Group Theory

Solutions to Problems in Homework Assignment 13

Spring, 2020

1. The basis of the real Lie algebra $L = \mathrm{sl}(2, \mathbb{R})$ is given by

$$b_1 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},\ b_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ b_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Find the representation matrices of the basis elements b_1, b_2, and b_3 in the adjoint representation.

In order to find the representation matrices of the basis elements b_1, b_2, and b_3 in the adjoint representation, we need the commutation relations between these basis elements. For $[b_1, b_2]$, we have

$$\begin{aligned}
[b_1, b_2] &= b_1 b_2 - b_2 b_1 \\
&= \frac{1}{4}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = b_3.
\end{aligned}$$

For $[b_2, b_3]$, we have

$$\begin{aligned}
[b_2, b_3] &= b_2 b_3 - b_3 b_2 \\
&= \frac{1}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = b_1.
\end{aligned}$$

For $[b_3, b_1]$, we have

$$\begin{aligned}
[b_3, b_1] &= b_3 b_1 - b_1 b_3 \\
&= \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -b_2.
\end{aligned}$$

From

$$[b_1, b_1] = 0,\ [b_1, b_2] = b_3,\ [b_1, b_3] = b_2,$$

we have

$$\begin{aligned}
&[\mathrm{ad}(b_1)]_{11} = 0,\ [\mathrm{ad}(b_1)]_{21} = 0,\ [\mathrm{ad}(b_1)]_{31} = 0, \\
&[\mathrm{ad}(b_1)]_{12} = 0,\ [\mathrm{ad}(b_1)]_{22} = 0,\ [\mathrm{ad}(b_1)]_{32} = 1, \\
&[\mathrm{ad}(b_1)]_{13} = 0,\ [\mathrm{ad}(b_1)]_{23} = 1,\ [\mathrm{ad}(b_1)]_{33} = 0.
\end{aligned}$$

Thus,

$$\mathrm{ad}(b_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

From

$$[b_2, b_1] = -b_3,\ [b_2, b_2] = 0,\ [b_2, b_3] = b_1,$$

we have

$$\begin{aligned}
&[\mathrm{ad}(b_2)]_{11} = 0,\ [\mathrm{ad}(b_2)]_{21} = 0,\ [\mathrm{ad}(b_2)]_{31} = -1,\\
&[\mathrm{ad}(b_2)]_{12} = 0,\ [\mathrm{ad}(b_2)]_{22} = 0,\ [\mathrm{ad}(b_2)]_{32} = 0,\\
&[\mathrm{ad}(b_2)]_{13} = 1,\ [\mathrm{ad}(b_2)]_{23} = 0,\ [\mathrm{ad}(b_2)]_{33} = 0.
\end{aligned}$$

Thus,

$$\mathrm{ad}(b_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

From

$$[b_3, b_1] = -b_2,\ [b_3, b_2] = -b_1,\ [b_3, b_3] = 0,$$

we have

$$\begin{aligned}
&[\mathrm{ad}(b_3)]_{11} = 0,\ [\mathrm{ad}(b_3)]_{21} = -1,\ [\mathrm{ad}(b_3)]_{31} = 0,\\
&[\mathrm{ad}(b_3)]_{12} = -1,\ [\mathrm{ad}(b_3)]_{22} = 0,\ [\mathrm{ad}(b_3)]_{32} = 0,\\
&[\mathrm{ad}(b_3)]_{13} = 0,\ [\mathrm{ad}(b_3)]_{23} = 0,\ [\mathrm{ad}(b_3)]_{33} = 0.
\end{aligned}$$

Thus,

$$\mathrm{ad}(b_3) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

2. Continue from the previous problem. Find the Killing forms $B(b_p, b_q)$ for $p, q = 1, 2, 3$.

For $B(b_1, b_1)$, we have

$$B(b_1, b_1) = \mathrm{tr}\big[\mathrm{ad}(b_1)^2\big] = \mathrm{tr}\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^2\right] = \mathrm{tr}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2.$$

For $B(b_1, b_2) = B(b_2, b_1)$, we have

$$B(b_1, b_2) = B(b_2, b_1) = \mathrm{tr}\big[\mathrm{ad}(b_1)\mathrm{ad}(b_2)\big] = \mathrm{tr}\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\right] = \mathrm{tr}\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

For $B(b_1, b_3) = B(b_3, b_1)$, we have

$$B(b_1, b_3) = B(b_3, b_1) = \mathrm{tr}\big[\mathrm{ad}(b_1)\mathrm{ad}(b_3)\big] = \mathrm{tr}\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = \mathrm{tr}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = 0.$$

For $B(b_2, b_2)$, we have

$$B(b_2, b_2) = \mathrm{tr}\big[\mathrm{ad}(b_2)^2\big] = \mathrm{tr}\left[\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}^2\right] = \mathrm{tr}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -2.$$

For $B(b_2, b_3) = B(b_3, b_2)$, we have

$$B(b_2, b_3) = B(b_3, b_2) = \mathrm{tr}\big[\mathrm{ad}(b_2)\mathrm{ad}(b_3)\big] = \mathrm{tr}\left[\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = \mathrm{tr}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = 0.$$

For $B(b_3, b_3)$, we have

$$B(b_3, b_3) = \mathrm{tr}\big[\mathrm{ad}(b_3)^2\big] = \mathrm{tr}\left[\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2\right] = \mathrm{tr}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2.$$

In summary, we have obtained

$$\begin{aligned} &B(b_1, b_1) = 2,\ B(b_2, b_2) = -2,\ B(b_3, b_3) = 2, \\ &B(b_p, b_q) = 0,\ p \neq q = 1, 2, 3. \end{aligned}$$

3. Choose the basis of the semi-simple complex Lie algebra $\tilde{L}$ such that each basis element is a member of some subspace $\tilde{L}_\gamma$. The adjoint representation matrix $\mathrm{ad}(h)$ of each element h in the Cartan subalgebra H is a diagonal matrix with zero diagonal elements corresponding to the basis elements of $\tilde{L}_0 = H$ and with diagonal element $\gamma(h)$ corresponding to each basis element of $\tilde{L}_\gamma$ (for $\gamma \in \Delta$). Show that

$$B(h, h') = \sum_{\gamma\in\Delta} (\dim \tilde{L}_\gamma)\gamma(h)\gamma(h') \text{ for all } h, h' \in H.$$

According to the statement of the problem, we have

$$\mathrm{ad}(h) = \begin{pmatrix} 0_{\dim H} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \gamma_1(h)1_{\dim \tilde{L}_{\gamma_1}} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \gamma_2(h)1_{\dim \tilde{L}_{\gamma_2}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \gamma_n(h)1_{\dim \tilde{L}_{\gamma_n}} \end{pmatrix},$$

where $0_{\dim H}$ is a $\dim H \times \dim H$ zero matrix, $1_{\dim \tilde{L}_{\gamma_i}}$ is a $\dim \tilde{L}_{\gamma_i} \times \dim \tilde{L}_{\gamma_i}$ unit matrix. The 0's are appropriate zero matrices.

According to the definition of the Killing form, we have

$$\begin{aligned} B(h, h') &= \mathrm{tr}\big[\mathrm{ad}(h)\mathrm{ad}(h')\big] = \sum_{jk}\big[\mathrm{ad}(h)\big]_{jk}\big[\mathrm{ad}(h')\big]_{kj} = \sum_{jk}\big[\mathrm{ad}(h)\big]_{jj}\delta_{jk}\big[\mathrm{ad}(h')\big]_{kk}\delta_{kj} \\ &= \sum_{j}\big[\mathrm{ad}(h)\big]_{jj}\big[\mathrm{ad}(h')\big]_{jj} = \sum_{i}\big(\dim \tilde{L}_{\gamma_i}\big)\gamma_i(h)\gamma_i(h') = \sum_{\gamma\in\Delta}\big(\dim \tilde{L}_\gamma\big)\gamma(h)\gamma(h'). \end{aligned}$$

4. Consider the complexification $\tilde{L}$ of $L = \mathrm{su}(2)$. The basis elements are given by

$$a_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},\ a_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ a_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The basis of the Cartan subalgebra H is given by $h_1 = a_3$. Verify that the two non-zero roots are α_1 and $-\alpha_1$ with $\alpha_1(h_1) = i$.

The basis element in the subspace of the non-zero root α_1 must be a linear combination of a_1 and a_2. Let

$$a_1' = a_1 + \kappa a_2,$$

where κ is a constant. The coefficient of unity for a_1 is allowed because the equation $[h_1, a_1'] = \alpha_1(h_1)a_1'$ holds no matter what non-zero coefficient a_1' on both sides is multiplied with. The commutator between $h_1 = a_3$ and a_1' can be evaluated as follows

$$[h_1, a_1'] = [a_3, a_1 + \kappa a_2] = [a_3, a_1] + \kappa[a_3, a_2] = -a_2 + \kappa a_1.$$

From $[h_1, a_1'] = \alpha_1(h_1)a_1'$, we have

$$-a_2 + \kappa a_1 = \alpha_1(h_1)(a_1 + \kappa a_2)$$

from which we have

$$\begin{aligned}\kappa &= \alpha_1(h_1),\\ -1 &= \kappa\alpha_1(h_1).\end{aligned}$$

The two solutions to the above equations are $\kappa = \alpha_1(h_1) = \pm 1$. We take $\alpha_1(h_1) = i$. The other solution corresponds to the non-zero root $-\alpha_1$. We thus have

$$\begin{aligned}[h_1, a_1 + ia_2] &= i(a_1 + ia_2),\\ [h_1, a_1 - ia_2] &= -i(a_1 - ia_2).\end{aligned}$$

Thus, the two non-zero roots are respectively α_1 and $-\alpha_1$ with $\alpha_1(h_1) = i$.

5. Continue from the previous problem. Find the values of $B(h_1, h_1)$ and $\langle\alpha_1, \alpha_1\rangle$.

From

$$[h_1, a_1] = -a_2,\ [h_1, a_2] = a_1,\ [h_1, a_3] = 0,$$

we have

$$\begin{aligned}&\left[\mathrm{ad}(h_1)\right]_{11} = 0,\ \left[\mathrm{ad}(h_1)\right]_{21} = -1,\ \left[\mathrm{ad}(h_1)\right]_{31} = 0,\\ &\left[\mathrm{ad}(h_1)\right]_{12} = 1,\ \left[\mathrm{ad}(h_1)\right]_{22} = 0,\ \left[\mathrm{ad}(h_1)\right]_{32} = 0,\\ &\left[\mathrm{ad}(h_1)\right]_{13} = 0,\ \left[\mathrm{ad}(h_1)\right]_{23} = 0,\ \left[\mathrm{ad}(h_1)\right]_{33} = 0.\end{aligned}$$

Thus,

$$\mathrm{ad}(h_1) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Killing form is then given by

$$B(h_1, h_1) = \mathrm{tr}\left\{\left[\mathrm{ad}(h_1)\right]^2\right\} = \mathrm{tr}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 = \mathrm{tr}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -2.$$

Since the Cartan subalgebra H is one-dimensional, h_{α_1} is proportional to $h_1 = a_3$. Let

$$h_{\alpha_1} = \kappa h_1.$$

We then have

$$\begin{aligned}B(h_{\alpha_1}, h_{\alpha_1}) &= \kappa^2 B(h_1, h_1) = -2\kappa^2,\\ \alpha_1(h_{\alpha_1}) &= \kappa\alpha_1(h_1) = i\kappa.\end{aligned}$$

From $B(h_{\alpha_1}, h_{\alpha_1}) = \alpha_1(h_{\alpha_1})$, we obtain

$$-2\kappa^2 = i\kappa.$$

In consideration that κ can not be zero, we have

$$\kappa = -\frac{1}{2}i.$$

We then have

$$B(h_{\alpha_1}, h_{\alpha_1}) = -2\left(\frac{1}{2}i\right)^2 = \frac{1}{2}.$$

From $\langle \alpha_1, \alpha_1 \rangle = B(h_{\alpha_1}, h_{\alpha_1})$, we have

$$\langle \alpha_1, \alpha_1 \rangle = \frac{1}{2}.$$