

**Problem 1 (15 pts) Score:** \_\_\_\_\_. Classify each of the following signals as an energy signal or as a power signal by calculating E (energy) or P (power). Note: the parameters involved are positive constants.

a)  $x(t) = e^{-\alpha|t|} \cos \pi t$ ,

b)  $x(t) = \Pi(t-3) \cos 3\pi t$ ,  $\left( \Pi(t) = \begin{cases} 1, & |t| \leq 0.5 \\ 0, & \text{otherwise} \end{cases} \right)$

c)  $x(t) = |t|$ ,

d)  $x(t) = \sum_{n=-\infty}^{\infty} \Lambda(\frac{t}{2} - n)$ ,  $\left( \Lambda(t) = \begin{cases} 1 - |t|, & |t| \leq 1 \\ 0, & \text{otherwise} \end{cases} \right)$

e)  $x(t) = e^{j2\pi 3t} u(t)$ ,  $\left( u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \text{otherwise} \end{cases} \right)$

**Solution:** a) The energy of the signal:

$$\begin{aligned} E &= \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} e^{-2\alpha|t|} \cos^2 \pi t dt \\ &= 2 \int_0^{+\infty} e^{-2\alpha t} \cos^2 \pi t dt \\ &= \int_0^{+\infty} e^{-2\alpha t} (\cos 2\pi t + 1) dt \\ &= \int_0^{+\infty} e^{-2\alpha t} \operatorname{Re}[e^{i2\pi t}] dt - \frac{1}{2\alpha} e^{-2\alpha t} \Big|_0^{+\infty} \\ &= \operatorname{Re} \left[ \int_0^{+\infty} e^{(-2\alpha + i2\pi)t} dt \right] + \frac{1}{2\alpha} \\ &= \operatorname{Re} \left[ \frac{1}{2\alpha - i2\pi} \right] + \frac{1}{2\alpha} \\ &= \frac{\alpha}{2(\alpha^2 + \pi^2)} + \frac{1}{2\alpha} < +\infty. \end{aligned} \tag{1}$$

Therefore, the signal is **an energy signal**.

b) The energy of the signal:

$$\begin{aligned} E &= \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |\Pi(t-3) \cos 3\pi t|^2 dt \\ &= \int_{5/2}^{7/2} \cos^2 3\pi t dt \\ &= \frac{1}{2} \int_{5/2}^{7/2} (\cos 6\pi t + 1) dt \\ &= \frac{1}{2} < +\infty. \end{aligned}$$

Therefore, the signal is **an energy signal**.

c) The average power of the signal is

$$P = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} |t|^2 dt = \lim_{T \rightarrow +\infty} \frac{2}{T} \int_0^{T/2} t^2 dt = \lim_{T \rightarrow +\infty} \frac{T^2}{12} = +\infty. \tag{2}$$

Therefore, the signal is **neither an energy signal nor a power signal**.

d) The signal function can be written as

$$x(t) = 1. \quad (3)$$

The average power of the signal:

$$P = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt = 1, \quad (4)$$

$$\Rightarrow 0 < P < +\infty. \quad (5)$$

Therefore, the signal is **a power signal**.

e) The average power of the signal:

$$\begin{aligned} P &= \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{T/2} |e^{j6\pi t} u(t)|^2 dt \\ &= \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^{T/2} |e^{j6\pi t}|^2 dt \\ &= \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^{T/2} dt \\ &= \frac{1}{2}. \end{aligned} \quad (6)$$

$$\Rightarrow 0 < P < +\infty. \quad (7)$$

Therefore, the signal is **a power signal**.

□

**Problem 2 (20 pts) Score:** \_\_\_\_\_. Calculate the Fourier transform and the energy of the following signals.

a)  $x_1(t) = 5 \operatorname{sinc}(2t) e^{j2\pi 3t}$

b)  $x_2(t) = \operatorname{sinc}^2(t-1)$

c)  $x_a(t) = x_1(t) + x_2(-t)$

d)  $x_b(t) = x_1(-t) + x_2(t)$

e)  $x_c(t) = 2x_1(t) \cos 6\pi t + x_2(t) e^{j6\pi t}$

**Solution:** a) We first look for the Fourier transform of  $\frac{1}{\pi t}$  (see reference at <sup>1</sup>). Consider such a function:

$$f_\alpha(t) = \begin{cases} e^{-2\pi\alpha t}, & t > 0 \\ 0, & t = 0 \\ -e^{2\pi\alpha t}, & t < 0 \end{cases} \quad (8)$$

where  $\alpha > 0$ . The Fourier transform of the above function is

$$\mathcal{F}[f_\alpha(t)] = \int_{-\infty}^{+\infty} f(t) e^{-j2\pi f\tau} d\tau = - \int_{-\infty}^0 e^{2\pi(\alpha-jf)\tau} d\tau + \int_0^{+\infty} e^{-2\pi(\alpha+jf)\tau} d\tau$$

<sup>1</sup><https://math.stackexchange.com/questions/1033870/does-the-fourier-transform-exist-for-ft-1-t>

$$= -\frac{1}{2\pi(\alpha - jf)} + \frac{1}{2\pi(\alpha + jf)} = -\frac{2jf}{2\pi(\alpha^2 + f^2)}. \quad (9)$$

Since the sign function is the limit of  $f_\alpha(t)$  when  $a \rightarrow 0$ :

$$\text{sgn}(t) = \begin{cases} 1, & t > 0, \\ 0, & t = 0, \\ -1, & t < 0, \end{cases} = \lim_{\alpha \rightarrow 0} f_\alpha(t), \quad (10)$$

by taking the limit  $\alpha \rightarrow 0$ , we get the Fourier transform of the sign function:

$$\mathcal{F}[\text{sgn}(t)] = \lim_{\alpha \rightarrow 0} [f_\alpha(t)] = \frac{1}{j\pi f}. \quad (11)$$

Using the duality property, the Fourier transform of  $\frac{1}{\pi t}$  is

$$\mathcal{F}\left[\frac{1}{\pi t}\right] = -j \text{sgn}(f). \quad (12)$$

Then, using the multiplication property, the Fourier transform of the sinc function is

$$\begin{aligned} \mathcal{F}[\text{sinc}(t)] &= \mathcal{F}\left[\frac{\sin(\pi t)}{\pi t}\right] = \mathcal{F}[\sin(\pi t)] * \mathcal{F}\left[\frac{1}{\pi t}\right] \\ &= \frac{1}{2j} \left[ \delta\left(f - \frac{1}{2}\right) - \delta\left(f + \frac{1}{2}\right) \right] * [-j \text{sgn}(f)] \\ &= -\frac{1}{2} \int_{-\infty}^{+\infty} \left[ \delta\left(\nu - \frac{1}{2}\right) - \delta\left(\nu + \frac{1}{2}\right) \right] \text{sgn}(f - \nu) d\nu \\ &= -\frac{1}{2} \left[ \text{sgn}\left(f - \frac{1}{2}\right) - \text{sgn}\left(f + \frac{1}{2}\right) \right] \\ &= \Pi(f). \end{aligned} \quad (13)$$

Finally, using the scaling shifting property, we have

$$\mathcal{F}[\text{sinc}(2t)] = \frac{1}{2} \Pi\left(\frac{f}{2}\right). \quad (14)$$

And using the frequency shifting property, we get the Fourier transform of  $x_1(t)$ :

$$\mathcal{F}[x_1(t)] = \mathcal{F}[5 \text{sinc}(2t)e^{j2\pi 3t}] = \frac{5}{2} \Pi\left(\frac{f-3}{2}\right). \quad (15)$$

The energy of  $x_1(t)$  is

$$E = \int_{-\infty}^{+\infty} |\mathcal{F}[x_1(t)]|^2 df = \int_{-\infty}^{+\infty} \left| \frac{5}{2} \Pi\left(\frac{f-3}{2}\right) \right|^2 df = \frac{25}{2}. \quad (16)$$

b) Using the time shifting property, we have

$$\mathcal{F}[\text{sinc}(t-1)] = \Pi(f)e^{-j2\pi f}. \quad (17)$$

Using the multiplication property, we get the Fourier transform of  $x_2(t)$ :

$$\begin{aligned} \mathcal{F}[x_2(t)] &= \mathcal{F}[\text{sinc}^2(t-1)] = \mathcal{F}[\text{sinc}(t-1)] * \mathcal{F}[\text{sinc}(t-1)] \\ &= \int_{-\infty}^{+\infty} \Pi(\nu)e^{-j2\pi\nu} \Pi(f-\nu)e^{-j2\pi(f-\nu)} d\nu \end{aligned}$$

$$\begin{aligned}
&= \int_{-\infty}^{+\infty} \Pi(\nu) \Pi(f - \nu) e^{-j2\pi f} d\nu \\
&= \Lambda(f) e^{-j2\pi f}.
\end{aligned} \tag{18}$$

The energy of  $x_2(t)$  is

$$\begin{aligned}
E &= \int_{-\infty}^{+\infty} |\mathcal{F}[x_2(t)]|^2 df = \int_{-\infty}^{+\infty} |\Lambda(f) e^{-j2\pi f}|^2 df \\
&= \int_{-\infty}^{+\infty} |\Lambda(f)|^2 df \\
&= 2 \int_0^1 (1 - f)^2 df \\
&= \frac{2}{3}.
\end{aligned}$$

c) Using the superposition and the scaling properties, the Fourier transform of  $x_a(t)$  is

$$\mathcal{F}[x_a(t)] = \mathcal{F}[x_1(t) + x_2(-t)] = \mathcal{F}[x_1(t)] + \mathcal{F}[x_2(-t)] = \frac{5}{2} \Pi\left(\frac{f-3}{2}\right) + \Lambda(f) e^{-j2\pi f}. \tag{19}$$

The energy of  $x_a(t)$  is

$$\begin{aligned}
E &= \int_{-\infty}^{+\infty} |\mathcal{F}[x_a(t)]|^2 df = \int_{-\infty}^{+\infty} \left| \frac{5}{2} \Pi\left(\frac{f-3}{2}\right) + \Lambda(f) e^{-j2\pi f} \right|^2 df \\
&= \int_2^4 \left| \frac{5}{2} \Pi\left(\frac{f-3}{2}\right) \right|^2 df + \int_{-1}^1 |\Lambda(f)|^2 df \\
&= \int_2^4 \frac{25}{4} df + 2 \int_0^1 (1 - f)^2 df \\
&= \frac{25}{2} + \frac{2}{3} = \frac{79}{6}.
\end{aligned} \tag{20}$$

d) Using the superposition and the scaling properties, the Fourier transform of  $x_b(t)$  is

$$\mathcal{F}[x_b(t)] = \mathcal{F}[x_1(-t) + x_2(t)] = \mathcal{F}[x_1(-t)] + \mathcal{F}[x_2(t)] = \frac{5}{2} \Pi\left(\frac{f-3}{2}\right) + \Lambda(f) e^{-j2\pi f}. \tag{21}$$

Since the Fourier transform of  $x_b(t)$  is the same as  $x_a(t)$ . The energy of  $x_b(t)$  is also the same as that of  $x_a(t)$ :

$$E = \frac{79}{6}. \tag{22}$$

e) The Fourier transform of the first term is

$$\begin{aligned}
\mathcal{F}[2x_1(t) \cos 6\pi t] &= \mathcal{F}[x_1(t)(e^{j2\pi 3t} + e^{-j2\pi 3t})] = \mathcal{F}[x_1(t)e^{j2\pi 3t}] + \mathcal{F}[x_1(t)e^{-j2\pi 3t}] \\
&= \frac{5}{2} \left[ \Pi\left(\frac{f}{2} - 3\right) + \Pi\left(\frac{f}{2}\right) \right].
\end{aligned} \tag{23}$$

The Fourier transform of the second term is

$$\mathcal{F}[x_2(t)e^{j6\pi t}] = \mathcal{F}[x_2(t)e^{j2\pi 3t}] = \Lambda(f - 3) e^{-j2\pi(f-3)}. \tag{24}$$

Therefore, the Fourier transform of  $x_c(t)$  is

$$\mathcal{F}[x_c(t)] = \mathcal{F}[2x_1(t) \cos 6\pi t + x_2(t)e^{j6\pi t}] = \mathcal{F}[2x_1(t) \cos 6\pi t] + \mathcal{F}[x_2(t)e^{j6\pi t}]$$

$$= \frac{5}{2} \left[ \Pi \left( \frac{f}{2} - 3 \right) + \Pi \left( \frac{f}{2} \right) \right] + \Lambda(f-3)e^{-j2\pi(f-3)}. \quad (25)$$

The energy of  $x_c(t)$  is

$$\begin{aligned} E &= \int_{-\infty}^{+\infty} |\mathcal{F}[x_c(t)]|^2 df = \int_{-\infty}^{+\infty} \left| \frac{5}{2} \left[ \Pi \left( \frac{f}{2} - 3 \right) + \Pi \left( \frac{f}{2} \right) \right] + \Lambda(f-3)e^{-j2\pi(f-3)} \right|^2 df \\ &= \int_5^7 \left| \frac{5}{2} \Pi \left( \frac{f}{2} - 3 \right) \right|^2 df + \int_{-1}^1 \left| \frac{5}{2} \Pi \left( \frac{f}{2} \right) \right|^2 df + \int_2^4 |\Lambda(f-3)e^{-j2\pi(f-3)}|^2 df \\ &= \int_5^7 \frac{25}{4} df + \int_{-1}^1 \frac{25}{4} df + 2 \int_3^4 (4-f)^2 df \\ &= \frac{25}{2} + \frac{25}{2} + \frac{2}{3} = \frac{77}{3}. \end{aligned} \quad (26)$$

□

**Problem 3 (10 pts) Score:** \_\_\_\_\_. Calculate the convolution of the following signal.

a)  $y(t) = e^{-|t|} * \Pi(t-2)$

b)  $y(t) = \text{sgn}(t) * \Lambda(t-2)$

**Solution:** a) The convolution can be written as

$$\begin{aligned} y(t) &= e^{-|t|} * \Pi(t-2) = \int_{-\infty}^{+\infty} e^{-|\tau|} \Pi(t-\tau-2) d\tau \\ &= \int_{t-\frac{5}{2}}^{t-\frac{3}{2}} e^{-|\tau|} d\tau. \end{aligned} \quad (27)$$

If  $t \geq \frac{5}{2}$ ,

$$y(t) = \int_{t-\frac{5}{2}}^{t-\frac{3}{2}} e^{-\tau} d\tau = e^{\frac{5}{2}-t} - e^{\frac{3}{2}-t}. \quad (28)$$

If  $\frac{3}{2} \leq t < \frac{5}{2}$ ,

$$y(t) = \int_{t-\frac{5}{2}}^0 e^{\tau} d\tau + \int_0^{t-\frac{3}{2}} e^{-\tau} d\tau = 2 - e^{t-\frac{5}{2}} - e^{t-\frac{3}{2}}. \quad (29)$$

If  $t < \frac{3}{2}$ ,

$$y(t) = \int_{t-\frac{5}{2}}^{t-\frac{3}{2}} e^{\tau} d\tau = e^{t-\frac{3}{2}} - e^{t-\frac{5}{2}}. \quad (30)$$

Therefore, in general,

$$y(t) = \begin{cases} e^{\frac{5}{2}-t} - e^{\frac{3}{2}-t}, & t \geq \frac{5}{2}, \\ 2 - e^{t-\frac{5}{2}} - e^{t-\frac{3}{2}}, & \frac{3}{2} \leq t < \frac{5}{2}, \\ e^{t-\frac{3}{2}} - e^{t-\frac{5}{2}}, & t < \frac{3}{2}. \end{cases} \quad (31)$$

b) The convolution can be written as

$$y(t) = \text{sgn}(t) * \Lambda(t-2) = \int_{-\infty}^{+\infty} \text{sgn}(\tau) \Lambda(t-\tau-2) d\tau$$

$$= \int_{t-\frac{5}{2}}^{t-\frac{3}{2}} \operatorname{sgn}(\tau) d\tau. \quad (32)$$

If  $t \geq \frac{5}{2}$ ,

$$y(t) = \int_{t-\frac{5}{2}}^{t-\frac{3}{2}} d\tau = 1. \quad (33)$$

If  $\frac{3}{2} < t \leq \frac{5}{2}$ ,

$$y(t) = \int_{t-\frac{5}{2}}^0 (-1) d\tau + \int_0^{t-\frac{3}{2}} d\tau = 2t - 4. \quad (34)$$

If  $t < \frac{3}{2}$ ,

$$y(t) = \int_{t-\frac{5}{2}}^{t-\frac{3}{2}} (-1) d\tau = -1. \quad (35)$$

Therefore, in general,

$$y(t) = \begin{cases} 1, & t \geq \frac{5}{2}, \\ 2t - 4, & \frac{3}{2} \leq t < \frac{5}{2}, \\ -1, & t < \frac{3}{2}. \end{cases} \quad (36)$$

□

**Problem 4 (20 pts) Score:** \_\_\_\_\_. Calculate the Fourier transform of the following periodic signal.

a)  $\sum_{n=-\infty}^{\infty} \Lambda\left(\frac{t}{2} - 2n\right)$

b)  $\left[\sum_{n=-\infty}^{\infty} \delta(t - 2n)\right] * \left[\Pi\left(\frac{t}{2}\right) \cos(2\pi t)\right]$

**Solution:** a) The Fourier series of the signal is

$$\sum_{n=-\infty}^{\infty} \Lambda\left(\frac{t}{2} - 2n\right) = \sum_{n=-\infty}^{\infty} c_n e^{jn2\pi\frac{t}{4}}, \quad (37)$$

where

$$\begin{aligned} c_n &= \frac{1}{4} \int_{-2}^2 \sum_{n=-\infty}^{\infty} \Lambda\left(\frac{\tau}{2} - 2n\right) e^{-jn2\pi\frac{\tau}{4}} d\tau \\ &= \frac{1}{2} \int_0^2 \left(1 - \frac{\tau}{2}\right) \cos\left(n2\pi\frac{\tau}{4}\right) d\tau \\ &= \frac{\sin(n\pi)}{n\pi} - \frac{1}{4} \frac{1}{n2\pi\frac{1}{4}} \int_0^2 \tau d\left[\sin\left(n2\pi\frac{\tau}{4}\right)\right] \\ &= \frac{\sin(n\pi)}{n\pi} - \frac{\sin(n\pi)}{n\pi} + \frac{1}{4} \frac{1}{n2\pi\frac{1}{4}} \int_0^2 \sin\left(n2\pi\frac{\tau}{4}\right) d\tau \\ &= \frac{1 - \cos(n\pi)}{(n\pi)^2}. \end{aligned} \quad (38)$$

Note that  $c_n = 2$  when  $n = 0$ . The fourier transform of the signal is

$$\mathcal{F}\left[\sum_{n=-\infty}^{\infty} \Lambda\left(\frac{t}{2} - 2n\right)\right] = \mathcal{F}\left[\sum_{n=-\infty}^{\infty} \frac{1 - \cos(n\pi)}{(n\pi)^2} e^{jn2\pi\frac{t}{4}}\right]$$

$$\begin{aligned}
&= \sum_{n=-\infty}^{\infty} \frac{1 - \cos(n\pi)}{(n\pi)^2} \mathcal{F} \left[ e^{-j2\pi(f - \frac{n}{4})t} \right] \\
&= \sum_{n=-\infty}^{\infty} \frac{1 - \cos(n\pi)}{(n\pi)^2} \delta(f - \frac{n}{4}).
\end{aligned} \tag{39}$$

b) The Fourier series of  $\sum_{n=-\infty}^{\infty} \delta(t - 2n)$  is

$$\sum_{n=-\infty}^{\infty} \delta(t - 2n) = \sum_{n=-\infty}^{\infty} c_n e^{jn2\pi \frac{t}{2}}, \tag{40}$$

where

$$c_n = \frac{1}{2} \int_{-1}^1 \sum_{n=-\infty}^{\infty} \delta(t - 2n) e^{-jn2\pi \frac{\tau}{2}} d\tau = \frac{1}{2}. \tag{41}$$

The Fourier transform of  $\sum_{n=-\infty}^{\infty} \delta(t - 2n)$  is

$$\mathcal{F} \left[ \sum_{n=-\infty}^{\infty} \delta(t - 2n) \right] = \mathcal{F} \left[ \sum_{n=-\infty}^{\infty} \frac{1}{2} e^{jn2\pi \frac{t}{2}} \right] = \sum_{n=-\infty}^{\infty} \frac{1}{2} \mathcal{F} \left[ e^{jn2\pi \frac{t}{2}} \right] = \frac{1}{2} \sum_{n=-\infty}^{\infty} \delta \left( f - \frac{n}{2} \right). \tag{42}$$

The Fourier transform of  $\Pi \left( \frac{t}{2} \right) \cos(2\pi t)$  is

$$\begin{aligned}
&\mathcal{F} \left[ \Pi \left( \frac{t}{2} \right) \cos(2\pi t) \right] = \int_{-\infty}^{+\infty} \Pi \left( \frac{\tau}{2} \right) \cos(2\pi \tau) e^{-j2\pi f \tau} d\tau \\
&= 2 \int_0^1 \cos(2\pi \tau) \cos(2\pi f \tau) d\tau \\
&= \int_0^1 \{ \cos[2\pi(f+1)\tau] + \cos[2\pi(f-1)\tau] \} d\tau \\
&= \frac{\sin[2\pi(f+1)]}{2\pi(f+1)} + \frac{\sin[2\pi(f-1)]}{2\pi(f-1)}.
\end{aligned}$$

Using the time convolution property, we get the Fourier transform of the signal

$$\begin{aligned}
\mathcal{F} \left\{ \left[ \sum_{n=-\infty}^{\infty} \delta(t - 2n) \right] * \left[ \Pi \left( \frac{t}{2} \right) \cos(2\pi t) \right] \right\} &= \mathcal{F} \left[ \sum_{n=-\infty}^{\infty} \delta(t - 2n) \right] \mathcal{F} \left[ \Pi \left( \frac{t}{2} \right) \cos(2\pi t) \right] \\
&= \frac{1}{2} \sum_{n=-\infty}^{\infty} \delta \left( f - \frac{n}{2} \right) \left\{ \frac{\sin[2\pi(f+1)]}{2\pi(f+1)} + \frac{\sin[2\pi(f-1)]}{2\pi(f-1)} \right\}.
\end{aligned} \tag{43}$$

□

**Problem 5 (15 pts) Score:** \_\_\_\_\_. **Determine** the range of permissible cutoff frequencies for the ideal lowpass filter used to reconstruct the following signal

$$x(t) = 4 \cos^2(200\pi t) \cos(1000\pi t)$$

Which is sampled at 2000 samples per second. Sketch  $X(f)$  and  $X_\delta(f)$  (spectrum after the sampling). Find the minimum allowable sampling frequency.

**Solution:** The signal can be written as

$$\begin{aligned}
x(t) &= 4 \cos^2(200\pi t) \cos(1000\pi t) = 2[1 + \cos(400\pi t)] \cos(1000\pi t) \\
&= 2 \cos(1000\pi t) + \cos(1400\pi t) + \cos(600\pi t).
\end{aligned} \tag{44}$$

The spectrum of the signal is

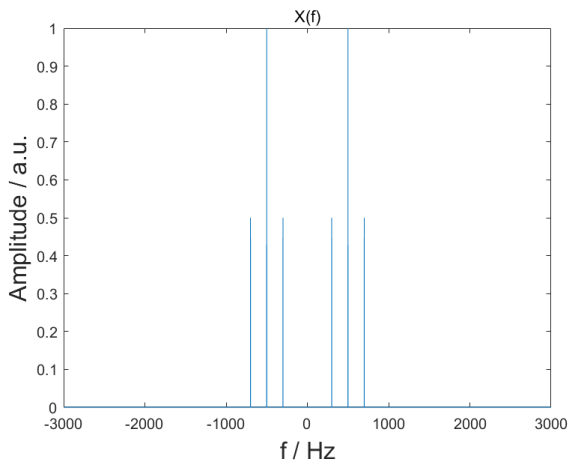
$$\begin{aligned} X(f) &= \mathcal{F}[x(t)] = \mathcal{F}[2 \cos(1000\pi t) + \cos(1400\pi t) + \cos(600\pi t)] \\ &= \delta(f - 500) + \delta(f + 500) + \frac{1}{2}\delta(f - 700) + \frac{1}{2}\delta(f + 700) + \frac{1}{2}\delta(f - 300) + \frac{1}{2}\delta(f + 300), \end{aligned} \quad (45)$$

as shown in figure 1(a). Sampling at 2000 samples per second means that the sampling frequency is  $f_s = 2000$  Hz.

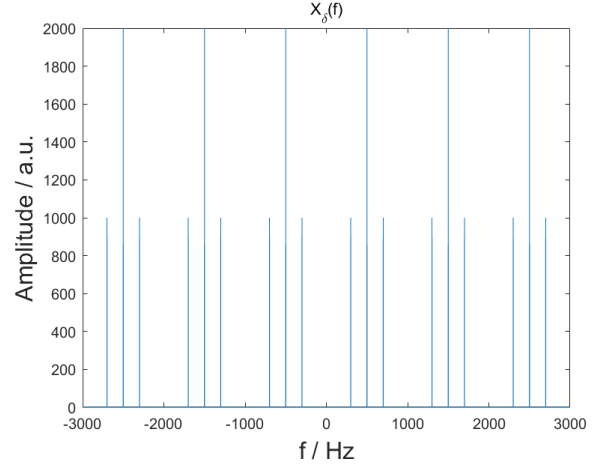
The spectrum after the sampling is

$$\begin{aligned} X_\delta(f) &= 2000 \sum_{n=-\infty}^{\infty} X(f - 2000n) \\ &= 2000 \sum_{n=-\infty}^{\infty} \left[ \delta(f - 2000n - 500) + \delta(f - 2000n + 500) + \frac{1}{2}\delta(f - 2000n - 700) + \frac{1}{2}\delta(f - 2000n + 700) \right. \\ &\quad \left. + \frac{1}{2}\delta(f - 2000n - 300) + \frac{1}{2}\delta(f - 2000n + 300) \right], \end{aligned} \quad (46)$$

as shown in figure 1(b). According to the sampling theorem, the range of permissible cutoff frequencies for the ideal



(a) Original spectrum of the signal:  $X(f)$



(b) Spectrum after the sampling:  $X_\delta(f)$

Figure 1:

lowpass filter to reconstruct the signal is  $700\text{Hz} < f_c < 1300\text{Hz}$  and the minimum allowable sampling frequency is 1400Hz. □

**Problem 6 Score:** \_\_\_\_\_. 1) Express the spectrum  $Y(f)$  of

$$y(t) = x(t) \cos(400\pi t) + \hat{x}(t) \sin(400\pi t)$$

using the spectrum  $X(f)$  of  $x(t)$ , where  $\hat{x}(t)$  is the Hilbert transform of  $x(t)$ . (5 pts)

2) if  $x(t) = \text{sinc}(t)$ , sketch  $Y(f)$ . (5 pts)

**Solution:** 1) The spectrum of  $y(t)$  is

$$\begin{aligned} Y(f) &= \mathcal{F}[y(t)] = \mathcal{F}[x(t) \cos(400\pi t) + \hat{x}(t) \sin(400\pi t)] \\ &= \mathcal{F}\left[x(t) \frac{e^{j400\pi t} + e^{-j400\pi t}}{2}\right] + \mathcal{F}\left\{\mathcal{F}^{-1}[-j \text{sgn}(f)X(f)] \frac{e^{j400\pi t} - e^{-j400\pi t}}{2j}\right\} \\ &= \frac{1}{2}X(f - 200) + \frac{1}{2}X(f + 200) - \frac{1}{2}\text{sgn}(f - 200)X(f - 200) + \frac{1}{2}\text{sgn}(f + 200)X(f + 200) \end{aligned} \quad (47)$$



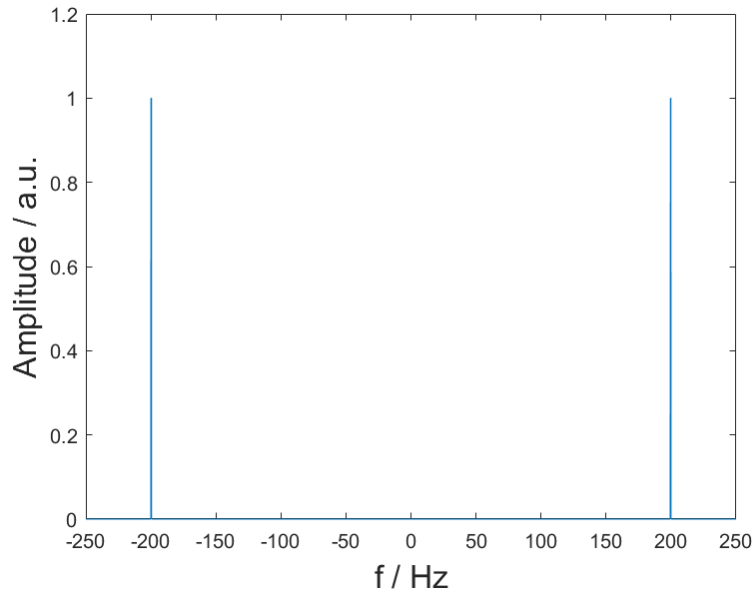
2) If  $x(t) = \text{sinc}(t)$ , then

$$X(f) = \Pi(f) \quad (48)$$

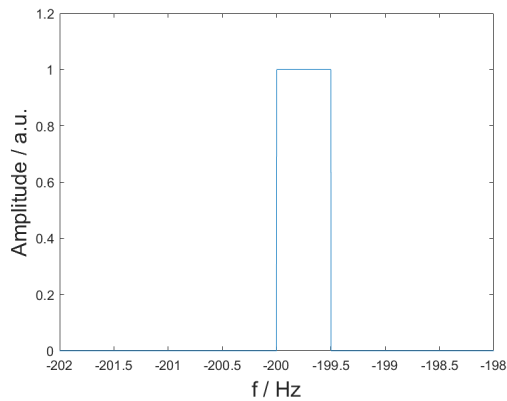
and

$$Y(f) = \frac{1}{2}\Pi(f - 200) + \frac{1}{2}\Pi(f + 200) - \frac{1}{2}\text{sgn}(f - 200)\Pi(f - 200) + \frac{1}{2}\text{sgn}(f + 200)\Pi(f + 200), \quad (49)$$

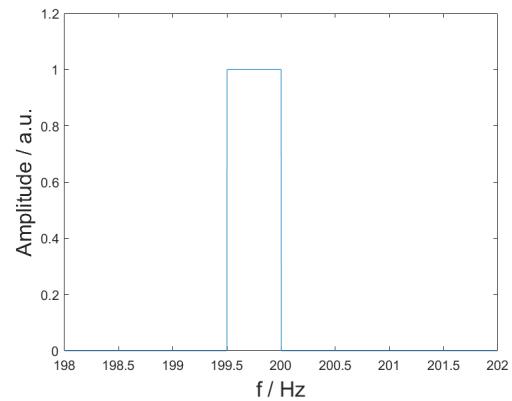
as shown in figure 2.



(a) Overall view of  $Y(f)$



(b)  $Y(f)$  near  $-200$



(c)  $Y(f)$  near  $+200$

Figure 2: Spectrum of  $y(t)$ :  $Y(f)$

□

**Problem 7 Score:** \_\_\_\_\_. Consider  $x(t) = 2\cos(60\pi t)$ , the reference frequency  $f_0 = 40$  Hz. Calculate the following signals.

- The Hilbert transform of  $x(t)$ , i.e.,  $\hat{x}(t)$ .
- The analytic signal  $x_p(t)$ .

c) The complex envelope of  $x(t)$ , i.e.,  $\tilde{x}(t)$ .

d) The inphase and quadrature component of  $x(t)$ , i.e.,  $x_R(t)$  and  $x_I(t)$ .

(Please refer to Lecture 2, Slide 36 or Page 88 of reference textbook, we will learn this in the next class. I am sorry for the lagging.)

e) Determine and plot the spectrum of the following signals:

i.  $x_1(t) = \frac{2}{3}x(t) + \frac{1}{3}j\hat{x}(t)$

ii.  $x_2(t) = \left[\frac{1}{5}x(t) + \frac{4}{5}j\hat{x}(t)\right] e^{j2\pi f_0 t}$

**Solution:** a) The Fourier spectrum of  $x(t)$  is

$$X(f) = \mathcal{F}[x(t)] = \delta(f - 30) + \delta(f + 30).$$

The spectrum of  $x(t)$  after Hilbert transform is

$$\hat{X}(f) = -j \operatorname{sgn}(f) X(f) = \delta(f - 30)e^{-j\pi/2} + \delta(f + 30)e^{j\pi/2}. \quad (50)$$

The Hilbert transform of  $x(t)$  is

$$\begin{aligned} \hat{x}(t) &= \mathcal{F}^{-1}[\hat{X}(f)] = \mathcal{F}[\delta(f - 30)e^{-j\pi/2} + \delta(f + 30)e^{j\pi/2}] \\ &= e^{j60\pi t} e^{-j\pi/2} + e^{-j60\pi t} e^{j\pi/2} \\ &= 2 \sin(60\pi t). \end{aligned} \quad (51)$$

b) The analytic signal of  $x(t)$  is

$$x_p(t) = x(t) + j\hat{x}(t) = 2 \cos(60\pi t) + j2 \sin(60\pi t) = 2e^{j60\pi t}. \quad (52)$$

c) The complex envelope of  $x(t)$  is

$$\tilde{x}(t) = x_p(t) e^{-j2\pi 40t} = 2e^{-j20\pi t}. \quad (53)$$

d) The inphase component of  $x(t)$  is

$$x_R(t) = \operatorname{Re}[\tilde{x}(t)] = 2 \cos(20\pi t). \quad (54)$$

The quadrature component of  $x(t)$  is

$$x_I(t) = \operatorname{Im}[\tilde{x}(t)] = 2 \sin(20\pi t). \quad (55)$$

e) i. The spectrum of  $x_1(t)$  is

$$\begin{aligned} X_1(f) &= \mathcal{F}[x_1(t)] = \mathcal{F}\left[\frac{2}{3}x(t) + \frac{1}{3}j\hat{x}(t)\right] \\ &= \frac{2}{3}X(f) + \frac{1}{3}j\hat{X}(f) \\ &= \frac{2}{3}[\delta(f - 30) + \delta(f + 30)] + \frac{1}{3}j[\delta(f - 30)e^{-j\pi/2} + \delta(f + 30)e^{j\pi/2}] \\ &= \delta(f - 30) + \frac{1}{3}\delta(f + 30). \end{aligned} \quad (56)$$

Its amplitude spectrum is

$$|X_1(f)| = \delta(f - 30) + \frac{1}{3}\delta(f + 30), \quad (57)$$

and angular spectrum is

$$\theta_1(f) = 0. \quad (58)$$

as shown in figure 3.

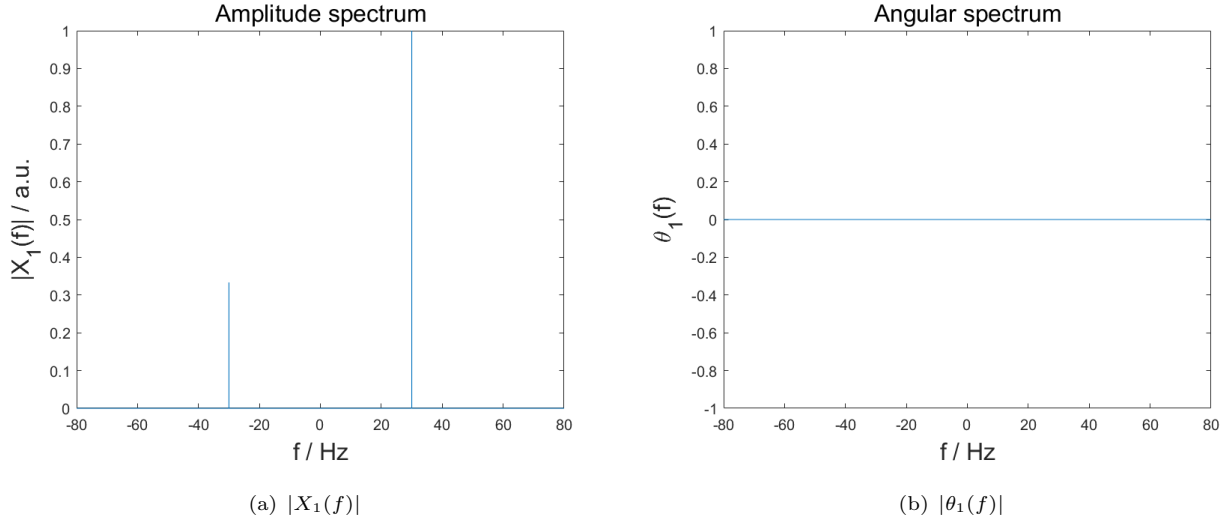


Figure 3: Spectrum of  $x_1(t)$

ii. The Fourier transform of  $\left[\frac{1}{5}x(t) + \frac{4}{5}j\hat{x}(t)\right]$  is

$$\begin{aligned} \mathcal{F}\left[\frac{1}{5}x(t) + \frac{4}{5}j\hat{x}(t)\right] &= \frac{1}{5}X(f) + \frac{4}{5}j\hat{X}(f) \\ &= \frac{1}{5}[\delta(f - 30) + \delta(f + 30)] + \frac{4}{5}j[\delta(f - 30)e^{-j\pi/2} + \delta(f + 30)e^{j\pi/2}] \\ &= \delta(f - 30) - \frac{3}{5}\delta(f + 30). \end{aligned} \quad (59)$$

Using the frequency shifting property, we get the spectrum of  $x_2(t)$ :

$$X_2(f) = \mathcal{F}[x_2(t)] = \mathcal{F}\left\{\left[\frac{1}{5}x(t) + \frac{4}{5}j\hat{x}(t)\right]e^{j2\pi f_0 t}\right\} = \delta(f - f_0 - 30) - \frac{3}{5}\delta(f - f_0 + 30). \quad (60)$$

Its amplitude spectrum is

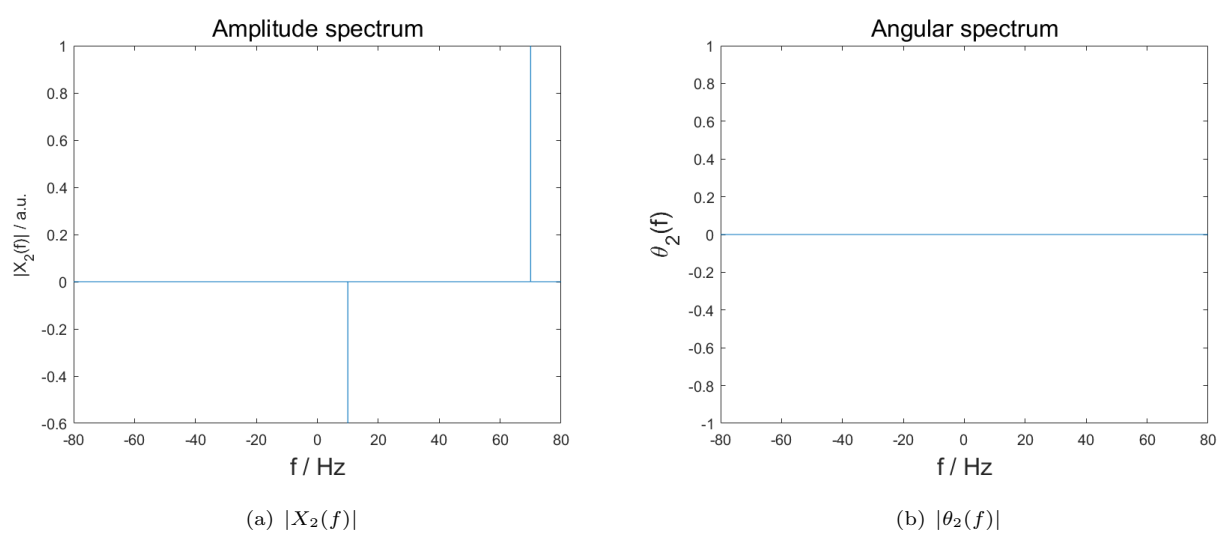
$$|X_2(f)| = \delta(f - f_0 - 30) - \frac{3}{5}\delta(f - f_0 + 30) = \delta(f - 70) - \frac{3}{5}\delta(f - 10), \quad (61)$$

and angular spectrum is

$$\theta_2(f) = 0, \quad (62)$$

as shown in figure 4.

□

Figure 4: Spectrum of  $x_2(t)$