Group Theory
Homework Assignment 13
Spring, 2020

1. The basis of the real Lie algebra $L = \mathrm{sl}(2, \mathbb{R})$ is given by
$$b_1 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},\ b_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ b_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
Find the representation matrices of the basis elements b_1, b_2, and b_3 in the adjoint representation.

2. Continue from the previous problem. Find the Killing forms $B(b_p, b_q)$ for $p, q = 1, 2, 3$.

3. Choose the basis of the semi-simple complex Lie algebra $\tilde{L}$ such that each basis element is a member of some subspace $\tilde{L}_\gamma$. The adjoint representation matrix $\mathrm{ad}(h)$ of each element h in the Cartan subalgebra H is a diagonal matrix with zero diagonal elements corresponding to the basis elements of $\tilde{L}_0 = H$ and with diagonal element $\gamma(h)$ corresponding to each basis element of $\tilde{L}_\gamma$ (for $\gamma \in \Delta$). Show that
$$B(h, h') = \sum_{\gamma \in \Delta} (\dim \tilde{L}_\gamma)\gamma(h)\gamma(h') \text{ for all } h, h' \in H.$$

4. Consider the complexification $\tilde{L}$ of $L = \mathrm{su}(2)$. The basis elements are given by
$$a_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},\ a_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ a_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$
The basis of the Cartan subalgebra H is given by $h_1 = a_3$. Verify that the two non-zero roots are α_1 and $-\alpha_1$ with $\alpha_1(h_1) = i$.

5. Continue from the previous problem. Find the values of $B(h_1, h_1)$ and $\langle \alpha_1, \alpha_1 \rangle$.