Group Theory
PHYS2102
Spring, 2020

Assignment 05

Due Time : 8:15, April 8, 2020 (Wednesday)

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Score : ________

Problem 1 Score: ________. The element of the group $G_1 = \{E, a_2, a_3, \cdots, a_{g_1}\}$ commute with the group $G_2 = \{E, b_2, b_3, \cdots, b_{g_2}\}$. That is, $a_i b_j = b_j a_i$ for $i = 1, 2, \cdots, g_1$, and $j = 1, 2, \cdots, g_2$. Here $a_1 = E$ and $b_1 = E$. Show that the direct product of G_1 and G_2, $G_1 \otimes G_2 = \{a_i b_j; i = 1, 2, \cdots, g_1, j = 1, 2, \cdots, g_2\}$, is a group.

Solution: The direct product of G_1 and G_2, $G_1 \otimes G_2 = \{a_i b_j; i = 1, 2, \cdots, g_1, j = 1, 2, \cdots, g_2\}$, satisfies all the four group axioms:

1. **Closure**: For two arbitrary elements $a_i b_j$ and $a_k b_l$ in $G_1 \otimes G_2$, their product is $(a_i b_j)(a_k b_l) = (a_i a_k)(b_j b_l)$. Since a_i and a_k are two elements of group G_1, $a_i a_k$ is an element of G_1. Since b_j and b_l are two elements of G_2, $b_j b_l$ is an element of G_2. Since $a_i a_k$ is an element of G_1 and $b_j b_l$ is an element of G_2, $(a_i b_j)(a_k b_l) = (a_i a_k)(b_j b_l)$ is an elements of $G_1 \otimes G_2$.
2. **Associativity**: For any three elements $a_i b_j$, $a_k b_l$, $a_m b_n$ of $G_1 \otimes G_2$, using the commutativity of G_1 and G_2, we have $[(a_i b_j)(a_k b_l)](a_m b_n) = a_i a_k a_m b_j b_l b_n = (a_i b_j)[(a_k b_l)(a_m b_n)]$.
3. **Existence of the identity element**: The identity element of $G_1 \otimes G_2$ is EE, since $(a_i b_j)(EE) = (a_i E)(b_j E) = a_i b_j = (E a_i)(E b_j) = (EE)(a_i b_j)$ for every element $a_i b_j$ in $G_1 \otimes G_2$.
4. **Existence of inverse elements**: For each element $a_i b_j$ of G, its inverse is $a_i^{-1} b_j^{-1}$. This is because $(a_i b_j)(a_i^{-1} b_j^{-1}) = (a_i a_i^{-1})(b_j b_j^{-1}) = EE = (a_i^{-1} a_i)(b_j^{-1} b_j) = (a_i^{-1} b_j^{-1})(a_i b_j)$. Since a_i is an element of G_1, a_i^{-1} is an element of G_1. Since b_j is an element of G_2, b_j^{-1} is an element of G_2. Since a_i^{-1} is an element of G_1 and b_j is an element of G_2, $a_i^{-1} b_j^{-1}$ is in $G_1 \otimes G_2$.

Therefore, $G_1 \otimes G_2$ is a group. $\square$

Problem 2 Score: ________. Show that if two matrices A and B are orthogonal, then their direct product $A \otimes B$ is also orthogonal matrix.

Solution: Suppose that the dimensions of the two matrices A and B are $m \times m$ and $n \times n$, respectively. Since the A and B are orthogonal, we have

$$A^T A = I_m, \tag{1}$$
$$B^T B = I_n, \tag{2}$$

where I_m and I_n is $m \times m$ and $n\times$ identity matrices, respectively, or

$$(A^T A)_{jk} = \sum_{l=1}^{m} A_{lj} A_{lk} = \delta_{jk}, \tag{3}$$
$$(A A^T)_{jk} = \sum_{l=1}^{m} A_{jl} A_{kl} = \delta_{jk}, \tag{4}$$
$$(B^T B)_{st} = \sum_{r=1}^{n} B_{rs} B_{rt} = \delta_{st}, \tag{5}$$
$$(B B^T)_{st} = \sum_{r=1}^{n} B_{sr} B_{tr} = \delta_{st}. \tag{6}$$

Now let's check that whether $(A \otimes B)^T (A \otimes B) = (A \otimes B)(A \otimes B)^T = I_{mn}$ holds.

$$\begin{aligned}
[(A \otimes B)^T (A \otimes B)]_{js,kt} =& \sum_{l=1}^{m} \sum_{r=1}^{n} (A \otimes B)_{lr,js} (A \otimes B)_{lr,kt} \\
=& \sum_{l=1}^{m} \sum_{r=1}^{n} (A_{lj} B_{rs})(A_{lk} B_{rt}) \\
=& \left[\sum_{l=1}^{m} A_{lj} A_{lk}\right] \left[\sum_{r=1}^{n} B_{rs} B_{rt}\right] \\
=& \delta_{jk} \delta_{st},
\end{aligned} \tag{7}$$

which means

$$(A \otimes B)^T (A \otimes B) = I_{mn}. \tag{8}$$

Similarly,

$$\begin{aligned}(A\otimes B)(A\otimes B)^T=&\sum_{l=1}^m\sum_{r=1}^n(A\otimes B)_{js,lr}(A\times B)_{kt,lr}\\=&\sum_{l=1}^m\sum_{r=1}^n(A_{jl}B_{sr})(A_{kl}B_{tr})\\=&\left[\sum_{l=1}^m A_{jl}A_{kl}\right]\left[\sum_{r=1}^n B_{sr}B_{st}\right]\\=&\delta_{jk}\delta_{st},\end{aligned}\tag{9}$$

which means

$$(A\otimes B)(A\otimes B)^T=I_{mn}.\tag{10}$$

Since $(A\otimes B)^T(A\otimes B)=(A\otimes B)(A\otimes B)^T=I_{mn}$, $A\otimes B$ is an orthogonal matrix. □

Problem 3 Score: _________. The character table of D_3 is given by

	$C_1=\{E\}$	$C_2=\{D,F\}$	$C_3=\{A,B,C\}$
Γ^1	1	1	1
Γ^2	1	1	-1
Γ^3	2	-1	0

Find the character table of $D_3\otimes D_3$.

Solution: The direct product is

$$\begin{aligned}D_3\otimes D_3=&\{(T_1,T_2);T_1,T_2\in D_3\}\\=&\{(E,E),(E,D),(E,F),(E,A),(E,B),(E,C),(D,E),(D,D),(D,F),(D,A),(D,B),(D,C),\\&(F,E),(F,D),(F,F),(F,A),(F,B),(F,C),(A,E),(A,D),(A,F),(A,A),(A,B),(A,C),\qquad(11)\\&(B,E),(B,D),(B,F),(B,A),(B,B),(B,C),(C,E),(C,D),(C,F),(C,A),(C,B),(C,C)\}.\qquad(12)\end{aligned}$$

First, we construct the classes of $D_3\otimes D_3$. As an instance, we constructing a class from (D,F):
For $X=(E,E),(E,D),(E,F),(D,E),(D,D),(D,F),(F,E),(F,D),(F,F)$,

$$X(D,F)X^{-1}=(D,F).\tag{13}$$

For $X=(E,A),(E,B),(E,C),(D,A),(D,B),(D,C),(F,A),(F,B),(F,C)$,

$$X(D,F)X^{-1}=(D,D).\tag{14}$$

For $X=(A,E),(A,D),(A,F),(B,E),(B,D),(B,F),(C,E),(C,D),(C,F)$,

$$X(D,F)X^{-1}=(F,D).\tag{15}$$

For $X=(A,A),(A,B),(A,C),(B,A),(B,B),(B,C),(C,A),(C,B),(C,C)$,

$$X(D,F)X^{-1}=(F,F).\tag{16}$$

The class of $D_3\otimes D_3$ constructed from (D,F) is $\{(D,D),(D,F),(F,D),(F,F)\}$.
Using the similar method, we can construct all the classes of $D_3\otimes D_3$:

$\{(E,E)\}$, $\{(E,D),(E,F)\}$, $\{(E,A),(E,B),(E,C)\}$,
$\{(D,E),(F,E)\}$, $\{(D,D),(D,F),(F,D),(F,F)\}$, $\{(D,A),(D,B),(D,C),(F,A),(F,B),(F,C)\}$,
$\{(A,E),(B,E),(C,E)\}$, $\{(A,D),(A,F),(B,D),(B,F),(C,D),(C,F)\}$, $\{(A,A),(A,B),(A,C),(B,A),(B,B),(B,C),(C,A),(C,B),(C,C)\}$.

We find that the classes of $D_3\otimes D_3$ are exactly the direct products of the classes of D_3:

$$\begin{matrix}C_1\otimes C_1, & C_1\otimes C_2, & C_1\otimes C_3,\\ C_2\otimes C_1, & C_2\otimes C_2, & C_2\otimes C_3,\\ C_3\otimes C_1, & C_3\otimes C_2, & C_3\otimes C_3.\end{matrix}$$

Next, we find the representations of $D_3 \otimes D_3$. The representations of the direct product group $D_3 \otimes D_3$ are defined to be the direct products of representations of the group D_3:

$$\begin{aligned}
&\Gamma^{11}((T_1,T_2)) = \Gamma^1(T_1)\otimes\Gamma^1(T_2), && \Gamma^{12}((T_1,T_2)) = \Gamma^1(T_1)\otimes\Gamma(T_2), && \Gamma^{13}((T_1,T_2)) = \Gamma^1(T_1)\otimes\Gamma^3(T_2),\\
&\Gamma^{21}((T_1,T_2)) = \Gamma^2(T_1)\otimes\Gamma^1(T_2), && \Gamma^{22}((T_1,T_2)) = \Gamma^2(T_1)\otimes\Gamma^2(T_2), && \Gamma^{23}((T_1,T_2)) = \Gamma^2(T_1)\otimes\Gamma^3(T_2),\\
&\Gamma^{31}((T_1,T_2)) = \Gamma^3(T_1)\otimes\Gamma^1(T_2), && \Gamma^{32}((T_1,T_2)) = \Gamma^3(T_1)\otimes\Gamma^2(T_2), && \Gamma^{33}((T_1,T_2)) = \Gamma^3(T_1)\otimes\Gamma^3(T_2).
\end{aligned}$$

Lastly, we construct the character table of $D_3 \otimes D_3$. Using that the characters in the direct product representations are equal to the products of the characters in the representations involved in the direct products, we construct the character table of $D_3 \otimes D_3$, as shown in table 1. □

Table 1: The character table of $C_3 \otimes C_3$.

	$C_1\otimes C_1$	$C_1\otimes C_2$	$C_1\otimes C_3$	$C_2\otimes C_1$	$C_2\otimes C_2$	$C_2\otimes C_3$	$C_3\otimes C_1$	$C_3\otimes C_2$	$C_3\otimes C_3$
Γ^{11}	1	1	1	1	1	1	1	1	1
Γ^{12}	1	1	−1	1	1	−1	1	1	−1
Γ^{13}	2	−1	0	2	−1	0	2	−1	0
Γ^{21}	1	1	1	1	1	1	−1	−1	−1
Γ^{22}	1	1	−1	1	1	−1	−1	−1	1
Γ^{23}	2	−1	0	2	−1	0	−2	1	0
Γ^{31}	2	2	2	−1	−1	−1	0	0	0
Γ^{32}	2	2	−2	2	2	−2	−2	−2	2
Γ^{33}	4	−2	0	−2	1	0	0	0	0

Problem 4 Score: _________. Show that the direct-product representation $\Gamma_1 \otimes \Gamma_2$ is an irreducible representation of $G_1 \otimes G_2$ if Γ_1 and Γ_2 are irreducible representations of G_1 and G_2 respectively.

Solution: First, let's prove that the direct product representation $\Gamma = \Gamma_1 \otimes \Gamma_2$ is a representation of $G_1 \otimes G_2$. Under the multiplication operation

$$\begin{aligned}
\Gamma((T_1,T_2))\Gamma((T_1',T_2')) =&[\Gamma_1(T_1)\otimes\Gamma_2(T_2)][\Gamma_1(T_1')\otimes\Gamma_2(T_2')] = [\Gamma_1(T_1)\Gamma_1(T_1')]\otimes[\Gamma_2(T_2)\Gamma(T_2')]\\
=&\Gamma_1(T_1T_1')\otimes\Gamma_2(T_2T_2') = \Gamma((T_1T_1',T_2T_2')) = \Gamma((T_1,T_2)(T_1',T_2')),
\end{aligned} \tag{17}$$

$\Gamma = \Gamma_1 \otimes \Gamma_2$ satisfies all the four group axioms:

1. **Closure**: For two arbitrary elements (T_1,T_2) and (T_1',T_2') in $G_1\otimes G_2$, $\Gamma((T_1,T_2)),\Gamma((T_1',T_2'))\in\Gamma$ and $(T_1,T_2)(T_1',T_2') = (T_1T_1',T_2T_2')\in G_1\otimes G_2$, so $\Gamma((T_1,T_2))\Gamma((T_1',T_2')) = \Gamma((T_1,T_2)(T_1',T_2'))\in\Gamma$, which means that Γ possesses the closure property.

2. **Associativity**: For three arbitrary elements $\Gamma((T_1,T_2))$, $\Gamma((T_1',T_2'))$, $\Gamma((T_1'',T_2''))$ in Γ, we have
$[\Gamma((T_1,T_2))\Gamma((T_1',T_2'))]\Gamma((T_1'',T_2'')) = \Gamma((T_1,T_2)(T_1',T_2'))\Gamma((T_1'',T_2'')) = \Gamma((T_1,T_2)(T_1',T_2')(T_1'',T_2''))$
$= \Gamma((T_1,T_2))\Gamma((T_1',T_2')(T_1'',T_2'')) = \Gamma((T_1,T_2))[\Gamma((T_1',T_2'))\Gamma((T_1'',T_2''))]$.

3. **Existence of the identity element**: The identity element in Γ is $\Gamma((E,E)$, since $\Gamma((E,E))\Gamma((T_1,T_2)) = \Gamma((E,E)(T_1,T_2)) = \Gamma((ET_1,ET_2)) = \Gamma((T_1,T_2))$ and $\Gamma((T_1,T_2))\Gamma((E,E)) = \Gamma((T_1,T_2)(E,E)) = \Gamma((T_1E,T_2E))$.

4. **Existence of inverse elements**: The inverse element of any arbitrary element $\Gamma((T_1,T_2))$ in Γ is $\Gamma((T_1^{-1},T_2^{-1}))$, since $\Gamma((T_1,T_2))\Gamma((T_1^{-1},T_2^{-1})) = \Gamma((T_1,T_2)(T_1^{-1},T_2^{-1})) = \Gamma((T_1T_1^{-1},T_2T_2^{-1})) = \Gamma((E,E))$
and $\Gamma((T_1^{-1},T_2^{-1}))\Gamma((T_1,T_2)) = \Gamma((T_1^{-1},T_2^{-1})) = \Gamma((T_1^{-1}T_1,T_2^{-1}T_2)) = \Gamma((E,E))$ and $\Gamma((T_1^{-1},T_2^{-1}))\in\Gamma$.

Therefore, $\Gamma = \Gamma_1 \otimes \Gamma_2$ is a group and thus is a representation of $G_1 \otimes G_2$

Next, let's prove that $\Gamma = \Gamma_1 \otimes \Gamma_2$ is irreducible. Suppose the characters in representations Γ_1 and Γ_2 are $\chi_1(T_1) = \mathrm{Tr}\ \Gamma_1(T_1)$ and $\chi_2(T_2) = \mathrm{Tr}\ \Gamma_2(T_2)$ respectively and the order of G_1 and G_2 are g_1 and g_2 respectively. The character in representation $\Gamma = \Gamma_1 \otimes \Gamma_2$ is

$$\begin{aligned}
\chi((T_1,T_2)) =&\mathrm{Tr}\ \Gamma((T_1,T_2))\\
=&\sum_{js}\Gamma((T_1,T_2))_{js,js}\\
=&\sum_{js}\Gamma_1(T_1)_{jj}\Gamma_2(T_2)_{ss}
\end{aligned}$$

$$= \left(\sum_j \Gamma_1(T_1)_{jj}\right)\left(\sum_s \Gamma_2(T_2)_{ss}\right) = \chi_1(T_1)\chi_2(T_2). \tag{18}$$

Since Γ_1 and Γ_2 are irreducible, we have

$$\sum_{T_1\in G_1} |\chi_1(T_1)|^2 = g_1, \tag{19}$$

$$\sum_{T_2\in G_2} |\chi_2(T_2)|^2 = g_2. \tag{20}$$

The value of $\sum_{(T_1,T_2)\in G_1\otimes G_2} |\chi(T_1,T_2)|^2$ is

$$\sum_{(T_1,T_2)\in G_1\otimes G_2} |\chi((T_1,T_2))|^2 = \left[\sum_{T_1\in G_1} |\chi_1(T_1)|^2\right]\left[\sum_{T_2\in G_2} |\chi_2(T_2)|^2\right] = g_1 g_2, \tag{21}$$

which is exactly the order of $\Gamma_1 \otimes \Gamma_2$.

Therefore, the direct-product representation $\Gamma = \Gamma_1 \otimes \Gamma$ is an irreducible representation of $\Gamma_1 \otimes G_2$. □

Problem 5 Score: ________. Rotations in two dimensions can be parameterized by

$$R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

(a) Show that $R(\varphi_1)R(\varphi_2) = R(\varphi_1 + \varphi_2)$.

(b) Show that $R(\varphi) = e^{\varphi a_1}$, where

$$a_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Solution: (a)

$$\begin{aligned} R(\varphi_1)R(\varphi_2) &= \begin{pmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{pmatrix}\begin{pmatrix} \cos\varphi_2 & -\sin\varphi_2 \\ \sin\varphi_2 & \cos\varphi_2 \end{pmatrix} \\ &= \begin{pmatrix} \cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 & -\cos\varphi_1\sin\varphi_2 - \sin\varphi_1\cos\varphi_2 \\ \sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2 & -\sin\varphi_1\sin\varphi_2 + \cos\varphi_1\cos\varphi_2 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\varphi_1+\varphi_2) & -\sin(\varphi_1+\varphi_2) \\ \sin(\varphi_1+\varphi_2) & \cos(\varphi_1+\varphi_2) \end{pmatrix} = R(\varphi_1+\varphi_2). \end{aligned} \tag{22}$$

(b)

$$\begin{aligned} e^{\varphi a_1} =& \sum_{k=0}^{\infty} \frac{1}{k!}(\varphi a_1)^k \\ =& \sum_{k=0}^{\infty} \frac{1}{(4k)!}\varphi^{4k} a_1^{4k} + \sum_{k=0}^{\infty} \frac{1}{(4k+1)!}\varphi^{4k+1} a_1^{4k+1} + \sum_{k=0}^{\infty} \frac{1}{(4k+2)!}\varphi^{4k+2} a_1^{4k+2} + \sum_{k=0}^{\infty} \frac{1}{(4k+3)!}\varphi^{4k+3} a_1^{4k+3} \\ =& \sum_{k=0}^{\infty} \frac{1}{(4k)!}\varphi^{4k}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{k=0}^{\infty} \frac{1}{(4k+1)!}\varphi^{4k+1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \sum_{k=0}^{\infty} \frac{1}{(4k+2)!}\varphi^{4k+2}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ &+ \sum_{k=0}^{\infty} \frac{1}{(4k+3)!}\varphi^{4k+3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ =& \begin{pmatrix} \sum_{k=0}^{\infty} \frac{1}{(4k)!}\varphi^{4k} - \sum_{k=0}^{\infty} \frac{1}{(4k+2)!}\frac{1}{(4k+2)!}\varphi^{4k+2} & -\sum_{k=0}^{\infty} \frac{1}{(4k+1)!}\varphi^{4k+1} + \sum_{k=0}^{\infty} \frac{1}{(4k+3)!}\varphi^{4k+3} \\ \sum_{k=0}^{\infty} \frac{1}{(4k+1)!}\varphi^{4k+1} - \sum_{k=0}^{\infty} \frac{1}{(4k+3)!}\varphi^{4k+3} & \sum_{k=0}^{\infty} \frac{1}{(4k)!}\varphi^{4k} - \sum_{k=0}^{\infty} \frac{1}{(4k+2)!}\frac{1}{(4k+2)!}\varphi^{4k+2} \end{pmatrix} \\ =& \begin{pmatrix} 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} + \cdots & -\frac{x}{1} + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots \\ \frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots & 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} + \cdots \end{pmatrix} \\ =& \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = R(\varphi). \end{aligned} \tag{23}$$

□