Group Theory

Solutions to Problems in Homework Assignment 10

Spring, 2020

1. Simplify the following permutations into the product of cycles without any common object.

   (a) $\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}2 & 3\end{pmatrix}\begin{pmatrix}1 & 2\end{pmatrix}$.

   (b) $\begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}1 & 3 & 4\end{pmatrix}\begin{pmatrix}3 & 2 & 1\end{pmatrix}$.

   (c) $\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix}^{-1}$.

   (d) $\begin{pmatrix}1 & 2 & 4 & 5\end{pmatrix}\begin{pmatrix}4 & 3 & 2 & 6\end{pmatrix}$.

   (e) $\begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}4 & 2 & 6\end{pmatrix}\begin{pmatrix}3 & 4 & 5 & 6\end{pmatrix}$.

(a) Utilizing the connecting and cutting technique, we obtain

$$\begin{aligned}
(1\ \ 2)(2\ \ 3)(1\ \ 2) &= (1\ \ 2\ \ 3)(1\ \ 2) \\
&= (3\ \ 1\ \ 2)(1\ \ 2) \\
&= (3\ \ 1)(1\ \ 2)(1\ \ 2) \\
&= (3\ \ 1) = (1\ \ 3)\,,
\end{aligned}$$

where we have made use of $(1\ \ 2)(1\ \ 2) = E$.

(b) Utilizing the connecting and cutting technique, we obtain

$$\begin{aligned}
(1\ \ 2\ \ 3)(1\ \ 3\ \ 4)(3\ \ 2\ \ 1) &= (2\ \ 3\ \ 1)(1\ \ 3\ \ 4)(3\ \ 2\ \ 1) \\
&= (2\ \ 3)(3\ \ 1)(1\ \ 3)(3\ \ 4)(3\ \ 2\ \ 1) \\
&= (2\ \ 3)(3\ \ 4)(3\ \ 2\ \ 1) \\
&= (2\ \ 3\ \ 4)(3\ \ 2\ \ 1) \\
&= (4\ \ 2\ \ 3)(3\ \ 2\ \ 1) \\
&= (4\ \ 2)(2\ \ 3)(3\ \ 2)(2\ \ 1) \\
&= (4\ \ 2)(2\ \ 1) \\
&= (4\ \ 2\ \ 1)\,,
\end{aligned}$$

where we have made use of $(3\ \ 1)(1\ \ 3) = E$ and $(2\ \ 3)(3\ \ 2) = E$.

(c) In the two-line notation, the permutation represented by the cycle $(1\ \ 2\ \ 3\ \ 4)$ is given by

$$(1\ \ 2\ \ 3\ \ 4) = \begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 4 & 1\end{pmatrix}.$$

The inverse of $(1\ \ 2\ \ 3\ \ 4)$ can be obtained by exchange the two rows in the above expression of $(1\ \ 2\ \ 3\ \ 4)$. Thus, the inverse of $(1\ \ 2\ \ 3\ \ 4)$ is given by

$$\begin{aligned}
(1\ \ 2\ \ 3\ \ 4)^{-1} &= \begin{pmatrix}2 & 3 & 4 & 1\\ 1 & 2 & 3 & 4\end{pmatrix} \\
&= (2\ \ 1\ \ 4\ \ 3) \\
&= (4\ \ 3\ \ 2\ \ 1)\,.
\end{aligned}$$

(d) Utilizing the connecting and cutting technique, we obtain

$$
\begin{aligned}
(1\ 2\ 4\ 5)(4\ 3\ 2\ 6) &= (5\ 1\ 2\ 4)(4\ 3\ 2\ 6) \\
&= (5\ 1\ 2)(2\ 4)(4\ 3)(3\ 2\ 6) \\
&= (5\ 1\ 2)(2\ 4\ 3)(3\ 2\ 6) \\
&= (5\ 1\ 2)(4\ 3\ 2)(3\ 2\ 6) \\
&= (5\ 1\ 2)(4\ 3)(3\ 2)(3\ 2)(2\ 6) \\
&= (5\ 1\ 2)(4\ 3)(2\ 6) \\
&= (5\ 1\ 2)(2\ 6)(4\ 3) \\
&= (5\ 1\ 2\ 6)(4\ 3) \\
&= (6\ 5\ 1\ 2)(4\ 3)\,,
\end{aligned}
$$

where we have made use of $(3\ 2)(3\ 2) = E$ and $(4\ 3)(2\ 6) = (2\ 6)(4\ 3)$.

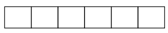
(e) Utilizing the connecting and cutting technique, we obtain

$$
\begin{aligned}
(1\ 2\ 3)(4\ 2\ 6)(3\ 4\ 5\ 6) &= (3\ 1\ 2)(2\ 6\ 4)(3\ 4\ 5\ 6) \\
&= (3\ 1\ 2\ 6\ 4)(3\ 4\ 5\ 6) \\
&= (1\ 2\ 6\ 4\ 3)(3\ 4\ 5\ 6) \\
&= (1\ 2\ 6\ 4)(4\ 3)(3\ 4)(4\ 5\ 6) \\
&= (1\ 2\ 6\ 4)(4\ 5\ 6) \\
&= (1\ 2\ 6\ 4)(6\ 4\ 5) \\
&= (1\ 2\ 6)(6\ 4)(6\ 4)(4\ 5) \\
&= (1\ 2\ 6)(4\ 5) \\
&= (6\ 1\ 2)(4\ 5)\,,
\end{aligned}
$$

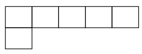
where we have made use of $(4\ 3)(3\ 4) = E$ and $(6\ 4)(6\ 4) = E$.

2. Write down all the Young patterns of the permutation group S_6 from the largest to the smallest.

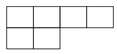
The Young patterns of the permutation group S_6 in the order from the largest to the smallest are given by



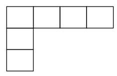
[6]



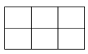
[5, 1]



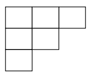
[4, 2]



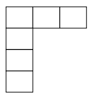
[4, 1,1]



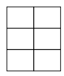
[3, 3]



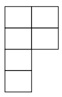
[3, 2, 1]



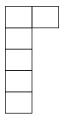
[3, 1, 1, 1]



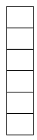
[2, 2, 2]



[2, 2, 1, 1]



[2, 1, 1, 1, 1]



[1, 1, 1, 1, 1, 1]

3. Using the hook rule, calculate the number $d_{[3,2,1,1]}(S_7)$ of the standard Young tableaux for the Young pattern $[3, 2, 1, 1]$ of the permutation group S_7.

Here we calculate the numbers $d_{[\lambda]}(S_7)$ of the standard Young tableaux for all the Young patterns $[\lambda]$ of the permutation groups S_7. All the regular Young patterns of S_7 with the boxes filled with the hook numbers are given in the following.

$$\begin{array}{ccccccc} 7&6&5&4&3&2&1 \end{array} \qquad \begin{array}{cccccc} 7&5&4&3&2&1 \\ 1&&&&& \end{array} \qquad \begin{array}{ccccc} 6&5&3&2&1 \\ 2&1&&& \end{array} \qquad \begin{array}{ccccc} 7&4&3&2&1 \\ 2&&&& \\ 1&&&& \end{array}$$

[7] [6,1] [5,2] [5,1,1]

$$\begin{array}{cccc} 5&4&3&1 \\ 3&2&1& \end{array} \qquad \begin{array}{cccc} 6&4&2&1 \\ 3&1&& \\ 1&&& \end{array} \qquad \begin{array}{cccc} 7&3&2&1 \\ 3&&& \\ 2&&& \\ 1&&& \end{array} \qquad \begin{array}{ccc} 5&3&2 \\ 4&2&1 \\ 1&& \end{array}$$

[4,3] [4,2,1] [4,1,1,1] [3,3,1]

$$\begin{array}{ccc} 5&4&1 \\ 3&2& \\ 2&1& \end{array} \qquad \begin{array}{ccc} 6&3&1 \\ 4&1& \\ 2&& \\ 1&& \end{array} \qquad \begin{array}{ccc} 7&2&1 \\ 4&& \\ 3&& \\ 2&& \\ 1&& \end{array} \qquad \begin{array}{cc} 5&3 \\ 4&2 \\ 3&1 \\ 1& \end{array}$$

[3,2,2] [3,2,1,1] [3,1,1,1,1] [2,2,2,1]

$$\begin{array}{cc} 6&2 \\ 5&1 \\ 3& \\ 2& \\ 1& \end{array} \qquad \begin{array}{cc} 7&1 \\ 5& \\ 4& \\ 3& \\ 2& \\ 1& \end{array} \qquad \begin{array}{c} 7 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{array}$$

[2,2,1,1,1] [2,1,1,1,1,1] [1,1,1,1,1,1,1]

(a) **Young pattern** $[7]$**.** For the Young pattern $[7]$, the number of the standard Young tableaux is given by

$$d_{[7]}(S_7) = \frac{7!}{Y_h^{[7]}} = \frac{7!}{7!} = 1.$$

(b) **Young pattern** $[6,1]$**.** For the Young pattern $[6,1]$, the number of the standard Young tableaux is given by

$$d_{[6,1]}(S_7) = \frac{7!}{Y_h^{[6,1]}} = \frac{7!}{7 \times 5 \times 4 \times 3 \times 2} = 6.$$

(c) **Young pattern** $[5,2]$**.** For the Young pattern $[5,2]$, the number of the standard Young tableaux is given by

$$d_{[5,2]}(S_7) = \frac{7!}{Y_h^{[5,2]}} = \frac{7!}{6 \times 5 \times 3 \times 2 \times 2} = 14.$$

(d) **Young pattern** $[5,1,1]$**.** For the Young pattern $[5,1,1] = [5,1^2]$, the number of the standard Young tableaux is given by

$$d_{[5,1^2]}(S_7) = \frac{7!}{Y_h^{[5,1^2]}} = \frac{7!}{7 \times 4 \times 3 \times 2 \times 2} = 15.$$

(e) **Young pattern** $[4,3]$**.** For the Young pattern $[4,3]$, the number of the standard Young tableaux is given by

$$d_{[4,3]}(S_7) = \frac{7!}{Y_h^{[4,3]}} = \frac{7!}{5 \times 4 \times 3 \times 3 \times 2} = 14.$$

(f) **Young pattern** $[4,2,1]$**.** For the Young pattern $[4,2,1]$, the number of the standard Young tableaux is given by

$$d_{[4,2,1]}(S_7) = \frac{7!}{Y_h^{[4,2,1]}} = \frac{7!}{6 \times 4 \times 2 \times 3} = 35.$$

(g) **Young pattern** $[4,1,1,1]$**.** For the Young pattern $[4,1,1,1]=[4,1^3]$, the number of the standard Young tableaux is given by

$$d_{[4,1^3]}(S_7)=\frac{7!}{Y_h^{[4,1^3]}}=\frac{7!}{7\times 3\times 2\times 3\times 2}=20.$$

(h) **Young pattern** $[3,3,1]$**.** For the Young pattern $[3,3,1]=[3^2,1]$, the number of the standard Young tableaux is given by

$$d_{[3^2,1]}(S_7)=\frac{7!}{Y_h^{[3^2,1]}}=\frac{7!}{5\times 3\times 2\times 4\times 2}=21.$$

(i) **Young pattern** $[3,2,2]$**.** For the Young pattern $[3,2,2]=[3,2^2]$, the number of the standard Young tableaux is given by

$$d_{[3,2^2]}(S_7)=\frac{7!}{Y_h^{[3,2^2]}}=\frac{7!}{5\times 4\times 3\times 2\times 2}=21.$$

(j) **Young pattern** $[3,2,1,1]$**.** For the Young pattern $[3,2,1,1]=[3,2,1^2]$, the number of the standard Young tableaux is given by

$$d_{[3,2,1^2]}(S_7)=\frac{7!}{Y_h^{[3,2,1^2]}}=\frac{7!}{6\times 3\times 4\times 2}=35.$$

(k) **Young pattern** $[3,1,1,1,1]$**.** For the Young pattern $[3,1,1,1,1]=[3,1^4]$, the number of the standard Young tableaux is given by

$$d_{[3,1^4]}(S_7)=\frac{7!}{Y_h^{[3,1^4]}}=\frac{7!}{7\times 2\times 4\times 3\times 2}=15.$$

(l) **Young pattern** $[2,2,2,1]$**.** For the Young pattern $[2,2,2,1]=[2^3,1]$, the number of the standard Young tableaux is given by

$$d_{[2^3,1]}(S_7)=\frac{7!}{Y_h^{[2^3,1]}}=\frac{7!}{5\times 3\times 4\times 2\times 3}=14.$$

(m) **Young pattern** $[2,2,1,1,1]$**.** For the Young pattern $[2,2,1,1,1]=[2^2,1^3]$, the number of the standard Young tableaux is given by

$$d_{[2^2,1^3]}(S_7)=\frac{7!}{Y_h^{[2^2,1^3]}}=\frac{7!}{6\times 2\times 5\times 3\times 2}=14.$$

(n) **Young pattern** $[2,1,1,1,1,1]$**.** For the Young pattern $[2,1,1,1,1,1]=[2,1^5]$, the number of the standard Young tableaux is given by

$$d_{[2,1^5]}(S_7)=\frac{7!}{Y_h^{[2,1^5]}}=\frac{7!}{7\times 5\times 4\times 3\times 2}=6.$$

(o) **Young pattern** $[1,1,1,1,1,1,1]$**.** For the Young pattern $[1,1,1,1,1,1,1]=[1^7]$, the number of the standard Young tableaux is given by

$$d_{[1^7]}(S_7)=\frac{7!}{Y_h^{[1^7]}}=\frac{7!}{7!}=1.$$

In summary, the values of $d_{[\lambda]}(S_7)$ for all the $[\lambda]$ are given in the following table.

$[\lambda]$	$[7]$	$[6,1]$	$[5,2]$	$[5,1^2]$	$[4,3]$	$[4,2,1]$	$[4,1^3]$	$[3^2,1]$	$[3,2^2]$	$[3,2,1^2]$	$[3,1^4]$	$[2^3,1]$	$[2^2,1^3]$	$[2,1^5]$	$[1^7]$
$d_{[\lambda]}(S_7)$	1	6	14	15	14	35	20	21	21	35	15	14	14	6	1

4. Write down the Young operator corresponding to the following Young tableau.

1	2
3	4

The horizontal permutations and the horizontal Young operator are given by

- P_1**:** E, $\begin{pmatrix}1 & 2\end{pmatrix}$.
- P_2**:** E, $\begin{pmatrix}3 & 4\end{pmatrix}$.
- P**:** E, $\begin{pmatrix}1 & 2\end{pmatrix}$, $\begin{pmatrix}3 & 4\end{pmatrix}$, $\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix}$.
- $\mathcal{P}$**:** $\mathcal{P} = E + \begin{pmatrix}1 & 2\end{pmatrix} + \begin{pmatrix}3 & 4\end{pmatrix} + \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix}$.

The vertical permutations and the vertical Young operator are given by

- Q_1**:** E, $\begin{pmatrix}1 & 3\end{pmatrix}$.
- Q_2**:** E, $\begin{pmatrix}2 & 4\end{pmatrix}$.
- Q**:** E, $\begin{pmatrix}1 & 3\end{pmatrix}$, $\begin{pmatrix}2 & 4\end{pmatrix}$, $\begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix}$.
- $\mathcal{Q}$**:** $\mathcal{Q} = E - \begin{pmatrix}1 & 3\end{pmatrix} - \begin{pmatrix}2 & 4\end{pmatrix} + \begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix}$.

The Young operator is then given by

$$\begin{aligned}
\mathcal{Y} &= \mathcal{P}\mathcal{Q} \\
&= \left[E + \begin{pmatrix}1 & 2\end{pmatrix} + \begin{pmatrix}3 & 4\end{pmatrix} + \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix}\right]\left[E - \begin{pmatrix}1 & 3\end{pmatrix} - \begin{pmatrix}2 & 4\end{pmatrix} + \begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix}\right] \\
&= E + \begin{pmatrix}1 & 2\end{pmatrix} + \begin{pmatrix}3 & 4\end{pmatrix} + \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix} \\
&\quad - \begin{pmatrix}1 & 3\end{pmatrix} - \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix} - \begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix} - \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix} \\
&\quad - \begin{pmatrix}2 & 4\end{pmatrix} - \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix} - \begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix} - \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix} \\
&\quad + \begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix} + \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix} + \begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix} + \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix} \\
&= E + \begin{pmatrix}1 & 2\end{pmatrix} + \begin{pmatrix}3 & 4\end{pmatrix} + \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix} \\
&\quad - \begin{pmatrix}1 & 3\end{pmatrix} - \begin{pmatrix}3 & 2 & 1\end{pmatrix} - \begin{pmatrix}4 & 3 & 1\end{pmatrix} - \begin{pmatrix}4 & 3 & 2 & 1\end{pmatrix} \\
&\quad - \begin{pmatrix}2 & 4\end{pmatrix} - \begin{pmatrix}1 & 2 & 4\end{pmatrix} - \begin{pmatrix}4 & 2 & 3\end{pmatrix} - \begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix} \\
&\quad + \begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix} + \begin{pmatrix}4 & 1 & 3 & 2\end{pmatrix} + \begin{pmatrix}4 & 2 & 3 & 1\end{pmatrix} + \begin{pmatrix}1 & 4\end{pmatrix}\begin{pmatrix}2 & 3\end{pmatrix}.
\end{aligned}$$

5. Write down the permutation R_{12} transforming the Young tableau $\mathcal{Y}_2$ to the Young tableau $\mathcal{Y}_1$.

$\mathcal{Y}_1$:

1	2	3
4		

$\mathcal{Y}_2$:

1	2	4
3		

Show that $\mathcal{P}_1 R_{12} = R_{12}\mathcal{P}_2$, $\mathcal{Q}_1 R_{12} = R_{12}\mathcal{Q}_2$, and $\mathcal{Y}_1 R_{12} = R_{12}\mathcal{Y}_2$.

Putting the digits in $\mathcal{Y}_2$ on the first line of R_{12} and those in $\mathcal{Y}_1$ on the second line of R_{12}, we have

$$R_{12} = \begin{pmatrix}1 & 2 & 4 & 3 \\ 1 & 2 & 3 & 4\end{pmatrix} = \begin{pmatrix}4 & 3\end{pmatrix}.$$

For $\mathcal{Y}_1$, we have

- $P_{1,1}$: E, $\begin{pmatrix}1 & 2\end{pmatrix}$, $\begin{pmatrix}1 & 3\end{pmatrix}$, $\begin{pmatrix}2 & 3\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\end{pmatrix}$, $\begin{pmatrix}3 & 2 & 1\end{pmatrix}$.
- P_1: E, $\begin{pmatrix}1 & 2\end{pmatrix}$, $\begin{pmatrix}1 & 3\end{pmatrix}$, $\begin{pmatrix}2 & 3\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\end{pmatrix}$, $\begin{pmatrix}3 & 2 & 1\end{pmatrix}$.
- $\mathcal{P}_1$: $\mathcal{P}_1 = E + \begin{pmatrix}1 & 2\end{pmatrix} + \begin{pmatrix}1 & 3\end{pmatrix} + \begin{pmatrix}2 & 3\end{pmatrix} + \begin{pmatrix}1 & 2 & 3\end{pmatrix} + \begin{pmatrix}3 & 2 & 1\end{pmatrix}$.
- $Q_{1,1}$: E, $\begin{pmatrix}1 & 4\end{pmatrix}$.
- Q_1: E, $\begin{pmatrix}1 & 4\end{pmatrix}$.
- $\mathcal{Q}_1$: $\mathcal{Q}_1 = E - \begin{pmatrix}1 & 4\end{pmatrix}$.
- $\mathcal{Y}_1$:

$$
\begin{aligned}
\mathcal{Y}_1 = \mathcal{P}_1\mathcal{Q}_1 &= \Big[E + \begin{pmatrix}1 & 2\end{pmatrix} + \begin{pmatrix}1 & 3\end{pmatrix} + \begin{pmatrix}2 & 3\end{pmatrix} + \begin{pmatrix}1 & 2 & 3\end{pmatrix} + \begin{pmatrix}3 & 2 & 1\end{pmatrix}\Big]\Big[E - \begin{pmatrix}1 & 4\end{pmatrix}\Big] \\
&= E + \begin{pmatrix}1 & 2\end{pmatrix} + \begin{pmatrix}1 & 3\end{pmatrix} + \begin{pmatrix}2 & 3\end{pmatrix} + \begin{pmatrix}1 & 2 & 3\end{pmatrix} + \begin{pmatrix}3 & 2 & 1\end{pmatrix} \\
&\quad - \begin{pmatrix}1 & 4\end{pmatrix} - \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}1 & 4\end{pmatrix} - \begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}1 & 4\end{pmatrix} - \begin{pmatrix}2 & 3\end{pmatrix}\begin{pmatrix}1 & 4\end{pmatrix} - \begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}1 & 4\end{pmatrix} - \begin{pmatrix}3 & 2 & 1\end{pmatrix}\begin{pmatrix}1 & 4\end{pmatrix} \\
&= E + \begin{pmatrix}1 & 2\end{pmatrix} + \begin{pmatrix}1 & 3\end{pmatrix} + \begin{pmatrix}2 & 3\end{pmatrix} + \begin{pmatrix}1 & 2 & 3\end{pmatrix} + \begin{pmatrix}3 & 2 & 1\end{pmatrix} \\
&\quad - \begin{pmatrix}1 & 4\end{pmatrix} - \begin{pmatrix}4 & 2 & 1\end{pmatrix} - \begin{pmatrix}4 & 3 & 1\end{pmatrix} - \begin{pmatrix}2 & 3\end{pmatrix}\begin{pmatrix}1 & 4\end{pmatrix} - \begin{pmatrix}4 & 2 & 3 & 1\end{pmatrix} - \begin{pmatrix}4 & 3 & 2 & 1\end{pmatrix}.
\end{aligned}
$$

For $\mathcal{Y}_2$, we have

- $P_{2,1}$: E, $\begin{pmatrix}1 & 2\end{pmatrix}$, $\begin{pmatrix}1 & 4\end{pmatrix}$, $\begin{pmatrix}2 & 4\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 4\end{pmatrix}$, $\begin{pmatrix}4 & 2 & 1\end{pmatrix}$.
- P_2: E, $\begin{pmatrix}1 & 2\end{pmatrix}$, $\begin{pmatrix}1 & 4\end{pmatrix}$, $\begin{pmatrix}2 & 4\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 4\end{pmatrix}$, $\begin{pmatrix}4 & 2 & 1\end{pmatrix}$.
- $\mathcal{P}_2$: $\mathcal{P}_2 = E + \begin{pmatrix}1 & 2\end{pmatrix} + \begin{pmatrix}1 & 4\end{pmatrix} + \begin{pmatrix}2 & 4\end{pmatrix} + \begin{pmatrix}1 & 2 & 4\end{pmatrix} + \begin{pmatrix}4 & 2 & 1\end{pmatrix}$.
- $Q_{2,1}$: E, $\begin{pmatrix}1 & 3\end{pmatrix}$.
- Q_2: E, $\begin{pmatrix}1 & 3\end{pmatrix}$.
- $\mathcal{Q}_2$: $\mathcal{Q}_2 = E - \begin{pmatrix}1 & 3\end{pmatrix}$.
- $\mathcal{Y}_2$:

$$
\begin{aligned}
\mathcal{Y}_2 = \mathcal{P}_2\mathcal{Q}_2 &= \Big[E + \begin{pmatrix}1 & 2\end{pmatrix} + \begin{pmatrix}1 & 4\end{pmatrix} + \begin{pmatrix}2 & 4\end{pmatrix} + \begin{pmatrix}1 & 2 & 4\end{pmatrix} + \begin{pmatrix}4 & 2 & 1\end{pmatrix}\Big]\Big[E - \begin{pmatrix}1 & 3\end{pmatrix}\Big] \\
&= E + \begin{pmatrix}1 & 2\end{pmatrix} + \begin{pmatrix}1 & 4\end{pmatrix} + \begin{pmatrix}2 & 4\end{pmatrix} + \begin{pmatrix}1 & 2 & 4\end{pmatrix} + \begin{pmatrix}4 & 2 & 1\end{pmatrix} \\
&\quad - \begin{pmatrix}1 & 3\end{pmatrix} - \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix} - \begin{pmatrix}1 & 4\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix} - \begin{pmatrix}2 & 4\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix} - \begin{pmatrix}1 & 2 & 4\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix} - \begin{pmatrix}4 & 2 & 1\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix} \\
&= E + \begin{pmatrix}1 & 2\end{pmatrix} + \begin{pmatrix}1 & 4\end{pmatrix} + \begin{pmatrix}2 & 4\end{pmatrix} + \begin{pmatrix}1 & 2 & 4\end{pmatrix} + \begin{pmatrix}4 & 2 & 1\end{pmatrix} \\
&\quad - \begin{pmatrix}1 & 3\end{pmatrix} - \begin{pmatrix}3 & 2 & 1\end{pmatrix} - \begin{pmatrix}4 & 1 & 3\end{pmatrix} - \begin{pmatrix}2 & 4\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix} - \begin{pmatrix}4 & 1 & 3 & 2\end{pmatrix} - \begin{pmatrix}4 & 2 & 1 & 3\end{pmatrix}.
\end{aligned}
$$

For $\mathcal{P}_1 R_{12}$, we have

$$
\begin{aligned}
\mathcal{P}_1 R_{12} &= \Big[E + \begin{pmatrix}1 & 2\end{pmatrix} + \begin{pmatrix}1 & 3\end{pmatrix} + \begin{pmatrix}2 & 3\end{pmatrix} + \begin{pmatrix}1 & 2 & 3\end{pmatrix} + \begin{pmatrix}3 & 2 & 1\end{pmatrix}\Big]\begin{pmatrix}4 & 3\end{pmatrix} \\
&= \begin{pmatrix}4 & 3\end{pmatrix} + \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}4 & 3\end{pmatrix} + \begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}4 & 3\end{pmatrix} + \begin{pmatrix}2 & 3\end{pmatrix}\begin{pmatrix}4 & 3\end{pmatrix} + \begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}4 & 3\end{pmatrix} + \begin{pmatrix}3 & 2 & 1\end{pmatrix}\begin{pmatrix}4 & 3\end{pmatrix} \\
&= \begin{pmatrix}4 & 3\end{pmatrix} + \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}4 & 3\end{pmatrix} + \begin{pmatrix}4 & 1 & 3\end{pmatrix} + \begin{pmatrix}4 & 2 & 3\end{pmatrix} + \begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix} + \begin{pmatrix}4 & 2 & 1 & 3\end{pmatrix}.
\end{aligned}
$$

For $R_{12}\mathcal{P}_2$, we have

$$\begin{aligned}
R_{12}\mathcal{P}_2 &= \begin{pmatrix}4 & 3\end{pmatrix}\Big[E + \begin{pmatrix}1 & 2\end{pmatrix} + \begin{pmatrix}1 & 4\end{pmatrix} + \begin{pmatrix}2 & 4\end{pmatrix} + \begin{pmatrix}1 & 2 & 4\end{pmatrix} + \begin{pmatrix}4 & 2 & 1\end{pmatrix}\Big]\\
&= \begin{pmatrix}4 & 3\end{pmatrix} + \begin{pmatrix}4 & 3\end{pmatrix}\begin{pmatrix}1 & 2\end{pmatrix} + \begin{pmatrix}4 & 3\end{pmatrix}\begin{pmatrix}1 & 4\end{pmatrix} + \begin{pmatrix}4 & 3\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix} + \begin{pmatrix}4 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 4\end{pmatrix} + \begin{pmatrix}4 & 3\end{pmatrix}\begin{pmatrix}4 & 2 & 1\end{pmatrix}\\
&= \begin{pmatrix}4 & 3\end{pmatrix} + \begin{pmatrix}4 & 3\end{pmatrix}\begin{pmatrix}1 & 2\end{pmatrix} + \begin{pmatrix}4 & 1 & 3\end{pmatrix} + \begin{pmatrix}4 & 2 & 3\end{pmatrix} + \begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix} + \begin{pmatrix}4 & 2 & 1 & 3\end{pmatrix}.
\end{aligned}$$

From the above two results, we see that $\mathcal{P}_1 R_{12} = R_{12}\mathcal{P}_2$.
For $\mathcal{Q}_1 R_{12}$, we have

$$\begin{aligned}
\mathcal{Q}_1 R_{12} &= \Big[E - \begin{pmatrix}1 & 4\end{pmatrix}\Big]\begin{pmatrix}4 & 3\end{pmatrix}\\
&= \begin{pmatrix}4 & 3\end{pmatrix} - \begin{pmatrix}1 & 4\end{pmatrix}\begin{pmatrix}4 & 3\end{pmatrix}\\
&= \begin{pmatrix}4 & 3\end{pmatrix} - \begin{pmatrix}4 & 3 & 1\end{pmatrix}.
\end{aligned}$$

For $R_{12}\mathcal{Q}_2$, we have

$$\begin{aligned}
R_{12}\mathcal{Q}_2 &= \begin{pmatrix}4 & 3\end{pmatrix}\Big[E - \begin{pmatrix}1 & 3\end{pmatrix}\Big]\\
&= \begin{pmatrix}4 & 3\end{pmatrix} - \begin{pmatrix}4 & 3\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix}\\
&= \begin{pmatrix}4 & 3\end{pmatrix} - \begin{pmatrix}4 & 3 & 1\end{pmatrix}.
\end{aligned}$$

From the above two results, we see that $\mathcal{Q}_1 R_{12} = R_{12}\mathcal{Q}_2$.
Multiplying $\mathcal{P}_1$ from the left on both sides of $\mathcal{Q}_1 R_{12} = R_{12}\mathcal{Q}_2$, we have

$$\begin{aligned}
\mathcal{P}_1\mathcal{Q}_1 R_{12} &= \mathcal{P}_1 R_{12}\mathcal{Q}_2\\
&= R_{12}\mathcal{P}_2\mathcal{Q}_2,
\end{aligned}$$

where we have made use of $\mathcal{P}_1 R_{12} = R_{12}\mathcal{P}_2$. Making use of $\mathcal{Y}_1 = \mathcal{P}_1\mathcal{Q}_1$ and $\mathcal{Y}_2 = \mathcal{P}_2\mathcal{Q}_2$, we obtain from the above result

$$\mathcal{Y}_1 R_{12} = R_{12}\mathcal{Y}_2.$$

We can also directly verify that $\mathcal{Y}_1 R_{12} = R_{12}\mathcal{Y}_2$ by making use of the above-obtained expressions for $\mathcal{Y}_1$ and $\mathcal{Y}_2$. For $\mathcal{Y}_1 R_{12}$, we have

$$\begin{aligned}
\mathcal{Y}_1 R_{12} &= \Big[E + \begin{pmatrix}1 & 2\end{pmatrix} + \begin{pmatrix}1 & 3\end{pmatrix} + \begin{pmatrix}2 & 3\end{pmatrix} + \begin{pmatrix}1 & 2 & 3\end{pmatrix} + \begin{pmatrix}3 & 2 & 1\end{pmatrix}\\
&\quad - \begin{pmatrix}1 & 4\end{pmatrix} - \begin{pmatrix}4 & 2 & 1\end{pmatrix} - \begin{pmatrix}4 & 3 & 1\end{pmatrix} - \begin{pmatrix}2 & 3\end{pmatrix}\begin{pmatrix}1 & 4\end{pmatrix} - \begin{pmatrix}4 & 2 & 3 & 1\end{pmatrix} - \begin{pmatrix}4 & 3 & 2 & 1\end{pmatrix}\Big]\begin{pmatrix}4 & 3\end{pmatrix}\\
&= \begin{pmatrix}4 & 3\end{pmatrix} + \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}4 & 3\end{pmatrix} + \begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}4 & 3\end{pmatrix} + \begin{pmatrix}2 & 3\end{pmatrix}\begin{pmatrix}4 & 3\end{pmatrix} + \begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}4 & 3\end{pmatrix} + \begin{pmatrix}3 & 2 & 1\end{pmatrix}\begin{pmatrix}4 & 3\end{pmatrix}\\
&\quad - \begin{pmatrix}1 & 4\end{pmatrix}\begin{pmatrix}4 & 3\end{pmatrix} - \begin{pmatrix}4 & 2 & 1\end{pmatrix}\begin{pmatrix}4 & 3\end{pmatrix} - \begin{pmatrix}4 & 3 & 1\end{pmatrix}\begin{pmatrix}4 & 3\end{pmatrix} - \begin{pmatrix}2 & 3\end{pmatrix}\begin{pmatrix}1 & 4\end{pmatrix}\begin{pmatrix}4 & 3\end{pmatrix} - \begin{pmatrix}4 & 2 & 3 & 1\end{pmatrix}\begin{pmatrix}4 & 3\end{pmatrix}\\
&\quad - \begin{pmatrix}4 & 3 & 2 & 1\end{pmatrix}\begin{pmatrix}4 & 3\end{pmatrix}\\
&= \begin{pmatrix}4 & 3\end{pmatrix} + \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}4 & 3\end{pmatrix} + \begin{pmatrix}4 & 1 & 3\end{pmatrix} + \begin{pmatrix}4 & 2 & 3\end{pmatrix} + \begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix} + \begin{pmatrix}4 & 2 & 1 & 3\end{pmatrix}\\
&\quad - \begin{pmatrix}4 & 3 & 1\end{pmatrix} - \begin{pmatrix}4 & 3 & 2 & 1\end{pmatrix} - \begin{pmatrix}1 & 4\end{pmatrix} - \begin{pmatrix}4 & 2 & 3 & 1\end{pmatrix} - \begin{pmatrix}2 & 3\end{pmatrix}\begin{pmatrix}1 & 4\end{pmatrix} - \begin{pmatrix}4 & 2 & 1\end{pmatrix}.
\end{aligned}$$

For $R_{12}\mathcal{Y}_2$, we have

$$\begin{aligned}
R_{12}\mathcal{Y}_2 &= \begin{pmatrix}4 & 3\end{pmatrix} \Big[E + \begin{pmatrix}1 & 2\end{pmatrix} + \begin{pmatrix}1 & 4\end{pmatrix} + \begin{pmatrix}2 & 4\end{pmatrix} + \begin{pmatrix}1 & 2 & 4\end{pmatrix} + \begin{pmatrix}4 & 2 & 1\end{pmatrix} \\
&\quad - \begin{pmatrix}1 & 3\end{pmatrix} - \begin{pmatrix}3 & 2 & 1\end{pmatrix} - \begin{pmatrix}4 & 1 & 3\end{pmatrix} - \begin{pmatrix}2 & 4\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix} - \begin{pmatrix}4 & 1 & 3 & 2\end{pmatrix} - \begin{pmatrix}4 & 2 & 1 & 3\end{pmatrix}\Big] \\
&= \begin{pmatrix}4 & 3\end{pmatrix} + \begin{pmatrix}4 & 3\end{pmatrix}\begin{pmatrix}1 & 2\end{pmatrix} + \begin{pmatrix}4 & 3\end{pmatrix}\begin{pmatrix}1 & 4\end{pmatrix} + \begin{pmatrix}4 & 3\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix} + \begin{pmatrix}4 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 4\end{pmatrix} + \begin{pmatrix}4 & 3\end{pmatrix}\begin{pmatrix}4 & 2 & 1\end{pmatrix} \\
&\quad - \begin{pmatrix}4 & 3\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix} - \begin{pmatrix}4 & 3\end{pmatrix}\begin{pmatrix}3 & 2 & 1\end{pmatrix} - \begin{pmatrix}4 & 3\end{pmatrix}\begin{pmatrix}4 & 1 & 3\end{pmatrix} - \begin{pmatrix}4 & 3\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix} - \begin{pmatrix}4 & 3\end{pmatrix}\begin{pmatrix}4 & 1 & 3 & 2\end{pmatrix} \\
&\quad - \begin{pmatrix}4 & 3\end{pmatrix}\begin{pmatrix}4 & 2 & 1 & 3\end{pmatrix} \\
&= \begin{pmatrix}4 & 3\end{pmatrix} + \begin{pmatrix}4 & 3\end{pmatrix}\begin{pmatrix}1 & 2\end{pmatrix} + \begin{pmatrix}4 & 1 & 3\end{pmatrix} + \begin{pmatrix}4 & 2 & 3\end{pmatrix} + \begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix} + \begin{pmatrix}4 & 2 & 1 & 3\end{pmatrix} \\
&\quad - \begin{pmatrix}4 & 3 & 1\end{pmatrix} - \begin{pmatrix}4 & 3 & 2 & 1\end{pmatrix} - \begin{pmatrix}1 & 4\end{pmatrix} - \begin{pmatrix}4 & 2 & 3 & 1\end{pmatrix} - \begin{pmatrix}1 & 4\end{pmatrix}\begin{pmatrix}2 & 3\end{pmatrix} - \begin{pmatrix}4 & 2 & 1\end{pmatrix}.
\end{aligned}$$

From the above two results, we see that $\mathcal{Y}_1 R_{12} = R_{12}\mathcal{Y}_2$.