Group Theory
Homework Assignment 05
Spring, 2020

1. The elements of the group $G_1 = \{E, a_2, a_3, \cdots, a_{g_1}\}$ commute with the elements of the group $G_2 = \{E, b_2, b_3, \cdots, b_{g_2}\}$. That is, $a_i b_j = b_j a_i$ for $i = 1, 2, \cdots, g_1$ and $j = 1, 2, \cdots, g_2$. Here $a_1 = E$ and $b_1 = E$. Show that the direct product of G_1 and G_2, $G_1 \otimes G_2 = \{a_i a_j; i = 1, 2, \cdots, g_1, j = 1, 2, \cdots, g_2\}$, is a group.

2. Show that if two matrices A and B are orthogonal, then their direct product $A \otimes B$ is also an orthogonal matrix.

3. The character table of D_3 is given by

	$C_1 = \{E\}$	$C_2 = \{D, F\}$	$C_3 = \{A, B, C\}$
Γ^1	1	1	1
Γ^2	1	1	-1
Γ^3	2	-1	0

   Find the character table of $D_3 \otimes D_3$.

4. Show that the direct-product representation $\Gamma_1 \otimes \Gamma_2$ is an irreducible representation of $G_1 \otimes G_2$ if Γ_1 and Γ_2 are irreducible representations of G_1 and G_2 respectively.

5. Rotations in two dimensions can be parameterized by

$$R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

   (a) Show that $R(\varphi_1)R(\varphi_2) = R(\varphi_1 + \varphi_2)$.

   (b) Show that $R(\varphi) = e^{\varphi a_1}$, where

$$a_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$