Group Theory

Solutions to Problems in Homework Assignment 02

Spring, 2018

1. Show that the intersection S of two invariant subgroups S_1 and S_2 of a group G is an invariant subgroup.

Let T be in $S = S_1 \cap S_2$. Then, $T \in S_1$ and $T \in S_2$. Let X be an arbitrary element of G. Because S_1 and S_2 are invariant subgroups of G, we have

$$XTX^{-1} \in S_1,\ XTX^{-1} \in S_2.$$

Because XTX^{-1} is in both S_1 and S_2, XTX^{-1} is in S,

$$XTX^{-1} \in S.$$

Therefore, S is an invariant subgroup of G.

2. The multiplication table of a finite group G is given by

	E	A	B	C	D	F	I	J	K	L	M	N
E	E	A	B	C	D	F	I	J	K	L	M	N
A	A	E	F	I	J	B	C	D	M	N	K	L
B	B	F	A	K	L	E	M	N	I	J	C	D
C	C	I	L	A	K	N	E	M	J	F	D	B
D	D	J	K	L	A	M	N	E	F	I	B	C
F	F	B	E	M	N	A	K	L	C	D	I	J
I	I	C	N	E	M	L	A	K	D	B	J	F
J	J	D	M	N	E	K	L	A	B	C	F	I
K	K	M	J	F	I	D	B	C	N	E	L	A
L	L	N	I	J	F	C	D	B	E	M	A	K
M	M	K	D	B	C	J	F	I	L	A	N	E
N	N	L	C	D	B	I	J	F	A	K	E	M

(a) Find the inverse of each element of G.

(b) Find the elements in each class of G.

(c) Find all invariant subgroups of G.

(a) To find the inverse of an element T, we first find the element E in the row or column of the multiplication table indexed by T. Then the column or row index of E is the inverse of T. In this manner, the inverses of all the elements of G are found to be given by

$$E^{-1} = E,\ A^{-1} = A,\ B^{-1} = F,\ C^{-1} = I,\ D^{-1} = J,\ F^{-1} = B,$$
$$I^{-1} = C,\ J^{-1} = D,\ K^{-1} = L,\ L^{-1} = K,\ M^{-1} = N,\ N^{-1} = M.$$

(b) We now find all the classes by evaluating $X^{-1}TX$ for a given T and every $X \in G$.

i. $C_1 = \{E\}$.
Since $XEX^{-1} = E\ \forall X \in G$, E is obviously in a class by itself.

ii. $C_2 = \{A\}$.
Let $T = A$. We have

$$\begin{aligned}
&B^{-1}AB = FAB = BB = A, && C^{-1}AC = IAC = CC = A, && D^{-1}AD = JAD = DD = A,\\
&F^{-1}AF = BAF = FF = A, && I^{-1}AI = CAI = II = A, && J^{-1}AJ = DAJ = JJ = A,\\
&K^{-1}AK = LAK = NK = A, && L^{-1}AL = KAL = ML = A, && M^{-1}AM = NAM = LM = A,\\
&N^{-1}AN = MAN = KN = A.
\end{aligned}$$

iii. $C_3 = \{B, C, D\}$.
Let $T = B$. We have

$$
\begin{aligned}
&A^{-1}BA = ABA = FA = B, && C^{-1}BC = IBC = NC = D, && D^{-1}BD = JBD = MD = C,\\
&F^{-1}BF = BBF = AF = B, && I^{-1}BI = CBI = LI = D, && J^{-1}BJ = DBJ = KJ = C,\\
&K^{-1}BK = LBK = IK = D, && L^{-1}BL = KBL = JL = C, && M^{-1}BM = NBM = CM = D,\\
&N^{-1}BN = MBN = DN = C.
\end{aligned}
$$

iv. $C_4 = \{F, I, J\}$.
Let $T = F$. We have

$$
\begin{aligned}
&A^{-1}FA = AFA = BA = F, && B^{-1}FB = FFB = AB = F, && C^{-1}FC = IFC = LC = J,\\
&D^{-1}FD = JFD = KD = I, && I^{-1}FI = CFI = NI = J, && J^{-1}FJ = DFJ = MJ = I,\\
&K^{-1}FK = LFK = CK = J, && L^{-1}FL = KFL = DL = I, && M^{-1}FM = NFM = IM = J,\\
&N^{-1}FN = MFN = JN = I.
\end{aligned}
$$

v. $C_5 = \{K, L\}$.
Let $T = K$. We have

$$
\begin{aligned}
&A^{-1}KA = AKA = MA = K, && B^{-1}KB = FKB = CB = L, && C^{-1}KC = IKC = DC = L,\\
&D^{-1}KD = JKD = BD = L, && F^{-1}KF = BKF = IF = L, && I^{-1}KI = CKI = JI = L,\\
&J^{-1}KJ = DKJ = FJ = L, && L^{-1}KL = KKL = NL = K, && M^{-1}KM = NKM = AM = K,\\
&N^{-1}KN = MKN = LN = K.
\end{aligned}
$$

vi. $C_6 = \{M, N\}$.
Let $T = M$. We have

$$
\begin{aligned}
&A^{-1}MA = AMA = KA = M, && B^{-1}MB = FMB = ICB = N, && C^{-1}MC = IMC = JC = N,\\
&D^{-1}MD = JMD = FD = N, && F^{-1}MF = BMF = CF = N, && I^{-1}MI = CMI = DI = N,\\
&J^{-1}MJ = DMJ = BJ = N, && K^{-1}MK = LMK = AK = M, && L^{-1}ML = KML = LL = M,\\
&N^{-1}MN = MMN = NN = M.
\end{aligned}
$$

(c) We know that an invariant subgroup contains the whole class(es) and that its order divides the order of the group. Let us examine the subsets that satisfy these conditions.

Subset	Number of elements	Is a subgroup?	Is an invariant subgroup?
$\{E\}$	1	Yes	Yes
$\{E, A\}$	2	Yes	Yes
$\{E, K, L\}$	3	No, $K^2 = N, \cdots$	No
$\{E, M, N\}$	3	Yes	Yes
$\{E, B, C, D\}$	4	No, $BC = K, \cdots$	No
$\{E, F, I, J\}$	4	No, $FI = K, \cdots$	No
$\{E, A, K, L\}$	4	No, $AK = M, \cdots$	No
$\{E, A, M, N\}$	4	No, $AM = K, \cdots$	No
$\{E, B, C, D, K, L\}$	6	No, $B^2 = A, \cdots$	No
$\{E, B, C, D, M, N\}$	6	No, $B^2 = A, \cdots$	No
$\{E, F, I, J, K, L\}$	6	No, $F^2 = A, \cdots$	No
$\{E, F, I, J, M, N\}$	6	No, $F^2 = A, \cdots$	No
$\{E, A, K, L, M, N\}$	6	Yes	Yes
$\{E, A, B, C, D, F, I, J, K, L, M, N\}$	12	Yes	Yes

In summary, the following subsets of the elements of G are invariant subgroups of G.

$$
\begin{aligned}
&\{E\},\\
&\{E, A\},\\
&\{E, M, N\},\\
&\{E, A, K, L, M, N\},\\
&\{E, A, B, C, D, F, I, J, K, L, M, N\}.
\end{aligned}
$$

Note that the first and last invariant subgroups are trivial subgroups of G.

3. Consider the group D_3.

   (a) List all the classes of D_3.

   (b) Find the right and left cosets of the subgroup $S = \{E, A\}$ of D_3.

The group D_3 is given by $D_3 = \{E, D, F, A, B, C\}$. The multiplication table of D_3 is given by

	E	D	F	A	B	C
E	E	D	F	A	B	C
D	D	F	E	B	C	A
F	F	E	D	C	A	B
A	A	C	B	E	F	D
B	B	A	C	D	E	F
C	C	B	A	F	D	E

The inverses of all the elements of D_3 are given by

$$E^{-1} = E,\ D^{-1} = F,\ F^{-1} = D,\ A^{-1} = A,\ B^{-1} = B,\ C^{-1} = C.$$

(a) We find all the classes of D_3 by evaluating $X^{-1}TX$ for a given T and every $X \in D_3$.

   i. $C_1 = \{E\}$.
   Since $X^{-1}EX = E\ \forall X \in D_3$, E is obviously in a class by itself.

   ii. $C_2 = \{D, F\}$.
   Let $T = D$. We have

$$F^{-1}DF = DDF = FF = D,\ A^{-1}DA = ADA = CA = F,$$
$$B^{-1}DB = BDB = AB = F,\ C^{-1}DC = CDC = BC = F.$$

   iii. $C_3 = \{A, B, C\}$.
   Let $T = A$. We have

$$D^{-1}AD = FAD = CD = B,\ F^{-1}AF = DAF = CF = A,$$
$$B^{-1}AB = BAB = DB = C,\ C^{-1}AC = CAC = FC = B.$$

(b) Find the right and left cosets of the subgroup $S = \{E, A\}$ of D_3.

   i. The right cosets of the subgroup $S = \{E, A\}$ of D_3 are given by

$$SE = SA = \{E, A\},$$
$$SD = SC = \{C, D\},$$
$$SF = SB = \{B, F\}.$$

   Detailed calculations are as follows

$$SE = \{EE, AE\} = \{E, A\},$$
$$SD = \{ED, AD\} = \{D, C\},$$
$$SF = \{EF, AF\} = \{F, B\},$$
$$SA = \{EA, AA\} = \{A, E\},$$
$$SB = \{EB, AB\} = \{B, F\},$$
$$SC = \{EC, AC\} = \{C, D\}.$$

   ii. The left cosets of the subgroup $S = \{E, A\}$ of D_3 are given by

$$ES = AS = \{E, A\},$$
$$DS = BS = \{B, D\},$$
$$FS = CS = \{C, F\}.$$

Detailed calculations are as follows

$$\begin{aligned} ES &= \{EE, EA\} = \{E, A\}, \\ DS &= \{DE, DA\} = \{D, B\}, \\ FS &= \{FE, FA\} = \{F, C\}, \\ AS &= \{AE, AA\} = \{A, E\}, \\ BS &= \{BE, BA\} = \{B, D\}, \\ CS &= \{CE, CA\} = \{C, F\}. \end{aligned}$$

4. For the group D_3 and its invariant subgroup $S = \{E, D, F\}$, find the factor group D_3/S. Construct the multiplication table for the factor group.

The right cosets of the invariant subgroup $S = \{E, D, F\}$ of D_3 are given by

$$\begin{aligned} SE = SD = SF &= \{E, D, F\}, \\ SA = SB = SC &= \{A, B, C\}. \end{aligned}$$

Detailed calculations are as follows

$$\begin{aligned} SE &= \{EE, DE, FE\} = \{E, D, F\}, \\ SD &= \{ED, DD, FD\} = \{D, F, E\}, \\ SF &= \{EF, DF, FF\} = \{F, E, D\}, \\ SA &= \{EA, DA, FA\} = \{A, B, C\}, \\ SB &= \{EB, DB, FB\} = \{B, C, A\}, \\ SC &= \{EC, DC, FC\} = \{C, A, B\}. \end{aligned}$$

The factor group D_3/S is of order two and its two elements are given by

$$SE = \{E, D, F\} \text{ and } SA = \{A, B, C\}.$$

To construct the multiplication table of D_3/S, we evaluate the following products

$$\begin{aligned} SE \cdot SE = S(EE) = SE &\Rightarrow \{E, D, F\} \cdot \{E, D, F\} = \{E, D, F\}, \\ SE \cdot SA = S(EA) = SA &\Rightarrow \{E, D, F\} \cdot \{A, B, C\} = \{A, B, C\}, \\ SA \cdot SE = S(AE) = SA &\Rightarrow \{A, B, C\} \cdot \{E, D, F\} = \{A, B, C\}, \\ SA \cdot SA = S(AA) = SE &\Rightarrow \{A, B, C\} \cdot \{A, B, C\} = \{E, D, F\}. \end{aligned}$$

From the above calculations, we see that $SE = \{E, D, F\}$ is the identity element of the factor group D_3/S and that the inverse of $SA = \{A, B, C\}$ is itself. The multiplication table of D_3/S is then given by

	$\{E, D, F\}$	$\{A, B, C\}$
$\{E, D, F\}$	$\{E, D, F\}$	$\{A, B, C\}$
$\{A, B, C\}$	$\{A, B, C\}$	$\{E, D, F\}$

5. Consider $C_6 = \{E,\, a,\, a^2,\, a^3,\, a^4,\, a^5\}$ and its two subgroups $S_1 = \{E,\, a^3\}$ and $S_2 = \{E,\, a^2,\, a^4\}$. Show that $C_6 = S_1 \otimes S_2$.

The elements of $S_1 \otimes S_2$ are given by

$$(E, E),\ (E, a^2),\ (E, a^4),\ (a^3, E),\ (a^3, a^2),\ (a^3, a^4).$$

We now introduce a mapping ϕ given by

$$\phi(a_1, a_2) = a_1 a_2,$$

where (a_1, a_2) is an element of $S_1 \otimes S_2$. The result of the mapping is the product of the pair in an element of $S_1 \otimes S_2$ under the multiplication operation of the group C_6. Through the mapping ϕ, the direct product group $S_1 \otimes S_2$ is mapped onto a group with the following elements

$$
\begin{aligned}
&\phi(E, E) = EE = E, \\
&\phi(E, a^2) = Ea^2 = a^2, \\
&\phi(E, a^4) = Ea^4 = a^4, \\
&\phi(a^3, E) = a^3 E = a^3, \\
&\phi(a^3, a^2) = a^3 a^2 = a^5, \\
&\phi(a^3, a^4) = a^3 a^4 = a^6 a = Ea = a,
\end{aligned}
$$

where $a^6 = E$ has been used. We see that the mapping ϕ yields exactly all the elements of the group C_6 and is one-to-one. Thus, C_6 is the direct product group of S_1 with S_2. That is, $C_6 = S_1 \otimes S_2$.