1. 设 $G(\vec r,\vec r')$ 对 $\vec r$ 的傅立叶变换为 $\tilde G(\vec k,\vec r')$

则其逆变换为 $G(\vec r,\vec r')=\frac{1}{(2\pi)^3}\iiint \tilde G(\vec k,\vec r')\exp(i\vec k\cdot\vec r)d\vec k$

$$=\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\tilde G(\vec k,\vec r')\exp[i(k_x x+k_y y+k_z z)]dk_x dk_y dk_z \quad ①$$

又有结论 $\delta^3(\vec x)=\frac{1}{(2\pi)^3}\iiint \exp(i\vec k\cdot\vec x)d\vec k=\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp[i(k_x x+k_y y+k_z z)]dk_x dk_y dk_z \quad ②$

其中设 $k'^2=k_x^2+k_y^2+k_z^2$

将①②分别代入原方程得

$$\frac{(\nabla^2+k^2)}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\tilde G(\vec k',\vec r')\exp[i(k_x x+k_y y+k_z z)]dk_x dk_y dk_z$$

$$=-\frac{1}{(2\pi)^3\varepsilon_0}\iiint\exp[i\vec k'\cdot(\vec r-\vec r')]d\vec k'$$

$$\Rightarrow\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\nabla^2+k^2)\tilde G(\vec k',\vec r')\exp[i(k_x x+k_y y+k_z z)]dk_x dk_y dk_z$$

$$=-\frac{1}{(2\pi)^3\varepsilon_0}\iiint\exp[i\vec k'\cdot(\vec r-\vec r')]d\vec k'$$

$$\Rightarrow\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\{(-k_x^2-k_y^2-k_z^2)\tilde G(\vec k',\vec r')\exp[i(k_x x+k_y y+k_z z)]$$
$$+k^2\tilde G(\vec k',\vec r')\exp[i(k_x x+k_y y+k_z z)]\}dk_x dk_y dk_z$$

$$=-\frac{1}{(2\pi)^3\varepsilon_0}\iiint\exp[i\vec k'\cdot(\vec r-\vec r')]d\vec k'$$

$$\Rightarrow\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(-k'^2+k^2)\tilde G(\vec k',\vec r')\exp[i(k_x x+k_y y+k_z z)]dk_x dk_y dk_z$$

$$=-\frac{1}{(2\pi)^3\varepsilon_0}\iiint\exp[i\vec k'\cdot(\vec r-\vec r')]d\vec k'$$

两边同时对 $\vec r$ 作傅里叶变换得

$$(-k'^2+k^2)\tilde G(\vec k',\vec r')=-\frac{\exp(-i\vec k'\cdot\vec r')}{\varepsilon_0}\Longrightarrow\tilde G(\vec k',\vec r')=-\frac{\exp(-i\vec k'\cdot\vec r')}{\varepsilon_0}$$

利用式①逆变换

$$G(\vec r,\vec r')=\frac{1}{(2\pi)^3\varepsilon_0}\iiint\frac{\exp(-i\vec k'\cdot\vec r')}{k'^2-k^2}\exp(i\vec k'\cdot\vec r)d\vec r$$

以 $(\vec r-\vec r')$ 方向为 $z$ 轴方向，转换到球坐标中

$$G(\vec r,\vec r')=\frac{1}{(2\pi)^3\varepsilon_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{+\infty}\frac{\exp[ik'(\vec r-\vec r')]}{k'^2-k^2}k'^2\sin\theta\, dk\, d\theta\, d\varphi$$

$$=\frac{1}{(2\pi)^3\varepsilon_0}\int_0^{2\pi}d\varphi\int_0^{\infty}\frac{k'^2}{k'^2-k^2}dk'\int_0^{\pi}\exp(ik'|\vec r-\vec r'|\cos\theta)\sin\theta\, d\theta$$

$$=\frac{1}{(2\pi)^2\varepsilon_0}\int_0^{\infty}\frac{k'^2}{k'^2-k^2}dk'\int_0^{\pi}-\exp(ik'|\vec r-\vec r'|\cos\theta)\,d\cos\theta$$

$$\overset{t=\cos\theta}{=\!=} \frac{1}{(2\pi)^2\varepsilon_0}\int_0^{\infty}\frac{k'^2}{k'^2-k^2}dk'\int_{-1}^{1}\exp(ik'|\vec{r}-\vec{r}'|t)$$

$$= \frac{1}{(2\pi)^2\varepsilon_0}\int_0^{\infty}\frac{k'^2}{k'^2-k^2}\,\frac{\exp(ik'|\vec{r}-\vec{r}'|t)\Big|_{-1}^{1}}{ik'|\vec{r}-\vec{r}'|}dk'$$

$$= \frac{1}{i(2\pi)^2\varepsilon_0|\vec{r}-\vec{r}'|}\int_0^{\infty}\frac{k'\left[\exp(ik'|\vec{r}-\vec{r}'|)-\exp(-ik'|\vec{r}-\vec{r}'|)\right]}{k'^2-k^2}dk'$$

$$= \frac{1}{i(2\pi)^2\varepsilon_0|\vec{r}-\vec{r}'|}\int_{-\infty}^{+\infty}\frac{k'\exp(ik'|\vec{r}-\vec{r}'|)}{k'^2-k^2}dk$$

$$= \frac{1}{i(2\pi)^2\varepsilon_0|\vec{r}-\vec{r}'|}2\pi i\,\mathrm{Res}\left[\frac{k'\exp(ik'|\vec{r}-\vec{r}'|)}{k'^2-k^2},k\right]$$ (留数定理)

$$= \frac{1}{i(2\pi)^2\varepsilon_0|\vec{r}-\vec{r}'|}2\pi i\lim_{k'\to k}\frac{k'\exp(ik'|\vec{r}-\vec{r}'|)}{k'^2-k^2}(k'-k)$$

$$= \frac{1}{i(2\pi)^2\varepsilon_0|\vec{r}-\vec{r}'|}2\pi i\,\frac{k\exp(ik|\vec{r}-\vec{r}'|)}{k+k}$$

$$= \frac{\exp(ik|\vec{r}-\vec{r}'|)}{4\pi\varepsilon_0|\vec{r}-\vec{r}'|}$$

2. $\frac{1}{2}\pi i^{\nu+1}[J_\nu(ix)+iN_\nu(ix)]$

$$=\frac{\pi}{2}i^{\nu+1}\left[J_\nu(ix)+i\frac{J_\nu(ix)\cos\nu\pi-J_{-\nu}(ix)}{\sin\nu\pi}\right]$$

$$=\frac{\pi}{2}i^{\nu+1}\left[J_\nu(ix)+\frac{J_\nu(ix)\cos\nu\pi-J_{-\nu}(ix)}{-i\sin\nu\pi}\right]$$

$$=\frac{\pi}{2}i^{\nu+1}\frac{J_\nu(ix)\cos\nu\pi-iJ_\nu(ix)\sin\nu\pi-J_{-\nu}(ix)}{-i\sin\nu\pi}$$

$$=\frac{\pi}{2}i^{\nu+2}\frac{J_\nu(ix)e^{-i\nu\pi}-J_{-\nu}(ix)}{\sin\nu\pi}$$

$$=\frac{\pi}{2}i^{\nu+2}\frac{J_\nu(ix)[e^{-i\pi}]^{\nu}-J_{-\nu}(ix)}{\sin\nu\pi}$$

$$=\frac{\pi}{2}i^{\nu+2}\frac{J_\nu(ix)(-1)^{\nu}-J_{-\nu}(ix)}{\sin\nu\pi}.$$

$$=\frac{\pi}{2}\frac{(-i)^{\nu}i^2J_\nu(ix)-i^{\nu}i^2J_{-\nu}(ix)}{\sin\nu\pi}.$$

$$=\frac{\pi}{2}\frac{i^{\nu}J_{-\nu}(ix)-(-i)^{\nu}J_\nu(ix)}{\sin\nu\pi}.$$

而根据第二类虚宗量贝塞数的定义

$$K_\nu(x)\triangleq\frac{\pi}{2}\frac{I_{-\nu}(x)-I_\nu(x)}{\sin\nu\pi}$$

$$=\frac{\pi}{2}\frac{(-i)^{-\nu}J_{-\nu}(ix)-(-i)^{\nu}J_\nu(ix)}{\sin\nu\pi}$$

$$=\frac{\pi}{2}\frac{i^{\nu}J_{-\nu}(ix)-(-i)^{\nu}J_\nu(ix)}{\sin\nu\pi}$$

故 $\frac{1}{2}\pi i^{\nu+1}[J_\nu(ix)+iN_\nu(ix)]=K_\nu(x)$

再根据Hankel函数的定义 $H_\nu^{(1)}(x)\triangleq J_\nu(x)+iN_\nu(x)$

有 $K_\nu(x)=\frac{\pi}{2}i^{\nu+1}H_\nu^{(1)}(x)=\frac{\pi}{2}i^{\nu+1}[J_\nu(ix)+iN_\nu(ix)]$.

当$\nu$为整数$n$时,取极限 $\lim\limits_{\nu\to n}K_\nu(x)=\lim\limits_{\nu\to n}\frac{\pi}{2}i^{\nu+1}H_\nu^{(1)}(x)=\lim\limits_{\nu\to n}\frac{\pi}{2}i^{\nu+1}[J_\nu(ix)+iN_\nu(ix)]$

得 $K_n(x)=\frac{\pi}{2}i^{n+1}H_n^{(1)}(x)=\frac{\pi}{2}i^{n+1}[J_n(ix)+iN_n(ix)]$.

3. (1) $\int x J_2(x)dx$

利用递推公式 $J_2(x)=-J_0(x)+\frac{2}{x}J_1(x)$

得 $\int xJ_2(x)dx=-\int xJ_0(x)dx+2\int J_1(x)dx$

$$=-xJ_1(x)-2J_0(x)$$

(2) $\int x^4 J_1(x)dx$

$$\int x^4J_1(x)dx=\int x^2[x^2J_1(x)]dx=\int x^2 d[x^2J_2(x)]$$

$$=x^4J_2(x)-2\int x^3J_2(x)dx$$

$$=x^4J_2(x)-2x^3J_3(x).$$

代入 $J_2(x)=\frac{2}{x}J_1(x)-J_0(x)$

和 $J_3(x)=\frac{4}{x}J_2(x)-J_1(x)=\frac{8}{x^2}J_1(x)-\frac{4}{x}J_0(x)-J_1(x)$

得 $\int x^4J_1(x)dx=x^4[\frac{2}{x}J_1(x)-J_0(x)]-2x^3[\frac{8}{x^2}J_1(x)-\frac{4}{x}J_0(x)-J_1(x)]$

$$=(-x^4+8x^2)J_0(x)+4(x^3-16x)J_1(x)$$

(3) $\int_0^R J_0(x)\cos x\,dx$

$$\int_0^R J_0(x)\cos x\,dx=xJ_0(x)\cos x\Big|_0^R-\int_0^R x\,d[J_0(x)\cos x]$$

$$=xJ_0(x)\cos x\Big|_0^R-\int_0^R x[-J_1(x)\cos x-J_0(x)\sin x]dx$$

$$=xJ_0(x)\cos x\Big|_0^R+\int_0^R xJ_1(x)\cos x\,dx+\int_0^R xJ_0(x)\sin x\,dx$$

$$=xJ_0(x)\cos x\Big|_0^R+\int_0^R xJ_1(x)\cos x\,dx+\int_0^R \sin x\,d[xJ_1(x)]$$

$$=xJ_0(x)\cos x\Big|_0^R+\int_0^R xJ_1(x)\cos x\,dx+xJ_1(x)\sin x\Big|_0^R-\int_0^R xJ_1(x)\,d(\sin x)$$

$$=xJ_0(x)\cos x\Big|_0^R+\int_0^R xJ_1(x)\cos x\,dx+xJ_1(x)\sin x\Big|_0^R-\int_0^R xJ_1(x)\cos x\,dx$$

$$=RJ_0(R)\cos R+RJ_1(R)\sin R$$

(4) $3J_0'(x)+4J_0'''(x)$

$J_0'(x)=-J_1(x)$

$J_0'''(x)=-\frac{d^2}{dx^2}J_1(x)=-\frac{1}{2}\frac{d}{dx}[J_0(x)-J_2(x)]=\frac{1}{2}J_1(x)+\frac{1}{2}\frac{d}{dx}[J_2(x)]$

$$=\frac{1}{2}J_1(x)+\frac{1}{4}J_1(x)+\frac{1}{4}J_3(x)=\frac{3}{4}J_1(x)+\frac{1}{4}J_3(x)$$

代入原式得 $3J_0'(x)+4J_0'''(x)=J_3(x)$

4. $J_\nu(x)$ 和 $J_{-\nu}(x)$ 是贝塞尔方程 $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\frac{x^2-\nu^2}{x^2}y=0$ 的两个线性独立的解

即 $$\begin{cases} \frac{d^2J_\nu(x)}{dx^2}+\frac{1}{x}\frac{dJ_\nu(x)}{dx}+\frac{x^2-\nu^2}{x^2}J_\nu(x)=0 & ① \\ \frac{d^2J_{-\nu}(x)}{dx^2}+\frac{1}{x}\frac{dJ_{-\nu}(x)}{dx}+\frac{x^2-\nu^2}{x^2}J_{-\nu}(x)=0 & ② \end{cases}$$

① $\cdot J_{-\nu}(x)-$ ② $\cdot J_\nu(x)$ 得

$$J_{-\nu}(x)\frac{d^2J_\nu(x)}{dx^2}+\frac{1}{x}J_{-\nu}(x)\frac{dJ_\nu(x)}{dx}-J_\nu(x)\frac{d^2J_{-\nu}(x)}{dx^2}-\frac{1}{x}J_\nu(x)\frac{dJ_{-\nu}(x)}{dx}=0$$

$$\Longrightarrow \left[J_{-\nu}(x)\frac{d^2J_\nu(x)}{dx^2}+\frac{dJ_{-\nu}(x)}{dx}\frac{dJ_\nu(x)}{dx}-J_\nu(x)\frac{d^2J_{-\nu}(x)}{dx^2}-\frac{dJ_\nu(x)}{dx}\frac{dJ_{-\nu}(x)}{dx}\right]$$
$$+\frac{1}{x}\left[J_{-\nu}(x)\frac{dJ_\nu(x)}{dx}-J_\nu(x)\frac{dJ_{-\nu}(x)}{dx}\right]=0$$

$$\Longrightarrow \frac{d}{dx}\left[J_\nu(x)\frac{dJ_{-\nu}(x)}{dx}-J_{-\nu}(x)\frac{dJ_\nu(x)}{dx}\right]+\frac{1}{x}\left[J_\nu(x)\frac{dJ_{-\nu}(x)}{dx}-J_{-\nu}(x)\frac{dJ_\nu(x)}{dx}\right]$$
$$=0$$

$$\Longrightarrow \frac{d}{dx}W[J_\nu(x),J_{-\nu}(x)]=-\frac{1}{x}W[J_\nu(x),J_{-\nu}(x)]$$

$$\Longrightarrow \ln W[J_\nu(x),J_{-\nu}(x)]=-\int\frac{1}{x}dx=-\ln x+B_1$$

$$\Longrightarrow W[J_\nu(x),J_{-\nu}(x)]=e^{-\ln x+B_1}=\frac{C_1}{x}$$

令 $x\to 0$, $$W[J_\nu(x),J_{-\nu}(x)]=\begin{vmatrix} \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu & \frac{1}{\Gamma(-\nu+1)}\left(\frac{x}{2}\right)^{-\nu} \\ \frac{1}{2\Gamma(\nu)}\left(\frac{x}{2}\right)^\nu & \frac{1}{2\Gamma(-\nu)}\left(\frac{x}{2}\right)^{-\nu-1} \end{vmatrix}$$

$$=\frac{1}{x}\left[\frac{1}{\Gamma(\nu+1)\Gamma(-\nu)}-\frac{1}{\Gamma(-\nu+1)\Gamma(\nu)}\right]$$

$$=\frac{1}{x}\left[\frac{1}{\nu\Gamma(\nu)\Gamma(-\nu)}+\frac{1}{\nu\Gamma(-\nu)\Gamma(\nu)}\right]$$

$$=-\frac{2}{x}\frac{1}{\Gamma(\nu)\Gamma(1-\nu)}=-\frac{2}{x}\frac{\sin\pi\nu}{\pi}$$ (余元公式)

$$\Longrightarrow C_1=-\frac{2\sin\pi\nu}{\pi}$$

$$\therefore W[J_\nu(x),J_{-\nu}(x)]=-\frac{2\sin\pi\nu}{\pi x}$$

同理，$J_\nu(x)$ 和 $Y_\nu(x)$ 也是贝塞尔方程 $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\frac{x^2-\nu^2}{x^2}y=0$ 的两个线性独立的解

即
$$\begin{cases}\dfrac{d^2J_\nu(x)}{dx^2}+\dfrac{1}{x}\dfrac{dJ_\nu(x)}{dx}+\dfrac{x^2-\nu^2}{x^2}J_\nu(x)=0 & ③\\[2mm] \dfrac{d^2Y_\nu(x)}{dx^2}+\dfrac{1}{x}\dfrac{dY_\nu(x)}{dx}+\dfrac{x^2-\nu^2}{x^2}Y_\nu(x)=0 & ④\end{cases}$$

③ $\cdot Y_\nu(x)-$ ④ $\cdot J_\nu(x)$ 得

$$Y_\nu(x)\frac{d^2J_\nu(x)}{dx^2}+\frac{1}{x}Y_\nu(x)\frac{dJ_\nu(x)}{dx}-J_\nu(x)\frac{d^2Y_\nu(x)}{dx^2}-\frac{1}{x}J_\nu(x)\frac{dY_\nu(x)}{dx}=0$$

$$\Longrightarrow\left[Y_\nu(x)\frac{d^2J_\nu(x)}{dx^2}+\frac{dY_\nu(x)}{dx}\frac{dJ_\nu(x)}{dx}-J_\nu(x)\frac{d^2Y_\nu(x)}{dx^2}-\frac{dJ_\nu(x)}{dx}\frac{dY_\nu(x)}{dx}\right]$$
$$+\frac{1}{x}\left[Y_\nu(x)\frac{dJ_\nu(x)}{dx}-J_\nu(x)\frac{dY_\nu(x)}{dx}\right]=0$$

$$\Longrightarrow\frac{d}{dx}\left[J_\nu(x)\frac{dY_\nu(x)}{dx}-Y_\nu(x)\frac{dJ_\nu(x)}{dx}\right]+\frac{1}{x}\left[J_\nu(x)\frac{dY_\nu(x)}{dx}-Y_\nu(x)\frac{dJ_\nu(x)}{dx}\right]$$
$$=0$$

$$\Longrightarrow\frac{d}{dx}W[J_\nu(x),Y_\nu(x)]=-\frac{1}{x}W[J_\nu(x),Y_\nu(x)]$$

$$\Longrightarrow\ln W[J_\nu(x),Y_\nu(x)]=-\int\frac{1}{x}dx=-\ln x+B_2$$

$$\Longrightarrow W[J_\nu(x),Y_\nu(x)]=e^{-\ln x+B_2}=\frac{C_2}{x}$$

令 $x\to+\infty$，
$$W[J_\nu(x),Y_\nu(x)]=\begin{vmatrix}\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\pi}{4}-\frac{\nu\pi}{2}\right) & \dfrac{\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\pi}{4}-\frac{\nu\pi}{2}\right)\cos\nu\pi-\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\pi}{4}+\frac{\nu\pi}{2}\right)}{\sin\nu\pi}\\ -\sqrt{\frac{2}{\pi x}}\sin\left(x-\frac{\pi}{4}-\frac{\nu\pi}{2}\right) & \dfrac{-\sqrt{\frac{2}{\pi x}}\sin\left(x-\frac{\pi}{4}-\frac{\nu\pi}{2}\right)\cos\nu\pi+\sqrt{\frac{2}{\pi x}}\sin\left(x-\frac{\pi}{4}+\frac{\nu\pi}{2}\right)}{\sin\nu\pi}\end{vmatrix}$$

$$=\frac{\frac{2}{\pi x}\left[\cos\left(x-\frac{\pi}{4}-\frac{\nu\pi}{2}\right)\sin\left(x-\frac{\pi}{4}+\frac{\nu\pi}{2}\right)-\sin\left(x-\frac{\pi}{4}-\frac{\nu\pi}{2}\right)\cos\left(x-\frac{\pi}{4}+\frac{\nu\pi}{2}\right)\right]}{\sin\nu\pi}$$

$$=\frac{\frac{2}{\pi x}\sin\left(x-\frac{\pi}{4}+\frac{\nu\pi}{2}-x+\frac{\pi}{4}+\frac{\nu\pi}{2}\right)}{\sin\nu\pi}=\frac{2}{\pi x}$$

$$\Longrightarrow C_2=\frac{2}{\pi}$$

$$\therefore W[J_\nu(x),Y_\nu(x)]=\frac{2}{\pi x}$$

(1) $\int \frac{dx}{xJ_\nu^2(x)}$

$$\int \frac{dx}{xJ_\nu^2(x)} = \int -\frac{\pi W[J_\nu(x), J_{-\nu}(x)]}{2\sin\pi\nu\, J_\nu^2(x)} dx$$

$$= -\frac{\pi}{2\sin\pi\nu} \int \frac{J_\nu(x)J_{-\nu}'(x) - J_\nu'(x)J_{-\nu}(x)}{J_\nu^2(x)} dx$$

$$= -\frac{\pi}{2\sin\pi\nu} \int d\left[\frac{J_{-\nu}(x)}{J_\nu(x)}\right]$$

$$= -\frac{\pi J_{-\nu}(x)}{2J_\nu(x)\sin\pi\nu} + C$$

(2) $\int \frac{dx}{xY_\nu^2(x)}$

$$\int \frac{dx}{xY_\nu^2(x)} = \int \frac{\pi W[J_\nu(x), Y_\nu(x)]}{2Y_\nu^2(x)} dx$$

$$= -\frac{\pi}{2} \int \frac{Y_\nu(x)J_\nu'(x) - Y_\nu'(x)J_\nu(x)}{Y_\nu^2(x)} dx$$

$$= -\frac{\pi}{2}\int d\left[\frac{J_\nu(x)}{Y_\nu(x)}\right]$$

$$= -\frac{\pi J_\nu(x)}{2Y_\nu(x)} + C$$

(3) $\int \frac{dx}{xJ_\nu(x)Y_\nu(x)}$

$$\int \frac{dx}{xJ_\nu(x)Y_\nu(x)} = \int \frac{\pi}{2} \frac{W[J_\nu(x), Y_\nu(x)]}{J_\nu(x)Y_\nu(x)} dx$$

$$= \int \frac{\pi}{2} \frac{J_\nu(x)Y_\nu'(x) - J_\nu'(x)Y_\nu(x)}{J_\nu(x)Y_\nu(x)} dx$$

$$= \frac{\pi}{2}\left[\int \frac{Y_\nu'(x)}{Y_\nu(x)} dx - \int \frac{J_\nu'(x)}{J_\nu(x)} dx\right]$$

$$= \frac{\pi}{2}\left[\int \frac{dY_\nu(x)}{Y_\nu(x)} - \int \frac{dJ_\nu(x)}{J_\nu(x)}\right]$$

$$= \frac{\pi}{2} \ln \frac{Y_\nu(x)}{J_\nu(x)} + C$$

5. 方程两边同乘 $\frac{1}{\gamma^2 z^{\alpha-2}}$ 有

$$\frac{u''}{\gamma^2 z^{\alpha-2}} + \frac{1-2\alpha}{\gamma^2 z^{\alpha-1}} u' + \left[\beta^2 z^{2\gamma-\alpha} + \frac{\alpha^2}{\gamma^2 z^{\alpha}} - \frac{\nu^2}{z^{\alpha}}\right] u = 0$$

$$\Longrightarrow \frac{1}{\gamma^2}\frac{u''}{z^{\alpha-2}} + \frac{-\gamma-2\alpha+1+\gamma}{\gamma^2}\frac{u'}{z^{\alpha-1}} + \frac{\alpha(\gamma+\alpha)-\alpha\gamma}{\gamma^2 z^{\alpha}} u + \left[\beta^2 (z^{\gamma})^2 - \nu^2\right]\frac{u}{z^{\alpha}} = 0$$

$$\Longrightarrow \frac{1}{\gamma^2}\left[\frac{u''}{z^{\alpha-2}} + (-\gamma-2\alpha+1)\frac{u'}{z^{\alpha-1}} + \frac{\alpha(\gamma+\alpha)}{z^{\alpha}} u\right] + \frac{1}{\gamma z^{\gamma-1}}\left(\frac{u'}{z^{\alpha}} - \frac{\alpha u}{z^{\alpha+1}}\right) + \left[\beta^2 (z^{\gamma})^2 - \nu^2\right]\frac{u}{z^{\alpha}} = 0$$

其中 $\frac{1}{\gamma^2}\left[\frac{u''}{z^{\alpha-2}} + (-\gamma-2\alpha+1)\frac{u'}{z^{\alpha-1}} + \frac{\alpha(\gamma+\alpha)}{z^{\alpha}} u\right]$

$$= \frac{1}{\gamma^2}\left[\frac{u''}{z^{\alpha-2}} - (\gamma+\alpha-1)\frac{u'}{z^{\alpha-1}} - \frac{\alpha u'}{z^{\alpha-1}} + \frac{\alpha(\gamma+\alpha)}{z^{\alpha}} u\right].$$

$$= \frac{(z^{\gamma})^2}{\gamma}\frac{1}{\gamma z^{\gamma-1}}\left[\frac{u''}{z^{\gamma+\alpha-1}} - (\gamma+\alpha-1)\frac{u'}{z^{\gamma+\alpha}} - \frac{\alpha u'}{z^{\gamma+\alpha}} + \frac{\alpha(\gamma+\alpha)u}{z^{\gamma+\alpha+1}}\right].$$

$$= \frac{(z^{\gamma})^2}{\gamma}\frac{1}{\frac{d(z^{\gamma})}{dz}}\frac{d}{dz}\left[\frac{u'}{z^{\gamma+\alpha-1}} - \frac{\alpha u}{z^{\gamma+\alpha}}\right]$$

$$= \frac{(z^{\gamma})^2}{\gamma}\frac{d}{dz^{\gamma}}\left[\frac{u'}{z^{\gamma+\alpha-1}} - \frac{\alpha u}{z^{\gamma+\alpha}}\right].$$

$$= (z^{\gamma})^2\frac{d}{dz^{\gamma}}\left[\frac{1}{\gamma z^{\gamma-1}}\left(\frac{u'}{z^{\alpha}} - \frac{\alpha u}{z^{\alpha+1}}\right)\right]$$

$$= (z^{\gamma})^2\frac{d}{dz^{\gamma}}\left[\frac{1}{\frac{dz^{\gamma}}{dz}}\frac{d}{dz}\left(\frac{u}{z^{\alpha}}\right)\right]$$

$$= (z^{\gamma})^2\frac{d^2}{(dz^{\gamma})^2}\left(\frac{u}{z^{\alpha}}\right) \quad ①$$

$$\frac{1}{\gamma z^{\gamma-1}}\left(\frac{u'}{z^{\alpha}} - \frac{\alpha u}{z^{\alpha+1}}\right) = \frac{1}{\frac{dz^{\gamma}}{dz}}\frac{d}{dz}\left(\frac{u}{z^{\alpha}}\right) = \frac{d}{dz^{\gamma}}\left(\frac{u}{z^{\alpha}}\right) \quad ②$$

将①②同代入方程中有

$$(z^{\gamma})^2\frac{d^2}{(dz^{\gamma})^2}\left(\frac{u}{z^{\alpha}}\right) + \frac{d}{dz^{\gamma}}\left(\frac{u}{z^{\alpha}}\right) + \left[\beta^2(z^{\gamma})^2 - \nu^2\right]\frac{u}{z^2} = 0$$

记 $x = z^{\gamma}$, $y = \frac{u}{z^{\alpha}}$，有

$$x^2\frac{d^2}{dx^2}y + \frac{d}{dx}y + \left[\beta^2 x^2 - \nu^2\right]y = 0$$

这是一个参数形式的贝塞尔函数，其通解为

$$y = C_1 J_\nu(\beta x) + C_2 Y_\nu(\beta x) \quad ③$$

将 $x=z^{\gamma}$, $y=\frac{u}{z^{\alpha}}$ 同代入③中得到原函数的通解:

$$\frac{u}{z^{\alpha}}=C_1J_{\nu}(\beta z^{\gamma})+C_2Y_{\nu}(\beta z^{\gamma})$$

$$\Longrightarrow u=C_1z^{\alpha}J_{\nu}(\beta z^{\gamma})+C_2z^{\alpha}Y_{\nu}(\beta z^{\gamma})$$

1. $u''+az^{b}u=0$

在此方程中 $\begin{cases}1-2\alpha=0\\(\beta\gamma)^2=1\\(z^{\gamma-1})^2=z^{b}\\\alpha^2-\gamma^2\nu^2=0\end{cases}\Longrightarrow\begin{cases}\alpha=\frac{1}{2}\\\gamma=\frac{b+1}{2}\\\beta=\frac{2}{b+1}\\\nu=\frac{1}{b+1}\end{cases}$

故其通解为 $u=C_1z^{\frac{1}{2}}J_{\frac{1}{b+1}}(\frac{2}{b+1}z^{\frac{b+1}{2}})+C_2z^{\frac{1}{2}}Y_{\frac{1}{b+1}}(\frac{2}{b+1}z^{\frac{b+1}{2}})$

2. $zu''-3u'+zu=0$

$\Longrightarrow u''-\frac{3}{z}u'+u=0$

在此方程中 $\begin{cases}1-2\alpha=-3\\(\beta\gamma)^2=1\\(z^{\gamma-1})^2=z\\\alpha^2-\gamma^2\nu^2=0\end{cases}\Longrightarrow\begin{cases}\alpha=2\\\gamma=\frac{3}{2}\\\beta=\frac{2}{3}\\\nu=\frac{16}{9}\end{cases}$

故其通解为 $u=C_1z^{2}J_{\frac{16}{9}}(\frac{2}{3}z^{\frac{3}{2}})+C_2z^{2}Y_{\frac{16}{9}}(\frac{2}{3}z^{\frac{3}{2}})$

6. 分离变量 $u(\rho,t)=R(\rho)T(t)$

代入原方程得 $\frac{\partial T}{kT\partial t}=\frac{1}{\rho R}\frac{\partial}{\partial\rho}(\rho\frac{\partial R}{\partial\rho})=-\lambda^2$

其中λ是分离常数

$$\Longrightarrow\begin{cases}T'+k\lambda^2T=0 & ① \\ \frac{\partial^2R}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial R}{\partial\rho}+\lambda^2R=0 & ②\end{cases}$$

由①得 $T(t)=Ce^{-k\lambda^2t}$

②为0阶贝塞尔方程，其通解为

$$R(\rho)=AJ_0(\lambda\rho)+BY_0(\lambda\rho)$$

$\because u|_{\rho=0}$ 有界而 $Y_0(0)$ 无界

$\therefore B=0 \quad R(\rho)=AJ_0(\lambda\rho)$.

$\because \frac{\partial u}{\partial\rho}|_{\rho=a}=T\frac{\partial R}{\partial\rho}|_{\rho=a}=0 \Longrightarrow \frac{\partial J_0(\lambda\rho)}{\partial\rho}|_{\rho=a}=0$

$\therefore \lambda a=\mu'_{0m}$ 是 $J_0'(x)$ 的第m个零点（其实也是 $J_1(x)$ 的零点，$\mu'_{00}=0$）

综上：方程本征解为 $u_m(\rho,t)=A_mJ_0(\mu'_{0m}\frac{\rho}{a})e^{-k\lambda^2t}$

一般解为 $u(\rho,t)=\sum_{m=0}^{\infty}A_mJ_0(\mu'_{0m}\frac{\rho}{a})e^{-k\lambda^2t}$

利用初始条件 $u|_{t=0}=u_0(1-\frac{\rho^2}{a^2})$ 有

$$u_0(1-\frac{\rho^2}{a^2})=\sum_{m=0}^{\infty}A_mJ_0(\mu'_{0m}\frac{\rho}{a})=A_0+\sum_{m=1}^{\infty}A_mJ_0(\mu'_{0m}\frac{\rho}{a}).$$

在区间 $[0,a]$ 上展开的系数

这表明 $A_m$ 是 $u_0(1-\frac{\rho^2}{a^2})$ 按 $J_0(\mu'_{0m}\frac{\rho}{a})$

$$A_0=\frac{2}{a^2J_0^2(\mu'_{00})}\int_0^a\rho u_0(1-\frac{\rho^2}{a^2})J_0(\mu'_{00}\frac{\rho}{a})d\rho$$

$$=\frac{2u_0}{a^2}\int_0^a(\rho-\frac{\rho^3}{a^2})d\rho=\frac{u_0}{2}$$

$$A_m=\frac{2}{a^2J_0^2(\mu'_{0m})}\int_0^a\rho u_0(1-\frac{\rho^2}{a^2})J_0(\mu'_{0m}\frac{\rho}{a})d\rho$$

$$=\frac{2u_0}{a^2J_0^2(\mu'_{0m})}[\int_0^a\rho J_0(\mu'_{0m}\frac{\rho}{a})d\rho-\int_0^a\frac{\rho^3}{a^2}J_0(\mu'_{0m}\frac{\rho}{a})d\rho]$$

$$=\frac{2u_0}{a^2J_0^2(\mu'_{0m})}\left\{\frac{a}{\mu'_{0m}}\int_0^a d[\rho J_1(\mu'_{0m}\frac{\rho}{a})]-\frac{1}{a\mu'_{0m}}\int_0^a\rho^2d[\rho J_1(\mu'_{0m}\frac{\rho}{a})]\right\}$$

$$=\frac{2u_0}{a^2\mu'_{0m}J_0^2(\mu'_{0m})}\left\{a\rho J_1(\mu'_{0m}\frac{\rho}{a})\Big|_0^a-\frac{\rho^3}{a}J_1(\mu'_{0m}\frac{\rho}{a})\Big|_0^a+\frac{1}{a}\int_0^a\rho J_1(\mu'_{0m}\frac{\rho}{a})d\rho^2\right\}$$

$$= \frac{4u_0}{a^3 \mu_{0m}' J_0^2(\mu_{0m}')} \int_0^a \rho^2 J_1(\mu_{0m}' \frac{\rho}{a}) d\rho$$

$$= \frac{4u_0}{a^2 \mu_{0m}'^2 J_0^2(\mu_{0m}')} \int_0^a d[\rho^2 J_2(\mu_{0m}' \frac{\rho}{a})]$$

$$= \frac{4u_0}{a^2 \mu_{0m}'^2 J_0^2(\mu_{0m}')} \rho^2 J_2(\mu_{0m}' \frac{\rho}{a}) \Big|_0^a$$

$$= \frac{4u_0 J_2(\mu_{0m}')}{\mu_{0m}'^2 J_0^2(\mu_{0m}')}$$

根据递推公式 $J_0(x) + J_2(x) = \frac{2}{x} J_1(x) \Rightarrow J_2(\mu_{0m}') = \frac{2}{x} J_1(\mu_{0m}') - J_0(\mu_{0m}')$

$$= - J_0(\mu_{0m}')$$

$\therefore A_m = - \frac{4u_0}{\mu_{0m}'^2 J_0(\mu_{0m}')}$

综上：柱体内温度分布与变化为

$$u = \frac{u_0}{2} - 4u_0 \sum_{m=1}^{\infty} \frac{1}{\mu_{0m}' J_0(\mu_{0m}')} J_0(\mu_{0m}' \frac{\rho}{a}) e^{-k\left(\frac{\mu_{0m}'}{a}\right)^2 t}$$

当时间足够长时柱体达到稳定温度 $\lim\limits_{t \to +\infty} u = \frac{u_0}{2}$

7. $\begin{cases} \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{\partial^2 u}{\partial z^2}=0 \\ u|_{z=0}=0,\quad u|_{z=h}=0 \\ u|_{\rho=0}\text{ 有界},\quad u|_{\rho=a}=u_0\sin(2\pi\frac{z}{h}) \end{cases}$

分离变量，设 $u(\rho,z)=R(\rho)Z(z)$

代入原方程得 $-\frac{Z''}{Z}=\frac{R''+\frac{1}{\rho}R'}{R}=\lambda^2$

$$\Longrightarrow\begin{cases} Z''+\lambda^2 Z=0 & ① \\ R''+\frac{1}{\rho}R'-\lambda^2 R=0 & ② \end{cases}$$

由①得 $Z(z)=C\sin(\lambda z)+D\cos(\lambda z)$

由边界条件 $u|_{z=0}=0$ 得 $D=0$

由边界条件 $u|_{z=h}=0$ 得 $\lambda_n=\frac{n\pi}{h}$ $\Longrightarrow Z(z)=C\sin(\frac{n\pi}{h}z)$

②为0阶虚宗量贝塞尔方程，其通解为

$R(\rho)=A_0I_0(\lambda\rho)+BK_0(\lambda\rho)$

$\because u|_{\rho=0}$ 有界而 $K_0(0)$ 无界

$\therefore B=0\qquad R(\rho)=AI_0(\lambda\rho)=A_0I_0(\frac{n\pi}{h}\rho)$

故原方程的本征解为 $u_n(\rho,z)=A_n\sin(\frac{n\pi}{h}z)I_0(\frac{n\pi}{h}\rho)$

一般解为 $u(\rho,z)=\sum_{n=1}^{\infty}A_n\sin(\frac{n\pi}{h}z)I_0(\frac{n\pi}{h}\rho)$.

利用边界条件 $u|_{\rho=a}=u_0\sin(2\pi\frac{z}{h})$

$$\sum_{n=1}^{\infty}A_n\sin(\frac{n\pi}{h}z)I_0(\frac{n\pi a}{h})=u_0\sin(2\pi\frac{z}{h}).$$

$$\Longrightarrow A_2=\frac{u_0}{I_0(\frac{2\pi a}{h})}$$

$$A_n=0\quad(n\neq 2).$$

故 $u(\rho,z)=\frac{u_0}{I(\frac{2\pi a}{h})}\sin(\frac{n\pi}{h}z)\,I_0(\frac{2\pi}{h}\rho)$

8. Rodrigues公式: $P_l(x)=\frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$

$$\int_{-1}^{1}(1+x)^k P_l(x)\,dx=\int_{-1}^{1}(1+x)^k\frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l\,dx$$

$$=\frac{1}{2^l l!}\int_{-1}^{1}(1+x)^l\, d\left[\frac{d^{l-1}}{dx^{l-1}}(x^2-1)^l\right]$$

$$=\left[\frac{1}{2^l l!}(1+x)^k\frac{d^{l-1}}{dx^{l-1}}(x^2-1)^l\right]\Big|_{-1}^{1}-\frac{1}{2^l l!}\int_{-1}^{1}\frac{d^{l-1}}{dx^{l-1}}\left[(x^2-1)^l\right]d\left[(1+x)^k\right]$$

(由于 $\frac{d^{l-1}}{dx^{l-1}}(x^2-1)^l$ 中含有 $(x^2-1)$ 因子，故第一项为0)

$$=-\frac{k}{2^l l!}\int_{-1}^{1}(1+x)^{k-1}\frac{d^{l-1}}{dx^{l-1}}(x^2-1)^l\,dx$$

$$\cdots\cdots$$

$$=(-1)^l\frac{k!}{2^l l!(k-l)!}\int_{-1}^{1}(1+x)^{k-l}(x^2-1)^l\,dx$$

$$=\frac{k!}{2^l l!(k-l)!}\int_{-1}^{1}(1+x)^k(1-x)^l\,dx$$

$$\overset{t=\frac{x+1}{2}}{=\!=}\frac{k!}{2^l l!(k-l)!}\int_0^1(2t)^k(2-2t)^l\,d(2t-1)$$

$$=\frac{2^{k+1}k!}{l!(k-l)!}\int_0^1 t^{(k+1)-1}(1-t)^{(l+1)-1}\,dt$$

$$=\frac{2^{l+1}k!}{l!(k-l)!}B(k+1,l+1)$$

Beta函数

$$=\frac{2^{l+1}k!}{l!(k-l)!}\frac{\Gamma(k+1)\Gamma(l+1)}{\Gamma(k+l+2)}$$

$$=\frac{2^{l+1}k!}{l!(k-l)!}\frac{k!\,l!}{(k+l+1)!}$$

$$=\frac{2^{l+1}(k!)^2}{(k-l)!(k+l+1)!}$$

当 $k<l$ 时 $\int_{-1}^{1}(1+x)^k P_l(x)\,dx=(-1)^l\frac{k!}{2^l l!(k-l)!}\int_{-1}^{1}\frac{d^l}{dx^l}\left[(1+x)^k\right](x^2-1)^l\,dx$

$$=0$$

9. $\begin{cases} \nabla^2 u = 0, \ a<r<b \\ u|_{r=a} = u_0, \ u|_{r=b} = u_0 \cos^2\theta \end{cases}$

拉普拉斯方程在轴对称条件下的通解为

$$u(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

利用边界条件:

$$\begin{cases} u|_{r=a} = \sum_{l=0}^{\infty}\left(A_l a^l + \frac{B_l}{a^{l+1}}\right) P_l(\cos\theta) = u_0 = u_0 P_0(\cos\theta) + 0\cdot P_2(\cos\theta) \\ u|_{r=b} = \sum_{l=0}^{\infty}\left(A_l b^l + \frac{B_l}{b^{l+1}}\right) P_l(\cos\theta) = u_0\cos^2\theta = \frac{1}{3}u_0 P_0(\cos\theta) + \frac{2}{3}u_0 P_2(\cos\theta) \end{cases}$$

$$\xRightarrow{\text{比较系数}} \begin{cases} A_0 + \frac{B_0}{a} = u_0 \\ A_0 + \frac{B_0}{b} = \frac{1}{3}u_0 \\ A_2 a^2 + \frac{B_2}{a^3} = 0 \\ A_2 b^2 + \frac{B_2}{b^3} = \frac{2}{3}u_0 \\ A_l = 0, \ B_l = 0 \ (l\neq 0, l\neq 2) \end{cases} \Longrightarrow \begin{cases} A_0 = \frac{(-3a+b)u_0}{3(b-a)} \\ B_0 = \frac{2abu_0}{3(b-a)} \\ A_2 = \frac{2b^3u_0}{3(b^5-a^5)} \\ B_2 = -\frac{2a^5b^3u_0}{3(b^5-a^5)} \\ A_l = 0, \ B_l = 0 \ (l\neq 0, l\neq 2) \end{cases}$$

$$\therefore u(r,\theta) = \frac{(-3a+b)u_0}{3(b-a)} + \frac{2abu_0}{3(b-a)r} + \frac{2a^2b^3u_0}{3(b^5-a^5)}\left[\left(\frac{r}{a}\right)^2 - \left(\frac{a}{r}\right)^3\right] P_2(\cos\theta)$$

10. (1) $(\sin\theta - 2\cos^2\theta)\cos^2\varphi$

$= \frac{1}{2}(\sin\theta - 2\cos^2\theta) + \frac{1}{2}(\sin\theta - 2\cos^2\theta)\cos 2\varphi$

$= \frac{1}{2}(\sin\theta - 2\cos^2\theta) + \frac{1}{2}(\sin\theta - 2\cos^2\theta)(e^{2i\varphi} + e^{-2i\varphi})$

设 $\sin\theta - 2\cos^2\theta = \sum_{l=0}^{\infty} A_l P_l(\cos\theta)$

其中 $A_l = \frac{2l+1}{2}\int_{-1}^{1}(\sin\theta - 2\cos^2\theta)P_l(\cos\theta)\,d\cos\theta$

当 $l$ 为偶数. $A_l = -(2l+1)\int_{-1}^{1}\cos^2\theta P_l(\cos\theta)\,d\cos\theta$

$= -(2l+1)\int_{-1}^{1} x^2 P_l(x)\,dx$

$= -(2l+1)\int_{-1}^{1}\left[\frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)\right]P_l(x)\,dx$

$= \begin{cases} -\frac{2}{3}, & l=0 \\ -\frac{4}{3}, & l=2 \\ 0 & \text{otherwise} \end{cases}$

当 $l$ 为奇数. $A_l = \frac{2l+1}{2}\int_{-1}^{1}\sin\theta P_l(\cos\theta)\,d\cos\theta$

$= (2l+1)\int_{0}^{1}\sqrt{1-x^2}P_l(x)\,dx$

设 $\sin\theta - 2\cos^2\theta = \sum_{l=0}^{\infty} B_l P_l^2(\cos\theta)$

其中 $B_l = \frac{2l+1}{2}\frac{(l+2)!}{(l-2)!}\int_{-1}^{1}(\sin\theta - 2\cos^2\theta)P_l^2(\cos\theta)\,d\cos\theta$

当 $l$ 为偶数, $B_l = -(2l+1)\frac{(l+2)!}{(l-2)!}\int_{-1}^{1}\cos^2\theta P_l^2(\cos\theta)\,d\cos\theta$

$= -(2l+1)\frac{(l+2)!}{(l-2)!}\int_{-1}^{1}x^2 P_l^2(x)\,dx$

当 $l$ 为奇数, $B_l = \frac{2l+1}{2}\frac{(l+2)!}{(l-2)!}\int_{-1}^{1}\sin\theta P_l^2(\cos\theta)\,d\cos\theta$

$= (2l+1)\frac{(l+2)!}{(l-2)!}\int_{-1}^{1}\sqrt{1-x^2}P_l^2(x)\,dx$

$\therefore (\sin\theta - 2\cos^2\theta)\cos^2\varphi = \frac{1}{2}\sum_{l=0}^{\infty}\sqrt{\frac{4\pi}{2l+1}}A_l Y_l^0(\theta,\varphi) + \frac{1}{2}\sum_{l=0}^{\infty}\sqrt{\frac{4\pi}{2l+1}\frac{(l+2)!}{(l-2)!}}B_l\left[Y_l^2(\theta,\varphi) + Y_l^{-2}(\theta,\varphi)\right]$

其中 $A_l = \begin{cases} -\frac{2}{3}, & l=0 \\ -\frac{4}{3}, & l=2 \\ 0, & l \text{ 为偶数}, l \neq 0,2 \\ (2l+1)\int_0^1\sqrt{1-x^2}P_l(x)\,dx, & l \text{ 为奇数} \end{cases}$

$B_l = \begin{cases} -(2l+1)\frac{(l+2)!}{(l-2)!}\int_{-1}^{1}x^2 P_l^2(x)\,dx, & x \text{ 为偶数} \\ (2l+1)\frac{(l+2)!}{(l-2)!}\int_{-1}^{1}\sqrt{1-x^2}P_l^2(x)\,dx, & x \text{ 为奇数} \end{cases}$

(2) $(1-2\sin\theta)\cos\theta\cos\varphi$

$$=\frac{1}{2}(1-2\sin\theta)\cos\theta\left(e^{i\varphi}+e^{-i\varphi}\right)$$

设 $(1-2\sin\theta)\cos\theta=\sum_{l=0}^{\infty}C_l P_l^1(\cos\theta)$.

其中 $C_l=\frac{2l+1}{2}\frac{(l+1)!}{(l-1)!}\int_{-1}^{1}(1-2\sin\theta)\cos\theta P_l^1(\cos\theta)\,d\cos\theta$

当$l$为奇数，$C_l=-(2l+1)\frac{(l+1)!}{(l-1)!}\int_{-1}^{1}\sin\theta\cos\theta P_l^1(\cos\theta)\,d\cos\theta$

$$=-2(2l+1)\frac{(l+1)!}{(l-1)!}\int_0^1\sqrt{1-x^2}\,x P_l^1(x)\,dx.$$

当$l$为偶数，$C_l=\frac{2l+1}{2}\frac{(l+1)!}{(l-1)!}\int_{-1}^{1}\cos\theta P_l^1(\cos\theta)\,d\cos\theta$

$$=(2l+1)\frac{(l+1)!}{(l-1)!}\int_0^1 x P_l^1(x)\,dx.$$

$$\therefore (1-2\sin\theta)\cos\theta\cos\varphi=\frac{1}{2}\sum_{l=0}^{\infty}\sqrt{\frac{4\pi}{2l+1}\frac{(l+1)!}{(l-1)!}}\,C_l\left[Y_l^1(x)+Y_l^{-1}(x)\right]$$

其中 $C_l=\begin{cases}-2(2l+1)\frac{(l+1)!}{(l-1)!}\int_0^1\sqrt{1-x^2}\,xP_l^1(x)\,dx, & l\text{为奇数}\\(2l+1)\frac{(l+1)!}{(l-1)!}\int_0^1 xP_l^1(x)\,dx, & l\text{为偶数}\end{cases}$

11. $\begin{cases} \nabla^2 u = A + Br^2 \sin 2\theta \cos\varphi \\ u|_{r=a} = 0 \end{cases}$

设方程通解为 $u(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=0}^{l} P_l^m(\cos\theta)[R_{l,m}(r)\sin m\varphi + S_{l,m}(r)\cos m\varphi]$

代入方程得 $\sum_{l=0}^{\infty}\sum_{m=0}^{l} \frac{1}{r^2}\left[\frac{d}{dr}\left(r^2\frac{dR_{l,m}}{dr}\right) - l(l+1)R_{l,m}\right]P_l^m(\cos\theta)\sin m\varphi$

$$+ \sum_{l=0}^{\infty}\sum_{m=0}^{l} \frac{1}{r^2}\left[\frac{d}{dr}\left(r^2\frac{dS_{l,m}}{dr}\right) - l(l+1)S_{l,m}\right]P_l^m(\cos\theta)\cos m\varphi$$

$$= A + Br^2\sin 2\theta\cos\varphi = AP_0^0(\cos\theta) - \frac{2}{3}Br^2 P_2^1(\cos\theta)\cos\varphi.$$

比较系数得 $\frac{d}{dr}\left(r^2\frac{dR_{l,m}}{dr}\right) - l(l+1)R_{l,m} = 0.$

$$\frac{d}{dr}\left(r^2\frac{dS_{0,0}}{dr}\right) = Ar^2$$

$$\frac{d}{dr}\left(r^2\frac{dS_{2,1}}{dr}\right) - 6S_{2,1} = -\frac{2}{3}Br^4$$

$\frac{d}{dr}\left(r^2\frac{S_{l,m}}{dr}\right) - l(l+1)S_{l,m} = 0$ (除 $(l,m)=(0,0)$ 和 $(l,m)=(2,1)$ 外)

由边界条件 $R_{l,m}(a)=0$, $S_{l,m}(a)=0$

由 $u|_{r=0}$ 有界，$R_{l,m}(0)$ 有界，$S_{l,m}(0)$ 有界.

$\therefore$ $R_{l,m}(r)=0.$

$S_{0,0}(r) = \frac{A}{6}(r^2-a^2)$

$S_{2,1}(r) = \frac{Br^2}{21}(a^2-r^2)$

$S_{l,m}(r)=0$ (除 $(l,m)=(0,0)$ 和 $(l,m)=(2,1)$ 外)

$\therefore u(r,\theta,\varphi) = \frac{A}{6}(r^2-a^2) + \frac{B}{21}r^2(a^2-r^2)P_2^1(\cos\theta)\cos\varphi$

$$= \frac{A}{6}(r^2-a^2) + \frac{B}{14}r^2(r^2-a^2)\sin 2\theta\cos\varphi.$$