Assignment 06

Due Time : 8:15, April 15, 2020 (Wednesday)

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Score : ________

Problem 1 Score: ________. The basis elements of the real Lie algebra $L = \mathrm{so}(3)$ are given by

$$a_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Show explicitly that these basis elements possess the following properties.

(a) The basis elements a_1, a_2, and a_3 obey the commutation relations

$$\begin{aligned} [a_1, a_2] =& a_1 a_2 - a_2 a_1 = -a_3, \\ [a_2, a_3] =& a_2 a_3 - a_3 a_2 = -a_1, \\ [a_3, a_1] =& a_3 a_1 - a_1 a_3 = -a_2. \end{aligned}$$

(b) The basis elements a_1, a_2, and a_3 are anti-Hermitian,

$$a_1^\dagger = -a_1, \quad a_2^\dagger = -a_2, \quad a_3^\dagger = -a_3.$$

Solution: (a) The basis elements a_1, a_2, and a_3 obey the commutation relations:

$$\begin{aligned} [a_1, a_2] =& a_1 a_2 - a_2 a_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\ =& \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -a_3, \qquad (1) \\ [a_2, a_3] =& a_2 a_3 - a_3 a_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ =& \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = -a_1, \qquad (2) \\ [a_3, a_1] =& a_3 a_1 - a_1 a_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ =& \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = -a_2. \qquad (3) \end{aligned}$$

(b) The basis elements a_1, a_2, and a_3 are anti-Hermitian:

$$a_1^\ddagger = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = -a_1, \qquad (4)$$

$$a_2^\ddagger = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = -a_2, \qquad (5)$$

$$a_3^\ddagger = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -a_3. \qquad (6)$$

□

Problem 2 Score: ________. The scalar transformation operators $Q(a_1)$, $Q(a_2)$, and $Q(a_3)$ for the real Lie algebra so(3) are found to be given by

$$Q(a_1) = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad Q(a_2) = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad Q(a_3) = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

Show that $[Q(a_1), Q(a_2)] = -Q(a_3)$, $[Q(a_2), Q(a_3)] = -Q(a_1)$, and $[Q(a_3), Q(a_1)] = -Q(a_2)$.

Solution: Applying $[Q(a_1),Q(a_2)]$, $[Q(a_2),Q(a_3)]$, and $[Q(a_3),Q(a_1)]$ to an arbitrary function $f(\vec{r})$ of $\vec{r}$, we get

$$\begin{aligned}
[Q(a_1),Q(a_2)]f(\vec{r})=&\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)\left(z\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial z}\right)-\left(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right)\left(y\frac{\partial f}{\partial z}-z\frac{\partial f}{\partial y}\right)\\
=&y\frac{\partial f}{\partial x}+yz\frac{\partial^2 f}{\partial z\partial x}-z^2\frac{\partial^2 f}{\partial y\partial x}-yx\frac{\partial^2 f}{\partial z^2}+zx\frac{\partial^2 f}{\partial y\partial z}\\
&-zy\frac{\partial^2 f}{\partial x\partial z}+xy\frac{\partial^2 f}{\partial z^2}+z^2\frac{\partial^2 f}{\partial x\partial y}-x\frac{\partial f}{\partial y}-xz\frac{\partial^2 f}{\partial z\partial y}\\
=&y\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial y}=\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)f(\vec{r})=-Q(a_3)f(\vec{r}). \qquad (7)\\
[Q(a_2),Q(a_3)]f(\vec{r})=&\left(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right)\left(x\frac{\partial f}{\partial y}-y\frac{\partial f}{\partial x}\right)-\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\left(z\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial z}\right)\\
=&z\frac{\partial f}{\partial y}+zx\frac{\partial^2 f}{\partial x\partial y}-x^2\frac{\partial^2 f}{\partial z\partial y}-zy\frac{\partial^2 f}{\partial x^2}+xy\frac{\partial^2 f}{\partial z\partial x}\\
&-xz\frac{\partial^2 f}{\partial y\partial x}+yz\frac{\partial^2 f}{\partial x^2}+x^2\frac{\partial^2 f}{\partial y\partial z}-y\frac{\partial f}{\partial z}-yx\frac{\partial^2 f}{\partial x\partial z}\\
=&z\frac{\partial f}{\partial y}-y\frac{\partial f}{\partial z}=\left(z\frac{\partial}{\partial y}-y\frac{\partial}{\partial z}\right)f(\vec{r})=-Q(a_1)f(\vec{r}), \qquad (8)\\
[Q(a_3),Q(a_1)]=&\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\left(y\frac{\partial f}{\partial z}-z\frac{\partial f}{\partial y}\right)-\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)\left(x\frac{\partial f}{\partial y}-y\frac{\partial f}{\partial x}\right)\\
=&x\frac{\partial f}{\partial z}+xy\frac{\partial^2 f}{\partial y\partial z}-y^2\frac{\partial^2 f}{\partial x\partial z}-xz\frac{\partial^2 f}{\partial y^2}+yz\frac{\partial^2 f}{\partial x\partial y}\\
&-yx\frac{\partial^2 f}{\partial z\partial y}+zx\frac{\partial^2 f}{\partial y^2}+y^2\frac{\partial^2 f}{\partial z\partial x}-z\frac{\partial f}{\partial x}-zy\frac{\partial^2 f}{\partial y\partial x}\\
=&x\frac{\partial f}{\partial z}-z\frac{\partial f}{\partial x}=\left(x\frac{\partial}{\partial z}-z\frac{\partial}{\partial x}\right)f(\vec{r})=-Q(a_2)f(\vec{r}). \qquad (9)
\end{aligned}$$

Due to the arbitrariness of $f(\vec{r})$, we have

$$[Q(a_1),Q(a_2)]=-Q(a_3), \qquad (10)$$
$$[Q(a_2),Q(a_3)]=-Q(a_1), \qquad (11)$$
$$[Q(a_3),Q(a_1)]=-Q(a_2). \qquad (12)$$

□

Problem 3 Score: _________. The generators of the real Lie algebra $L=\mathrm{su}(2)$ are given by

$$a_1=\frac{1}{2}\begin{pmatrix}0&i\\i&0\end{pmatrix},\quad a_2=\frac{1}{2}\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad a_3=\frac{1}{2}\begin{pmatrix}i&0\\0&-i\end{pmatrix}.$$

Show explicitly that a_1, a_2, and a_3 obey the commutation relations

$$\begin{aligned}
[a_1,a_2]=&a_1a_2-a_2a_1=-a_3,\\
[a_2,a_3]=&a_2a_3-a_3a_2=-a_1,\\
[a_3,a_1]=&a_3a_1-a_1a_3=-a_2.
\end{aligned}$$

Solution: a_1, a_2, and a_3 obey the commutation relations:

$$\begin{aligned}
[a_1,a_2]=&a_1a_2-a_2a_1=\frac{1}{4}\begin{pmatrix}0&i\\i&0\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}-\frac{1}{4}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}0&i\\i&0\end{pmatrix}=\frac{1}{4}\begin{pmatrix}-i&0\\0&i\end{pmatrix}-\frac{1}{4}\begin{pmatrix}i&0\\0&-i\end{pmatrix}\\
=&\frac{1}{2}\begin{pmatrix}-i&0\\0&i\end{pmatrix}=-a_3, \qquad (13)\\
[a_2,a_3]=&a_2a_3-a_3a_2=\frac{1}{4}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}i&0\\0&-i\end{pmatrix}-\frac{1}{4}\begin{pmatrix}i&0\\0&-i\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}=\frac{1}{4}\begin{pmatrix}0&-i\\-i&0\end{pmatrix}-\frac{1}{4}\begin{pmatrix}0&i\\i&0\end{pmatrix}\\
=&\frac{1}{2}\begin{pmatrix}0&-i\\-i&0\end{pmatrix}=-a_1, \qquad (14)
\end{aligned}$$

$$
\begin{aligned}
[a_3, a_1] =& a_3 a_1 - a_1 a_3 = \frac{1}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
=& \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -a_2.
\end{aligned} \tag{15}
$$

□

Problem 4 Score: _________. The generators of the real Lie algebra $L = \mathrm{su}(2)$ in the above problem can expressed in terms of the following Pauli matrices

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

Show that the Pauli matrices possess the following properties.

(a) $\sigma_1^2 = \sigma_2^2 = \sigma_3^2$.

(b) $\sigma_1\sigma_2 = -\sigma_2\sigma_1 = i\sigma_3$, $\sigma_2\sigma_3 = -\sigma_3\sigma_2 = i\sigma_1$, $\sigma_3\sigma_1 = -\sigma_1\sigma_3 = i\sigma_2$.

Solution: (a) Since

$$
\sigma_1^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{16}
$$

$$
\sigma_2^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{17}
$$

$$
\sigma_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{18}
$$

we have

$$
\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{19}
$$

(b) Since

$$
\sigma_1\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3, \tag{20}
$$

$$
-\sigma_2\sigma_1 = -\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = i\sigma_3, \tag{21}
$$

we have

$$
\sigma_1\sigma_2 = -\sigma_2\sigma_1 = i\sigma_3. \tag{22}
$$

Since

$$
\sigma_2\sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\sigma_1, \tag{23}
$$

$$
-\sigma_3\sigma_2 = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = i\sigma_1, \tag{24}
$$

we have

$$
\sigma_2\sigma_3 = -\sigma_3\sigma_2 = i\sigma_1. \tag{25}
$$

Since

$$
\sigma_3\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2, \tag{26}
$$

$$
-\sigma_1\sigma_3 = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = i\sigma_2, \tag{27}
$$

we have

$$
\sigma_3\sigma_1 = -\sigma_1\sigma_3 = i\sigma_2. \tag{28}
$$

□

Problem 5 Score: ________. Let $\vec{n}=(n_1,n_2,n_3)$ be a unit vector specifying a direction in three dimensional space.

(a) Evaluate $(\vec{\sigma}\cdot\vec{n})^2$ with $\vec{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$.

(b) Evaluate $e^{i(\vec{\sigma}\cdot\vec{n})\omega/2}$.

Solution: (a)

$$\begin{aligned}(\vec{\sigma}\cdot\vec{n})^2 =&(n_1\sigma_1+n_2\sigma_2+n_3\sigma_3)^2\\ =&n_1^2\sigma_1^2+n_2^2\sigma_2^2+n_3^2\sigma_3^2+n_1n_2(\sigma_1\sigma_2+\sigma_2\sigma_1)+n_2n_3(\sigma_2\sigma_3+\sigma_3\sigma_2)+n_3n_1(\sigma_3\sigma_1+\sigma_1\sigma_3)\\ =&(n_1^2+n_2^2+n_3^2)\begin{pmatrix}1&0\\0&1\end{pmatrix}+n_1n_2\times 0+n_2n_3\times 0+n_3n_1\times 0\\ =&\begin{pmatrix}1&0\\0&1\end{pmatrix}.\end{aligned}\tag{29}$$

(b) From

$$(\vec{\sigma}\cdot\vec{n})^2=\begin{pmatrix}1&0\\0&1\end{pmatrix},\tag{30}$$

we know that

$$(\vec{\sigma}\cdot\vec{n})^j=\begin{cases}\vec{\sigma}\cdot\vec{n}=\begin{pmatrix}n_3&n_1-in_2\\n_1+in_2&-n_3\end{pmatrix}, & j\text{ is odd,}\\ \begin{pmatrix}1&0\\0&1\end{pmatrix}, & j\text{ is even.}\end{cases}\tag{31}$$

Then

$$\begin{aligned}e^{i(\vec{\sigma}\cdot\vec{n})\omega/2} =&1+\sum_{j=1}^{\infty}\frac{[i(\vec{\sigma}\cdot\vec{n})\omega/2]^j}{j!}\\ =&1+\sum_{j=0}^{\infty}\frac{i^{2j+1}(\vec{\sigma}\cdot\vec{n})^{2j+1}(\omega/2)^{2j+1}}{(2j+1)!}+\sum_{j=1}^{\infty}\frac{i^{2j}(\vec{\sigma}\cdot\vec{n})^{2j}(\omega/2)^{2j}}{(2j)!}\\ =&1+(\vec{\sigma}\cdot\vec{n})\sum_{j=0}^{\infty}\frac{i(-1)^j(\omega/2)^{2j+1}}{(2j+1)!}+\begin{pmatrix}1&0\\0&1\end{pmatrix}\sum_{j=1}^{\infty}\frac{(-1)^j(\omega/2)^{2j}}{(2j)!}\\ =&1+i\begin{pmatrix}n_3&n_1-in_2\\n_1+in_2&-n_3\end{pmatrix}\sin(\omega/2)+\begin{pmatrix}1&0\\0&1\end{pmatrix}[\cos(\omega/2)-1]\\ =&\begin{pmatrix}\cos(\omega/2)+in_3\sin(\omega/2)&(n_2+in_1)\sin(\omega/2)\\(-n_2+in_1)\sin(\omega/2)&\cos(\omega/2)-in_3\sin(\omega/2)\end{pmatrix}.\end{aligned}\tag{32}$$

□