

# Assignment 3

Due time : 10:15, Oct 9, 2020 (Friday)

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Grade : \_\_\_\_\_

**Problem 1 (Autocorrelation and cross-correlation function, 30 pts) Score: \_\_\_\_\_.** The random process are given by  $X(t) = n(t) + A \cos(2\pi f_0 t + \theta)$ ,  $Y(t) = n(t) + A \sin(2\pi f_0 t + \theta)$ , where  $A$  and  $f_0$  are positive constants and  $\theta$  is a random variable uniformly distributed in the interval  $[-\pi, \pi]$ . The first term  $n(t)$  represents a stationary random noise process with autocorrelation function  $R_n(\tau) = B\Lambda(\tau) + C$ , where  $B$  and  $C$  are positive constants. We further assume the random process  $n(t)$  and  $A \cos(2\pi f_0 t + \theta)$  are uncorrelated,  $n(t)$  and  $A \sin(2\pi f_0 t + \theta)$  are also uncorrelated.

- 1) Find the autocorrelation functions of  $X(t)$  and  $Y(t)$ , respectively.
- 2) Find the cross-correlation function of  $X(t)$  and  $Y(t)$ .
- 3) Find the power spectral densities of  $X(t)$  and  $Y(t)$ , respectively.
- 4) Find the cross power spectral density of  $X(t)$  and  $Y(t)$ .
- 5) Find the total power of  $X(t)$  and  $Y(t)$ , respectively.
- 6) Find the DC powers of  $X(t)$  and  $Y(t)$ , respectively.

(Hint:  $\Lambda(\tau) = \begin{cases} 1 - |\tau|, & |\tau| \leq 1 \\ 0, & \text{otherwise} \end{cases}$ , the DC power of  $X(t)$  is  $\overline{X(t)}^2 = E^2[X(t)]$ )

**Solution:** 1) The autocorrelation function of  $X(t)$  is

$$\begin{aligned}
 R_X(t, t + \tau) &= E[X(t)X(t + \tau)] = E\{[n(t) + A \cos(2\pi f_0 t + \theta)][n(t + \tau) + A \cos(2\pi f_0(t + \tau) + \theta)]\} \\
 &= E[n(t)n(t + \tau)] + E[n(t)A \cos(2\pi f_0 t + \theta)] \\
 &\quad + E[A \cos(2\pi f_0 t + \theta)n(t + \tau)] + E[A \cos(2\pi f_0 t + \theta)A \cos(2\pi f_0(t + \tau) + \theta)] \\
 &\quad (\because n(t) \text{ and } A \cos(2\pi f_0 t + \theta) \text{ are uncorrelated}) \\
 &= R_X(\tau) + E[n(t)]E[A \cos(2\pi f_0(t + \tau) + \theta)] \\
 &\quad + E[A \cos(2\pi f_0 t + \theta)]E[n(t + \tau)] + A^2 E[\cos(2\pi f_0 t + \theta)\cos(2\pi f_0(t + \tau) + \theta)] \\
 &= B\Lambda(\tau) + C + \frac{A^2}{2} \int_{-\pi}^{\pi} [\cos(4\pi f_0 t + 2\pi f_0 \tau + 2\theta) + \cos(2\pi f_0 \tau)] \frac{1}{2\pi} d\theta \\
 &= B\Lambda(\tau) + C + \frac{A^2}{2} \cos(2\pi f_0 \tau).
 \end{aligned} \tag{1}$$

Similarly, the autocorrelation function of  $Y(t)$  is

$$\begin{aligned}
 R_Y(t, t + \tau) &= E[Y(t)Y(t + \tau)] = E\{[n(t) + A \sin(2\pi f_0 t + \theta)][n(t + \tau) + A \sin(2\pi f_0(t + \tau) + \theta)]\} \\
 &= E[X(t)X(t + \tau)] + E[n(t)A \sin(2\pi f_0 t + \theta)] \\
 &\quad + E[A \sin(2\pi f_0 t + \theta)n(t + \tau)] + E[A \sin(2\pi f_0 t + \theta)A \sin(2\pi f_0(t + \tau) + \theta)] \\
 &\quad (\because n(t) \text{ and } A \sin(2\pi f_0 t + \theta) \text{ are uncorrelated}) \\
 &= R_n(\tau) + E[n(t)]E[A \sin(2\pi f_0 t + \theta)] \\
 &\quad + E[A \sin(2\pi f_0 t + \theta)]E[n(t + \tau)] + A^2 E[\sin(2\pi f_0 t + \theta)\sin(2\pi f_0(t + \tau) + \theta)] \\
 &= B\Lambda(\tau) + C + \frac{A^2}{2} \int_{-\pi}^{\pi} [\cos(2\pi f_0 \tau) - \cos(4\pi f_0 t + 2\pi f_0 \tau + 2\theta)] \frac{1}{2\pi} d\theta \\
 &= B\Lambda(\tau) + C + \frac{A^2}{2} \cos(2\pi f_0 \tau).
 \end{aligned} \tag{2}$$

2) The cross-correlation function of  $X(t)$  and  $Y(t)$  is

$$\begin{aligned}
 R_{X,Y}(t, \tau) &= E[X(t)Y(t + \tau)] = E\{[n(t) + A \cos(2\pi f_0 t + \theta)][n(t + \tau) + A \sin(2\pi f_0(t + \tau) + \theta)]\} \\
 &= E[n(t)n(t + \tau)] + E[n(t)A \sin(2\pi f_0(t + \tau) + \theta)] \\
 &\quad + E[A \cos(2\pi f_0 t + \theta)n(t + \tau)] + E[A \cos(2\pi f_0 t + \theta)A \sin(2\pi f_0(t + \tau) + \theta)] \\
 &\quad (\because n(t) \text{ and } A \cos(2\pi f_0 t + \theta) \text{ are uncorrelated, } n(t) \text{ and } A \sin(2\pi f_0 t + \theta) \text{ are uncorrelated}) \\
 &= R_n(\tau) + E[n(t)]E[A \sin(2\pi f_0(t + \tau) + \theta)] \\
 &\quad + E[A \cos(2\pi f_0 t + \theta)]E[n(t + \tau)] + A^2 E[\cos(2\pi f_0 t + \theta)\sin(2\pi f_0(t + \tau) + \theta)] \\
 &= B\Lambda(\tau) + C + \frac{A^2}{2} \int_{-\pi}^{\pi} [\sin(4\pi f_0 t + 2\pi f_0 \tau + 2\theta) + \sin(2\pi f_0 \tau)] \frac{1}{2\pi} d\theta \\
 &= B\Lambda(\tau) + C + \frac{A^2}{2} \sin(2\pi f_0 \tau).
 \end{aligned} \tag{3}$$

3) The first-order statistics of  $X(t)$

$$E[X(t)] = E[n(t) + A \cos(2\pi f_0 t + \theta)] = E[n(t)] + AE[\cos(2\pi f_0 t + \theta)] = 0 + 0 = 0, \tag{4}$$

$$E\{|X(t) - E[X(t)]|^2\} = E[X^2(t)] - E^2[X(t)] = E[X^2(t)] = R_X(t, t) = B + C + \frac{A^2}{2}, \tag{5}$$

are not dependent on  $t$ , and as obtained in 1), the second-order statistics of  $X(t)$  only depends on the gap, so  $X(t)$  is wide-sense stationary. According to Wiener-Khinchine, the power spectral density of  $X(t)$  is

$$\begin{aligned}
 S_X(t) &= \mathcal{F}[R_X(\tau)] = \mathcal{F}\left[B\Lambda(\tau) + C + \frac{A^2}{2} \cos(2\pi f_0 \tau)\right] \\
 &= B \operatorname{sinc}^2 f + C\delta(f) + \frac{A^2}{4} [\delta(f - f_0) + \delta(f + f_0)].
 \end{aligned} \tag{6}$$

Similarly, the first-order statistics of  $Y(t)$

$$E[Y(t)] = E[n(t) + A \sin(2\pi f_0 t + \theta)] = E[n(t)] + AE[\sin(2\pi f_0 t + \theta)] = 0 + 0 = 0, \tag{7}$$

$$E\{|Y(t) - E[Y(t)]|^2\} = E[Y^2(t)] - E^2[Y(t)] = E[Y^2(t)] = R_Y(t, t) = B + C + \frac{A^2}{2}, \tag{8}$$

are not dependent on  $t$ , and as obtained in 2), the second-order statistics of  $Y(t)$  only depends on the gap, so  $Y(t)$  is wide-sense stationary. According to Wiener-Khinchine theorem, the power spectral density of  $Y(t)$  is

$$\begin{aligned}
 S_Y(t) &= \mathcal{F}[R_Y(\tau)] = \mathcal{F}\left[B\Lambda(\tau) + C + \frac{A^2}{2} \cos(2\pi f_0 \tau)\right] \\
 &= B \operatorname{sinc}^2 f + C\delta(f) + \frac{A^2}{4} [\delta(f - f_0) + \delta(f + f_0)].
 \end{aligned} \tag{9}$$

4) As we obtained in 3), both  $X(t)$  and  $Y(t)$  are wide-sense stationary. The cross power of  $X(t)$  and  $Y(t)$  is

$$\begin{aligned}
 S_{XY}(f) &= \mathcal{F}[R_{XY}(\tau)] = \mathcal{F}\left[B\Lambda(\tau) + C + \frac{A^2}{2} \sin(2\pi f_0 \tau)\right] \\
 &= B \operatorname{sinc}^2 f + C\delta(f) + \frac{A^2}{4j} [\delta(f - f_0) - \delta(f + f_0)].
 \end{aligned} \tag{10}$$

5) The total power of  $X(t)$  is

$$P_X = E[X^2(t)] = R_X(\tau = 0) = B + C + \frac{A^2}{2}. \tag{11}$$

Similarly, the total power of  $Y(t)$  is

$$P_Y = E[Y^2(t)] = R_Y(\tau = 0) = B + C + \frac{A^2}{2}. \tag{12}$$

- 6) As obtained in 3), the mean of  $X(t)$  is  $E[X(t)] = 0$ , so the DC power of  $X(t)$  is

$$\overline{X(t)}^2 = E^2[X(t)] = 0. \quad (13)$$

Similarly, as obtained in 3), the mean of  $Y(t)$  is  $E[Y(t)] = 0$ , so the DC power of  $Y(t)$  is

$$\overline{Y(t)}^2 = E^2[Y(t)] = 0. \quad (14)$$

□

**Problem 2 (Gaussian random process transmission through a linear system, 30 pts) Score: \_\_\_\_\_.**

The input to a lowpass filter with impulse response  $h(t) = \exp(-10t)u(t)$  is white, Gaussian noise with two-sided power spectral density of 2 W/Hz. Obtain the following:

- 1) The mean of the output.
- 2) The power spectral density of the output.
- 3) The autocorrelation function of the output.
- 4) The probability density function of the output at an arbitrary time  $t_1$ .
- 5) The joint probability density function of the output at times  $t_1$  and  $t_1 + 2$ .
- 6) Find the noise equivalent bandwidth of the filter.

(Hint:  $\mathcal{F}[\exp(-\alpha t)u(t), \alpha > 0] = \frac{1}{\alpha + j2\pi f}$ ,  $\mathcal{F}[\exp(-\alpha |t|), \alpha > 0] = \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$ )

**Solution:**

- 1) The output is

$$Y(t) = n_w(t) * h(t) = \int_{-\infty}^{+\infty} h(\tau) n_w(t - \tau) d\tau. \quad (15)$$

The mean of the output is

$$\begin{aligned} E[Y(t)] &= E \left[ \int_{-\infty}^{+\infty} h(\tau) n_w(t - \tau) d\tau \right] \\ &= \int_{-\infty}^{+\infty} h(\tau) E[n_w(t - \tau)] d\tau \\ &\quad (\because n_w(t) \text{ is a white, Gaussian noise}) \\ &= \int_{-\infty}^{+\infty} h(\tau) \cdot 0 d\tau \\ &= 0. \end{aligned} \quad (16)$$

- 2) The spectral of the response of the lowpass filter is

$$H(f) = \mathcal{F}[h(t)] = \mathcal{F}[\exp(-10t)u(t)] = \frac{1}{10 + i2\pi f}. \quad (17)$$

The power spectral density of the output is

$$S_{n_w}(f) = |H(f)|^2 S_{n_w}(f) = \frac{1}{100 + 4\pi^2 f^2} \frac{N_0}{2} = \frac{1}{50 + 2\pi^2 f^2} \quad (\text{W/Hz}). \quad (18)$$

3) The input is white, Gaussian noise, and thus, stationary, so the autocorrelation function of the output is

$$\begin{aligned}
 R_Y(\tau) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(\tau_1)h(\tau_2)R_{n_w}(t - \tau_1 + \tau_2) d\tau_1 d\tau_2 \\
 &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(\tau_1)h(\tau_2) \frac{N_0}{2} \delta(\tau - \tau_1 + \tau_2) d\tau_1 d\tau_2 \\
 &= \frac{N_0}{2} \int_{-\infty}^{+\infty} h(\tau + \tau_2)h(\tau_2) d\tau_2 \\
 &= \frac{N_0}{2} \int_{-\infty}^{+\infty} \exp[-10(\tau + \tau_2)]u(\tau + \tau_2)\exp(-10\tau_2)u(\tau_2) d\tau_2 \\
 &= \frac{N_0}{2} \exp(-10\tau) \int_{\max\{0, -\tau\}}^{+\infty} \exp(-20\tau_2) d\tau_2 \\
 &= \frac{N_0}{2} \exp(-10\tau) \frac{1}{20} \exp[-20 \max\{0, -\tau\}] \\
 &= \frac{1}{10} \exp(-10|\tau|) \quad (\text{W}^2/\text{Hz}^2).
 \end{aligned} \tag{19}$$

4) Since the input is a Gaussian noise and the lowpass filter is a linear system, the output is a Gaussian random process. As obtained in 1), the mean of the output is 0. The variance of the output is

$$\sigma_Y^2 = E[Y^2(t)] = R_Y(0) = \frac{1}{10}. \tag{20}$$

The probability density function of the output at an arbitrary time  $t_1$  is

$$f_Y = \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left\{-\frac{1}{2\sigma_Y^2}(y - E[Y(t)])^2\right\} = \sqrt{\frac{5}{\pi}} \exp(-5y^2). \tag{21}$$

5) The autocovariance of  $Y(t)$  and  $Y(t+2)$  is

$$\sigma_{Y(t), Y(t+2)} = E\{[Y(t) - E[Y(t)]][Y(t+2) - E[Y(t+2)]]\} = E[Y(t)Y(t+2)] = R_Y(2) = \frac{1}{10} \exp(-20). \tag{22}$$

The correlation coefficient of  $Y(t)$  and  $Y(t+2)$  is

$$\rho_{Y(t), Y(t+2)} = \frac{\sigma_{Y(t), Y(t+2)}}{\sigma_{Y(t)}\sigma_{Y(t+2)}} = \frac{\frac{1}{10} \exp(-20)}{\frac{1}{10} \times \frac{1}{10}} = 10 \exp(-20). \tag{23}$$

The joint probability density function of the output at times  $t_1$  and  $t_1 + 2$  is

$$\begin{aligned}
 f_{Y(t), Y(t+2)}(y_1, y_2) &= \frac{1}{2\pi\sigma_{Y(t)}\sigma_{Y(t+2)}\sqrt{1 - \rho_{Y(t), Y(t+2)}^2}} \times \\
 &\exp\left\{-\frac{1}{2(1 - \rho_{Y(t), Y(t+2)}^2)} \left[ \frac{(y_1 - E[Y(t)])^2}{\sigma_{Y(t)}^2} - \frac{2\rho_{Y(t), Y(t+2)}(y_1 - E[Y(t)])(y_2 - E[Y(t+2)])}{\sigma_{Y(t)}\sigma_{Y(t+2)}} + \frac{(y_2 - E[Y(t+2)])^2}{\sigma_{Y(t+2)}^2} \right] \right\} \\
 &= \frac{50}{\pi\sqrt{1 - 100\exp(-40)}} \exp\left\{-\frac{100\exp(40)}{2[\exp(40) - 100]} [y_1^2 - 20\exp(-20)y_1y_2 + y_2^2]\right\}.
 \end{aligned} \tag{24}$$

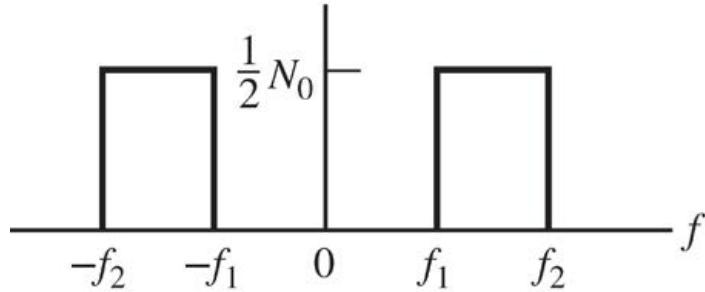
6) The noise equivalent bandwidth of the filter is

$$B_N = \frac{\int_0^{+\infty} |H(f)|^2 df}{H_0^2} = \frac{\int_0^{+\infty} \frac{1}{100+4\pi^2f^2} df}{1/100} = \frac{5}{\pi} \arctan\left(\frac{\pi}{5}x\right) \Big|_0^{+\infty} = \frac{5}{2}. \tag{25}$$

□

**Problem 3 (Narrowband noise, 40pts) Score: \_\_\_\_\_.** Noise  $n(t)$  has the power spectral density shown in the figure 1. We write  $n(t) = n_c(t) \cos(2\pi f_0 t + \theta) - n_s(t) \sin(2\pi f_0 t + \theta)$ , find and plot  $S_{n_c}(f)$ ,  $S_{n_s}(f)$  and  $S_{n_c n_s}(f)$  for the following case.

- 1)  $f_0 = f_1$
- 2)  $f_0 = f_2$
- 3)  $f_0 = (f_1 + f_2)/2$
- 4) For which of these cases are  $n_c(t)$  and  $n_s(t)$  uncorrelated.

Figure 1:  $S_n(f)$ 

**Solution:** The power spectral density of  $n(t)$  can be written as

$$S_n(f) = \frac{N_0}{2} \left[ \Pi\left(\frac{f - \frac{f_1+f_2}{2}}{f_2 - f_1}\right) + \Pi\left(\frac{f + \frac{f_1+f_2}{2}}{f_2 - f_1}\right) \right]. \quad (26)$$

- 1) For  $f_0 = f_1$ ,

$$\begin{aligned} S_{n_c}(f) &= S_{n_s}(f) = \text{LP}[S_n(f - f_0) + S_n(f + f_0)] = \frac{N_0}{2} \left[ \Pi\left(\frac{f + \frac{f_2-f_1}{2}}{f_2 - f_1}\right) + \Pi\left(\frac{f - \frac{f_2-f_1}{2}}{f_2 - f_1}\right) \right] \\ &= \frac{N_0}{2} \Pi\left(\frac{f}{2(f_2 - f_1)}\right), \end{aligned} \quad (27)$$

as shown in figure 3(a).

$$S_{n_c n_s}(f) = j \text{LP}[S_n(f - f_0) - S_n(f + f_0)] = j \frac{N_0}{2} \left[ \Pi\left(\frac{f + \frac{f_2-f_1}{2}}{f_2 - f_1}\right) - \Pi\left(\frac{f - \frac{f_2-f_1}{2}}{f_2 - f_1}\right) \right], \quad (28)$$

as shown in figure 3(b).

- 2) For  $f_0 = f_2$ ,

$$\begin{aligned} S_{n_c}(f) &= S_{n_s}(f) = \text{LP}[S_n(f - f_0) + S_n(f + f_0)] = \frac{N_0}{2} \left[ \Pi\left(\frac{f - \frac{f_2-f_1}{2}}{f_2 - f_1}\right) + \Pi\left(\frac{f + \frac{f_2-f_1}{2}}{f_2 - f_1}\right) \right] \\ &= \frac{N_0}{2} \Pi\left(\frac{f}{2(f_2 - f_1)}\right), \end{aligned} \quad (29)$$

as shown in figure 4(a).

$$S_{n_c n_s}(f) = j \text{LP}[S_n(f - f_0) - S_n(f + f_0)] = j \frac{N_0}{2} \left[ \Pi\left(\frac{f - \frac{f_2-f_1}{2}}{f_2 - f_1}\right) - \Pi\left(\frac{f + \frac{f_2-f_1}{2}}{f_2 - f_1}\right) \right], \quad (30)$$

as shown in figure 4(b).

3) For  $f_0 = \frac{f_1+f_2}{2}$ ,

$$S_{n_c}(f) = S_{n_s}(f) = \text{LP}[S_n(f - f_0) + S_n(f + f_0)] = N_0 \Pi \left( \frac{f}{f_2 - f_1} \right), \quad (31)$$

as shown in figure 5(a).

$$S_{n_c n_s}(f) = j \text{LP}[S_n(f - f_0) - S_n(f + f_0)] = 0, \quad (32)$$

as shown in figure 5(b).

- 4) For cases **1), 2) and 3)**,  $n_c(t)$  and  $n_s(t)$  are uncorrelated. This is because, for all the cases above, the cross power spectral density is pure imaginary, so that the correlation function of  $n_s(t)$  and  $n_c(t)$ ,  $R_{n_s n_c}(\tau) = \mathcal{F}^{-1}[S_{n_s n_c}(f)]$ , is an odd function and thus  $R_{n_s n_c}(0) = 0$ . Therefore, for all the cases above,  $n_c(t)$  and  $n_s(t)$  are uncorrelated.

□

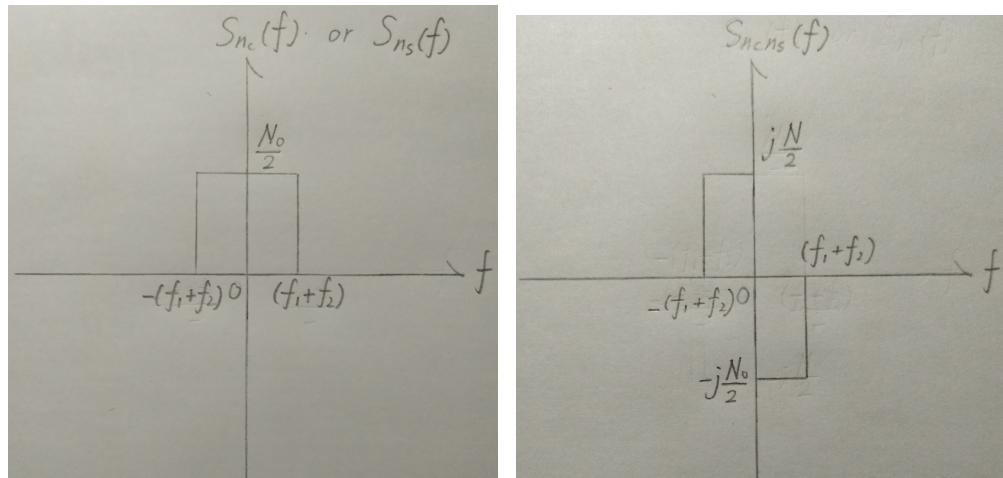
(a)  $S_{n_c}(f)$  or  $S_{n_s}(f)$ (b)  $S_{n_c n_s}(f)$ 

Figure 2: Case 1)

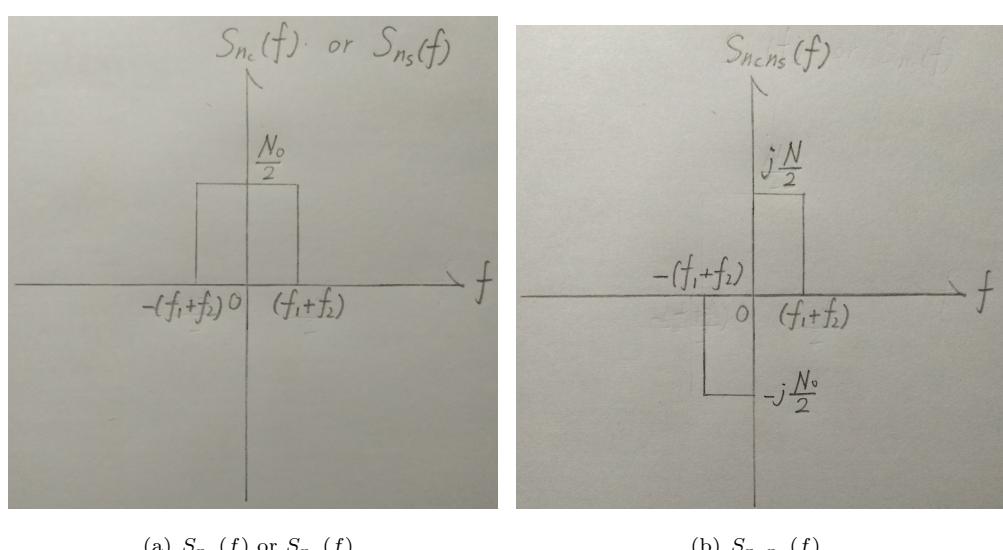
(a)  $S_{n_c}(f)$  or  $S_{n_s}(f)$ (b)  $S_{n_c n_s}(f)$ 

Figure 3: Case 2)

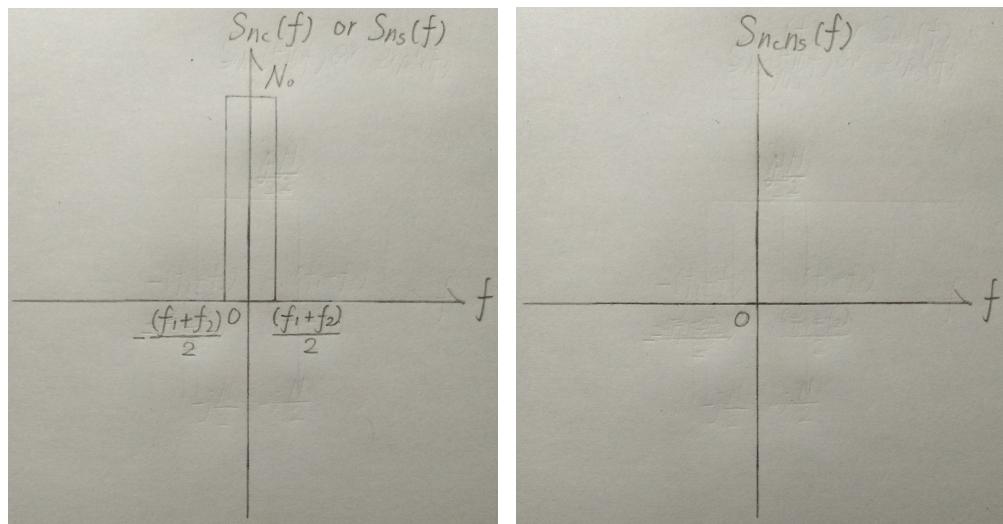
(a)  $S_{n_c}(f)$  or  $S_{n_s}(f)$ (b)  $S_{n_c n_s}(f)$ 

Figure 4: Case 3)