Group Theory

Solutions to the Problems in Homework Assignment 12

Spring, 2020

The analytic homomorphic mapping ϕ of SU(2) onto SO(3) is given by

$$\phi(u)_{jk} = \frac{1}{2}\,\mathrm{tr}\big(\sigma_j u \sigma_k u^{-1}\big),\ u \in \mathrm{SU}(2),\ j,k = 1,2,3.$$

Note that the Pauli matrices σ_1, σ_2, and σ_3 are also denoted respectively by σ_x, σ_y, and σ_z.

1. Verify explicitly that a rotation about the Oz-axis is indeed obtained from the given mapping for $u = e^{-i\theta\sigma_z/2}$.

Making use of $\sigma_\ell^2 = 1$ with $\ell = 1$, 2 or 3, we have

$$u_\ell = e^{-i\theta\sigma_\ell/2} = \cos(\theta/2) - i\sigma_\ell \sin(\theta/2).$$

We then have

$$\begin{aligned}
\phi(u_\ell)_{jk} &= \frac{1}{2}\,\mathrm{tr}\big\{\sigma_j\big[\cos(\theta/2) - i\sigma_\ell\sin(\theta/2)\big]\sigma_k\big[\cos(\theta/2) + i\sigma_\ell\sin(\theta/2)\big]\big\}\\
&= \frac{1}{2}\,\mathrm{tr}\big\{\big[\sigma_j\cos(\theta/2) - i\sigma_j\sigma_\ell\sin(\theta/2)\big]\big[\sigma_k\cos(\theta/2) + i\sigma_k\sigma_\ell\sin(\theta/2)\big]\big\}\\
&= \frac{1}{2}\,\mathrm{tr}\big[\sigma_j\sigma_k\cos^2(\theta/2) + i\sigma_j\sigma_k\sigma_\ell\sin(\theta/2)\cos(\theta/2) - i\sigma_j\sigma_\ell\sigma_k\sin(\theta/2)\cos(\theta/2) + \sigma_j\sigma_\ell\sigma_k\sigma_\ell\sin^2(\theta/2)\big].
\end{aligned}$$

For $\mathrm{tr}(\sigma_j\sigma_k)$, we have

$$\mathrm{tr}(\sigma_j\sigma_k) = 2\delta_{jk}.$$

For $\mathrm{tr}(\sigma_j\sigma_k\sigma_\ell)$, we have

$$\mathrm{tr}(\sigma_j\sigma_k\sigma_\ell) = i\sum_{m=1}^{3}\epsilon_{jkm}\,\mathrm{tr}(\sigma_m\sigma_\ell) = 2i\sum_{m=1}^{3}\epsilon_{jkm}\delta_{m\ell} = 2i\epsilon_{jk\ell}.$$

For $\mathrm{tr}(\sigma_j\sigma_\ell\sigma_k)$, we have

$$\mathrm{tr}(\sigma_j\sigma_\ell\sigma_k) = 2i\epsilon_{j\ell k} = -2i\epsilon_{jk\ell}.$$

For $\mathrm{tr}(\sigma_j\sigma_\ell\sigma_k\sigma_\ell)$, we have

$$\mathrm{tr}(\sigma_j\sigma_\ell\sigma_k\sigma_\ell) = \delta_{k\ell}\,\mathrm{tr}(\sigma_j\sigma_k) - (1-\delta_{k\ell})\,\mathrm{tr}(\sigma_j\sigma_k) = 2\delta_{jk}(2\delta_{k\ell} - 1).$$

We thus have

$$\phi(u_\ell)_{jk} = \big[\cos^2(\theta/2)\delta_{jk} - \sin\theta\,\epsilon_{jk\ell} + \sin^2(\theta/2)\delta_{jk}(2\delta_{k\ell} - 1)\big].$$

For $\ell = 3$, we have

$$\phi(u_3)_{jk} = \big[\cos^2(\theta/2)\delta_{jk} - \sin\theta\,\epsilon_{jk3} + \sin^2(\theta/2)\delta_{jk}(2\delta_{k3} - 1)\big].$$

$\phi(u_3)$ is then given by

$$\phi(u_3) = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}.$$

The above transformation matrix is for a rotation about the z-axis through an angle of θ. Therefore, a rotation about the Oz-axis is indeed obtained from the given mapping for $u = e^{-i\theta\sigma_z/2}$. Note that the rotation here is an active rotation.

2. Verify explicitly that a rotation about the Oy-axis is indeed obtained from the given mapping for $u = e^{-i\theta\sigma_y/2}$.

For $\ell = 2$, we have

$$\phi(u_2)_{jk} = \left[\cos^2(\theta/2)\delta_{jk} - \sin\theta\epsilon_{jk2} + \sin^2(\theta/2)\delta_{jk}(2\delta_{k2} - 1)\right].$$

$\phi(u_2)$ is then given by

$$\phi(u_2) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}.$$

The above transformation matrix is for a rotation about the y-axis through an angle of θ. Therefore, a rotation about the Oy-axis is indeed obtained from the given mapping for $u = e^{-i\theta\sigma_y/2}$. Again, the rotation here is an active rotation.

For completeness, here we also consider the mapping for $u = e^{-i\theta\sigma_x/2}$. For $\ell = 1$, we have

$$\phi(u_1)_{jk} = \left[\cos^2(\theta/2)\delta_{jk} - \sin\theta\epsilon_{jk1} + \sin^2(\theta/2)\delta_{jk}(2\delta_{k1} - 1)\right].$$

$\phi(u_1)$ is then given by

$$\phi(u_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

The above transformation matrix is for a rotation about the x-axis through an angle of θ. Therefore, a rotation about the Ox-axis is indeed obtained from the given mapping for $u = e^{-i\theta\sigma_x/2}$. Again, the rotation here is an active rotation.

3. Show that the analytic isomorphic mapping of the real Lie algebra su(2) onto the real Lie algebra so(3) is given by

$$\psi(a)_{jk} = \frac{1}{2}\operatorname{tr}\big(\sigma_j[a, \sigma_k]\big),\ a \in \mathrm{su}(2),\ j, k = 1, 2, 3.$$

For $u = e^{ta}$, we have $u^{-1} = e^{-ta}$ and

$$\phi(e^{ta})_{jk} = \frac{1}{2}\operatorname{tr}\big(\sigma_j e^{ta}\sigma_k e^{-ta}\big).$$

Differentiating $\phi(e^{ta})_{jk}$ with respect to t and then taking the $t \to 0$ limit, we obtain

$$\begin{aligned}
\psi(a)_{jk} &= \lim_{t\to 0}\frac{d\phi(e^{ta})_{jk}}{dt} \\
&= \frac{1}{2}\lim_{t\to 0}\operatorname{tr}\big(\sigma_j e^{ta} a\sigma_k e^{-ta} - \sigma_j e^{ta}\sigma_k a e^{-ta}\big) \\
&= \frac{1}{2}\operatorname{tr}\big(\sigma_j a\sigma_k - \sigma_j\sigma_k a\big) \\
&= \frac{1}{2}\operatorname{tr}\big(\sigma_j[a, \sigma_k]\big).
\end{aligned}$$

4. Without using the explicit matrix representations of the Pauli matrices, show that $\psi(a_p)_{jk} = \epsilon_{pjk}$ for $a_p = i\sigma_p/2$ with $p = 1, 2, 3$.

Hint: $[\sigma_p, \sigma_k] = 2i\sum_{\ell=1}^{3}\epsilon_{pk\ell}\sigma_\ell$.

For $a = a_p = i\sigma_p/2$, we have

$$\psi(a_p)_{jk} = \frac{i}{4}\,\mathrm{tr}\big(\sigma_j[a_p, \sigma_k]\big),\ a \in \mathrm{su}(2)$$

Making use of $[\sigma_p, \sigma_k] = 2i\sum_{\ell=1}^{3}\epsilon_{pk\ell}\sigma_\ell$, we have

$$\psi(a_p)_{jk} = -\frac{1}{2}\sum_{\ell=1}^{3}\epsilon_{pk\ell}\,\mathrm{tr}\big(\sigma_j\sigma_\ell\big) = -\frac{1}{2}\sum_{\ell=1}^{3}\epsilon_{pk\ell}\cdot 2\delta_{j\ell} = -\epsilon_{pkj} = \epsilon_{pjk}.$$

5. Using the definition of ϵ_{pjk}, show that

$$\psi(a_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},\ \psi(a_2) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ \psi(a_3) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can obtain the matrix elements by making use of the antisymmetric property of ϵ_{pjk}. For $\psi(a_1)$, we have

$$\begin{aligned}
&\psi(a_1)_{11} = \epsilon_{111} = 0,\ \psi(a_1)_{12} = \epsilon_{112} = 0,\ \psi(a_1)_{13} = \epsilon_{113} = 0,\\
&\psi(a_1)_{21} = \epsilon_{121} = 0,\ \psi(a_1)_{22} = \epsilon_{122} = 0,\ \psi(a_1)_{23} = \epsilon_{123} = 1,\\
&\psi(a_1)_{31} = \epsilon_{131} = 0,\ \psi(a_1)_{32} = \epsilon_{132} = -1,\ \psi(a_1)_{33} = \epsilon_{133} = 0.
\end{aligned}$$

Thus,

$$\psi(a_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

For $\psi(a_2)$, we have

$$\begin{aligned}
&\psi(a_2)_{11} = \epsilon_{211} = 0,\ \psi(a_2)_{12} = \epsilon_{212} = 0,\ \psi(a_2)_{13} = \epsilon_{213} = -1,\\
&\psi(a_2)_{21} = \epsilon_{221} = 0,\ \psi(a_2)_{22} = \epsilon_{222} = 0,\ \psi(a_2)_{23} = \epsilon_{223} = 0,\\
&\psi(a_2)_{31} = \epsilon_{231} = 1,\ \psi(a_2)_{32} = \epsilon_{232} = 0,\ \psi(a_2)_{33} = \epsilon_{233} = 0.
\end{aligned}$$

Thus,

$$\psi(a_2) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

For $\psi(a_3)$, we have

$$\begin{aligned}
&\psi(a_3)_{11} = \epsilon_{311} = 0,\ \psi(a_3)_{12} = \epsilon_{312} = 1,\ \psi(a_3)_{13} = \epsilon_{313} = 0,\\
&\psi(a_3)_{21} = \epsilon_{321} = -1,\ \psi(a_3)_{22} = \epsilon_{322} = 0,\ \psi(a_3)_{23} = \epsilon_{323} = 0,\\
&\psi(a_3)_{31} = \epsilon_{331} = 0,\ \psi(a_3)_{32} = \epsilon_{332} = 0,\ \psi(a_3)_{33} = \epsilon_{333} = 0.
\end{aligned}$$

Thus,

$$\psi(a_3) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$