Group Theory
Homework Assignment 03
Spring, 2018

1. Γ is a faithful representation of a non-Abelian group G. If the representation matrix of each element in the group is transformed as in the following, determine whether the resultant set of matrices forms a representation of the group G.
   (a) $\Gamma(T)^\dagger$ (Hermitian conjugate).
   (b) $\Gamma(T)^t$ (transpose).
   (c) $\Gamma(T)^{-1}$ (inverse).
   (d) $\Gamma(T)^*$ (complex conjugate).
   (e) $(\Gamma(T)^{-1})^\dagger$ (Hermitian conjugate of the inverse).
   (f) $\det\Gamma(T)$ (determinant).
   (g) $\mathrm{Tr}\,\Gamma(T)$ (trace).

2. A two-dimensional representation of $C_2 = \{E, a\}$ is given by
$$\Gamma(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \Gamma(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
   Find the similarity transformation that reduces the above two-dimensional representation of C_2 into the direct sum of two irreducible one-dimensional representations.

3. Consider the following two-dimensional representation Γ of the group $G = \{E, a, b\}$ of order $g = 3$
$$\Gamma(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \Gamma(a) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix},\ \Gamma(b) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}.$$
   (a) Check the orthogonality relation
$$\frac{1}{g}\sum_{T\in G}\Gamma(T)^*_{jk}\Gamma(T)_{st} = \frac{1}{d}\delta_{js}\delta_{kt}$$
   for all the possible combinations of $j, k, s,$ and t. Note that $j, k, s, t = 1, 2$ and that $d = 2$.
   (b) Is the representation Γ reducible?

4. Show that the sum of the characters of all the elements of a finite group in an irreducible representation except the identity representation is zero.

5. Consider the group $G = \{E, a, b, b^2, b^3, b^4, b^5, ab, ab^2, ab^3, ab^4, ab^5\}$ with $a^2 = b^6 = E$ and $a^{-1}ba = b^{-1}$.
   (a) Find all the elements in each class of G.
   (b) Γ^1 and Γ^2 are two representations of G. In the representation Γ^1, $\Gamma^1(a)$ and $\Gamma^1(b)$ are respectively given by
$$\Gamma^1(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \Gamma^1(b) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$$
   with $\omega = e^{i2\pi/3}$. In the representation Γ^2, $\Gamma^2(a)$ and $\Gamma^2(b)$ are respectively given by
$$\Gamma^2(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \Gamma^2(b) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$
   Find the partial character table of G with entries only for the representations Γ^1 and Γ^2.
   (c) Are the representations Γ^1 and Γ^2 equivalent?
   (d) Is the representation Γ^1 reducible?
   (e) Is the representation Γ^2 reducible?