Assignment 1

Due time : 10:15, Sept 18, 2020 (Friday)

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Grade : ________

Problem 1 (15 pts) Score: ________. Classify each of the following signals as an energy signal or as a power signal by calculating E (energy) or P (power). Note: the parameters involved are positive constants.

a) $x(t) = e^{-\alpha|t|}\cos\pi t$,

b) $x(t) = \Pi(t-3)\cos 3\pi t$, $\left(\Pi(t) = \begin{cases} 1, & |t| \leq 0.5 \\ 0, & \text{otherwise} \end{cases}\right)$

c) $x(t) = |t|$,

d) $x(t) = \sum_{n=-\infty}^{\infty} \Lambda(\frac{t}{2} - n)$, $\left(\Lambda(t) = \begin{cases} 1 - |t|, & |t| \\ 0, & \text{otherwise} \end{cases}\right)$

e) $x(t) = e^{j2\pi 3t}u(t)$, $\left(u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}\right)$

Solution: a) The energy of the signal:

$$\begin{aligned}
E = & \int_{-\infty}^{+\infty} |x(t)|^2\,dt = \int_{-\infty}^{+\infty} e^{-2\alpha|t|}\cos^2\pi t\,dt \\
= & 2\int_0^{+\infty} e^{-2\alpha t}\cos^2\pi t\,dt \\
= & \int_0^{+\infty} e^{-2\alpha t}(\cos 2\pi t + 1)\,dt \\
= & \int_0^{+\infty} e^{-2\alpha t}\,\mathrm{Re}\left[e^{i2\pi t}\right]dt - \frac{1}{2\alpha}e^{-2\alpha t}\Big|_0^{+\infty} \\
= & \mathrm{Re}\left[\int_0^{+\infty} e^{(-2\alpha + i2\pi)t}\,dt\right] + \frac{1}{2\alpha} \\
= & \mathrm{Re}\left[\frac{1}{2\alpha - i2\pi}\right] + \frac{1}{2\alpha} \\
= & \frac{\alpha}{2(\alpha^2 + \pi^2)} + \frac{1}{2\alpha} < +\infty.
\end{aligned} \tag{1}$$

Therefore, the signal is **an energy signal**.

b) The energy of the signal:

$$\begin{aligned}
E = & \int_{-\infty}^{+\infty} |x(t)|^2\,dt = \int_{-\infty}^{+\infty} |\Pi(t-3)\cos 3\pi t|^2\,dt \\
= & \int_{5/2}^{7/2} \cos^2 3\pi t\,dt \\
= & \frac{1}{2}\int_{5/2}^{7/2} (\cos 6\pi t + 1)\,dt \\
= & \frac{1}{2} < +\infty.
\end{aligned}$$

Therefore, the signal is **an energy signal**.

c) The average power of the signal is

$$P = \lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2\,dt = \lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}|t|^2\,dt = \lim_{T\to+\infty}\frac{2}{T}\int_0^{T/2}t^2\,dt = \lim_{T\to+\infty}\frac{T^2}{12} = +\infty. \tag{2}$$

Therefore, the signal is **neither an energy signal nor a power signal**.

d) The signal function can be written as

$$x(t) = 1. \tag{3}$$

The average power of the signal:

$$P = \lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt = \lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt = 1, \tag{4}$$

$$\Longrightarrow 0 < P < +\infty. \tag{5}$$

Therefore, the signal is **a power signal**.

e) The average power of the signal:

$$\begin{aligned} P =& \lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt = \lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| e^{j6\pi t} u(t) \right|^2 dt \\ =& \lim_{T\to+\infty} \frac{1}{T} \int_{0}^{T/2} \left| e^{j6\pi t} \right|^2 dt \\ =& \lim_{T\to+\infty} \frac{1}{T} \int_{0}^{T/2} dt \\ =& \frac{1}{2}. \end{aligned} \tag{6}$$

$$\Longrightarrow 0 < P < +\infty. \tag{7}$$

Therefore, the signal is **a power signal**.

□

Problem 2 (20 pts) Score: ________. Calculate **the Fourier transform and the energy** of the following signals.

a) $x_1(t) = 5\,\mathrm{sinc}\,(2t) e^{j2\pi 3t}$

b) $x_2(t) = \mathrm{sinc}^2(t-1)$

c) $x_a(t) = x_1(t) + x_2(-t)$

d) $x_b(t) = x_1(-t) + x_2(t)$

e) $x_c(t) = 2x_1(t)\cos 6\pi t + x_2(t) e^{j6\pi t}$

Solution: a) We first look for the Fourier transform of $\frac{1}{\pi t}$ (see reference at [1]). Consider such a function:

$$f_\alpha(t) = \begin{cases} e^{-2\pi\alpha t}, & t > 0 \\ 0, & t = 0 \\ -e^{2\pi\alpha t}, & t < 0 \end{cases} \tag{8}$$

where $\alpha > 0$. The Fourier transform of the above function is

$$\mathscr{F}[f_\alpha(t)] = \int_{-\infty}^{+\infty} f(t) e^{-j2\pi f\tau} \, d\tau = -\int_{-\infty}^{0} e^{2\pi(\alpha - jf)\tau} \, d\tau + \int_{0}^{+\infty} e^{-2\pi(\alpha+if)\tau} \, d\tau$$

[1] https://math.stackexchange.com/questions/1033870/does-the-fourier-transform-exist-for-ft-1-t

$$= -\frac{1}{2\pi(\alpha - jf)} + \frac{1}{2\pi(\alpha + if)} = -\frac{2jf}{2\pi(\alpha^2 + f^2)}. \tag{9}$$

Since the sign function is the limit of $f_\alpha(t)$ when $a \to 0$:

$$\operatorname{sgn}(t) = \begin{cases} 1, & t > 0, \\ 0, & t = 0, \\ -1, & t < 0, \end{cases} = \lim_{\alpha \to 0} f_\alpha(t), \tag{10}$$

by taking the limit $\alpha \to 0$, we get the Fourier transform of the sign function:

$$\mathscr{F}[\operatorname{sgn}(t)] = \lim_{\alpha \to 0} [f_\alpha(t)] = \frac{1}{j\pi f}. \tag{11}$$

Using the duality property, the Fourier transform of $\frac{1}{\pi t}$ is

$$\mathscr{F}[\frac{1}{\pi t}] = -j \operatorname{sgn}(f). \tag{12}$$

Then, using the multiplication property, the Fourier transform of the sinc function is

$$\begin{aligned} \mathscr{F}[\operatorname{sinc}(t)] =& \mathscr{F}\left[\frac{\sin(\pi t)}{\pi t}\right] = \mathscr{F}[\sin(\pi t)] * \mathscr{F}\left[\frac{1}{\pi t}\right] \\ =& \frac{1}{2j}\left[\delta\left(f - \frac{1}{2}\right) - \delta\left(f + \frac{1}{2}\right)\right] * [-j \operatorname{sgn}(f)] \\ =& -\frac{1}{2}\int_{-\infty}^{+\infty}\left[\delta\left(\nu - \frac{1}{2}\right) - \delta\left(\nu + \frac{1}{2}\right)\right] \operatorname{sgn}(f - \nu)\, d\nu \\ =& -\frac{1}{2}\left[\operatorname{sgn}\left(f - \frac{1}{2}\right) - \operatorname{sgn}\left(f + \frac{1}{2}\right)\right] \\ =& \Pi(f). \end{aligned} \tag{13}$$

Finally, using the scaling shifting property, we have

$$\mathscr{F}[\operatorname{sinc}(2t)] = \frac{1}{2}\Pi\left(\frac{f}{2}\right). \tag{14}$$

And using the frequency shifting property, we get the Fourier transform of $x_1(t)$:

$$\mathscr{F}[x_1(t)] = \mathscr{F}[5\operatorname{sinc}(2t)e^{j2\pi 3t}] = \frac{5}{2}\Pi\left(\frac{f-3}{2}\right). \tag{15}$$

The energy of $x_1(t)$ is

$$E = \int_{-\infty}^{+\infty} |\mathscr{F}[x_1(t)]|^2\, df = \int_{-\infty}^{+\infty}\left|\frac{5}{2}\Pi\left(\frac{f-3}{2}\right)\right|^2 df = \frac{25}{2}. \tag{16}$$

b) Using the time shifting property, we have

$$\mathscr{F}[\operatorname{sinc}(t-1)] = \Pi(f)e^{-j2\pi f}. \tag{17}$$

Using the multiplication property, we get the Fourier transform of $x_2(t)$:

$$\begin{aligned} \mathscr{F}[x_2(t)] =& \mathscr{F}[\operatorname{sinc}^2(t-1)] = \mathscr{F}[\operatorname{sinc}(t-1)] * \mathscr{F}[\operatorname{sinc}(t-1)] \\ =& \int_{-\infty}^{+\infty} \Pi(\nu)e^{-j2\pi\nu}\Pi(f-\nu)e^{-j2\pi(f-\nu)}\, d\nu \end{aligned}$$

$$
\begin{aligned}
&=\int_{-\infty}^{+\infty} \Pi(\nu)\Pi(f-\nu)e^{-j2\pi f}\,d\nu \\
&=\Lambda(f)e^{-j2\pi f}. \qquad (18)
\end{aligned}
$$

The energy of $x_2(t)$ is

$$
\begin{aligned}
E=&\int_{-\infty}^{+\infty} |\mathscr{F}[x_2(t)]|^2\,df=\int_{-\infty}^{+\infty} \left|\Lambda(f)e^{-j2\pi f}\right|^2 df \\
=&\int_{-\infty}^{+\infty} |\Lambda(f)|^2\,df \\
=&2\int_0^1 (1-f)^2\,df \\
=&\frac{2}{3}.
\end{aligned}
$$

c) Using the superposition and the scaling properties, the Fourier transform of $x_a(t)$ is

$$
\mathscr{F}[x_a(t)]=\mathscr{F}[x_1(t)+x_2(-t)]=\mathscr{F}[x_1(t)]+\mathscr{F}[x_2(-t)]=\frac{5}{2}\Pi\left(\frac{f-3}{2}\right)+\Lambda(f)e^{-j2\pi f}. \qquad (19)
$$

The energy of $x_a(t)$ is

$$
\begin{aligned}
E=&\int_{-\infty}^{+\infty} |\mathscr{F}[x_a(t)]|^2\,df=\int_{-\infty}^{+\infty} \left|\frac{5}{2}\Pi\left(\frac{f-3}{2}\right)+\Lambda(f)e^{-j2\pi f}\right|^2 df \\
=&\int_2^4 \left|\frac{5}{2}\Pi\left(\frac{f-3}{2}\right)\right|^2 df+\int_{-1}^1 |\Lambda(f)|^2\,df \\
=&\int_2^4 \frac{25}{4}\,df+2\int_0^1 (1-f)^2\,df \\
=&\frac{25}{2}+\frac{2}{3}=\frac{79}{6}. \qquad (20)
\end{aligned}
$$

d) Using the superposition and the scaling properties, the Fourier transform of $x_b(t)$ is

$$
\mathscr{F}[x_b(t)]=\mathscr{F}[x_1(-t)+x_2(t)]=\mathscr{F}[x_1(-t)]+\mathscr{F}[x_2(t)]=\frac{5}{2}\Pi\left(\frac{-f-3}{2}\right)+\Lambda(f)e^{-j2\pi f}. \qquad (21)
$$

~~Since the Fourier transform of $x_b(t)$ is the same as $x_a(t)$.~~ The energy of $x_b(t)$ is ~~also the same as that of $x_a(t)$~~:

$$
E=\frac{79}{6}. \qquad (22)
$$

e) The Fourier transform of the first term is

$$
\begin{aligned}
\mathscr{F}[2x_1(t)\cos 6\pi t]=&\mathscr{F}\left[x_1(t)(e^{j2\pi 3t}+e^{-j2\pi 3t})\right]=\mathscr{F}\left[x_1(t)e^{j2\pi 3t}\right]+\mathscr{F}\left[x_1(t)e^{-j2\pi 3t}\right] \\
=&\frac{5}{2}\left[\Pi\left(\frac{f}{2}-3\right)+\Pi\left(\frac{f}{2}\right)\right]. \qquad (23)
\end{aligned}
$$

The Fourier transform of the second term is

$$
\mathscr{F}\left[x_2(t)e^{j6\pi t}\right]=\mathscr{F}\left[x_2(t)e^{j2\pi 3t}\right]=\Lambda(f-3)e^{-j2\pi(f-3)}. \qquad (24)
$$

Therefore, the Fourier transform of $x_c(t)$ is

$$
\mathscr{F}[x_c(t)]=\mathscr{F}[2x_1(t)\cos 6\pi t+x_2(t)e^{j6\pi t}]=\mathscr{F}[2x_1(t)\cos 6\pi t]+\mathscr{F}\left[x_2(t)e^{j6\pi t}\right]
$$

$$=\frac{5}{2}\left[\Pi\left(\frac{f}{2}-3\right)+\Pi\left(\frac{f}{2}\right)\right]+\Lambda(f-3)e^{-j2\pi(f-3)}. \tag{25}$$

The energy of $x_c(t)$ is

$$\begin{aligned}E=&\int_{-\infty}^{+\infty}|\mathscr{F}[x_c(t)]|^2\,df=\int_{-\infty}^{+\infty}\left|\frac{5}{2}\left[\Pi\left(\frac{f}{2}-3\right)+\Pi\left(\frac{f}{2}\right)\right]+\Lambda(f-3)e^{-j2\pi(f-3)}\right|^2\,df\\=&\int_5^7\left|\frac{5}{2}\Pi\left(\frac{f}{2}-3\right)\right|^2\,df+\int_{-1}^1\left|\frac{5}{2}\Pi\left(\frac{f}{2}\right)\right|^2\,df+\int_2^4\left|\Lambda(f-3)e^{-j2\pi(f-3)}\right|^2\,df\\=&\int_5^7\frac{25}{4}\,df+\int_{-1}^1\frac{25}{4}\,df+2\int_3^4(4-f)^2\,df\\=&\frac{25}{2}+\frac{25}{2}+\frac{2}{3}=\frac{77}{3}.\end{aligned} \tag{26}$$

□

Problem 3 (10 pts) Score: _________. Calculate the convolution of the following signal.

a) $y(t)=e^{-|t|}*\Pi(t-2)$

b) $y(t)=\operatorname{sgn}(t)*\Lambda(t-2)$

Solution: a) The convolution can be written as

$$\begin{aligned}y(t)=&e^{-|t|}*\Pi(t-2)=\int_{-\infty}^{+\infty}e^{-|\tau|}\Pi(t-\tau-2)\,d\tau\\=&\int_{t-\frac{5}{2}}^{t-\frac{3}{2}}e^{-|\tau|}\,d\tau.\end{aligned} \tag{27}$$

If $t\geq\frac{5}{2}$,

$$y(t)=\int_{t-\frac{5}{2}}^{t-\frac{3}{2}}e^{-\tau}\,d\tau=e^{\frac{5}{2}-t}-e^{\frac{3}{2}-t}. \tag{28}$$

If $\frac{3}{2}\leq t<\frac{5}{2}$,

$$y(t)=\int_{t-\frac{5}{2}}^{0}e^{\tau}\,d\tau+\int_0^{t-\frac{3}{2}}e^{-\tau}\,d\tau=2-e^{t-\frac{5}{2}}-e^{t-\frac{3}{2}}. \tag{29}$$

If $t<\frac{3}{2}$,

$$y(t)=\int_{t-\frac{5}{2}}^{t-\frac{3}{2}}e^{\tau}\,d\tau=e^{t-\frac{3}{2}}-e^{t-\frac{5}{2}}. \tag{30}$$

Therefore, in general,

$$y(t)=\begin{cases}e^{\frac{5}{2}-t}-e^{\frac{3}{2}-t}, & t\geq\frac{5}{2},\\ 2-e^{t-\frac{5}{2}}-e^{t-\frac{3}{2}}, & \frac{3}{2}\leq t<\frac{3}{2},\\ e^{t-\frac{3}{2}}-e^{t-\frac{5}{2}}, & t<\frac{3}{2}.\end{cases} \tag{31}$$

b) The convolution can be written as

$$y(t)=\operatorname{sgn}(t)*\Lambda(t-2)=\int_{-\infty}^{+\infty}\operatorname{sgn}(\tau)\Lambda(t-\tau-2)\,d\tau$$

$$=\underline{\int_{t-\frac{5}{2}}^{t-\frac{3}{2}} \operatorname{sgn}(\tau)\, d\tau.} \tag{32}$$

$$=-\int_{-\infty}^{0} \Lambda(t-\tau-2)\, d\tau + \int_{0}^{+\infty} \Lambda(t-\tau-2)\, d\tau. \tag{33}$$

If $t \geq 3$,

$$\begin{aligned} y(t) =& \int_{0}^{+\infty} \Lambda(t-\tau-2)\, d\tau \\ =& \int_{t-3}^{t-2} (t-\tau-1)\, d\tau + \int_{t-2}^{t-1} (3-t+\tau)\, d\tau \\ =& 1. \end{aligned} \tag{34}$$

If $2 \leq t < 3$,

$$\begin{aligned} y(t) =& -\int_{-\infty}^{0} \Lambda(t-\tau-2)\, d\tau + \int_{0}^{+\infty} \Lambda(t-\tau-2)\, d\tau \\ =& -\int_{t-3}^{0} (t-\tau-1)\, d\tau + \int_{0}^{t-2} (t-\tau-1)\, d\tau + \int_{t-2}^{t-1} (3-t-\tau)\, d\tau \\ =& -t^2+6t-8. \end{aligned} \tag{35}$$

If $1 \leq t < 2$,

$$\begin{aligned} y(t) =& -\int_{t-3}^{0} \Lambda(t-\tau-2)\, d\tau + \int_{0}^{+\infty} \Lambda(t-\tau-2)\, d\tau \\ =& -\int_{t-3}^{t-2} (t-\tau-1) - \int_{t-2}^{0} (3-t-\tau)\, d\tau + \int_{0}^{t-1} (3-t-\tau)\, d\tau \\ =& t^2-2t. \end{aligned} \tag{36}$$

If $t < 1$,

$$\begin{aligned} y(t) =& -\int_{-\infty}^{0} \Lambda(t-\tau-2)\, d\tau \\ =& -\int_{t-3}^{t-2} (t-\tau-1)\, d\tau - \int_{t-2}^{t-1} (3-t-\tau)\, d\tau \\ =& -1. \end{aligned} \tag{37}$$

Therefore, in general,

$$y(t) = \begin{cases} 1, & t \geq 3, \\ -t^2+6t-8, & 2 \leq t < 3, \\ t^2-2t, & 1 \leq t < 2, \\ -1, & t < 3. \end{cases} \tag{38}$$

□

Problem 4 (20 pts) Score: __________. Calculate the Fourier transform of the following periodic signal.

a) $\sum_{n=-\infty}^{\infty} \Lambda(\frac{t}{2}-2n)$

b) $\left[\sum_{n=-\infty}^{\infty} \delta(t-2n)\right] * \left[\Pi(\frac{t}{2}) \cos(2\pi t)\right]$

Solution: a) The Fourier series of the signal is

$$\sum_{n=-\infty}^{\infty}\Lambda\left(\frac{t}{2}-2n\right)=\sum_{n=-\infty}^{\infty}c_n e^{jn2\pi\frac{t}{4}}, \tag{39}$$

where

$$\begin{aligned}
c_n=&\frac{1}{4}\int_{-2}^{2}\sum_{n=-\infty}^{\infty}\Lambda\left(\frac{\tau}{2}-2n\right)e^{-jn2\pi\frac{\tau}{4}}\,d\tau\\
=&\frac{1}{2}\int_0^2\left(1-\frac{\tau}{2}\right)\cos\left(n2\pi\frac{\tau}{4}\right)\,d\tau\\
=&\frac{\sin(n\pi)}{n\pi}-\frac{1}{4}\frac{1}{n2\pi\frac{1}{4}}\int_0^2\tau\,d\left[\sin\left(n2\pi\frac{\tau}{4}\right)\right]\\
=&\frac{\sin(n\pi)}{n\pi}-\frac{\sin(n\pi)}{n\pi}+\frac{1}{4}\frac{1}{n2\pi\frac{1}{4}}\int_0^2\sin\left(n2\pi\frac{\tau}{4}\right)\,d\tau\\
=&\frac{1-\cos(n\pi)}{(n\pi)^2}.
\end{aligned} \tag{40}$$

Note that $c_n=2$ when $n=0$. The fourier transform of the signal is

$$\begin{aligned}
\mathscr{F}\left[\sum_{n=-\infty}^{\infty}\Lambda\left(\frac{t}{2}-2n\right)\right]=&\mathscr{F}\left[\sum_{n=-\infty}^{\infty}\frac{1-\cos(n\pi)}{(n\pi)^2}e^{jn2\pi\frac{t}{4}}\right]\\
=&\sum_{n=-\infty}^{\infty}\frac{1-\cos(n\pi)}{(n\pi)^2}\mathscr{F}\left[e^{-j2\pi(f-\frac{n}{4})t}\right]\\
=&\sum_{n=-\infty}^{\infty}\frac{1-\cos(n\pi)}{(n\pi)^2}\delta(f-\frac{n}{4}).
\end{aligned} \tag{41}$$

b) The Fourier series of $\sum_{n=-\infty}^{\infty}\delta(t-2n)$ is

$$\sum_{n=-\infty}^{\infty}\delta(t-2n)=\sum_{n=-\infty}^{\infty}c_n e^{jn2\pi\frac{t}{2}}, \tag{42}$$

where

$$c_n=\frac{1}{2}\int_{-1}^{1}\sum_{n=-\infty}^{\infty}\delta(t-2n)e^{-jn2\pi\frac{\tau}{2}}\,d\tau=\frac{1}{2}. \tag{43}$$

The Fourier transform of $\sum_{n=-\infty}^{\infty}\delta(t-2n)$ is

$$\mathscr{F}\left[\sum_{n=-\infty}^{\infty}\delta(t-2n)\right]=\mathscr{F}\left[\sum_{n=-\infty}^{\infty}\frac{1}{2}e^{jn2\pi\frac{t}{2}}\right]=\sum_{n=-\infty}^{\infty}\frac{1}{2}\mathscr{F}\left[e^{jn2\pi\frac{t}{2}}\right]=\frac{1}{2}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{2}\right). \tag{44}$$

The Fourier transform of $\Pi\left(\frac{t}{2}\right)\cos(2\pi t)$ is

$$\begin{aligned}
&\mathscr{F}\left[\Pi\left(\frac{t}{2}\right)\cos(2\pi t)\right]=\int_{-\infty}^{+\infty}\Pi\left(\frac{\tau}{2}\right)\cos(2\pi\tau)e^{-j2\pi f\tau}\,d\tau\\
=&2\int_0^1\cos(2\pi\tau)\cos(2\pi f\tau)\,d\tau\\
=&\int_0^1\{\cos[2\pi(f+1)\tau]+\cos[2\pi(f-1)\tau]\}\,d\tau\\
=&\frac{\sin[2\pi(f+1)]}{2\pi(f+1)}+\frac{\sin[2\pi(f-1)]}{2\pi(f-1)}.
\end{aligned}$$

Using the time convolution property, we get the Fourier transform of the signal

$$\begin{aligned}\mathscr{F}\left\{\left[\sum_{n=-\infty}^{\infty}\delta(t-2n)\right]*\left[\Pi\left(\frac{t}{2}\right)\cos(2\pi t)\right]\right\}=&\mathscr{F}\left[\sum_{n=-\infty}^{\infty}\delta(t-2n)\right]\mathscr{F}\left[\Pi\left(\frac{t}{2}\right)\cos(2\pi t)\right]\\=&\frac{1}{2}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{2}\right)\left\{\frac{\sin[2\pi(f+1)]}{2\pi(f+1)}+\frac{\sin[2\pi(f-1)]}{2\pi(f-1)}\right\}.\end{aligned}\tag{45}$$

□

Problem 5 (15 pts) Score: _________. Determine the range of permissible cutoff frequencies for the ideal lowpass filter used to reconstruct the following signal

$$x(t)=4\cos^2(200\pi t)\cos(1000\pi t)$$

Which is sampled at 2000 samples per second. Sketch $X(f)$ and $X_\delta(f)$ (spectrum after the sampling). Find the minimum allowable sampling frequency.

Solution: The signal can be written as

$$\begin{aligned}x(t)=&4\cos^2(200\pi t)\cos(1000\pi t)=2[1+\cos(400\pi t)]\cos(1000\pi t)\\=&2\cos(1000\pi t)+\cos(1400\pi t)+\cos(600\pi t).\end{aligned}\tag{46}$$

The spectrum of the signal is

$$\begin{aligned}X(f)=&\mathscr{F}[x(t)]=\mathscr{F}[2\cos(1000\pi t)+\cos(1400\pi t)+\cos(600\pi t)]\\=&\delta(f-500)+\delta(f+500)+\frac{1}{2}\delta(f-700)+\frac{1}{2}\delta(f+700)+\frac{1}{2}\delta(f-300)+\frac{1}{2}\delta(f+300),\end{aligned}\tag{47}$$

as shown in figure 1(a). Sampling at 2000 samples per second means that the sampling frequency is $f_s=2000$ Hz. The spectrum after the sampling is

$$\begin{aligned}X_\delta(f)=&2000\sum_{n=-\infty}^{\infty}X(f-2000n)\\=&2000\sum_{n=-\infty}^{\infty}\left[\delta(f-2000n-500)+\delta(f-2000n+500)+\frac{1}{2}\delta(f-2000n-700)+\frac{1}{2}\delta(f-2000n+700)\right.\\&\left.+\frac{1}{2}\delta(f-2000n-300)+\frac{1}{2}\delta(f-2000n+300)\right],\end{aligned}\tag{48}$$

as shown in figure 1(b). According to the sampling theorem, the range of permissible cutoff frequencies for the ideal lowpass filter to reconstruct the signal is $700\text{Hz}<f_c<1300\text{Hz}$ and the minimum allowable sampling frequency is 1400Hz.

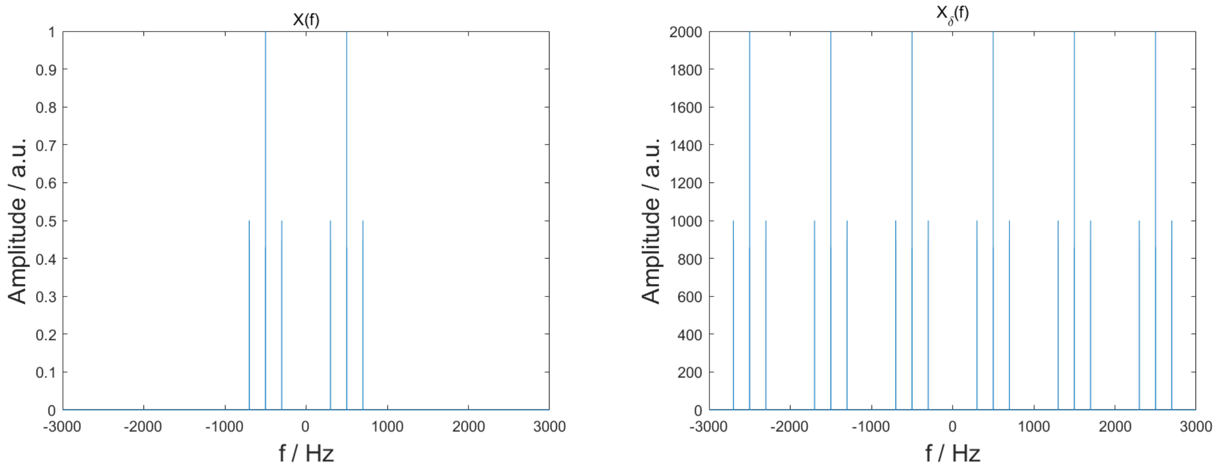
□

Problem 6 Score: _________. 1) Express the spectrum $Y(f)$ of

$$y(t)=x(t)\cos(400\pi t)+\hat{x}(t)\sin(400\pi t)$$

using the spectrum $X(f)$ of $x(t)$, where $\hat{x}(t)$ is the Hilbert transform of $x(t)$. (5 pts)

2) if $x(t)=\text{sinc}\,(t)$, sketch $Y(f)$. (5 pts)



(a) Original spetrum of the signal: $X(f)$

(b) Spectrum after the sampling: $X_\delta(f)$

Figure 1:

Solution: 1) The spectrum of $y(t)$ is

$$\begin{aligned}Y(f) =&\mathscr{F}[y(t)] = \mathscr{F}[x(t)\cos(400\pi t) + \hat{x}(t)\sin(400\pi t)]\\=&\mathscr{F}\left[x(t)\frac{e^{j400\pi t}+e^{-j400\pi t}}{2}\right] + \mathscr{F}\left\{\mathscr{F}^{-1}[-j\,\mathrm{sgn}\,(f)X(f)]\frac{e^{j400\pi t}-e^{-j400\pi t}}{2j}\right\}\\=&\frac{1}{2}X(f-200)+\frac{1}{2}X(f+200)-\frac{1}{2}\,\mathrm{sgn}\,(f-200)X(f-200)+\frac{1}{2}\,\mathrm{sgn}\,(f+200)X(f+200)\end{aligned} \tag{49}$$

2) If $x(t) = \mathrm{sinc}\,(t)$, then

$$X(f) = \Pi(f) \tag{50}$$

and

$$Y(f) = \frac{1}{2}\Pi(f-200)+\frac{1}{2}\Pi(f+200)-\frac{1}{2}\,\mathrm{sgn}\,(f-200)\Pi(f-200)+\frac{1}{2}\,\mathrm{sgn}\,(f+200)\Pi(f+200), \tag{51}$$

as shown in figure 2.

□

Problem 7 Score: ________. Consider $x(t) = 2\cos(60\pi t)$, the reference frequency $f_0 = 40$ Hz. Calculate the following signals.

a) The Hilbert transform of $x(t)$, i.e., $\hat{x}(t)$.

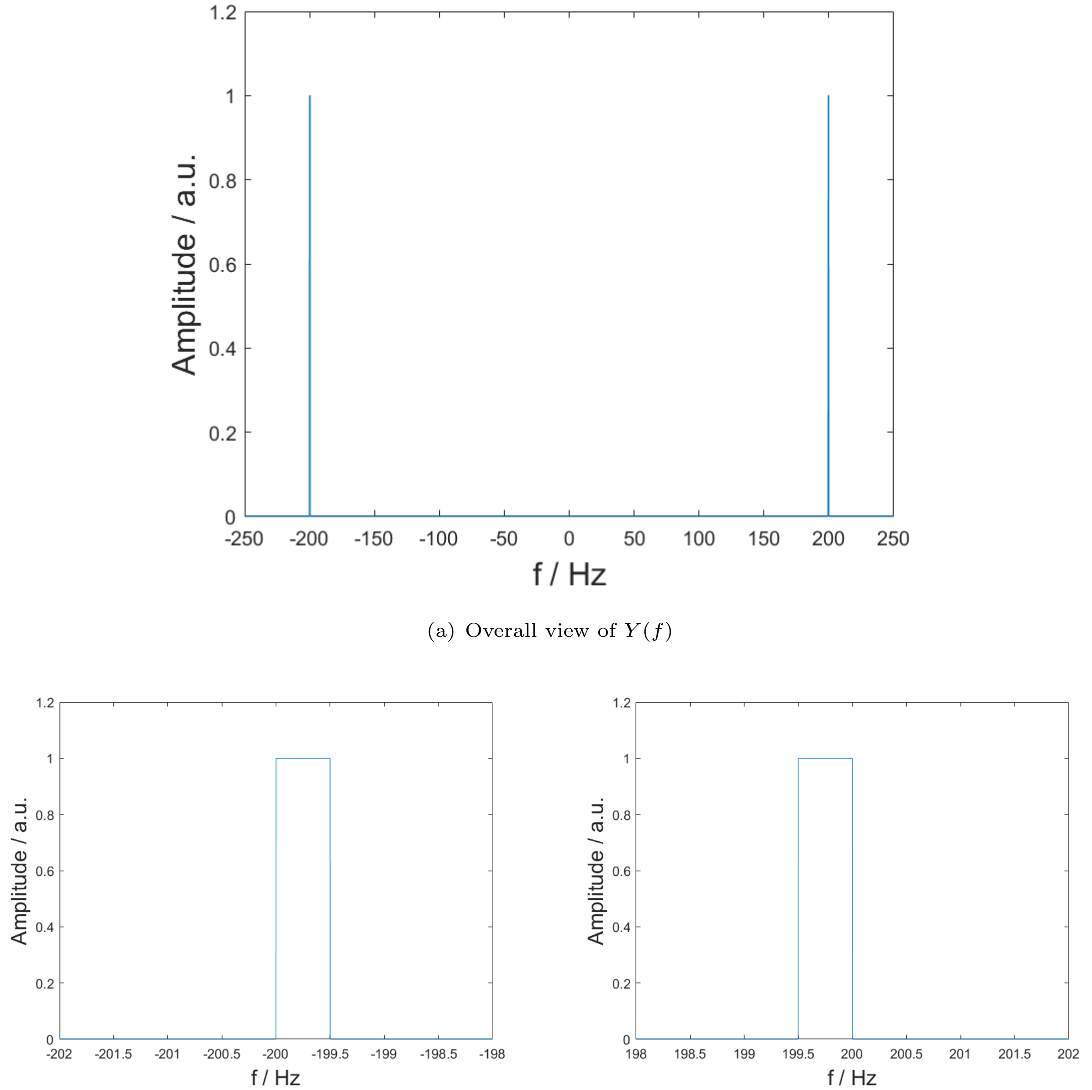
b) The analytic signal $x_p(t)$.

c) The complex envelope of $x(t)$, i.e., $\tilde{x}(t)$.

d) The inphase and quadrature component of $x(t)$, i.e., $x_R(t)$ and $x_I(t)$.
（Please refer to Lecture 2, Slide 36 or Page 88 of reference textbook, we will learn this in the next class. I am sorry for the lagging.）

e) Determine and plot the spectrum of the following signals:

i. $x_1(t) = \frac{2}{3}x(t) + \frac{1}{3}j\hat{x}(t)$



(a) Overall view of $Y(f)$

(b) $Y(f)$ near -200

(c) $Y(f)$ near $+200$

Figure 2: Spectrum of $y(t)$: $Y(f)$

ii. $x_2(t) = \left[\frac{1}{5}x(t) + \frac{4}{5}j\hat{x}(t)\right] e^{j2\pi f_0 t}$

Solution: a) The Fourier spectrum of $x(t)$ is

$$X(f) = \mathscr{F}[x(t)] = \delta(f-30) + \delta(f+30).$$

The spectrum of $x(t)$ after Hilbert transform is

$$\hat{X}(f) = -j\,\mathrm{sgn}\,(f)X(f) = \delta(f-30)e^{-j\pi/2} + \delta(f+30)e^{j\pi/2}. \tag{52}$$

The Hilbert transform of $x(t)$ is

$$\begin{aligned}\hat{x}(t) =& \mathscr{F}^{-1}[\hat{X}(f)] = \mathscr{F}[\delta(f-30)e^{-j\pi/2} + \delta(f+30)e^{j\pi/2}]\\ =& e^{j60\pi t}e^{-j\pi/2} + e^{-j60\pi t}e^{j\pi/2}\\ =& 2\sin(60\pi t).\end{aligned} \tag{53}$$

b) The analytic signal of $x(t)$ is

$$x_p(t) = x(t) + j\hat{x}(t) = 2\cos(60\pi t) + j2\sin(60\pi t) = 2e^{j60\pi t}. \tag{54}$$

c) The complex envelope of $x(t)$ is

$$\tilde{x}(t) = x_p(t)e^{-j2\pi 40t} = 2e^{-j20\pi t}. \tag{55}$$

d) The inphase component of $x(t)$ is

$$x_R(t) = \operatorname{Re}[\tilde{x}(t)] = 2\cos(20\pi t). \tag{56}$$

The quadrature component of $x(t)$ is

$$x_I(t) = \operatorname{Im}[\tilde{x}(t)] = -2\sin(20\pi t). \tag{57}$$

e) i. The spectrum of $x_1(t)$ is

$$\begin{aligned}
X_1(f) =&\mathscr{F}[x_1(t)] = \mathscr{F}\left[\frac{2}{3}x(t) + \frac{1}{3}j\hat{x}(t)\right]\\
=&\frac{2}{3}X(f) + \frac{1}{3}j\hat{X}(f)\\
=&\frac{2}{3}[\delta(f-30)+\delta(f+30)] + \frac{1}{3}j\left[\delta(f-30)e^{-j\pi/2} + \delta(f+30)e^{j\pi/2}\right]\\
=&\delta(f-30) + \frac{1}{3}\delta(f+30).
\end{aligned} \tag{58}$$

Its amplitude spectrum is

$$|X_1(f)| = \delta(f-30) + \frac{1}{3}\delta(f+30), \tag{59}$$

and angular spectrum is

$$\theta_1(f) = 0. \tag{60}$$

as shown in figure 3.

ii. The Fourier transform of $\left[\frac{1}{5}x(t) + \frac{4}{5}j\hat{x}(t)\right]$ is

$$\begin{aligned}
\mathscr{F}\left[\frac{1}{5}x(t) + \frac{4}{5}j\hat{x}(t)\right] =&\frac{1}{5}X(f) + \frac{4}{5}j\hat{X}(f)\\
=&\frac{1}{5}[\delta(f-30)+\delta(f+30)] + \frac{4}{5}j\left[\delta(f-30)e^{-j\pi/2} + \delta(f+30)e^{j\pi/2}\right]\\
=&\delta(f-30) - \frac{3}{5}\delta(f+30).
\end{aligned} \tag{61}$$

Using the frequency shifting property, we get the spectrum of $x_2(t)$:

$$X_1(f) = \mathscr{F}[x_2(t)] = \mathscr{F}\left\{\left[\frac{1}{5}x(t) + \frac{4}{5}j\hat{x}(t)\right]e^{j2\pi f_0 t}\right\} = \delta(f-f_0-30) - \frac{3}{5}\delta(f-f_0+30). \tag{62}$$

Its amplitude spectrum is

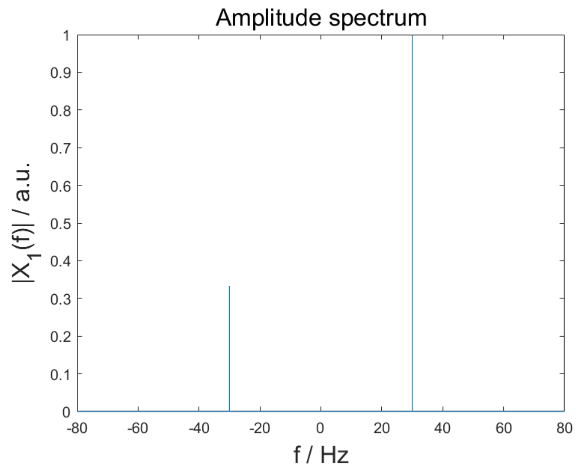
$$|X_1(f)| = \delta(f-f_0-30) - \frac{3}{5}\delta(f-f_0+30) = \delta(f-70) - \frac{3}{5}\delta(f-10), \tag{63}$$

and angular spectrum is

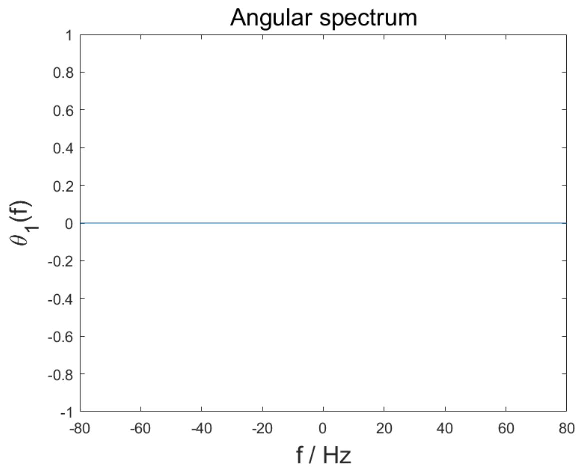
$$\theta_2(f) = 0, \tag{64}$$

as shown in figure 4.

□

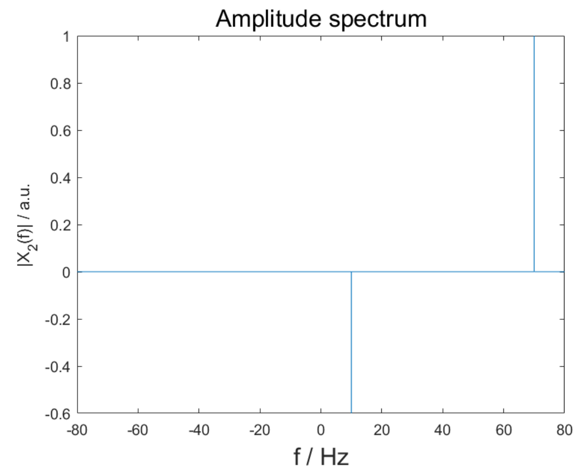


(a) $|X_1(f)|$

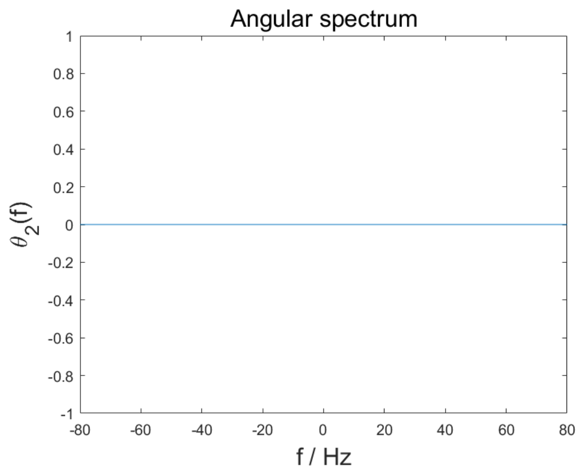


(b) $|\theta_1(f)|$

Figure 3: Spectrum of $x_1(t)$



(a) $|X_2(f)|$



(b) $|\theta_2(f)|$

Figure 4: Spectrum of $x_2(t)$