}$, $x \geq 0$, $y \geq 0$. Find the following.

- 1) Find the constant A .
- 2) Find the marginal pdfs of X and Y , $f_X(x)$ and $f_Y(y)$.
- 3) Are X and Y statistically independent? Justify your answer.
- 4) Find the probability density function of $Z = X + Y$.

Solution: 1) As a property of pdf,

$$\begin{aligned} \int_0^{+\infty} \int_0^{+\infty} f(x, y) dx dy &= \int_0^{+\infty} \int_0^{+\infty} Axye^{-2(x+y)} dx dy = A \left[\int_0^{+\infty} xe^{-2x} dx \right] \left[\int_0^{+\infty} ye^{-2y} dy \right] \\ &= A \left[\int_0^{+\infty} xe^{-2x} dx \right]^2 = A \left[-\frac{1}{2} \int_0^{+\infty} x d(e^{-2x}) \right]^2 \\ &= A \left[-\frac{1}{2} (xe^{-2x}) \Big|_0^{+\infty} + \frac{1}{2} \int_0^{+\infty} e^{-2x} dx \right]^2 \\ &= A \left[\frac{1}{4} \right]^2 = \frac{A}{16} = 1, \end{aligned}$$

which gives us that

$$A = 16. \quad (4)$$

- 2) The marginal pdf of X is

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{+\infty} f(x, \beta) d\beta = \int_{-\infty}^{+\infty} 16x\beta e^{-2(x+\beta)} d\beta \\ &= 16xe^{-2x} \int_{-\infty}^{+\infty} \beta e^{-2\beta} d\beta \\ &= 4xe^{-2x}, \end{aligned} \quad (5)$$

and similarly, the marginal pdf of Y is

$$f_Y(y) = \int_{-\infty}^{+\infty} f(\alpha, y) d\alpha = \int_{-\infty}^{+\infty} 16\alpha y e^{-2(\alpha+y)} d\alpha = 4ye^{-2y}.$$

3) X and Y are statistically independent, since

$$f(x, y) = 16xye^{-2(x+y)} = (4xe^{-2x})(4ye^{-2y}) = f_X(x)f_Y(y), \quad \text{for all } x \geq 0, y \geq 0. \quad (6)$$

4) Since X and Y are independent, the pdf of $Z = X + Y$ is the convolution of pdfs of X and Y :

$$\begin{aligned} f_Z(z) &= \int_0^z f_X(z-u)f_Y(u) du \\ &= \int_0^z 4(z-u)e^{-2(z-u)} \cdot 4ue^{-2u} du \\ &= 16e^{-2z} \int_0^z (z-u)u du \\ &= \frac{8}{3}z^3e^{-2z}. \end{aligned} \quad (7)$$

□

Problem 3 (Gaussian random variables, 20pts) Score: _____. Assume two random variables X and Y are jointly Gaussian with mean $m_x = 1$, $m_y = 2$, and variances $\sigma_x^2 = 1$, $\sigma_y^2 = 4$.

- 1) Write down the expressions for their marginal pdfs.
- 2) Assume the correlation coefficient is $\rho_{XY} = 0.5$, a new random variables is defined as $Z = X + Y$, find the expression for the pdf of Z .
- 3) Express the following probabilities using the Q function.
 - a) $P(|X| \leq 5)$
 - b) $P(-2 < Y \leq 12)$

Solution: 1) The expression for marginal pdf of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left[-\frac{1}{2\sigma_x^2}(x - m_x)^2\right] = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(x - 1)^2\right], \quad (8)$$

and the expression for marginal pdf of Y is

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}} \exp\left[-\frac{1}{2\sigma_y^2}(y - m_y)^2\right] = \frac{1}{\sqrt{2\pi} \cdot 2} \exp\left[-\frac{1}{2 \times 4}(y - 2)^2\right]. \quad (9)$$

2) The jointly pdf of X and Y is

$$\begin{aligned} f_{XY}(x, y) &= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho_{XY}^2}} \exp\left\{-\frac{1}{2(1-\rho_{XY}^2)}\left[\frac{(x-m_x)^2}{\sigma_x^2} - \frac{2\rho(x-m_x)(y-m_y)}{\sigma_x\sigma_y} + \frac{(y-m_y)^2}{\sigma_y^2}\right]\right\} \\ &= \frac{1}{2\pi\sqrt{3}} \exp\left\{-\frac{1}{2 \times \frac{3}{4}}\left[(x-1)^2 - \frac{(x-1)(y-2)}{2} + \frac{(y-2)^2}{4}\right]\right\}. \end{aligned} \quad (10)$$

The expression of the pdf of Z is

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{+\infty} f_{XY}(u, z-u) du \\ &= \int_{-\infty}^{+\infty} \frac{1}{2\pi\sqrt{3}} \exp\left\{-\frac{1}{2 \times \frac{3}{4}}\left[(u-1)^2 - \frac{(u-1)(z-u-2)}{2} + \frac{(z-u-2)^2}{4}\right]\right\} du \\ &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi} \cdot 7} \frac{1}{\sqrt{2\pi \cdot \frac{3}{7}}} \exp\left\{-\frac{1}{2 \times \frac{3}{7}}\left[\left(u - \frac{2}{7}\left(z + \frac{1}{2}\right)\right)^2 + \frac{3}{49}(z-3)^2\right]\right\} \\ &= \frac{1}{\sqrt{2\pi}7} \exp\left\{-\frac{1}{14}(z-3)^2\right\}. \end{aligned} \quad (11)$$

3) a)

$$\begin{aligned}
 P(|X| \leq 5) &= P(-5 \leq X \leq 5) = 1 - P(X > 5) - P(X < -5) = 1 - P(X > 5) - P(X > 7) \\
 &= 1 - Q(5 - 1) - Q(7 - 1) = 1 - Q(4) - Q(6).
 \end{aligned} \tag{12}$$

b)

$$\begin{aligned}
 P(-2 < Y \leq 12) &= 1 - P(Y > 12) - P(Y < -2) = 1 - P(Y > 12) - P(Y > 6) \\
 &= 1 - Q\left(\frac{12-2}{2}\right) - P\left(\frac{6-2}{2}\right) = 1 - Q(5) - Q(2).
 \end{aligned} \tag{13}$$

□

Problem 4 (Random Process: probability, 20pts) Score: _____. A fair die is thrown. Depending on the number of spots on the up face, the following random process are generated.

1) Sketch several examples of sample functions of each case. ($A > 0$ is a constant).

$$\begin{aligned}
 \text{a) } X(t, \zeta) &= \begin{cases} 2A, & 1 \text{ or } 2 \text{ spots up} \\ 0, & 3 \text{ or } 4 \text{ spots up} \\ -2A, & 5 \text{ or } 6 \text{ spots up} \end{cases} \\
 \text{b) } X(t, \zeta) &= \begin{cases} 2A, & 1 \text{ or } 2 \text{ spots up} \\ At, & 3 \text{ or } 4 \text{ spots up} \\ -At, & 5 \text{ or } 6 \text{ spots up} \end{cases}
 \end{aligned}$$

2) What are the following probabilities for each case (case a and case b in 1))?

- i. $P(X(4) \geq A)$ ($X(4)$ means $X(t=4, \zeta)$, which is a random variable.)
- ii. $P(X(2) \leq 0)$ ($X(2)$ means $X(t=2, \zeta)$, which is a random variable.)

Solution: 1) a) As shown in figure 1(a).

b) As shown in figure 1(b).

2) i. For case a,

$$P(X(4) \geq A) = \frac{1}{3}. \tag{14}$$

For case b,

$$P(X(4) \geq A) = \frac{2}{3}. \tag{15}$$

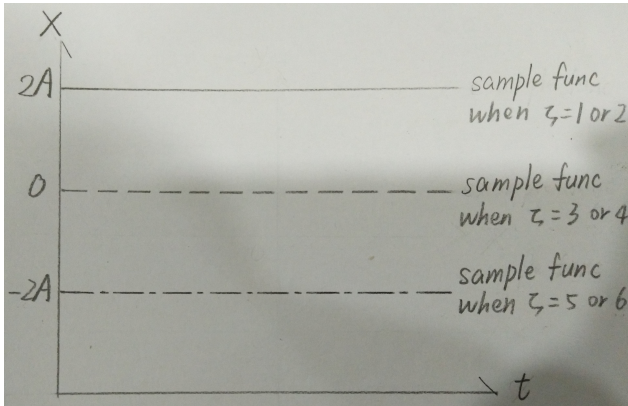
ii. For case a,

$$P(X(2) \leq 0) = \frac{2}{3}. \tag{16}$$

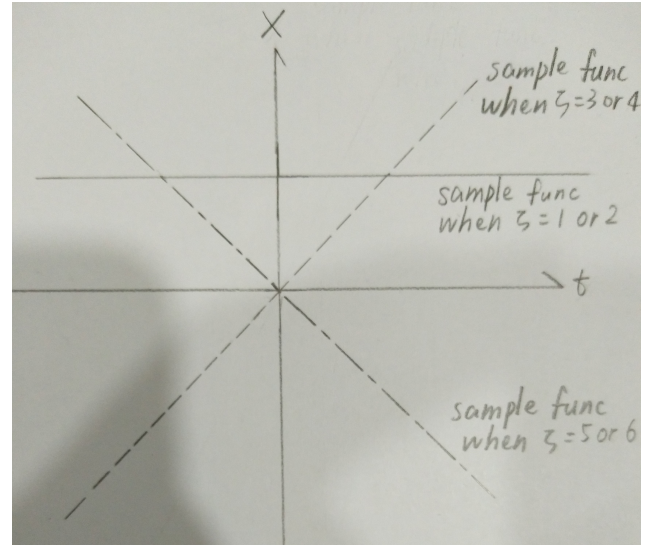
For case b,

$$P(X(2) \leq 0) = \frac{1}{3}. \tag{17}$$

□



(a) Sample functions of a)



(b) Sample functions of b)

Figure 1:

Problem 5 (Random Process: mean, variance, autocorrelation, 30 pts) Score: _____. Let the sample function of a random process be given by $X(t) = A \cos 2\pi f_0 t$, where f_0 is fixed and A has the pdf

$$f_A(a) = \frac{1}{\sqrt{2\pi\sigma_a^2}} e^{-\frac{|a-\mu_a|^2}{2\sigma_a^2}}.$$

The mean and variance of A are given by $\mu_a = 0$, $\sigma_a^2 = 4$.

- 1) Find the mean and variance of random process $X(t)$ at time t_0 .
(Hint: ensemble-mean; Note that $\cos 2\pi f_0 t_0$ is just a constant.)
- 2) Find the autocorrelation function of $X(t)$.
- 3) Is $X(t)$ stationary?
- 4) Find the time average mean and autocorrelation function of $X(t)$.
(Hint1: the time average mean can be calculated as $\langle X(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) dt$, the time average autocorrelation function can be calculated as $\langle X(t)X(t+\tau) \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t)x(t+\tau) dt$
Hint2: For periodic signal, $\langle X(t) \rangle = \frac{1}{T} \int_0^T x(t) dt$, and $\langle X(t)X(t+\tau) \rangle = \frac{1}{T} \int_0^T x(t)x(t+\tau) dt$)
- 5) Is the $X(t)$ ergodic?

Solution: 1) The mean of random process $X(t)$ at time t_0 is

$$E[X(t_0)] = E[X(t = t_0)] = \int_{-\infty}^{+\infty} a \cos(2\pi f_0 t_0) \frac{1}{\sqrt{2\pi\sigma_a^2}} e^{-\frac{|a-\mu_a|^2}{2\sigma_a^2}} da = 0. \quad (18)$$

The variance of random process $X(t)$ at time t_0 is

$$\sigma_X^2(t_0) = E \left[\left| X(t_0) - \overline{X(t_0)} \right|^2 \right] = E[X(t_0)^2] = E[A^2] \cos^2(2\pi f_0 t_0) = \sigma_a^2 \cos^2(2\pi f_0 t_0) = 4 \cos^2(2\pi f_0 t_0). \quad (19)$$

- 2) Let $t_1 = t$, $t_2 = t + \tau$. The autocorrelation function of $X(t)$ is

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = E[A \cos(2\pi f_0 t) A \cos(2\pi f_0(t + \tau))]$$

$$\begin{aligned}
&= E[A^2] \cos(2\pi f_0 t) \cos[2\pi f_0(t + \tau)] \\
&= \sigma_a^2 \cos(2\pi f_0 t) \cos[2\pi f_0(t + \tau)] \\
&= 4 \cos(2\pi f_0 t) \cos[2\pi f_0(t + \tau)].
\end{aligned} \tag{20}$$

3) **No**, since the second-order statistics of $X(t)$ does not depend only on the time gap τ , but also on the begin time t .

4) $X(t)$ is a periodic signal. Its time mean is

$$\langle X(t) \rangle = \frac{1}{T} \int_0^T x(t) dt = f_0 \int_0^{1/f_0} A \cos(2\pi f_0 t) dt = 0. \tag{21}$$

and its autocorrelation function is

$$\begin{aligned}
\langle X(t)X(t + \tau) \rangle &= \frac{1}{T} \int_0^T x(t)x(t + \tau) dt = f_0 \int_0^{1/f_0} A \cos(2\pi f_0 t) A \cos(2\pi f_0(t + \tau)) dt \\
&= A^2 f_0 \frac{1}{2} \int_0^{1/f_0} \cos[2\pi f_0(2t + \tau)] + \cos[2\pi f_0 \tau] dt \\
&= \frac{A^2}{2} \cos(2\pi f_0 \tau).
\end{aligned} \tag{22}$$

5) **No**, since $R_X(t, t + \tau) \neq \langle X(t)X(t + \tau) \rangle$.

□