Group Theory
PHYS2102
Spring, 2020

Assignment 10

Due Time : 8:15, May 27, 2020 (Wednesday)

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Score : _____

Problem 1 Score: _____. Simplify the following permutations into the product of the cycles without any common object.

(a) $(1\quad 2)(2\quad 3)(1\quad 2)$.

(b) $(1\quad 2\quad 3)(1\quad 3\quad 4)(3\quad 2\quad 1)$.

(c) $(1\quad 2\quad 3\quad 4)^{-1}$.

(d) $(1\quad 2\quad 4\quad 5)(4\quad 3\quad 2\quad 6)$.

(e) $(1\quad 2\quad 3)(4\quad 2\quad 6)(3\quad 4\quad 5\quad 6)$.

Solution: (a)

$$(1\quad 2)(2\quad 3)(1\quad 2)=(1\quad 2\quad 3)(1\quad 2)=(3\quad 1\quad 2)(1\quad 2)=(3\quad 1)(1\quad 2)(1\quad 2)=(3\quad 1). \tag{1}$$

(b)

$$\begin{aligned}(1\quad 2\quad 3)(1\quad 3\quad 4)(3\quad 2\quad 1)=&(2\quad 3\quad 1)(1\quad 3\quad 4)(3\quad 2\quad 1)=(2\quad 3)(3\quad 1)(1\quad 3)(3\quad 4)(3\quad 2\quad 1)\\ =&(2\quad 3)(3\quad 4)(3\quad 2\quad 1)=(2\quad 3)(4\quad 3)(3\quad 2\quad 1)=(2\quad 3)(4\quad 3\quad 2\quad 1)\\ =&(2\quad 3)(3\quad 2\quad 1\quad 4)=(2\quad 3)(3\quad 2)(2\quad 1\quad 4)=(2\quad 1\quad 4).\end{aligned} \tag{2}$$

(c)

$$\because (1\quad 2\quad 3\quad 4)^4=E, \tag{3}$$

$$\begin{aligned}\therefore (1\quad 2\quad 3\quad 4)^{-1}=&(1\quad 2\quad 3\quad 4)^3=(3\quad 4\quad 1\quad 2)(1\quad 2\quad 3\quad 4)(3\quad 4\quad 1\quad 2)\\ =&(3\quad 4\quad 1)(1\quad 2)(1\quad 2)(2\quad 3)(3\quad 4)(3\quad 4)(4\quad 1\quad 2)=(3\quad 4\quad 1)(2\quad 3)(4\quad 1\quad 2)\\ =&(4\quad 1\quad 3)(3\quad 2)(4\quad 1\quad 2)=(4\quad 1\quad 3\quad 2)(4\quad 1\quad 2)\\ =&(3\quad 2\quad 4\quad 1)(4\quad 1\quad 2)=(3\quad 2\quad 4)(4\quad 1)(4\quad 1)(1\quad 2)\\ =&(3\quad 2\quad 4)(1\quad 2)=(4\quad 3\quad 2)(2\quad 1)=(4\quad 3\quad 2\quad 1).\end{aligned} \tag{4}$$

(d)

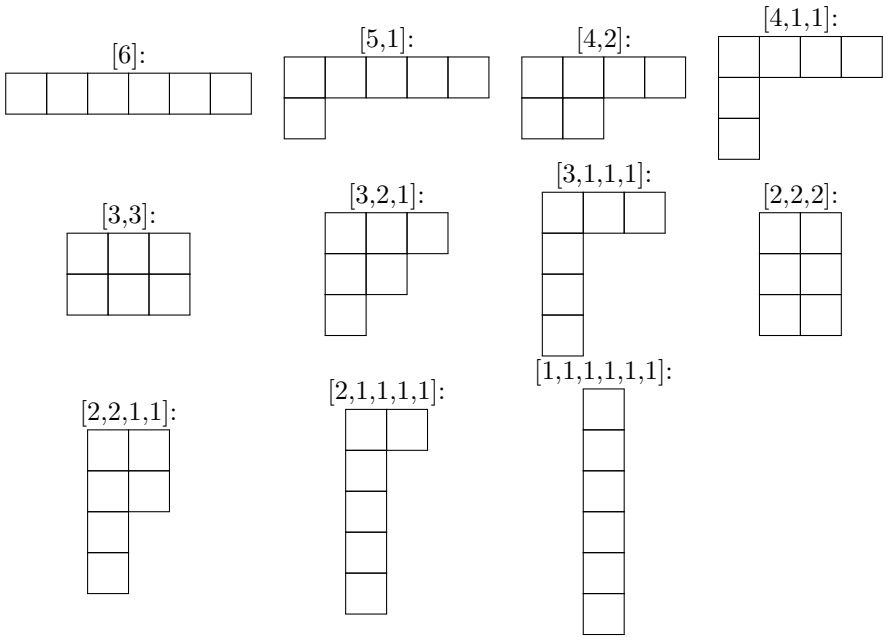
$$\begin{aligned}(1\quad 2\quad 4\quad 5)(4\quad 3\quad 2\quad 6)=&(5\quad 1\quad 2\quad 4)(4\quad 3\quad 2\quad 6)=(5\quad 1\quad 2)(2\quad 4)(4\quad 3\quad 2)(2\quad 6)\\ =&(5\quad 1\quad 2)(2\quad 4)(2\quad 4\quad 3)(2\quad 6)\\ =&(5\quad 1\quad 2)(2\quad 4)(2\quad 4)(4\quad 3)(2\quad 6)\\ =&(5\quad 1\quad 2)(4\quad 3)(2\quad 6)=(5\quad 1\quad 2)(2\quad 6)(4\quad 3)\\ =&(5\quad 1\quad 2\quad 6)(4\quad 3).\end{aligned} \tag{5}$$

(e)

$$\begin{aligned}(1\quad 2\quad 3)(4\quad 2\quad 6)(3\quad 4\quad 5\quad 6)=&(3\quad 1\quad 2)(2\quad 6\quad 4)(3\quad 4\quad 5\quad 6)\\ =&(3\quad 1\quad 2\quad 6\quad 4)(3\quad 4\quad 5\quad 6)\\ =&(1\quad 2\quad 6\quad 4\quad 3)(3\quad 4\quad 5\quad 6)\\ =&(1\quad 2\quad 6\quad 4)(4\quad 3)(3\quad 4)(4\quad 5\quad 6)\\ =&(1\quad 2\quad 6\quad 4)(4\quad 5\quad 6)\\ =&(1\quad 2\quad 6\quad 4)(6\quad 4\quad 5)\\ =&(1\quad 2\quad 6)(6\quad 4)(6\quad 4)(4\quad 5)\\ =&(1\quad 2\quad 6)(4\quad 5).\end{aligned} \tag{6}$$

□

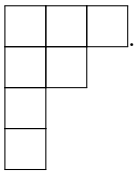
Problem 2 Score: _____. Write down all the Young patterns of the permutation group S_6 from the largest to the smallest.



Solution: Young pattern of the group S_6 from the largest to the smallest: □

Problem 3 Score: _____. Using the hook rule, calculate the number $d_{[3,2,1,1]}(S_7)$ of the standard Young tableaux for the Young pattern $[3,2,1,1]$ of the permutation group 7.

Solution: The Young pattern [3,2,1,1] is

 .

The corresponding Hook numbers of the boxes are

6	3	1
4	1	
2		
1		

.

The product of the Hook numbers is

$$Y_h^{[3,2,1,1]} = \prod_{ij} h_{ij} = 6 \times 3 \times 4 \times 2. \tag{7}$$

Then

$$d_{[3,2,1,1]}(S_7) = \frac{7!}{Y_h^{[3,2,1,1]}} = 35. \tag{8}$$

□

Problem 4 Score: _____. Write down the Young operator corresponding to the following tableau.

1	2
3	4

Solution: Horizontal permutations:

$$P_1 : E, (1 \quad 2). \tag{9}$$

$$P_2 : E, (3 \quad 4). \tag{10}$$

$$P = \prod_j P_j : E, (1 \quad 2), (3 \quad 4), (1 \quad 2)(3 \quad 4). \tag{11}$$

Horizontal operator:

$$\mathcal{P} = \sum P = E + (1 \quad 2) + (3 \quad 4) + (1 \quad 2)(3 \quad 4). \tag{12}$$

Vertical permutations:

$$Q_1 : E, (1 \quad 3). \tag{13}$$

$$Q_2 : E, (2 \quad 4). \tag{14}$$

$$Q = \prod_k Q_k : E, (1 \quad 3), (2 \quad 4), (1 \quad 3)(2 \quad 4). \tag{15}$$

Vertical operator:

$$\mathcal{Q} = \sum \delta(Q) Q = E - (1 \quad 3) - (2 \quad 4) + (1 \quad 3)(2 \quad 4). \tag{16}$$

Young operator:

$$\begin{aligned} \mathcal{Y} =& \mathcal{P}\mathcal{Q} = [E + (1 \quad 2) + (3 \quad 4) + (1 \quad 2)(3 \quad 4)][E - (1 \quad 3) - (2 \quad 4) + (1 \quad 3)(2 \quad 4)] \qquad (17) \\ =& E + (1 \quad 2) + (3 \quad 4) + (1 \quad 2)(3 \quad 4) \\ & - (1 \quad 3) - (2 \quad 1 \quad 3) - (4 \quad 3 \quad 1) - (2 \quad 1 \quad 4 \quad 3) \\ & - (2 \quad 4) - (1 \quad 2 \quad 4) - (3 \quad 4 \quad 2) - (1 \quad 2 \quad 3 \quad 4) \\ & + (1 \quad 3)(2 \quad 4) + (1 \quad 3 \quad 2 \quad 4) + (3 \quad 1 \quad 4 \quad 2) + (1 \quad 4)(3 \quad 2). \qquad (18) \end{aligned}$$

□

Problem 5 Score: _____. Write down the permutation R_{12} transforming the Young tableau $\mathcal{Y}_2$ to the Young tableau $\mathcal{Y}_1$.

$$\mathcal{Y}_1 : \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \qquad \mathcal{Y}_2 : \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}$$

Show that $\mathcal{P}_1 R_{12} = R_{12}\mathcal{P}_2$, $\mathcal{Q}_1 R_{12} = R_{12}\mathcal{Q}_2$, and $\mathcal{Y}_1 R_{12} = R_{12}\mathcal{Y}_2$.

Solution: The permutation transforming the Young tableau $\mathcal{Y}_2$ to the Young tableau $\mathcal{Y}_1$ is

$$R_{12} = \begin{pmatrix} 1 & 2 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} = (3 \quad 4). \tag{19}$$

The horizontal permutations of the first Young tableau:

$$P_{1,1} : E, (1 \quad 2), (1 \quad 3), (2 \quad 3), (1 \quad 2 \quad 3), (3 \quad 2 \quad 1). \tag{20}$$

$$P_{1,2} : E. \tag{21}$$

$$P_1 = \prod_j P_{1,j} : E, (1 \quad 2), (1 \quad 3), (2 \quad 3), (1 \quad 2 \quad 3), (3 \quad 2 \quad 1). \tag{22}$$

The horizontal operator of the first Young tableau:

$$\mathcal{P}_1 = \sum P_1 = E + (1 \quad 2) + (1 \quad 3) + (2 \quad 3) + (1 \quad 2 \quad 3) + (3 \quad 2 \quad 1). \tag{23}$$

The horizontal permutations of the second Young tableau:

$$P_{2,1} : E, (1 \quad 2), (1 \quad 4), (2 \quad 4), (1 \quad 2 \quad 4), (4 \quad 2 \quad 1). \tag{24}$$

$$P_{2,2} : E. \tag{25}$$

$$P_2 = \prod_j P_{2,j} : E, (1 \quad 2), (1 \quad 4), (2 \quad 4), (1 \quad 2 \quad 4), (4 \quad 2 \quad 1). \tag{26}$$

The horizontal operator of the second Young tableau:

$$\mathcal{P}_2 = \sum P_2 = E + (1 \quad 2) + (1 \quad 4) + (2 \quad 4) + (1 \quad 2 \quad 4) + (4 \quad 2 \quad 1). \tag{27}$$

Since

$$\mathcal{P}_1 R_{12} = (3 \quad 4) + (1 \quad 2)(3 \quad 4) + (1 \quad 3 \quad 4) + (2 \quad 3 \quad 4) + (1 \quad 2 \quad 3 \quad 4) + (2 \quad 1 \quad 3 \quad 4), \tag{28}$$

and

$$R_{12}\mathcal{P}_2 = (3 \quad 4) + (3 \quad 4)(1 \quad 2) + (3 \quad 4 \quad 1) + (3 \quad 4 \quad 2) + (3 \quad 4 \quad 1 \quad 2) + (3 \quad 4 \quad 2 \quad 1), \tag{29}$$

we have

$$\mathcal{P}_1 R_{12} = R_{12}\mathcal{P}_2. \tag{30}$$

The vertical permutations of the first Young tableau:

$$Q_{1,1} : E, (1 \quad 4). \tag{31}$$
$$Q_{1,2} : E. \tag{32}$$
$$Q_{1,3} : E. \tag{33}$$
$$Q_1 = \prod_k Q_{1,k} : E, (1 \quad 4). \tag{34}$$

The vertical operator of the first Young tableau:

$$\mathcal{Q}_1 = \sum \delta(Q_1) Q_1 = E - (1 \quad 4). \tag{35}$$

The vertical permutations of the second tableau:

$$Q_{2,1} : E, (1 \quad 3). \tag{36}$$
$$Q_{2,2} : E. \tag{37}$$
$$Q_{2,3} : E. \tag{38}$$
$$Q_2 = \prod_k Q_{2,k} : E, (1 \quad 3). \tag{39}$$

The vertical permutations of the second Young tableau:

$$\mathcal{Q}_2 = \sum \delta(Q_2) Q_2 = E - (1 \quad 3). \tag{40}$$

Since

$$\mathcal{Q}_1 R_{12} = (3 \quad 4) - (1 \quad 4 \quad 3), \tag{41}$$

and

$$R_{12}\mathcal{Q}_2 = (3 \quad 4) - (4 \quad 3 \quad 1), \tag{42}$$

we have

$$\mathcal{Q}_1 R_{12} = R_{12}\mathcal{Q}_2. \tag{43}$$

The Young operator of the first Young tableau:

$$\begin{aligned}\mathcal{Y}_1 =& \mathcal{P}_1\mathcal{Q}_1 = E + (1 \quad 2) + (1 \quad 3) + (2 \quad 3) + (1 \quad 2 \quad 3) + (3 \quad 2 \quad 1) \\ & - (1 \quad 4) - (2 \quad 1 \quad 4) - (3 \quad 1 \quad 4) - (2 \quad 3)(1 \quad 4) - (2 \quad 3 \quad 1 \quad 4) - (3 \quad 2 \quad 1 \quad 4).\end{aligned} \tag{44}$$

The Young operator of the second Young tableau:

$$\begin{aligned}\mathcal{Y}_2 =& \mathcal{P}_2\mathcal{Q}_2 = E + (1 \quad 2) + (1 \quad 4) + (2 \quad 4) + (1 \quad 2 \quad 4) + (4 \quad 2 \quad 1) \\ & - (1 \quad 3) - (2 \quad 1 \quad 3) + (4 \quad 1 \quad 3) + (2 \quad 4)(1 \quad 3) + (2 \quad 4 \quad 1 \quad 3) + (4 \quad 2 \quad 1 \quad 3).\end{aligned} \tag{45}$$

Since

$$\begin{aligned}\mathcal{Y}_1 R_{12} =&(3\quad 4)+(1\quad 2)(3\quad 4)+(1\quad 3\quad 4)+(2\quad 3\quad 4)+(1\quad 2\quad 3\quad 4)+(2\quad 1\quad 3\quad 4) &(46)\\ &-(1\quad 4\quad 3)-(2\quad 1\quad 4\quad 3)-(1\quad 4)-(2\quad 3\quad 1\quad 4)-(2\quad 3)(1\quad 4)-(2\quad 1\quad 4), &(47)\end{aligned}$$

and

$$\begin{aligned}R_{12}\mathcal{Y}_2 =&(3\quad 4)+(3\quad 4)(1\quad 2)+(3\quad 4\quad 1)+(3\quad 4\quad 2)+(3\quad 4\quad 1\quad 2)+(3\quad 4\quad 2\quad 1) &(48)\\ &-(4\quad 3\quad 1)-(4\quad 3\quad 2\quad 1)-(4\quad 1)-(4\quad 2\quad 3\quad 1)-(4\quad 1)(3\quad 2)-(4\quad 2\quad 1), &(49)\end{aligned}$$

we have

$$\mathcal{Y}_1 R_{12} = R_{12}\mathcal{Y}_2. \tag{50}$$

□