Group Theory

Solutions to the Problems in Homework Assignment 07

Spring, 2020

Consider a particle of mass μ confined to a square in two dimensions whose vertices are located at $(z, x) = (1, 1)$, $(1, -1)$, $(-1, -1)$, and $(-1, 1)$ on the zOx plane. The potential is zero within the square and infinite on the edges of the square. The eigenfunctions $\psi_{mn}(z, x)$ of the Hamiltonian of the particle are of the form

$$\psi_{mn}(z,x) \propto \begin{cases} \cos(k_m z)\cos(k_n x), \text{if both } m \text{ and } n \text{ are odd,} \\ \cos(k_m z)\sin(k_n x), \text{if } m \text{ is odd but } n \text{ is even,} \\ \sin(k_m z)\cos(k_n x), \text{if } m \text{ is even but } n \text{ is odd,} \\ \sin(k_m z)\sin(k_n x), \text{if both } m \text{ and } n \text{ are even,} \end{cases}$$

where $k_m = m\pi/2$, $k_n = n\pi/2$, and m and n are positive integers. The corresponding eigenvalues are given by

$$E_{mn} = \frac{\pi^2\hbar^2}{8\mu}(m^2+n^2).$$

The symmetry group of the Hamiltonian H_0 is D_4 whose character table is given by

	$C_1=\{E\}$	$C_2=\{C_{2x}, C_{2z}\}$	$C_3=\{C_{2y}\}$	$C_4=\{C_{4y}, C_{4y}^{-1}\}$	$C_5=\{C_{2c}, C_{2d}\}$
Γ^1	1	1	1	1	1
Γ^2	1	1	1	-1	-1
Γ^3	1	-1	1	1	-1
Γ^4	1	-1	1	-1	1
Γ^5	2	0	-2	0	0

1. For which irreducible representations do the eigenfunctions $\psi_{11}(z, x)$ and $\psi_{22}(z, x)$ form bases respectively?

Through inspection, we find the following transformation matrices for the elements of D_4 in two dimensions with the position vector of a point given by (z, x)

$$R(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, R(C_{2x}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, R(C_{2y}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, R(C_{2z}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$R(C_{4y}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, R(C_{4y}^{-1}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, R(C_{2c}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, R(C_{2d}) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

The transformation generated by the element T of D_4 on a basis function $\phi_i^p(z, x)$ in representation Γ^p is given by

$$Q(T)\phi_i^p(z,x) = \phi_i^p(z',x') = \sum_{j=1}^{d_p} \Gamma^p(T)_{ji}\phi_j^p(z,x),$$

where

$$\begin{pmatrix} z' \\ x' \end{pmatrix} = R(T)\begin{pmatrix} z \\ x \end{pmatrix}.$$

From the above transformation matrices of the elements of D_4, we have

$$\begin{pmatrix} z' \\ x' \end{pmatrix} = R(E)\begin{pmatrix} z \\ x \end{pmatrix} = \begin{pmatrix} z \\ x \end{pmatrix}, \quad \begin{pmatrix} z' \\ x' \end{pmatrix} = R(C_{2x})\begin{pmatrix} z \\ x \end{pmatrix} = \begin{pmatrix} -z \\ x \end{pmatrix}, \quad \begin{pmatrix} z' \\ x' \end{pmatrix} = R(C_{2y})\begin{pmatrix} z \\ x \end{pmatrix} = \begin{pmatrix} -z \\ -x \end{pmatrix},$$

$$\begin{pmatrix} z' \\ x' \end{pmatrix} = R(C_{2z})\begin{pmatrix} z \\ x \end{pmatrix} = \begin{pmatrix} z \\ -x \end{pmatrix}, \quad \begin{pmatrix} z' \\ x' \end{pmatrix} = R(C_{4y})\begin{pmatrix} z \\ x \end{pmatrix} = \begin{pmatrix} x \\ -z \end{pmatrix}, \quad \begin{pmatrix} z' \\ x' \end{pmatrix} = R(C_{4y}^{-1})\begin{pmatrix} z \\ x \end{pmatrix} = \begin{pmatrix} -x \\ z \end{pmatrix},$$

$$\begin{pmatrix} z' \\ x' \end{pmatrix} = R(C_{2c})\begin{pmatrix} z \\ x \end{pmatrix} = \begin{pmatrix} x \\ z \end{pmatrix}, \quad \begin{pmatrix} z' \\ x' \end{pmatrix} = R(C_{2d})\begin{pmatrix} z \\ x \end{pmatrix} = \begin{pmatrix} -x \\ -z \end{pmatrix}.$$

For $\psi_{11}(z,x) \propto \cos(\pi z/2)\cos(\pi x/2)$, we have $Q(T)\psi_{11}(z,x) = \psi_{11}(z,x)$ for all $T \in D_4$. Since $\psi_{11}(z,x)$ is a complete basis set by itself, $\psi_{11}(z,x)$ is the basis of a one-dimensional irreducible representation. Noting that $Q(T)\psi_{11}(z,x) = \psi_{11}(z,x)$ for all $T \in D_4$, we see that the characters of the elements are all equal to one. Thus, $\psi_{11}(z,x)$ forms the basis for the one-dimensional identity representation Γ^1.

For $\psi_{22}(z,x) \propto \sin(\pi z)\sin(\pi x)$, we have

$$\begin{aligned}
&Q(E)\psi_{22}(z,x) = \psi_{22}(z,x), && Q(C_{2x})\psi_{22}(z,x) = -\psi_{22}(z,x), && Q(C_{2y})\psi_{22}(z,x) = \psi_{22}(z,x),\\
&Q(C_{2z})\psi_{22}(z,x) = -\psi_{22}(z,x), && Q(C_{4y})\psi_{22}(z,x) = -\psi_{22}(z,x), && Q(C_{4y}^{-1})\psi_{22}(z,x) = -\psi_{22}(z,x),\\
&Q(C_{2c})\psi_{22}(z,x) = \psi_{22}(z,x), && Q(C_{2d})\psi_{22}(z,x) = \psi_{22}(z,x).
\end{aligned}$$

Since $\psi_{22}(z,x)$ is a complete basis set by itself, $\psi_{22}(z,x)$ is the basis of a one-dimensional irreducible representation. From the above results, we can infer the characters of the elements of D_4. We have

$$\chi(E) = 1,\ \chi(C_{2x}) = -1,\ \chi(C_{2y}) = 1,\ \chi(C_{2z}) = -1,\ \chi(C_{4y}) = -1,\ \chi(C_{4y}^{-1}) = -1,\ \chi(C_{2c}) = 1,\ \chi(C_{2d}) = 1.$$

Comparing the above characters of the elements with those in the given character table, we see that $\psi_{22}(z,x)$ forms the basis for the one-dimensional parity representation Γ^4.

2. Find the matrices representing all the elements of D_4 in the space spanned by the degenerate eigenfunctions $\psi_{12}(z,x)$ and $\psi_{21}(z,x)$. And then calculate the characters for all the classes of D_4 in this representation. For which irreducible representation do $\psi_{12}(z,x)$ and $\psi_{21}(z,x)$ form a basis?

Note that $\psi_{12}(z,x) \propto \cos(\pi z/2)\sin(\pi x)$ and $\psi_{21}(z,x) \propto \sin(\pi z)\cos(\pi x/2)$. For $\psi_{12}(z,x) \propto \cos(\pi z/2)\sin(\pi x)$, we have

$$\begin{aligned}
&Q(E)\psi_{12}(z,x) = \psi_{12}(z,x), && Q(C_{2x})\psi_{12}(z,x) = \psi_{12}(z,x), && Q(C_{2y})\psi_{12}(z,x) = -\psi_{12}(z,x),\\
&Q(C_{2z})\psi_{12}(z,x) = -\psi_{12}(z,x), && Q(C_{4y})\psi_{12}(z,x) = -\psi_{21}(z,x), && Q(C_{4y}^{-1})\psi_{12}(z,x) = \psi_{21}(z,x),\\
&Q(C_{2c})\psi_{12}(z,x) = \psi_{21}(z,x), && Q(C_{2d})\psi_{12}(z,x) = -\psi_{21}(z,x).
\end{aligned}$$

For $\psi_{21}(z,x) \propto \sin(\pi z)\cos(\pi x/2)$, we have

$$\begin{aligned}
&Q(E)\psi_{21}(z,x) = \psi_{21}(z,x), && Q(C_{2x})\psi_{21}(z,x) = -\psi_{21}(z,x), && Q(C_{2y})\psi_{21}(z,x) = -\psi_{21}(z,x),\\
&Q(C_{2z})\psi_{21}(z,x) = \psi_{21}(z,x), && Q(C_{4y})\psi_{21}(z,x) = \psi_{12}(z,x), && Q(C_{4y}^{-1})\psi_{21}(z,x) = -\psi_{12}(z,x),\\
&Q(C_{2c})\psi_{21}(z,x) = \psi_{12}(z,x), && Q(C_{2d})\psi_{21}(z,x) = -\psi_{12}(z,x).
\end{aligned}$$

Let $\phi_1 = \psi_{12}(z,x)$ and $\phi_2 = \psi_{21}(z,x)$. Making use of

$$Q(T)\phi_i^p(z,x) = \phi_i^p(z',x') = \sum_{j=1}^{d_p}\Gamma^p(T)_{ji}\phi_j^p(z,x),$$

we have

$$\begin{aligned}
&\Gamma_{11}(E) = 1, && \Gamma_{21}(E) = 0, && \Gamma_{11}(C_{2x}) = 1, && \Gamma_{21}(C_{2x}) = 0, && \Gamma_{11}(C_{2y}) = -1, && \Gamma_{21}(C_{2y}) = 0,\\
&\Gamma_{11}(C_{2z}) = -1, && \Gamma_{21}(C_{2z}) = 0, && \Gamma_{11}(C_{4y}) = 0, && \Gamma_{21}(C_{4y}) = -1, && \Gamma_{11}(C_{4y}^{-1}) = 0, && \Gamma_{21}(C_{4y}^{-1}) = 1,\\
&\Gamma_{11}(C_{2c}) = 0, && \Gamma_{21}(C_{2c}) = 1, && \Gamma_{11}(C_{2d}) = 0, && \Gamma_{21}(C_{2d}) = -1,
\end{aligned}$$

and

$$\begin{aligned}
&\Gamma_{12}(E) = 0, && \Gamma_{22}(E) = 1, && \Gamma_{12}(C_{2x}) = 0, && \Gamma_{22}(C_{2x}) = -1, && \Gamma_{12}(C_{2y}) = 0, && \Gamma_{22}(C_{2y}) = -1,\\
&\Gamma_{12}(C_{2z}) = 0, && \Gamma_{22}(C_{2z}) = 1, && \Gamma_{12}(C_{4y}) = 1, && \Gamma_{22}(C_{4y}) = 0, && \Gamma_{12}(C_{4y}^{-1}) = -1, && \Gamma_{22}(C_{4y}^{-1}) = 0,\\
&\Gamma_{12}(C_{2c}) = 1, && \Gamma_{22}(C_{2c}) = 0, && \Gamma_{12}(C_{2d}) = -1, && \Gamma_{22}(C_{2d}) = 0,
\end{aligned}$$

From the above results, we obtain the following matrices representing all the elements

$$\begin{aligned}
&\Gamma(E) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, && \Gamma(C_{2x}) = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, && \Gamma(C_{2y}) = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, && \Gamma(C_{2z}) = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},\\
&\Gamma(C_{4y}) = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, && \Gamma(C_{4y}^{-1}) = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, && \Gamma(C_{2c}) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, && \Gamma(C_{2d}) = \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}.
\end{aligned}$$

The characters are given by

	$C_1=\{E\}$	$C_2=\{C_{2x},C_{2z}\}$	$C_3=\{C_{2y}\}$	$C_4=\{C_{4y},C_{4y}^{-1}\}$	$C_5=\{C_{2c},C_{2d}\}$
Γ	2	0	-2	0	0

From the given character table, we see that the above character system is identical with that of the irreducible representation Γ^5. Thus, $\psi_{12}(z,x)$ and $\psi_{21}(z,x)$ form a basis for the two-dimensional coordinate irreducible representation Γ^5.

3. What is the degeneracy corresponding to $(m=6, n=7)$ and $(m=2, n=9)$? Is this degeneracy normal or accidental?

For $(m=6, n=7)$, the energy is given by

$$E_{6,7} = \frac{\pi^2\hbar^2}{8m}(6^2+7^2) = \frac{85\pi^2\hbar^2}{8m}.$$

For $(m=2, n=9)$, the energy is given by

$$E_{6,7} = \frac{\pi^2\hbar^2}{8m}(2^2+9^2) = \frac{85\pi^2\hbar^2}{8m}.$$

Thus, the energy $85\pi^2\hbar^2/8m$ is four-fold degenerate. Because the elements of D_4 transform $\psi_{6,7}(z,x)$ and $\psi_{7,6}(z,x)$ only between them, $\psi_{6,7}(z,x)$ and $\psi_{7,6}(z,x)$ give rise to a two-fold degeneracy of the energy $85\pi^2\hbar^2/8m$. Similarly, the elements of D_4 transform $\psi_{2,9}(z,x)$ and $\psi_{9,2}(z,x)$ only between them. $\psi_{2,9}(z,x)$ and $\psi_{9,2}(z,x)$ also give rise to a two-fold degeneracy of the energy $85\pi^2\hbar^2/8m$. Since the elements of D_4 can not transform $\psi_{6,7}(z,x)$ and $\psi_{7,6}(z,x)$ into $\psi_{2,9}(z,x)$ or $\psi_{9,2}(z,x)$, the four-fold degeneracy is accidental.

4. Find the matrices representing all the elements of D_4 in the space spanned by the degenerate eigenfunctions $\psi_{mn}(z,x)$ and $\psi_{nm}(z,x)$. Here both m and n are odd integers but they are not equal. And then calculate the characters for all the classes of D_4 in this representation. Is this representation reducible or irreducible? If this representation is reducible, write it as a direct sum of irreducible representations.

For odd m and n, we have $\psi_{mn}(z,x) \propto \cos(m\pi z/2)\cos(n\pi x/2)$ and $\psi_{nm}(z,x) \propto \cos(n\pi z/2)\cos(m\pi x/2)$.
The elements of D_4 transform $\psi_{mn}(z,x)$ as follows

$$\begin{aligned}
&Q(E)\psi_{mn}(z,x) = \psi_{mn}(z,x), \quad Q(C_{2x})\psi_{mn}(z,x) = \psi_{mn}(z,x), \; Q(C_{2y})\psi_{mn}(z,x) = \psi_{mn}(z,x),\\
&Q(C_{2z})\psi_{mn}(z,x) = \psi_{mn}(z,x), \; Q(C_{4y})\psi_{mn}(z,x) = \psi_{nm}(z,x), \; Q(C_{4y}^{-1})\psi_{mn}(z,x) = \psi_{nm}(z,x),\\
&Q(C_{2c})\psi_{mn}(z,x) = \psi_{nm}(z,x), \; Q(C_{2d})\psi_{mn}(z,x) = \psi_{nm}(z,x).
\end{aligned}$$

The elements of D_4 transform $\psi_{nm}(z,x)$ as follows

$$\begin{aligned}
&Q(E)\psi_{nm}(z,x) = \psi_{nm}(z,x), \quad Q(C_{2x})\psi_{nm}(z,x) = \psi_{nm}(z,x), \; Q(C_{2y})\psi_{nm}(z,x) = \psi_{nm}(z,x),\\
&Q(C_{2z})\psi_{nm}(z,x) = \psi_{nm}(z,x), \; Q(C_{4y})\psi_{nm}(z,x) = \psi_{mn}(z,x), \; Q(C_{4y}^{-1})\psi_{nm}(z,x) = \psi_{mn}(z,x),\\
&Q(C_{2c})\psi_{nm}(z,x) = \psi_{mn}(z,x), \; Q(C_{2d})\psi_{nm}(z,x) = \psi_{mn}(z,x).
\end{aligned}$$

Making use of

$$Q(T)\phi_i^p(z,x) = \phi_i^p(z',x') = \sum_{j=1}^{d_p} \Gamma^p(T)_{ji}\phi_j^p(z,x)$$

with $\phi_1 = \psi_{mn}$ and $\phi_2 = \psi_{nm}$, we have

$$\begin{aligned}
&\Gamma_{11}(E) = 1, \quad \Gamma_{21}(E) = 0, \quad \Gamma_{11}(C_{2x}) = 1, \; \Gamma_{21}(C_{2x}) = 0, \; \Gamma_{11}(C_{2y}) = 1, \; \Gamma_{21}(C_{2y}) = 0,\\
&\Gamma_{11}(C_{2z}) = 1, \; \Gamma_{21}(C_{2z}) = 0, \; \Gamma_{11}(C_{4y}) = 0, \; \Gamma_{21}(C_{4y}) = 1, \; \Gamma_{11}(C_{4y}^{-1}) = 0, \; \Gamma_{21}(C_{4y}^{-1}) = 1,\\
&\Gamma_{11}(C_{2c}) = 0, \; \Gamma_{21}(C_{2c}) = 1, \; \Gamma_{11}(C_{2d}) = 0, \; \Gamma_{21}(C_{2d}) = 1,
\end{aligned}$$

and

$$\Gamma_{12}(E)=0,\quad \Gamma_{22}(E)=1,\quad \Gamma_{12}(C_{2x})=0,\ \Gamma_{22}(C_{2x})=1,\ \Gamma_{12}(C_{2y})=0,\ \Gamma_{22}(C_{2y})=1,$$
$$\Gamma_{12}(C_{2z})=0,\ \Gamma_{22}(C_{2z})=1,\ \Gamma_{12}(C_{4y})=1,\ \Gamma_{22}(C_{4y})=0,\ \Gamma_{12}(C_{4y}^{-1})=1,\ \Gamma_{22}(C_{4y}^{-1})=0,$$
$$\Gamma_{12}(C_{2c})=1,\ \Gamma_{22}(C_{2c})=0,\ \Gamma_{12}(C_{2d})=1,\ \Gamma_{22}(C_{2d})=0.$$

From the above results, we obtain the following matrices representing all the elements

$$\Gamma(E)=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \Gamma(C_{2x})=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \Gamma(C_{2y})=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \Gamma(C_{2z})=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$
$$\Gamma(C_{4y})=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \Gamma(C_{4y}^{-1})=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \Gamma(C_{2c})=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \Gamma(C_{2d})=\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

The characters are given by

	$C_1=\{E\}$	$C_2=\{C_{2x},C_{2z}\}$	$C_3=\{C_{2y}\}$	$C_4=\{C_{4y},C_{4y}^{-1}\}$	$C_5=\{C_{2c},C_{2d}\}$
Γ	2	2	2	0	0

Because the above character system is not identical with the character system of any irreducible representation of D_4, the representation Γ is reducible. For the convenience of reducing the Γ representation, we list in the following the characters for all the irreducible representations of D_4 and the representation Γ.

	$C_1=\{E\}$	$C_2=\{C_{2x},C_{2z}\}$	$C_3=\{C_{2y}\}$	$C_4=\{C_{4y},C_{4y}^{-1}\}$	$C_5=\{C_{2c},C_{2d}\}$
Γ^1	1	1	1	1	1
Γ^2	1	1	1	-1	-1
Γ^3	1	-1	1	1	-1
Γ^4	1	-1	1	-1	1
Γ^5	2	0	-2	0	0
Γ	2	2	2	0	0

We now calculate the number of times that each irreducible representation occurs in the reduction of Γ. For Γ^1, we have

$$n_1=\frac{1}{g}\sum_{T\in D_4}\chi(T)\chi^1(T)^*=\frac{1}{8}\big(2\times 1+2\times 2\times 1+2\times 1\big)=1.$$

For Γ^2, we have

$$n_2=\frac{1}{g}\sum_{T\in D_4}\chi(T)\chi^2(T)^*=\frac{1}{8}\big(2\times 1+2\times 2\times 1+2\times 1\big)=1.$$

For Γ^3, we have

$$n_3=\frac{1}{g}\sum_{T\in D_4}\chi(T)\chi^3(T)^*=\frac{1}{8}\big[2\times 1+2\times 2\times(-1)+2\times 1\big]=0.$$

For Γ^4, we have

$$n_4=\frac{1}{g}\sum_{T\in D_4}\chi(T)\chi^4(T)^*=\frac{1}{8}\big[2\times 1+2\times 2\times(-1)+2\times 1\big]=0.$$

For Γ^5, we have

$$n_5=\frac{1}{g}\sum_{T\in D_4}\chi(T)\chi^5(T)^*=\frac{1}{8}\big[2\times 2+2\times(-2)\big]=0.$$

Thus,

$$\Gamma\approx\Gamma^1\oplus\Gamma^2.$$

5. Consider the case in which the particle is subject to an interaction given by Ax with A a constant.

(a) For which irreducible representation of D_4 is x an irreducible tensor operator?

(b) Consider the transitions caused by the interaction. If the particle is initially in the state $\psi_{mn}(z,x)$ or $\psi_{nm}(z,x)$ with m and n respectively even and odd integers, through reducing the direct product of irreducible representations find the irreducible representations which the allowed final states transform as.

(a) The transformation matrices for the elements of D_4 are given by

$$R(E)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},R(C_{2x})=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},R(C_{2y})=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix},R(C_{2z})=\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix},$$

$$R(C_{4y})=\begin{pmatrix}0&0&-1\\0&1&0\\1&0&0\end{pmatrix},R(C_{4y}^{-1})=\begin{pmatrix}0&0&1\\0&1&0\\-1&0&0\end{pmatrix},R(C_{2c})=\begin{pmatrix}0&0&1\\0&-1&0\\1&0&0\end{pmatrix},R(C_{2d})=\begin{pmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{pmatrix}.$$

x, y, and z are transformed by the rotation corresponding to the element X of D_4 as follows

$$\begin{aligned}Q(X)xQ(X)^{-1}&=R_{11}(X)x+R_{21}(X)y+R_{31}(X)z,\\Q(X)yQ(X)^{-1}&=R_{12}(X)x+R_{22}(X)y+R_{32}(X)z,\\Q(X)zQ(X)^{-1}&=R_{13}(X)x+R_{23}(X)y+R_{33}(X)z.\end{aligned}$$

Noticing that $R(X)_{21}=R(X)_{23}=0$ for all $X\in D_4$, we have

$$\begin{aligned}Q(X)xQ(X)^{-1}&=R_{11}(X)x+R_{31}(X)z,\\Q(X)zQ(X)^{-1}&=R_{13}(X)x+R_{33}(X)z.\end{aligned}$$

Let $T_1=x$ and $T_2=z$. We can then rewrite the above transformation in the form

$$Q(X)T_jQ(X)^{-1}=\sum_{k=1}^{2}\Gamma(X)_{kj}T_k,$$

where

$$\Gamma(X)=\begin{pmatrix}R_{11}(X)&R_{31}(X)\\R_{13}(X)&R_{33}(X)\end{pmatrix}.$$

From the above-given transformation matrices for the elements of D_4, we have the following matrices for all the elements of D_4 in the representation Γ

$$\Gamma(E)=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \Gamma(C_{2x})=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\ \Gamma(C_{2y})=\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\ \Gamma(C_{2z})=\begin{pmatrix}-1&0\\0&1\end{pmatrix},$$
$$\Gamma(C_{4y})=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\ \Gamma(C_{4y}^{-1})=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\ \Gamma(C_{2c})=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \Gamma(C_{2d})=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}.$$

The characters of the classes of D_4 in the representation Γ are given by

	$C_1=\{E\}$	$C_2=\{C_{2x},C_{2z}\}$	$C_3=\{C_{2y}\}$	$C_4=\{C_{4y},C_{4y}^{-1}\}$	$C_5=\{C_{2c},C_{2d}\}$
Γ	2	0	-2	0	0

Comparing the characters in the representation Γ with those in the irreducible representation Γ^5, we see that the representation Γ is equivalent to the irreducible representation Γ^5. Hence, x and z are a set of irreducible tensor operators for the irreducible representation Γ^5. Therefore, x is an irreducible tensor operator for the irreducible representation Γ^5.

(b) From the solution to Problem 2 in the above, we know that the degenerate eigenfunctions $\psi_{mn}(z, x)$ and $\psi_{nm}(z, x)$ with m and n respectively even and odd integers form a basis for Γ^5. Thus, $\psi_{mn}(z, x)$ and $\psi_{nm}(z, x)$ transform as Γ^5. Because x is an irreducible tensor operator for the irreducible representation Γ^5 and $\psi_{mn}(z, x)$ and $\psi_{nm}(z, x)$ transform as Γ^5, we must consider the direct product $\Gamma^5 \otimes \Gamma^5$ to find the allowed final states. The characters for $\Gamma^5 \otimes \Gamma^5$ are given by

	$C_1=\{E\}$	$C_2=\{C_{2x}, C_{2z}\}$	$C_3=\{C_{2y}\}$	$C_4=\{C_{4y}, C_{4y}^{-1}\}$	$C_5=\{C_{2c}, C_{2d}\}$
$\Gamma^5 \otimes \Gamma^5$	4	0	4	0	0

Using

$$n_{pq}^r = \frac{1}{g} \sum_{T \in G} \chi^p(T) \chi^q(T) \chi^r(T)^*,$$

we have

$$\begin{aligned}
n_{55}^1 &= \frac{1}{8}(4+4) = 1, \\
n_{55}^2 &= \frac{1}{8}(4+4) = 1, \\
n_{55}^3 &= \frac{1}{8}(4+4) = 1, \\
n_{55}^4 &= \frac{1}{8}(4+4) = 1, \\
n_{55}^5 &= \frac{1}{8}(8-8) = 0.
\end{aligned}$$

We thus have

$$\Gamma^5 \otimes \Gamma^5 \approx \Gamma^1 \oplus \Gamma^2 \oplus \Gamma^3 \oplus \Gamma^4.$$

The eigenfunctions that transform as Γ^1 through Γ^4 may be the final states while the eigenfunctions that transform as Γ^5 can not be the final states.