Group Theory

Solutions to the Problems in Homework Assignment 03

Spring, 2020

1. Γ is a faithful representation of a non-Abelian group G. If the representation matrix of each element in the group is transformed as in the following, determine whether the resultant set of matrices forms a representation of the group G.

   (a) $\Gamma(T)^\dagger$ (Hermitian conjugate).
   (b) $\Gamma(T)^t$ (transpose).
   (c) $\Gamma(T)^{-1}$ (inverse).
   (d) $\Gamma(T)^*$ (complex conjugate).
   (e) $(\Gamma(T)^{-1})^\dagger$ (Hermitian conjugate of the inverse).
   (f) $\det\Gamma(T)$ (determinant).
   (g) $\mathrm{Tr}\,\Gamma(T)$ (trace).

(a) For $T_1, T_2 \in G$, we have

$$\Gamma(T_1T_2)^\dagger = \big[\,\Gamma(T_1)\Gamma(T_2)\,\big]^\dagger = \Gamma(T_2)^\dagger\Gamma(T_1)^\dagger \neq \Gamma(T_1)^\dagger\Gamma(T_2)^\dagger.$$

Thus, the set of matrices $\Gamma(T)^\dagger$ does not form a representation of G.

(b) For $T_1, T_2 \in G$, we have

$$\Gamma(T_1T_2)^t = \big[\,\Gamma(T_1)\Gamma(T_2)\,\big]^t = \Gamma(T_2)^t\Gamma(T_1)^t \neq \Gamma(T_1)^t\Gamma(T_2)^t.$$

Thus, the set of matrices $\Gamma(T)^t$ does not form a representation of G.

(c) For $T_1, T_2 \in G$, we have

$$\Gamma(T_1T_2)^{-1} = \big[\,\Gamma(T_1)\Gamma(T_2)\,\big]^{-1} = \Gamma(T_2)^{-1}\Gamma(T_1)^{-1} \neq \Gamma(T_1)^{-1}\Gamma(T_2)^{-1}.$$

Thus, the set of matrices $\Gamma(T)^{-1}$ does not form a representation of G.

(d) For $T_1, T_2 \in G$, we have

$$\Gamma(T_1T_2)^* = \big[\,\Gamma(T_1)\Gamma(T_2)\,\big]^* = \Gamma(T_1)^*\Gamma(T_2)^*.$$

Thus, the set of matrices $\Gamma(T)^*$ forms a representation of G.

(e) For $T_1, T_2 \in G$, we have

$$\big[\,\Gamma(T_1T_2)^{-1}\,\big]^\dagger = \Big\{\big[\,\Gamma(T_1)\Gamma(T_2)\,\big]^{-1}\Big\}^\dagger = \big[\,\Gamma(T_2)^{-1}\Gamma(T_1)^{-1}\,\big]^\dagger = \big[\,\Gamma(T_1)^{-1}\,\big]^\dagger\big[\,\Gamma(T_2)^{-1}\,\big]^\dagger.$$

Thus, the set of matrices $\big[\,\Gamma(T)^{-1}\,\big]^\dagger$ forms a representation of G.

(f) For $T_1, T_2 \in G$, we have

$$\det\Gamma(T_1T_2) = \det\big[\,\Gamma(T_1)\Gamma(T_2)\,\big] = \det\Gamma(T_1)\det\Gamma(T_2).$$

Thus, the set of values $\det\Gamma(T)$ forms a representation of G.

(g) For $T_1, T_2 \in G$, we have

$$\mathrm{tr}\,\Gamma(T_1T_2) = \mathrm{tr}\big[\,\Gamma(T_1)\Gamma(T_2)\,\big] = \sum_{ij}\Gamma(T_1)_{ij}\Gamma(T_2)_{ji} \neq \left(\sum_i \Gamma(T_1)_{ii}\right)\left(\sum_j \Gamma(T_2)_{jj}\right) = \mathrm{tr}\,\Gamma(T_1)\Gamma(T_2).$$

Thus, the set of values $\mathrm{tr}\,\Gamma(T)$ does not form a representation of G.

2. A two-dimensional representation of $C_2 = \{E, a\}$ is given by

$$\Gamma(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \Gamma(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find the similarity transformation that reduces the above two-dimensional representation of C_2 into the direct sum of two irreducible one-dimensional representations.

To reduce $\Gamma(a)$, we first diagonalize it. We use λ to denote the eigenvalue of $\Gamma(a)$ and $\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ to denote its corresponding eigenvector. The eigenequation of $\Gamma(a)$, $\Gamma(a)\psi = \lambda\psi$, reads

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

That is,

$$\begin{aligned} &-\lambda\alpha + \beta = 0, \\ &\alpha - \lambda\beta = 0. \end{aligned}$$

The necessary and sufficient condition for the existence of nontrivial solutions is given by the following secular equation

$$\begin{aligned} &\det\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0, \\ &\lambda^2 - 1 = 0. \end{aligned}$$

Thus, the eigenvalues are $\lambda_{1,2} = \pm 1$. For $\lambda_1 = +1$, we have

$$\begin{aligned} &-\alpha + \beta = 0, \\ &\alpha - \beta = 0. \end{aligned}$$

Thus, $\alpha = \beta$. The *normalized* eigenvector corresponding to $\lambda_1 = +1$ is then given by

$$\psi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Similarly, The *normalized* eigenvector corresponding to $\lambda_2 = -1$ is given by

$$\psi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The matrix in the similarity transformation that diagonalizes $\Gamma(a)$ is then given by

$$S = \begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The inverse of S is given by

$$S^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Using S, we have

$$S^{-1}\Gamma(E)S = \Gamma(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ S^{-1}\Gamma(a)S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus, the similarity transformation given by S indeed reduces the given two-dimensional representation of C_2 into the direct sum of two irreducible one-dimensional representations.

3. Consider the following two-dimensional representation Γ of the group $G = \{E, a, b\}$ of order $g = 3$

$$\Gamma(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \Gamma(a) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \; \Gamma(b) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}.$$

(a) Check the orthogonality relation

$$\frac{1}{g}\sum_{T\in G}\Gamma(T)^*_{jk}\Gamma(T)_{st} = \frac{1}{d}\delta_{js}\delta_{kt}$$

for all the possible combinations of j, k, s, and t. Note that $j, k, s, t = 1, 2$ and that $d = 2$.

(b) Is the representation Γ reducible?

(a) We list the results respectively for $j = s = 1, 2$ and $k = t = 1, 2$, $j = s = 1, 2$ and $k \neq t = 1, 2$, $j \neq s = 1, 2$ and $k = t = 1, 2$, and $j \neq s = 1, 2$ and $k \neq t = 1, 2$.
For $j = s = 1, 2$ and $k = t = 1, 2$, we have

$$\sum_{T\in G}\Gamma(T)^*_{11}\Gamma(T)_{11} = 1^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 = \frac{3}{2} = \frac{g}{d},$$

$$\sum_{T\in G}\Gamma(T)^*_{12}\Gamma(T)_{12} = 0^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{2} = \frac{g}{d},$$

$$\sum_{T\in G}\Gamma(T)^*_{21}\Gamma(T)_{21} = 0^2 + \left(\frac{-\sqrt{3}}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{2} = \frac{g}{d},$$

$$\sum_{T\in G}\Gamma(T)^*_{22}\Gamma(T)_{22} = 1^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 = \frac{3}{2} = \frac{g}{d}.$$

We see that the orthogonality relation is satisfied for $j = s = 1, 2$ and $k = t = 1, 2$.
For $j = s = 1, 2$ and $k \neq t = 1, 2$, we have

$$\sum_{T\in G}\Gamma(T)^*_{11}\Gamma(T)_{12} = \sum_{T\in G}\Gamma(T)^*_{12}\Gamma(T)_{11} = 1\times 0 + \left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) = 0,$$

$$\sum_{T\in G}\Gamma(T)^*_{21}\Gamma(T)_{22} = \sum_{T\in G}\Gamma(T)^*_{22}\Gamma(T)_{21} = 0\times 1 + \left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right) = 0.$$

We see that the orthogonality relation is satisfied for $j = s = 1, 2$ and $k \neq t = 1, 2$.
For $j \neq s = 1, 2$ and $k = t = 1, 2$, we have

$$\sum_{T\in G}\Gamma(T)^*_{11}\Gamma(T)_{21} = \sum_{T\in G}\Gamma(T)^*_{21}\Gamma(T)_{11} = 1\times 0 + \left(-\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = 0,$$

$$\sum_{T\in G}\Gamma(T)^*_{12}\Gamma(T)_{22} = \sum_{T\in G}\Gamma(T)^*_{22}\Gamma(T)_{12} = 0\times 1 + \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right) + \left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right) = 0.$$

We see that the orthogonality relation is satisfied for $j \neq s = 1, 2$ and $k = t = 1, 2$.
For $j \neq s = 1, 2$ and $k \neq t = 1, 2$.

$$\sum_{T\in G}\Gamma(T)^*_{11}\Gamma(T)_{22} = \sum_{T\in G}\Gamma(T)^*_{22}\Gamma(T)_{11} = 1\times 1 + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) = \frac{3}{2} \neq 0,$$

$$\sum_{T\in G}\Gamma(T)^*_{12}\Gamma(T)_{21} = \sum_{T\in G}\Gamma(T)^*_{21}\Gamma(T)_{12} = 0\times 0 + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{-\sqrt{3}}{2}\right) + \left(\frac{-\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = -\frac{3}{2} \neq 0.$$

We see that the orthogonality relation is not satisfied for $j \neq s = 1, 2$ and $k \neq t = 1, 2$.

(b) In consideration that the orthogonality relation is not satisfied for $j \neq s = 1, 2$ and $k \neq t = 1, 2$, we conclude that the representation Γ is reducible.

4. Show that the sum of the characters of all the elements of a finite group in an irreducible representation except the identity representation is zero.

Setting the irreducible representation Γ^p to be the identity representation Γ^1 with $\chi^1(T) = 1\ \forall T \in G$ in the first orthogonality theorem for characters,

$$\frac{1}{g} \sum_{T \in G} \chi^p(T)^* \chi^q(T) = \delta_{pq},$$

we have

$$\frac{1}{g} \sum_{T \in G} \chi^q(T) = \delta_{1q}.$$

For $q \neq 1$, we have

$$\sum_{T \in G} \chi^q(T) = 0, \ \ q \neq 1.$$

Thus, the sum of the characters of all the elements of a finite group in an irreducible representation except the identity representation is zero.

5. Consider the group $G = \{E, a, b, b^2, b^3, b^4, b^5, ab, ab^2, ab^3, ab^4, ab^5\}$ with $a^2 = b^6 = E$ and $a^{-1}ba = b^{-1}$.

(a) Find all the elements in each class of G.

(b) Γ^1 and Γ^2 are two representations of G. In the representation Γ^1, $\Gamma^1(a)$ and $\Gamma^1(b)$ are respectively given by

$$\Gamma^1(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \Gamma^1(b) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$$

with $\omega = e^{i2\pi/3}$. In the representation Γ^2, $\Gamma^2(a)$ and $\Gamma^2(b)$ are respectively given by

$$\Gamma^2(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \Gamma^2(b) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Find the partial character table of G with entries only for the representations Γ^1 and Γ^2.

(c) Are the representations Γ^1 and Γ^2 equivalent?

(d) Is the representation Γ^1 reducible?

(e) Is the representation Γ^2 reducible?

(a) The multiplication table of the group is found to be given by

	E	a	b	b^2	b^3	b^4	b^5	ab	ab^2	ab^3	ab^4	ab^5
E	E	a	b	b^2	b^3	b^4	b^5	ab	ab^2	ab^3	ab^4	ab^5
a	a	E	ab	ab^2	ab^3	ab^4	ab^5	b	b^2	b^3	b^4	b^5
b	b	ab^5	b^2	b^3	b^4	b^5	E	a	ab	ab^2	ab^3	ab^4
b^2	b^2	ab^4	b^3	b^4	b^5	E	b	ab^5	a	ab	ab^2	ab^3
b^3	b^3	ab^3	b^4	b^5	E	b	b^2	ab^4	ab^5	a	ab	ab^2
b^4	b^4	ab^2	b^5	E	b	b^2	b^3	ab^3	ab^4	ab^5	a	ab
b^5	b^5	ab	E	b	b^2	b^3	b^4	ab^2	ab^3	ab^4	ab^5	a
ab	ab	b^5	ab^2	ab^3	ab^4	ab^5	a	E	b	b^2	b^3	b^4
ab^2	ab^2	b^4	ab^3	ab^4	ab^5	a	ab	b^5	E	b	b^2	b^3
ab^3	ab^3	b^3	ab^4	ab^5	a	ab	ab^2	b^4	b^5	E	b	b^2
ab^4	ab^4	b^2	ab^5	E	ab	ab^2	ab^3	b^3	b^4	b^5	E	b
ab^5	ab^5	b	a	ab	ab^2	ab^3	ab^4	b^2	b^3	b^4	b^5	E

From the above multiplication table, we can easily infer that the inverses of all the elements of G are respectively given by

$$E^{-1} = E,\ a^{-1} = a,\ b^{-1} = b^5,\ (b^2)^{-1} = b^4,\ (b^3)^{-1} = b^3,\ (b^4)^{-1} = b^2,\ (b^5)^{-1} = b,$$
$$(ab)^{-1} = ab,\ (ab^2)^{-1} = ab^2,\ (ab^3)^{-1} = ab^3,\ (ab^4)^{-1} = ab^4,\ (ab^5)^{-1} = ab^5.$$

We now find the classes of G.

i. $C_1 = \{E\}$.
Since E commutes with all the elements of G, it is in a class by itself.

ii. $C_2 = \{b^3\}$.
From the element b^3 of G, we have

$$a^{-1}b^3a = b^3,\ b^{-1}b^3b = (b^2)^{-1}b^3b^2 = (b^4)^{-1}b^3b^4 = (b^5)^{-1}b^3b^5 = b^3,$$
$$(ab)^{-1}b^3ab = (ab^2)^{-1}b^3ab^2 = (ab^3)^{-1}b^3ab^3 = (ab^4)^{-1}b^3ab^4 = (ab^5)^{-1}b^3b^5 = b^3.$$

iii. $C_3 = \{b, b^5\}$.
From the element b of G, we have

$$a^{-1}ba = b^5,\ (b^2)^{-1}bb^2 = (b^3)^{-1}bb^3 = (b^4)^{-1}bb^4 = (b^5)^{-1}bb^5 = b,$$
$$(ab)^{-1}bab = (ab^2)^{-1}bab^2 = (ab^3)^{-1}bab^3 = (ab^4)^{-1}bab^4 = (ab^5)^{-1}ab^5 = b^5.$$

iv. $C_4 = \{b^2, b^4\}$.
From the element b^2 of G, we have

$$a^{-1}b^2a = b^4,\ b^{-1}b^2b = (b^3)^{-1}b^2b^3 = (b^4)^{-1}b^2b^4 = (b^5)^{-1}b^2b^5 = b^2,$$
$$(ab)^{-1}b^2ab = (ab^2)^{-1}b^2ab^2 = (ab^3)^{-1}b^2ab^3 = (ab^4)^{-1}b^2ab^4 = (ab^5)^{-1}b^2b^5 = b^4.$$

v. $C_5 = \{a, ab^2, ab^4\}$.
From the element a of G, we have

$$b^{-1}ab = ab^2,\ (b^2)^{-1}ab^2 = ab^4,\ (b^3)^{-1}ab^3 = a,\ (b^4)^{-1}ab^4 = ab^2,\ (b^5)^{-1}ab^5 = ab^4,\ (ab)^{-1}a(ab) = ab^2,$$
$$(ab^2)^{-1}a(ab^2) = ab^4,\ (ab^3)^{-1}a(ab^3) = a,\ (ab^4)^{-1}a(ab^4) = ab^2,\ (ab^5)^{-1}a(ab^5) = ab^4.$$

vi. $C_6 = \{ab, ab^3, ab^5\}$.
From the element ab of G, we have

$$a^{-1}aba = ab^5,\ b^{-1}abb = ab^3,\ (b^2)^{-1}abb^2 = ab^5,\ (b^3)^{-1}abb^3 = ab,\ (b^4)^{-1}abb^4 = ab^3,\ (b^5)^{-1}abb^5 = ab^5,$$
$$(ab^2)^{-1}abab^2 = ab^3,\ (ab^3)^{-1}abab^3 = ab^5,\ (ab^4)^{-1}abab^4 = ab,\ (ab^5)^{-1}abb^5 = ab^3.$$

In consideration that G has six classes, we see that G has six inequivalent irreducible representations. Let n_p be the dimension of the pth inequivalent irreducible representation. From

$$\sum_{p=1}^{n_C} n_p^2 = g$$

with $N_C = 6$ and $g = 12$, we see that four irreducible representations are one-dimensional and two irreducible representations are two-dimensional.

(b) In consideration that the two representations Γ^1 and Γ^2 are both two-dimensional, we have $\chi^1(E) = \chi^2(E) = 2$. We thus have

$$\chi^1(C_1) = \chi^2(C_1) = 2.$$

From the given expressions of $\Gamma^{1,2}(a)$ and $\Gamma^{1,2}(b)$, we can infer the following characters of classes C_3 and

C_5 in the two representations Γ^1 and Γ^2

$$\chi^1(C_3) = \operatorname{tr}\Gamma^1(b) = \operatorname{tr}\begin{pmatrix}\omega & 0 \\ 0 & \omega^{-1}\end{pmatrix} = \omega + \omega^{-1} = e^{i2\pi/3} + e^{-i2\pi/3} = 2\cos(2\pi/3) = -1,$$
$$\chi^2(C_3) = \operatorname{tr}\Gamma^2(b) = \operatorname{tr}\begin{pmatrix}-1 & 0 \\ 0 & 1\end{pmatrix} = 0,$$
$$\chi^1(C_5) = \operatorname{tr}\Gamma^1(a) = \operatorname{tr}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} = 0,$$
$$\chi^2(C_5) = \operatorname{tr}\Gamma^2(a) = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix} = 0.$$

To find $\chi^{1,2}(C_2)$, we make use of $\Gamma^{1,2}(b^3) = \Gamma^{1,2}(b)^3$. We have

$$\Gamma^1(b^3) = \Gamma^1(b)^3 = \begin{pmatrix}\omega & 0 \\ 0 & \omega^{-1}\end{pmatrix}^3 = \begin{pmatrix}\omega^3 & 0 \\ 0 & \omega^{-3}\end{pmatrix} = \begin{pmatrix}e^{i2\pi} & 0 \\ 0 & e^{-i2\pi}\end{pmatrix} = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix},$$
$$\Gamma^2(b^3) = \Gamma^2(b)^3 = \begin{pmatrix}-1 & 0 \\ 0 & 1\end{pmatrix}^3 = \begin{pmatrix}-1 & 0 \\ 0 & 1\end{pmatrix}.$$

We thus have

$$\chi^1(C_2) = \operatorname{tr}\Gamma^1(b^3) = \operatorname{tr}\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} = 2,$$
$$\chi^2(C_2) = \operatorname{tr}\Gamma^2(b^3) = \begin{pmatrix}-1 & 0 \\ 0 & 1\end{pmatrix} = 0.$$

To find $\chi^{1,2}(C_4)$, we make use of $\Gamma^{1,2}(b^2) = \Gamma^{1,2}(b)^2$. We have

$$\Gamma^1(b^2) = \Gamma^1(b)^2 = \begin{pmatrix}\omega & 0 \\ 0 & \omega^{-1}\end{pmatrix}^2 = \begin{pmatrix}\omega^2 & 0 \\ 0 & \omega^{-2}\end{pmatrix} = \begin{pmatrix}e^{i4\pi/3} & 0 \\ 0 & e^{-i4\pi/3}\end{pmatrix} = \begin{pmatrix}-e^{i\pi/3} & 0 \\ 0 & -e^{-i\pi/3}\end{pmatrix},$$
$$\Gamma^2(b^2) = \Gamma^2(b)^2 = \begin{pmatrix}-1 & 0 \\ 0 & 1\end{pmatrix}^2 = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}.$$

We thus have

$$\chi^1(C_4) = \operatorname{tr}\Gamma^1(b^2) = \operatorname{tr}\begin{pmatrix}-e^{i\pi/3} & 0 \\ 0 & -e^{-i\pi/3}\end{pmatrix} = -e^{i\pi/3} - e^{-i\pi/3} = -2\cos(\pi/3) = -1,$$
$$\chi^2(C_4) = \operatorname{tr}\Gamma^2(b^2) = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} = 2.$$

To find $\chi^{1,2}(C_6)$, we make use of $\Gamma^{1,2}(ab) = \Gamma^{1,2}(a)\Gamma^{1,2}(b)$. We have

$$\Gamma^1(ab) = \Gamma^1(a)\Gamma^1(b) = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\omega & 0 \\ 0 & \omega^{-1}\end{pmatrix} = \begin{pmatrix}0 & \omega^{-1} \\ \omega & 0\end{pmatrix} = \begin{pmatrix}0 & e^{-i2\pi/3} \\ e^{i2\pi/3} & 0\end{pmatrix},$$
$$\Gamma^2(ab) = \Gamma^2(a)\Gamma^2(b) = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}-1 & 0 \\ 0 & 1\end{pmatrix} = \begin{pmatrix}-1 & 0 \\ 0 & -1\end{pmatrix}.$$

We thus have

$$\chi^1(C_6) = \operatorname{tr}\Gamma^1(ab) = \operatorname{tr}\begin{pmatrix}0 & e^{-i2\pi/3} \\ e^{i2\pi/3} & 0\end{pmatrix} = 0,$$
$$\chi^2(C_6) = \operatorname{tr}\Gamma^2(ab) = \begin{pmatrix}-1 & 0 \\ 0 & -1\end{pmatrix} = -2.$$

Collecting the above-obtained results, we have the following partial character table for G.

	$C_1 = \{E\}$	$C_2 = \{b^3\}$	$C_3 = \{b, b^5\}$	$C_4 = \{b^2, b^4\}$	$C_5 = \{a, ab^2, ab^4\}$	$C_6 = \{ab, ab^3, ab^5\}$
Γ^1	2	2	-1	-1	0	0
Γ^2	2	0	0	2	0	-2

(c) Because Γ^1 and Γ^2 possess different character systems, they are not equivalent.

(d) To determine whether the representation Γ^1 is reducible, we make use of the following necessary and sufficient condition for a representation of G to be irreducible

$$\frac{1}{g}\sum_{T\in G} |\chi(T)|^2 = 1.$$

For Γ^1, we have

$$\frac{1}{g}\sum_{T\in G} |\chi^1(T)|^2 = \frac{1}{12}\left[\, 2^2 + 2^2 + 2\times(-1)^2 + 2\times(-1)^2 \,\right] = 1.$$

Thus, Γ^1 is irreducible.

(e) For Γ^2, we have

$$\frac{1}{g}\sum_{T\in G} |\chi^2(T)|^2 = \frac{1}{12}\left[\, 2^2 + 2\times 2^2 + 3\times(-2)^2 \,\right] = 2 \neq 1.$$

Thus, Γ^2 is reducible.