Group Theory
PHYS2102
Spring, 2020

Assignment 11

Due Time : 8:15, June 10, 2020 (Wednesday)

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Score : ________

Consider the permutation group S_4.

Problem 1 Score: ________. Consider some of the properties of S_4.

(a) What are the classes in S_4?

(b) What are the inequivalent irreducible representations of S_4?

(c) Write down all the Young tableaux in each irreducible representation of S_4. What is the dimension of each irreducible representation of S_4?

Solution: (a) We use partitions to denote the classes of S_4:

$$(4),\quad (3,1),\quad (2,2),\quad (2,1,1),\quad (1,1,1,1).$$

where the class (4) is

$$\{(1\quad 2\quad 3\quad 4),(1\quad 2\quad 4\quad 3),(1\quad 3\quad 2\quad 4),(1\quad 3\quad 4\quad 2),(1\quad 4\quad 2\quad 3),(1\quad 4\quad 3\quad 2)\},$$

the class $(3,1)$ is

$$\{(1\quad 2\quad 3),(3\quad 2\quad 1),(1\quad 2\quad 4),(4\quad 2\quad 1),(1\quad 3\quad 4),(4\quad 3\quad 1),(2\quad 3\quad 4),(4\quad 3\quad 2)\},$$

the class $(2,2)$ is

$$\{(1\quad 2)(3\quad 4),(1\quad 3)(2\quad 4),(1\quad 4)\quad(2\quad 3)\},$$

the class $(2,1,1)$ is

$$\{(1\quad 2),(1\quad 3),(1\quad 4),(2\quad 3),(2\quad 4),(3\quad 4)\},$$

and the class $(1,1,1,1)$ is

$$\{E\}.$$

(b) We use partitions to denote the irreducible representations of S_4:

$$[4],\quad [3,1],\quad [2,2],\quad [2,1,1],\quad [1,1,1,1],$$

which correspond to the representations:

$$\Gamma^{[4]},\quad \Gamma^{[3,1]},\quad \Gamma^{[2,2]}\quad \Gamma^{[2,1,1]},\quad \Gamma^{[1,1,1,1]}.$$

(c) (The Young tableaux we talk about here are all standard Young tableaux.) The Young tableau in $\Gamma^{[4]}$ are

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4\\ \hline\end{array}.$$

The dimension of $\Gamma^{[4]}$ is

$$d_{[4]}(S_4)=\frac{4!}{4\times 3\times 2\times 1}=1. \tag{1}$$

The Young tableaux in $\Gamma^{[3,1]}$ are

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3\\ \hline 4 \\ \cline{1-1}\end{array},\quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4\\ \hline 3 \\ \cline{1-1}\end{array},\quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4\\ \hline 2 \\ \cline{1-1}\end{array}.$$

The dimension of $\Gamma^{[3,1]}$ is

$$d_{[3,1]}=\frac{4!}{4\times 2\times 1\times 1}=3. \tag{2}$$

The Young tableaux in $\Gamma^{[2,2]}$ are

$$\begin{array}{|c|c|}\hline 1 & 2\\ \hline 3 & 4\\ \hline\end{array},\quad \begin{array}{|c|c|}\hline 1 & 3\\ \hline 2 & 4\\ \hline\end{array}.$$

The dimension of $\Gamma^{[2,2]}$ is

$$d_{[2,2]}=\frac{4!}{3\times 2\times 2\times 1}=2. \tag{3}$$

The Young tableaux in $\Gamma^{[2,1,1]}$ are

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array},\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array},\quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

The dimension of $\Gamma^{[2,1,1]}$ is

$$d_{[2,1,1]} = \frac{4!}{4\times 1\times 2\times 1} = 3. \tag{4}$$

The Young tableaux in $\Gamma^{[1,1,1,1]}$ are

$$\begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}$$

$$d_{[1,1,1,1]} = \frac{4!}{4\times 3\times 2\times 1} = 1. \tag{5}$$

□

Problem 2 Score: _________. Consider the Young operators in the irreducible representation $[3,1]$ of S_4.

(a) Write down all the Young tableaux and the corresponding Young operators $\mathcal{Y}_\mu^{[3,1]}$'s in the irreducible representation $[3,1]$ of S_4.

(b) Argue that all the Young operators in the irreducible representation $[3,1]$ of S_4 are orthogonal.

Solution: (a) The Young tableaux in $\Gamma^{[3,1]}$ are

$$\mathcal{Y}_1^{[3,1]} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array},\quad \mathcal{Y}_2^{[3,1]} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array},\quad \mathcal{Y}_3^{[3,1]} = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}.$$

The horizontal permutations of $\mathcal{Y}_1^{[3,1]}$:

$$P_{1,1} : E, (1\quad 2), (1\quad 3), (2\quad 3), (1\quad 2\quad 3), (3\quad 2\quad 1), \tag{6}$$
$$P_{1,2} : E, \tag{7}$$
$$P_1 = \prod_j P_{1,j} : E, (1\quad 2), (1\quad 3), (2\quad 3), (1\quad 2\quad 3), (3\quad 2\quad 1). \tag{8}$$

The horizontal operator of $\mathcal{Y}_1^{[3,1]}$:

$$\mathcal{P}_1 = \sum P_1 = E + (1\quad 2) + (1\quad 3) + (2\quad 3) + (1\quad 2\quad 3) + (3\quad 2\quad 1). \tag{9}$$

The vertical permutations of $\mathcal{Y}_1^{[3,1]}$:

$$Q_{1,1} : E, (1\quad 4), \tag{10}$$
$$Q_{1,2} : E, \tag{11}$$
$$Q_{1,3} : E, \tag{12}$$
$$Q_1 = \prod_k Q_{1,k} : E, (1\quad 4). \tag{13}$$

The vertical operator of $\mathcal{Y}_1^{[3,1]}$:

$$\mathcal{Q}_1 = \sum \delta(Q_1) Q_1 = E - (1\quad 4). \tag{14}$$

The Young operator of $\mathcal{Y}_1^{[3,1]}$:

$$\mathcal{Y}_1^{[3,1]} = \mathcal{P}_1\mathcal{Q}_1 = E + (1\quad 2) + (1\quad 3) + (2\quad 3) + (1\quad 2\quad 3) + (3\quad 2\quad 1)$$

$$-(1\quad 4)-(2\quad 1\quad 4)-(3\quad 1\quad 4)-(2\quad 3)(1\quad 4)-(2\quad 3\quad 1\quad 4)-(3\quad 2\quad 1\quad 4). \tag{15}$$

The horizontal permutations of $\mathcal{Y}_2^{[3,1]}$:

$$P_{2,1}: E, (1\quad 2), (1\quad 2), (2\quad 4), (1\quad 2\quad 4), (4\quad 2\quad 1), \tag{16}$$
$$P_{2,2}: E, \tag{17}$$
$$P_2 = \prod_j P_{2,j}: E, (1\quad 2), (1\quad 2), (2\quad 4), (1\quad 2\quad 4), (4\quad 2\quad 1). \tag{18}$$

The horizontal operator of $\mathcal{Y}_2^{[3,1]}$:

$$\mathcal{P}_2 = \sum P_2 = E + (1\quad 2) + (1\quad 2) + (2\quad 4) + (1\quad 2\quad 4) + (4\quad 2\quad 1) \tag{19}$$

The vertical permutations of $\mathcal{Y}_2^{[3,1]}$:

$$Q_{2,1}: E, (1\quad 3), \tag{20}$$
$$Q_{2,2}: E, \tag{21}$$
$$Q_{2,3}: E, \tag{22}$$
$$Q_2 = \prod_k Q_{2,k}: E, (1\quad 3). \tag{23}$$

The vertical operator of $\mathcal{Y}_2^{[3,1]}$:

$$\mathcal{Q}_2 = \sum \delta(Q_2) Q_2 = E - (1\quad 3). \tag{24}$$

The Young operator of $\mathcal{Y}_2^{[3,1]}$:

$$\begin{aligned}\mathcal{Y}_2^{[3,1]} =& \mathcal{P}_1 \mathcal{Q}_1 = E + (1\quad 2) + (1\quad 4) + (2\quad 4) + (1\quad 2\quad 4) + (4\quad 2\quad 1)\\ &-(1\quad 3)-(2\quad 1\quad 3)-(4\quad 1\quad 3)-(2\quad 4)(1\quad 3)-(2\quad 4\quad 1\quad 3)-(4\quad 2\quad 1\quad 3).\end{aligned} \tag{25}$$

The horizontal permutations of $\mathcal{Y}_3^{[3,1]}$:

$$P_{3,1}: E, (1\quad 3), (1\quad 4), (3\quad 4), (1\quad 3\quad 4), (4\quad 3\quad 1), \tag{26}$$
$$P_{3,2}: E, \tag{27}$$
$$P_3 = \prod_j P_{3,j}: E, (1\quad 3), (1\quad 4), (3\quad 4), (1\quad 3\quad 4), (4\quad 3\quad 1). \tag{28}$$

The horizontal operator of $\mathcal{Y}_3^{[3,1]}$:

$$\mathcal{P}_3 = \sum P_3 = E + (1\quad 3) + (1\quad 4) + (3\quad 4) + (1\quad 3\quad 4) + (4\quad 3\quad 1). \tag{29}$$

The vertical permutations of $\mathcal{Y}_3^{[3,1]}$:

$$Q_{3,1}: E, (1\quad 2), \tag{30}$$
$$Q_{3,2}: E, \tag{31}$$
$$Q_3 = \prod Q_{3,k}: E, (1\quad 2). \tag{32}$$

The vertical operator of $\mathcal{Y}_3^{[3,1]}$:

$$\mathcal{Q}_3 = \sum \delta(Q_3) Q_3 = E - (1\quad 2). \tag{33}$$

The Young operator of $\mathcal{Y}_3^{[3,1]}$:

$$\mathcal{Y}_3^{[3,1]} = \mathcal{P}_3 \mathcal{Q}_3 = \mathcal{P}_1 \mathcal{Q}_1 = E + (1\quad 3) + (1\quad 4) + (3\quad 4) + (1\quad 3\quad 4) + (4\quad 3\quad 1) \tag{34}$$
$$-(1\quad 2)-(3\quad 1\quad 2)-(4\quad 1\quad 2)-(3\quad 4)(1\quad 2)-(3\quad 4\quad 1\quad 2)-(4\quad 3\quad 1\quad 2). \tag{35}$$

(b) Since there are always two digits in row of $\mathcal{Y}_\mu^{[3,1]}$ occur in the same column of $\mathcal{Y}_\nu^{[3,1]}$ for $\mu,\nu=1,2,3$ and $\mu\neq\nu$, $\mathcal{Y}_\mu^{[3,1]}\mathcal{Y}_\nu^{[3,1]}$, all the Young operators in the irreducible representation $[3,1]$ of S_4 are orthogonal.
We can also prove this conclusion by calculate the product $\mathcal{Y}_\mu^{[3,1]}\mathcal{Y}_\nu^{[3,1]}$ directly:
Since

$$
\begin{aligned}
&\mathcal{Y}_1^{[3,1]}\mathcal{Y}_2^{[3,1]}\\
=&[E+(1\ 2)+(1\ 3)+(2\ 3)+(1\ 2\ 3)+(3\ 2\ 1)-(1\ 4)-(2\ 1\ 4)-(3\ 1\ 4)-(2\ 3)(1\ 4)-(2\ 3\ 1\ 4)-(3\ 2\ 1\ 4)]\\
&[E+(1\ 2)+(1\ 4)+(2\ 4)+(1\ 2\ 4)+(4\ 2\ 1)-(1\ 3)-(2\ 1\ 3)-(4\ 1\ 3)-(2\ 4)(1\ 3)-(2\ 4\ 1\ 3)-(4\ 2\ 1\ 3)]\\
=&E+(1\ 2)+(1\ 4)+(2\ 4)+(1\ 2\ 4)+(4\ 2\ 1)-(1\ 3)-(2\ 1\ 3)-(4\ 1\ 3)-(2\ 4)(1\ 3)-(2\ 4\ 1\ 3)-(4\ 2\ 1\ 3)\\
&+(1\ 2)+(1\ 2)(1\ 2)+(1\ 2)(1\ 4)+(1\ 2)(2\ 4)+(1\ 2)(1\ 2\ 4)+(1\ 2)(4\ 2\ 1)-(1\ 2)(1\ 3)-(1\ 2)(2\ 1\ 3)-(1\ 2)(4\ 1\ 3)-(1\ 2)(2\ 4)(1\ 3)-(1\ 2)(2\ 4\ 1\ 3)\\
&\quad-(1\ 2)(4\ 2\ 1\ 3)\\
&+(1\ 3)+(1\ 3)(1\ 2)+(1\ 3)(1\ 4)+(1\ 3)(2\ 4)+(1\ 3)(1\ 2\ 4)+(1\ 3)(4\ 2\ 1)-(1\ 3)(1\ 3)-(1\ 3)(2\ 1\ 3)-(1\ 3)(4\ 1\ 3)-(1\ 3)(2\ 4)(1\ 3)-(1\ 3)(2\ 4\ 1\ 3)\\
&\quad-(1\ 3)(4\ 2\ 1\ 3)\\
&+(2\ 3)+(2\ 3)(1\ 2)+(2\ 3)(1\ 4)+(2\ 3)(2\ 4)+(2\ 3)(1\ 2\ 4)+(2\ 3)(4\ 2\ 1)-(2\ 3)(1\ 3)-(2\ 3)(2\ 1\ 3)-(2\ 3)(4\ 1\ 3)-(2\ 3)(2\ 4)(1\ 3)-(2\ 3)(2\ 4\ 1\ 3)\\
&\quad-(2\ 3)(4\ 2\ 1\ 3)\\
&+(1\ 2\ 3)+(1\ 2\ 3)(1\ 2)+(1\ 2\ 3)(1\ 4)+(1\ 2\ 3)(2\ 4)+(1\ 2\ 3)(1\ 2\ 4)+(1\ 2\ 3)(4\ 2\ 1)-(1\ 2\ 3)(1\ 3)-(1\ 2\ 3)(2\ 1\ 3)-(1\ 2\ 3)(4\ 1\ 3)-(1\ 2\ 3)(2\ 4)(1\ 3)\\
&\quad-(1\ 2\ 3)(2\ 4\ 1\ 3)-(1\ 2\ 3)(4\ 2\ 1\ 3)\\
&+(3\ 2\ 1)+(3\ 2\ 1)(1\ 2)+(3\ 2\ 1)(1\ 4)+(3\ 2\ 1)(2\ 4)+(3\ 2\ 1)(1\ 2\ 4)+(3\ 2\ 1)(4\ 2\ 1)-(3\ 2\ 1)(1\ 3)-(3\ 2\ 1)(2\ 1\ 3)-(3\ 2\ 1)(4\ 1\ 3)-(3\ 2\ 1)(2\ 4)(1\ 3)\\
&\quad-(3\ 2\ 1)(2\ 4\ 1\ 3)-(3\ 2\ 1)(4\ 2\ 1\ 3)\\
&-(1\ 4)-(1\ 4)(1\ 2)-(1\ 4)(1\ 4)-(1\ 4)(2\ 4)-(1\ 4)(1\ 2\ 4)-(1\ 4)(4\ 2\ 1)+(1\ 4)(1\ 3)+(1\ 4)(2\ 1\ 3)+(1\ 4)(4\ 1\ 3)+(1\ 4)(2\ 4)(1\ 3)+(1\ 4)(2\ 4\ 1\ 3)\\
&\quad+(1\ 4)(4\ 2\ 1\ 3)\\
&-(2\ 1\ 4)-(2\ 1\ 4)(1\ 2)-(2\ 1\ 4)(1\ 4)-(2\ 1\ 4)(2\ 4)-(2\ 1\ 4)(1\ 2\ 4)-(2\ 1\ 4)(4\ 2\ 1)+(2\ 1\ 4)(1\ 3)+(2\ 1\ 4)(2\ 1\ 3)+(2\ 1\ 4)(4\ 1\ 3)+(2\ 1\ 4)(2\ 4)(1\ 3)\\
&\quad+(2\ 1\ 4)(2\ 4\ 1\ 3)+(2\ 1\ 4)(4\ 2\ 1\ 3)\\
&-(3\ 1\ 4)-(3\ 1\ 4)(1\ 2)-(3\ 1\ 4)(1\ 4)-(3\ 1\ 4)(2\ 4)-(3\ 1\ 4)(1\ 2\ 4)-(3\ 1\ 4)(4\ 2\ 1)+(3\ 1\ 4)(1\ 3)+(3\ 1\ 4)(2\ 1\ 3)+(3\ 1\ 4)(4\ 1\ 3)+(3\ 1\ 4)(2\ 4)(1\ 3)\\
&\quad+(3\ 1\ 4)(2\ 4\ 1\ 3)+(3\ 1\ 4)(4\ 2\ 1\ 3)\\
&-(2\ 3)(1\ 4)-(2\ 3)(1\ 4)(1\ 2)-(2\ 3)(1\ 4)(1\ 4)-(2\ 3)(1\ 4)(2\ 4)-(2\ 3)(1\ 4)(1\ 2\ 4)-(2\ 3)(1\ 4)(4\ 2\ 1)+(2\ 3)(1\ 4)(1\ 3)+(2\ 3)(1\ 4)(2\ 1\ 3)\\
&\quad+(2\ 3)(1\ 4)(4\ 1\ 3)+(2\ 3)(1\ 4)(2\ 4)(1\ 3)+(2\ 3)(1\ 4)(2\ 4\ 1\ 3)+(2\ 3)(1\ 4)(4\ 2\ 1\ 3)\\
&-(2\ 3\ 1\ 4)-(2\ 3\ 1\ 4)(1\ 2)-(2\ 3\ 1\ 4)(1\ 4)-(2\ 3\ 1\ 4)(2\ 4)-(2\ 3\ 1\ 4)(1\ 2\ 4)-(2\ 3\ 1\ 4)(4\ 2\ 1)+(2\ 3\ 1\ 4)(1\ 3)+(2\ 3\ 1\ 4)(2\ 1\ 3)+(2\ 3\ 1\ 4)(4\ 1\ 3)\\
&\quad+(2\ 3\ 1\ 4)(2\ 4)(1\ 3)+(2\ 3\ 1\ 4)(2\ 4\ 1\ 3)+(2\ 3\ 1\ 4)(4\ 2\ 1\ 3)\\
&-(3\ 2\ 1\ 4)-(3\ 2\ 1\ 4)(1\ 2)-(3\ 2\ 1\ 4)(1\ 4)-(3\ 2\ 1\ 4)(2\ 4)-(3\ 2\ 1\ 4)(1\ 2\ 4)-(3\ 2\ 1\ 4)(4\ 2\ 1)+(3\ 2\ 1\ 4)(1\ 3)+(3\ 2\ 1\ 4)(2\ 1\ 3)+(3\ 2\ 1\ 4)(4\ 1\ 3)\\
&\quad+(3\ 2\ 1\ 4)(2\ 4)(1\ 3)+(3\ 2\ 1\ 4)(2\ 4\ 1\ 3)+(3\ 2\ 1\ 4)(4\ 2\ 1\ 3)\\
=&0,
\end{aligned}
\tag{36}
$$

and

$$
\begin{aligned}
&\mathcal{Y}_2^{[3,1]}\mathcal{Y}_1^{[3,1]}\\
=&[E+(1\ 2)+(1\ 4)+(2\ 4)+(1\ 2\ 4)+(4\ 2\ 1)-(1\ 3)-(2\ 1\ 3)-(4\ 1\ 3)-(2\ 4)(1\ 3)-(2\ 4\ 1\ 3)-(4\ 2\ 1\ 3)]\\
&[E+(1\ 2)+(1\ 3)+(2\ 3)+(1\ 2\ 3)+(3\ 2\ 1)-(1\ 4)-(2\ 1\ 4)-(3\ 1\ 4)-(2\ 3)(1\ 4)-(2\ 3\ 1\ 4)-(3\ 2\ 1\ 4)]\\
=&E+(1\ 2)+(1\ 3)+(2\ 3)+(1\ 2\ 3)+(3\ 2\ 1)-(1\ 4)-(2\ 1\ 4)-(3\ 1\ 4)-(2\ 3)(1\ 4)-(2\ 3\ 1\ 4)-(3\ 2\ 1\ 4)\\
&+(1\ 2)+(1\ 2)(1\ 2)+(1\ 2)(1\ 3)+(1\ 2)(2\ 3)+(1\ 2)(1\ 2\ 3)+(1\ 2)(3\ 2\ 1)-(1\ 2)(1\ 4)-(1\ 2)(2\ 1\ 4)-(1\ 2)(3\ 1\ 4)-(1\ 2)(2\ 3)(1\ 4)-(1\ 2)(2\ 3\ 1\ 4)\\
&\quad-(1\ 2)(3\ 2\ 1\ 4)\\
&+(1\ 4)+(1\ 4)(1\ 2)+(1\ 4)(1\ 3)+(1\ 4)(2\ 3)+(1\ 4)(1\ 2\ 3)+(1\ 4)(3\ 2\ 1)-(1\ 4)(1\ 4)-(1\ 4)(2\ 1\ 4)-(1\ 4)(3\ 1\ 4)-(1\ 4)(2\ 3)(1\ 4)-(1\ 4)(2\ 3\ 1\ 4)\\
&\quad-(1\ 4)(3\ 2\ 1\ 4)\\
&+(2\ 4)+(2\ 4)(1\ 2)+(2\ 4)(1\ 3)+(2\ 4)(2\ 3)+(2\ 4)(1\ 2\ 3)+(2\ 4)(3\ 2\ 1)-(2\ 4)(1\ 4)-(2\ 4)(2\ 1\ 4)-(2\ 4)(3\ 1\ 4)-(2\ 4)(2\ 3)(1\ 4)-(2\ 4)(2\ 3\ 1\ 4)\\
&\quad-(2\ 4)(3\ 2\ 1\ 4)\\
&+(1\ 2\ 4)+(1\ 2\ 4)(1\ 2)+(1\ 2\ 4)(1\ 3)+(1\ 2\ 4)(2\ 3)+(1\ 2\ 4)(1\ 2\ 3)+(1\ 2\ 4)(3\ 2\ 1)-(1\ 2\ 4)(1\ 4)-(1\ 2\ 4)(2\ 1\ 4)-(1\ 2\ 4)(3\ 1\ 4)-(1\ 2\ 4)(2\ 3)(1\ 4)\\
&\quad-(1\ 2\ 4)(2\ 3\ 1\ 4)-(1\ 2\ 4)(3\ 2\ 1\ 4)\\
&+(4\ 2\ 1)+(4\ 2\ 1)(1\ 2)+(4\ 2\ 1)(1\ 3)+(4\ 2\ 1)(2\ 3)+(4\ 2\ 1)(1\ 2\ 3)+(4\ 2\ 1)(3\ 2\ 1)-(4\ 2\ 1)(1\ 4)-(4\ 2\ 1)(2\ 1\ 4)-(4\ 2\ 1)(3\ 1\ 4)-(4\ 2\ 1)(2\ 3)(1\ 4)\\
&\quad-(4\ 2\ 1)(2\ 3\ 1\ 4)-(4\ 2\ 1)(3\ 2\ 1\ 4)\\
&-(1\ 3)-(1\ 3)(1\ 2)-(1\ 3)(1\ 3)-(1\ 3)(2\ 3)-(1\ 3)(1\ 2\ 3)-(1\ 3)(3\ 2\ 1)+(1\ 3)(1\ 4)+(1\ 3)(2\ 1\ 4)+(1\ 3)(3\ 1\ 4)+(1\ 3)(2\ 3)(1\ 4)+(1\ 3)(2\ 3\ 1\ 4)\\
&\quad+(1\ 3)(3\ 2\ 1\ 4)\\
&-(2\ 1\ 3)-(2\ 1\ 3)(1\ 2)-(2\ 1\ 3)(1\ 3)-(2\ 1\ 3)(2\ 3)-(2\ 1\ 3)(1\ 2\ 3)-(2\ 1\ 3)(3\ 2\ 1)+(2\ 1\ 3)(1\ 4)+(2\ 1\ 3)(2\ 1\ 4)+(2\ 1\ 3)(3\ 1\ 4)+(2\ 1\ 3)(2\ 3)(1\ 4)\\
&+(2\ 1\ 3)(2\ 3\ 1\ 4)+(2\ 1\ 3)(3\ 2\ 1\ 4)\\
&-(4\ 1\ 3)-(4\ 1\ 3)(1\ 2)-(4\ 1\ 3)(1\ 3)-(4\ 1\ 3)(2\ 3)-(4\ 1\ 3)(1\ 2\ 3)-(4\ 1\ 3)(3\ 2\ 1)+(4\ 1\ 3)(1\ 4)+(4\ 1\ 3)(2\ 1\ 4)+(4\ 1\ 3)(3\ 1\ 4)+(4\ 1\ 3)(2\ 3)(1\ 4)\\
&+(4\ 1\ 3)(2\ 3\ 1\ 4)+(4\ 1\ 3)(3\ 2\ 1\ 4)\\
&-(2\ 4)(1\ 3)-(2\ 4)(1\ 3)(1\ 2)-(2\ 4)(1\ 3)(1\ 3)-(2\ 4)(1\ 3)(2\ 3)-(2\ 4)(1\ 3)(1\ 2\ 3)-(2\ 4)(1\ 3)(3\ 2\ 1)+(2\ 4)(1\ 3)(1\ 4)+(2\ 4)(1\ 3)(2\ 1\ 4)\\
&+(2\ 4)(1\ 3)(3\ 1\ 4)+(2\ 4)(1\ 3)(2\ 3)(1\ 4)+(2\ 4)(1\ 3)(2\ 3\ 1\ 4)+(2\ 4)(1\ 3)(3\ 2\ 1\ 4)\\
&-(2\ 4\ 1\ 3)-(2\ 4\ 1\ 3)(1\ 2)-(2\ 4\ 1\ 3)(1\ 3)-(2\ 4\ 1\ 3)(2\ 3)-(2\ 4\ 1\ 3)(1\ 2\ 3)-(2\ 4\ 1\ 3)(3\ 2\ 1)+(2\ 4\ 1\ 3)(1\ 4)+(2\ 4\ 1\ 3)(2\ 1\ 4)+(2\ 4\ 1\ 3)(3\ 1\ 4)\\
&\quad+(2\ 4\ 1\ 3)(2\ 3)(1\ 4)+(2\ 4\ 1\ 3)(2\ 3\ 1\ 4)+(2\ 4\ 1\ 3)(3\ 2\ 1\ 4)
\end{aligned}
$$

$$
\begin{aligned}
&-(4\,2\,1\,3)-(4\,2\,1\,3)(1\,2)-(4\,2\,1\,3)(1\,3)-(4\,2\,1\,3)(2\,3)-(4\,2\,1\,3)(1\,2\,3)-(4\,2\,1\,3)(3\,2\,1)+(4\,2\,1\,3)(1\,4)+(4\,2\,1\,3)(2\,1\,4)+(4\,2\,1\,3)(3\,1\,4)\\
&\quad+(4\,2\,1\,3)(2\,3)(1\,4)+(4\,2\,1\,3)(2\,3\,1\,4)+(4\,2\,1\,3)(3\,2\,1\,4)\\
=&0,
\end{aligned}\tag{37}
$$

$\mathcal{Y}_1^{[3,1]}$ and $\mathcal{Y}_2^{[3,1]}$ are orthogonal.
Since

$$
\begin{aligned}
&\mathcal{Y}_2^{[3,1]}\mathcal{Y}_3^{[3,1]}\\
=&[E+(1\quad 2)+(1\quad 4)+(2\quad 4)+(1\quad 2\quad 4)+(4\quad 2\quad 1)-(1\quad 3)-(2\quad 1\quad 3)-(4\quad 1\quad 3)-(2\quad 4)(1\quad 3)-(2\quad 4\quad 1\quad 3)-(4\quad 2\quad 1\quad 3)]\\
&[E+(1\quad 3)+(1\quad 4)+(3\quad 4)+(1\quad 3\quad 4)+(4\quad 3\quad 1)-(1\quad 2)-(3\quad 1\quad 2)-(4\quad 1\quad 2)-(3\quad 4)(1\quad 2)-(3\quad 4\quad 1\quad 2)-(4\quad 3\quad 1\quad 2)]\\
=&E+(1\,3)+(1\,4)+(3\,4)+(1\,3\,4)+(4\,3\,1)-(1\,2)-(3\,1\,2)-(4\,1\,2)-(3\,4)(1\,2)-(3\,4\,1\,2)-(4\,3\,1\,2)\\
&+(1\,2)+(1\,2)(1\,3)+(1\,2)(1\,4)+(1\,2)(3\,4)+(1\,2)(1\,3\,4)+(1\,2)(4\,3\,1)-(1\,2)(1\,2)-(1\,2)(3\,1\,2)-(1\,2)(4\,1\,2)-(1\,2)(3\,4)(1\,2)-(1\,2)(3\,4\,1\,2)\\
&\quad-(1\,2)(4\,3\,1\,2)\\
&+(1\,4)+(1\,4)(1\,3)+(1\,4)(1\,4)+(1\,4)(3\,4)+(1\,4)(1\,3\,4)+(1\,4)(4\,3\,1)-(1\,4)(1\,2)-(1\,4)(3\,1\,2)-(1\,4)(4\,1\,2)-(1\,4)(3\,4)(1\,2)-(1\,4)(3\,4\,1\,2)\\
&\quad-(1\,4)(4\,3\,1\,2)\\
&+(2\,4)+(2\,4)(1\,3)+(2\,4)(1\,4)+(2\,4)(3\,4)+(2\,4)(1\,3\,4)+(2\,4)(4\,3\,1)-(2\,4)(1\,2)-(2\,4)(3\,1\,2)-(2\,4)(4\,1\,2)-(2\,4)(3\,4)(1\,2)-(2\,4)(3\,4\,1\,2)\\
&\quad-(2\,4)(4\,3\,1\,2)\\
&+(1\,2\,4)+(1\,2\,4)(1\,3)+(1\,2\,4)(1\,4)+(1\,2\,4)(3\,4)+(1\,2\,4)(1\,3\,4)+(1\,2\,4)(4\,3\,1)-(1\,2\,4)(1\,2)-(1\,2\,4)(3\,1\,2)-(1\,2\,4)(4\,1\,2)-(1\,2\,4)(3\,4)(1\,2)\\
&\quad-(1\,2\,4)(3\,4\,1\,2)-(1\,2\,4)(4\,3\,1\,2)\\
&+(4\,2\,1)+(4\,2\,1)(1\,3)+(4\,2\,1)(1\,4)+(4\,2\,1)(3\,4)+(4\,2\,1)(1\,3\,4)+(4\,2\,1)(4\,3\,1)-(4\,2\,1)(1\,2)-(4\,2\,1)(3\,1\,2)-(4\,2\,1)(4\,1\,2)-(4\,2\,1)(3\,4)(1\,2)\\
&\quad-(4\,2\,1)(3\,4\,1\,2)-(4\,2\,1)(4\,3\,1\,2)\\
&-(1\,3)-(1\,3)(1\,3)-(1\,3)(1\,4)-(1\,3)(3\,4)-(1\,3)(1\,3\,4)-(1\,3)(4\,3\,1)+(1\,3)(1\,2)+(1\,3)(3\,1\,2)+(1\,3)(4\,1\,2)+(1\,3)(3\,4)(1\,2)+(1\,3)(3\,4\,1\,2)\\
&\quad+(1\,3)(4\,3\,1\,2)\\
&-(2\,1\,3)-(2\,1\,3)(1\,3)-(2\,1\,3)(1\,4)-(2\,1\,3)(3\,4)-(2\,1\,3)(1\,3\,4)-(2\,1\,3)(4\,3\,1)+(2\,1\,3)(1\,2)+(2\,1\,3)(3\,1\,2)+(2\,1\,3)(4\,1\,2)+(2\,1\,3)(3\,4)(1\,2)\\
&\quad+(2\,1\,3)(3\,4\,1\,2)+(2\,1\,3)(4\,3\,1\,2)\\
&-(4\,1\,3)-(4\,1\,3)(1\,3)-(4\,1\,3)(1\,4)-(4\,1\,3)(3\,4)-(4\,1\,3)(1\,3\,4)-(4\,1\,3)(4\,3\,1)+(4\,1\,3)(1\,2)+(4\,1\,3)(3\,1\,2)+(4\,1\,3)(4\,1\,2)+(4\,1\,3)(3\,4)(1\,2)\\
&\quad+(4\,1\,3)(3\,4\,1\,2)+(4\,1\,3)(4\,3\,1\,2)\\
&-(2\,4)(1\,3)-(2\,4)(1\,3)(1\,3)-(2\,4)(1\,3)(1\,4)-(2\,4)(1\,3)(3\,4)-(2\,4)(1\,3)(1\,3\,4)-(2\,4)(1\,3)(4\,3\,1)+(2\,4)(1\,3)(1\,2)+(2\,4)(1\,3)(3\,1\,2)\\
&\quad+(2\,4)(1\,3)(4\,1\,2)+(2\,4)(1\,3)(3\,4)(1\,2)+(2\,4)(1\,3)(3\,4\,1\,2)+(2\,4)(1\,3)(4\,3\,1\,2)\\
&-(2\,4\,1\,3)-(2\,4\,1\,3)(1\,3)-(2\,4\,1\,3)(1\,4)-(2\,4\,1\,3)(3\,4)-(2\,4\,1\,3)(1\,3\,4)-(2\,4\,1\,3)(4\,3\,1)+(2\,4\,1\,3)(1\,2)+(2\,4\,1\,3)(3\,1\,2)+(2\,4\,1\,3)(4\,1\,2)\\
&\quad+(2\,4\,1\,3)(3\,4)(1\,2)+(2\,4\,1\,3)(3\,4\,1\,2)+(2\,4\,1\,3)(4\,3\,1\,2)\\
&-(4\,2\,1\,3)-(4\,2\,1\,3)(1\,3)-(4\,2\,1\,3)(1\,4)-(4\,2\,1\,3)(3\,4)-(4\,2\,1\,3)(1\,3\,4)-(4\,2\,1\,3)(4\,3\,1)+(4\,2\,1\,3)(1\,2)+(4\,2\,1\,3)(3\,1\,2)+(4\,2\,1\,3)(4\,1\,2)\\
&\quad+(4\,2\,1\,3)(3\,4)(1\,2)+(4\,2\,1\,3)(3\,4\,1\,2)+(4\,2\,1\,3)(4\,3\,1\,2)\\
=&0,
\end{aligned}\tag{38}
$$

and

$$
\begin{aligned}
&\mathcal{Y}_3^{[3,1]}\mathcal{Y}_2^{[3,1]}\\
=&[E+(1\quad 3)+(1\quad 4)+(3\quad 4)+(1\quad 3\quad 4)+(4\quad 3\quad 1)-(1\quad 2)-(3\quad 1\quad 2)-(4\quad 1\quad 2)-(3\quad 4)(1\quad 2)-(3\quad 4\quad 1\quad 2)-(4\quad 3\quad 1\quad 2)]\\
&[E+(1\quad 2)+(1\quad 4)+(2\quad 4)+(1\quad 2\quad 4)+(4\quad 2\quad 1)-(1\quad 3)-(2\quad 1\quad 3)-(4\quad 1\quad 3)-(2\quad 4)(1\quad 3)-(2\quad 4\quad 1\quad 3)-(4\quad 2\quad 1\quad 3)]\\
=&E+(1\,2)+(1\,4)+(2\,4)+(1\,2\,4)+(4\,2\,1)-(1\,3)-(2\,1\,3)-(4\,1\,3)-(2\,4)(1\,3)-(2\,4\,1\,3)-(4\,2\,1\,3)\\
&+(1\,3)+(1\,3)(1\,2)+(1\,3)(1\,4)+(1\,3)(2\,4)+(1\,3)(1\,2\,4)+(1\,3)(4\,2\,1)-(1\,3)(1\,3)-(1\,3)(2\,1\,3)-(1\,3)(4\,1\,3)-(1\,3)(2\,4)(1\,3)-(1\,3)(2\,4\,1\,3)\\
&\quad-(1\,3)(4\,2\,1\,3)\\
&+(1\,4)+(1\,4)(1\,2)+(1\,4)(1\,4)+(1\,4)(2\,4)+(1\,4)(1\,2\,4)+(1\,4)(4\,2\,1)-(1\,4)(1\,3)-(1\,4)(2\,1\,3)-(1\,4)(4\,1\,3)-(1\,4)(2\,4)(1\,3)-(1\,4)(2\,4\,1\,3)\\
&\quad-(1\,4)(4\,2\,1\,3)\\
&+(3\,4)+(3\,4)(1\,2)+(3\,4)(1\,4)+(3\,4)(2\,4)+(3\,4)(1\,2\,4)+(3\,4)(4\,2\,1)-(3\,4)(1\,3)-(3\,4)(2\,1\,3)-(3\,4)(4\,1\,3)-(3\,4)(2\,4)(1\,3)-(3\,4)(2\,4\,1\,3)\\
&\quad-(3\,4)(4\,2\,1\,3)\\
&+(1\,3\,4)+(1\,3\,4)(1\,2)+(1\,3\,4)(1\,4)+(1\,3\,4)(2\,4)+(1\,3\,4)(1\,2\,4)+(1\,3\,4)(4\,2\,1)-(1\,3\,4)(1\,3)-(1\,3\,4)(2\,1\,3)-(1\,3\,4)(4\,1\,3)-(1\,3\,4)(2\,4)(1\,3)\\
&\quad-(1\,3\,4)(2\,4\,1\,3)-(1\,3\,4)(4\,2\,1\,3)\\
&+(4\,3\,1)+(4\,3\,1)(1\,2)+(4\,3\,1)(1\,4)+(4\,3\,1)(2\,4)+(4\,3\,1)(1\,2\,4)+(4\,3\,1)(4\,2\,1)-(4\,3\,1)(1\,3)-(4\,3\,1)(2\,1\,3)-(4\,3\,1)(4\,1\,3)-(4\,3\,1)(2\,4)(1\,3)\\
&\quad-(4\,3\,1)(2\,4\,1\,3)-(4\,3\,1)(4\,2\,1\,3)\\
&-(1\,2)-(1\,2)(1\,2)-(1\,2)(1\,4)-(1\,2)(2\,4)-(1\,2)(1\,2\,4)-(1\,2)(4\,2\,1)+(1\,2)(1\,3)+(1\,2)(2\,1\,3)+(1\,2)(4\,1\,3)+(1\,2)(2\,4)(1\,3)+(1\,2)(2\,4\,1\,3)\\
&\quad+(1\,2)(4\,2\,1\,3)\\
&-(3\,1\,2)-(3\,1\,2)(1\,2)-(3\,1\,2)(1\,4)-(3\,1\,2)(2\,4)-(3\,1\,2)(1\,2\,4)-(3\,1\,2)(4\,2\,1)+(3\,1\,2)(1\,3)+(3\,1\,2)(2\,1\,3)+(3\,1\,2)(4\,1\,3)+(3\,1\,2)(2\,4)(1\,3)\\
&\quad+(3\,1\,2)(2\,4\,1\,3)+(3\,1\,2)(4\,2\,1\,3)\\
&-(4\,1\,2)-(4\,1\,2)(1\,2)-(4\,1\,2)(1\,4)-(4\,1\,2)(2\,4)-(4\,1\,2)(1\,2\,4)-(4\,1\,2)(4\,2\,1)+(4\,1\,2)(1\,3)+(4\,1\,2)(2\,1\,3)+(4\,1\,2)(4\,1\,3)+(4\,1\,2)(2\,4)(1\,3)\\
&\quad+(4\,1\,2)(2\,4\,1\,3)+(4\,1\,2)(4\,2\,1\,3)\\
&-(3\,4)(1\,2)-(3\,4)(1\,2)(1\,2)-(3\,4)(1\,2)(1\,4)-(3\,4)(1\,2)(2\,4)-(3\,4)(1\,2)(1\,2\,4)-(3\,4)(1\,2)(4\,2\,1)+(3\,4)(1\,2)(1\,3)+(3\,4)(1\,2)(2\,1\,3)\\
&\quad+(3\,4)(1\,2)(4\,1\,3)+(3\,4)(1\,2)(2\,4)(1\,3)+(3\,4)(1\,2)(2\,4\,1\,3)+(3\,4)(1\,2)(4\,2\,1\,3)
\end{aligned}
$$

$$
\begin{aligned}
&-(3412)-(3412)(12)-(3412)(14)-(3412)(24)-(3412)(124)-(3412)(421)+(3412)(13)+(3412)(213)+(3412)(413)\\
&\quad+(3412)(24)(13)+(3412)(2413)+(3412)(4213)\\
&-(4312)-(4312)(12)-(4312)(14)-(4312)(24)-(4312)(124)-(4312)(421)+(4312)(13)+(4312)(213)+(4312)(413)\\
&\quad+(4312)(24)(13)+(4312)(2413)+(4312)(4213)\\
=&0, \qquad (39)
\end{aligned}
$$

$\mathcal{Y}_2^{[3,1]}$ and $\mathcal{Y}_3^{[3,1]}$ are orthogonal.
Since

$$
\begin{aligned}
&\mathcal{Y}_1^{[3,1]}\mathcal{Y}_3^{[3,1]}\\
=&[E+(1\ \ 2)+(1\ \ 3)+(2\ \ 3)+(1\ \ 2\ \ 3)+(3\ \ 2\ \ 1)-(1\ \ 4)-(2\ \ 1\ \ 4)-(3\ \ 1\ \ 4)-(2\ \ 3)(1\ \ 4)-(2\ \ 3\ \ 1\ \ 4)-(3\ \ 2\ \ 1\ \ 4)]\\
&[E+(1\ \ 3)+(1\ \ 4)+(3\ \ 4)+(1\ \ 3\ \ 4)+(4\ \ 3\ \ 1)-(1\ \ 2)-(3\ \ 1\ \ 2)-(4\ \ 1\ \ 2)-(3\ \ 4)(1\ \ 2)-(3\ \ 4\ \ 1\ \ 2)-(4\ \ 3\ \ 1\ \ 2)]\\
=&E+(13)+(14)+(34)+(134)+(431)-(12)-(312)-(412)-(34)(12)-(3412)-(4312)\\
&+(12)+(12)(13)+(12)(14)+(12)(34)+(12)(134)+(12)(431)-(12)(12)-(12)(312)-(12)(412)-(12)(34)(12)-(12)(3412)\\
&\quad-(12)(4312)\\
&+(13)+(13)(13)+(13)(14)+(13)(34)+(13)(134)+(13)(431)-(13)(12)-(13)(312)-(13)(412)-(13)(34)(12)-(13)(3412)\\
&\quad-(13)(4312)\\
&+(23)+(23)(13)+(23)(14)+(23)(34)+(23)(134)+(23)(431)-(23)(12)-(23)(312)-(23)(412)-(23)(34)(12)-(23)(3412)\\
&\quad-(23)(4312)\\
&+(123)+(123)(13)+(123)(14)+(123)(34)+(123)(134)+(123)(431)-(123)(12)-(123)(312)-(123)(412)-(123)(34)(12)\\
&\quad-(123)(3412)-(123)(4312)\\
&+(321)+(321)(13)+(321)(14)+(321)(34)+(321)(134)+(321)(431)-(321)(12)-(321)(312)-(321)(412)-(321)(34)(12)\\
&\quad-(321)(3412)-(321)(4312)\\
&-(14)-(14)(13)-(14)(14)-(14)(34)-(14)(134)-(14)(431)+(14)(12)+(14)(312)+(14)(412)+(14)(34)(12)+(14)(3412)\\
&\quad+(14)(4312)\\
&-(214)-(214)(13)-(214)(14)-(214)(34)-(214)(134)-(214)(431)+(214)(12)+(214)(312)+(214)(412)+(214)(34)(12)\\
&\quad+(214)(3412)+(214)(4312)\\
&-(314)-(314)(13)-(314)(14)-(314)(34)-(314)(134)-(314)(431)+(314)(12)+(314)(312)+(314)(412)+(314)(34)(12)\\
&\quad+(314)(3412)+(314)(4312)\\
&-(23)(14)-(23)(14)(13)-(23)(14)(14)-(23)(14)(34)-(23)(14)(134)-(23)(14)(431)+(23)(14)(12)+(23)(14)(312)\\
&\quad+(23)(14)(412)+(23)(14)(34)(12)+(23)(14)(3412)+(23)(14)(4312)\\
&-(2314)-(2314)(13)-(2314)(14)-(2314)(34)-(2314)(134)-(2314)(431)+(2314)(12)+(2314)(312)+(2314)(412)\\
&\quad+(2314)(34)(12)+(2314)(3412)+(2314)(4312)\\
&-(3214)-(3214)(13)-(3214)(14)-(3214)(34)-(3214)(134)-(3214)(431)+(3214)(12)+(3214)(312)+(3214)(412)\\
&\quad+(3214)(34)(12)+(3214)(3412)+(3214)(4312)\\
=&0, \qquad (40)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{Y}_1^{[3,1]}\mathcal{Y}_3^{[3,1]}\\
=&[E+(1\ \ 3)+(1\ \ 4)+(3\ \ 4)+(1\ \ 3\ \ 4)+(4\ \ 3\ \ 1)-(1\ \ 2)-(3\ \ 1\ \ 2)-(4\ \ 1\ \ 2)-(3\ \ 4)(1\ \ 2)-(3\ \ 4\ \ 1\ \ 2)-(4\ \ 3\ \ 1\ \ 2)]\\
&[E+(1\ \ 2)+(1\ \ 3)+(2\ \ 3)+(1\ \ 2\ \ 3)+(3\ \ 2\ \ 1)-(1\ \ 4)-(2\ \ 1\ \ 4)-(3\ \ 1\ \ 4)-(2\ \ 3)(1\ \ 4)-(2\ \ 3\ \ 1\ \ 4)-(3\ \ 2\ \ 1\ \ 4)]\\
=&E+(12)+(13)+(23)+(123)+(321)-(14)-(214)-(314)-(23)(14)-(2314)-(3214)\\
&+(13)+(13)(12)+(13)(13)+(13)(23)+(13)(123)+(13)(321)-(13)(14)-(13)(214)-(13)(314)-(13)(23)(14)-(13)(2314)\\
&\quad-(13)(3214)\\
&+(14)+(14)(12)+(14)(13)+(14)(23)+(14)(123)+(14)(321)-(14)(14)-(14)(214)-(14)(314)-(14)(23)(14)-(14)(2314)\\
&\quad-(14)(3214)\\
&+(34)+(34)(12)+(34)(13)+(34)(23)+(34)(123)+(34)(321)-(34)(14)-(34)(214)-(34)(314)-(34)(23)(14)-(34)(2314)\\
&\quad-(34)(3214)\\
&+(134)+(134)(12)+(134)(13)+(134)(23)+(134)(123)+(134)(321)-(134)(14)-(134)(214)-(134)(314)-(134)(23)(14)\\
&\quad-(134)(2314)-(134)(3214)\\
&+(431)+(431)(12)+(431)(13)+(431)(23)+(431)(123)+(431)(321)-(431)(14)-(431)(214)-(431)(314)-(431)(23)(14)\\
&\quad-(431)(2314)-(431)(3214)\\
&-(12)-(12)(12)-(12)(13)-(12)(23)-(12)(123)-(12)(321)+(12)(14)+(12)(214)+(12)(314)+(12)(23)(14)+(12)(2314)\\
&\quad+(12)(3214)\\
&-(312)-(312)(12)-(312)(13)-(312)(23)-(312)(123)-(312)(321)+(312)(14)+(312)(214)+(312)(314)+(312)(23)(14)\\
&\quad+(312)(2314)+(312)(3214)\\
&-(412)-(412)(12)-(412)(13)-(412)(23)-(412)(123)-(412)(321)+(412)(14)+(412)(214)+(412)(314)+(412)(23)(14)\\
&\quad+(412)(2314)+(412)(3214)
\end{aligned}
$$

$$
\begin{aligned}
&- (3\,4)(1\,2) - (3\,4)(1\,2)(1\,2) - (3\,4)(1\,2)(1\,3) - (3\,4)(1\,2)(2\,3) - (3\,4)(1\,2)(1\,2\,3) - (3\,4)(1\,2)(3\,2\,1) + (3\,4)(1\,2)(1\,4) + (3\,4)(1\,2)(2\,1\,4)\\
&\quad + (3\,4)(1\,2)(3\,1\,4) + (3\,4)(1\,2)(2\,3)(1\,4) + (3\,4)(1\,2)(2\,3\,1\,4) + (3\,4)(1\,2)(3\,2\,1\,4)\\
&- (3\,4\,1\,2) - (3\,4\,1\,2)(1\,2) - (3\,4\,1\,2)(1\,3) - (3\,4\,1\,2)(2\,3) - (3\,4\,1\,2)(1\,2\,3) - (3\,4\,1\,2)(3\,2\,1) + (3\,4\,1\,2)(1\,4) + (3\,4\,1\,2)(2\,1\,4) + (3\,4\,1\,2)(3\,1\,4)\\
&\quad + (3\,4\,1\,2)(2\,3)(1\,4) + (3\,4\,1\,2)(2\,3\,1\,4) + (3\,4\,1\,2)(3\,2\,1\,4)\\
&- (4\,3\,1\,2) - (4\,3\,1\,2)(1\,2) - (4\,3\,1\,2)(1\,3) - (4\,3\,1\,2)(2\,3) - (4\,3\,1\,2)(1\,2\,3) - (4\,3\,1\,2)(3\,2\,1) + (4\,3\,1\,2)(1\,4) + (4\,3\,1\,2)(2\,1\,4) + (4\,3\,1\,2)(3\,1\,4)\\
&\quad + (4\,3\,1\,2)(2\,3)(1\,4) + (4\,3\,1\,2)(2\,3\,1\,4) + (4\,3\,1\,2)(3\,2\,1\,4)\\
=&0, \qquad (41)
\end{aligned}
$$

$\mathcal{Y}_1^{[3,1]}$ and $\mathcal{Y}_3^{[3,1]}$ are orthogonal.
Therefore, all the Young operators in the irreducible representation $[3,1]$ of S_4 are orthogonal.

□

Problem 3 Score: ________. Write down all the standard basis vectors in the irreducible representations $[3,1]$ of S_4.

Solution: The permutation transforming $\mathcal{Y}_1^{[3,1]}$ to $\mathcal{Y}_2^{[3,1]}$ is $R_{12} = (3\quad 4)$.
The permutation transforming $\mathcal{Y}_2^{[3,1]}$ to $\mathcal{Y}_1^{[3,1]}$ is $R_{21} = (3\quad 4)$.
The permutation transforming $\mathcal{Y}_2^{[3,1]}$ to $\mathcal{Y}_3^{[3,1]}$ is $R_{23} = (2\quad 3)$.
The permutation transforming $\mathcal{Y}_3^{[3,1]}$ to $\mathcal{Y}_2^{[3,1]}$ is $R_{32} = (2\quad 3)$.
The permutation transforming $\mathcal{Y}_1^{[3,1]}$ to $\mathcal{Y}_3^{[3,1]}$ is $R_{13} = (2\quad 3\quad 4)$.
The permutation transforming $\mathcal{Y}_3^{[3,1]}$ to $\mathcal{Y}_1^{[3,1]}$ is $R_{31} = (4\quad 3\quad 2)$.
The standard basis vectors in the irreducible representation $[3,1]$ of S_4:

$$
b_{11}^{[3,1]} = e_1^{[3,1]} = \frac{3}{4!}\mathcal{Y}_1^{[3,1]} = \frac{1}{8}[E + (1\,2) + (1\,3) + (2\,3) + (1\,2\,3) + (3\,2\,1) - (1\,4) - (2\,1\,4) - (3\,1\,4) - (2\,3)(1\,4) - (2\,3\,1\,4) - (3\,2\,1\,4)]. \qquad (42)
$$

$$
\begin{aligned}
b_{21}^{[3,1]} =& R_{21}e_1^{[3,1]}\\
=&\frac{1}{8}[(3\,4) + (3\,4)(1\,2) + (4\,3\,2) + (4\,3\,1\,2) + (4\,3\,2\,1) - (3\,4\,1) - (3\,4\,2\,1) - (1\,3) - (3\,2\,4\,1) - (2\,4)(3\,1) - (3\,2\,1)], \qquad (43)\\
b_{31}^{[3,1]} =& R_{31}e_1^{[3,1]}\\
=&\frac{1}{8}[(4\,3\,2) + (4\,3\,2\,1) + (2\,4\,3\,1) + (4\,3) + (4\,3\,1) + (4\,3)(2\,1) + (4\,2\,1\,3) - (3\,2\,4\,1) - (3\,2\,1) - (2\,4)(3\,1) - (3\,4\,1) - (3\,1) - (4\,3)(2\,1)], \qquad (44)
\end{aligned}
$$

$$
b_{22}^{[3,1]} = e_2^{[3,1]} = \frac{3}{4!}\mathcal{Y}_2^{[3,1]} = \frac{1}{8}[E + (1\,2) + (1\,4) + (2\,4) + (1\,2\,4) + (4\,2\,1) - (1\,3) - (2\,1\,3) - (4\,1\,3) - (2\,4)(1\,3) - (2\,4\,1\,3) - (4\,2\,1\,3)], \qquad (45)
$$

$$
\begin{aligned}
b_{12}^{[3,1]} =& R_{12}e_2^{[3,1]}\\
=&\frac{1}{8}[(3\,4) + (3\,4)(1\,2) + (3\,4\,1) + (3\,4\,2) + (3\,4\,1\,2) + (3\,4\,2\,1) - (4\,3\,1) - (4\,3\,2\,1) - (4\,1) - (4\,2\,3\,1) - (4\,1)(3\,2) - (4\,2\,1)], \qquad (46)\\
b_{32}^{[3,1]} =& R_{32}e_2^{[3,1]}\\
=&\frac{1}{8}[(2\,3) + (3\,2\,1) + (2\,3)(1\,4) + (3\,2\,4) + (3\,2\,4\,1) + (3\,2\,1\,4) - (2\,3\,1) - (2\,1) - (2\,3\,4\,1) - (2\,4\,3\,1) - (2\,4\,1) - (2\,1)(3\,4)], \qquad (47)
\end{aligned}
$$

$$
b_{33}^{[3,1]} = e_3^{[3,1]} = \frac{3}{4!}\mathcal{Y}_3^{[3,1]} = \frac{1}{8}[E + (1\,3) + (1\,4) + (3\,4) + (1\,3\,4) + (4\,3\,1) - (1\,2) - (3\,1\,2) - (4\,1\,2) - (3\,4)(1\,2) - (3\,4\,1\,2) - (4\,3\,1\,2)], \qquad (48)
$$

$$
\begin{aligned}
b_{13}^{[3,1]} =& R_{13}e_3^{[3,1]}\\
=&\frac{1}{8}[(2\,3\,4) + (4\,2\,3\,1) + (2\,3\,4\,1) + (2\,3) + (2\,3)(4\,1) + (2\,3\,1) - (3\,4\,2\,1) - (4\,2)(3\,1) - (3\,4\,1) - (3\,2\,1) - (3\,2\,4\,1) - (3\,1)], \qquad (49)\\
b_{23}^{[3,1]} =& R_{23}e_3^{[3,1]}\\
=&\frac{1}{8}[(2\,3) + (2\,3\,1) + (2\,3)(1\,4) + (2\,3\,4) + (2\,3\,4\,1) + (2\,3\,1\,4) - (3\,2\,1) - (3\,1) - (3\,2\,4\,1) - (3\,4\,2\,1) - (3\,4\,1) - (3\,1)(2\,4)]. \qquad (50)
\end{aligned}
$$

□

Problem 4 Score: ________. Write down all the $Q_{\nu k}$'s in the irreducible representations $[3,1]$ of S_4. Find all the $\mathcal{Y}'$'s using $\mathcal{Y}' = Q_{\nu k}\mathcal{Y}_{\nu k}Q_{\nu k}^{-1}$. Find all the $\mathcal{Y}_\mu(S)$'s in the irreducible representation $[3,1]$ of S_4 for $S = (1\ 2\ 3\ 4)$ using $\mathcal{Y}_\nu(S) = S\mathcal{Y}_\nu^{[3,1]}S^{-1}$.

Solution: Since all the Young operators are orthogonal, $\mathcal{Y}_\mu^{[3,1]}\mathcal{Y}_\nu^{[3,1]} = 0$ for $\mu,\nu = 1,2,3$ and $\mu \neq \nu$, we have

$$
P_{\mu\nu} = 0, \quad \mu,\nu = 1,2,3, \mu = \neq \nu. \qquad (51)
$$

Using $y_{d_{[\lambda]}} = E$ and $y_\nu = E - \sum_{\rho=\nu+1}^{d_{[\lambda]}} P_{\nu\rho} y_\rho$ for $\nu < d_{[\lambda]}$, we have

$$y_5 = y_4 = y_3 = y_2 = y_1 = E. \tag{52}$$

Using $y_v = \sum_k \delta_k T_k$, we have

$$y_5 = E : \delta_1 = 1, T_1 = E, \tag{53}$$
$$y_4 = E : \delta_1 = 1, T_1 = E, \tag{54}$$
$$y_3 = E : \delta_1 = 1, T_1 = E, \tag{55}$$
$$y_2 = E : \delta_1 = 1, T_1 = E, \tag{56}$$
$$y_1 = E : \delta_1 = 1, T_1 = E. \tag{57}$$

Using $\mathcal{Y}_{\nu k} = T_k^{-1} \mathcal{Y}_\nu^{[\lambda]} T_k$, we have

$$\mathcal{Y}_{11} = \mathcal{Y}_1^{[3,1]} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}, \tag{58}$$

$$\mathcal{Y}_{21} = \mathcal{Y}_2^{[3,1]} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}, \tag{59}$$

$$\mathcal{Y}_{31} = \mathcal{Y}_3^{[3,1]} = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}. \tag{60}$$

The vertical permutations of $\mathcal{Y}_{\nu 1}$'s for $\nu = 1, 2, 3$ are equal to the vertical permutations of $\mathcal{Y}_\nu^{[3,1]}$, as shown in table 1, line 4. Using $\mathcal{Y}' = Q_{\nu k} \mathcal{Y}_{\nu k} Q_{\nu k}^{-1}$, we have all the $\mathcal{Y}'$'s, as shown in 1, line 5. For $S = (1 \quad 2 \quad 3 \quad 4)$, using $\mathcal{Y}_\nu(S) = S \mathcal{Y}_\nu^{[3,1]} S^{-1}$,

Table 1:

	$\nu = 1$, $k = 1$	$\nu = 2$, $k = 1$	$\nu = 3$, $k = 1$
$\mathcal{Y}_{\nu k}$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$
$Q_{\nu k}$	$E, (1\,4)$	$E, (1\,3)$	$E, (1\,2)$
$\mathcal{Y}'$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$, $\begin{array}{\|c\|c\|c\|}\hline 4 & 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$, $\begin{array}{\|c\|c\|c\|}\hline 3 & 2 & 4 \\ \hline 1 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$, $\begin{array}{\|c\|c\|c\|}\hline 2 & 3 & 4 \\ \hline 1 \\ \cline{1-1}\end{array}$

we have

$$\mathcal{Y}_1(S) = S\mathcal{Y}_2^{[3,1]} S^{-1} = \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 1 \\ \cline{1-1}\end{array}, \tag{61}$$

$$\mathcal{Y}_2(S) = S\mathcal{Y}_2^{[3,1]} S^{-1} = \begin{array}{|c|c|c|}\hline 2 & 3 & 1 \\ \hline 4 \\ \cline{1-1}\end{array}, \tag{62}$$

$$\mathcal{Y}_3(S) = S\mathcal{Y}_3^{[3,1]} S^{-1} = \begin{array}{|c|c|c|}\hline 2 & 4 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}. \tag{63}$$

□

Problem 5 Score: __________. Construct the table for $A^\mu_{\nu k}(S)$ for $S = (1\ 2\ 3\ 4)$ with $\mathcal{Y}_\mu(S)$ labeling the columns and $\sum_k \delta_k \mathcal{Y}_{\nu k}$ labeling the rows. Write down the representation matrix of $S = (1\ 2\ 3\ 4)$.

Solution: The $Q_{\nu k}$ are shown in table 2.

Table 2:

		$\nu=1$, $k=1$	$\nu=2$, $k=1$	$\nu=3$, $k=1$
$\mathcal{Y}_{\nu k}$		$\begin{ytableau} 1 & 2 & 3 \\ 4 \end{ytableau}$	$\begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau}$	$\begin{ytableau} 1 & 3 & 4 \\ 2 \end{ytableau}$
$\mathcal{Y}'$		$\begin{ytableau} 1 & 2 & 3 \\ 4 \end{ytableau}$, $\begin{ytableau} 4 & 2 & 3 \\ 1 \end{ytableau}$	$\begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau}$, $\begin{ytableau} 3 & 2 & 4 \\ 1 \end{ytableau}$	$\begin{ytableau} 1 & 3 & 4 \\ 2 \end{ytableau}$, $\begin{ytableau} 2 & 3 & 4 \\ 1 \end{ytableau}$
$\mathcal{Y}_\mu(S)$	$\begin{ytableau} 2 & 3 & 4 \\ 1 \end{ytableau}$	-1	-1	-1
	$\begin{ytableau} 2 & 3 & 1 \\ 4 \end{ytableau}$	1	0	0
	$\begin{ytableau} 2 & 4 & 1 \\ 3 \end{ytableau}$	0	1	0

The table for $A^\mu_{\nu k}(S)$ for $S=(1\ 2\ 3\ 4)$ with $\mathcal{Y}_\mu(S)$ labeling the columns and $\sum_k \delta_k \mathcal{Y}_{\nu k}$ labeling the rows are shown in table 3.

Table 3:

$\sum_k \delta \mathcal{Y}_{\nu k}$ \ $\mathcal{Y}_\mu(S)$	$\begin{ytableau} 2 & 3 & 4 \\ 1 \end{ytableau}$	$\begin{ytableau} 2 & 3 & 1 \\ 4 \end{ytableau}$	$\begin{ytableau} 2 & 4 & 1 \\ 3 \end{ytableau}$
$\begin{ytableau} 1 & 2 & 3 \\ 4 \end{ytableau}$	-1	1	0
$\begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau}$	-1	0	1
$\begin{ytableau} 1 & 3 & 4 \\ 2 \end{ytableau}$	-1	0	0

Therefore, the representation matrix of $S=(1\quad 2\quad 3\quad 4)$ is

$$\Gamma^{[3,1]}[(1\,2\,3\,4)]=\begin{pmatrix}-1 & 1 & 0\\ -1 & 0 & 1\\ -1 & 0 & 0\end{pmatrix}. \tag{64}$$

$\square$