

第 1 题 得分: _____. 计算二元对称信道的信道容量.

解: 对二元对称信道, 设信源发送信号为 $X \in \{0, 1\}$, 经信道传输后信宿接收信号为 $Y \in \{0, 1\}$, 信道如图 1 所示:

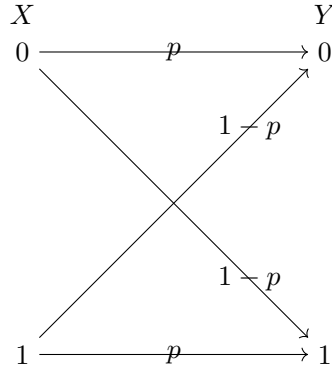


图 1: 二元对称信道

即

$$P(Y = 0 | X = 0) = P(Y = 1 | X = 1) = p, \quad (1)$$

$$P(Y = 1 | X = 0) = P(Y = 0 | X = 1) = 1 - p. \quad (2)$$

X 与 Y 的互信息量为

$$I(X; Y) = H(Y) - H(Y | X), \quad (3)$$

其中 Y 的信息熵为

$$\begin{aligned} H(Y) &= -P(Y = 0) \log_2 P(Y = 0) - P(Y = 1) \log_2 P(Y = 1) \\ &= -P(Y = 0) \log_2 P(Y = 0) - [1 - P(Y = 0)] \log_2 [1 - P(Y = 0)] \\ &\leq -\frac{1}{2} \log_2 \frac{1}{2} - \frac{1}{2} \log_2 \frac{1}{2} = 1, \end{aligned} \quad (4)$$

而 $Y | X$ 的条件熵

$$\begin{aligned} H(Y | X) &= -P(X = 0)[P(Y = 0 | X = 0) \log_2 P(Y = 0 | X = 0) + P(Y = 1 | X = 0) \log_2 P(Y = 1 | X = 0)] \\ &\quad - P(X = 1)[P(Y = 0 | X = 1) \log_2 P(Y = 0 | X = 1) + P(Y = 1 | X = 1) \log_2 P(Y = 1 | X = 1)] \\ &= -P(X = 0)[p \log_2 p + (1 - p) \log_2 (1 - p)] - P(X = 1)[(1 - p) \log_2 (1 - p) + p \log_2 p] \\ &= -p \log_2 p - (1 - p) \log_2 (1 - p). \end{aligned} \quad (5)$$

故二元对称信道的信道容量为

$$C = \max_{P(X=0)} I(X; Y) = 1 + p \log_2 p + (1 - p) \log_2 (1 - p). \quad (6)$$

□

第 2 题 得分: _____. 空间 H 中存在两组正交归一化态 $\{|\psi_i\rangle\}$ 、 $\{|\tilde{\psi}_i\rangle\}$, 则存在么正变换 U , 使得 $U|\psi_i\rangle = |\tilde{\psi}_i\rangle$, 试构造该 U 变换.

解: 构造

$$U = \sum_i |\tilde{\psi}_i\rangle\langle\psi_i|, \quad (7)$$

则 U 满足变换

$$U|\psi_i\rangle = \sum_j |\tilde{\psi}_j\rangle\langle\psi_j|\psi_i\rangle = \sum_j \delta_{ij} |\tilde{\psi}_j\rangle = |\tilde{\psi}_i\rangle, \quad \forall i, \quad (8)$$

且 U 是幺正的:

$$U^\dagger U = UU^\dagger = \left(\sum_i |\psi_i\rangle\langle\tilde{\psi}_i| \right) \left(\sum_j |\tilde{\psi}_j\rangle\langle\psi_j| \right) = \sum_{ij} \delta_{ij} |\psi_i\rangle\langle\psi_j| = \sum_i |\psi_i\rangle\langle\psi_i| = I. \quad (9)$$

□

第 3 题 得分: _____. 空间 H 中存在两组归一化态 $\{|\psi_i\rangle\}$ 、 $\{|\tilde{\psi}_i\rangle\}$, 它们满足: $\forall i, j$, 有 $\langle\psi_i|\psi_j\rangle = \langle\tilde{\psi}_i|\tilde{\psi}_j\rangle$. 请证明, 则存在 U , 使得 $U|\psi_i\rangle = |\tilde{\psi}_i\rangle$, 并构造出该 U 变换.

解: 设

$$U = \sum_{jk} a_{jk} |\tilde{\psi}_j\rangle\langle\psi_k|. \quad (10)$$

要使 $U|\psi_i\rangle = |\tilde{\psi}_i\rangle$, 即需

$$U|\psi_i\rangle = \sum_{jk} a_{jk} |\tilde{\psi}_j\rangle\langle\psi_k|\psi_i\rangle = \sum_j \left(\sum_k a_{jk} \langle\psi_k|\psi_i\rangle \right) |\tilde{\psi}_j\rangle = |\tilde{\psi}_i\rangle, \quad \forall i, \quad (11)$$

$$\implies \sum_k a_{jk} \langle\psi_k|\psi_i\rangle = \delta_{ij}, \quad \forall i, j, \quad (12)$$

$$\implies AB = I \quad (13)$$

其中矩阵

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad (14)$$

$$B = \begin{bmatrix} \langle\psi_1|\psi_1\rangle & \langle\psi_1|\psi_2\rangle & \cdots & \langle\psi_1|\psi_n\rangle \\ \langle\psi_2|\psi_1\rangle & \langle\psi_2|\psi_2\rangle & \cdots & \langle\psi_2|\psi_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\psi_n|\psi_1\rangle & \langle\psi_n|\psi_2\rangle & \cdots & \langle\psi_n|\psi_n\rangle \end{bmatrix}. \quad (15)$$

因此我们只需由上式构造矩阵 B , 再计算

$$A = B^{-1}, \quad (16)$$

然后构造

$$U = \sum_{jk} a_{jk} |\tilde{\psi}_j\rangle\langle\psi_k| \quad (17)$$

即可满足 $U|\psi_i\rangle = |\tilde{\psi}_i\rangle$. □

第 4 题 得分: _____. 对两比特态 $|\phi\rangle = \frac{1}{\sqrt{2}}|0\rangle_A \left(\frac{1}{2}|0\rangle_B + \frac{\sqrt{3}}{2}|1\rangle_B\right) + \frac{1}{\sqrt{2}}|1\rangle_A \left(\frac{\sqrt{3}}{2}|0\rangle_B + \frac{1}{2}|1\rangle_B\right)$,

i) 求约化密度矩阵 ρ_A, ρ_B ;

ii) 求 $|\phi\rangle$ 的 Schmidt 分解形式.

解: 1) 复合系统的密度矩阵为

$$\begin{aligned}\rho_{AB} &= |\phi\rangle\langle\phi| \\ &= \left[\frac{1}{\sqrt{2}}|0\rangle_A \left(\frac{1}{2}|0\rangle_B + \frac{\sqrt{3}}{2}|1\rangle_B\right) + \frac{1}{\sqrt{2}}|1\rangle_A \left(\frac{\sqrt{3}}{2}|0\rangle_B + \frac{1}{2}|1\rangle_B\right) \right] \left[\frac{1}{\sqrt{2}}\langle 0|_A \left(\frac{1}{2}\langle 0|_B + \frac{\sqrt{3}}{2}\langle 1|_B\right) + \frac{1}{\sqrt{2}}\langle 1|_A \left(\frac{\sqrt{3}}{2}\langle 0|_B + \frac{1}{2}\langle 1|_B\right) \right] \\ &= \frac{1}{8}|0\rangle_A\langle 0|_B\langle 0|_A\langle 0|_B + \frac{\sqrt{3}}{8}|0\rangle_A\langle 0|_B\langle 0|_A\langle 1|_B + \frac{\sqrt{3}}{8}|0\rangle_A\langle 0|_B\langle 1|_A\langle 0|_B + \frac{1}{8}|0\rangle_A\langle 0|_B\langle 1|_A\langle 1|_B \\ &\quad + \frac{\sqrt{3}}{8}|0\rangle_A\langle 1|_B\langle 0|_A\langle 0|_B + \frac{3}{8}|0\rangle_A\langle 1|_B\langle 0|_A\langle 1|_B + \frac{3}{8}|0\rangle_A\langle 1|_B\langle 1|_A\langle 0|_B + \frac{\sqrt{3}}{8}|0\rangle_A\langle 1|_B\langle 1|_A\langle 1|_B \\ &\quad + \frac{\sqrt{3}}{8}|1\rangle_A\langle 0|_B\langle 0|_A\langle 0|_B + \frac{3}{8}|1\rangle_A\langle 0|_B\langle 0|_A\langle 1|_B + \frac{3}{8}|1\rangle_A\langle 0|_B\langle 1|_A\langle 0|_B + \frac{\sqrt{3}}{8}|1\rangle_A\langle 0|_B\langle 1|_A\langle 1|_B \\ &\quad + \frac{1}{8}|1\rangle_A\langle 1|_B\langle 0|_A\langle 0|_B + \frac{\sqrt{3}}{8}|1\rangle_A\langle 1|_B\langle 0|_A\langle 1|_B + \frac{\sqrt{3}}{8}|1\rangle_A\langle 1|_B\langle 1|_A\langle 0|_B + \frac{1}{8}|1\rangle_A\langle 1|_B\langle 1|_A\langle 1|_B.\end{aligned}\quad (18)$$

约化密度矩阵

$$\rho_A = \text{Tr}_B[\rho_{AB}] = \sum_{i=0,1} \langle i|_B(|\phi\rangle\langle\phi|)|i\rangle_B = \frac{1}{2}|0\rangle_A\langle 0|_A + \frac{\sqrt{3}}{4}|0\rangle_A\langle 1|_A + \frac{\sqrt{3}}{4}|1\rangle_A\langle 0|_A + \frac{1}{2}|1\rangle_A\langle 1|_A, \quad (19)$$

$$\rho_B = \text{Tr}_A[\rho_{AB}] = \sum_{i=0,1} \langle i|_A(|\phi\rangle\langle\phi|)|i\rangle_A = \frac{1}{2}|0\rangle_B\langle 0|_B + \frac{\sqrt{3}}{4}|0\rangle_B\langle 1|_B + \frac{\sqrt{3}}{4}|1\rangle_B\langle 0|_B + \frac{1}{2}|1\rangle_B\langle 1|_B. \quad (20)$$

(2) $|\phi\rangle$ 可表为

$$|\phi\rangle = \sum_{m,n \in \{0,1\}} a_{mn}|m\rangle_A|n\rangle_B, \quad (21)$$

其中矩阵

$$a = \begin{bmatrix} \frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{4} \\ \frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{4} \end{bmatrix}. \quad (22)$$

对 a 做奇异值分解, 得

$$a = u d v, \quad (23)$$

其中

$$u = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}, \quad (24)$$

$$d = \begin{bmatrix} \frac{\sqrt{2}-\sqrt{6}}{4} & 0 \\ 0 & \frac{\sqrt{2}+\sqrt{6}}{4} \end{bmatrix}, \quad (25)$$

$$v = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}. \quad (26)$$

令子系统 A 和子系统 B 的正交基

$$|\psi_i\rangle_A = \sum_{j=0,1} u_{ji}|j\rangle, \quad (27)$$

$$|\psi_i\rangle_B = \sum_{k=0,1} v_{ik} |k\rangle, \quad (28)$$

以及

$$\lambda_i = d_{ii}, \quad (29)$$

即

$$|\psi_1\rangle_A = \frac{\sqrt{2}}{2} |0\rangle_A - \frac{\sqrt{2}}{2} |1\rangle_A, \quad |\psi_2\rangle_A = \frac{\sqrt{2}}{2} |0\rangle_A + \frac{\sqrt{2}}{2} |1\rangle_A, \quad (30)$$

$$|\psi_1\rangle_B = \frac{\sqrt{2}}{2} |0\rangle_B - \frac{\sqrt{2}}{2} |1\rangle_B, \quad |\psi_2\rangle_B = \frac{\sqrt{2}}{2} |0\rangle_B + \frac{\sqrt{2}}{2} |1\rangle_B, \quad (31)$$

$$\lambda_1 = \frac{\sqrt{2} - \sqrt{6}}{4}, \quad \lambda_2 = \frac{\sqrt{2} + \sqrt{6}}{4}, \quad (32)$$

从而得到 $|\phi\rangle$ 的 Schmidt 分解式为

$$\begin{aligned} |\phi\rangle &= \sum_{i=1}^2 \lambda_i |\psi_i\rangle_A |\psi_i\rangle_B \\ &= \frac{\sqrt{2} - \sqrt{6}}{4} \left(\frac{\sqrt{2}}{2} |0\rangle_A - \frac{\sqrt{2}}{2} |1\rangle_A \right) \left(\frac{\sqrt{2}}{2} |0\rangle_B - \frac{\sqrt{2}}{2} |1\rangle_B \right) + \frac{\sqrt{2} + \sqrt{6}}{4} \left(\frac{\sqrt{2}}{2} |0\rangle_A + \frac{\sqrt{2}}{2} |1\rangle_A \right) \left(\frac{\sqrt{2}}{2} |0\rangle_B + \frac{\sqrt{2}}{2} |1\rangle_B \right). \end{aligned} \quad (33)$$

□

第 5 题 得分: _____. 对三粒子系统纯态 $|\phi_{ABC}\rangle$, 在空间 $H_A \otimes H_B \otimes H_C$ 中是否存在 H_A, H_B, H_C 中的正交基, 使得 $|\phi_{ABC}\rangle = \sum_i \sqrt{p_i} |i_A\rangle \otimes |i_B\rangle \otimes |i_C\rangle$ 一定成立? 给出理由.

解: 不一定. 理由如下:

对三粒子系统纯态 $|\phi_{ABC}\rangle$, 利用 Schmidt 分解总是可以得到

$$|\phi_{ABC}\rangle = \sum_i \sqrt{p_i} |i_A\rangle \otimes |i_{BC}\rangle. \quad (34)$$

对于 $|i_{BC}\rangle$, 利用 Schmidt 分解得

$$|i_{BC}\rangle = \sum_j \sqrt{p_{ij}} |j_{i,B}\rangle \otimes |j_{i,C}\rangle, \quad (35)$$

从而

$$|\phi_{ABC}\rangle = \sum_i \sqrt{p_i} |i_A\rangle \otimes \sum_j \sqrt{p_{ij}} |j_{i,B}\rangle \otimes |j_{i,C}\rangle \quad (36)$$

因此, $|\phi_{ABC}\rangle$ 无法写成 $|\phi_{ABC}\rangle = \sum_i \sqrt{p_i} |i_A\rangle \otimes |i_B\rangle \otimes |i_C\rangle$ 的形式.

只有当对任意 i , $|i_{BC}\rangle$ 的分解式 (式 35) 中的求和都只有一项, 才能保证 $|\phi_{ABC}\rangle = \sum_i \sqrt{p_i} |i_A\rangle \otimes |i_B\rangle \otimes |i_C\rangle$ 的分解形式成立. □