Assignment 02

Due Time : 8:15, March 18, 2020 (Wednesday)

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Score : ________

Problem 1 (Problem title) Score: ________. Show that the intersection S of two invariant subgroups S_1 and S_2 of a group G is an invariant subgroup.

Solution: First, we prove that S is a subgroup of G. Since S_1 and S_2 are two subgroups of G and S is the intersection of S_1 and S_2, S is a subset of G. S satisfies all the four group axioms:

1. **Closure**: Since S is the intersection of S_1 and S_2, if T and R are the elements of S, then T and R are elements of both S_1 and S_2. From the closure of S_1 and S_2, we have $TR \in S_1$ and $TR \in S_2$, so $TR \in S$.
2. **Associativity**: Since S is a subset of group G, the associativity of S is automatically satisfied.
3. **Existence of identity element**: Since S_1 and S_2 are subgroups, the identity element E is in both S_1 and S_2. Then E is also in their intersection S.
4. **Existence of inverse element**: Since S is the intersection of S_1 and S_2, each element T of S is in both S_1 and S_2. Because S_1 and S_2 are subgroups, the inverse T^{-1} of T is also in both S_1 and S_2. Then T^{-1} is in $S = S_1 \cap S_2$.

In this way, S is a subgroup of G.

Then we prove that $XTX^{-1} \in S = S_1 \cap S_2$ holds for every $T \in S$ and every $X \in G$. Since S is the intersection of S_1 and S_2, we have $T \in S_1$ and $T \in S_2$ for every $T \in S$. Because S_1 and S_2 are two invariant subgroups of G, we have $XTX^{-1} \in S_1$ and $XTX^{-1} \in S_2$ for every X in G. In this way, $XTX^{-1} \in S = S_1 \cap S_2$ holds for every $T \in S$ and every $X \in G$.

Therefore, S is an invariant subgroup of G. □

Problem 2 Score: ________. The multiplication table of a finite group G is given by

	E	A	B	C	D	F	I	J	K	L	M	N
E	E	A	B	C	D	F	I	J	K	L	M	N
A	A	E	F	I	J	B	C	D	M	N	K	L
B	B	F	A	K	L	E	M	N	I	J	C	D
C	C	I	L	A	K	N	E	M	J	F	D	B
D	D	J	K	L	A	M	N	E	F	I	B	C
F	F	B	E	M	N	A	K	L	C	D	I	J
I	I	C	N	E	M	L	A	K	D	B	J	F
J	J	D	M	N	E	K	L	A	B	C	F	I
K	K	M	J	F	I	D	B	C	N	E	L	A
L	L	N	I	J	F	C	D	B	E	M	A	K
M	M	K	D	B	C	J	F	I	L	A	N	E
N	N	L	C	D	B	I	J	F	A	K	E	M

(a) Find the inverse of each element of G.

(b) Find the elements in each class of G.

(c) Find all invariant subgroups of G.

Solution: (a) The inverse of each element of G:

$$\begin{aligned} E^{-1} &= E, & A^{-1} &= A, & B^{-1} &= F, \\ C^{-1} &= I, & D^{-1} &= J, & F^{-1} &= B, \\ I^{-1} &= C, & J^{-1} &= D, & K^{-1} &= L, \\ L^{-1} &= K, & M^{-1} &= N, & N^{-1} &= M. \end{aligned}$$

(b) Constructing a class from A: for $X = E, A, B, C, D, F, I, J, K, L, M, N$,

$$XAX^{-1} = A \tag{1}$$

The class of G constructed from A is $\{A\}$.
Using the similar method, we construct all the classes of G:

$$\{E\}, \quad \{A\}, \quad \{B, C, D\}, \quad \{F, I, J\}, \quad \{K, L\}, \quad \{M, N\}.$$

Table 1: The subgroups of G.

order	Subgroup(s)
1	$\{E\}$
2	$\{E, A\}$
3	$\{E, M, N\}$
4	$\{E, A, B, C, D\}$, $\{E, A, C, I\}$, $\{E, A, D, J\}$
6	$\{E, A, K, L, M, N\}$
12	G

(c) A subgroup of G is an invariant subgroup if and only if it consist entirely of complete classes of G. The subgroups of G are shown in table 1.
The invariant subgroups of G are

$$\{E\}, \quad \{E, A\}, \quad \{E, M, N\}, \quad \{E, A, B, C, D\}, \quad \{E, A, K, L, M, N\}, \quad G$$

□

Problem 3 Score: _________. Consider the group D_3.

(a) List all the classes of D_3.

(b) Find the right and left cosets of the subgroup $S = \{E, A\}$ of D_3.

Solution: (a) The group D_3 is

$$D_3 = \{E, D, F, A, B, C\}, \tag{2}$$

where E is the identity element, D is the rotation in the plane about the center of the equilateral triangle through $2\pi/3$, F is the rotation in the plane about the center of the triangle through $4\pi/3$, and A, B, C are the rotations about the three axis of symmetry of the triangle. The multiplication table of D_3 is shown in table . The inverse of

Table 2: The multiplication table of D_3.

	E	D	F	A	B	C
E	E	D	F	A	B	C
D	D	F	E	B	C	A
F	F	E	D	C	A	B
A	A	C	B	E	F	D
B	B	A	C	D	E	F
C	C	B	A	F	D	E

each element of D_3 is

$$E^{-1} = E, \qquad D^{-1} = F, \qquad F^{-1} = D,$$
$$A^{-1} = A, \qquad B^{-1} = B, \qquad C^{-1} = C.$$

Constructing a class from D: for $X = E, D, F$,

$$X^{-1}DX = D. \tag{3}$$

For $X = A, B, C$,

$$X^{-1}DX = F. \tag{4}$$

The class of D_3 constructed form D is $\{D, F\}$.
Using the similar method, we construct all the classes of D_3:

$$\{E\}, \quad \{D, F\}, \quad \{A, B, C\}. \tag{5}$$

(b) The right cosets of the subgroup $S = \{E, A\}$ of D_3 is

$$SE = SA = \{E, A\},$$
$$SD = SC = \{D, C\},$$

$$SF = SB = \{F, B\}.$$

The left cosets of the subgroup S is

$$\begin{aligned} ES =& AS = \{E, A\}, \\ DS =& BS = \{D, B\}, \\ FS =& CS = \{F, C\}. \end{aligned}$$

□

Problem 4 Score: ________. For the group D_3 and its invariant subgroup $S = \{E, D, F\}$, find the factor group D_3/S. Consider the multiplication table for the factor group.

Solution: The right coset of the invariant subgroup $S = \{E, D, F\}$ of D_3 is

$$\begin{aligned} SE =& SD = SF = \{E, D, F\}, \\ SA =& SB = SC = \{A, B, C\}. \end{aligned}$$

The factor group D_3/S is $\{SE, SA\}$. Under the multiplication operation of $ST_1 \cdot ST_2 = S(T_1T_2)$, the multiplication table of the factor group is shown in table 3. □

Table 3: The multiplication table of D_3/S.

	SE	SA
SE	SE	SA
SA	SA	SA

Problem 5 Score: ________. Consider $C_6 = \{E, a, a^2, a^3, a^4, a^5\}$ and its two subgroups $S_1 = \{E, a^3\}$ and $S_2 = \{E, a^2, a^4\}$. Show that $C_6 = S_1 \otimes S_2$.

Solution: The two subgroups S_1 and S_2 satisfy the following three conditions:

1. The elements of S_1 commute with the elements of S_2, $S^m S^n = S^{m+n} = S^{n+m} = S^n S^m$.
2. S_1 and S_2 have only the identity element E in common.
3. Every element of G' can be written as a product of an element of S_1 with an element of S_2,

$$\begin{aligned} E =& EE, \\ a =& a^3 a^4, \\ a^2 =& Ea^2, \\ a^3 =& a^3 E, \\ a^4 =& Ea^4, \\ a^5 =& a^3 a^2. \end{aligned}$$

Therefore, $C_6 \cong S_1 \times S_2$. □