Group Theory
Homework Assignment 06
Spring, 2020

1. The basis elements of the real Lie algebra $L = \mathrm{so}(3)$ are given by

$$a_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},\ a_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ a_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

   Show explicitly that these basis elements possess the following properties.

   (a) The basis elements a_1, a_2, and a_3 obey the commutation relations

$$\begin{aligned} [a_1, a_2] &= a_1 a_2 - a_2 a_1 = -a_3, \\ [a_2, a_3] &= a_2 a_3 - a_3 a_2 = -a_1, \\ [a_3, a_1] &= a_3 a_1 - a_1 a_3 = -a_2. \end{aligned}$$

   (b) The basis elements a_1, a_2, and a_3 are anti-Hermitian,

$$a_1^\dagger = -a_1,\ a_2^\dagger = -a_2,\ a_3^\dagger = -a_3.$$

2. The scalar transformation operators $Q(a_1)$, $Q(a_2)$, and $Q(a_3)$ for the real Lie algebra $\mathrm{so}(3)$ are found to be given by

$$Q(a_1) = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},\ Q(a_2) = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},\ Q(a_3) = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

   Show that $[Q(a_1), Q(a_2)] = -Q(a_3)$, $[Q(a_2), Q(a_3)] = -Q(a_1)$, and $[Q(a_3), Q(a_1)] = -Q(a_2)$.

3. The generators of the real Lie algebra $L = \mathrm{su}(2)$ are given by

$$a_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},\ a_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ a_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

   Show explicitly that a_1, a_2, and a_3 obey the commutation relations

$$\begin{aligned} [a_1, a_2] &= a_1 a_2 - a_2 a_1 = -a_3, \\ [a_2, a_3] &= a_2 a_3 - a_3 a_2 = -a_1, \\ [a_3, a_1] &= a_3 a_1 - a_1 a_3 = -a_2. \end{aligned}$$

4. The generators of the real Lie algebra $L = \mathrm{su}(2)$ in the above problem can be expressed in terms of the following Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   Show that the Pauli matrices possess the following properties.

   (a) $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1$.

   (b) $\sigma_1\sigma_2 = -\sigma_2\sigma_1 = i\sigma_3,\ \sigma_2\sigma_3 = -\sigma_3\sigma_2 = i\sigma_1,\ \sigma_3\sigma_1 = -\sigma_1\sigma_3 = i\sigma_2$.

5. Let $\vec{n} = (n_1, n_2, n_3)$ be a unit vector specifying a direction in three-dimensional space.

   (a) Evaluate $(\vec{\sigma}\cdot\vec{n})^2$ with $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$.

   (b) Evaluate $e^{i(\vec{\sigma}\cdot\vec{n})\omega/2}$.