Group Theory

Solutions to the Problems in Homework Assignment 06

Spring, 2020

1. The basis elements of the real Lie algebra $L = \mathrm{so}(3)$ are given by

$$a_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},\ a_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ a_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Show explicitly that these basis elements possess the following properties.

(a) The basis elements a_1, a_2, and a_3 obey the commutation relations

$$\begin{aligned} [a_1, a_2] &= a_1 a_2 - a_2 a_1 = -a_3, \\ [a_2, a_3] &= a_2 a_3 - a_3 a_2 = -a_1, \\ [a_3, a_1] &= a_3 a_1 - a_1 a_3 = -a_2. \end{aligned}$$

(b) The basis elements a_1, a_2, and a_3 are anti-Hermitian,

$$a_1^\dagger = -a_1,\ a_2^\dagger = -a_2,\ a_3^\dagger = -a_3.$$

(a) For $[a_1, a_2]$, we have

$$\begin{aligned} [a_1, a_2] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -a_3. \end{aligned}$$

For $[a_2, a_3]$, we have

$$\begin{aligned} [a_2, a_3] &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = -a_1. \end{aligned}$$

For $[a_3, a_1]$, we have

$$\begin{aligned} [a_3, a_1] &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = -a_2. \end{aligned}$$

(b) Since a_1, a_2, and a_3 are all real matrices, their Hermitian conjugates are equal to their transposes. For a_1, we have

$$a_1^\dagger = a_1^t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}^t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = -a_1.$$

For a_2, we have

$$a_2^\dagger = a_2^t = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}^t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = -a_2.$$

For a_3, we have

$$a_3^\dagger = a_3^t = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^t = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -a_3.$$

2. The scalar transformation operators $Q(a_1)$, $Q(a_2)$, and $Q(a_3)$ for the real Lie algebra so(3) are found to be given by

$$Q(a_1) = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},\ Q(a_2) = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},\ Q(a_3) = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

Show that $[Q(a_1), Q(a_2)] = -Q(a_3)$, $[Q(a_2), Q(a_3)] = -Q(a_1)$, and $[Q(a_3), Q(a_1)] = -Q(a_2)$.

Let $[Q(a_1), Q(a_2)]$ act on an arbitrary well-behaved function $f(\vec{r})$. We have

$$\begin{aligned}
[Q(a_1), Q(a_2)]f &= \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}\right) - \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right) \\
&= y\frac{\partial f}{\partial x} + yz\frac{\partial^2 f}{\partial z\partial x} - xy\frac{\partial^2 f}{\partial z^2} - z^2\frac{\partial^2 f}{\partial y\partial x} + zx\frac{\partial^2 f}{\partial y\partial z} \\
&\quad - yz\frac{\partial^2 f}{\partial x\partial z} + z^2\frac{\partial^2 f}{\partial x\partial y} + xy\frac{\partial^2 f}{\partial z^2} - x\frac{\partial f}{\partial y} - zx\frac{\partial^2 f}{\partial z\partial y} \\
&= y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y} = -Q(a_3)f.
\end{aligned}$$

Because $f(\vec{r})$ is arbitrary, we have

$$[Q(a_1), Q(a_2)] = -Q(a_3).$$

Let $[Q(a_2), Q(a_3)]$ act on an arbitrary well-behaved function $f(\vec{r})$. We have

$$\begin{aligned}
[Q(a_2), Q(a_3)]f &= \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\right) - \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\left(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}\right) \\
&= z\frac{\partial f}{\partial y} + zx\frac{\partial^2 f}{\partial x\partial y} - yz\frac{\partial^2 f}{\partial x^2} - x^2\frac{\partial^2 f}{\partial z\partial y} + xy\frac{\partial^2 f}{\partial z\partial x} \\
&\quad - zx\frac{\partial^2 f}{\partial y\partial x} + x^2\frac{\partial^2 f}{\partial y\partial z} + yz\frac{\partial^2 f}{\partial x^2} - y\frac{\partial f}{\partial z} - xy\frac{\partial^2 f}{\partial x\partial z} \\
&= z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z} = -Q(a_1)f
\end{aligned}$$

Because $f(\vec{r})$ is arbitrary, we have

$$[Q(a_2), Q(a_3)] = -Q(a_1).$$

Let $[Q(a_3), Q(a_1)]$ act on an arbitrary well-behaved function $f(\vec{r})$. We have

$$\begin{aligned}
[Q(a_3), Q(a_1)]f &= \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right) - \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\right) \\
&= x\frac{\partial f}{\partial z} + xy\frac{\partial^2 f}{\partial y\partial z} - zx\frac{\partial^2 f}{\partial y^2} - y^2\frac{\partial^2 f}{\partial x\partial z} + yz\frac{\partial^2 f}{\partial x\partial y} \\
&\quad - xy\frac{\partial^2 f}{\partial z\partial y} + y^2\frac{\partial^2 f}{\partial z\partial x} + zx\frac{\partial^2 f}{\partial y^2} - z\frac{\partial f}{\partial x} - yz\frac{\partial^2 f}{\partial y\partial x} \\
&= x\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial x} = -Q(a_2)f.
\end{aligned}$$

Because $f(\vec{r})$ is arbitrary, we have

$$[Q(a_3), Q(a_1)] = -Q(a_2).$$

3. The generators of the real Lie algebra $L = \mathrm{su}(2)$ are given by

$$a_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},\ a_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ a_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Show explicitly that a_1, a_2, and a_3 obey the commutation relations

$$\begin{aligned}
[a_1, a_2] &= a_1a_2 - a_2a_1 = -a_3, \\
[a_2, a_3] &= a_2a_3 - a_3a_2 = -a_1, \\
[a_3, a_1] &= a_3a_1 - a_1a_3 = -a_2.
\end{aligned}$$

For $[a_1, a_2]$, we have

$$\begin{aligned}
[a_1, a_2] = a_1a_2 - a_2a_1 &= \frac{1}{4}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} - \frac{1}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = -a_3.
\end{aligned}$$

For $[a_2, a_3]$, we have

$$\begin{aligned}
[a_2, a_3] = a_2a_3 - a_3a_2 &= \frac{1}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \frac{1}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = -a_1.
\end{aligned}$$

For $[a_3, a_1]$, we have

$$\begin{aligned}
[a_3, a_1] = a_3a_1 - a_1a_3 &= \frac{1}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\sigma_2.
\end{aligned}$$

4. The generators of the real Lie algebra $L = \mathrm{su}(2)$ in the above problem can be expressed in terms of the following Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that the Pauli matrices possess the following properties.

(a) $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1$.

(b) $\sigma_1\sigma_2 = -\sigma_2\sigma_1 = i\sigma_3,\ \sigma_2\sigma_3 = -\sigma_3\sigma_2 = i\sigma_1,\ \sigma_3\sigma_1 = -\sigma_1\sigma_3 = i\sigma_2.$

(a) For σ_1^2, we have

$$\sigma_1^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

For σ_2^2, we have

$$\sigma_2^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

For σ_3^2, we have

$$\sigma_3^2 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = 1.$$

(b) For $\sigma_1\sigma_2$, we have

$$\sigma_1\sigma_2 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = i\sigma_3.$$

For $\sigma_2\sigma_1$, we have

$$\sigma_2\sigma_1 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix} = -i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = -i\sigma_3.$$

We thus have

$$\sigma_1\sigma_2 = -\sigma_2\sigma_1 = i\sigma_3$$

and

$$\sigma_1\sigma_2 + \sigma_2\sigma_1 = 0.$$

For $\sigma_2\sigma_3$, we have

$$\sigma_2\sigma_3 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}0 & i\\ i & 0\end{pmatrix} = i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = i\sigma_1.$$

For $\sigma_3\sigma_2$, we have

$$\sigma_3\sigma_2 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix} = -i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = -i\sigma_1.$$

We thus have

$$\sigma_2\sigma_3 = -\sigma_3\sigma_2 = i\sigma_1$$

and

$$\sigma_2\sigma_3 + \sigma_3\sigma_2 = 0.$$

For $\sigma_3\sigma_1$, we have

$$\sigma_3\sigma_1 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = i\sigma_2.$$

For $\sigma_1\sigma_3$, we have

$$\sigma_1\sigma_3 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = -i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = -i\sigma_2.$$

We thus have

$$\sigma_3\sigma_1 = -\sigma_1\sigma_3 = i\sigma_2$$

and

$$\sigma_3\sigma_1 + \sigma_1\sigma_3 = 0.$$

5. Let $\vec{n} = (n_1, n_2, n_3)$ be a unit vector specifying a direction in three-dimensional space.

   (a) Evaluate $(\vec{\sigma} \cdot \vec{n})^2$ with $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$.

   (b) Evaluate $e^{i(\vec{\sigma}\cdot\vec{n})\omega/2}$.

(a) Making use of the properties of the Pauli matrices and $\vec{n}^2 = n_1^2 + n_2^2 + n_3^2 = 1$, we have

$$\begin{aligned}(\vec{\sigma} \cdot \vec{n})^2 &= (\sigma_1 n_1 + \sigma_2 n_2 + \sigma_3 n_3)(\sigma_1 n_1 + \sigma_2 n_2 + \sigma_3 n_3) \\ &= \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 + (\sigma_1\sigma_2 + \sigma_2\sigma_1) n_1 n_2 + (\sigma_2\sigma_3 + \sigma_3\sigma_2) n_2 n_3 + (\sigma_3\sigma_1 + \sigma_1\sigma_3) n_3 n_1 \\ &= n_1^2 + n_2^2 + n_3^2 + 0 \cdot n_1 n_2 + 0 \cdot n_2 n_3 + 0 \cdot n_3 n_1 \\ &= 1.\end{aligned}$$

(b) Expanding $e^{i(\vec{\sigma}\cdot\vec{n})\omega/2}$ and separating the even- and odd-order terms, we have

$$e^{i(\vec{\sigma}\cdot\vec{n})\omega/2} = \sum_{j=0}^{\infty} \frac{i^{2j}(\vec{\sigma}\cdot\vec{n})^{2j}(\omega/2)^{2j}}{(2j)!} + \sum_{j=0}^{\infty} \frac{i^{2j+1}(\vec{\sigma}\cdot\vec{n})^{2j+1}(\omega/2)^{2j+1}}{(2j+1)!}.$$

Making use of $i^{2j} = (-1)^j$ and $(\vec{\sigma}\cdot\vec{n})^2 = 1$, we have

$$e^{i(\vec{\sigma}\cdot\vec{n})\omega/2} = \sum_{j=0}^{\infty} \frac{(-1)^j(\omega/2)^{2j}}{(2j)!} + i(\vec{\sigma}\cdot\vec{n}) \sum_{j=0}^{\infty} \frac{(-1)^j(\omega/2)^{2j+1}}{(2j+1)!}.$$

Utilizing

$$\cos x = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{(2j)!},\ \sin x = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{(2j+1)!},$$

we have

$$e^{i(\vec{\sigma}\cdot\vec{n})\omega/2} = \cos(\omega/2) + i(\vec{\sigma}\cdot\vec{n})\sin(\omega/2).$$