Group Theory
Homework Assignment 07
Spring, 2020

Consider a particle of mass μ confined to a square in two dimensions whose vertices are located at $(z, x) = (1, 1)$, $(1, -1)$, $(-1, -1)$, and $(-1, 1)$ on the zOx plane. The potential is zero within the square and infinite on the edges of the square. The eigenfunctions $\psi_{mn}(z, x)$ of the Hamiltonian of the particle are of the form

$$\psi_{mn}(z,x) \propto \begin{cases} \cos(k_m z)\cos(k_n x), \text{if both } m \text{ and } n \text{ are odd,} \\ \cos(k_m z)\sin(k_n x), \text{if } m \text{ is odd but } n \text{ is even,} \\ \sin(k_m z)\cos(k_n x), \text{if } m \text{ is even but } n \text{ is odd,} \\ \sin(k_m z)\sin(k_n x), \text{if both } m \text{ and } n \text{ are even,} \end{cases}$$

where $k_m = m\pi/2$, $k_n = n\pi/2$, and m and n are positive integers. The corresponding eigenvalues are given by

$$E_{mn} = \frac{\pi^2\hbar^2}{8\mu}(m^2 + n^2).$$

The symmetry group of the Hamiltonian H_0 is D_4 whose character table is given by

	$C_1 = \{E\}$	$C_2 = \{C_{2x}, C_{2z}\}$	$C_3 = \{C_{2y}\}$	$C_4 = \{C_{4y}, C_{4y}^{-1}\}$	$C_5 = \{C_{2c}, C_{2d}\}$
Γ^1	1	1	1	1	1
Γ^2	1	1	1	-1	-1
Γ^3	1	-1	1	1	-1
Γ^4	1	-1	1	-1	1
Γ^5	2	0	-2	0	0

1. For which irreducible representations do the eigenfunctions $\psi_{11}(z, x)$ and $\psi_{22}(z, x)$ form bases respectively?

2. Find the matrices representing all the elements of D_4 in the space spanned by the degenerate eigenfunctions $\psi_{12}(z, x)$ and $\psi_{21}(z, x)$. And then calculate the characters for all the classes of D_4 in this representation. For which irreducible representation do $\psi_{12}(z, x)$ and $\psi_{21}(z, x)$ form a basis?

3. What is the degeneracy corresponding to $(m = 6, n = 7)$ and $(m = 2, n = 9)$? Is this degeneracy normal or accidental?

4. Find the matrices representing all the elements of D_4 in the space spanned by the degenerate eigenfunctions $\psi_{mn}(z, x)$ and $\psi_{nm}(z, x)$. Here both m and n are odd integers but they are not equal. And then calculate the characters for all the classes of D_4 in this representation. Is this representation reducible or irreducible? If this representation is reducible, write it as a direct sum of irreducible representations.

5. Consider the case in which the particle is subject to an interaction given by Ax with A a constant.

   (a) For which irreducible representation of D_4 is x an irreducible tensor operator?

   (b) Consider the transitions caused by the interaction. If the particle is initially in the state $\psi_{mn}(z, x)$ or $\psi_{nm}(z, x)$ with m and n respectively even and odd integers, through reducing the direct product of irreducible representations find the irreducible representations which the allowed final states transform as.