Introduction to System System
EE140
Fall, 2020

Assignment 2

Due time : 10:15, Sept 25, 2020 (Friday)

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Grade : ________

Problem 1 (Conditional probability, 10 pts) Score: ________. Given a binary communication channel where A =input and B =output, let $P(A)=0.3$, $P(B|A)=0.8$, and $P(\bar{B}|\bar{A})=0.6$. Find $P(A|B)$ and $P(\bar{A}|\bar{B})$.

Solution: Using the law of total probability, the probability of getting output B is

$$\begin{aligned}P(B)=&P(B|A)P(A)+P(B|\bar{A})P(\bar{A})=P(B|A)P(A)+[1-P(\bar{B}|\bar{A})][1-P(A)]\\=&0.8\times 0.3+(1-0.6)\times(1-0.3)=0.52.\end{aligned}\tag{1}$$

Using Bayes' theorem, we have

$$P(A|B)=\frac{P(B|A)P(A)}{P(B)}=\frac{0.8\times 0.3}{0.52}=\frac{6}{13}.\tag{2}$$

and

$$P(\bar{A}|\bar{B})=\frac{P(\bar{B}|\bar{A})P(\bar{A})}{P(\bar{B})}=\frac{P(\bar{B}|\bar{A})[1-P(A)]}{P(\bar{B})}=\frac{0.6\times(1-0.3)}{1-0.52}=\frac{7}{8}.\tag{3}$$

□

Problem 2 (Property of PDF, 20pts) Score: ________. The joint pdf of random variables X and Y is $f(x,y)=Axye^{-2(x+y)}$, $x\geq 0$, $y\geq 0$. Find the following.

1) Find the constant A.
2) Find the marginal pdfs of X and Y, $f_X(x)$ and $f_Y(y)$.
3) Are X and Y statistically independent? Justify your answer.
4) Find the probability density function of $Z=X+Y$.

Solution: 1) As a property of pdf,

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}f(x,y)\,dx\,dy=&\int_0^{+\infty}\int_0^{+\infty}Axye^{-2(x+y)}\,dx\,dy=A\left[\int_0^{+\infty}xe^{-2x}\,dx\right]\left[\int_0^{+\infty}ye^{-2y}\,dy\right]\\=&A\left[\int_0^{+\infty}xe^{-2x}\,dx\right]^2=A\left[-\frac{1}{2}\int_0^{+\infty}x\,d\left(e^{-2x}\right)\right]^2\\=&A\left[-\frac{1}{2}\left(xe^{-2x}\right)\Big|_0^{+\infty}+\frac{1}{2}\int_0^{+\infty}e^{-2x}\,dx\right]^2\\=&A\left[\frac{1}{4}\right]^2=\frac{A}{16}=1.\end{aligned}$$

which gives us that

$$A=16.\tag{4}$$

2) The marginal pdf of X is

$$\begin{aligned}f_X(x)=&\int_{-\infty}^{+\infty}f(x,\beta)\,d\beta=\int_{-\infty}^{+\infty}16x\beta e^{-2(x+\beta)}\,d\beta\\=&16xe^{-2x}\int_{-\infty}^{+\infty}\beta e^{-2\beta}\,d\beta\\=&4xe^{-2x},\end{aligned}\tag{5}$$

and similarly, the marginal pdf of Y is

$$f_Y(y)=\int_{-\infty}^{+\infty}f(\alpha,y)\,d\alpha=\int_{-\infty}^{+\infty}16\alpha ye^{-2(\alpha+y)}\,d\alpha=4ye^{-2y}.$$

3) X **and** Y **are statistically independent**, since

$$f(x,y) = 16xye^{-2(x+y)} = \left(4xe^{-2x}\right)\left(4ye^{-2y}\right) = f_X(x)f_Y(y), \qquad \text{for all } x \geq 0, y \geq 0. \tag{6}$$

4) Since X and Y are independent, the pdf of $Z = X + Y$ is the convolution of pdfs of X and Y:

$$\begin{aligned} f_Z(z) &= \int_0^z f_X(z-u)f_Y(u)\,du \\ &= \int_0^z 4(z-u)e^{-2(z-u)} \cdot 4ue^{-2u}\,du \\ &= 16e^{-2z}\int_0^z (z-u)u\,du \\ &= \frac{8}{3}z^3e^{-2z}. \end{aligned} \tag{7}$$

□

Problem 3 (Gaussian random variables, 20pts) Score: 20. Assume two random variables X and Y are jointly Gaussian with mean $m_x = 1$, $m_y = 2$, and variances $\sigma_x^2 = 1$, $\sigma_y^2 = 4$.

1) Write down the expressions for their marginal pdfs.
2) Assume the correlation coefficient is $\rho_{XY} = 0.5$, a new random variables is defined as $Z = X + Y$, find the expression for the pdf of Z.
3) Express the following probabilities using the Q function.
   a) $P(|X| \leq 5)$
   b) $P(-2 < Y \leq 12)$

Solution: 1) The expression for marginal pdf of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}}\exp\left[-\frac{1}{2\sigma_x^2}(x-m_x)^2\right] = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}(x-1)^2\right]. \tag{8}$$

and the expression for marginal pdf of Y is

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}}\exp\left[-\frac{1}{2\sigma_y^2}(y-m_y)^2\right] = \frac{1}{\sqrt{2\pi\cdot 4}}\exp\left[-\frac{1}{2\cdot 4}(y-2)^2\right]. \tag{9}$$

2) The jointly pdf of X and Y is

$$\begin{aligned} f_{XY}(x,y) &= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho_{XY}^2}}\exp\left\{-\frac{1}{2(1-\rho_{XY}^2)}\left[\frac{(x-m_x)^2}{\sigma_x^2} - \frac{2\rho(x-m_x)(y-m_y)}{\sigma_x\sigma_y} + \frac{(y-m_y)^2}{\sigma_y^2}\right]\right\} \\ &= \frac{1}{2\pi\sqrt{3}}\exp\left\{-\frac{1}{2\times\frac{3}{4}}\left[(x-1)^2 - \frac{(x-1)(y-2)}{2} + \frac{(y-2)^2}{4}\right]\right\}. \end{aligned} \tag{10}$$

The expression of the pdf of Z is

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{+\infty} f_{XY}(u, z-u)\,du \\ &= \int_{-\infty}^{+\infty} \frac{1}{2\pi\sqrt{3}}\exp\left\{-\frac{1}{2\times\frac{3}{4}}\left[(u-1)^2 - \frac{(u-1)(z-u-2)}{2} + \frac{(z-u-2)^2}{4}\right]\right\}du \\ &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\cdot 7}}\frac{1}{\sqrt{2\pi\frac{3}{7}}}\exp\left\{-\frac{1}{2\times\frac{3}{7}}\left[\left(u-\frac{2}{7}\left(z+\frac{1}{2}\right)\right)^2 + \frac{3}{49}(z-3)^2\right]\right\} \\ &= \frac{1}{\sqrt{2\pi 7}}\exp\left\{-\frac{1}{14}(z-3)^2\right\}. \end{aligned} \tag{11}$$

Z is also gussian distribution

$Z \sim N(\mu, \sigma^2)$

$\therefore f_Z(z) = \dots$

3) a)

$$
\begin{aligned}
P(|X| \leq 5) =& P(-5 \leq X \leq 5) = 1 - P(X > 5) - P(X < -5) = 1 - P(X > 5) - P(X > 7) \\
=& 1 - Q(5-1) - Q(7-1) = 1 - Q(4) - Q(6).
\end{aligned} \tag{12}
$$

b)

$$
\begin{aligned}
P(-2 < Y \leq 12) =& 1 - P(Y > 12) - P(Y < -2) = 1 - P(Y > 12) - P(Y > 6) \\
=& 1 - Q\left(\frac{12-2}{2}\right) - P\left(\frac{6-2}{2}\right) = 1 - Q(5) - Q(2).
\end{aligned} \tag{13}
$$

□

Problem 4 (Random Process: probability, 20pts) Score: 20 . A fair die is thrown. Depending on the number of spots on the up face, the following random process are generated.

1) Sketch several examples of sample functions of each case. ($A > 0$ is a constant).

a) $X(t, \zeta) = \begin{cases} 2A, & \text{1 or 2 spots up} \\ 0, & \text{3 or 4 spots up} \\ -2A, & \text{5 or 6 spots up} \end{cases}$

b) $X(t, \zeta) = \begin{cases} 2A, & \text{1 or 2 spots up} \\ At, & \text{3 or 4 spots up} \\ -At, & \text{5 or 6 spots up} \end{cases}$

2) What are the following probabilities for each case (case a and case b in 1))?

i. $P(X(4) \geq A)$ ($X(4)$ means $X(t=4, \zeta)$, which is a random variable.)

ii. $P(X(2) \leq 0)$ ($X(2)$ means $X(t=2, \zeta)$, which is a random variable.)

Solution: 1) a) As shown in figure 1(a).

b) As shown in figure 1(b).

2) i. For case a,

$$P(X(4) \geq A) = \frac{1}{3}. \tag{14}$$

For case b,

$$P(X(4) \geq A) = \frac{2}{3}. \tag{15}$$

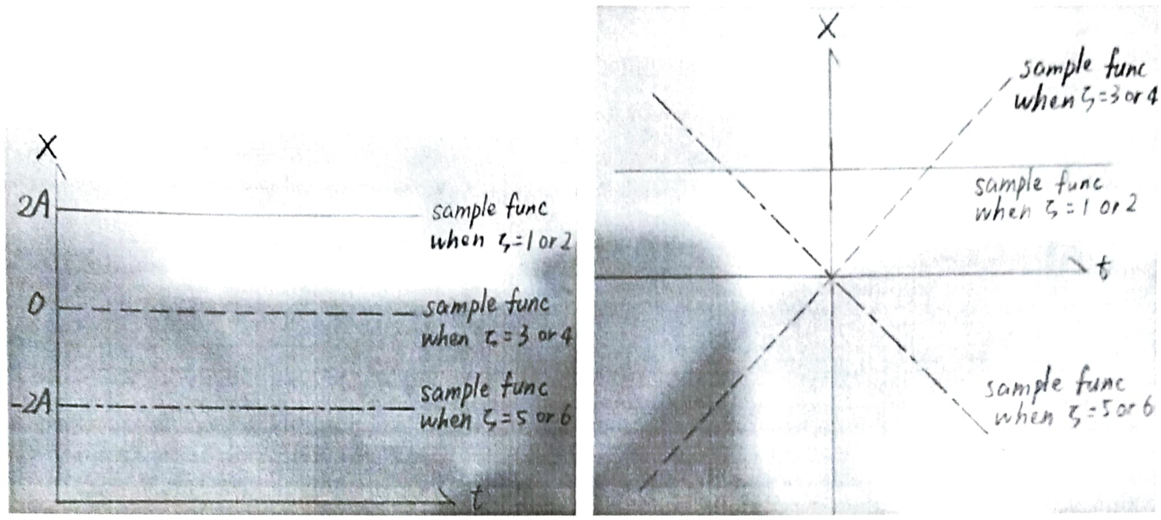
ii. For case a,

$$P(X(2) \leq 0) = \frac{2}{3}. \tag{16}$$

For case b,

$$P(X(2) \leq 0) = \frac{1}{3}. \tag{17}$$

□



(a) Sample functions of a) (b) Sample functions of b)

Figure 1:

Problem 5 (Random Process: mean, variance, autocorrelation, 30 pts) Score: 30 **.** Let the sample function of a random process be given by $X(t) = A\cos 2\pi f_0 t$, where f_0 is fixed and A has the pdf

$$f_A(a) = \frac{1}{\sqrt{2\pi\sigma_a^2}} e^{-\frac{|a-\mu_a|^2}{2\sigma_a^2}}.$$

The mean and variance of A are given by $\mu_a = 0$, $\sigma_a^2 = 4$.

1) Find the mean and variance of random process $X(t)$ at time t_0.
   (Hint: ensemble-mean: Note that $\cos 2\pi f_0 t_0$ is just a constant.)

2) Find the autocorrelation function of $X(t)$.

3) Is $X(t)$ stationary?

4) Find the time average mean and autocorrelation function of $X(t)$.
   (Hint1: the time average mean can be calculated as $\langle X(t)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)\,dt$, the time average auto-correlation function can be calculated as $\langle X(t)X(t+\tau)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)\,dt$
   Hint2: For periodic signal, $\langle X(t)\rangle = \frac{1}{T}\int_0^T x(t)\,dt$, and $\langle X(t)X(t+\tau)\rangle = \frac{1}{T}\int_0^T x(t)x(t+\tau)\,dt$)

5) Is the $X(t)$ ergodic?

Solution: 1) The mean of random process $X(t)$ at time t_0 is

$$E(X(t_0)) = E[X(t=t_0)] = \int_{-\infty}^{+\infty} a\cos(2\pi f_0 t_0)\frac{1}{\sqrt{2\pi\sigma_a^2}} e^{-\frac{|a-\mu_a|^2}{2\sigma_a^2}}\,da = 0. \tag{18}$$

The variance of random process $X(t)$ at time t_0 is

$$\sigma_X^2(t_0) = E\left[\left|X(t_0) - \overline{X(t_0)}\right|^2\right] = E[X(t_0)^2] = E[A^2]\cos^2(2\pi f_0 t_0) = \sigma_a^2\cos^2(2\pi f_0 t_0) = 4\cos^2(2\pi f_0 t_0). \tag{19}$$

2) Let $t_1 = t$, $t_2 = t+\tau$. The autocorrelation function of $X(t)$ is

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = E[A\cos(2\pi f_0 t)A\cos(2\pi f_0(t+\tau))]$$

$$
\begin{aligned}
&= E[A^2]\cos(2\pi f_0 t)\cos[2\pi f_0(t+\tau)] \\
&= \sigma_a^2 \cos(2\pi f_0 t)\cos[2\pi f_0(t+\tau)] \\
&= 4\cos(2\pi f_0 t)\cos[2\pi f_0(t+\tau)].
\end{aligned} \tag{20}
$$

3) **No**, since the second-order statistics of $X(t)$ does not depend only on the time gap τ, but also on the begin time t.

4) $X(t)$ is a periodic signal. Its time mean is

$$
\begin{aligned}
\langle X(t)\rangle &= \frac{1}{T}\int_0^T x(t)\,dt = f_0\int_0^{1/f_0} A\cos(2\pi f_0 t)\,dt \\
&= 0.
\end{aligned} \tag{21}
$$

and its autocorrelation function is

$$
\begin{aligned}
\langle X(t)X(t+\tau)\rangle &= \frac{1}{T}\int_0^T x(t)x(t+\tau)\,dt = f_0\int_0^{1/f_0} A\cos(2\pi f_0 t)A\cos(2\pi f_0(t+\tau))\,dt \\
&= A^2 f_0 \frac{1}{2}\int_0^{1/f_0} \cos[2\pi f_0(2t+\tau)] + \cos[2\pi f_0\tau]\,dt \\
&= \frac{A^2}{2}\cos(2\pi f_0\tau).
\end{aligned} \tag{22}
$$

5) **No**, since $R_X(t, t+\tau) \neq \langle X(t)X(t+\tau)\rangle$.

□