Homework Assignment 2

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November 14, 2021

1. Answer to H1:

   i)
   $|\psi\rangle$ can be taken as $|0\rangle$(considering the line between (0,0,0) and $|\psi\rangle$ to be the z-axis on the Bloch sphere), $|\Phi\rangle$ can be expressed as

   $$|\Phi\rangle = cos\frac{\theta}{2}|0\rangle + e^{i\phi}sin\frac{\theta}{2}|1\rangle$$

   Which leads to

   $$\bar{F} = \langle|\langle\Phi|\psi\rangle|^2\rangle = \langle cos^2\frac{\theta}{2}\rangle = \frac{\int\int cos^2\frac{\theta}{2}sin\theta d\theta d\phi}{\int\int sin\theta d\theta d\phi} = \frac{1}{2}\int_0^{\pi}(\frac{sin\theta}{2} + \frac{sin(2\theta)}{4})d\theta = \frac{1}{2}$$

   ii)
   let $P_\uparrow = |0\rangle\langle 0|$ and$P_\downarrow = |1\rangle\langle 1|$, and $|\psi\rangle$ be

   $$|\psi\rangle = cos\frac{\theta}{2}|0\rangle + e^{i\phi}sin\frac{\theta}{2}|1\rangle$$

   We see that:

   $$\rho = P_\uparrow cos^2\frac{\theta}{2} + P_\downarrow sin^2\frac{\theta}{2}$$

   Then we have the average by integrating out θ and ϕ on Bloch sphere in the expected value $\langle\psi|\rho|\psi\rangle$:

   $$\langle\langle\psi|\rho|\psi\rangle\rangle = \frac{\int\int(cos^4\frac{\theta}{2} + sin^4\frac{\theta}{2})sin\theta d\theta d\phi}{\int\int sin\theta d\theta d\phi} = -\frac{1}{2}\int_0^{\pi}(\frac{1}{2} + \frac{cos^2\theta}{2})dcos\theta = \frac{2}{3}$$

2. Answer to H2:
   Firstly, we do direct sum expand on the vector:

   $$|u_1\rangle = \begin{pmatrix} \sqrt{\frac{2}{3}} \\ 0 \\ a \end{pmatrix}, |u_2\rangle = \begin{pmatrix} -\sqrt{\frac{1}{6}} \\ \frac{\sqrt{2}}{2} \\ b \end{pmatrix}, |u_3\rangle = \begin{pmatrix} -\sqrt{\frac{1}{6}} \\ -\frac{\sqrt{2}}{2} \\ c \end{pmatrix}$$

Then we consider the orthogonality between those vectors, and we have:

$$ab = bc = ac = \frac{1}{3}$$

Let

$$a = b = c = \sqrt{\frac{1}{3}}$$

Secondly, we do the direct product expansion

$$|\psi_1\rangle = \begin{pmatrix} \sqrt{\frac{2}{3}} \\ 0 \\ \sqrt{\frac{1}{3}} \\ 0 \end{pmatrix}, |\psi_2\rangle = \begin{pmatrix} -\sqrt{\frac{1}{6}} \\ \frac{\sqrt{2}}{2} \\ \sqrt{\frac{1}{3}} \\ 0 \end{pmatrix}, |\psi_3\rangle = \begin{pmatrix} -\sqrt{\frac{1}{6}} \\ -\frac{\sqrt{2}}{2} \\ \sqrt{\frac{1}{3}} \\ 0 \end{pmatrix}$$

And we add another vector

$$|\psi\rangle_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

which leads to

$$\begin{pmatrix} \sqrt{\frac{2}{3}} \\ 0 \\ \sqrt{\frac{1}{3}} \\ 0 \end{pmatrix} \begin{pmatrix} -\sqrt{\frac{1}{6}} \\ \frac{\sqrt{2}}{2} \\ \sqrt{\frac{1}{3}} \\ 0 \end{pmatrix} \begin{pmatrix} -\sqrt{\frac{1}{6}} \\ -\frac{\sqrt{2}}{2} \\ \sqrt{\frac{1}{3}} \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

3. Answer to S1:
   i)

$$|\psi\rangle_{AB} = \frac{\sqrt{3}+1}{4}(|00\rangle + |11\rangle) + \frac{\sqrt{3}-1}{4}(|10\rangle + |01\rangle)$$

We have the density matrix:

$$\rho_{AB} = |\psi\rangle_{AB\,AB}\langle\psi|$$

Then we trace off B states(or alternatively ,trace A):

$$\rho_A = Tr_B(\rho_{AB}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

From which we can see that the ρ given in the subject does not match ρ_A, since the unitary operation keeps the eigenvalues unchanged ,we check the eigenvalues of ρ_A to see if those matches the ones of ρ:

$$\lambda_1 = \frac{3}{4}, \lambda_2 = \frac{1}{4}$$

The results tell us those two matrix can indeed be transformed into each other with a unitary operation, which can be constructed using the eigenvectors of ρ_A:

$$U_A = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

ii)

$$|\psi\rangle_{ABC} = \frac{\sqrt{7}}{4}(|000\rangle + |010\rangle) + \frac{1}{4}(|101\rangle - |111\rangle)$$

Likewise, we trace off C qubit(or A or B) to get the reduced density matrix:

$$\rho_{AB} = Tr_C(|\psi\rangle_{ABCABC}\langle\psi|) = \begin{pmatrix} \frac{7}{16} & \frac{7}{16} & 0 & 0 \\ \frac{7}{16} & \frac{7}{16} & 0 & 0 \\ 0 & 0 & \frac{1}{16} & -\frac{1}{16} \\ 0 & 0 & -\frac{1}{16} & \frac{1}{16} \end{pmatrix}$$

And the calculation of eigenvalues show that:

$$\lambda_{\rho_{AB}} = (\frac{3}{8}, \frac{1}{8}, 0, 0), \lambda_\rho = (\frac{13}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16})$$

Which means that we can not get ρ be unitarily operating ρ_{AB}.

4. Answer to S2:
   a)Since the starting state of B is $|0\rangle$, we can get the measurement operators like this:

$$M_k =_B \langle k|U_{AB}|0\rangle_B$$

As a result, for

$$U_1 = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$

And

$$U_2 = \frac{1}{\sqrt{2}}(X \otimes I + Y \otimes X)$$

We have as follows

$$M_1^0 = |0\rangle_A\langle 0|, \quad M_1^1 = |1\rangle_A\langle 1|$$

And

$$M_2^0 = \frac{\sqrt{2}}{2}X, \quad M_2^1 = \frac{\sqrt{2}}{2}Y$$

Hence the measurement may be wrote as

$$\xi_1(\rho_A) = M_1^0\rho_A M_1^{0\dagger} + M_1^1\rho_A M_1^{1\dagger}$$

And

$$\xi_2(\rho_A) = M_2^0\rho_A M_2^{0\dagger} + M_2^1\rho_A M_2^{1\dagger}$$

And we can test the summation

$$\Sigma_{i=0,1}M_k^{i\dagger}M_k^i = I_A \quad for \quad k = 1, 2$$

b)Firstly, we have

$$U = U_1U_2 = \frac{1}{\sqrt{2}}(|0\rangle\langle 1| \otimes I + |1\rangle\langle 0| \otimes X - i|0\rangle\langle 1| \otimes X + i|1\rangle\langle 0| \otimes I)$$

From which we can obtain the operators likewise:

$$M^0 = \frac{1}{\sqrt{2}}(|1\rangle_A\langle 0| - i|0\rangle_A\langle 1|), \quad M^1 = \frac{1}{\sqrt{2}}(|0\rangle_A\langle 1| + i|1\rangle_A\langle 0|$$

And the measurement shall be written as

$$\xi(\rho_A) = M^0\rho_A M^{0\dagger} + M^1\rho_A M^{1\dagger}$$

And likewise, the summation:

$$\Sigma_{i=0,1}M^{i\dagger}M^i = I_A$$

Now we have

$$\xi_1\xi_2(\rho_A) = \xi_1(\frac{1}{2}X\rho X + \frac{1}{2}Y\rho Y) = |0\rangle\langle 1|\rho_A|1\rangle\langle 0| + |1\rangle\langle 0|\rho_A|0\rangle\langle 1|$$

Which clearly differs from $\xi(\rho_A)$.

5. Answer to S3:
Note: When you try to test the linearity of $\xi(\rho)$, you must make sure that $\lambda_1\rho_1 + \lambda_2\rho_2$ is also a density matrix, and satisfies $tr(\lambda_1\rho_1 + \lambda_2\rho_2) = 1$, which means $\lambda_1 + \lambda_2 = 1$.
a)When $d = 2$. Constructing an operation by letting:

$$M_0 = b_0 I \quad M_1 = b_1\sigma_1 \quad M_2 = b_2\sigma_2 \quad M_3 = b_3\sigma_3$$

And expressing ρ as

$$\rho = \frac{1}{2}(I + \vec{n}\vec{\sigma})$$

Considering the relation:

$$\sigma_i\sigma_j\sigma_i = -\sigma_i \quad for \quad i \neq j$$

Hence:

$$\Sigma_{i=0}^3 M_i^\dagger \rho M_i =$$

$$\frac{1}{2}(b_0^2 + b_1^2 + b_2^2 + b_3^2)I + \frac{1}{2}(b_0^2 + b_1^2 - b_2^2 - b_3^2)n_1\sigma_1 +$$

$$\frac{1}{2}(b_0^2 - b_1^2 + b_2^2 - b_3^2)n_2\sigma_2 +$$

$$\frac{1}{2}(b_0^2 - b_1^2 - b_2^2 + b_3^2)n_3\sigma_3$$

$$= \frac{I}{2} + \frac{1-p}{2}\vec{n}\vec{\sigma})$$

Which leads to

$$b_0^2 + b_1^2 + b_2^2 + b_3^2 = 1 \quad b_0^2 + b_1^2 - b_2^2 - b_3^2 = b_0^2 - b_1^2 + b_2^2 - b_3^2 = b_0^2 - b_1^2 - b_2^2 + b_3^2 = 1 - p$$

Hence we have:

$$b_0 = \sqrt{\frac{4-3p}{4}}, b_1 = b_2 = b_3 = \sqrt{\frac{p}{4}}$$

So we have the operation:

$$\xi(\rho) = \frac{4-3p}{4} I\rho I + \frac{p}{4}(\sigma_1\rho\sigma_1 + \sigma_2\rho\sigma_2 + \sigma_3\rho\sigma_3)$$

b)When $d = 3$. We can constructing the operation by using the Gell−Mann Matrices λ_i for $i = 1, 2, ..., 8$, following the same steps, we have

$$\Sigma_i \lambda_i \rho \lambda_i^\dagger = 2I - \frac{2}{3}\rho$$

Then we have

$$\xi(\rho) = \frac{p}{6}\Sigma_{i=1}^{8} \lambda_i\rho\lambda_i + (1 - \frac{8p}{9}) I\rho I$$

Which gives

$$M_i = \sqrt{\frac{p}{6}}, for \quad i = 1, 2, ...8, \quad M_0 = \sqrt{1 - \frac{8p}{9}} I$$

Alternative solutions: Let $M_0 = \sqrt{1-p}I$ and $M_{i,j} = \sqrt{\frac{p}{d}}|i\rangle\langle j|$, we see that

$$M_0\rho M_0 + \Sigma_{i,j=1}^{d} M_{i,j}\rho M_{i,j}^\dagger = (1-p)\rho + \frac{p}{d}\Sigma_{i,j=1}^{d}|i\rangle\langle j|\rho|j\rangle\langle i| =$$

$$(1-p)\rho + \frac{p}{d}\Sigma_{i=1}^{d}|i\rangle\langle i|tr(\rho)$$

$$= (1-p)\rho + \frac{p}{d}I$$

6. Answer to S8:
a)Sufficiency: if each M_i in $(M_i, i = 1, 2, ...m)$ satisfies $M_i M_i^\dagger = M_i^\dagger M_i = I$, from the summation:

$$I = \Sigma_{i=1}^{m} M_i^\dagger M_i = mI$$

We can see that $m = 1$, Then the operation may be written as

$$\xi(\rho) = M\rho M^\dagger$$

Which can be undone by

$$\xi^{-1}(\rho) = M^\dagger \rho M$$

Hence the operation is indeed reversible.

b)necessity: if the operation is reversible, which means that the original density matrix ρ can be retrieved by another operation:

$$\xi_1(\rho) = \Sigma_{i=1}^{n} N_i\rho N_i^\dagger$$

So that

$$\xi_2(\rho) = \xi_1(\xi(\rho)) = \Sigma_{j=1}^{n}\Sigma_{i=1}^{m} N_j M_i \rho M_i^\dagger N_j^\dagger = \rho$$

Where ξ_2 is another operation on ρ with Kraus operator $N_j M_i$ Satisfying

$$\Sigma_{j=1}^{n}\Sigma_{i=1}^{m} M_i^{\dagger} N_j^{\dagger} N_j M_i = \Sigma_{i=1}^{m} M_i^{\dagger} M_i = I$$

That keeps an arbitrary ρ unchanged, so ξ_2 is simply:

$$\xi_2(\rho) = I(\rho) = I\rho I$$

Which means that the Kraus operators $N_j M_i = \lambda_{j,i} I$ with $\lambda_{j,i}$ satisfying

$$\Sigma_{i,j}|\lambda_{j,i}|^2 = 1$$

Since

$$\Sigma_{i,j}|\lambda_{j,i}|^2 I = \Sigma_{i,j} M_i^{\dagger} N_j^{\dagger} N_j M_i = I.$$

From the relation $\Sigma_j N_j^{\dagger} N_j = I$ We have

$$M_a^{\dagger} M_b = \Sigma_j M_a^{\dagger} N_j^{\dagger} N_j M_b = \Sigma_j \lambda_{j,a}^{\star} \lambda_{j,b} I = \gamma_{a,b} I$$

And then directly:

$$M_i^{\dagger} = \gamma_{i,i} M_i^{-1}$$

Also

$$M_i M_i^{\dagger} M_j = \gamma_{i,i} M_j = \gamma_{i,j} M_i$$

From which we can tell all the operators are unitary and alike except a coefficient factor, hence the necessity.

7. Answer to S9:

$$U(\theta, \vec{n}) = cos\frac{\theta}{2} I - isin\frac{\theta}{2}\vec{n}\vec{\sigma}$$

And

$$U_A \otimes U_B = cos^2\frac{\theta}{2} I_A \otimes I_B - isin\frac{\theta}{2}cos\frac{\theta}{2}(\vec{n}\vec{\sigma}_A \otimes I_B + I_A \otimes \vec{n}\vec{\sigma}_B) - sin^2\frac{\theta}{2}(\vec{n}\vec{\sigma}_A) \otimes (\vec{n}\vec{\sigma}_B)$$

Since:

$$cos^2\frac{\theta}{2} I_A \otimes I_B|\psi^-\rangle = cos^2\frac{\theta}{2}|\psi^-\rangle$$

And

$$\sigma_i \otimes \sigma_i|\psi^-\rangle = -|\psi^-\rangle$$

$$(\sigma_i \otimes \sigma_j + \sigma_i \otimes \sigma_j)|\psi^-\rangle = 0 \quad for \quad i \neq j$$

Which leads to

$$(\vec{n}\vec{\sigma}_A) \otimes (\vec{n}\vec{\sigma}_B)|\psi^-\rangle = -|\psi^-\rangle$$

We have that:

$$(\vec{n}\vec{\sigma}_A \otimes I_B + I_A \otimes \vec{n}\vec{\sigma}_B)|\psi^-\rangle =$$
$$\vec{n}\vec{\sigma}_A \otimes I_B|\psi^-\rangle - I_A \otimes \vec{n}\vec{\sigma}_B \vec{n}\vec{\sigma}_A \otimes \vec{n}\vec{\sigma}_B|\psi^-\rangle =$$
$$\vec{n}\vec{\sigma}_A \otimes I_B|\psi^-\rangle - \vec{n}\vec{\sigma}_A \otimes I_B|\psi^-\rangle = 0$$

So what is left:

$$cos^2\frac{\theta}{2}|\psi^-\rangle - sin^2\frac{\theta}{2}(\vec{n}\vec{\sigma}_A) \otimes (\vec{n}\vec{\sigma}_B)|\psi^-\rangle = cos^2\frac{\theta}{2}|\psi^-\rangle + sin^2\frac{\theta}{2}|\psi^-\rangle = |\psi^-\rangle$$

We have used the relation $\sigma_a\sigma_b = \delta_{ab} + i\epsilon_{abc}\sigma_c$ above.