# Electrodynamics Homework_2

陈稼霖 45875852

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## 1

Let

$$r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

be the distance between the point of the source $\vec{x}'$ and the point to measure the field $\vec{x}$. Furthermore, let us define $\vec{r} = \vec{x}' - \vec{x}$. Show the following relations

### 1.1

$$\nabla r = -\nabla' r = \frac{\vec{r}}{r}$$

**pf:**

$$\begin{aligned}
\nabla r =& \frac{\partial\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}{\partial x}\vec{i} + \frac{\partial\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}{\partial y}\vec{j} \\
&+ \frac{\partial\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}{\partial z}\vec{k} \\
=& \frac{x-x'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}\vec{i} + \frac{y-y'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}\vec{j} \\
&+ \frac{z-z'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}\vec{k} \\
=& \frac{(x-x')\vec{i} + (y-y')\vec{j} + (z-z')\vec{k}}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \\
=& \frac{\vec{r}}{r}
\end{aligned}$$

$$
\begin{aligned}
-\nabla' r =& -\frac{\partial\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}{\partial x'}\vec{i}-\frac{\partial\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}{\partial y'}\vec{j} \\
& -\frac{\partial\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}{\partial z'}\vec{k} \\
=&\frac{x-x'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\vec{i}+\frac{y-y'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\vec{j} \\
& +\frac{z-z'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\vec{k} \\
=&\frac{(x-x')\vec{i}+(y-y')\vec{j}+(z-z')\vec{k}}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \\
=&\frac{\vec{r}}{r}
\end{aligned}
$$

$$\therefore \nabla r = -\nabla' r = \frac{\vec{r}}{r}$$

## 1.2

$$\nabla\frac{1}{r} = -\nabla'\frac{1}{r} = -\frac{\vec{r}}{r^3}$$

**pf:**

$$
\begin{aligned}
\nabla\frac{1}{r} =& \frac{\partial\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}}{\partial x}\vec{i}+\frac{\partial\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}}{\partial y}\vec{j} \\
& +\frac{\partial\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}}{\partial z}\vec{k} \\
=& -\frac{x-x'}{[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{3}{2}}}\vec{i}-\frac{y-y'}{[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{3}{2}}}\vec{j} \\
& -\frac{z-z'}{[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{3}{2}}}\vec{k} \\
=& -\frac{(x-x')\vec{i}+(y-y')\vec{j}+(z-z')\vec{k}}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}} \\
=& -\frac{\vec{r}}{r^3}
\end{aligned}
$$

$$
\begin{aligned}
-\nabla'\frac{1}{r} =& -\frac{\partial\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}}{\partial x'}\vec{i}-\frac{\partial\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}}{\partial y'}\vec{j} \\
& -\frac{\partial\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}}{\partial z'}\vec{k} \\
=& -\frac{x-x'}{[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{3}{2}}}\vec{i}-\frac{y-y'}{[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{3}{2}}}\vec{j} \\
& -\frac{z-z'}{[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{3}{2}}}\vec{k} \\
=& -\frac{(x-x')\vec{i}+(y-y')\vec{j}+(z-z')\vec{k}}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}} \\
=& -\frac{\vec{r}}{r^3}
\end{aligned}
$$

$$\therefore \nabla\frac{1}{r} = -\nabla'\frac{1}{r} = -\frac{\vec{r}}{r^3}$$

### 1.3

$$\nabla \times \frac{\vec{r}}{r^3} = 0$$

**pf:**

$$\nabla \times \frac{\vec{r}}{r^3} = (\nabla \frac{1}{r^3}) \times \vec{r} + \frac{1}{r^3} \nabla \times \vec{r}$$

where

$$\begin{aligned}
\nabla \frac{1}{r^3} =& \frac{\partial \frac{1}{[(x-x')^2+(y-y')^+(z-z')^2]^{3/2}}}{\partial x} \vec{i} + \frac{\partial \frac{1}{[(x-x')^2+(y-y')^+(z-z')^2]^{3/2}}}{\partial y} \vec{j} \\
&+ \frac{\partial \frac{1}{[(x-x')^2+(y-y')^+(z-z')^2]^{3/2}}}{\partial z} \vec{k} \\
=& - \frac{3(x-x')}{[(x-x')^2 + (y-y')^+ (z-z')^2]^{5/2}} \vec{i} - \frac{3(y-y')}{[(x-x')^2 + (y-y')^+ (z-z')^2]^{5/2}} \vec{j} \\
&- \frac{3(z-z')}{[(x-x')^2 + (y-y')^+ (z-z')^2]^{5/2}} \vec{k} \\
=& -3 \frac{(x-x')\vec{i} + (y-y')\vec{j} + (z-z')\vec{k}}{[(x-x')^2 + (y-y')^+ (z-z')^2]^{5/2}} \\
=& - \frac{3\vec{r}}{r^5}
\end{aligned}$$

and

$$\begin{aligned}
\nabla \times \vec{r} =& (\frac{\partial(z-z')}{\partial y} - \frac{\partial(y-y')}{\partial z})\vec{i} + (\frac{\partial(x-x')}{\partial z} - \frac{\partial(z-z')}{\partial x})\vec{j} + (\frac{\partial(y-y')}{\partial x} - \frac{\partial(x-x')}{\partial y})\vec{k} \\
=& \vec{0}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\nabla \times \frac{\vec{r}}{r^3} =& -3 \frac{\vec{r}}{r^5} \times \vec{r} + \vec{r} \\
=& \vec{0}
\end{aligned}$$

### 1.4

$$\nabla \cdot \frac{\vec{r}}{r^3} = -\nabla' \cdot \frac{\vec{r}}{r^3} = 0$$

**pf:**

$$\begin{aligned}
\nabla \cdot \frac{\vec{r}}{r^3} =& (\nabla \frac{1}{r^3}) \cdot \vec{r} + \frac{1}{r^3} \nabla \cdot \vec{r} \\
=& - \frac{3\hat{r}}{r^4} \cdot \vec{r} + \frac{3}{r^3} \\
=& 0
\end{aligned}$$

where

$$\nabla\frac{1}{r^3} = -\frac{3\vec{r}}{r^5}$$

and

$$\nabla\cdot\vec{r} = \frac{\partial(x-x')}{\partial x} + \frac{\partial(y-y')}{\partial y} + \frac{\partial(z-z')}{\partial z} = 3$$

So we have

$$\begin{aligned}\nabla\cdot\frac{\vec{r}}{r^3} =& -\frac{3\hat{r}}{r^5}\cdot\vec{r} + \frac{3}{r^3}\\ =&0\end{aligned}$$

Similarly,

$$\begin{aligned}-\nabla'\cdot\frac{\vec{r}}{r^3} =& -(\nabla'\frac{1}{r^3})\cdot\vec{r} - \frac{1}{r^3}\nabla'\cdot\vec{r}\\ =& -\frac{3\hat{r}}{r^5}\cdot\vec{r} + \frac{3}{r^3}\\ =&0\end{aligned}$$

Therefore,

$$\nabla\cdot\frac{\vec{r}}{r^3} = -\nabla'\cdot\frac{\vec{r}}{r^3} = 0$$

# 2

Show that the interaction between two fixed current loops obeys Newton's third law.

**pf:** Suppose the two fixed current loops are marked as 1 and 2 respectively. The Ampere force on a short part of the current loop 2 is

$$d\vec{F}_{12} = I_2 d\vec{l}_2 \times \vec{B}_1 = I_2 d\vec{l}_2 \times (\frac{\mu_0}{4\pi}\oint_{L_1}\frac{I_1 d\vec{l}_1\times\vec{r}_{12}}{r_{12}^3})$$

The total Ampere force on the current loop 2 is

$$\begin{aligned}F_{12} =& \oint_{L_2} d\vec{F}_2 = \frac{\mu_0 I_1 I_2}{4\pi}\oint_{L_2}\oint_{L_1}\frac{d\vec{l}_2\times(d\vec{l}_1\times\vec{r}_{12})}{r_{12}^3}\\ =&\frac{\mu_0 I_1 I_2}{4\pi}\oint_{L_2}\oint_{L_1}\frac{(d\vec{l}_2\cdot\vec{r}_{12})d\vec{l}_1 - (d\vec{l}_2\cdot d\vec{l}_1)\vec{r}_{12}}{r_{12}^3}\end{aligned}$$

where

$$\begin{aligned}\oint_{L_2}\oint_{L_1}\frac{(d\vec{l}_2\cdot\vec{r}_{12})d\vec{l}_1}{r_{12}^3} =& \oint_{L_1} d\vec{l}_1\oint_{L_2}\frac{\vec{r}_{12}\cdot d\vec{l}_2}{r_{12}^3}\\ =&\oint_{L_1} d\vec{l}_1\iint_{S_2}\nabla\times(\frac{\vec{r}_12}{r_{12}^3})\cdot d\vec{S}_2\\ =&\oint_{L_1} d\vec{l}_1\iint_{S_2}\vec{0}\cdot d\vec{S}_2 = 0\end{aligned}$$

So we have

$$F_{12} = -\frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_2} \oint_{L_1} \frac{(d\vec{l_2} \cdot d\vec{l_1})\vec{r}_{12}}{r_{12}^3}$$

Similarly, the Ampere force on current loop 1 is

$$F_{21} = -\frac{\mu_0 I_2 I_1}{4\pi} \oint_{L_1} \oint_{L_2} \frac{(d\vec{l_1} \cdot d\vec{l_2})\vec{r}_{21}}{r_{21}^3} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_2} \oint_{L_1} \frac{(d\vec{l_2} \cdot d\vec{l_1})\vec{r}_{12}}{r_{12}^3}$$

Therefore,

$$\vec{F}_{12} = -\vec{F}_{21}$$

which means the interaction between two fixed current loops obeys Newton's third law.

# 3

Use the equation below to find the related equation for the conduction current $\vec{J} = n_f e\vec{v}$. Solve this equation for $\vec{E}(t) = \vec{E}_0\delta(t)$ if $\vec{J}(t < 0) = 0$. What is $\vec{J}$ immediately after $t = 0$? Connect this with the sum rule.

$$\frac{d\vec{v}}{dt} = -\gamma\vec{v} + \frac{e}{m}\vec{E}$$

**Sol**: Multiply both side of the equation with $e^{\gamma t}$ and move the items with $\vec{v}$ to the left side to get

$$\begin{aligned} e^{\gamma t}\frac{d\vec{v}}{dt} + \gamma e^{\gamma t}\vec{v} &= \frac{e}{m}e^{\gamma t}\vec{E} \\ \Longrightarrow \frac{d(e^{\gamma t}\vec{v})}{dt} &= \frac{e}{m}e^{\gamma t}\vec{E} \end{aligned}$$

Integrate both side of the equation about $t$ from $-\infty$ to $t$ to get

$$\begin{aligned} e^{\gamma t}\vec{v}(t) &= \frac{e}{m}\int_{-\infty}^{t} e^{\gamma\tau}\vec{E}(\tau)d\tau \\ \Longrightarrow \vec{v}(t) &= \frac{e}{m} \cdot e^{-\gamma t}\int_{-\infty}^{t} e^{\gamma\tau}\vec{E}(\tau)d\tau \end{aligned}$$

The related equation for the conduction current is

$$\vec{J}(t) = n_f e\vec{v}(t) = \frac{n_f e^2}{m} \cdot e^{-\gamma t}\int_{-\infty}^{t} e^{\gamma\tau}\vec{E}(\tau)d\tau$$

The electricity density immediately after $t = 0$ is

$$\vec{J}(0^+) = \frac{n_f e^2}{m} \cdot e^{-\gamma t}\int_{-\infty}^{0^+} e^{\gamma\tau}E_0\delta(\tau)d\tau = \frac{n_f e^2 E_0}{m}$$