第二次作业

截止日期：2021. 11. 3（周三）

姓名 : 陈 稼 霖
学号 : SA21038052
成绩 : __________

第 6 题 得分: _________. 设 $|\psi\rangle$ 为量子比特态, 在 Bloch 球面上均匀随机分布.

i) 随机地猜想一个态 $|\phi\rangle$, 求猜测态相对于 $|\psi\rangle$ 的平均保真度 $\bar{F} = \langle|\langle\phi|\psi\rangle|^2\rangle$.

ii) 对此量子态做正交测量 $\{P_\uparrow, P_\downarrow\}$, $P_\uparrow + P_\downarrow = I$. 测量后系统被制备到: $\rho = p_\uparrow\langle\psi|P_\uparrow|\psi\rangle + p_\downarrow\langle\psi|P_\downarrow|\psi\rangle$, 求 ρ 与原来的态 $|\psi\rangle$ 的平均保真度. ($\bar{F} = \langle\langle\psi|\rho|\psi\rangle\rangle$)

解: i) 设

$$|\psi\rangle = \cos\frac{\theta_1}{2}|0\rangle + e^{i\varphi_1}\sin\frac{\theta_1}{2}|1\rangle, \tag{1}$$
$$|\phi\rangle = \cos\frac{\theta_2}{2}|0\rangle + e^{i\varphi_2}\sin\frac{\theta_2}{2}|1\rangle. \tag{2}$$

态 $|\phi\rangle$ 相对于态 $|\psi\rangle$ 的保真度为

$$\begin{aligned}
F = |\langle\phi|\psi\rangle|^2 =& \left|\left(\cos\frac{\theta_2}{2}\langle 0| + e^{-i\varphi_2}\sin\frac{\theta_2}{2}\right)\left(\cos\frac{\theta_1}{2}|0\rangle + e^{i\varphi_1}\sin\frac{\theta_1}{2}\right)\right|^2\\
=&\left|\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + e^{i(\varphi_1-\varphi_2)}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right|^2\\
=&\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} + \sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} + 2\cos(\varphi_1-\varphi_2)\sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_2}{2}\\
=&\left(\frac{1+\cos\theta_1}{2}\right)\left(\frac{1+\cos\theta_2}{2}\right) + \left(\frac{1-\cos\theta_1}{2}\right)\left(\frac{1-\cos\theta_2}{2}\right) + 2\cos(\varphi_1-\varphi_2)\frac{1}{4}\sin\theta_1\sin\theta_2\\
=&\frac{1}{2}(1+\cos\theta_1\cos\theta_2) + \frac{1}{2}\cos(\varphi_1-\varphi_2)\sin\theta_1\sin\theta_2.
\end{aligned}$$

平均保真度为

$$\begin{aligned}
\bar{F} =& \langle|\langle\phi|\psi\rangle|^2\rangle = \frac{1}{2}\int_0^\pi \sin\theta_2\,\mathrm{d}\theta_2\,\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\varphi_2\ |\langle\phi|\psi\rangle|^2\\
=&\frac{1}{2}\int_0^\pi \sin\theta_2\,\mathrm{d}\theta_2\,\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\varphi_2\,\frac{1}{2}(1+\cos\theta_1\cos\theta_2)\\
=&\frac{1}{2}\int_0^\pi \sin\theta_2\,\mathrm{d}\theta_2\,\frac{1}{2}(1+\cos\theta_1\cos\theta_2)\\
=&\frac{1}{2}.
\end{aligned}\tag{3}$$

ii) 算符 $P_\uparrow$ 和 $P_\downarrow$ 可表为

$$P_\uparrow = |0\rangle\langle 0|, \tag{4}$$
$$P_\downarrow = |1\rangle\langle 1|. \tag{5}$$

测量后系统被制备到

$$\begin{aligned}
\rho =& P_\uparrow\langle\psi|P_\uparrow|\psi\rangle + P_\downarrow\langle\psi|P_\downarrow|\psi\rangle\\
=& |0\rangle\langle 0|\left(\cos\frac{\theta_1}{2}\langle 0| + e^{-i\varphi_1}\sin\frac{\theta_1}{2}\langle 1|\right)|0\rangle\langle 0|\left(\cos\frac{\theta_1}{2}|0\rangle + e^{i\varphi_1}\sin\frac{\theta_1}{2}|1\rangle\right)\\
&+ |1\rangle\langle 1|\left(\cos\frac{\theta_1}{2}\langle 0| + e^{-i\varphi_1}\sin\frac{\theta_1}{2}\langle 1|\right)|1\rangle\langle 1|\left(\cos\frac{\theta_1}{2}|0\rangle + e^{i\varphi_1}\sin\frac{\theta_1}{2}|1\rangle\right)\\
=& \cos^2\frac{\theta_1}{2}|0\rangle\langle 0| + \sin^2\frac{\theta_1}{2}|1\rangle\langle 1|.
\end{aligned}\tag{6}$$

测量后的密度矩阵与原来的态 $|\psi\rangle$ 的保真度为

$$\begin{aligned}\bar{F}=&\langle\psi|\rho|\psi\rangle=\left(\cos\frac{\theta_1}{2}\langle 0|+e^{-i\varphi_1}\sin\frac{\theta_1}{2}\langle 1|\right)\left(\cos^2\frac{\theta_1}{2}|0\rangle\langle 0|+\sin^2\frac{\theta_1}{2}|1\rangle\langle 1|\right)\left(\cos\frac{\theta_1}{2}|0\rangle+e^{i\varphi_1}\sin\frac{\theta_1}{2}|1\rangle\right)\\=&\cos^4\frac{\theta_1}{2}+\sin^4\frac{\theta_1}{2}.\end{aligned}\tag{7}$$

平均保证度为

$$\begin{aligned}\bar{F}=&\langle\langle\psi|\rho|\psi\rangle\rangle=\frac{1}{2}\int_0^\pi\sin\theta_1\,\mathrm{d}\theta\,\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\varphi\,\langle\psi|\rho|\psi\rangle \qquad (8)\\=&\frac{1}{2}\int_0^\pi\sin\theta_1\,\mathrm{d}\theta_1\,\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\varphi\,\left(\cos^4\frac{\theta_1}{2}+\sin^4\frac{\theta_1}{2}\right)\\=&\frac{1}{2}\int_0^\pi\sin\theta_1\,\mathrm{d}\theta_1\,\left(\cos^4\frac{\theta_1}{2}+\sin^4\frac{\theta_1}{2}\right)\\=&\frac{1}{2}\int_0^\pi\sin\theta_1\,\mathrm{d}\theta_1\,\left[\left(\frac{1+\cos\theta_1}{2}\right)^2+\left(\frac{1-\cos\theta_1}{2}\right)^2\right]\\=&\frac{1}{2}\int_0^\pi\sin\theta_1\,\mathrm{d}\theta_1\,\frac{1}{2}(1+\cos^2\theta_1)\\=&\frac{1}{4}\int_{-1}^1\mathrm{d}(\cos\theta_1)\,(1+2\cos^2\theta_1)\\=&\frac{2}{3}. \qquad (9)\end{aligned}$$

□

第 7 题 得分: ________. $|\psi_1\rangle=|0\rangle$, $|\psi_2\rangle=-\frac{1}{2}|0\rangle+\frac{\sqrt{3}}{2}|1\rangle$, $|\psi_3\rangle=-\frac{1}{2}|0\rangle-\frac{\sqrt{3}}{2}|1\rangle$. 现令 $F_i=\frac{2}{3}|\psi_i\rangle\langle\psi_i|$, 则 $\{F_a\}_{a=1,2,3}$ 构成二维空间中的 POVM. 现引入一个辅助的 qubit, 试在拓展空间中实施一个正交测量, 从而实现此 POVM.

解: 记欲实现 POVM 的子系统为 A, 引入辅助子系统 B, 本征基为 $\{|0\rangle_B,|1\rangle_B\}$, 令 $\rho_B=|0\rangle_B\langle 0|$, 则拓展空间中的系统状态为

$$\rho_{AB}=\rho_A\otimes|0\rangle_B\langle 0|.\tag{10}$$

首先将该二维空间中的 POVM 拓展成三维空间中的正交测量, 即:

$$|\psi_1\rangle=|0\rangle=\begin{pmatrix}1\\0\end{pmatrix}\qquad\longrightarrow\qquad|u_1\rangle=\begin{pmatrix}\sqrt{\frac{2}{3}}\\0\\\sqrt{\frac{1}{3}}\end{pmatrix},\tag{11}$$

$$|\psi_2\rangle=-\frac{1}{2}|0\rangle+\frac{\sqrt{3}}{2}|1\rangle=\begin{pmatrix}-\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix}\qquad\longrightarrow\qquad|u_2\rangle=\begin{pmatrix}-\frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{3}}\end{pmatrix},\tag{12}$$

$$|\psi_3\rangle=-\frac{1}{2}|0\rangle-\frac{\sqrt{3}}{2}|1\rangle=\begin{pmatrix}-\frac{1}{2}\\-\frac{\sqrt{3}}{2}\end{pmatrix}\qquad\longrightarrow\qquad|u_3\rangle=\begin{pmatrix}-\frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{3}}\end{pmatrix},\tag{13}$$

再将三维空间中的正交测量拓展至四维空间:

$$\begin{pmatrix}|u_1\rangle & |u_2\rangle & |u_3\rangle\end{pmatrix}=\begin{pmatrix}\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}}\\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{pmatrix}\longrightarrow\begin{pmatrix}|\Phi_1\rangle_{AB} & |\Phi_2\rangle_{AB} & |\Phi_3\rangle_{AB} & |\Phi_4\rangle_{AB}\end{pmatrix}=\begin{pmatrix}\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0\\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\tag{14}$$

即

$$|\Phi_1\rangle_{AB}=\sqrt{\frac{2}{3}}|0\rangle_A|0\rangle_B+\sqrt{\frac{1}{3}}|0\rangle_A|1\rangle_A,\tag{15}$$

$$|\Phi_2\rangle_{AB}=-\frac{1}{\sqrt{6}}|0\rangle_A|0\rangle_B+\frac{1}{\sqrt{2}}|1\rangle_A|0\rangle_B+\frac{1}{\sqrt{3}}|0\rangle_A|1\rangle_B,\tag{16}$$

$$|\Phi_3\rangle_{AB}=-\frac{1}{\sqrt{6}}|0\rangle_A|0\rangle_B-\frac{1}{\sqrt{2}}|1\rangle_A|0\rangle_B+\frac{1}{\sqrt{3}}|0\rangle_A|1\rangle_B,\tag{17}$$

$$|\Phi_4\rangle_{AB}=|1\rangle_A|1\rangle_B.\tag{18}$$

拓展空间中的正交测量为 $\{E_a=|\Phi_a\rangle_{AB}\langle\Phi_a|\}_{a=1,2,3,4}$.

我们可以来稍微验证一下, 首先 $\{|\Phi_a\rangle_{AB}\}$ 构成拓展空间 $H_A\otimes H_B$ 中的一组正交归一基, 故 $\{E_a\}_{a=1,2,3,4}$ 是拓展空间 $H_A\otimes H_B$ 中的正交测量; 其次, 易证

$$\mathrm{Tr}(E_a\rho)=\mathrm{Tr}_A(F_a\rho_A),\quad a=1,2,3,\tag{19}$$

因此, 在拓展空间中实施正交测量 $\{E_a\}_{a=1,2,3}$, 在子空间 H_A 中等价于实施 POVM $\{F_a\}_{a=1,2,3}$. □

第 1 补充习 题 得分: __________. 判定下列组合中, 纯态是否是相应混态的纯化态. 如果是, 求出其对应纯态的 Schmidt 分解形式; 如果不是, 是否存在单方的局域幺正操作, 将其变换成到相应混合量子态的纯化态?

(a) $\{\rho=\frac{3}{4}|0\rangle\langle 0|+\frac{1}{4}|1\rangle\langle 1|,|\psi\rangle=\frac{\sqrt{3}+1}{4}(|00\rangle+|11\rangle)+\frac{\sqrt{3}-1}{4}(|01\rangle+|10\rangle)\}$

(b) $\{\rho=\frac{3}{4}|\phi^+\rangle\langle\phi^+|+\frac{1}{16}I\otimes I,|\psi\rangle=\frac{\sqrt{7}}{4}(|000\rangle+|010\rangle)+\frac{1}{4}(|101\rangle-|111\rangle)\}$

解: (a) 将复合系统纯态中的两个 qubit 依次标号为 A, B. 复合系统纯态对应的密度矩阵为

$$\rho_{AB}=|\psi\rangle\langle\psi|=\left[\frac{\sqrt{3}+1}{4}(|00\rangle+|11\rangle)+\frac{\sqrt{3}-1}{4}(|01\rangle+|10\rangle)\right]\left[\frac{\sqrt{3}+1}{4}(\langle 00|+\langle 11|)+\frac{\sqrt{3}-1}{4}(\langle 01|+\langle 10|)\right]\tag{20}$$

对其关于 A 求偏迹得

$$\begin{aligned}&\mathrm{Tr}_A(\rho_{AB})=\mathrm{Tr}_A(|\psi\rangle\langle\psi|)\\&=\sum_{i=0}^{1}{}_A\langle i|\left[\frac{\sqrt{3}+1}{4}(|00\rangle+|11\rangle)+\frac{\sqrt{3}-1}{4}(|01\rangle+|10\rangle)\right]\left[\frac{\sqrt{3}+1}{4}(\langle 00|+\langle 11|)+\frac{\sqrt{3}-1}{4}(\langle 01|+\langle 10|)\right]|i\rangle_A\\&=\frac{1}{2}|0\rangle\langle 0|+\frac{1}{4}|0\rangle\langle 1|+\frac{1}{4}|1\rangle\langle 0|+\frac{1}{2}|1\rangle\langle 1|.\end{aligned}\tag{21}$$

关于 B 求偏迹得

$$\begin{aligned}&\mathrm{Tr}_B(\rho_{AB})=\mathrm{Tr}_B(|\psi\rangle\langle\psi|)\\&=\sum_{i=0}^{1}{}_B\langle i|\left[\frac{\sqrt{3}+1}{4}(|00\rangle+|11\rangle)+\frac{\sqrt{3}-1}{4}(|01\rangle+|10\rangle)\right]\left[\frac{\sqrt{3}+1}{4}(\langle 00|+\langle 11|)+\frac{\sqrt{3}-1}{4}(\langle 01|+\langle 10|)\right]|i\rangle_B\end{aligned}$$

$$=\frac{1}{2}|0\rangle\langle 0|+\frac{1}{4}|0\rangle\langle 1|+\frac{1}{4}|1\rangle\langle 0|+\frac{1}{2}|1\rangle\langle 1|, \tag{22}$$

$\mathrm{Tr}_A(\rho_{AB})\neq\rho$, $\mathrm{Tr}_B(\rho_{AB})\neq\rho$, 故 $|\psi\rangle$ 不是 ρ 的纯化态.

设对 A 的局域幺正操作 $U_A\otimes I_B$, 将其作用于纯态 $|\psi\rangle$ 上后, 再对 B 求偏迹, 有

$$\begin{aligned}&\mathrm{Tr}_B((U_A\otimes I_B)\rho_{AB}(U_A\otimes I_B)^\dagger)\\=&\mathrm{Tr}_B((U_A\otimes I_B)\rho_{AB}(U_A^{-1}\otimes I_B) \qquad (23)\\=&U_A\,\mathrm{Tr}_B(\rho_{AB})U_A^{-1}. \qquad (24)\end{aligned}$$

注意到 ρ 为对角阵而 $\mathrm{Tr}_B(\rho_{AB})$ 具有与 ρ 相同的本征值: $\frac{3}{4}$, $\frac{1}{4}$, $\mathrm{Tr}_B(\rho_{AB})$ 的本征矢为 $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$, $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$, 故可由令

$$U_A=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}=H_A, \tag{25}$$

使得 $\mathrm{Tr}_B\left((U_A\otimes I_B)\rho_{AB}(U_A\otimes I_B)^{-1}\right)=\rho$. 因此, 存在单方的局域幺正操作 $H_A\otimes I_B=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\times\begin{pmatrix}1&0\\0&1\end{pmatrix}$, 可以将纯态 $|\psi\rangle$ 变换成混态 ρ 的纯化态.

(b) 将复合系统中的三个 qubit 依次标号为 A, B, C. 复合系统纯态对应的密度矩阵为

$$\rho_{ABC}=|\psi\rangle\langle\psi|=\left[\frac{\sqrt{7}}{4}(|000\rangle+|010\rangle)+\frac{1}{4}(|101\rangle-|111\rangle)\right]\left[\frac{\sqrt{7}}{4}(\langle 000|+\langle 010|)+\frac{1}{4}(\langle 101|-\langle 111|)\right], \tag{26}$$

对其关于 A 求偏迹得

$$\begin{aligned}&\mathrm{Tr}_A(\rho_{ABC})=\mathrm{Tr}_A(|\psi\rangle\langle\psi|)\\=&\sum_{i=0}^{1}{}_A\langle i|\left[\frac{\sqrt{7}}{4}(|000\rangle+|010\rangle)+\frac{1}{4}(|101\rangle-|111\rangle)\right]\left[\frac{\sqrt{7}}{4}(\langle 000|+\langle 010|)+\frac{1}{4}(\langle 101|-\langle 111|)\right]|i\rangle_A\\=&\frac{1}{16}\begin{pmatrix}7&0&7&0\\0&1&0&-1\\7&0&7&0\\0&-1&0&1\end{pmatrix}. \qquad (27)\end{aligned}$$

关于 B 求偏迹得

$$\begin{aligned}&\mathrm{Tr}_B(\rho_{ABC})=\mathrm{Tr}_B(|\psi\rangle\langle\psi|)\\=&\sum_{i=0}^{1}{}_B\langle i|\left[\frac{\sqrt{7}}{4}(|000\rangle+|010\rangle)+\frac{1}{4}(|101\rangle-|111\rangle)\right]\left[\frac{\sqrt{7}}{4}(\langle 000|+\langle 010|)+\frac{1}{4}(\langle 101|-\langle 111|)\right]|i\rangle_B\\=&\frac{1}{16}\begin{pmatrix}14&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&2\end{pmatrix}. \qquad (28)\end{aligned}$$

关于 C 求偏迹得

$$\mathrm{Tr}_C(\rho_{ABC})=\mathrm{Tr}_C(|\psi\rangle\langle\psi|)$$

$$
\begin{aligned}
&=\sum_{i=0}^{1}{}_C\langle i|\left[\frac{\sqrt{7}}{4}(|000\rangle+|010\rangle)+\frac{1}{4}(|101\rangle-|111\rangle)\right]\left[\frac{\sqrt{7}}{4}(\langle 000|+\langle 010|)+\frac{1}{4}(\langle 101|-\langle 111|)\right]|i\rangle_C\\
&=\frac{1}{16}\begin{pmatrix}7&7&0&0\\7&7&0&0\\0&0&1&-1\\0&0&-1&1\end{pmatrix}.
\end{aligned}\tag{29}
$$

而题设中给出的混合态密度矩阵为

$$
\begin{aligned}
\rho=&\frac{3}{4}|\phi^+\rangle\langle\phi^+|+\frac{1}{16}I\otimes I=\frac{3}{4}\frac{1}{\sqrt{2}}(|0\rangle|0\rangle+|1\rangle|1\rangle)\frac{1}{\sqrt{2}}(\langle 0|\langle 0|+\langle 1|\langle 1|)+\frac{1}{16}I\otimes I\\
=&\frac{3}{8}\begin{pmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix}+\frac{1}{16}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=\frac{1}{16}\begin{pmatrix}7&0&0&6\\0&1&0&0\\0&0&1&0\\6&0&0&7\end{pmatrix}.
\end{aligned}\tag{30}
$$

$\mathrm{Tr}_A(\rho_{ABC})\neq\rho$, $\mathrm{Tr}_B(\rho_{ABC})\neq\rho$, $\mathrm{Tr}_C(\rho_{ABC})\neq\rho$, 故 $|\psi\rangle$ <u>不是</u> ρ 的纯化态.

由于 $\mathrm{Tr}_A(\rho_{ABC})$, $\mathrm{Tr}_B(\rho_{ABC})$, $\mathrm{Tr}_C(\rho_{ABC})$ 的本征值均为 $\frac{7}{8},\frac{1}{8},0,0$, 不同于与 ρ 的本征值 $\frac{13}{16},\frac{1}{16},\frac{1}{16},\frac{1}{16}$, 故<u>不存在</u>单方的局域幺正操作, 将纯态 $|\psi\rangle$ 变换成到相应混合量子态 ρ 的纯化态.

□

第 2 补充习 题 得分: _________. 现有一个主系统 A 和一个辅助系统 B 组成的联合量子比特系统 $H_A\otimes H_B$, 分别作用下面的联合幺正操作: $U_1=|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes X$, $U_2=\frac{1}{\sqrt{2}}(X\otimes I+Y\otimes X)$, 其中 X,Y,Z 分别对应三个泡利矩阵, 假定辅助系统的初始态为 $|0\rangle$,

a) 试分别写出 U_1 和 U_2 在主系统中的算符和表示;

b) 如果考虑联合作用 $U=U_1U_2$, 取同样的辅助系统的初始态为 $|0\rangle$, 写出其算符和形式; 并验证该算符和是否对应 U_1 和 U_2 各自对应超算符 ξ_1 和 ξ_2 的联合 $\xi=\xi_1\xi_2$.

解: a) U_1 的 Kraus 算符为

$$
M_0^{(1)}={}_B\langle 0|U_1|0\rangle_B=|0\rangle\langle 0|,\tag{31}
$$
$$
M_1^{(1)}={}_B\langle 1|U_1|0\rangle_B=|1\rangle\langle 1|.\tag{32}
$$

U_1 在主系统中的算符和表示为

$$
M_0^{(1)\dagger}M_0^{(1)}+M_0^{(2)\dagger}M_0^{(2)}=(|0\rangle\langle 0|)(|0\rangle\langle 0|)+(|1\rangle\langle 1|)(|1\rangle\langle 1|)=|0\rangle\langle 0|+|1\rangle\langle 1|=I_A.\tag{33}
$$

U_2 的 Kraus 算符为

$$
M_0^{(2)}={}_B\langle 0|U_2|0\rangle_B=\frac{1}{\sqrt{2}}X,\tag{34}
$$
$$
M_1^{(2)}={}_B\langle 1|U_2|0\rangle_B=\frac{1}{\sqrt{2}}Y.\tag{35}
$$

U_2 在主系统中的算符和表示为

$$
M_0^{(2)\dagger}M_0^{(2)}+M_1^{(2)\dagger}M_1^{(2)}=\frac{1}{2}(XX+YY)=I_A.\tag{36}
$$

b) 复合变换

$$\begin{aligned}U =&U_1U_2 = (|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes X)\frac{1}{\sqrt{2}}(X\otimes I + Y\otimes X)\\ =&\frac{1}{\sqrt{2}}(|0\rangle\langle 1|\otimes I - i|0\rangle\langle 1|\otimes X + |1\rangle\langle 0|\otimes X + i|1\rangle\langle 0|\otimes I)\end{aligned}\tag{37}$$

的 Kraus 算符为

$$M_0 ={}_B\langle 0|U|0\rangle_B = \frac{1}{\sqrt{2}}(|0\rangle\langle 1| + i|1\rangle\langle 0|),\tag{38}$$

$$M_1 ={}_B\langle 1|U|0\rangle_B = \frac{1}{\sqrt{2}}(|1\rangle\langle 0| - i|0\rangle\langle 1|).\tag{39}$$

U 的算符和表示为

$$\begin{aligned}M_0^\dagger M_0 + M_1^\dagger M_1 =&\frac{1}{2}(|1\rangle\langle 0| - i|0\rangle\langle 1|)(|0\rangle\langle 1| + i|1\rangle\langle 0|) + \frac{1}{2}(|0\rangle\langle 1| + i|1\rangle\langle 0|)(|1\rangle\langle 0| - i|0\rangle\langle 1|)\\ =&|0\rangle\langle 0| + |1\rangle\langle 1| = I_A.\end{aligned}\tag{40}$$

联合超算符 $\xi = \xi_1\xi_2$ 作用在主系统的密度矩阵上:

$$\begin{aligned}\$(\rho_A) =&\$_1(\$_2(\rho_A)) = \sum_{i=0}^{1} M_i^{(1)}\left(\sum_{j=0}^{1} M_j^{(2)}\rho_A M_j^{(2)\dagger}\right)M_i^{(1)\dagger}\\ =&\frac{1}{2}(|0\rangle\langle 0|)(X\rho_A X + Y\rho_A Y)(|0\rangle\langle 0|) + \frac{1}{2}(|1\rangle\langle 1|)(X\rho_A X + Y\rho_A Y)(|1\rangle\langle 1|)\\ =&|1\rangle\langle 0|\rho_A|0\rangle\langle 1| + |0\rangle\langle 1|\rho_A|1\rangle\langle 0|,\end{aligned}\tag{41}$$

其算符和表示为

$$(|1\rangle\langle 0|)(|0\rangle\langle 1|) + (|0\rangle\langle 1|)(|1\rangle\langle 0|) = |1\rangle\langle 1| + |0\rangle\langle 0| = I_A.\tag{42}$$

$U = U_1U_2$ 对主系统密度矩阵的作用不同于联合超算符 $\$ = \$_1\$_2$ 对主系统的作用, 故算符 $U = U_1U_2$ 并不对应联合超算符 $\$ = \$_1\$_2$.

□

第 3 补充习 题 得分: ________. 假定有一个超算符演化满足 $\xi(\rho) = \frac{p}{d}I + (1-p)\rho$, 其中 p 为小于等于 1 的实数, d 表示系统的维数, 试在 $d = 2$ 时, 构造出该演化的算符和形式. 如果 $d = 3$, 该如何构造?

解: 当 $d = 2$ 时, 该超算符演化可表为

$$\begin{aligned}\$(\rho) =&\frac{p}{d}I + (1-p)\rho\\ =&\frac{p}{2}[|0\rangle\langle 0| + |1\rangle\langle 1|] + (\sqrt{1-p}I)\rho(\sqrt{1-p}I)\\ =&\frac{p}{2}[|0\rangle 1\langle 0| + |1\rangle 1\langle 1|] + (\sqrt{1-p}I)\rho(\sqrt{1-p}I)\\ =&\frac{p}{2}[|0\rangle(\langle 0|\rho|0\rangle + \langle 1|\rho|1\rangle)\langle 0| + |1\rangle(\langle 0|\rho|0\rangle + \langle 1|\rho|1\rangle)\langle 1|] + (\sqrt{1-p}I)\rho(\sqrt{1-p}I)\\ =&\left(\sqrt{\frac{p}{2}}|0\rangle\langle 0|\right)\rho\left(\sqrt{\frac{p}{2}}|0\rangle\langle 0|\right) + \left(\sqrt{\frac{p}{2}}|0\rangle\langle 1|\right)\rho\left(\sqrt{\frac{p}{2}}|1\rangle\langle 0|\right) + \left(\sqrt{\frac{p}{2}}|1\rangle\langle 0|\right)\rho\left(\sqrt{\frac{p}{2}}|0\rangle\langle 1|\right)\\ &+\left(\sqrt{\frac{p}{2}}|1\rangle\langle 1|\right)\rho\left(\sqrt{\frac{p}{2}}|1\rangle\langle 1|\right) + (\sqrt{1-p}I)\rho(\sqrt{1-p}I).\end{aligned}\tag{43}$$

该演化的算符和形式为

$$
\begin{aligned}
&\left(\sqrt{\frac{p}{2}}|0\rangle\langle 0|\right)\left(\sqrt{\frac{p}{2}}|0\rangle\langle 0|\right)+\left(\sqrt{\frac{p}{2}}|1\rangle\langle 0|\right)\left(\sqrt{\frac{p}{2}}|0\rangle\langle 1|\right)+\left(\sqrt{\frac{p}{2}}|0\rangle\langle 1|\right)\left(\sqrt{\frac{p}{2}}|1\rangle\langle 0|\right)\\
&+\left(\sqrt{\frac{p}{2}}|1\rangle\langle 1|\right)\left(\sqrt{\frac{p}{2}}|1\rangle\langle 1|\right)+(\sqrt{1-p}I)(\sqrt{1-p}I)\\
=&p|0\rangle\langle 0|+p|1\rangle\langle 1|+(1-p)I=I. \qquad (44)
\end{aligned}
$$

当 $d=3$ 时, 该超算符演化可表为

$$
\begin{aligned}
\$(\rho)=&\frac{p}{d}I+(1-p)\rho\\
=&\frac{p}{3}[|0\rangle\langle 0|+|1\rangle\langle 1|+|2\rangle\langle 2|]+(\sqrt{1-p}I)\rho(\sqrt{1-p}I)\\
=&\frac{p}{3}[|0\rangle 1\langle 0|+|1\rangle 1\langle 1|+|2\rangle 1\langle 2|]+(\sqrt{1-p}I)\rho(\sqrt{1-p}I)\\
=&\frac{p}{3}[|0\rangle(\langle 0|\rho|0\rangle+\langle 1|\rho|1\rangle+\langle 2|\rho|2\rangle)\langle 0|+|1\rangle(\langle 0|\rho|0\rangle+\langle 1|\rho|1\rangle+\langle 2|\rho|2\rangle)\langle 1|+|2\rangle(\langle 0|\rho|0\rangle+\langle 1|\rho|1\rangle+\langle 2|\rho|2\rangle)\langle 2|]\\
&+(\sqrt{1-p}I)\rho(\sqrt{1-p}I)\\
=&\left(\sqrt{\frac{p}{3}}|0\rangle\langle 0|\right)\rho\left(\sqrt{\frac{p}{3}}|0\rangle\langle 0|\right)+\left(\sqrt{\frac{p}{3}}|0\rangle\langle 1|\right)\rho\left(\sqrt{\frac{p}{3}}|1\rangle\langle 0|\right)+\left(\sqrt{\frac{p}{3}}|0\rangle\langle 2|\right)\rho\left(\sqrt{\frac{p}{3}}|2\rangle\langle 0|\right)\\
&+\left(\sqrt{\frac{p}{3}}|1\rangle\langle 0|\right)\rho\left(\sqrt{\frac{p}{3}}|0\rangle\langle 1|\right)+\left(\sqrt{\frac{p}{3}}|1\rangle\langle 1|\right)\rho\left(\sqrt{\frac{p}{3}}|1\rangle\langle 1|\right)+\left(\sqrt{\frac{p}{3}}|1\rangle\langle 2|\right)\rho\left(\sqrt{\frac{p}{3}}|2\rangle\langle 1|\right)\\
&+\left(\sqrt{\frac{p}{3}}|2\rangle\langle 0|\right)\rho\left(\sqrt{\frac{p}{3}}|0\rangle\langle 2|\right)+\left(\sqrt{\frac{p}{3}}|2\rangle\langle 1|\right)\rho\left(\sqrt{\frac{p}{3}}|1\rangle\langle 2|\right)+\left(\sqrt{\frac{p}{3}}|2\rangle\langle 2|\right)\rho\left(\sqrt{\frac{p}{3}}|2\rangle\langle 2|\right)\\
&+(\sqrt{1-p}I)\rho(\sqrt{1-p}I).
\end{aligned}
$$

该演化的算符和形式为

$$
\begin{aligned}
&\left(\sqrt{\frac{p}{3}}|0\rangle\langle 0|\right)\left(\sqrt{\frac{p}{3}}|0\rangle\langle 0|\right)+\left(\sqrt{\frac{p}{3}}|1\rangle\langle 0|\right)\left(\sqrt{\frac{p}{3}}|0\rangle\langle 1|\right)+\left(\sqrt{\frac{p}{3}}|2\rangle\langle 0|\right)\left(\sqrt{\frac{p}{3}}|0\rangle\langle 2|\right)\\
+&\left(\sqrt{\frac{p}{3}}|0\rangle\langle 1|\right)\left(\sqrt{\frac{p}{3}}|1\rangle\langle 0|\right)+\left(\sqrt{\frac{p}{3}}|1\rangle\langle 1|\right)\left(\sqrt{\frac{p}{3}}|1\rangle\langle 1|\right)+\left(\sqrt{\frac{p}{3}}|2\rangle\langle 1|\right)\left(\sqrt{\frac{p}{3}}|1\rangle\langle 2|\right)\\
+&\left(\sqrt{\frac{p}{3}}|0\rangle\langle 2|\right)\left(\sqrt{\frac{p}{3}}|2\rangle\langle 0|\right)+\left(\sqrt{\frac{p}{3}}|1\rangle\langle 2|\right)\left(\sqrt{\frac{p}{3}}|2\rangle\langle 1|\right)+\left(\sqrt{\frac{p}{3}}|2\rangle\langle 2|\right)\left(\sqrt{\frac{p}{3}}|2\rangle\langle 2|\right)+(\sqrt{1-p}I)(\sqrt{1-p}I)\\
=&p|0\rangle\langle 0|+p|1\rangle\langle 1|+p|2\rangle\langle 2|+(1-p)I=I. \qquad (45)
\end{aligned}
$$

□

第 8 题 得分: _________. 证明超算符仅在幺正条件下才是可逆的.

证: 充分性: 给定超算符 $\$(\rho)=\sum_{i=1}^{m}M_i\rho M_i^\dagger$, 其中 Kraus 算符均为幺正的, 即 $M_i^\dagger M_i=I$. 超算符满足

$$\sum_{i=1}^{m}M_i^\dagger M_i=I, \qquad (46)$$

$$\Longrightarrow\sum_{i=1}^{m}I=mI=I, \qquad (47)$$

$$\Longrightarrow m=1. \qquad (48)$$

从而 $\$(\rho)=M\rho M^\dagger$, 其中 $M^\dagger M=I$. 我们可以构造这一超算符的逆:

$$\$(\rho)=M^\dagger\rho M, \qquad (49)$$

其满足

$$\$^{-1}(\$(\rho))=M^\dagger M\rho M^\dagger M=\rho. \tag{50}$$

必要性: 当存在超算符

$$\$(\rho)=\sum_{i=1}^m M_i\rho M_i^\dagger \tag{51}$$

的逆

$$\$_1(\rho)=\sum_{i=1}^n N_i\rho N_i^\dagger. \tag{52}$$

超算符及其逆依次作用于密度矩阵

$$\$_2(\rho)=\$_1(\$(\rho))=\sum_{j=1}^n\sum_{i=1}^m N_jM_i\rho M_i^\dagger N_j^\dagger=\rho=I\rho I. \tag{53}$$

这里 $\$_2$ 可视为另一作用在密度矩阵上的超算符, 对应的 Kraus 算符为 N_jM_i, 满足

$$\sum_{j=1}^n\sum_{i=1}^m M_i^\dagger N_j^\dagger N_jM_i=I=\sum_{i=1}^m M_i^\dagger M_i, \tag{54}$$

$$\Longrightarrow\sum_{j=1}^n N_j^\dagger N_j=I. \tag{55}$$

这意味着 Kraus 算符与单位算符 I 仅差一系数,

$$N_jM_i=\lambda_{ji}I, \tag{56}$$

且

$$\sum_{ij}\left|\lambda_{ji}\right|^2=1. \tag{57}$$

利用上面得到的 $\sum_{j=1}^n N_j^\dagger N_j=I$, 有

$$M_a^\dagger M_b=\sum_j M_a^\dagger N_j^\dagger N_jM_b=\sum_j\lambda_{ja}^*\lambda_{jb}I=\gamma_{ab}I, \tag{58}$$

$$\Longrightarrow M_i^\dagger=\gamma_{ii}M_i^{-1}. \tag{59}$$

又有

$$M_iM_i^\dagger M_j=\gamma_{ii}M_j=\gamma_{ij}M_i. \tag{60}$$

得证?? □

第 9 题 得分: __________. 证明 $|\psi^-\rangle=\frac{1}{\sqrt{2}}(|0\rangle|1\rangle-|1\rangle|0\rangle)$ 在 $U(\theta,\vec{n})\otimes U(\theta,\vec{n})$ 下是不变的.

证: 已知

$$U(\theta,\vec{n})=\exp(-i\theta\vec{n}\cdot\vec{\sigma}/2)=\cos\left(\frac{\theta}{2}\right)-i\sin\left(\frac{\theta}{2}\right)\vec{n}\cdot\vec{\sigma}=\cos\left(\frac{\theta}{2}\right)I-i\sin\left(\frac{\theta}{2}\right)(n_xX+n_yY+n_zZ), \tag{61}$$

$$\Longrightarrow U(\theta,\vec{n})|0\rangle=\left(\cos\frac{\theta}{2}-in_z\sin\frac{\theta}{2}\right)|0\rangle-i\sin\frac{\theta}{2}\left(n_x-in_y\right)|1\rangle, \tag{62}$$

$$U(\theta,\vec{n})|1\rangle=-i\sin\frac{\theta}{2}(n_x+in_y)|0\rangle+\left(\cos\frac{\theta}{2}+in_z\sin\frac{\theta}{2}\right)|1\rangle. \tag{63}$$

将 $U(\theta,\vec{n})\otimes U(\theta,\vec{n})$ 作用在 $|\psi^-\rangle$ 上, 可得

$$\begin{aligned}
&U(\theta,\vec{n})\otimes U(\theta,\vec{n})|\psi^-\rangle=U(\theta,\vec{n})\otimes U(\theta,\vec{n})\frac{1}{\sqrt{2}}(|0\rangle|1\rangle-|1\rangle|0\rangle)\\
=&\frac{1}{\sqrt{2}}\left\{\left[\left(\cos\frac{\theta}{2}-in_z\sin\frac{\theta}{2}\right)|0\rangle-i\sin\frac{\theta}{2}(n_x-in_y)|1\rangle\right]\left[-i\sin\frac{\theta}{2}(n_x+in_y)|0\rangle+\left(\cos\frac{\theta}{2}+in_z\sin\frac{\theta}{2}\right)|1\rangle\right]\right.\\
&\left.-\left[-i\sin\frac{\theta}{2}(n_x+in_y)|0\rangle+\left(\cos\frac{\theta}{2}+in_z\sin\frac{\theta}{2}\right)|1\rangle\right]\left[\left(\cos\frac{\theta}{2}-in_z\sin\frac{\theta}{2}\right)|0\rangle-i\sin\frac{\theta}{2}(n_x-in_y)|1\rangle\right]\right\}\\
=&\frac{1}{\sqrt{2}}\left\{\left[\left(\cos\frac{\theta}{2}-in_z\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2}+in_z\sin\frac{\theta}{2}\right)+\sin^2\frac{\theta}{2}(n_x+in_y)(n_x-in_y)\right]|0\rangle|1\rangle\right.\\
&\left.+\left[-\sin^2\frac{\theta}{2}(n_x-in_y)(n_x+in_y)-\left(\cos\frac{\theta}{2}+in_z\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2}-in_z\sin\frac{\theta}{2}\right)\right]|1\rangle|0\rangle\right\}\\
=&\frac{1}{\sqrt{2}}\left\{\left[\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}(n_x^2+n_y^2+n_z^2)\right]|0\rangle|1\rangle-\left[\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}(n_x^2+n_y^2+n_z^2)\right]|1\rangle|0\rangle\right\}\\
=&\frac{1}{\sqrt{2}}(|0\rangle|1\rangle-|1\rangle|0\rangle),
\end{aligned} \tag{64}$$

故 $|\psi^-\rangle$ 在 $U(\theta,\vec{n})\otimes U(\theta,\vec{n})$ 下是不变的. $\square$