Assignment 03

Due Time : 8:15, March 25, 2020 (Wednesday)

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Score : ________

Problem 1 Score: ________. Γ is a faithful representation of a non-Abelian group G. If the representation of each element in the group is transformed as in the following, determine whether the resultant set of matrices forms a representation of the group G.

(a) $\Gamma(T)^\dagger$ (Hermitian conjugate).

(b) $\Gamma(T)^t$ (transpose).

(c) $\Gamma(T)^{-1}$ (inverse).

(d) $\Gamma(T)^*$ (complex conjugate).

(e) $(\Gamma(T)^{-1})^\dagger$ (Hermitian conjugate of the inverse).

(f) $\det\Gamma(T)$ (determinant).

(g) Tr Γ (trace).

Solution: (a) $\Gamma(T)^\dagger$ does **not** form a representation of G.
Proof: For two arbitrary elements T_1 and T_2 in G, $\Gamma(T_1)^\dagger\Gamma(T_2)^\dagger=[\Gamma(T_2)\Gamma(T_1)]^\dagger=\Gamma(T_2T_1)^\dagger$. Because G is a non-Abelian group, T_2T_1 is not necessarily equal to T_1T_2. As a result, $\Gamma(T_1)^\dagger\Gamma(T_2)^\dagger=\Gamma(T_2T_1)^\dagger\neq\Gamma(T_1T_2)^\dagger$, so $\Gamma(T)\dagger$ does not form a representation of G.

(b) $\Gamma(T)^t$ does **not** form a representation of G.
Proof: For two arbitrary elements T_1 and T_2 in G, $\Gamma(T_1)^t\Gamma(T_2)^t=[\Gamma(T_2)\Gamma(T_1)]^t=\Gamma(T_2T_1)^t$. Because G is a non-Abelian group, T_2T_1 is not necessarily equal to T_1T_2. As a result, $\Gamma(T_1)^t\Gamma(T_2)^t=\Gamma(T_2T_1)^t\neq\Gamma(T_1T_2)^t$, so $\Gamma(T)^t$ does not form a representation of G.

(c) $\Gamma(T)^{-1}$ does **not** form a representation of G.
Proof: For two arbitrary elements T_1 and T_2 in G, $\Gamma(T_1)^{-1}\Gamma^{-1}(T_2)=[\Gamma(T_2)\Gamma(T_1)]^{-1}=\Gamma(T_2T_1)^{-1}$. Because G is a non-Abelian group, T_2T_1 is not necessarily equal to T_1T_2. As a result, $\Gamma(T_1)^{-1}\Gamma(T_2)^{-1}\neq\Gamma(T_2T_1)^{-1}\neq\Gamma(T_1T_2)^{-2}$, so $\Gamma(T)^{-1}$ does not form a representation of G.

(d) $\Gamma(T)^*$ forms a representation of G.
Proof: For two arbitrary elements T_1 and T_2 in G, $\Gamma(T_1)^*\Gamma(T_2)^*[\Gamma(T_1)\Gamma(T_2)]^*=\Gamma(T_1T_2)^*$, so $\Gamma(T)^*$ forms a representation of G.

(e) $(\Gamma(T)^{-1})^\dagger$ forms a representation of G.
For two arbitrary elements T_1 and T_2 in G, $(\Gamma(T_1)^{-1})^\dagger(\Gamma(T_2)^{-1})^\dagger=(\Gamma(T_2)^{-1}\Gamma(T_1)^{-1})^\dagger=((\Gamma(T_1)\Gamma(T_2))^{-1})^\dagger=[\Gamma(T_1T_2)^{-1}]^\dagger$, so $(\Gamma(T)^{-1})^\dagger$ forms a representation of G.

(f) $\det\Gamma(T)$ forms a representation of G.
Proof: For two arbitrary elements T_1 and T_2 in G, $\det(\Gamma(T_1))\det(\Gamma(T_2))=\det(\Gamma(T_1)\Gamma(T_2))=\det(\Gamma(T_1T_2))$, so $\det\Gamma(T)$ forms a representation of G.

(g) Tr $\Gamma(T)$ does **not** form a representation of G.
Proof: For two arbitrary elements T_1 and T_2 in G, Tr $(\Gamma(T_1T_2))=$ Tr $(\Gamma(T_1)\Gamma(T_2))$. Since the trace of the product is not necessarily equal to the product of the traces, Tr $(\Gamma(T_1T_2))=$ Tr $(\Gamma(T_1)\Gamma(T_2))\neq$ Tr $(\Gamma(T_1))$Tr $(\Gamma(T_2))$. As a result, Tr $\Gamma(T)$ does not form a representation of G.

□

Problem 2 Score: ________. A two-dimensional representation of $C_2=\{E,a\}$ is given by

$$\Gamma(E)=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad\Gamma(a)=\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

Find the similarity transformation that reduces the above two-dimensional representation of C_2 into the direct sum of two irreducible one-dimensional representation.

Solution: To find the similarity transformation that reduce Γ into the direct sum of two irreducible one-dimensional representation is to diagonalize Γ: $\Gamma''=S^{-1}\Gamma(T)S$. Since for any invertible matrix S, $S^{-1}\Gamma(E)S=S^{-1}S=S^{-1}S=1_2$, which is already a diagonal matrix, we only need to find S that diagonalize $\Gamma(a)$.
The characteristic equation of $\Gamma(a)$ is

$$\det[\Gamma(a)-\lambda 1_2]=\begin{vmatrix}-\lambda&1\\1&-\lambda\end{vmatrix}=\lambda^2-1=0.\tag{1}$$

Solving the above characteristic equation, we get the two eigenvalues of $\Gamma(a)$:

$$\lambda_1 = 1, \quad \lambda_2 = -1. \tag{2}$$

Suppose the eigenvector is $(x \quad y)^T$. For eigenvalue $\lambda_1 = 1$, we have

$$\Gamma(a)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3}$$

Solving the above eigenfunction and normalizing the solution, we get the eigenvector corresponding to $\lambda_1 = 1$:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{4}$$

For eigenvalue $\lambda_2 = -1$, we have

$$\Gamma(a)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_2 \begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix}. \tag{5}$$

Solving the above eigenfunction and normalizing the solution, we get the eigenvector corresponding to $\lambda_2 = -1$:

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{6}$$

We choose the transformation matrix as

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{7}$$

which is equal to its inverse:

$$S^{-1} = S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{8}$$

Now we make the following similarity transformation to reduce the original representation of C_2 into the direct sum of two irreducible one-dimensional representation:

$$\Gamma''(E) = S^{-1}\Gamma(E)S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \Gamma''_{11}(E) \oplus \Gamma''_{22}(E), \tag{9}$$

$$\Gamma''(a) = S^{-1}\Gamma(a)S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \Gamma''_{11}(a) \oplus \Gamma''_{22}(a). \tag{10}$$

where

$$\Gamma_{11}(E) = 1, \qquad \Gamma_{11}(a) = 1, \tag{11}$$

$$\Gamma_{22}(E) = 1, \qquad \Gamma_{22}(a) = -1. \tag{12}$$

□

Problem 3 Score: _________. Consider the following two-dimensional representation Γ of the group $G = \{E, a, b\}$ of order $g = 3$.

$$\Gamma(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma(a) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \quad \Gamma(b) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

(a) Check the orthogonality relation

$$\frac{1}{g}\sum_{T \in G}\Gamma(T)^*_{jk}\Gamma(T)_{st} = \frac{1}{d}\delta_{js}\delta_{kt} \tag{13}$$

for all the possible combinations of j, k, s, and t. Note that $j, k, s, t = 1, 2$ and that $d = 2$.

(b) Is the representation Γ reducible?

Solution: (a)

$$\frac{1}{g}\sum_{T \in G}\Gamma(T)^*_{11}\Gamma(T)_{11} = \frac{1}{3}[1 \times 1 + \frac{1}{2}(-1) \times \frac{1}{2}(-1) + \frac{1}{2}(-1) \times \frac{1}{2}(-1)] = \frac{1}{2} = \frac{1}{d}\delta_{11}\delta_{11}, \tag{14}$$

$$\frac{1}{g}\sum_{T\in G}\Gamma(T)^*_{11}\Gamma(T)_{12}=\frac{1}{3}[1\times 0+\frac{1}{2}(-1)\times\frac{1}{2}\sqrt{3}+\frac{1}{2}(-1)\times\frac{1}{2}(-\sqrt{3})]=0=\frac{1}{d}\delta_{11}\delta_{12},\tag{15}$$
$$\frac{1}{g}\sum_{T\in G}\Gamma(T)^*_{11}\Gamma(T)_{21}=\frac{1}{3}[1\times 0+\frac{1}{2}(-1)\times\frac{1}{2}(-\sqrt{3})+\frac{1}{2}(-1)\times\frac{1}{2}\sqrt{3}]=0=\frac{1}{d}\delta_{12}\delta_{11},\tag{16}$$
$$\frac{1}{g}\sum_{T\in G}\Gamma(T)^*_{11}\Gamma(T)_{22}=\frac{1}{3}[1\times 1+\frac{1}{2}(-1)\times\frac{1}{2}(-1)+\frac{1}{2}(-1)\times\frac{1}{2}(-1)]=\frac{1}{2}\neq\frac{1}{d}\delta_{12}\delta_{12}=0,\tag{17}$$
$$\frac{1}{g}\sum_{T\in G}\Gamma(T)^*_{12}\Gamma(T)_{11}=\frac{1}{3}[0\times 1+\frac{1}{2}\sqrt{3}\times\frac{1}{2}(-1)+\frac{1}{2}(-\sqrt{3})\times\frac{1}{2}(-1)]=0=\frac{1}{d}\delta_{11}\delta_{21},\tag{18}$$
$$\frac{1}{g}\sum_{T\in G}\Gamma(T)^*_{12}\Gamma(T)_{12}=\frac{1}{3}[0\times 0+\frac{1}{2}\sqrt{3}\times\frac{1}{2}\sqrt{3}+\frac{1}{2}(-\sqrt{3})\times\frac{1}{2}(-\sqrt{3})]=\frac{1}{2}=\frac{1}{d}\delta_{11}\delta_{22},\tag{19}$$
$$\frac{1}{g}\sum_{T\in G}\Gamma(T)^*_{12}\Gamma(T)_{21}=\frac{1}{3}[0\times 0+\frac{1}{2}\sqrt{3}\times\frac{1}{2}(-\sqrt{3})+\frac{1}{2}(-\sqrt{3})\times\frac{1}{2}\sqrt{3}]=-\frac{1}{2}\neq\frac{1}{d}\delta_{12}\delta_{21}=0,\tag{20}$$
$$\frac{1}{g}\sum_{T\in G}\Gamma(T)^*_{12}\Gamma(T)_{22}=\frac{1}{3}[0\times 1+\frac{1}{2}\sqrt{3}\times\frac{1}{2}(-1)+\frac{1}{2}(-\sqrt{3})\times\frac{1}{2}(-1)]=0=\frac{1}{d}\delta_{12}\delta_{22},\tag{21}$$
$$\frac{1}{g}\sum_{T\in G}\Gamma(T)^*_{21}\Gamma(T)_{11}=\frac{1}{3}[0\times 1+\frac{1}{2}(-\sqrt{3})\times\frac{1}{2}(-1)+\frac{1}{2}\sqrt{3}\times\frac{1}{2}(-1)]=0=\frac{1}{d}\delta_{21}\delta_{11},\tag{22}$$
$$\frac{1}{g}\sum_{T\in G}\Gamma(T)^*_{21}\Gamma(T)_{12}=\frac{1}{3}[0\times 0+\frac{1}{2}(-\sqrt{3})\times\frac{1}{2}\sqrt{3}+\frac{1}{2}\sqrt{3}\times\frac{1}{2}(-\sqrt{3})]=-\frac{1}{2}\neq\frac{1}{d}\delta_{21}\delta_{12}=0,\tag{23}$$
$$\frac{1}{g}\sum_{T\in G}\Gamma(T)^*_{21}\Gamma(T)_{21}=\frac{1}{3}[0\times 0+\frac{1}{2}(-\sqrt{3})\times\frac{1}{2}(-\sqrt{3})+\frac{1}{2}\sqrt{3}\times\frac{1}{2}\sqrt{3}]=\frac{1}{2}=\frac{1}{d}\delta_{22}\delta_{11},\tag{24}$$
$$\frac{1}{g}\sum_{T\in G}\Gamma(T)^*_{21}\Gamma(T)_{22}=\frac{1}{3}[0\times 1+\frac{1}{2}(-\sqrt{3})\times\frac{1}{2}(-1)+\frac{1}{2}\sqrt{3}\times\frac{1}{2}(-1)]=0=\frac{1}{d}\delta_{22}\delta_{12},\tag{25}$$
$$\frac{1}{g}\sum_{T\in G}\Gamma(T)^*_{22}\Gamma(T)_{11}=\frac{1}{3}[1\times 1+\frac{1}{2}(-1)\times\frac{1}{2}(-1)+\frac{1}{2}(-1)\times\frac{1}{2}(-1)]=\frac{1}{2}\neq\frac{1}{d}\delta_{21}\delta_{21}=0,\tag{26}$$
$$\frac{1}{g}\sum_{T\in G}\Gamma(T)^*_{22}\Gamma(T)_{12}=\frac{1}{g}[1\times 0+\frac{1}{2}(-1)\times\frac{1}{2}\sqrt{3}+\frac{1}{2}(-1)\times\frac{1}{2}(-\sqrt{3})]=0=\frac{1}{d}\delta_{21}\delta_{22},\tag{27}$$
$$\frac{1}{g}\sum_{T\in G}\Gamma(T)^*_{22}\Gamma(T)_{21}=\frac{1}{g}[1\times 0+\frac{1}{2}(-1)\times\frac{1}{2}(-\sqrt{3})+\frac{1}{2}(-1)\times\frac{1}{2}\sqrt{3}]=0=\frac{1}{d}\delta_{22}\delta_{21},\tag{28}$$
$$\frac{1}{g}\sum_{T\in G}\Gamma(T)^*_{22}\Gamma(T)_{22}=\frac{1}{g}[1\times 1+\frac{1}{2}(-1)\times\frac{1}{2}(-1)+\frac{1}{2}(-1)\times\frac{1}{2}(-1)]=\frac{1}{2}=\frac{1}{d}\delta_{22}\delta_{22}.\tag{29}$$

Therefore, the orthogonality relation does **not** holds for all the combinations of j, k, s, and t.

(b) Since

$$\Gamma(E)\Gamma(E)=\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\tag{30}$$
$$\Gamma(a)\Gamma(a)^\dagger=\frac{1}{2}\begin{pmatrix}-1&\sqrt{3}\\-\sqrt{3}&-1\end{pmatrix}\frac{1}{2}\begin{pmatrix}-1&-\sqrt{3}\\\sqrt{3}&-1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\tag{31}$$
$$\Gamma(b)\Gamma(b)^\dagger=\frac{1}{2}\begin{pmatrix}-1&-\sqrt{3}\\\sqrt{3}&-1\end{pmatrix}\frac{1}{2}\begin{pmatrix}-1&\sqrt{3}\\-\sqrt{3}&-1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}.\tag{32}$$

Γ is a unitary representation of G. Because the orthogonality relation does not holds for all the combinations of j, k, s, and t, the representation Γ is reducible.
Actually, we can choose transformation matrix

$$S=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\i&-i\end{pmatrix},\tag{33}$$

whose inverse is

$$S^{-1}=\frac{1}{2}\begin{pmatrix}1&-i\\1&i\end{pmatrix},\tag{34}$$

so that Γ can be transformed into a completely reducible representation

$$\begin{aligned}
\Gamma''(E) =&S^{-1}\Gamma(E)S = 1_2 = \Gamma''_{11}(E)\oplus\Gamma''_{22}(E), &(35)\\
\Gamma''(a) =&S^{-1}\Gamma(a)S = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}\frac{1}{2}\begin{pmatrix}-1 & \sqrt{3}\\ -\sqrt{3} & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-1+\sqrt{3}i & 0\\ 0 & -1-\sqrt{3}\end{pmatrix} = \Gamma''_{11}(a)\oplus\Gamma''_{22}(a), &(36)\\
\Gamma''(b) =&S^{-1}\Gamma(a)S = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}\frac{1}{2}\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-1-\sqrt{3}i & 0\\ 0 & -1+\sqrt{3}\end{pmatrix} = \Gamma''_{11}(b)\oplus\Gamma''_{22}(b). &(37)
\end{aligned}$$

where

$$\begin{aligned}
\Gamma''_{11}(E) =&1, & \Gamma''_{11}(a) =& -1+\sqrt{3}i, & \Gamma''_{22}(b) =& -1-\sqrt{3}i, &(38)\\
\Gamma''_{22}(E) =&1, & \Gamma''_{22}(a) =& -1-\sqrt{3}i, & \Gamma''_{22}(b) =& -1+\sqrt{3}i. &(39)
\end{aligned}$$

□

Problem 4 Score: _________. Show that the sum of the characters of all the elements of a finite group in an irreducible representation except the identity representation is zero.

Solution: Notations:

- G: a finite group of order g.
- Γ^1: the identity representation of G.
- Γ^p, $p\neq 1$: an arbitrary irreducible representation of G that is not equivalent to the identity representation.
- $\chi^p(T)$: the character of an element T of G in Γ^p.

The sum of the characters of all the elements of a finite group in an irreducible representation except the identity representation is

$$\begin{aligned}
\sum_{T\in G}\chi^p(T) =& \sum_{T\in G}\chi^p(T)\cdot 1\\
=&\sum_{T\in G}\chi^p(T)\chi^1(T)\\
&\text{(using orthogonality relation for characters)}\\
=&g\delta_{p1}\\
=&0. &(40)
\end{aligned}$$

□

Problem 5 Score: _________. Consider the group $G=\{E,a,b,b^2,b^3,b^4,b^5,ab,ab^2,ab,ab^2,ab^3,ab^4,ab^5\}$ with $a^2=b^6=E$ and $a^{-1}ba=b^{-1}$.

(a) Find all the elements in each class of G.

(b) Γ^1 and Γ^2 are two representation of G. In the representation Γ^1, $\Gamma^1(a)$ and $\Gamma^1(b)$ are respectively given by

$$\Gamma^1(a)=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \Gamma^1(b)=\begin{pmatrix}\omega & 0\\ 0 & \omega^{-1}\end{pmatrix}$$

with $\omega=e^{i2\pi/3}$. In the representation Γ^2, $\Gamma^2(a)$ and $\Gamma^2(b)$ are respectively given by

$$\Gamma^2(a)=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \Gamma^2(b)=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$$

Find the partial character table of G with entries only for the representation Γ^1 and Γ^2.

(c) Are the representation Γ^1 and Γ^2 equivalent?

Table 1: Multiplication table of G.

	E	a	b	b^2	b^3	b^4	b^5	ab	ab^2	ab^3	ab^4	ab^5
E	E	a	b	b^2	b^3	b^4	b^5	ab	ab^2	ab^3	ab^4	ab^5
a	a	E	ab	ab^2	ab^3	ab^4	ab^5	b	b^2	b^3	b^4	b^5
b	b	ab^5	b^2	b^3	b^4	b^5	E	a	ab	ab^2	ab^3	ab^4
b^2	b^2	ab^4	b^3	b^4	b^5	E	b	ab^5	a	ab	ab^2	ab^3
b^3	b^3	ab^3	b^4	b^5	E	b	b^2	ab^4	ab^5	a	ab	ab^2
b^4	b^4	ab^2	b^5	E	b	b^2	b^3	ab^3	ab^4	ab^5	a	ab
b^5	b^5	ab	E	b	b^2	b^3	b^4	ab^2	ab^3	ab^4	ab^5	a
ab	ab	b^5	ab^2	ab^3	ab^4	ab^5	a	E	b	b^2	b^3	b^4
ab^2	ab^2	b^4	ab^3	ab^4	ab^5	a	ab	b^5	E	b	b^2	b^3
ab^3	ab^3	b^3	ab^4	ab^5	a	ab	ab^2	b^4	b^5	E	b	b^2
ab^4	ab^4	b^2	ab^5	a	ab	ab^2	ab^3	b^3	b^4	b^5	E	b
ab^5	ab^5	b	a	ab	ab^2	ab^3	ab^4	b^2	b^3	b^4	b^5	E

(d) Is the representation Γ^1 reducible?

(e) Is the representation Γ^2 reducible?

Solution: (a) Construct the multiplication table of G, as shown in table 1. The inverse of every element in G are shown as following:

$$E^{-1} = E, \quad a^{-1} = a, \quad b^{-1} = b^5, \quad (b^2)^{-1} = b^4, \quad (b^3)^{-1} = b^3, \quad (b^4)^{-1} = b^2, \tag{41}$$
$$(b^5)^{-1} = b, \quad (ab)^{-1} = ab, \quad (ab^2) = ab^2, \quad (ab^3)^{-1} = ab^3, \quad (ab^4)^{-1} = ab^4, \quad (ab^5)^{-1} = ab^5. \tag{42}$$

Constructing a class from a: For $X = E, a, b^3, ab^3$,

$$XaX^{-1} = a. \tag{43}$$

For $X = b, b^4, ab^2, ab^5$,

$$XaX^{-1} = ab^4. \tag{44}$$

For $X = b^2, b^5, ab, ab^4$,

$$XaX^{-1} = ab^2. \tag{45}$$

The class of G constructed from a is $\{a, ab^2, ab^4\}$.
Using the similar method, we construct all the classes of G:

$$\mathcal{C}_1 = \{E\}, \quad \mathcal{C}_2 = \{a, ab^2, ab^4\}, \quad \mathcal{C}_3 = \{b, b^5\}, \quad \mathcal{C}_4 = \{b^2, b^4\}, \quad \mathcal{C}_5 = \{b^3\}, \quad \mathcal{C}_6 = \{ab, ab^3, ab^5\}.$$

(b) In the representation Γ^1,

$$\Gamma^1(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \chi^1(E) = 2, \tag{46}$$
$$\Gamma^1(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \chi^1(a) = 0, \tag{47}$$
$$\Gamma^1(b) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \qquad \chi^1(b) = -1, \tag{48}$$
$$\Gamma^1(b^2) = \Gamma^1(b)^2 = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^{-2} \end{pmatrix}, \qquad \chi^1(b^2) = -1, \tag{49}$$
$$\Gamma^1(b^3) = \Gamma^1(b)^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \chi^1(b^3) = 2, \tag{50}$$
$$\Gamma^1(b^4) = \Gamma^1(b)^4 = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \qquad \chi^1(b^4) = -1, \tag{51}$$
$$\Gamma^1(b^5) = \Gamma^1(b)^5 = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^{-2} \end{pmatrix}, \qquad \chi^1(b^5) = -1, \tag{52}$$
$$\Gamma^1(ab) = \Gamma^1(a)\Gamma^1(b) = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix}, \qquad \chi^1(ab) = 0, \tag{53}$$

$$\Gamma^1(ab^2) = \Gamma^1(a)\Gamma^1(b)^2 = \begin{pmatrix} 0 & \omega^{-2} \\ \omega^2 & 0 \end{pmatrix}, \qquad \chi^1(ab^2) = 0, \tag{54}$$

$$\Gamma^1(ab^3) = \Gamma^1(a)\Gamma^1(b)^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \chi^1(ab^3) = 0, \tag{55}$$

$$\Gamma^1(ab^4) = \Gamma^1(a)\Gamma^1(b)^4 = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix}, \qquad \chi^1(ab^2) = 0, \tag{56}$$

$$\Gamma^1(ab^5) = \Gamma^1(a)\Gamma^1(b)^5 = \begin{pmatrix} 0 & \omega^{-2} \\ \omega^2 & 0 \end{pmatrix}, \qquad \chi^1(ab^2) = 0. \tag{57}$$

In the representation Γ^2,

$$\Gamma^2(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \chi^2(E) = 2, \tag{58}$$

$$\Gamma^2(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \chi^2(a) = 0, \tag{59}$$

$$\Gamma^2(b) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \chi^2(b) = 0, \tag{60}$$

$$\Gamma^2(b^2) = \Gamma^2(b)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \chi^2(b^2) = 2, \tag{61}$$

$$\Gamma^2(b^3) = \Gamma^2(b)^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \chi^2(b^3) = 0, \tag{62}$$

$$\Gamma^2(b^4) = \Gamma^2(b)^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \chi^2(b^4) = 2, \tag{63}$$

$$\Gamma^2(b^5) = \Gamma^2(b)^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \chi^2(b^5) = 0, \tag{64}$$

$$\Gamma^2(ab) = \Gamma^2(a)\Gamma^2(b) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \chi^2(ab) = -2, \tag{65}$$

$$\Gamma^2(ab^2) = \Gamma^2(a)\Gamma^2(b)^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \chi^2(ab^2) = 0, \tag{66}$$

$$\Gamma^2(ab^3) = \Gamma^2(a)\Gamma^2(b)^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \chi^2(ab^3) = -2, \tag{67}$$

$$\Gamma^2(ab^4) = \Gamma^2(a)\Gamma^2(b)^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \chi^2(ab^2) = 0, \tag{68}$$

$$\Gamma^2(ab^5) = \Gamma^2(a)\Gamma^2(b)^5 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \chi^2(ab^2) = -2. \tag{69}$$

The partial character table of G with entries only for the representation Γ^1 and Γ^2 is shown in table 2.

Table 2: Partial character table of G.

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$
Γ^1	2	0	-1	-1	2	0
Γ^2	2	0	0	2	0	-2

(c) As shown above, $\Gamma^2(T)$'s are all diagonal matrices, so Γ is a completely reducible representation of G. Let's try diagonalizing Γ^1 and see if it is the same with Γ^2. The characteristic equation of $\Gamma^1(a)$ is

$$\det(\Gamma^1(a) - \lambda 1_2) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0. \tag{70}$$

Solving the above characteristic equation, we get the two eigenvalues of $\Gamma^1(a)$:

$$\lambda_1 = 1, \quad \lambda_2 = -1. \tag{71}$$

Suppose the eigenvector is $\begin{pmatrix} x & y \end{pmatrix}^T$. For eigenvalue $\lambda_1 = 1$, we have

$$\Gamma^1(a)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{72}$$

Solving the above eigenfunction and normalizing the solution, we get the eigenvector corresponding to $\lambda_1 = 1$:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{73}$$

For eigenvalue $\lambda_2 = -1$, we have

$$\Gamma^1(a) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}. \tag{74}$$

Solving the above eigenfunction and normalizing the solution, we get the eigenvector corresponding to $\lambda_2 = -1$:

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{75}$$

We choose the transformation matrices as

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{76}$$

which is equal to its inverse:

$$S^{-1} = S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{77}$$

By making the following transformation

$$\begin{aligned} \Gamma^{1''}(a) =& S^{-1}\Gamma^1(a)S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & (78) \\ \Gamma^{1''}(b) =& S^{-1}\Gamma^1(b)S = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, & (79) \\ \Gamma^{1''}(ab) =& S^{-1}\Gamma^1(b)S = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix} & (80) \end{aligned}$$

we find that $\Gamma^{1''}(b) \neq \Gamma^2(b)$ and $\Gamma^1(ab)$ can even not be transformed in the form of a reducible representation, so the representation Γ^1 and Γ^2 is **not** equivalent.

(d) As shown in (c), Γ^1 is **not** reducible.

(e) As mentioned in (c), Γ^2 is a completely reducible representation of G.

□