Assignment 12

Due Time : 8:15, June 13, 2020 (Wednesday)

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Score : ________

The analytic homomorphic mapping ϕ of SU(2) onto SO(3) is given by

$$\phi(u)_{jk} = \frac{1}{2}\mathrm{tr}(\sigma_j u \sigma_k u^{-1}), \quad u =\in \mathrm{SU}(2), \quad j,k = 1,2,3.$$

Note that the Pauli matrices σ_1, σ_2, and σ_3 are also denoted respectively by σ_x, σ_y, and σ_z.

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Problem 1 Score: ________. Verify explicitly that a rotation about the Oz-axis is indeed obtained from the given mapping for $u = e^{-i\theta\sigma_z/2}$.

Solution: The matrix of counterclockwise rotation about the Oz-axis through angle θ is

$$R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

We then verify that the matrix obtained from the given mapping ϕ for $u = e^{-i\theta\sigma_z/2}$ is equal to the above rotation matrix.

$$u = e^{-i\theta\sigma_z/2} = 1 + \sum_{j=1}^{\infty} \frac{(-i\theta\sigma_z/2)^j}{j!}. \tag{2}$$

Since

$$\sigma_z^j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^j = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & j \text{ is odd}, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & j \text{ is even}, \end{cases} \tag{3}$$

we have

$$\begin{aligned} u =& 1 + \sum_{j=1}^{\infty} \frac{(-i\theta\sigma_z/2)^j}{j!} \\ =& 1 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sum_{j=0}^{\infty} \frac{(-i)^{2j+1}(\theta/2)^{2j+1}}{(2j+1)!} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sum_{j=1}^{\infty} \frac{(-i)^{2j}(\theta/2)^{2j}}{(2j)!} \\ =& 1 + (-i) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sum_{j=0}^{\infty} \frac{(-1)^j(\theta/2)^{2j+1}}{(2j+1)!} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sum_{j=1}^{\infty} \frac{(-1)^j(\theta/2)^{2j}}{(2j)!} \\ =& 1 + (-i) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sin\frac{\theta}{2} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\cos\frac{\theta}{2} - 1) \\ =& \begin{pmatrix} \cos\frac{\theta}{2} - i\sin\frac{\theta}{2} & 0 \\ 0 & \cos\frac{\theta}{2} + i\sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}, \end{aligned} \tag{4}$$

and

$$u^{-1} = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}. \tag{5}$$

Now we calculate the elements of the matrix in SO(3) corresponding to u:

$$\begin{aligned} \phi(u)_{11} =& \frac{1}{2}\mathrm{tr}(\sigma_1 u \sigma_1 u^{-1}) \\ =& \frac{1}{2}\mathrm{tr}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}\right] \\ =& \frac{1}{2}\mathrm{tr}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \cos\theta, \end{aligned} \tag{6}$$

$$\phi(u)_{12} = \frac{1}{2}\mathrm{tr}(\sigma_1 u \sigma_2 u^{-1})$$

$$
\begin{aligned}
&=\frac{1}{2}\mathrm{tr}\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}e^{-i\theta/2} & 0\\ 0 & e^{i\theta/2}\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}e^{i\theta/2} & 0\\ 0 & e^{-i\theta/2}\end{pmatrix}\right]\\
&=\frac{1}{2}\mathrm{tr}\begin{pmatrix}ie^{i\theta} & 0\\ 0 & -ie^{-i\theta}\end{pmatrix}=-\sin\theta,
\end{aligned}\tag{7}
$$

$$
\begin{aligned}
\phi(u)_{13}=&\frac{1}{2}\mathrm{tr}(\sigma_1 u\sigma_3 u^{-1})\\
=&\frac{1}{2}\mathrm{tr}\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}e^{-i\theta/2} & 0\\ 0 & e^{i\theta/2}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}e^{i\theta/2} & 0\\ 0 & e^{-i\theta/2}\end{pmatrix}\right]\\
=&\frac{1}{2}\mathrm{tr}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}=0,
\end{aligned}\tag{8}
$$

$$
\begin{aligned}
\phi(u)_{21}=&\frac{1}{2}\mathrm{tr}(\sigma_2 u\sigma_1 u^{-1})\\
=&\frac{1}{2}\mathrm{tr}\left[\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}e^{-i\theta/2} & 0\\ 0 & e^{i\theta/2}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}e^{i\theta/2} & 0\\ 0 & e^{-i\theta/2}\end{pmatrix}\right]\\
=&\frac{1}{2}\mathrm{tr}\begin{pmatrix}-ie^{i\theta} & 0\\ 0 & ie^{-i\theta}\end{pmatrix}=\sin\theta,
\end{aligned}\tag{9}
$$

$$
\begin{aligned}
\phi(u)_{22}=&\frac{1}{2}\mathrm{tr}(\sigma_2 u\sigma_2 u^{-1})\\
=&\frac{1}{2}\mathrm{tr}\left[\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}e^{-i\theta/2} & 0\\ 0 & e^{i\theta/2}\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}e^{i\theta/2} & 0\\ 0 & e^{-i\theta/2}\end{pmatrix}\right]\\
=&\frac{1}{2}\mathrm{tr}\begin{pmatrix}e^{i\theta} & 0\\ 0 & e^{-i\theta}\end{pmatrix}=\cos\theta,
\end{aligned}\tag{10}
$$

$$
\begin{aligned}
\phi(u)_{23}=&\frac{1}{2}\mathrm{tr}(\sigma_2 u\sigma_3 u^{-1})\\
=&\frac{1}{2}\mathrm{tr}\left[\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}e^{-i\theta/2} & 0\\ 0 & e^{i\theta/2}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}e^{i\theta/2} & 0\\ 0 & e^{-i\theta/2}\end{pmatrix}\right]\\
=&\frac{1}{2}\mathrm{tr}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}=0,
\end{aligned}\tag{11}
$$

$$
\begin{aligned}
\phi(u)_{31}=&\frac{1}{2}\mathrm{tr}(\sigma_3 u\sigma_1 u^{-1})\\
=&\frac{1}{2}\mathrm{tr}\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}e^{-i\theta/2} & 0\\ 0 & e^{i\theta/2}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}e^{i\theta/2} & 0\\ 0 & e^{-i\theta/2}\end{pmatrix}\right]\\
=&\frac{1}{2}\mathrm{tr}\begin{pmatrix}0 & e^{-i\theta}\\ -e^{i\theta} & 0\end{pmatrix}=0,
\end{aligned}\tag{12}
$$

$$
\begin{aligned}
\phi(u)_{32}=&\frac{1}{2}\mathrm{tr}(\sigma_3 u\sigma_2 u^{-1})\\
=&\frac{1}{2}\mathrm{tr}\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}e^{-i\theta/2} & 0\\ 0 & e^{i\theta/2}\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}e^{i\theta/2} & 0\\ 0 & e^{-i\theta/2}\end{pmatrix}\right]\\
=&\frac{1}{2}\mathrm{tr}\begin{pmatrix}0 & -ie^{-i\theta}\\ ie^{i\theta} & 0\end{pmatrix}=0,
\end{aligned}\tag{13}
$$

$$
\begin{aligned}
\phi(u)_{33}=&\frac{1}{2}\mathrm{tr}(\sigma_3 u\sigma_3 u^{-1})\\
=&\frac{1}{2}\mathrm{tr}\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}e^{-i\theta/2} & 0\\ 0 & e^{i\theta/2}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}e^{i\theta/2} & 0\\ 0 & e^{-i\theta/2}\end{pmatrix}\right]
\end{aligned}
$$

$$=\frac{1}{2}\mathrm{tr}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=1. \tag{14}$$

The matrix in SO(3) corresponding to u is

$$\phi(u)=\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{15}$$

which is exactly the rotation matrix we write at the beginning. Therefore, a rotation about the Oz-axis is indeed obtained from the given mapping for $u=e^{-i\theta\sigma_z/2}$. □

Problem 2 Score: _________. Verify explicitly that a rotation about the Oy-axis is indeed obtained from the given mapping for $u=e^{-i\theta\sigma_y/2}$.

Solution: The matrix of counterclockwise rotation about the Oy-axis through angle θ is

$$R=\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}. \tag{16}$$

We then verify that the matrix obtained from the given mapping ϕ for $u=e^{-i\theta\sigma_y/2}$ is equal to the above rotation matrix.

$$u=e^{-i\theta\sigma_y/2}=1+\sum_{j=1}^{\infty}\frac{(-i\theta\sigma_y/2)^j}{j!}. \tag{17}$$

Since

$$\sigma_y^j=\begin{cases}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, & j \text{ is odd},\\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, & j \text{ is even},\end{cases} \tag{18}$$

we have

$$\begin{aligned}u=&1+\sum_{j=1}\frac{(-i\theta\sigma_y/2)^j}{j!}\\ =&1+\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\sum_{j=0}^{\infty}\frac{(-i)^{2j+1}(\theta/2)^{2j+1}}{(2j+1)}+\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\sum_{j=1}^{\infty}\frac{(-i)^{2j}(\theta/2)^{2j}}{(2j)!}\\ =&1+(-i)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\sum_{j=0}^{\infty}\frac{(-1)^{j}(\theta/2)^{2j+1}}{(2j+1)}+\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\sum_{j=1}^{\infty}\frac{(-1)^{j}(\theta/2)^{2j}}{(2j)!}\\ =&1+(-i)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\sin\frac{\theta}{2}+\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}(\cos\frac{\theta}{2}-1)\\ =&\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix},\end{aligned} \tag{19}$$

and

$$u^{-1}=\begin{pmatrix}\cos\frac{\theta}{2} & \sin\frac{\theta}{2}\\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}. \tag{20}$$

Now we calculate the elements of the matrix in SO(3) corresponding to u:

$$\begin{aligned}\phi(u)_{11}=&\frac{1}{2}\mathrm{tr}(\sigma_1 u\sigma_1 u^{-1})\\ =&\frac{1}{2}\mathrm{tr}\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2} & \sin\frac{\theta}{2}\\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}\right]\\ =&\frac{1}{2}\mathrm{tr}\begin{pmatrix}\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2} & 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\\ -2\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\end{pmatrix}=\frac{1}{2}\mathrm{tr}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}=\cos\theta,\end{aligned} \tag{21}$$

$$\begin{aligned}\phi(u)_{12}=&\frac{1}{2}\mathrm{tr}(\sigma_1u\sigma_2u^{-1})\\=&\frac{1}{2}\mathrm{tr}\left[\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}&-\sin\frac{\theta}{2}\\\sin\frac{\theta}{2}&\cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}&\sin\frac{\theta}{2}\\-\sin\frac{\theta}{2}&\cos\frac{\theta}{2}\end{pmatrix}\right]\\=&\frac{1}{2}\mathrm{tr}\begin{pmatrix}i&0\\0&-i\end{pmatrix}=0,\end{aligned}\tag{22}$$

$$\begin{aligned}\phi(u)_{13}=&\frac{1}{2}\mathrm{tr}(\sigma_1u\sigma_3u^{-1})\\=&\frac{1}{2}\mathrm{tr}\left[\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}&-\sin\frac{\theta}{2}\\\sin\frac{\theta}{2}&\cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}&\sin\frac{\theta}{2}\\-\sin\frac{\theta}{2}&\cos\frac{\theta}{2}\end{pmatrix}\right]\\=&\frac{1}{2}\mathrm{tr}\begin{pmatrix}2\sin\frac{\theta}{2}\cos\frac{\theta}{2}&\sin^2\frac{\theta}{2}-\cos^2\frac{\theta}{2}\\\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}&2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\end{pmatrix}=\frac{1}{2}\mathrm{tr}\begin{pmatrix}\sin\theta&-\cos\theta\\\cos\theta&\sin\theta\end{pmatrix}=\sin\theta,\end{aligned}\tag{23}$$

$$\begin{aligned}\phi(u)_{21}=&\frac{1}{2}\mathrm{tr}(\sigma_2u\sigma_1u^{-1})\\=&\frac{1}{2}\mathrm{tr}\left[\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}&-\sin\frac{\theta}{2}\\\sin\frac{\theta}{2}&\cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}&\sin\frac{\theta}{2}\\-\sin\frac{\theta}{2}&\cos\frac{\theta}{2}\end{pmatrix}\right]\\=&\frac{1}{2}\mathrm{tr}\begin{pmatrix}i(\sin^2\frac{\theta}{2}-i\cos^2\frac{\theta}{2})&-2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\\-2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}&i(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2})\end{pmatrix}=\frac{1}{2}\mathrm{tr}\begin{pmatrix}-i\cos\theta&-i\sin\theta\\-i\sin\theta&i\cos\theta\end{pmatrix}=0,\end{aligned}\tag{24}$$

$$\begin{aligned}\phi(u)_{22}=&\frac{1}{2}\mathrm{tr}(\sigma_2u\sigma_2u^{-1})\\=&\frac{1}{2}\mathrm{tr}\left[\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}&-\sin\frac{\theta}{2}\\\sin\frac{\theta}{2}&\cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}&\sin\frac{\theta}{2}\\-\sin\frac{\theta}{2}&\cos\frac{\theta}{2}\end{pmatrix}\right]\\=&\frac{1}{2}\mathrm{tr}\begin{pmatrix}1&0\\0&1\end{pmatrix}=1,\end{aligned}\tag{25}$$

$$\begin{aligned}\phi(u)_{23}=&\frac{1}{2}\mathrm{tr}(\sigma_2u\sigma_3u^{-1})\\=&\frac{1}{2}\mathrm{tr}\left[\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}&-\sin\frac{\theta}{2}\\\sin\frac{\theta}{2}&\cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}&\sin\frac{\theta}{2}\\-\sin\frac{\theta}{2}&\cos\frac{\theta}{2}\end{pmatrix}\right]\\=&\frac{1}{2}\mathrm{tr}\begin{pmatrix}-2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}&i\cos^2\frac{\theta}{2}-i\sin^2\frac{\theta}{2}\\i\cos^2\frac{\theta}{2}-i\sin^2\frac{\theta}{2}&2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}-i\sin\theta&i\cos\theta\\i\cos\theta&i\sin\theta\end{pmatrix}=0,\end{aligned}\tag{26}$$

$$\begin{aligned}\phi(u)_{31}=&\frac{1}{2}\mathrm{tr}(\sigma_3u\sigma_1u^{-1})\\=&\frac{1}{2}\mathrm{tr}\left[\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}&-\sin\frac{\theta}{2}\\\sin\frac{\theta}{2}&\cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}&\sin\frac{\theta}{2}\\-\sin\frac{\theta}{2}&\cos\frac{\theta}{2}\end{pmatrix}\right]\\=&\frac{1}{2}\mathrm{tr}\begin{pmatrix}-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}&\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\\\sin^2\frac{\theta}{2}-\cos^2\frac{\theta}{2}&-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\end{pmatrix}=\frac{1}{2}\mathrm{tr}\begin{pmatrix}-\sin\theta&\cos\theta\\-\cos\theta&-\sin\theta\end{pmatrix}=-\sin\theta,\end{aligned}\tag{27}$$

$$\begin{aligned}\phi(u)_{32}=&\frac{1}{2}\mathrm{tr}(\sigma_3u\sigma_2u^{-1})\\=&\frac{1}{2}\mathrm{tr}\left[\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}&-\sin\frac{\theta}{2}\\\sin\frac{\theta}{2}&\cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}&\sin\frac{\theta}{2}\\-\sin\frac{\theta}{2}&\cos\frac{\theta}{2}\end{pmatrix}\right]\\=&\frac{1}{2}\mathrm{tr}\begin{pmatrix}0&-i\\-i&0\end{pmatrix}=0,\end{aligned}\tag{28}$$

$$\phi(u)_{33}=\frac{1}{2}\mathrm{tr}(\sigma_3u\sigma_3u^{-1})$$

$$
\begin{aligned}
&=\frac{1}{2}\mathrm{tr}\left[\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2} & \sin\frac{\theta}{2}\\-\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}\right]\\
&=\frac{1}{2}\mathrm{tr}\begin{pmatrix}\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2} & 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\\-2\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\end{pmatrix}=\frac{1}{2}\mathrm{tr}\begin{pmatrix}\cos\theta & 2\sin\theta\\-2\sin\theta & \cos\theta\end{pmatrix}=\cos\theta.
\end{aligned}
\tag{29}
$$

The matrix in SO(3) corresponding to u is

$$
\phi(u)=\begin{pmatrix}\cos\theta & 0 & \sin\theta\\0 & 1 & 0\\-\sin\theta & 0 & \cos\theta\end{pmatrix},
\tag{30}
$$

which is exactly the rotation matrix we write at the beginning. Therefore, a rotation about the Oy-axis is indeed obtained from the given mapping for $u=e^{-i\theta\sigma_y/2}$. □

Problem 3 Score: ________. Show that the analytic isomorphic mapping of the real Lie algebra su(2) onto the real algebra so(3) is given by

$$
\psi(a)_{jk}=\frac{1}{2}\mathrm{tr}(\sigma_j[a,\sigma_k]),\quad a\in\mathrm{su}(2),\quad j,k=1,2,3.
$$

Solution: The analytic isomorphic mapping of the real Lie algebra su(2) onto the real algebra so(3) is

$$
\begin{aligned}
\psi(a)_{jk}=&\left.\frac{d\phi(e^{ta})}{dt}\right|_{t=0}\\
=&\left.\frac{d\left\{\frac{1}{2}\mathrm{tr}[\sigma_j e^{ta}\sigma_k(e^{ta})^{-1}]\right\}}{dt}\right|_{t=0}\\
=&\lim_{t\to 0}\frac{\frac{1}{2}\mathrm{tr}[\sigma_j e^{ta}\sigma_k(e^{ta})^{-1}]-\frac{1}{2}\mathrm{tr}[\sigma_j\sigma_k]}{t}\\
&(\text{using } e^{ta}b[e^{ta}]^{-1}=b+t[a,b]+\frac{1}{2}[a,[a,b]]+\cdots)\\
=&\lim_{t\to 0}\frac{\frac{1}{2}\mathrm{tr}\left\{\sigma_j\left(\sigma_k+t[a,\sigma_k]+\frac{1}{2}t^2[a,[a,\sigma_k]]+\cdots\right)\right\}-\frac{1}{2}\mathrm{tr}\left\{\sigma_j\sigma_k\right\}}{t}\\
=&\frac{1}{2}\mathrm{tr}(\sigma_j[a,\sigma_k]),
\end{aligned}
\tag{31}
$$

for $a\in\mathrm{su}(2)$, $j,k=1,2,3$. □

Problem 4 Score: ________. Without using the explicit matrix representations of the Pauli matrices, show that $\psi(a_p)_{jk}=\epsilon_{pjk}$ for $a_p=i\sigma_p/2$ with $p=1,2,3$.
Hint: $[\sigma_p,\sigma_k]=2i\sum_{l=1}^3\epsilon_{pkl}\sigma_l$.

Solution: Using the conclusion we obtained in the latter problem, we have

$$
\begin{aligned}
\psi(a_p)_{jk}=&\psi(i\sigma_p/2)=\frac{1}{2}\mathrm{tr}(\sigma_j[i\sigma_p/2,\sigma_k])=\frac{1}{2}\mathrm{tr}\left(\frac{i}{2}\sigma_j[\sigma_p,\sigma_k]\right)\\
=&\frac{1}{2}\mathrm{tr}\left(-\sigma_j\sum_{l=1}^3\epsilon_{pkl}\sigma_l\right)\\
&(\text{using } \sigma_a\sigma_b=\delta_{ab}I+i\sum_{c=1}^3\epsilon_{abc}\sigma_c,\ \text{where } I \text{ is the identity matrix})\\
=&\frac{1}{2}\mathrm{tr}\left(-\sum_{l=1}^3\epsilon_{pkl}\left(\delta_{jl}I+i\sum_{m=1}^3\epsilon_{jlm}\sigma_m\right)\right)\\
&(\text{using } \mathrm{tr}(I)=2,\ \text{and } \mathrm{tr}\sigma_m=0,\quad m=1,2,3)\\
=&-\sum_{l=1}^3\epsilon_{pkl}\delta_{lj}\\
=&-\epsilon_{pkj}\\
=&\epsilon_{pjk}.
\end{aligned}
\tag{32}
$$

□

Problem 5 Score: ________. Using the definition of ϵ_{pjk}, show that

$$\psi(a_1)=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix},\quad \psi(a_2)=\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix},\quad \psi(a_3)=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}.$$

Solution: Using the conclusion we obtained in the latter problem and the definition of ϵ_{pjk}, the matrix elements of $\psi(a_1)$ are

$$\begin{aligned}
\psi(a_1)_{11}=&\epsilon_{111}=0, &(33)\\
\psi(a_1)_{12}=&\epsilon_{112}=0, &(34)\\
\psi(a_1)_{13}=&\epsilon_{113}=0, &(35)\\
\psi(a_1)_{21}=&\epsilon_{121}=0, &(36)\\
\psi(a_1)_{22}=&\epsilon_{122}=0, &(37)\\
\psi(a_1)_{23}=&\epsilon_{123}=1, &(38)\\
\psi(a_1)_{31}=&\epsilon_{131}=0, &(39)\\
\psi(a_1)_{32}=&\epsilon_{132}=-1, &(40)\\
\psi(a_1)_{33}=&\epsilon_{133}=0, &(41)\\
& &(42)
\end{aligned}$$

so

$$\psi(a_1)=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}. \tag{43}$$

The matrix elements of $\psi(a_2)$ are

$$\begin{aligned}
\psi(a_2)_{11}=&\epsilon_{211}=0, &(44)\\
\psi(a_2)_{12}=&\epsilon_{212}=0, &(45)\\
\psi(a_2)_{13}=&\epsilon_{213}=-1, &(46)\\
\psi(a_2)_{21}=&\epsilon_{221}=0, &(47)\\
\psi(a_2)_{22}=&\epsilon_{222}=0, &(48)\\
\psi(a_2)_{23}=&\epsilon_{223}=0, &(49)\\
\psi(a_2)_{31}=&\epsilon_{231}=1, &(50)\\
\psi(a_2)_{32}=&\epsilon_{232}=0, &(51)\\
\psi(a_2)_{33}=&\epsilon_{233}=0, &(52)\\
& &(53)
\end{aligned}$$

so

$$\psi(a_2)=\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix}. \tag{54}$$

The matrix elements of $\psi(a_3)$ are

$$\begin{aligned}
\psi(a_3)_{11}=&\epsilon_{311}=0, &(55)\\
\psi(a_3)_{12}=&\epsilon_{312}=1, &(56)\\
\psi(a_3)_{13}=&\epsilon_{313}=0, &(57)\\
\psi(a_3)_{21}=&\epsilon_{321}=-1, &(58)\\
\psi(a_3)_{22}=&\epsilon_{322}=0, &(59)\\
\psi(a_3)_{23}=&\epsilon_{323}=0, &(60)\\
\psi(a_3)_{31}=&\epsilon_{331}=0, &(61)\\
\psi(a_3)_{32}=&\epsilon_{332}=0, &(62)
\end{aligned}$$

$$\psi(a_3)_{33} = \epsilon_{333} = 0, \tag{63}$$
$$\tag{64}$$

so

$$\psi(a_3) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{65}$$

□