Group Theory

Solutions to Problems in Homework Assignment 11

Spring, 2020

Consider the permutation group S_4.

1. Consider some of the properties of S_4.
   (a) What are the classes in S_4?
   (b) What are the inequivalent irreducible representations of S_4?
   (c) Write down all the Young tableaux in each irreducible representation of S_4. What is the dimension of each irreducible representation of S_4?

   (a) From the partition $\sum_{j=1}^{m} \ell_j = n$ with $n = 4$, we see that the classes in S_4 are $(4), (3,1), (2^2), (2,1^2), (1^4)$.
   (b) From the partition $\sum_{j=1}^{m} \ell_j = n$ with $n = 4$, we see that the inequivalent irreducible representations of S_4 are $[4], [3,1], [2^2], [2,1^2], [1^4]$.
   (c) The standard Young tableau in the irreducible representation $[4]$ is

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline\end{array}$$

   Because there is only one standard Young tableau in the irreducible representation $[4]$, the dimension of the irreducible representation $[4]$ is one.
   The standard Young tableaux in the irreducible representation $[3,1]$ are

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$$

   Because there are three standard Young tableau in the irreducible representation $[3,1]$, the dimension of the irreducible representation $[3,1]$ is three.
   The standard Young tableaux in the irreducible representation $[2^2]$ are

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$$

   Because there are two standard Young tableau in the irreducible representation $[2^2]$, the dimension of the irreducible representation $[2^2]$ is two.
   The standard Young tableaux in the irreducible representation $[2,1^2]$ are

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

   Because there are three standard Young tableau in the irreducible representation $[2,1^2]$, the dimension of the irreducible representation $[2,1^2]$ is three.
   The standard Young tableau in the irreducible representation $[1^4]$ is

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline\end{array}$$

   Because there is only one standard Young tableau in the irreducible representation $[1^4]$, the dimension of the irreducible representation $[1^4]$ is one.
   A summary on the inequivalent irreducible representations of S_4 is given in Table I.

2. Consider the Young operators in the irreducible representation $[3,1]$ of S_4.
   (a) Write down all the Young tableaux and the corresponding Young operators $\mathcal{Y}_{\mu}^{[3,1]}$'s in the irreducible representation $[3,1]$ of S_4.

TABLE I: Inequivalent irreducible representations of S_4

	Standard Young tableau(x)	Dimension
$[4]$	$\begin{array}{\|c\|c\|c\|c\|}\hline 1&2&3&4\\\hline\end{array}$	1
$[3,1]$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}$ $\begin{array}{\|c\|c\|c\|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}$ $\begin{array}{\|c\|c\|c\|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}$	3
$[2^2]$	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 3&4\\\hline\end{array}$ $\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2&4\\\hline\end{array}$	2
$[2,1^2]$	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}$ $\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array}$ $\begin{array}{\|c\|c\|}\hline 1&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}$	3
$[1^4]$	$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}$	1

(b) Argue that all the Young operators in the irreducible representation $[3,1]$ of S_4 are orthogonal.

(a) For the standard Young tableaux $\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}$, the horizontal permutations are

$$P_1 = E, (1\ \ 2), (1\ \ 3), (2\ \ 3), (1\ \ 2\ \ 3), (3\ \ 2\ \ 1),$$

the vertical permutations are

$$Q_1 = E, (1\ \ 4),$$

and the Young operator is

$$\begin{aligned}\mathcal{Y}_1 &= \big[\, E + (1\ \ 2) + (1\ \ 3) + (2\ \ 3) + (1\ \ 2\ \ 3) + (3\ \ 2\ \ 1)\,\big]\big[\, E - (1\ \ 4)\,\big]\\ &= E + (1\ \ 2) + (1\ \ 3) + (2\ \ 3) + (1\ \ 2\ \ 3) + (3\ \ 2\ \ 1)\\ &\quad - (1\ \ 4) - (1\ \ 2)(1\ \ 4) - (1\ \ 3)(1\ \ 4) - (2\ \ 3)(1\ \ 4) - (1\ \ 2\ \ 3)(1\ \ 4) - (3\ \ 2\ \ 1)(1\ \ 4)\\ &= E + (1\ \ 2) + (1\ \ 3) + (2\ \ 3) + (1\ \ 2\ \ 3) + (3\ \ 2\ \ 1)\\ &\quad - (1\ \ 4) - (2\ \ 1\ \ 4) - (3\ \ 1\ \ 4) - (2\ \ 3)(1\ \ 4) - (2\ \ 3\ \ 1\ \ 4) - (3\ \ 2\ \ 1\ \ 4).\end{aligned}$$

For the standard Young tableaux $\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}$, the horizontal permutations are

$$P_1 = E, (1\ \ 2), (2\ \ 4), (1\ \ 4), (1\ \ 2\ \ 4), (4\ \ 2\ \ 1),$$

the vertical permutations are

$$Q_1 = E, (1\ \ 3),$$

and the Young operator is

$$\begin{aligned}\mathcal{Y}_2 &= \big[\, E + (1\ \ 2) + (2\ \ 4) + (1\ \ 4) + (1\ \ 2\ \ 4) + (4\ \ 2\ \ 1)\,\big]\big[\, E - (1\ \ 3)\,\big]\\ &= E + (1\ \ 2) + (2\ \ 4) + (1\ \ 4) + (1\ \ 2\ \ 4) + (4\ \ 2\ \ 1)\\ &\quad - (1\ \ 3) - (1\ \ 2)(1\ \ 3) - (2\ \ 4)(1\ \ 3) - (1\ \ 4)(1\ \ 3) - (1\ \ 2\ \ 4)(1\ \ 3) - (4\ \ 2\ \ 1)(1\ \ 3)\\ &= E + (1\ \ 2) + (2\ \ 4) + (1\ \ 4) + (1\ \ 2\ \ 4) + (4\ \ 2\ \ 1)\\ &\quad - (1\ \ 3) - (2\ \ 1\ \ 3) - (2\ \ 4)(1\ \ 3) - (4\ \ 1\ \ 3) - (2\ \ 4\ \ 1\ \ 3) - (4\ \ 2\ \ 1\ \ 3).\end{aligned}$$

For the standard Young tableaux $\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$, the horizontal permutations are

$$P_1 = E, (1\ \ 3), (1\ \ 4), (3\ \ 4), (1\ \ 3\ \ 4), (4\ \ 3\ \ 1),$$

the vertical permutations are

$$Q_1 = E, (1\ \ 2),$$

and the Young operator is

$$\begin{aligned}
\mathcal{Y}_3 &= \big[\, E + (1\ \ 3) + (1\ \ 4) + (3\ \ 4) + (1\ \ 3\ \ 4) + (4\ \ 3\ \ 1) \,\big]\big[\, E - (1\ \ 2) \,\big] \\
&= E + (1\ \ 3) + (1\ \ 4) + (3\ \ 4) + (1\ \ 3\ \ 4) + (4\ \ 3\ \ 1) \\
&\quad - (1\ \ 2) - (1\ \ 3)(1\ \ 2) - (1\ \ 4)(1\ \ 2) - (3\ \ 4)(1\ \ 2) - (1\ \ 3\ \ 4)(1\ \ 2) - (4\ \ 3\ \ 1)(1\ \ 2) \\
&= E + (1\ \ 3) + (1\ \ 4) + (3\ \ 4) + (1\ \ 3\ \ 4) + (4\ \ 3\ \ 1) \\
&\quad - (1\ \ 2) - (3\ \ 1\ \ 2) - (4\ \ 1\ \ 2) - (3\ \ 4)(1\ \ 2) - (3\ \ 4\ \ 1\ \ 2) - (4\ \ 3\ \ 1\ \ 2).
\end{aligned}$$

(b) We use the following theorem and the corollary to argue that all the Young operators in the irreducible representation $[3,1]$ of S_4 are orthogonal.

[**Theorem**] If there exist two digits a and b in one row of a Young tableau $\mathcal{Y}$ which also occur in one column of a Young tableau $\mathcal{Y}'$, then $\mathcal{Y}'\mathcal{Y} = 0$.

[**Corollary**] For a given Young pattern, if a standard Young tableau $\mathcal{Y}'$ is larger than a standard Young tableau $\mathcal{Y}$, then $\mathcal{Y}'\mathcal{Y} = 0$.

From the Corollary, we have $\mathcal{Y}_3\mathcal{Y}_2 = \mathcal{Y}_3\mathcal{Y}_1 = \mathcal{Y}_2\mathcal{Y}_1 = 0$.

Comparing $\mathcal{Y}_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$ with $\mathcal{Y}_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$, we see that the digits 1 and 4 occur in one row of $\mathcal{Y}_2$ and one column of $\mathcal{Y}_1$. Hence, $\mathcal{Y}_1\mathcal{Y}_2 = 0$ according to the Theorem. From $\mathcal{Y}_2\mathcal{Y}_1 = \mathcal{Y}_1\mathcal{Y}_2 = 0$, we conclude that $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are orthogonal.

Comparing $\mathcal{Y}_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$ with $\mathcal{Y}_3 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$, we see that the digits 1 and 3 occur in one row of $\mathcal{Y}_3$ and one column of $\mathcal{Y}_2$. Hence, $\mathcal{Y}_2\mathcal{Y}_3 = 0$ according to the Theorem. From $\mathcal{Y}_3\mathcal{Y}_2 = \mathcal{Y}_2\mathcal{Y}_3 = 0$, we conclude that $\mathcal{Y}_2$ and $\mathcal{Y}_3$ are orthogonal.

Comparing $\mathcal{Y}_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$ with $\mathcal{Y}_3 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$, we see that the digits 1 and 4 occur in one row of $\mathcal{Y}_3$ and one column of $\mathcal{Y}_1$. Hence, $\mathcal{Y}_1\mathcal{Y}_3 = 0$ according to the Theorem. From $\mathcal{Y}_3\mathcal{Y}_1 = \mathcal{Y}_1\mathcal{Y}_3 = 0$, we conclude that $\mathcal{Y}_1$ and $\mathcal{Y}_3$ are orthogonal.

We have thus argued that all the Young operators in the irreducible representation $[3,1]$ of S_4 are orthogonal. The orthogonality of the Young operators in the irreducible representation $[3,1]$ of S_4 can be also verified directly by using the above-obtained expressions of the Young operators.

3. Write down all the standard basis vectors in the irreducible representation $[3,1]$ of S_4.

In consideration that all the Young operators in the irreducible representation $[3,1]$ of S_4 are orthogonal, the orthogonal primitive idempotents in the irreducible representation $[3,1]$ are given by

$$e_\mu^{[3,1]} = \frac{d_{[3,1]}}{4!}\mathcal{Y}_\mu = \frac{1}{8}\mathcal{Y}_\mu.$$

That is,

$$\begin{aligned}
e_1^{[3,1]} &= \frac{1}{8}\mathcal{Y}_1 \\
&= \frac{1}{8}\big[\, E + (1\ \ 2) + (1\ \ 3) + (2\ \ 3) + (1\ \ 2\ \ 3) + (3\ \ 2\ \ 1) \\
&\qquad - (1\ \ 4) - (2\ \ 1\ \ 4) - (3\ \ 1\ \ 4) - (2\ \ 3)(1\ \ 4) - (2\ \ 3\ \ 1\ \ 4) - (3\ \ 2\ \ 1\ \ 4) \,\big],
\end{aligned}$$

$$
\begin{aligned}
e_2^{[3,1]} &= \frac{1}{8}\mathcal{Y}_2 \\
&= \frac{1}{8}\big[\ E + (1\ \ 2) + (2\ \ 4) + (1\ \ 4) + (1\ \ 2\ \ 4) + (4\ \ 2\ \ 1) \\
&\quad - (1\ \ 3) - (2\ \ 1\ \ 3) - (2\ \ 4)(1\ \ 3) - (4\ \ 1\ \ 3) - (2\ \ 4\ \ 1\ \ 3) - (4\ \ 2\ \ 1\ \ 3)\ \big],
\end{aligned}
$$

$$
\begin{aligned}
e_3^{[3,1]} &= \frac{1}{8}\mathcal{Y}_3 \\
&= \frac{1}{8}\big[\ E + (1\ \ 3) + (1\ \ 4) + (3\ \ 4) + (1\ \ 3\ \ 4) + (4\ \ 3\ \ 1) \\
&\quad - (1\ \ 2) - (3\ \ 1\ \ 2) - (4\ \ 1\ \ 2) - (3\ \ 4)(1\ \ 2) - (3\ \ 4\ \ 1\ \ 2) - (4\ \ 3\ \ 1\ \ 2)\ \big].
\end{aligned}
$$

In terms of the orthogonal primitive idempotents $e_\mu^{[3,1]}$, the standard basis vectors $b_{\mu\nu}^{[3,1]}$ are given by

$$
b_{\mu\nu}^{[3,1]} = R_{\mu\nu} e_\nu^{[3,1]},\ \ \mu, \nu = 1, 2, 3.
$$

We now find $b_{\mu 1}^{[3,1]}$ for $\mu = 1, 2, 3$ from $e_1^{[3,1]}$. Making use of $R_{11} = E$,

$$
\begin{aligned}
R_{21} &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} = (3\ \ 4), \\
R_{31} &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} = (2\ \ 3\ \ 4),
\end{aligned}
$$

we have

$$
\begin{aligned}
b_{11}^{[3,1]} &= R_{22} e_1^{[3,1]} = e_1^{[3,1]} \\
&= \frac{1}{8}\big[\ E + (1\ \ 2) + (1\ \ 3) + (2\ \ 3) + (1\ \ 2\ \ 3) + (3\ \ 2\ \ 1) \\
&\quad - (1\ \ 4) - (2\ \ 1\ \ 4) - (3\ \ 1\ \ 4) - (2\ \ 3)(1\ \ 4) - (2\ \ 3\ \ 1\ \ 4) - (3\ \ 2\ \ 1\ \ 4)\ \big], \\
b_{21}^{[3,1]} &= R_{21} e_1^{[3,1]} \\
&= \frac{1}{8}\big[\ (3\ \ 4) + (3\ \ 4)(1\ \ 2) + (4\ \ 3\ \ 1) + (4\ \ 3\ \ 2) + (4\ \ 3\ \ 1\ \ 2) + (4\ \ 3\ \ 2\ \ 1) \\
&\quad - (3\ \ 4\ \ 1) - (3\ \ 4\ \ 2\ \ 1) - (3\ \ 1) - (3\ \ 2\ \ 4\ \ 1) - (2\ \ 4)(3\ \ 1) - (3\ \ 2\ \ 1)\ \big], \\
b_{31}^{[3,1]} &= R_{31} e_1^{[3,1]} \\
&= \frac{1}{8}\big[\ (2\ \ 3\ \ 4) + (3\ \ 4\ \ 2\ \ 1) + (4\ \ 2\ \ 3\ \ 1) + (2\ \ 4) + (2\ \ 4)(1\ \ 3) + (4\ \ 2\ \ 1) \\
&\quad - (2\ \ 3\ \ 4\ \ 1) - (3\ \ 4)(1\ \ 2) - (2\ \ 3\ \ 1) - (2\ \ 4\ \ 1) - (2\ \ 4\ \ 3\ \ 1) - (1\ \ 2)\ \big].
\end{aligned}
$$

We now find $b_{\mu 2}^{[3,1]}$ for $\mu = 1, 2, 3$ from $e_2^{[3,1]}$. Making use of $R_{22} = E$,

$$
\begin{aligned}
R_{12} &= \begin{pmatrix} 1 & 2 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} = (3\ \ 4), \\
R_{32} &= \begin{pmatrix} 1 & 2 & 4 & 3 \\ 1 & 3 & 4 & 2 \end{pmatrix} = (2\ \ 3),
\end{aligned}
$$

we have

$$\begin{aligned}
b_{12}^{[3,1]} &= R_{12}e_2^{[3,1]} \\
&= \frac{1}{8}\big[\,(3\ \ 4) + (3\ \ 4)(1\ \ 2) + (3\ \ 4\ \ 2) + (3\ \ 4\ \ 1) + (3\ \ 4\ \ 1\ \ 2) + (3\ \ 4\ \ 2\ \ 1) \\
&\quad - (4\ \ 3\ \ 1) - (4\ \ 3\ \ 2\ \ 1) - (4\ \ 2\ \ 3\ \ 1) - (1\ \ 4) - (1\ \ 4)(2\ \ 3) - (4\ \ 2\ \ 1)\,\big] \\
b_{22}^{[3,1]} &= R_{22}e_2^{[3,1]} = e_2^{[3,1]} \\
&= \frac{1}{8}\big[\,E + (1\ \ 2) + (2\ \ 4) + (1\ \ 4) + (1\ \ 2\ \ 4) + (4\ \ 2\ \ 1) \\
&\quad - (1\ \ 3) - (2\ \ 1\ \ 3) - (2\ \ 4)(1\ \ 3) - (4\ \ 1\ \ 3) - (2\ \ 4\ \ 1\ \ 3) - (4\ \ 2\ \ 1\ \ 3)\,\big], \\
b_{32}^{[3,1]} &= R_{32}e_2^{[3,1]} \\
&= \frac{1}{8}\big[\,(2\ \ 3) + (3\ \ 2\ \ 1) + (3\ \ 2\ \ 4) + (2\ \ 3)(1\ \ 4) + (3\ \ 2\ \ 4\ \ 1) + (3\ \ 2\ \ 1\ \ 4) \\
&\quad - (2\ \ 3\ \ 1) - (1\ \ 2) - (2\ \ 4\ \ 3\ \ 1) - (2\ \ 3\ \ 4\ \ 1) - (2\ \ 4\ \ 1) - (1\ \ 2)(3\ \ 4)\,\big].
\end{aligned}$$

We now find $b_{\mu 3}^{[3,1]}$ for $\mu = 1, 2, 3$ from $e_3^{[3,1]}$. Making use of $R_{33} = E$,

$$R_{13} = \begin{pmatrix} 1 & 3 & 4 & 2 \\ 1 & 2 & 3 & 4 \end{pmatrix} = (4\ \ 3\ \ 2),$$

$$R_{23} = \begin{pmatrix} 1 & 3 & 4 & 2 \\ 1 & 2 & 4 & 3 \end{pmatrix} = (2\ \ 3),$$

we have

$$\begin{aligned}
b_{13}^{[3,1]} &= R_{13}e_3^{[3,1]} \\
&= \frac{1}{8}\big[\,(4\ \ 3\ \ 2) + (2\ \ 4\ \ 3\ \ 1) + (3\ \ 2\ \ 4\ \ 1) + (2\ \ 4) + (2\ \ 4\ \ 2) + (2\ \ 4)(1\ \ 3) \\
&\quad - (4\ \ 3\ \ 2\ \ 1) - (4\ \ 3\ \ 1) - (2\ \ 3)(1\ \ 4) - (4\ \ 2\ \ 1) - (1\ \ 4) - (4\ \ 2\ \ 3\ \ 1)\,\big], \\
b_{23}^{[3,1]} &= R_{23}e_3^{[3,1]} \\
&= \frac{1}{8}\big[\,(2\ \ 3) + (2\ \ 3\ \ 1) + (2\ \ 3)(1\ \ 4) + (2\ \ 3\ \ 4) + (2\ \ 3\ \ 4\ \ 1) + (2\ \ 3\ \ 1\ \ 4) \\
&\quad - (3\ \ 2\ \ 1) - (1\ \ 3) - (3\ \ 2\ \ 4\ \ 1) - (3\ \ 4\ \ 2\ \ 1) - (3\ \ 4\ \ 1) - (2\ \ 4)(1\ \ 3)\,\big], \\
b_{33}^{[3,1]} &= R_{33}e_3^{[3,1]} = e_3^{[3,1]} \\
&= \frac{1}{8}\big[\,E + (1\ \ 3) + (1\ \ 4) + (3\ \ 4) + (1\ \ 3\ \ 4) + (4\ \ 3\ \ 1) \\
&\quad - (1\ \ 2) - (3\ \ 1\ \ 2) - (4\ \ 1\ \ 2) - (3\ \ 4)(1\ \ 2) - (3\ \ 4\ \ 1\ \ 2) - (4\ \ 3\ \ 1\ \ 2)\,\big].
\end{aligned}$$

We have thus obtained all the standard basis vectors.

4. Write down all the $Q_{\nu k}$'s in the irreducible representation $[3,1]$ of S_4. Find all the $\mathcal{Y}'$'s using $\mathcal{Y}' = Q_{\nu k}\mathcal{Y}_{\nu k}Q_{\nu k}^{-1}$. Find all the $\mathcal{Y}_\mu(S)$'s in the irreducible representation $[3,1]$ of S_4 for $S = (1\ \ 2\ \ 3\ \ 4)$ using $\mathcal{Y}_\mu(S) = S\mathcal{Y}_\mu^{[3,1]}S^{-1}$.

Since $y_1 = y_2 = y_3 = E$, we have $\delta_1 = 1$ and $T_1 = R$ for y_1, y_2, and y_3. Thus, $\mathcal{Y}_{\nu k} = \mathcal{Y}_\nu^{[3,1]}$ for $\nu = 1, 2, 3$ and $k = 1$. In Table II, we list all the $\mathcal{Y}_{\nu k}$'s and $Q_{\nu k}$'s.

All the $\mathcal{Y}_\mu(S)$'s in the irreducible representation $[3,1]$ of S_4 for $S = (1\ \ 2\ \ 3\ \ 4)$ are

$$\mathcal{Y}_1(S) = \begin{matrix} 2 & 3 & 4 \\ 1 & & \end{matrix}, \quad \mathcal{Y}_2(S) = \begin{matrix} 2 & 3 & 1 \\ 4 & & \end{matrix}, \quad \mathcal{Y}_3(S) = \begin{matrix} 2 & 4 & 1 \\ 3 & & \end{matrix}.$$

5. Construct the table for $A_{\nu k}^{\mu}(S)$ for $S = (1\ \ 2\ \ 3\ \ 4)$ with $\mathcal{Y}_\mu(S)$ labeling the columns and $\sum_k \delta_k \mathcal{Y}_{\nu k}$ labeling the rows. Write down the representation matrix of $S = (1\ \ 2\ \ 3\ \ 4)$.

TABLE II: $\mathcal{Y}_{\nu k}$'s, $Q_{\nu k}$'s, and $\mathcal{Y}'$ for the irreducible representation $[3,1]$ of S_4.

	$\nu=1$, $k=1$	$\nu=2$, $k=1$	$\nu=3$, $k=1$
$\mathcal{Y}_\nu^{[3,1]}$	1 2 3 4	1 2 4 3	1 3 4 2
$\mathcal{Y}_{\nu k}$	1 2 3 4	1 2 4 3	1 3 4 2
$Q_{\nu k}$	E, (1 4)	E, (1 3)	E, (1 2)
$\mathcal{Y}'$	1 2 3 4	1 2 4 3	1 3 4 2
	4 2 3 1	3 2 4 1	2 3 4 1

Table III is the table for $A^{\mu}_{\nu k}(S)$ for $S = (1\ \ 2\ \ 3\ \ 4)$.

TABLE III: $A^{\mu}_{\nu k}(S)$ for $S = (1\,2\,3\,4)$ in the irreducible representation $[3,1]$ of S_4.

$\sum_k \delta_k \mathcal{Y}_{\nu k}$	$\mathcal{Y}_\mu(S)$: 2 3 4 1	2 3 1 4	2 4 1 3
1 2 3 4	-1	1	0
1 2 4 3	-1	0	1
1 3 4 2	-1	0	0

From Table III, we see that the representation matrix of $S = (1\ \ 2\ \ 3\ \ 4)$ in the irreducible representation $[3,1]$ of S_4 is given by

$$D^{[3,1]}(S = (1\ \ 2\ \ 3\ \ 4)) = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$