Electrodynamics Homework 陈稼霖

1. (a) 设0时刻电荷密度分布(在以转动轴为z轴的球坐标系中)为

$$\rho(\vec{x},0)=\rho(r,\theta,\phi)$$

则t时刻电荷密度分布为

$$\rho(\vec{x},t)=\rho(r,\theta,\phi-\omega t)$$

多极矩 $q_{lm}(t)=\int r^l Y_{ml}^*(\theta,\phi)\rho(r,\theta,\phi-\omega t)d^3x$

$$\overset{\phi'=\phi-\omega t}{=\!=\!=}\int r^l Y_{ml}^*(\theta,\phi'+\omega t)\rho(r,\theta,\phi)d^3x$$

$$=\int r^l Y_{ml}^*(\theta,\phi+\omega t)\rho(r,\theta,\phi)d^3x$$

(根据球谐函数定义 $Y_{ml}^*=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{-im\phi}$)

$$=\int r^l Y_{ml}^*(\theta,\phi)e^{im\omega t}\rho(r,\theta,\phi)d^3x$$

$$=\bar{q}_{lm}e^{im\omega t}$$

其中 $\bar{q}_{lm}$ 是电荷静止时(或t=0时刻)的多极矩 $\bar{q}_{lm}=\int r^l Y_{ml}^*(\theta,\phi)\rho(\vec{x},t)d^3x$

但考虑到 $e^{im\omega t}$ 和 $e^{-im\omega t}$ 实部相等(且频率为 $m\omega$ 和 $-m\omega$ 实际上都是 $m\omega$ 的情况)

故实际的多极矩应为

$$q_{lm}(t)=\begin{cases}2\bar{q}_{lm}e^{im\omega t}, & m>0\\ \bar{q}_{lm}e^{im\omega t}=\bar{q}_{lm}, & m=0\\ 0, & m<0\end{cases}$$

(b) 对 $\rho(\vec{x},t)$ 作傅里叶级数展开

$$\rho(\vec{x},t)=\sum_{n=-\infty}^{+\infty}\rho_n(\vec{x})e^{-in\omega_0 t}$$

其中 $\rho_n(\vec{x})=\frac{1}{T}\int_0^T\rho(\vec{x},t)e^{in\omega_0 t}dt$

$\because \rho_{-n}(\vec{x})=\frac{1}{T}\int_0^T\rho(\vec{x},t)e^{i(-n)\omega_0 t}dt=\frac{1}{T}\int_0^T\rho(\vec{x},t)e^{-in\omega_0 t}dt=\rho_n^*(\vec{x})$

$\therefore \rho(\vec{x},t)=\rho_0(\vec{x})+\sum_{n=1}^{+\infty}\left[\rho_n(\vec{x})e^{-in\omega_0 t}+\rho_n^*(\vec{x})e^{in\omega_0 t}\right]$

$$=\rho_0(\vec{x})+\sum_{n=1}^{+\infty}\left[\rho_n(\vec{x})e^{-in\omega_0 t}+\left(\rho_n(\vec{x})e^{-in\omega_0 t}\right)^*\right]$$

$$=\rho_0(\vec{x})+\sum_{n=1}^{+\infty}\mathrm{Re}\left[2\rho_n(\vec{x})e^{-in\omega_0 t}\right]$$

从而可以得到与(a)中相同的多极矩

$$q_{ml}=\frac{2}{T}\int d^3x\int_0^T dt\,\rho(r,\theta,\phi-\omega_0 t)r^l Y_{lm}^*(\theta,\phi)e^{in\omega_0 t}$$

$$=\frac{2}{T}\int d^3x\int_0^T dt\,\rho(r,\theta,\phi)r^l Y_{lm}^*(\theta,\phi)e^{-in\omega_0 t}e^{in\omega_0 t}$$

$$=\delta_{m0}\left\{\int\rho(r,\theta,\phi)r^l Y_{lm}^*(\theta,\phi)d^3x\right\}+\sum_{n=1}^{+\infty}\delta_{mn}\left\{2\int\rho(r,\theta,\phi)r^l Y_{lm}^*(\theta,\phi)d^3x\right\}$$

(c) 对于这一旋转的电荷，其电荷密度分布为

$$\rho(\vec{x},t)=\frac{q}{R^2\sin\theta}\delta(r-R)\delta(\theta)\delta(\phi-\omega_0 t)$$

$$=\frac{q}{R^2}\delta(r-R)\delta(\cos\theta)\delta(\phi-\omega_0 t)$$

方法(a)：0时刻的电荷密度分布为

$$\rho(\vec{x},0)=\frac{q}{R^2}\delta(r-R)\delta(\cos\theta)\delta(\phi)$$

对应多极矩

$$\overline{q}_{lm}=\int r^l Y_{lm}^*(\theta,\phi)\rho(\vec{x},0)d^3x$$

$$=\int_0^{2\pi}\int_0^{\pi}\int_0^{+\infty} r^l Y_{lm}^*(\theta,\phi)\frac{q}{R^2}\delta(r-R)\delta(\cos\theta)\delta(\phi)r^2\sin\theta\, dr d\theta d\phi$$

$$=qR^l\int_0^{2\pi}\int_0^{\pi} Y_{lm}^*(\theta,\phi)\delta(\cos\theta)\delta(\phi)d(\cos\theta)d\phi$$

$$=qR^l Y_{lm}^*(\tfrac{\pi}{2},0)=qR^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(0)$$

含时的多极矩：

对 $l=0$：

$$q_{00}(t)=q\sqrt{\frac{1}{4\pi}\frac{(0-0)!}{(0+0)!}}P_0^0(0)=\sqrt{\frac{1}{4\pi}}q.$$

对 $l=1$：

$$q_{10}=qR\sqrt{\frac{2\times1+1}{4\pi}\frac{(1-0)!}{(1+0)!}}P_1^0(0)=0$$

$$q_{11}=2qR\sqrt{\frac{2\times1+1}{4\pi}\frac{(1-1)!}{(1+1)!}}P_1^1(0)=-\sqrt{\frac{3}{2\pi}}qR.\quad (q_{1,-1}=0).$$

方法(b) 对 $\rho(\vec{x},t)$ 作傅里叶级数展开

$$\rho(\vec{x},t)=\rho_0(\vec{x})+\sum_{n=1}^{\infty}Re[2\rho_n(\vec{x})e^{-in\omega_0 t}].$$

其中 $\rho_n(\vec{x})=\frac{\omega}{2\pi}\int_0^{2\pi/\omega_0}\frac{q}{R^2}\delta(r-R)\delta(\cos\theta)\delta(\phi-\omega_0 t)e^{in\omega_0 t}dt$

$$=\frac{q}{2\pi R^2}\delta(r-R)\delta(\cos\theta)e^{in\phi}$$

电荷密度分量 $\rho_n(\vec{x})$ 对多极矩 q_{lm} 的贡献为

$$q_{lm}[\rho_n]=\int r^l Y_{lm}^*(\theta,\phi)\rho_n(\vec{x})d^3x$$

$$=\int_0^{2\pi}\int_0^{\pi}\int_0^{+\infty} r^l Y_{lm}^*(\theta,\phi)\frac{q}{2\pi R^2}\delta(r-R)\delta(\cos\theta)e^{in\phi}r^2\sin\theta\, dr d\theta d\phi$$

$$=\frac{qR^l}{2\pi}\int_0^{\pi}\int_0^{+\infty}Y_{lm}^*(\theta,\phi)\delta(\cos\theta)e^{in\phi}d(\cos\theta)d\phi$$

$$=\frac{qR^l}{2\pi}\int_0^{+\infty}Y_{lm}^*(\tfrac{\pi}{2},\phi)e^{in\phi}d\phi$$

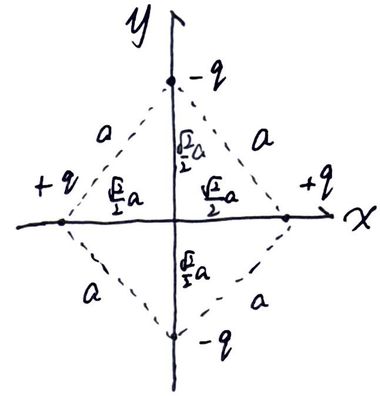
$$=qR^l Y_{lm}^*(\tfrac{\pi}{2},0)\delta_{mn}$$

同样可得 $q_{00}=qR^0 Y_{00}^*(\tfrac{\pi}{2},0)=\sqrt{\frac{1}{4\pi}}q$

$q_{10}=qR^1 Y_{10}^*(\tfrac{\pi}{2},0)=0$

$q_{11}=qR^1 Y_{11}^*(\tfrac{\pi}{2},0)+qR^1 Y_{1,-1}^*(\tfrac{\pi}{2},0)=-\sqrt{\frac{3}{2\pi}}qR\quad (q_{1,-1}=0)$

2. 设电荷在0时刻分布如图



则0时刻电荷密度分布为

$$\rho(\vec{x},0)=q\delta(z)\left\{\delta\left(x-\frac{a}{\sqrt{2}}\right)\delta(y)-\delta(x)\delta\left(y-\frac{a}{\sqrt{2}}\right)+\delta\left(x+\frac{a}{\sqrt{2}}\right)\delta(y)-\delta(x)\delta\left(y+\frac{a}{2}\right)\right\}$$

t时刻电荷密度分布变为

$$\rho(\vec{x},t)=q\delta(z)\left\{\delta\left(x-\frac{a}{\sqrt{2}}\cos\omega t\right)\delta\left(y-\frac{a}{\sqrt{2}}\sin\omega t\right)-\delta\left(x+\frac{a}{\sqrt{2}}\sin\omega t\right)\delta\left(y-\frac{a}{\sqrt{2}}\cos\omega t\right)+\delta\left(x+\frac{a}{\sqrt{2}}\cos\omega t\right)\delta\left(y+\frac{a}{\sqrt{2}}\sin\omega t\right)-\delta\left(x-\frac{a}{\sqrt{2}}\sin\omega t\right)\delta\left(y+\frac{a}{\sqrt{2}}\cos\omega t\right)\right\}.$$

四极矩

$$D_{11}=\int(3x^2-x^2-y^2-z^2)\rho(\vec{x},t)d^3x$$

$$=q\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(2x^2-y^2-z^2)\delta(z)\left\{\delta\left(x-\frac{a}{\sqrt{2}}\cos\omega t\right)\delta\left(y-\frac{a}{\sqrt{2}}\sin\omega t\right)-\delta\left(x+\frac{a}{\sqrt{2}}\sin\omega t\right)\delta\left(y-\frac{a}{\sqrt{2}}\cos\omega t\right)+\delta\left(x+\frac{a}{\sqrt{2}}\cos\omega t\right)\delta\left(y+\frac{a}{\sqrt{2}}\sin\omega t\right)-\delta\left(x-\frac{a}{\sqrt{2}}\sin\omega t\right)\delta\left(y+\frac{a}{\sqrt{2}}\cos\omega t\right)\right\}dxdydz$$

$$=3a^2q(\cos^2\omega t-\sin^2\omega t)=3a^2q\cos2\omega t$$

$$D_{12}=D_{21}=\int(3xy-x^2-y^2-z^2)\rho(\vec{x},t)d^3x$$

$$=q\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(3xy-x^2-y^2-z^2)\delta(z)\left\{\delta\left(x-\frac{a}{\sqrt{2}}\cos\omega t\right)\delta\left(y-\frac{a}{\sqrt{2}}\sin\omega t\right)-\delta\left(x+\frac{a}{\sqrt{2}}\sin\omega t\right)\delta\left(y-\frac{a}{\sqrt{2}}\cos\omega t\right)+\delta\left(x+\frac{a}{\sqrt{2}}\cos\omega t\right)\delta\left(y+\frac{a}{\sqrt{2}}\sin\omega t\right)-\delta\left(x-\frac{a}{\sqrt{2}}\sin\omega t\right)\delta\left(y+\frac{a}{\sqrt{2}}\cos\omega t\right)\right\}dxdydz.$$

$$=6aq^2\sin\omega t\cos\omega t=3aq^2\sin2\omega t.$$

$$D_{13}=D_{31}=D_{23}=D_{32}=D_{33}=0$$

$$D_{22}=\int(2y^2-x^2-z^2)\rho(\vec{x},t)d^3x$$

$$=-3a^2q\cos2\omega t.$$

$$D_{33}=\int(3z^2-x^2-y^2-z^2)\rho(\vec{x},t)d^3x$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(2z^2-x^2-y^2)\delta(z)\left\{\delta\left(x-\frac{a}{\sqrt{2}}\cos\omega t\right)\delta\left(y-\frac{a}{\sqrt{2}}\sin\omega t\right)-\delta\left(x+\frac{a}{\sqrt{2}}\sin\omega t\right)\delta\left(y-\frac{a}{\sqrt{2}}\cos\omega t\right)+\delta\left(x+\frac{a}{\sqrt{2}}\cos\omega t\right)\delta\left(y+\frac{a}{\sqrt{2}}\sin\omega t\right)-\delta\left(x-\frac{a}{\sqrt{2}}\sin\omega t\right)\delta\left(y+\frac{a}{\sqrt{2}}\cos\omega t\right)\right\}dxdydz.$$

$$=0$$

$$D_{13}=D_{31}=\int(3xz-x^2-y^2-z^2)\rho(\vec{x},t)d^3x$$

$$=q\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(3xz-x^2-y^2-z^2)\delta(z)\{\delta(x-\frac{a}{\sqrt{2}}\cos\omega t)\delta(y-\frac{a}{\sqrt{2}}\sin\omega t)$$

$$-\delta(x+\frac{a}{\sqrt{2}}\sin\omega t)\delta(y-\frac{a}{\sqrt{2}}\cos\omega t)$$

$$+\delta(x+\frac{a}{\sqrt{2}}\cos\omega t)\delta(y+\frac{a}{\sqrt{2}}\sin\omega t)$$

$$-\delta(x-\frac{a}{\sqrt{2}}\sin\omega t)\delta(y+\frac{a}{\sqrt{2}}\cos\omega t)\}dxdydz$$

$$=0$$

$$D_{23}=D_{32}=\int(3yz-x^2-y^2-z^2)\rho(\vec{x},t)d^3x$$

$$=0$$

$$\Rightarrow \boxed{\vec{D}(t)=3aq^2\begin{bmatrix}\cos 2\omega t & \sin 2\omega t & 0\\ \sin 2\omega t & -\cos 2\omega t & 0\\ 0 & 0 & 0\end{bmatrix}}$$

其复数形式为

$$\vec{D}(t)=3a^2qe^{-2i\omega t}\begin{bmatrix}1 & i & 0\\ i & -1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\because \vec{D}(t)=\vec{D}(0)e^{-2i\omega t}$$

$\boxed{\therefore \text{辐射频率为 } 2\omega}$

$$\vec{D}=\vec{e}_R\cdot\vec{D}=\begin{bmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\phi\end{bmatrix}\cdot 3a^2qe^{-2i\omega t}\begin{bmatrix}1 & i & 0\\ i & -1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$=3a^2qe^{-2i\omega t}[\sin\theta(\cos\phi+i\sin\phi)\ \ \sin\theta(i\cos\phi-\sin\phi).\ \ 0]$$

$$=3a^2q\sin\theta e^{-2i\omega t}(e^{i\phi}\hat{x}+ie^{i\phi}\hat{y}).$$

$$=3a^2q\sin\theta e^{i(\phi-2\omega t)}(\hat{x}+i\hat{y}).$$

$$\vec{e}_r\times\vec{D}=3a^2q\sin\theta e^{i(\phi-2\omega t)}\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ 1 & i & 0\end{vmatrix}$$

$$=-3a^2qi\sin\theta e^{i(\phi-2\omega t)}[\cos\theta(\hat{x}+i\hat{y})-\sin\theta e^{i\phi}\hat{z}]$$

辐射场

$$\vec{H} = -\frac{3ck^3}{24\pi}\frac{e^{ikr}}{r}\vec{e}_r\times\vec{D}$$

$$= \frac{ck^3qa^2}{8\pi}\frac{e^{ikr}}{r}\sin\theta\, e^{i(\phi-2\omega t)}\left(-\cos\theta\,\hat{x} - i\cos\theta\,\hat{y} + \sin\theta\, e^{i\phi}\hat{z}\right).$$

$$\vec{E} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\,\vec{H}\times\vec{e}_r$$

$$= \frac{\sqrt{\frac{\mu_0}{\varepsilon_0}}\,ck^3qa^2}{8\pi^2}\frac{e^{ikr}}{r}\sin\theta\, e^{i(\phi-2\omega t)}\Big[(-i\cos^2\theta - \sin^2\theta\sin\phi\, e^{i\phi})\hat{x} + (\sin^2\theta\cos\phi\, e^{i\phi} + \cos^2\theta)\hat{y} + (-\cos\theta\sin\theta\sin\phi - i\cos\theta\sin\theta\cos\phi)\hat{z}\Big].$$

$$= \frac{\sqrt{\frac{\mu_0}{\varepsilon_0}}\,ck^3qa^2}{8\pi^2}\frac{e^{ikr}}{r}\sin\theta\, e^{i(\phi-2\omega t)}\Big[(-i\cos^2\theta - \sin^2\theta\sin\phi\, e^{i\phi})\hat{x} + (\sin^2\theta\cos\phi\, e^{i\phi} + \cos^2\theta)\hat{y} - \sin\theta\cos\theta\, e^{i\phi}\hat{z}\Big]$$

功率角分布

$$\frac{dP}{d\Omega} = \frac{c^2\sqrt{\frac{\mu_0}{\varepsilon_0}}}{1152\pi^2}k^6\left[\vec{D}^*\cdot\vec{D} - |\vec{e}_r\cdot\vec{D}|^2\right]$$

$$= \frac{c^2\sqrt{\frac{\mu_0}{\varepsilon_0}}}{1152\pi^2}k^6\left[18a^4q^2\sin^2\theta - |3a^2q\sin^2\theta\, e^{2i\phi}|^2\right]$$

$$= \frac{c^2\sqrt{\frac{\mu_0}{\varepsilon_0}}\,k^6a^4q^2}{128\pi^2}\sin^2\theta(1+\cos^2\theta)$$

$$= \frac{\sqrt{\frac{\mu_0}{\varepsilon_0}}\,a^4q^2\omega^6}{2\pi^2c^4}\sin^2\theta(1+\cos^2\theta)$$

总功率

$$P = \int\frac{dP}{d\Omega}d\Omega = \int_0^{2\pi}d\phi\int_0^{\pi}\frac{\sqrt{\frac{\mu_0}{\varepsilon_0}}\,a^4q^2\omega^6}{2\pi^2c^4}\sin^2\theta(1+\cos^2\theta)\sin\theta\, d\theta$$

$$= 2\pi\int_{-1}^{1}\frac{\sqrt{\frac{\mu_0}{\varepsilon_0}}\,a^4q^2\omega^6}{2\pi^2c^4}(1-\cos^4\theta)\,d\cos\theta$$

$$= \frac{8\sqrt{\frac{\mu_0}{\varepsilon_0}}\,a^4q^2\omega^6}{5\pi c^2}$$

3. 以两半球壳顶点连线为z轴，电势满足拉普拉斯方程
球壳外

$$\nabla^2\varphi=0$$

在0时刻，其通解为

$$\varphi=\sum_{l=0}^{+\infty}\left(a_l r^l+\frac{b_l}{r^{l+1}}\right)P_l(\cos\theta)$$

由边界条件 $\varphi|_{r=+\infty}=0$ 得

$$a_l=0$$

$$\therefore \varphi=\sum_{l=0}^{+\infty}\frac{b_l}{r^{l+1}}P_l(\cos\theta)$$

其中 利用边界条件 $\varphi|_{r=R}=\begin{cases}V, & 0\le\cos\theta\le1\\ -V, & -1\le\cos\theta<0\end{cases}$

$$b_l=\frac{2l+1}{2}\int_{-1}^{1}\varphi(R,\cos\theta)P_l(\cos\theta)\,d\cos\theta$$

（由于 $\varphi(R,\cos\theta)$ 为关于 $\cos\theta$ 的奇函数）

$$=\begin{cases}0, & l\text{为偶数}\\ (2l+1)V\int_{-1}^{1}P_l(\cos\theta)\,d\cos\theta, & l\text{为奇数}\end{cases}$$

$\therefore \varphi$ 的第一项为

$$\varphi_1=\frac{b_1}{r^2}P_1(\cos\theta)$$

$$=\frac{3}{2}V\left(\frac{R}{r}\right)^2P_1(\cos\theta)=\frac{3}{2}VR^2\frac{r\cos\theta}{r^3}=\frac{3}{2}VR^2\frac{z}{r^3}$$

这就是系统的偶极矩所产生的电势 $\varphi_{偶极矩}=\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$

$$=\frac{1}{4\pi\varepsilon_0}\frac{|\vec{p}|\hat{z}\cdot\vec{r}}{r^3}$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{|\vec{p}|\,z}{r^3}$$

0时刻

$\therefore$ 偶极矩 $\vec{p}(0)=4\pi\varepsilon_0\left(\frac{3}{2}VR^2\right)\hat{z}=6\pi\varepsilon_0VR^2\hat{z}$

t时刻偶极矩 $\vec{p}(t)=6\pi\varepsilon_0VR^2e^{-i\omega t}$

磁场 $\vec{H}=\frac{ck^2}{4\pi}(\vec{r}\times\vec{p})\frac{e^{ikr}}{r}=-\frac{ck^2}{4\pi}6\pi\varepsilon_0VR^2\frac{e^{i(kr-\omega t)}}{r}\sin\theta\,\hat{\phi}$

$$=-\frac{3\sqrt{\frac{\varepsilon_0}{\mu_0}}Vk^2R^2}{2}\frac{e^{i(kr-\omega t)}}{r}\sin\theta\,\hat{\phi}$$

电场 $\vec{E} = -\sqrt{\frac{\mu_0}{\epsilon_0}}\hat{r}\times\vec{H} = -\frac{3}{2}Vk^2R^2\frac{e^{i(kr-\omega t)}}{r}\sin\theta\,\hat{\theta}$

功率角分布：$\frac{dP}{d\Omega} = \frac{1}{2}\mathrm{Re}[r^2\hat{r}\cdot\vec{E}\times\vec{H}^*]$

$$= \frac{1}{2}\mathrm{Re}\left[r^2\hat{r}\cdot\left[-\frac{3}{2}Vk^2R^2\frac{e^{i(kr-\omega t)}}{r}\sin\theta\,\hat{\theta}\right]\times\left[-\frac{3}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}V^2k^2R^2\frac{e^{-i(kr-\omega t)}}{r}\sin\theta\,\hat{\phi}\right]\right]$$

$$= \frac{9\sqrt{\frac{\epsilon_0}{\mu_0}}V^2k^4R^4}{8}\sin^2\theta$$

总功率 $P = \int\frac{9\sqrt{\frac{\epsilon_0}{\mu_0}}V^2k^4R^4}{8}\sin^2\theta\,d\Omega$

$$= \int_0^{2\pi}d\varphi\int_0^{\pi}\frac{9\sqrt{\frac{\epsilon_0}{\mu_0}}V^2k^4R^4}{8}\sin^3\theta\,d\Omega.$$

$$= 2\pi\int_{-1}^{1}\frac{9\sqrt{\frac{\epsilon_0}{\mu_0}}V^2k^4R^4}{8}(1-\cos^2\theta)\,d(\cos\theta)$$

$$= 3\pi\sqrt{\frac{\epsilon_0}{\mu_0}}V^2k^4R^4$$