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Intro to Communication System
EE140
Fall, 2020

Assignment 10

Due time : 10:15, Dec 18, 2020 (Friday)

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Grade : ________

Problem 1 (5.12, Orthogonal subspace) Score: ________. For any subspace $\mathcal{S}$ of an inner product space $\mathcal{V}$, define $\mathcal{S}^\perp$ as the set of vectors $\boldsymbol{v} \in \mathcal{V}$ that are orthogonal to all $\boldsymbol{w} \in \mathcal{S}$.

(a) Show that $\mathcal{S}^\perp$ is a subspace of $\mathcal{V}$.

(b) Assuming that $\mathcal{S}$ is finite-dimensional, show that any $\boldsymbol{u} \in \mathcal{V}$ can be uniquely decomposed into $\boldsymbol{u} = \boldsymbol{u}_{|\mathcal{S}} + \boldsymbol{u}_{\perp\mathcal{S}}$, where $\boldsymbol{u}_{|\mathcal{S}} \in \mathcal{S}$ and $\boldsymbol{u}_{\perp\mathcal{S}} \in \mathcal{S}^\perp$.

(c) Assuming that $\mathcal{V}$ is finite-dimensional, show that $\mathcal{V}$ has an orthonormal basis where some of the basis vectors form a basis for $\mathcal{S}$ and the remaining basis vectors form a basis for $\mathcal{S}^\perp$.

Solution: (a) $\mathcal{S}^\perp$ satisfies the following two conditions:

(i) $\mathbf{0} \in \mathcal{S}^\perp$, since $\mathbf{0} \in \mathcal{V}$ and $\mathbf{0} \cdot \boldsymbol{w} = 0\ \forall \boldsymbol{w} = \mathcal{S}$;

(ii) If $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathcal{S}^\perp$, i.e. $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathcal{V}$ and $\boldsymbol{v}_1 \cdot \boldsymbol{w} = 0$, $\boldsymbol{v}_2 \cdot \boldsymbol{w} = 0\ \forall \boldsymbol{w} \in \mathcal{S}$, then $\alpha\boldsymbol{v}_1 + \beta\boldsymbol{v}_2 \in \mathcal{V}$ and $(\alpha\boldsymbol{v}_1 + \beta\boldsymbol{v}_2) \cdot \boldsymbol{w} = \alpha\boldsymbol{v}_1 \cdot \boldsymbol{w} + \beta\boldsymbol{v}_2 \cdot \boldsymbol{w} = 0\ \forall \boldsymbol{w} \in \mathcal{S}$, so $\alpha\boldsymbol{v}_1 + \beta\boldsymbol{v}_2 \in \mathcal{S}^\perp$, where α, β are arbitrary scalars.

Therefore, $\mathcal{S}^\perp$ is a subspace of $\mathcal{V}$.

(b) According to Projection theorem (Theorem 5.3.1), since $\mathcal{S}$ is a subspace of the inner product space $\mathcal{V}$, for any $\boldsymbol{u} \in \boldsymbol{V}$, there is a unique vector $\boldsymbol{u}_{|\mathcal{S}} \in \mathcal{S}$ such that $(\boldsymbol{u} - \boldsymbol{u}_{|\mathcal{S}}) \cdot \boldsymbol{s} = 0\ \forall \boldsymbol{s} \in \mathcal{S}$ where $\boldsymbol{u} - \boldsymbol{u}_{|\mathcal{S}} = \boldsymbol{u}_{\perp\mathcal{S}} \in \mathcal{S}^\perp$. Therefore, any u $\in \mathcal{V}$ an be uniquely decomposed into $\boldsymbol{u} = \boldsymbol{u}_{|\mathcal{S}} + \boldsymbol{u}_{\perp\mathcal{S}}$, where $\boldsymbol{u}_{|\mathcal{S}} = \mathcal{S}$ and $\boldsymbol{u}_{\perp\mathcal{S}} \in \mathcal{S}^\perp$.

(c) Since $\mathcal{S}$ is a subspace, we can find an orthonormal basis of $\mathcal{S}$, say, $\{\boldsymbol{s}_k | k = 1, 2, \cdots, n_1\}$. For any $\boldsymbol{u}_{|\mathcal{S}} \in \mathcal{S}$, we can decompose it uniquely into the linear combination of $\{\boldsymbol{s}_k | k = 1, 2, \cdots, n_1\}$: $\boldsymbol{u}_{|\mathcal{S}} = \sum_{k=1}^{n_1} \alpha_j \boldsymbol{s}_j$. Similarly, since $\mathcal{S}^\perp$ is a subspace, we can find an orthonormal basis of $\mathcal{S}^\perp$, say, $\{\boldsymbol{t}_j | j = 1, 2, \cdots, n_2\}$. For any $\boldsymbol{u}_{\perp\mathcal{S}} \in \mathcal{S}$, we can decompose it uniquely into the linear combination of $\{\boldsymbol{t}_j | j = 1, 2, \cdots, n_2\}$: $\boldsymbol{u}_{\perp\mathcal{S}} = \sum_{j=1}^{n_2} \beta_j \boldsymbol{t}_j$. Now we prove that $\{\boldsymbol{s}_k | k = 1, 2, \cdots, n_1\} \cup \{\boldsymbol{t}_j | j = 1, 2, \cdots, n_2\}$ is a orthonormal basis of $\mathcal{V}$:

(i) $\{\boldsymbol{s}_k | k = 1, 2, \cdots, n_1\} \cup \{\boldsymbol{t}_j | j = 1, 2, \cdots, n_2\}$ is a basis of $\mathcal{S}$, since for any $\boldsymbol{u} \in \mathcal{V}$, we can first decompose it uniquely into $\boldsymbol{u} = \boldsymbol{u}_{|\mathcal{S}} + \boldsymbol{u}_{\perp\mathcal{S}}$, where $\boldsymbol{u}_{|\mathcal{S}} \in \mathcal{S}$ and $\boldsymbol{u}_{\perp\mathcal{S}} \in \mathcal{S}^\perp$, and then decompose it uniquely into $\boldsymbol{u} = \sum_{k=1}^{n_1} \alpha_k \boldsymbol{s}_k + \sum_{j=1}^{n_2} \beta_j \boldsymbol{t}_j$;

(ii) $\{\boldsymbol{s}_k | k = 1, 2, \cdots, n_1\} \cup \{\boldsymbol{t}_j | j = 1, 2, \cdots, n_2\}$ is orthonormal, since $\{\boldsymbol{s}_k | k = 1, 2, \cdots, n_1\}$ is orthonormal, $\{\boldsymbol{t}_j | j = 1, 2, \cdots, n_2\}$ is orthonormal, and for any $\boldsymbol{s}_k$ and $\boldsymbol{t}_j$, the definition of $\mathcal{S}^\perp$ requires that $\boldsymbol{s}_k \cdot \boldsymbol{t}_j = 0$.

Therefore, $\mathcal{V}$ has an orthonormal basis where some of the basis vectors form a basis for $\mathcal{S}$ and the remaining basis vectors form a basis for $\mathcal{S}^\perp$.

□

Problem 2 (5.13, Othonormal expansion) Score: ________. Expand the function $\operatorname{sinc}(3t/2)$ as an orthonormal expansion in the set of functions $\{\operatorname{sinc}(t-n); -\infty < n < \infty\}$.

Solution: $\operatorname{sinc}(3t/2)$ has the bandwidth of $\frac{1}{3}$, so it is band-limited in $\frac{1}{2}$. According to Sampling Theorem,

$$\operatorname{sinc}(3t/2) = \sum_{n=-\infty}^{+\infty} \sin(3n/2) \operatorname{sinc}(t-n) \tag{1}$$

□

Problem 3 (6.3) Score: ________. (a) Assume that the received signal in a 4-PAM system is $V_k = U_k + Z_k$, where U_k is the transmitted 4-PAM signal at time k. Let Z_k be independent of U_k and Gaussian with density $f_Z(z) = \sqrt{1/2\pi}\exp(-z^2/2)$. Assume that the receiver chooses the signal $\tilde{U}_k$ closest to V_k. (It is shown in Chapter 8 that this detection rule minimizes P_e for equiprobable signals.) Find the probability P_e (in terms of Gaussian integrals) that $U_k \neq \tilde{U}_k$.

(b) Evaluate the partial derivative of P_e with respect to the third signal point a_3 (i.e. the positive inner signal point) at the point where a_3 is equal to its value $d/2$ in standard 4-PAM and all other signal points are kept at 4-PAM values. [Hint. This does not require any calculation.]

Solution: (a) Suppose the signal constellation of the 4-PAM system is $\mathcal{A} = \{a_1 = -\frac{3}{2}d, a_2 = -\frac{d}{2}, a_3 = \frac{d}{2}, a_4 = \frac{3}{2}d\}$. If $U_k = a_1$ or $U_k = a_4$, the probability that $U_k \neq \tilde{U}_k$ is

$$P_e = \int_{\frac{d}{2}}^{+\infty} f_Z(z)\,\mathrm{d}z = Q\left(\frac{d}{2}\right), \tag{2}$$

where Q-function

$$Q(x) = \int_x^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,\mathrm{d}z \tag{3}$$

If $U_k = a_2$ or $U_k = a_3$, the probability that $U_k \neq \tilde{U}_k$ is

$$P_e = \int_{-\infty}^{-\frac{d}{2}} f_Z(z)\,\mathrm{d}z + \int_{\frac{d}{2}}^{+\infty} f_Z(z)\,\mathrm{d}z = 2\int_{\frac{d}{2}}^{+\infty} f_Z(z)\,\mathrm{d}z = 2Q\left(\frac{d}{2}\right). \tag{4}$$

(b) Suppose the third signal point is moved to $a_3' = \frac{d}{2} + \epsilon$. For $U_k = a_3'$, the probability that $U_k \neq \tilde{U}_k$ becomes

$$P_e' = \int_{-\infty}^{-\frac{d+\epsilon}{2}} f_Z(z)\,\mathrm{d}z + \int_{\frac{d-\epsilon}{2}}^{+\infty} f_Z(z)\,\mathrm{d}z = \int_{\frac{d+\epsilon}{2}}^{+\infty} f_Z(z)\,\mathrm{d}z + \int_{\frac{d-\epsilon}{2}}^{+\infty} f_Z(z)\,\mathrm{d}z = Q\left(\frac{d+\epsilon}{2}\right) + Q\left(\frac{d-\epsilon}{2}\right). \tag{5}$$

so the partial derivative of P_e with respect to the third signal points a_3 is

$$\frac{\partial P_e}{\partial a_3} = \lim_{\epsilon\to 0}\frac{P_e' - P_e}{\epsilon} = \lim_{\epsilon\to 0}\frac{Q\left(\frac{d+\epsilon}{2}\right) + Q\left(\frac{d-\epsilon}{2}\right) - 2Q\left(\frac{d}{2}\right)}{\epsilon} = 0. \tag{6}$$

□

Problem 4 (6.4, Nyquist) Score: ________. Suppose that the PAM modulated baseband waveform $u(t) = \sum_{k=-\infty}^{\infty} u_k p(t-kT)$ is received. That is, $u(t)$ is known, T is known, and $p(t)$ is known. We want to determine the signals $\{u_k\}$ from $u(t)$. Assume only linear operations can be used. That is, we wish to find some waveform $d_k(t)$ for each integer k such that $\int_{-\infty}^{\infty} u(t)d_k(t)\,\mathrm{d}t = u_k$.

(a) What properties must be satisfied by $d_k(t)$ such that the above equation is satisfied no matter what values are taken by the other signals, $\cdots, u_{k-2}, u_{k-1}, u_{k+1}, u_{k+2}, \cdots$? These properties should take the from of constrains on the inner products $\langle p(t-kT), d_j(t)\rangle$. Do not worry about convergence, interchange of limits, etc.

(b) Suppose you find a function $d_0(t)$ that satisfies these constrains for $k = 0$. Shown that, for each k, a function $d_k(t)$ satisfying these constrains can be found simply in terms of $d_0(t)$.

(c) What is the relationship between $d_0(t)$ and a function $q(t)$ that avoids intersymbol interference in the approach taken in Section 6.3 (i.e. a function $q(t)$ such that $p(t) * q(t)$ is ideal Nyquist)?

You have shown that the filter/sample approach in Section 6.3 is no less general than the arbitrary linear operation approach here. Note that, in the absence of noise and with a known constellation, it must be possible to retrieve the signals from the waveform using nonlinear operations even in the presence of intersymbol interference.

Solution: (a) In order that

$$\int_{-\infty}^{+\infty} u(t)d_j(t)\,\mathrm{d}t = \int_{-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} u_k p(t-kT)d_j(t)\,\mathrm{d}t = \sum_{k=-\infty}^{+\infty} u_k \int_{-\infty}^{+\infty} p(t-kT)d_j(t)\,\mathrm{d}t = u_j \quad \forall j, \tag{7}$$

$d_k(t)$ must satisfy that

$$\langle p(t-kT), d_j(t)\rangle = \int_{-\infty}^{+\infty} p(t-kT)d_j(t)\,\mathrm{d}t = \delta_{jk}, \tag{8}$$

where

$$\delta_{jk} \equiv \begin{cases} 1, & j=k; \\ 0, & j\neq k. \end{cases} \tag{9}$$

(b) Other $d_k(t)$ can be expressed in terms of $d_0(t)$ as

$$d_k(t) = d_0(t-kT). \tag{10}$$

Here shows the reason: If $d_0(t)$ satisfies that

$$\langle p(t-kT), d_0(t)\rangle = \int_{-\infty}^{+\infty} p(t-kT)d_0(t)\,\mathrm{d}t = \delta_{0k} \quad \forall k, \tag{11}$$

then for each j,

$$\begin{aligned}\langle p(t-kT), d_j(t)\rangle &= \int_{-\infty}^{+\infty} p(t-kT)d_j(t)\,\mathrm{d}t = \int_{-\infty}^{+\infty} p(t-kT)d_0(t-jT)\,\mathrm{d}t \\ &= \int_{-\infty}^{+\infty} p(t-(k-j)T)d_0(t)\,\mathrm{d}t = \delta_{jk} \quad \forall k. \end{aligned}\tag{12}$$

(c) The filtered waveform is

$$r(t) = \int_{-\infty}^{+\infty} u(\tau)q(\tau-t)\,\mathrm{d}\tau = \int_{-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} u_k p(\tau-kT)q(t-\tau)\,\mathrm{d}\tau = \sum_{k=-\infty}^{+\infty} u_k \int_{-\infty}^{+\infty} p(\tau-kT)q(t-\tau)\,\mathrm{d}\tau.$$

In order that sampled signal

$$r(jT) = \sum_{k=-\infty}^{+\infty} u_k \int_{-\infty}^{+\infty} p(\tau-kT)q(jT-\tau)\,\mathrm{d}\tau = u_j, \tag{13}$$

$q(t)$ must satisfies that

$$\langle p(t-kT), q(jT-t)\rangle = \int p(t-kT)q(jT-t)\,\mathrm{d}\tau = \delta_{jk}. \tag{14}$$

Comparing the above equation with equation (8), the relationship between $d_j(t)$ and $q(t)$ is

$$d_j(t) = q(jT-t). \tag{15}$$

For $j=0$, we have

$$d_0(t) = q(-t). \tag{16}$$

□

Problem 5 (6.5, Nyquist) Score: ________. Let $v(t)$ be a continuous $\mathcal{L}_2$ waveform with $v(0) = 1$ and define $g(t) = v(t)\,\mathrm{sinc}\,(t/T)$.

(a) Show that $g(t)$ is ideal Nyquist with interval T.

(b) Find $\hat{g}(f)$ as a function of $\hat{v}(f)$.

(c) Give a direct demonstration that $\hat{g}(f)$ satisfies the Nyquist criterion.

(d) If $v(t)$ is baseband-limited to B_b, what is $g(t)$ baseband-limited to?

Solution: (a) Since

$$g(0) = v(0)\,\mathrm{sinc}\,(0) = 1, \tag{17}$$

and

$$g(kT) = v(kT)\,\mathrm{sinc}\,(k) = 0 \quad \forall k \neq 0, \tag{18}$$

$g(t)$ is ideal Nyquist with interval T.

(b)

$$\hat{g}(t) = \mathcal{F}[v(t)\,\mathrm{sinc}\,(t/T)] = \hat{v}(f) * T\,\mathrm{rect}\,(Tf) = T\int_{-\infty}^{+\infty} \hat{v}(s)\,\mathrm{rect}\,(T(s-f))\,\mathrm{d}s = T\int_{f-\frac{1}{2T}}^{f+\frac{1}{2T}} \hat{v}(s)\,\mathrm{d}s. \tag{19}$$

(c) Since

$$\begin{aligned}\sum_k \hat{g}\left(f+\frac{k}{T}\right)\mathrm{rect}\,(fT) =& \sum_k T\int_{f-\frac{1}{2T}+\frac{k}{T}}^{f+\frac{1}{2T}+\frac{k}{T}} \hat{v}(s)\,\mathrm{d}s\,\mathrm{rect}\,(fT) = T\,\mathrm{rect}\,(fT)\int_{-\infty}^{+\infty}\hat{v}(s)\,\mathrm{d}s\\ =& T\,\mathrm{rect}\,(fT)v(t=0) = T\,\mathrm{rect}\,(fT),\end{aligned} \tag{20}$$

$\hat{g}(f)$ satisfies the Nyquist criterion.

(d) If $v(t)$ is baseband-limited to B_b, then $g(t)$ is baseband-limited to $B_b + \frac{1}{2T}$.

□

Problem 6 (6.6, Nyquist) Score: ________. Consider a PAM baseband system in which the modulator is defined by a signal interval T and a waveform $p(t)$, the channel is defined by a filter $h(t)$, and the receiver is defined by a a filter $q(t)$ which is sampled at T-spaced intervals. The received waveform, after the receiver filter $q(t)$, is then given by $r(t) = \sum_k u_k g(t-kT)$, where $g(t) = p(t) * h(t) * q(t)$.

(a) What properties must $g(t)$ have so that $r(kT) = u_k$ for all k and for all choices of input $\{u_k\}$? What is the Nyquist criterion for $\hat{g}(f)$?

(b) Now assume that $T = 1/2$ and that $p(t)$, $h(t)$, $q(t)$ and all their Fourier transforms are restricted to be real. Assume further that $\hat{p}(f)$ and $\hat{h}(f)$ are specified by Figure 1, i.e. by

$$\hat{p}(f) = \begin{cases} 1 & |f| \leq 0.5; \\ 1.5 - t & 0.5 < |f| \leq 1.5; \\ 0 & |f| > 1.5; \end{cases} \qquad \hat{h}(f) = \begin{cases} 1 & |f| \leq 0.75; \\ 0 & 0.75 < |f| \leq 1; \\ 1 & 1 < |f| \leq 1.25; \\ 0 & |f| > 1.25. \end{cases}$$

Is it possible to choose a receiver filter transform $\hat{q}(f)$ so that there is no intersymbol interference? If so, give such a $\hat{q}(f)$ and indicate the regions in which your solution is nonunique.

Figure 1:

(c) Redo part (b) with the modification that now $\hat{h}(f) = 1$ for $|f| \leq 0.75$ and $\hat{h}(f) = 0$ for $|f| > 0.75$.

(d) Explain the conditions on $\hat{p}(f)\hat{h}(f)$ under which intersymbol interference can be avoided by proper choice of $\hat{q}(f)$. (You may assume, as above, that $\hat{p}(f)$, $\hat{h}(f)$, $p(t)$, and $h(t)$ are all real.)

Solution: (a) In order that

$$r(kT) = \sum_j u_j g((k-j)T) = u_k, \tag{21}$$

$g(t)$ must have the property that

$$g(kt) = \begin{cases} 1, & k = 0; \\ 0, & k \neq 0. \end{cases} \tag{22}$$

(b) It is possible. Since

$$g(t) = p(t) * h(t) * q(t). \tag{23}$$

we have

$$\hat{g}(f) = \hat{p}(f)\hat{h}(f)\hat{q}(f). \tag{24}$$

Now

$$\hat{p}(f)\hat{h}(f) = \begin{cases} 1, & |f| \leq \frac{1}{2}; \\ 1.5 - |f|, & \frac{1}{2} < |f| \leq \frac{3}{4}; \\ 0, & \frac{3}{4} < |f| \leq 1; \\ 1.5 - |f|, & 1 < |f| \leq \frac{5}{4}; \\ 0, & |f| > \frac{5}{4}. \end{cases} \tag{25}$$

To avoid intersymbol interference, $g(t)$ must satisfies Nyquist criterion, i.e., $g(t)$ must be band-edge symmetric

$$\hat{g}(f) + g(2-f) = T = \frac{1}{2}, \quad 0 \leq f \leq 1. \tag{26}$$

so

$$\hat{g}(f) = \begin{cases} 1, & 0 \leq |f| \leq \frac{3}{4}; \\ 0, & \frac{3}{4} < |f| \leq 1; \\ 1, & 1 < |f| \leq \frac{5}{4}; \\ 0, & |f| > \frac{5}{4}. \end{cases} \tag{27}$$

To give such a $\hat{g}(f)$, we need $\hat{q}(f)$ satisfies

$$\hat{q}(f) = \begin{cases} \frac{1}{2}, & 0 \leq |f| \leq \frac{1}{2}; \\ \frac{1}{3-2|f|}, & \frac{1}{2} < |f| \leq \frac{3}{4}; \\ \frac{1}{3-2|f|}, & 1 < |f| \leq \frac{5}{4}, \end{cases} \tag{28}$$

where $\hat{q}(f)$ is nonunique in region $\frac{3}{4} < |f| \leq 1$ and $|f| > \frac{5}{4}$. One possible $q(f)$ is

$$\hat{q}(f) = \begin{cases} \frac{1}{2}, & 0 \leq |f| \leq \frac{1}{2}; \\ \frac{1}{3-2|f|}, & \frac{1}{2} < |f| \leq \frac{5}{4}; \\ 0, & |f| > \frac{5}{4}. \end{cases} \tag{29}$$

(c) If

$$\hat{h}(f) = \begin{cases} 1, & |f| \leq \frac{3}{4}; \\ 0, & |f| > \frac{3}{4}. \end{cases} \tag{30}$$

It is impossible to find $\hat{q}(f)$ so that there is no intersymbol interference. Here is the reason: Now

$$\hat{p}(f)\hat{h}(f) = \begin{cases} 1, & 0 \leq |f| \leq \frac{1}{2}; \\ 1.5 - |f|, & \frac{1}{2} < |f| \leq \frac{3}{4}; \\ 0, & |f| > \frac{3}{4}. \end{cases}, \tag{31}$$

so

$$\hat{g}(f) = 0, \quad \text{for } |f| > \frac{3}{4}. \tag{32}$$

Therefore, no matter how $\hat{q}(f)$ be, there is no way for $\hat{g}(f)$ to be band-edge symmetric and thus no way for intersymbol interference.

(d) Condition $\hat{p}(f)\hat{h}(f)$ under which intersymbol interference can be avoid by choice of $\hat{q}(f)$: *For any* $|f| \leq 1$, *if* $\hat{p}(f)\hat{h}(f) = 0$, *then* $\hat{p}(2-f)\hat{h}(2-f) \neq 0$.

□

Problem 7 (6.16, Passband expansion) Score: ________. Prove Theorem 6.6.1. [Hint. First show that the set of functions $\{\hat{\psi}_{k,1}(f)\}$ and $\{\hat{\psi}_{k,2}(f)\}$ are orthogonal with energy 2 by comparing the integral over negative frequencies with that over positive frequencies.] Indicate explicitly why you need $f_c > B/2$.

Theorem 6.6.1 *Let* $\{\theta_k(t) : k \in \mathbb{Z}\}$ *be an orthonormal set limited to the frequency band* $[-B/2, B/2]$. *Let* f_c *be greater than* $B/2$, *and for each* $k \in \mathbb{Z}$ *let*

$$\psi_{k,1}(t) = Re\left[2\theta_k(t)e^{2\pi i f_c t}\right],$$
$$\psi_{k,2}(t) = Im\left[-2\theta_k(t)e^{2\pi i f_c t}\right].$$

The set $\{\psi_{k,i}; k \in \mathbb{Z}, i \in \{1,2\}\}$ *is an orthogonal set of functions, each with energy* 2. *Furthermore, if* $u(f) = \sum_k u_k\theta_k(t)$, *then the corresponding passband function* $x(t) = 2\,Re\left[u(t)e^{2\pi i f_c t}\right]$ *is given by*

$$x(t) = \sum_k Re\left[u_k\right]\psi_{k,1}(t) + Im\left[u_k\right]\psi_{k,2}(t).$$

Solution:

$$\psi_{k,1}(t) = \theta_k(t)e^{2\pi i f_c t} + \theta_k^*(t)e^{-2\pi i f_c t}, \tag{33}$$
$$\psi_{k,2}(t) = i[\theta_k(t)e^{2\pi i f_c t} - \theta_k^*(t)e^{2\pi i f_c t}], \tag{34}$$

have Fourier transforms

$$
\begin{aligned}
\hat{\psi_{k,1}}(f) =&\hat{\theta}_k(f-f_c)+\hat{\theta}_k^*(-f-f_c), && (35)\\
\hat{\psi_{k,2}}(f) =&i[\hat{\theta}_k(f-f_c)-\hat{\theta}_k^*(-f-f_c)]. && (36)
\end{aligned}
$$

Since $\{\theta_k(t)\}$ is an orthonormal set, according to Parseval's theorem, we have

$$
\langle\theta_k(t),\theta_j(t)\rangle=\int_{-\infty}^{+\infty}\theta_k(t)\theta_k^*(t)\,\mathrm{d}t=\int_{-\infty}^{+\infty}\hat{\theta}_k(f)\hat{\theta}_j^*(f)\,\mathrm{d}f=\langle\hat{\theta}_k(f),\hat{\theta}_j(f)\rangle=\delta_{jk}.
$$

Now let's look at the inner products of the functions in $\{\psi_{k,i};k\in\mathbb{Z},i\in\{1,2\}\}$:

(i)

$$
\begin{aligned}
\langle\psi_{k,1}(t),\psi_{j,1}(t)\rangle=&\langle\hat{\psi}_{k,1}(f),\hat{\psi}_{j,1}(f)\rangle=\int_{-\infty}^{+\infty}[\hat{\theta}_k(f-f_c)+\hat{\theta}_k^*(-f-f_c)][\hat{\theta}_j(f-f_c)+\hat{\theta}_j(-f-f_c)]^*\,\mathrm{d}f\\
=&\int_{-\infty}^{+\infty}\hat{\theta}_k(f-f_c)\hat{\theta}_j^*(f-f_c)\,\mathrm{d}f+\int_{-\infty}^{+\infty}\hat{\theta}_k(f-f_c)\hat{\theta}_j(-f-f_c)\,\mathrm{d}f\\
&+\int_{-\infty}^{+\infty}\hat{\theta}_k^*(-f-f_c)\hat{\theta}_j^*(f-f_c)\,\mathrm{d}f+\int_{-\infty}^{+\infty}\hat{\theta}_k^*(-f-f_c)\hat{\theta}_j(-f-f_c)\,\mathrm{d}f. \qquad (37)
\end{aligned}
$$

Since $\{\theta_k(t):k\in\mathbb{Z}\}$ are limited to the frequency band $[-B/2,B/2]$, $\hat{\theta}_k(f-f_c)$ and $\hat{\theta}_j^*(f-f_c)$ are band-limited to $[f_c-B/2,f_c+B/2]$, and $\hat{\theta}_k^*(-f-f_c)$ and $\hat{\theta}_j(-f-f_c)$ are band-limited to $[-f_c-B/2,-f_c+B/2]$. Since $f>B/2$, the frequency band $[-f_c-B/2,-f_c+B/2]$ and $[f_c-B/2,f_c+B/2]$ do not overlap, so the integral $\int_{-\infty}^{+\infty}\hat{\theta}_k(f-f_c)\hat{\theta}_k(-f-f_c)\,\mathrm{d}f$ and $\int_{-\infty}^{+\infty}\hat{\theta}_k^*(-f-f_c)\hat{\theta}_j^*(f-f_c)\,\mathrm{d}f$ vanishes. In this way,

$$
\begin{aligned}
\langle\psi_{k,1}(t),\psi_{j,1}(t)\rangle=&\int_{-\infty}^{+\infty}\hat{\theta}_k(f-f_c)\hat{\theta}_j^*(f-f_c)\,\mathrm{d}f+\int_{-\infty}^{+\infty}\hat{\theta}_k^*(-f-f_c)\hat{\theta}_j(-f-f_c)\,\mathrm{d}f\\
=&\int_{-\infty}^{+\infty}\hat{\theta}_k(f)\hat{\theta}_j(f)\,\mathrm{d}f+\int_{-\infty}^{+\infty}\hat{\theta}_k^*(f)\hat{\theta}_j(f)\,\mathrm{d}f\\
=&2\delta_{jk}. \qquad (38)
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\langle\psi_{k,2}(t),\psi_{j,2}(t)\rangle=&\langle\hat{\psi}_{k,2}(f),\psi_{j,2}(f)\rangle=\int_{-\infty}^{+\infty}[\hat{\theta}_k(f-f_c)-\theta_k^*(-f-f_c)][\hat{\theta}_j(f-f_c)-\theta_j^*(-f-f_c)]^*\,\mathrm{d}f\\
=&\int_{-\infty}^{+\infty}\hat{\theta}_k(f-f_c)\hat{\theta}_j^*(f-f_c)\,\mathrm{d}f-\int_{-\infty}^{+\infty}\hat{\theta}_k(f-f_c)\hat{\theta}_j(-f-f_c)\,\mathrm{d}f\\
&-\int_{-\infty}^{+\infty}\hat{\theta}_k^*(-f-f_c)\hat{\theta}_j^*(f-f_c)\,\mathrm{d}f+\int_{-\infty}^{+\infty}\hat{\theta}_k^*(-f-f_c)\hat{\theta}_j(-f-f_c)\,\mathrm{d}f. \qquad (39)
\end{aligned}
$$

where $\int_{-\infty}^{+\infty}\hat{\theta}_k^(f-f_c)\hat{\theta}_j^(-f-f_c)\,\mathrm{d}f$ and $\int_{-\infty}^{+\infty}\hat{\theta}_k^*(-f-f_c)\hat{\theta}_j^*(f-f_c)\,\mathrm{d}f$ vanishes. In this way,

$$
\begin{aligned}
\langle\psi_{k,2}(t),\psi_{j,2}(t)\rangle=&\int_{-\infty}^{+\infty}\hat{\theta}_k(f-f_c)\hat{\theta}_j^*(f-f_c)\,\mathrm{d}f+\int_{-\infty}^{+\infty}\hat{\theta}_k^*(-f-f_c)\hat{\theta}_j(-f-f_c)\,\mathrm{d}f\\
=&\int_{-\infty}^{+\infty}\hat{\theta}_k(f)\hat{\theta}_j(f)\,\mathrm{d}f+\int_{-\infty}^{+\infty}\hat{\theta}_k^*(f)\hat{\theta}_j(f)\,\mathrm{d}f\\
=&2\delta_{jk}. \qquad (40)
\end{aligned}
$$

(iii)

$$
\langle\psi_{k,1}(t),\psi_{j,2}(t)\rangle=\langle\hat{\psi}_{k,1}(f),\hat{\psi}_{j,2}(f)\rangle=-i\int_{-\infty}^{+\infty}[\hat{\theta}_k(f-f_c)+\hat{\theta}_k^*(-f-f_c)][\hat{\theta}_j(f-f_c)-\hat{\theta}_j^*(-f-f_c)]^*\,\mathrm{d}f
$$

$$
\begin{aligned}
=& -i\int_{-\infty}^{+\infty}\hat{\theta}_k(f-f_c)\hat{\theta}_k^*(f-f_c)\,\mathrm{d}f+i\int_{-\infty}^{+\infty}\hat{\theta}_k(f-f_c)\hat{\theta}_j(-f-f_c)\,\mathrm{d}f\\
&-i\int_{-\infty}^{+\infty}\hat{\theta}_k^*(-f-f_c)\hat{\theta}_j^*(f-f_c)\,\mathrm{d}f+i\int_{-\infty}^{+\infty}\hat{\theta}_k^*(-f-f_c)\hat{\theta}_j(-f-f_c)\,\mathrm{d}f\\
=& -i\delta_{jk}+0-0+i\delta_{jk}\\
=&0.
\end{aligned}
\tag{41}
$$

In general,

$$
\langle\psi_{k,m}(t),\psi_{j,n}(t)\rangle=2\delta_{jk}\delta_{mn}. \tag{42}
$$

Therefore, the set of $\{\psi_{k,i};k\in\mathbb{Z},i\in\{1,2\}\}$ is an orthogonal set of functions, each with energy 2.

(As mentioned above, only if $f_c>B/2$, can we eliminate the terms, such as $\int_{-\infty}^{+\infty}\hat{\theta}_k(f-f_c)\hat{\theta}_k(-f-f_c)\,\mathrm{d}f$ and $\int_{-\infty}^{+\infty}\hat{\theta}_k^*(-f-f_c)\hat{\theta}_j(f-f_c)\,\mathrm{d}f$, and get the above inner products. This is why we need $f_c>B/2$.)

If $u(f)=\sum_k u_k\theta_k(t)$, the corresponding passband function $x(t)$ is

$$
\begin{aligned}
x(t)=&2\,\mathrm{Re}\,[u(t)e^{2\pi i f_c t}]=2\,\mathrm{Re}\,[\sum_k u_k\theta_k(t)e^{2\pi i f_c t}]\\
=&\sum_k\{2\,\mathrm{Re}\,[u_k]\,\mathrm{Re}\,[\theta_k(t)e^{2\pi i f_c t}]-2\,\mathrm{Im}\,[u_k]\,\mathrm{Im}\,[\theta_k(t)e^{2\pi i f_c t}]\}\\
=&\sum_k\ \mathrm{Re}\,[u_k]\psi_{k,1}(t)+\mathrm{Im}\,[u_k]\psi_{k,2}(t).
\end{aligned}
\tag{43}
$$

□