Group Theory
Homework Assignment 04
Spring, 2020

1. The multiplication table for the group $D_3 = \{E, D, F, A, B, C\}$ is given by

	E	D	F	A	B	C
E	E	D	F	A	B	C
D	D	F	E	B	C	A
F	F	E	D	C	A	B
A	A	C	B	E	F	D
B	B	A	C	D	E	F
C	C	B	A	F	D	E

   (a) Determine the dimensions of all the inequivalent irreducible representations of D_3.

   (b) Find the character table for D_3.

2. The transformation matrices in two-dimensional real space for the elements of the group $D_3 = \{E, D, F, A, B, C\}$ are given by

$$R(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, R(D) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, R(F) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix},$$
$$R(A) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, R(B) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}, R(C) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}.$$

   The basis functions of a carrier space for D_3 are given by $\psi_1(\vec{\rho}) = x^2F(\rho)$, $\psi_2(\vec{\rho}) = xyF(\rho)$, and $\psi_3(\vec{\rho}) = y^2F(\rho)$, where $F(\rho)$, a function of $\rho = \sqrt{x^2+y^2}$, ensures that the basis functions are normalizable. Using

$$Q(T)\psi_n(\vec{\rho}) = \psi_n(R(T)^{-1}\vec{\rho}) = \sum_{m=1}^{3} \Gamma(T)_{mn}\psi_m(\vec{\rho}),\ n = 1,\ 2,\ 3,$$

   find the representation matrix $\Gamma(F)$ of the element F of D_3. Here $Q(T)$ is the scalar transformation operator and $\vec{\rho}$ is the position vector of a point in two-dimensional real space.

3. Using the information given in the previous problem, find the representation matrix $\Gamma(B)$ of the element B of D_3.

4. Show that, if the projection operators P^p_{mn} and P^q_{jk} belong to two unitary irreducible representations Γ^p and Γ^q of G that are not equivalent if $p \neq q$ (but are identical if $p = q$), then $P^p_{mn}P^q_{jk} = \delta_{pq}\delta_{nj}P^q_{mk}$.

5. Choosing $\phi(\vec{r}) = (xy + yz)e^{-r}$, construct the basis functions for the two-dimensional irreducible representation Γ^5 of the crystallographic point group D_4.