数学物理方法II第一次作业

陈稼霖　　　45875852

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在极坐标系中写出速度$\frac{d\vec{r}}{dt}$和加速度$\frac{d\vec{v}}{dt}$，结果用$\vec{e}_r$、$\vec{e}_\theta$表示.
解：在极坐标系中速度为

$$\begin{aligned}\frac{d\vec{r}}{dt}=&\frac{d(r\vec{e}_r)}{dt}\\=&\frac{dr}{dt}\vec{e}_r+r\frac{d\vec{e}_r}{dt}\\=&\frac{dr}{dt}\vec{e}_r+r\frac{d\theta}{dt}\vec{e}_\theta\end{aligned}$$

加速度为

$$\begin{aligned}\frac{d\vec{v}}{dt}=&\frac{d(\frac{dr}{dt}\vec{e}_r+r\frac{d\theta}{dt}\vec{e}_\theta)}{dt}\\=&\frac{d^2r}{dt^2}\vec{e}_r+\frac{dr}{dt}\frac{d\vec{e}_r}{dt}+\frac{dr}{dt}\frac{d\theta}{dt}\vec{e}_\theta+r\frac{d^2\theta}{dt^2}\vec{e}_\theta+r\frac{d\theta}{dt}\frac{d\vec{e}_\theta}{dt}\\=&\frac{d^2r}{dt^2}\vec{e}_r+\frac{dr}{dt}\frac{d\theta}{dt}\vec{e}_\theta+\frac{dr}{dt}\frac{d\theta}{dt}\vec{e}_\theta+r\frac{d^2\theta}{dt^2}\vec{e}_\theta-r(\frac{d\theta}{dt})^2\vec{e}_r\\=&[\frac{d^2r}{dt^2}-r(\frac{d\theta}{dt})^2]\vec{e}_r+(2\frac{dr}{dt}\frac{d\theta}{dt}+r\frac{d^2\theta}{dt^2})\vec{e}_\theta\end{aligned}$$

2

设矢量$\vec{r}$为某个源点$\vec{x}'$指向场点$\vec{x}$的矢量，而r为源点到场点的距离.

$$\vec{r}=\vec{x}-\vec{x}'=\vec{e}_x(x-x')+\vec{e}_y(y-y')+\vec{e}_z(z-z')$$

请求出
(1) $\nabla\frac{1}{r}$
解：

$$\begin{aligned}\nabla\frac{1}{r}=&\frac{\partial\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}}{\partial x}\vec{e}_x+\frac{\partial\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}}{\partial y}\vec{e}_y\\&+\frac{\partial\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}}{\partial z}\vec{e}_z\end{aligned}$$

$$
\begin{aligned}
&=-\frac{x-x'}{[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{3}{2}}}\vec{e}_x-\frac{y-y'}{[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{3}{2}}}\vec{e}_y\\
&\quad-\frac{z-z'}{[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{3}{2}}}\vec{e}_z\\
&=-\frac{(x-x')\vec{e}_x+(y-y')\vec{e}_y+(z-z')\vec{e}_z}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}\\
&=-\frac{\vec{r}}{r^3}
\end{aligned}
$$

(2) $\nabla\times\frac{\vec{r}}{r^3}$

解：

$$
\nabla\times\frac{\vec{r}}{r^3}=(\nabla\frac{1}{r^3})\times\vec{r}+\frac{1}{r^3}\nabla\times\vec{r}
$$

式中

$$
\begin{aligned}
\nabla\frac{1}{r^3}=&\frac{\partial\frac{1}{[(x-x')^2+(y-y')^+(z-z')^2]^{3/2}}}{\partial x}\vec{e}_x+\frac{\partial\frac{1}{[(x-x')^2+(y-y')^+(z-z')^2]^{3/2}}}{\partial y}\vec{e}_y\\
&+\frac{\partial\frac{1}{[(x-x')^2+(y-y')^+(z-z')^2]^{3/2}}}{\partial z}\vec{e}_z\\
&=-\frac{3(x-x')}{[(x-x')^2+(y-y')^+(z-z')^2]^{5/2}}\vec{e}_x-\frac{3(y-y')}{[(x-x')^2+(y-y')^+(z-z')^2]^{5/2}}\vec{e}_y\\
&\quad-\frac{3(z-z')}{[(x-x')^2+(y-y')^+(z-z')^2]^{5/2}}\vec{e}_z\\
&=-3\frac{(x-x')\vec{e}_x+(y-y')\vec{e}_y+(z-z')\vec{e}_z}{[(x-x')^2+(y-y')^+(z-z')^2]^{5/2}}\\
&=-\frac{3\vec{r}}{r^5}
\end{aligned}
$$

且

$$
\begin{aligned}
\nabla\times\vec{r}=&(\frac{\partial(z-z')}{\partial y}-\frac{\partial(y-y')}{\partial z})\vec{e}_x+(\frac{\partial(x-x')}{\partial z}-\frac{\partial(z-z')}{\partial x})\vec{e}_y+(\frac{\partial(y-y')}{\partial x}-\frac{\partial(x-x')}{\partial y})\vec{e}_z\\
=&\vec{0}
\end{aligned}
$$

故

$$
\begin{aligned}
\nabla\times\frac{\vec{r}}{r^3}=&-3\frac{\vec{r}}{r^5}\times\vec{r}+\vec{r}\\
=&\vec{0}
\end{aligned}
$$

(3) $\nabla\times\vec{r}$

解：

$$
\begin{aligned}
\nabla\times\vec{r}=&(\frac{\partial(z-z')}{\partial y}-\frac{\partial(y-y')}{\partial z})\vec{e}_x+(\frac{\partial(x-x')}{\partial z}-\frac{\partial(z-z')}{\partial x})\vec{e}_y+(\frac{\partial(y-y')}{\partial x}-\frac{\partial(x-x')}{\partial y})\vec{e}_z\\
=&\vec{0}
\end{aligned}
$$

3

从杆的纵振动问题导出波动方程，其问题设置如下：
均匀细杆在外力作用下沿杆长方向作微小振动，设杆长方向为x轴，$u(x,t)$为x处的截面在t时刻沿杆长方向的位移. 其中理想化假设如下 i) 振动方向与杆的方向一致.
ii) 均匀细杆：同一横截面上各点的质量密度ρ，横截面面积S与杨氏模量Y（应力与应变值比值）都是常数.
iii) 杆有弹性，服从Hooke定律：即应力与相对伸长成正比.
iv) 外力与杆的方向一致，各点单位长度上的外力为$f_0(x,t)$，重力不计。并求出如下情况对应的边界条件：
(1) $x=0$处固定.
(2) $x=0$处受$G(t)$的横向外力.
解：对于杆在区间$[a+u(a,t),b+u(b,t)]$上的部分，其左右端分别受到张力

$$T_a = -YS\frac{u(a+\Delta x,t)-u(a,t)}{\Delta x} = -YS\frac{\partial u}{\partial x}|_{x=a}$$
$$T_b = YS\frac{u(b+\Delta x,t)-u(b,t)}{\Delta x}(b,t) = YS\frac{\partial u}{\partial x}|_{x=b}$$

根据牛顿第二定律得到波动方程

$$T_a+T_b+(b-a)f_0 = \rho(b-a)S\frac{\partial^2 u}{\partial t^2}$$
$$\Longrightarrow YS\frac{\frac{\partial u}{\partial x}|_{x=b}-\frac{\partial u}{\partial x}|_{x=a}}{b-a}+f_0 = \rho S\frac{\partial^2 u}{\partial t^2}$$
$$\Longrightarrow \frac{\partial^2 u}{\partial t^2}-\frac{Y}{\rho}\frac{\partial^2 u}{\partial x^2} = \frac{f_0}{\rho S}$$

(1) $u|_{x=0}=0$
(2) 根据牛顿第二定律有

$$\rho\Delta x S\frac{\partial^2 u}{\partial t^2} = G(t)+YS\frac{u(\Delta x,t)-u(0,t)}{\Delta x}+f_0(0,t)\Delta x$$

令$\Delta x\to 0$得到

$$\frac{\partial u}{\partial x}|_{x=0} = -\frac{g(t)}{YS}$$

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二维波动方程的推出：有一均匀的各向同性的弹性圆膜，四周固定。试列出膜的横振动方程与边界条件(设ρ_m为面密度，沿任何方向单位长度张力为T).
提示：在极坐标中进行微元分析，进而化为直角坐标下的波动方程.
解：设时刻t点$\boldsymbol{r}$偏离平衡位置$u(\boldsymbol{r},t)$，弹性圆膜半径为R. 原题图中所取膜面积元的质量为

$$\Delta m = \rho_m\cdot\Delta S = \rho_m\cdot\rho\Delta\phi\cdot\Delta\rho$$

在径向，根据牛二律有

$$\rho_m\cdot\rho\Delta\phi\cdot\Delta\rho\frac{\partial^2 u}{\partial t^2}=-T\rho\Delta\phi\sin\alpha|_{\rho}+T(\rho+\Delta\rho)\Delta\phi\sin\alpha|_{\rho+\Delta\rho}+T\Delta\rho\sin\beta|_{\phi}-T\Delta\rho\sin\beta|_{\phi+\Delta\phi}$$

在小振动下有如下近似

$$\sin\alpha\approx\frac{\partial u}{\partial\rho},\ \sin\beta\approx\frac{1}{\rho}\frac{\partial u}{\partial\phi}$$

故原方程可化为

$$\begin{aligned}\rho_m\rho\Delta\phi\Delta\rho\frac{\partial^2 u}{\partial t^2}=&T\Delta\phi[(\rho\frac{\partial u}{\partial\rho})|_{\rho+\Delta\rho}-(\rho\frac{\partial u}{\partial\rho})|_{\rho}]+T\Delta\rho\frac{1}{\rho}[\frac{\partial u}{\partial\phi}|_{\phi+\Delta\phi}-\frac{\partial u}{\partial\phi}|_{\phi}]\\=&T\Delta\phi\frac{\partial}{\partial\rho}(\rho\frac{\partial u}{\partial\rho})\Delta\rho+T\Delta\rho\frac{1}{\rho}\frac{\partial}{\partial\phi}(\frac{\partial u}{\partial\phi})\Delta\phi\\\Longrightarrow\frac{\partial^2 u}{\partial t^2}=&\frac{T}{\rho_m}[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial u}{\partial\rho})+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2}]\end{aligned}$$

由于四周固定，故

$$u|_{\rho=R}=0$$

转换到直角坐标系，振动方程及边界条件为

$$\begin{cases}\frac{\partial^2 u}{\partial t^2}=\frac{T}{\rho_m}\nabla^2 u\\ u|_{\sqrt{x^2+y^2}=R}=0\end{cases}$$

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将下列二阶偏微分方程化为标准形式.

(1) $\frac{\partial^2 u}{\partial x^2}+4\frac{\partial^2 u}{\partial x\partial y}+5\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}+2\frac{\partial u}{\partial y}=0$;

解：此方程的特征方程为

$$(\frac{dy}{dx})^2-4\frac{dy}{dx}+5=0$$

解得

$$y-4x=C_1,\ y-x=C_2$$

做变换

$$\xi=y-4x,\ \eta=y-x$$

原方程化为标准形式

$$\frac{\partial^2 u}{\partial\xi\partial\eta}=-\frac{\partial u}{\partial\xi}+\frac{1}{2}\frac{\partial u}{\partial\eta}$$

(2) $\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial y^2}+\frac{1}{2}\frac{\partial u}{\partial y}=0$;

解：此方程的特征方程为

$$(\frac{dy}{dx})^2+y=0$$

判别式为

$$\Delta = -y$$

当$\Delta > 0$即$y < 0$时，解得

$$x \pm 2(-y)^{1/2} = \gamma$$

做变换

$$\xi = x + 2(-y)^{1/2},\ \ \eta = x - 2(-y)^{1/2}$$

原方程化为标准形式

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

当$\Delta = 0$即$y = 0$时，原方程为

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\frac{\partial u}{\partial y} = 0$$

当$\Delta < 0$即$y > 0$时，特征方程解得

$$x \pm 2iy^{1/2} = \gamma$$

做变换

$$\xi = x,\ \ \eta = 2y^{1/2}$$

原方程化为标准形式

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = 0$$

(3) $\frac{\partial^2 u}{\partial x^2} - 2\cos x \frac{\partial^2 u}{\partial x \partial y} - (3 + \sin^2 x)\frac{\partial^2 u}{\partial y^2} - y - \frac{\partial u}{\partial y} = 0$
解：此方程的特征方程为

$$(\frac{dy}{dx}) + 2\cos x \frac{dy}{dx} - (3 + \sin^2 x) = 0$$

判别式为

$$\Delta = \cos^2 x + 3 + \sin^2 x = 4 > 0$$

解得

$$y \pm \sin x = \gamma$$

做变换

$$\xi = y + \sin x,\ \ \eta = y - \sin x$$

原方程化为标准形式

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = -\frac{1}{8}(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} + u)$$