Group Theory
Homework Assignment 12
Spring, 2020

The analytic homomorphic mapping ϕ of SU(2) onto SO(3) is given by

$$\phi(u)_{jk} = \frac{1}{2}\,\mathrm{tr}\big(\sigma_j u \sigma_k u^{-1}\big),\ u \in \mathrm{SU}(2),\ j,k = 1,2,3.$$

Note that the Pauli matrices σ_1, σ_2, and σ_3 are also denoted respectively by σ_x, σ_y, and σ_z.

1. Verify explicitly that a rotation about the Oz-axis is indeed obtained from the given mapping for $u = e^{-i\theta\sigma_z/2}$.
2. Verify explicitly that a rotation about the Oy-axis is indeed obtained from the given mapping for $u = e^{-i\theta\sigma_y/2}$.
3. Show that the analytic isomorphic mapping of the real Lie algebra su(2) onto the real Lie algebra so(3) is given by

$$\psi(a)_{jk} = \frac{1}{2}\,\mathrm{tr}\big(\sigma_j [a, \sigma_k]\big),\ a \in \mathrm{su}(2),\ j,k = 1,2,3.$$

4. Without using the explicit matrix representations of the Pauli matrices, show that $\psi(a_p)_{jk} = \epsilon_{pjk}$ for $a_p = i\sigma_p/2$ with $p = 1,2,3$.

   Hint: $[\sigma_p, \sigma_k] = 2i \sum_{\ell=1}^{3} \epsilon_{pk\ell}\sigma_\ell$.

5. Using the definition of ϵ_{pjk}, show that

$$\psi(a_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},\ \psi(a_2) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ \psi(a_3) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$