Group Theory

Solutions to the Problems in Homework Assignment 05

Spring, 2020

1. The elements of the group $G_1 = \{E, a_2, a_3, \cdots, a_{g_1}\}$ commute with the elements of the group $G_2 = \{E, b_2, b_3, \cdots, b_{g_2}\}$. That is, $a_i b_j = b_j a_i$ for $i = 1, 2, \cdots, g_1$ and $j = 1, 2, \cdots, g_2$. Here $a_1 = E$ and $b_1 = E$. Show that the direct product of G_1 and G_2, $G_1 \otimes G_2 = \{a_i b_j; i = 1, 2, \cdots, g_1, j = 1, 2, \cdots, g_2\}$, is a group.

(i) **Closure.** For the product of $a_i b_j$ and $a_k b_\ell \in G_1 \otimes G_2$, we have

$$(a_i b_j)(a_k b_\ell) = (a_i a_k)(b_j b_\ell) = a_m b_n \in G_1 \otimes G_2,$$

where we have made use of the facts that $a_i a_k = a_m \in G_1$ and that $b_j b_\ell = b_n \in G_2$.

(ii) **Associativity.** In consideration that $a_i b_j = b_j a_i$, we have

$$\begin{aligned}[(a_i b_j)(a_k b_\ell)](a_m b_n) &= (a_i a_k)(b_j b_\ell)(a_m b_n) = [(a_i a_k) a_m][(b_j b_\ell) b_n] = [a_i (a_k a_m)][b_j (b_\ell b_n)] \\ &= (a_i b_j)[(a_k a_m)(b_\ell b_n)] = (a_i b_j)[(a_k b_\ell)(a_m b_n)],\end{aligned}$$

where we have made use of the associativity laws for G_1 and G_2, respectively.

(iii) **Identity element**. Let E_a and E_b be the identity elements of G_1 and G_2, respectively. Making use of the fact that $a_i b_j = b_j a_i$, we have

$$(a_i b_j)(E_a E_b) = (a_i E_a)(b_j E_b) = (E_a a_i)(E_b b_j) = (E_a E_b)(a_i b_j).$$

Thus, $E_a E_b = E_b E_a$ is the identity element of $G_1 \otimes G_2$.

(iv) **Inverse elements**. Making use of the fact that $a_i b_j = b_j a_i$, we have

$$(a_i b_j)^{-1} = b_j^{-1} a_i^{-1} = a_i^{-1} b_j^{-1}.$$

Since

$$(a_i b_j)^{-1}(a_i b_j) = b_j^{-1}(a_i^{-1} a_i) b_j = b_j^{-1} E_a b_j = E_a (b_j^{-1} b_j) = E_a E_b$$

and

$$(a_i b_j)(a_i b_j)^{-1} = a_i (b_j b_j^{-1}) a_i^{-1} = a_i E_b a_i^{-1} = E_b (a_i a_i^{-1}) = E_b E_a = E_a E_b,$$

$b_j^{-1} a_i^{-1} = a_i^{-1} b_j^{-1}$ is the inverse of the element $a_i b_j$.

2. Show that if two matrices A and B are orthogonal, then their direct product $A \otimes B$ is also an orthogonal matrix.

Since A and B are orthogonal matrices, we have

$$AA^t = A^t A = 1, \; BB^t = B^t B = 1$$

or

$$\sum_k A_{ik} A_{jk} = \sum_k A_{ki} A_{kj} = \delta_{ij}, \; \sum_k B_{ik} B_{jk} = \sum_k B_{ki} B_{kj} = \delta_{ij}.$$

For $(A \otimes B)(A \otimes B)^t$, we have

$$\begin{aligned}\left[(A \otimes B)(A \otimes B)^t\right]_{ij,k\ell} &= \sum_{mn} (A \otimes B)_{ij,mn} (A \otimes B)^t_{mn,k\ell} = \sum_{mn} (A \otimes B)_{ij,mn} (A \otimes B)_{k\ell,mn} \\ &= \sum_{mn} A_{im} B_{jn} A_{km} B_{\ell n} = \left(\sum_m A_{im} A_{km}\right)\left(\sum_n B_{jn} B_{\ell n}\right) = \delta_{ik}\delta_{j\ell}.\end{aligned}$$

For $(A \otimes B)^t (A \otimes B)$, we have

$$\begin{aligned}\left[(A \otimes B)^t (A \otimes B)\right]_{ij,k\ell} &= \sum_{mn} (A \otimes B)^t_{ij,mn} (A \otimes B)_{mn,k\ell} = \sum_{mn} (A \otimes B)_{mn,ij} (A \otimes B)_{mn,k\ell} \\ &= \sum_{mn} A_{mi} B_{nj} A_{mk} B_{n\ell} = \left(\sum_m A_{mi} A_{mk} \right) \left(\sum_n B_{nj} B_{n\ell} \right) = \delta_{ik} \delta_{j\ell}.\end{aligned}$$

Thus, $A \otimes B$ is also an orthogonal matrix.

3. The character table of D_3 is given by

	$C_1 = \{E\}$	$C_2 = \{D, F\}$	$C_3 = \{A, B, C\}$
Γ^1	1	1	1
Γ^2	1	1	-1
Γ^3	2	-1	0

Find the character table of $D_3 \otimes D_3$.

We first find the classes of $D_3 \otimes D_3$. Let Γ^1 be an irreducible representation of D_3 on the left in $D_3 \otimes D_3$. Let Γ^2 be an irreducible representation of D_3 on the right in $D_3 \otimes D_3$. Let Γ be an irreducible representation of $D_3 \otimes D_3$. For $a \in D_3$ on the left in $D_3 \otimes D_3$ and $b \in D_3$ on the right in $D_3 \otimes D_3$, we have

$$\Gamma((a, b)) = \Gamma^1(a) \otimes \Gamma^2(b).$$

From

$$\begin{aligned}\Gamma((c, d))^{-1} \Gamma((a, b)) \Gamma((c, d)) &= \Gamma((c^{-1}, d^{-1})) \Gamma((a, b)) \Gamma((c, d)) \\ &= \left[\, \Gamma^1(c^{-1}) \otimes \Gamma^2(d^{-1}) \,\right] \left[\, \Gamma^1(a) \otimes \Gamma^2(b) \,\right] \left[\, \Gamma^1(c) \otimes \Gamma^2(d) \,\right] \\ &= \left[\, \Gamma^1(c^{-1}) \Gamma^1(a) \Gamma^1(c) \,\right] \otimes \left[\, \Gamma^2(d^{-1}) \Gamma^2(b) \Gamma^2(d) \,\right],\end{aligned}$$

we see that the classes of $D_3 \otimes D_3$ are direct products of the classes of D_3. Thus, there are nine classes for $D_3 \otimes D_3$, which indicates that there are nine inequivalent irreducible representations that are direct products of the representations of D_3. The character table of $D_3 \otimes D_3$ is given by

	$C_1 \otimes C_1$	$C_1 \otimes C_2$	$C_1 \otimes C_3$	$C_2 \otimes C_1$	$C_2 \otimes C_2$	$C_2 \otimes C_3$	$C_3 \otimes C_1$	$C_3 \otimes C_2$	$C_3 \otimes C_3$
$\Gamma^1 \otimes \Gamma^1$	1	1	1	1	1	1	1	1	1
$\Gamma^1 \otimes \Gamma^2$	1	1	-1	1	1	-1	1	1	-1
$\Gamma^1 \otimes \Gamma^3$	2	-1	0	2	-1	0	2	-1	0
$\Gamma^2 \otimes \Gamma^1$	1	1	1	1	1	1	-1	-1	-1
$\Gamma^2 \otimes \Gamma^2$	1	1	-1	1	1	-1	-1	-1	1
$\Gamma^2 \otimes \Gamma^3$	2	-1	0	2	-1	0	-2	1	0
$\Gamma^3 \otimes \Gamma^1$	2	2	2	-1	-1	-1	0	0	0
$\Gamma^3 \otimes \Gamma^2$	2	2	-2	-1	-1	1	0	0	0
$\Gamma^3 \otimes \Gamma^3$	4	-2	0	-2	1	0	0	0	0

4. Show that the direct-product representation $\Gamma_1 \otimes \Gamma_2$ is an irreducible representation of $G_1 \otimes G_2$ if Γ_1 and Γ_2 are irreducible representations of G_1 and G_2 respectively.

We first show that $\Gamma_1 \otimes \Gamma_2$ is a representation of $G_1 \otimes G_2$. It can be easily seen that the last three group axioms are obeyed. We now prove that the first group axiom on the closure property of a group is also obeyed. According to the definition of the direct product of two groups, we have for $a \in G_1$ and $b \in G_2$

$$\Gamma((a, b)) = \Gamma_1(a) \otimes \Gamma_2(b).$$

For $\Gamma((a,b))\Gamma((c,d))$ with $c \in G_1$ and $d \in G_2$, we have

$$\begin{aligned}\Gamma((a,b))\Gamma((c,d)) &= \big[\,\Gamma_1(a) \otimes \Gamma_2(b)\,\big]\big[\,\Gamma_1(c) \otimes \Gamma_2(d)\,\big] \\ &= \big[\,\Gamma_1(a)\Gamma_1(c)\,\big] \otimes \big[\,\Gamma_2(b)\Gamma_2(d)\,\big] \\ &= \Gamma_1(ac) \otimes \Gamma_2(bd) \\ &= \Gamma(ac, bd) = \Gamma((a,b)(c,d)).\end{aligned}$$

Thus, $\Gamma_1 \otimes \Gamma_2$ is a representation of $G_1 \otimes G_2$.

To further prove that $\Gamma_1 \otimes \Gamma_2$ is an irreducible representation if Γ_1 and Γ_2 are irreducible representations of G_1 and G_2 respectively, we make use of the following necessary and sufficient condition for a representation Γ of a finite group G to be irreducible

$$\frac{1}{g} \sum_{T \in G} |\chi(T)|^2 = 1.$$

We first find an expression for $\chi((a,b))$ with $a \in G_1$ and $b \in G_2$. Making use of $\Gamma((a,b)) = \Gamma_1(a) \otimes \Gamma_2(b)$, we have

$$\begin{aligned}\chi((a,b)) = \operatorname{tr}\Gamma((a,b)) &= \sum_{jk} \Gamma((a,b))_{jk,jk} = \sum_{jk} \Gamma_1(a)_{jj}\Gamma_2(a)_{kk} \\ &= \Big[\,\sum_j \Gamma_1(a)_{jj}\,\Big]\Big[\,\sum_k \Gamma_2(a)_{kk}\,\Big] = \chi^1(a)\chi^2(b).\end{aligned}$$

We then have

$$\begin{aligned}\frac{1}{g_1 g_2} \sum_{(a,b)} |\chi((a,b))|^2 &= \frac{1}{g_1 g_2} \sum_{(a,b)} |\chi^1(a)\chi^2(b)|^2 \\ &= \Big[\,\frac{1}{g_1} \sum_a |\chi^1(a)|^2\,\Big]\Big[\,\frac{1}{g_2} \sum_b |\chi^2(b)|^2\,\Big] \\ &= 1 \times 1 = 1,\end{aligned}$$

where we have made use of the fact that Γ_1 and Γ_2 are irreducible representations of G_1 and G_2 respectively. From the above result, we can conclude that $\Gamma_1 \otimes \Gamma_2$ is an irreducible representation of $G_1 \otimes G_2$.

5. Rotations in two dimensions can be parameterized by

$$R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

(a) Show that $R(\varphi_1)R(\varphi_2) = R(\varphi_1 + \varphi_2)$.

(b) Show that $R(\varphi) = e^{\varphi a_1}$, where

$$a_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(a) Making use of the given parameterization, we have

$$\begin{aligned}R(\varphi_1)R(\varphi_2) &= \begin{pmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{pmatrix}\begin{pmatrix} \cos\varphi_2 & -\sin\varphi_2 \\ \sin\varphi_2 & \cos\varphi_2 \end{pmatrix} \\ &= \begin{pmatrix} \cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 & -\cos\varphi_1\sin\varphi_2 - \sin\varphi_1\cos\varphi_2 \\ \sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2 & -\sin\varphi_1\sin\varphi_2 + \cos\varphi_1\cos\varphi_2 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\varphi_1+\varphi_2) & -\sin(\varphi_1+\varphi_2) \\ \sin(\varphi_1+\varphi_2) & \cos(\varphi_1+\varphi_2) \end{pmatrix} \\ &= R(\varphi_1+\varphi_2).\end{aligned}$$

(b) Expressing $e^{\varphi a_1}$ as a power series, we have

$$e^{\varphi a_1} = \sum_{k=0}^{\infty} \frac{1}{k!}(\alpha a_1)^k = 1 + \varphi a_1 + \frac{1}{2!}\varphi^2 a_1^2 + \frac{1}{3!}\varphi^3 a_1^3 + \cdots .$$

Evaluating the first few powers of a_1 , we have

$$\begin{aligned}
a_1^2 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
a_1^3 &= a_1^2 a_1 = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -a_1, \\
a_1^4 &= a_1^2 a_1^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
a_1^5 &= a_1^4 a_1 = a_1.
\end{aligned}$$

We can thus infer that

$$\begin{aligned}
&a_1^{2n} = (-1)^n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
&a_1^{2n+1} = (-1)^n a_1, \\
&n = 0, 1, 2, \cdots .
\end{aligned}$$

Breaking the above sum into a sum over terms of even powers in a_1 and a sum over terms of odd powers in a_1, we have

$$\begin{aligned}
e^{\varphi a_1} &= \sum_{n=0}^{\infty} \frac{(\alpha a_1)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(\alpha a_1)^{2n+1}}{(2n+1)!} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{(2n+1)!} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos\varphi + a_1 \sin\varphi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos\varphi + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sin\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.
\end{aligned}$$

Therefore, $R(\varphi) = e^{\varphi a_1}$.