Assignment 13

Due Time : 8:15, June 17, 2020 (Wednesday)

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Score : ________

Problem 1 Score: ________. The basis of the real algebra $L = \mathrm{sl}(2,\mathbb{R})$ is given by

$$b_1 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad b_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad b_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Find the representation matrices of the basis elements b_1, b_2 and b_3 in the adjoint representation.

Solution: Since

$$[b_1,b_1] =\frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = 0 = 0\cdot b_1 + 0\cdot b_2 + 0\cdot b_3, \tag{1}$$

$$[b_1,b_2] =\frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0\cdot b_1 + 0\cdot b_2 + 1\cdot b_3, \tag{2}$$

$$[b_1,b_3] =\frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0\cdot b_1 + 1\cdot b_2 + 0\cdot b_3, \tag{3}$$

the representation matrix of the basis element b_1 is

$$\mathrm{ad}(b_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{4}$$

Since

$$[b_2,b_1] =\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0\cdot b_1 + 0\cdot b_2 - 1\cdot b_3, \tag{5}$$

$$[b_2,b_2] =\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0 = 0\cdot b_1 + 0\cdot b_2 + 0\cdot b_3, \tag{6}$$

$$[b_2,b_3] =\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = 1\cdot b_1 + 0\cdot b_2 + 0\cdot b_3, \tag{7}$$

the representation matrix of the basis element b_2 is

$$\mathrm{ad}(b_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{8}$$

Since

$$[b_3,b_1] =\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0\cdot b_1 - 1\cdot b_2 + 0\cdot b_3, \tag{9}$$

$$[b_3,b_2] =\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1\cdot b_1 + 0\cdot b_2 + 0\cdot b_3, \tag{10}$$

$$[b_3,b_3] =\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0 = 0\cdot b_1 + 0\cdot b_2 + 0\cdot b_3, \tag{11}$$

the representation matrix of basis element b_3 is

$$\mathrm{ad}(b_3) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{12}$$

□

Problem 2 Score: ________. Continue from the previous problem. Find the Killing form $B(b_p, b_q)$ for $p, q = 1, 2, 3$.

Solution:

$$B(b_1,b_1) =\mathrm{tr}[\mathrm{ad}(b_1)\mathrm{ad}(b_1)] = \mathrm{tr}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2, \tag{13}$$

$$
\begin{aligned}
B(b_2,b_2) =& \mathrm{tr}[\mathrm{ad}(b_2)\mathrm{ad}(b_2)] = \mathrm{tr}\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix} = -2, && (14)\\
B(b_3,b_3) =& \mathrm{tr}[\mathrm{ad}(b_3)\mathrm{ad}(b_3)] = \mathrm{tr}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} = 2, && (15)\\
B(b_1,b_2) = B(b_2,b_1) =& \mathrm{tr}[\mathrm{ad}(b_1)\mathrm{ad}(b_2)] = \mathrm{tr}\begin{pmatrix}0 & 0 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = 0, && (16)\\
B(b_2,b_3) = B(b_3,b_2) =& \mathrm{tr}[\mathrm{ad}(b_2)\mathrm{ad}(b_3)] = \mathrm{tr}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\end{pmatrix} = 0, && (17)\\
B(b_3,b_1) = B(b_1,b_3) =& \mathrm{tr}[\mathrm{ad}(b_3)\mathrm{ad}(b_1)] = \mathrm{tr}\begin{pmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = 0. && (18)
\end{aligned}
$$

□

Problem 3 Score: _________. Choose the basis of the semi-simple complex Lie algebra $\tilde{L}$ such that each basis elements is a member of some subspace $\tilde{L}_\gamma$. The adjoint representation matrix $\mathrm{ad}(h)$ of each element h in Cartan subalgebra H is a diagonal matrix with zero diagonal elements corresponding to the basis elements of $\tilde{L}_0 = H$ and with diagonal element $\gamma(h)$ corresponding to each basis element of $\tilde{L}_\gamma$ (for $\gamma \in \Delta$). Show that

$$
B(h,h') = \sum_{\gamma\in\Delta}(\mathrm{dim}\tilde{L}_\gamma)\gamma(h)\gamma(h') \text{ for all } h, h' \in H.
$$

Solution:

$$
\begin{aligned}
B(h,h') =& \mathrm{tr}[\mathrm{ad}(h),\mathrm{ad}(h)'] = \sum_{jk}[\mathrm{ad}(h)]_{jk}[\mathrm{ad}(h')]_{kj}\\
=& \sum_{jk}[\mathrm{ad}(h)]_{jj}\delta_{jk}[\mathrm{ad}(h')]_{kk'}\delta_{kj} = \sum_j[\mathrm{ad}(h)]_{jj}[\mathrm{ad}(h')]_{jj}\\
=& \sum_i(\mathrm{dim}\tilde{L}_{\gamma i})\gamma_i(h)\gamma_i(h') = \sum_{\gamma\in\Delta}(\mathrm{dim}\tilde{L}_\gamma)\gamma(h)\gamma(h'). && (19)
\end{aligned}
$$

□

Problem 4 Score: _________. Consider the complexification $\tilde{L}$ of $L = \mathrm{su}(2)$. The basis elements are given by

$$
a_1 = \frac{1}{2}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}, \quad a_2 = \frac{1}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \quad a_3 = \frac{1}{2}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}.
$$

The basis of the Cartan subalgebra H is given by $h_1 = a_3$. Verify that the two non-zero roots are α_1 and $-\alpha_1$ with $\alpha_1(h_1) = i$.

Solution: Express the basis of $\tilde{L}$ as

$$
a'_\alpha = \mu a_1 + \nu a_2. \tag{20}
$$

Since

$$
\begin{aligned}
[h_1,a_1] = [a_3,a_1] =& a_3a_1 - a_1a_3 = \frac{1}{2}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}\frac{1}{2}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}\frac{1}{2}\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = -a_2, && (21)\\
[h_1,a_2] = [a_3,a_2] =& a_3a_2 - a_2a_3 = \frac{1}{2}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}\frac{1}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\frac{1}{2}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix} = a_1, && (22)
\end{aligned}
$$

we have

$$
[h_1, a'_\alpha] = -\mu a_2 + \nu a_1. \tag{23}
$$

According to the definition of none-zero root

$$[h_1, a'_\alpha] = \alpha(h_1) a'_\alpha \tag{24}$$

we have

$$[h_1, a'_\alpha] = -\mu\alpha(h_1)a_1 + \nu\alpha(h_1)a_2. \tag{25}$$

Due to the linear independence of a_1 and a_2, we have

$$\mu + \nu\alpha(h_1) = 0, \tag{26}$$

$$\mu\alpha(h_1) - \nu = 0. \tag{27}$$

Condition for non-trivial solution of μ and ν requires

$$\det\begin{vmatrix} 1 & \alpha(h_1) \\ \alpha(h_1) & -1 \end{vmatrix} = 0 \Longrightarrow \alpha(h_1) = \pm i. \tag{28}$$

Therefore, the two non-zero roots are α_1 and $-\alpha_1$ with $\alpha_1(h_1) = i$. □

Problem 5 Score: _________. Continue from the previous problem. Find the values of $B(h_1, h_1)$ and $\langle \alpha_1, \alpha_1 \rangle$.

Solution:

$$B(h_1, h_1) = \mathrm{tr}[\mathrm{ad}(h_1)\mathrm{ad}(h_1)] = \mathrm{tr}\left[\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = -2. \tag{29}$$

Continue from the previous problem,

$$\mu + \nu\alpha(h_1) = 0, \tag{30}$$

$$\mu\alpha(h_1) - \nu = 0. \tag{31}$$

For $\alpha(h_1) = i$, $\nu = i\mu$. Choose $\mu = 1$, we have $\nu = i$, and thus $a_\alpha = a_1 + ia_2$.
For $\alpha(h_1) = -i$, $\nu = -i\mu$. Choose $\mu = 1$, we have $\nu = -i$, and thus $a_{-\alpha_1} = a_1 - ia_2$.
Suppose $h_{\alpha_1} = \kappa h_1$, then

$$B(h_{\alpha_1}, h_{\alpha_2}) = \alpha_1(h_{\alpha_1}) \tag{32}$$

$$\Longrightarrow -2\kappa^2 = i\kappa \Longrightarrow \kappa = -\frac{1}{2}i, \tag{33}$$

$$\Longleftarrow h_{\alpha_1} = -\frac{1}{2}ih_1 = -\frac{1}{2}ia_3 = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{34}$$

$$\langle \alpha_1, \alpha_1 \rangle = B(h_{\alpha_1}, h_{\alpha_1}) = -\frac{1}{4}B(h_1, h_1) = -\frac{1}{4} \times (-2) = \frac{1}{2}. \tag{35}$$

□