第三次作业

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姓名：陈 稼 霖
学号：SA21038052
成绩：__________

第 1 题 得分: ________. 试证明相对熵纠缠度量在纯态情况下和 Von Neumann 熵是等价的. (求任意给定纯态 $|\psi_{AB}\rangle$ 和任意混合态 $\sum_i p_i\rho_i\otimes\sigma_i$ 中的最小相对熵 $S\left(|\psi_{AB}\rangle\langle\psi_{AB}\|\sum_i p_i\rho_i\otimes\sigma_i\right)$).

证: 记

$$\rho_{AB}=|\psi_{AB}\rangle\langle\psi_{AB}|. \tag{1}$$

纯态 $|\psi_{AB}\rangle$ 和混合态 $\sum_i p_i\rho_i\otimes\sigma_i$ 的相对熵为

$$\begin{aligned}S\left(\rho_{AB}\|\sum_i p_i\rho_i\otimes\sigma_i\right)=&\operatorname{Tr}\left(\rho_{AB}\left(\log_2\rho_{AB}-\log_2\sum_i p_i\rho_i\otimes\sigma_i\right)\right)\\=&\operatorname{Tr}(\rho_{AB}\log_2\rho_{AB})-\operatorname{Tr}\left(\rho_{AB}\log_2\sum_i p_i\rho_i\otimes\sigma_i\right)\\=&-S(\rho_{AB})-\operatorname{Tr}\left(\rho_{AB}\log_2\sum_i p_i\rho_i\otimes\sigma_i\right).\end{aligned} \tag{2}$$

而 $|\psi_{AB}\rangle$ 的 Von Neumann 熵为

$$S(\rho_{AB}\|\rho_A\otimes\rho_B)=S(\rho_A)+S(\rho_B)-S(\rho_{AB}). \tag{3}$$

要证 $\min_{\sum_i p_i\rho_i\otimes\sigma_i}S\left(|\psi_{AB}\rangle\langle\psi_{AB}\|\sum_i p_i\rho_i\otimes\sigma_i\right)=S(|\psi_{AB}\rangle\langle\psi_{AB}|)$, 即证 $\min\operatorname{Tr}\left(\rho_{AB}\log_2 p_i\rho_i\otimes\sigma_i\right)=S(\rho_A)+S(\rho_B)$.
对 $|\psi_{AB}\rangle$ 进行 Schmidt 分解得

$$|\psi_{AB}\rangle=\sum_i a_i|\psi_i\rangle_A|\psi_i\rangle_B, \tag{4}$$

其中 $\sum_i|a_i|^2=1$, $\{|\psi_i\rangle_A\}$ 和 $\{|\phi_i\rangle_B\}$ 分别是子系统 A 和 B 的正交归一基. □

第 2 题 得分: ________. 计算混合量子态 $\rho=p|\phi^+\rangle\langle\phi^+|+\frac{1-p}{4}I_{4\times4}$ 的纠缠 concurrence, 其中 $0\leq p\leq1$, $|\phi^+\rangle$ 是 Bell 态.

解: 混合量子态为

$$\begin{aligned}\rho=&p|\phi^+\rangle\langle\phi^+|+\frac{1-p}{4}I_{4\times4}\\=&p\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)\frac{1}{\sqrt{2}}(\langle00|+\langle11|)+\frac{1-p}{4}I_{4\times4}\\=&\frac{p}{2}(|00\rangle\langle00|+|00\rangle\langle11|+|11\rangle\langle00|+|11\rangle\langle11|)+\frac{1-p}{4}I_{4\times4}\\=&\frac{p}{2}\begin{pmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix}+\frac{1-p}{4}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}\frac{1+p}{4}&0&0&\frac{p}{2}\\0&\frac{1-p}{4}&0&0\\0&0&\frac{1-p}{4}&0\\\frac{p}{2}&0&0&\frac{1+p}{4}\end{pmatrix}.\end{aligned} \tag{5}$$

取该密度矩阵的复共轭, 并用泡利算符 σ_y 分别作用两个 qubit, 得

$$\begin{aligned}\tilde{\rho}=&(\sigma_y\otimes\sigma_y)\rho^*(\sigma_y\otimes\sigma_y)\\=&\left(\begin{pmatrix}0&-i\\i&0\end{pmatrix}\otimes\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right)\begin{pmatrix}\frac{1+p}{4}&0&0&\frac{p}{2}\\0&\frac{1-p}{4}&0&0\\0&0&\frac{1-p}{4}&0\\\frac{p}{2}&0&0&\frac{1+p}{4}\end{pmatrix}\left(\begin{pmatrix}0&-i\\i&0\end{pmatrix}\otimes\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right)\end{aligned}$$

$$
\begin{aligned}
&=\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}\begin{pmatrix}\frac{1+p}{4}&0&0&\frac{p}{2}\\0&\frac{1-p}{4}&0&0\\0&0&\frac{1-p}{4}&0\\\frac{p}{2}&0&0&\frac{1+p}{4}\end{pmatrix}\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}\\
&=\begin{pmatrix}\frac{1+p}{4}&0&0&\frac{p}{2}\\0&\frac{1-p}{4}&0&0\\0&0&\frac{1-p}{4}&0\\\frac{p}{2}&0&0&\frac{1+p}{4}\end{pmatrix}.
\end{aligned}\tag{6}
$$

对原混合量子态的密度矩阵进行奇异值分解得

$$
\rho=\begin{pmatrix}\frac{1+p}{4}&0&0&\frac{p}{2}\\0&\frac{1-p}{4}&0&0\\0&0&\frac{1-p}{4}&0\\\frac{p}{2}&0&0&\frac{1+p}{4}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0&0\\0&0&1&0\\0&0&0&1\\\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}&0&0\end{pmatrix}\begin{pmatrix}\frac{1+3p}{4}&0&0&0\\0&\frac{1-p}{4}&0&0\\0&0&\frac{1-p}{4}&0\\0&0&0&\frac{1-p}{4}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&0&0&-\frac{1}{\sqrt{2}}\\0&1&0&0\\0&0&1&0\end{pmatrix}\tag{7}
$$

故对其开方得

$$
\sqrt{\rho}=\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0&0\\0&0&1&0\\0&0&0&1\\\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}&0&0\end{pmatrix}\begin{pmatrix}\frac{\sqrt{1+3p}}{2}&0&0&0\\0&\frac{\sqrt{1-p}}{2}&0&0\\0&0&\frac{\sqrt{1-p}}{2}&0\\0&0&0&\frac{\sqrt{1-p}}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&0&0&-\frac{1}{\sqrt{2}}\\0&1&0&0\\0&0&1&0\end{pmatrix}\tag{8}
$$

R 矩阵为

$$
\begin{aligned}
R=&\sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}\\
=&\sqrt{\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0&0\\0&0&1&0\\0&0&0&1\\\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}&0&0\end{pmatrix}\begin{pmatrix}\frac{\sqrt{1+3p}}{2}&0&0&0\\0&\frac{\sqrt{1-p}}{2}&0&0\\0&0&\frac{\sqrt{1-p}}{2}&0\\0&0&0&\frac{\sqrt{1-p}}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&0&0&-\frac{1}{\sqrt{2}}\\0&1&0&0\\0&0&1&0\end{pmatrix}\begin{pmatrix}\frac{1+p}{4}&0&0&\frac{p}{2}\\0&\frac{1-p}{4}&0&0\\0&0&\frac{1-p}{4}&0\\\frac{p}{2}&0&0&\frac{1+p}{4}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0&0\\0&0&1&0\\0&0&0&1\\\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}&0&0\end{pmatrix}\begin{pmatrix}\frac{\sqrt{1+3p}}{2}&0&0&0\\0&\frac{\sqrt{1-p}}{2}&0&0\\0&0&\frac{\sqrt{1-p}}{2}&0\\0&0&0&\frac{\sqrt{1-p}}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&0&0&-\frac{1}{\sqrt{2}}\\0&1&0&0\\0&0&1&0\end{pmatrix}}\\
=&\sqrt{\begin{pmatrix}0&0&1&0\\0&0&0&1\\\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}&0&0\end{pmatrix}\begin{pmatrix}\left(\frac{1+3p}{4}\right)^2&0&0&0\\0&\left(\frac{1-p}{4}\right)^2&0&0\\0&0&\left(\frac{1-p}{4}\right)^2&0\\0&0&0&\left(\frac{1-p}{4}\right)^2\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&0&0&-\frac{1}{\sqrt{2}}\\0&1&0&0\\0&0&1&0\end{pmatrix}}\\
=&\begin{pmatrix}0&0&1&0\\0&0&0&1\\\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}&0&0\end{pmatrix}\begin{pmatrix}\frac{1+3p}{4}&0&0&0\\0&\frac{1-p}{4}&0&0\\0&0&\frac{1-p}{4}&0\\0&0&0&\frac{1-p}{4}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&0&0&-\frac{1}{\sqrt{2}}\\0&1&0&0\\0&0&1&0\end{pmatrix}\\
=&\begin{pmatrix}\frac{1+p}{4}&0&0&\frac{p}{2}\\0&\frac{1-p}{4}&0&0\\0&0&\frac{1-p}{4}&0\\\frac{p}{2}&0&0&\frac{1+p}{4}\end{pmatrix}.
\end{aligned}\tag{9}
$$

R 的本征值从大到小依次为 $\lambda_1=\frac{1+3p}{4}$, $\lambda_2=\frac{1-p}{4}$, $\lambda_3=\frac{1-p}{4}$, $\lambda_4=\frac{1-p}{4}$, 故 concurrence 为

$$
C(\rho)=\max\{0,\lambda_1-\lambda_2-\lambda_3-\lambda_4\}=\max\{0,\frac{3p-1}{2}\}=\begin{cases}0, & 0\leq p\leq\frac{1}{3},\\ \frac{3p-1}{2}, & \frac{1}{3}<p\leq 1.\end{cases}\tag{10}
$$

□

第 10 题 得分: ________. 证明 $S(\rho_A)+S(\rho_B)\leq S(\rho_{AC})+S(\rho_{BC})$.

证: [1]首先来证明一个引理:

[1]参考文献: Lieb, Elliott H., and Mary Beth Ruskai. "Proof of the strong subadditivity of quantum-mechanical entropy." *J. Math. Phys.* 19 (1973): 36-55.

映射 $f(\rho_{AC})=S(\mathrm{Tr}_C(\rho_{AC}))-S(\rho_{AC})$ 在 A 与 C 的复合 Hilbert 空间 $H_A\otimes H_C$ 上是凸的, 即

$$\begin{aligned}f(\rho_{AC})=&S(\mathrm{Tr}_C(\rho_{AC}))-S(\rho_{AC})\\ \leq&\alpha f(\rho_{A'C})+(1-\alpha)f(\rho_{A''C})=\alpha[S(\rho_{A'})-S(\rho_{A'C})]+(1-\alpha)[S(\rho_{A''})-S(\rho_{A''C})],\end{aligned}\tag{11}$$

其中 $\rho_{AC}=\alpha\rho_{A'C}+(1-\alpha)\rho_{A''C}$.

这里我们定义

$$\Delta=\alpha\,\mathrm{Tr}[\rho_{A'C}(-\log_2\rho_{A'C}+\log_2\rho_{A'}+\log_2\rho_{AC}-\log_2\rho_A)],\tag{12}$$

$$\Gamma=(1-\alpha)\,\mathrm{Tr}[\rho_{A''C}(-\log_2\rho_{A''C}+\log_2\rho_{A''}+\log_2\rho_{AC}-\log_2\rho_A)].\tag{13}$$

要证上述引理, 即转化为证

$$\Delta+\Gamma\leq 0.\tag{14}$$

利用 Klein 定理, 得

$$\Delta+\Gamma\leq\alpha\,\mathrm{Tr}[\exp(\log_2\rho_{A'}+\log_2\rho_{AC}-\log_2\rho_A)-\rho_{A'C}]+(1-\alpha)\,\mathrm{Tr}[\exp(\log_2\rho_{A''}+\log_2\rho_{AC}-\log_2\rho_A)-\rho_{A''C}],\tag{15}$$

再利用映射 $g(C)=\exp(K+\ln C)$ (其中 $K=\log_2\rho_{AC}-\log_2\rho_A$) 的上凸性 (即 $\alpha\,\mathrm{Tr}_{AC}[\exp(\ln\rho_{A'}+\ln\rho_{AC}-\log_2\rho_A)]+(1-\alpha)\,\mathrm{Tr}_{AC}[\exp(\log_2\rho_{A''}+\log_2\rho_{AC}-\log_2\rho_A)]\leq\mathrm{Tr}(\exp(\log_2\rho_A+\log_2)))$, 得

$$\Delta+\Gamma=\mathrm{Tr}[\exp(\log_2\rho_A+\log_2\rho_{AB}-\log_2\rho_A)-\rho_{AC}]=0.\tag{16}$$

引理证毕.

我们定义

$$G(\rho_{ABC})=S(\mathrm{Tr}_{BC}(\rho_{ABC}))+S(\mathrm{Tr}_{AC}(\rho_{ABC}))-S(\mathrm{Tr}_B(\rho_{ABC}))-S(\mathrm{Tr}_A(\rho_{ABC})),\tag{17}$$

利用上证得的引理, 这里 $S(\mathrm{Tr}_{BC}(\rho_{ABC}))-S(\mathrm{Tr}_B(\rho_{ABC}))$ 和 $S(\mathrm{Tr}_{AC}(\rho_{ABC}))-S(\mathrm{Tr}_A(\rho_{ABC}))$ 在 $H_A\otimes H_B\otimes H_C$ 上也是凸的, 故 $G(\rho_{ABC})$ 在 $H_A\otimes H_B\otimes H_C$ 上是凸的. 利用 Araki, Huzihiro, and Elliott H. Lieb. "Entropy inequalities." *Inequalities.* Springer, Berlin, Heidelberg, 2002. 47-57. 中的引理 3 得

$$G(\rho_{ABC})\leq 0,\tag{18}$$

故 $S(\rho_A)+S(\rho_B)\leq S(\rho_{AC})+S(\rho_{BC})$. □

第 11 题 得分: ________. 考虑 2-qubit 系统 $\rho_{AB}=\frac{1}{8}I\otimes I+\frac{1}{2}|\psi^-\rangle\langle\psi^-|$, 分别沿 $\vec{n}$, $\vec{m}$ 方向测 A, B 粒子的自旋. 其中 $\vec{m}\cdot\vec{n}=\cos\theta$, 则测量结果均为向上的联合概率是多少? 由 Peres-Horodeski 判据, 确定 ρ_{AB} 是否为可分量子态.

解: 2-qubit 系统的密度矩阵为

$$\begin{aligned}\rho_{AB}=&\frac{1}{8}I\otimes I+\frac{1}{2}|\psi^-\rangle\langle\psi^-|\\ =&\frac{1}{8}I\otimes I+\frac{1}{2}\frac{1}{\sqrt{2}}(|01\rangle-|10\rangle)\frac{1}{\sqrt{2}}(\langle 01|-\langle 10|)\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{8}I\times I+\frac{1}{4}(|01\rangle\langle 01|-|01\rangle\langle 10|-|10\rangle\langle 01|+|10\rangle\langle 10|)\\
=&\frac{1}{8}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}+\frac{1}{4}\begin{pmatrix}0&0&0&0\\0&1&-1&0\\0&-1&1&0\\0&0&0&0\end{pmatrix}\\
=&\frac{1}{8}\begin{pmatrix}1&0&0&0\\0&3&-2&0\\0&-2&3&0\\0&0&0&1\end{pmatrix}.
\end{aligned}\tag{19}
$$

利用第 9 题的结论, $|\psi^-\rangle$ 在 $U(\theta,\vec{n})\otimes U(\theta,\vec{n})$ 旋转变换下不变, 而 $\because(U(\theta,\vec{n})\otimes U(\theta,\vec{n}))I\times I(U(\theta,\vec{n})\otimes U(\theta,\vec{n}))^\dagger=(U(\theta,\vec{n})\otimes U(\theta,\vec{n}))I\times I(U(\theta,\vec{n})\otimes U(\theta,\vec{n}))^{-1}=I\otimes I$, $\therefore I\times I$ 也在 $U(\theta,\vec{n})\otimes U(\theta,\vec{n})$ 旋转变换下不变, 故我们不妨设 $\vec{n}$ 沿 $|0\rangle$ 方向, $\vec{m}$ 沿 $\cos\theta|0\rangle+e^{i\phi}\sin\theta|1\rangle$. 分别沿 $\vec{n}$ 和 $\vec{m}$ 测量 A, B 粒子的自旋, 测量结果均向上对应的投影矩阵为

$$
\begin{aligned}
P_{\uparrow\uparrow}=&|0\rangle\langle 0|\otimes\left(\cos\frac{\theta}{2}|0\rangle+e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right)\left(\cos\frac{\theta}{2}\langle 0|+e^{-i\phi}\sin\frac{\theta}{2}\langle 1|\right)\\
=&\cos^2\frac{\theta}{2}|00\rangle\langle 00|+e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2}|00\rangle\langle 01|+e^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2}|01\rangle\langle 00|+\sin^2\frac{\theta}{2}|01\rangle\langle 01|\\
=&\begin{pmatrix}\cos^2\frac{\theta}{2}&e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2}&0&0\\e^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2}&\sin^2\frac{\theta}{2}&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}
\end{aligned}\tag{20}
$$

测量结果均为向上的联合概率为

$$
\begin{aligned}
p_{\uparrow\uparrow}=\mathrm{Tr}(P_{\uparrow\uparrow}\rho_{AB})=\mathrm{Tr}&\left(\begin{pmatrix}\cos^2\frac{\theta}{2}&e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2}&0&0\\e^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2}&\sin^2\frac{\theta}{2}&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\frac{1}{8}\begin{pmatrix}1&0&0&0\\0&3&-2&0\\0&-2&3&0\\0&0&0&1\end{pmatrix}\right)\\
=\frac{1}{8}\mathrm{Tr}&\begin{pmatrix}\cos^2\frac{\theta}{2}&3e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2}&-2e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2}&0\\e^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2}&3\sin^2\frac{\theta}{2}&-2\sin^2\frac{\theta}{2}&0\\0&0&0&0\\0&0&0&0\end{pmatrix}=\frac{1}{8}\left(\cos^2\frac{\theta}{2}+3\sin^2\frac{\theta}{2}\right)=\frac{1}{8}+\frac{1}{4}\sin^2\frac{\theta}{2}.
\end{aligned}\tag{21}
$$

ρ_{AB} 的部分转置矩阵为

$$
\sigma_{AB}=\rho_{AB}^{T_B}=\frac{1}{8}\begin{pmatrix}1&0&0&-2\\0&3&0&0\\0&0&3&0\\-2&0&0&1\end{pmatrix},\tag{22}
$$

其特征值为 $\frac{3}{8},\frac{3}{8},\frac{3}{8},-\frac{1}{8}$, 其中有负特征值, 故 σ_{AB} 非半正定, 根据 Peres-Horodeski 判据, ρ_{AB} 不是可分量子态. $\square$