1. (1) $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(2x+\lambda)y=0$

(当 $x\neq 0$ 时) 原方程可化为 $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\frac{2}{x}y+\frac{\lambda}{x^2}y=0$

其中 $a(x)=\frac{1}{x}$　$b(x)=\frac{2}{x}$　$c(x)=\frac{1}{x^2}$

Sturm-Liouville 方程：$\frac{d}{dx}[e^{\int a(x)dx}\frac{dy}{dx}]+[b(x)e^{\int a(x)dx}]y+\lambda[c(x)e^{\int a(x)dx}]y=0$

即 $\frac{d}{dx}(|x|\frac{dy}{dx})+\frac{2|x|}{x}y+\lambda\frac{|x|}{x^2}y=0$

$\Longrightarrow \frac{d}{dx}(x\frac{dy}{dx})+2y+\frac{\lambda}{x}y=0$

(2) $x(1-x)\frac{d^2y}{dx^2}+(a-bx)\frac{dy}{dx}-\lambda y=0$

(当 $x\neq 0, x\neq 1$ 时) 原方程可化为 $\frac{d^2y}{dx^2}+\frac{a-bx}{x(1-x)}\frac{dy}{dx}-\frac{\lambda}{x(1-x)}y=0$

其中 $a(x)=\frac{a-bx}{x(1-x)}$　$b(x)=0$　$c(x)=-\frac{1}{x(1-x)}$

$=\frac{(a-b)x+a(1-x)}{x(1-x)}$

$=\frac{a-b}{1-x}+\frac{a}{x}$

$\int a(x)dx=(b-a)\ln|x-1|+a\ln|x|+C_1=\ln C|x-1|^{b-a}|x|^a$

Sturm-Liouville 方程: $\frac{d}{dx}[e^{\int a(x)dx}\frac{dy}{dx}]+[b(x)e^{\int a(x)dx}]y+\lambda[c(x)e^{\int a(x)dx}]y=0$

即 $\frac{d}{dx}(|x-1|^{b-a}|x|^a\frac{dy}{dx})+\lambda\frac{|x-1|^{b-a}|x|^a}{(x-1)x}y=0$

$\Longrightarrow \frac{d}{dx}[(x-1)^{b-a}x^a\frac{dy}{dx}]+\lambda(x-1)^{b-a-1}x^{a-1}y=0$

2. $\begin{cases} \dfrac{d}{dx}\left[p(x)\dfrac{dy}{dx}\right]+[\lambda\rho(x)-q(x)]y=0 \\ y(b)=a_{11}y(a)+a_{12}y'(a),\ y'(b)=a_{21}y(a)+a_{22}y'(a) \end{cases}$

其中 $p(a)=p(b)$

Q 因子： $Q=p(a)[y_n(a)y_m'(a)-y_m(a)y_n'(a)]$

$-p(b)[y_n(b)y_m'(b)-y_m(b)y_n'(b)]$

$=p(a)[y_n(a)y_m'(a)-y_m(a)y_n'(a)$

$-(a_{11}y_n(a)+a_{12}y_n'(a))(a_{21}y_m(a)+a_{22}y_m'(a))$

$+(a_{11}y_m(a)+a_{12}y_m'(a))(a_{21}y_n(a)+a_{22}y_n'(a))]$

$=p(a)\left(1-\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right)[y_n(a)y_m'(a)-y_m(a)y_n'(a)]$

当 $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=1$ 时，$Q=0$

从而对应不同本征值的本征函数正交.

3. 求证: (1) $\delta(\vec{r}-\vec{r}_0)=\frac{1}{r^2}\delta(r-r_0)\delta(\cos\theta-\cos\theta_0)\delta(\varphi-\varphi_0)$

左边 $=\delta(\vec{r}-\vec{r}_0)=\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$

$$=\begin{cases}+\infty, & x=x_0,\ y=y_0,\ z=z_0\\ 0, & \text{otherwise}\end{cases}$$

$$=\begin{cases}+\infty, & r=r_0,\ \theta=\theta_0,\ \varphi=\varphi_0\\ 0, & \text{otherwise}\end{cases}$$

右边 $=\frac{1}{r^2}\delta(r-r_0)\delta(\cos\theta-\cos\theta_0)\delta(\varphi-\varphi_0)$

$$=\begin{cases}+\infty, & r=r_0,\ \theta=\theta_0,\ \varphi=\varphi_0\\ 0, & \text{otherwise}\end{cases}$$

对左边积分 $=\iiint_{\mathbb{R}^3}\delta(\vec{r}-\vec{r}_0)d^3V=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)dxdydz$

$$=\int_{-\infty}^{+\infty}\delta(x-x_0)dx\int_{-\infty}^{+\infty}\delta(y-y_0)dy\int_{-\infty}^{+\infty}\delta(z-z_0)dz$$

$$=1$$

对右边积分 $=\iiint_{\mathbb{R}^3}\frac{1}{r^2}\delta(r-r_0)\delta(\cos\theta-\cos\theta_0)\delta(\varphi-\varphi_0)dV$

$$=\int_0^{2\pi}\int_0^{\pi}\int_0^{+\infty}\frac{1}{r^2}\delta(r-r_0)\delta(\cos\theta-\cos\theta_0)\delta(\varphi-\varphi_0)r^2\sin\theta\, dr\,d\theta\,d\varphi$$

$$=\int_0^{2\pi}\delta(r-r_0)dr\int_{-1}^{1}\delta(\cos\theta-\cos\theta_0)d\cos\theta\int_0^{+\infty}\delta(\varphi-\varphi_0)d\varphi$$

$$=1$$

综上可知 $\delta(\vec{r}-\vec{r}_0)=\frac{1}{r^2}\delta(r-r_0)\delta(\cos\theta-\cos\theta_0)\delta(\varphi-\varphi_0)$

(2) $\nabla^2\frac{1}{|\vec{r}-\vec{r}_0|}=-4\pi\delta(\vec{r}-\vec{r}_0)$

当 $\vec{r}\neq\vec{r}_0$ 时 $\nabla\frac{1}{|\vec{r}-\vec{r}_0|}=-\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3}$

$$\nabla^2\frac{1}{|\vec{r}-\vec{r}_0|}=-\nabla\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3}=0=-4\pi\delta(\vec{r}-\vec{r}_0)$$

$$\iiint_{\mathbb{R}^3}\nabla^2\frac{1}{|\vec{r}-\vec{r}_0|}dV=\iiint_{|\vec{r}-\vec{r}_0|<R}\nabla^2\frac{1}{|\vec{r}-\vec{r}_0|}dV=\oiint_{|\vec{r}-\vec{r}_0|=R}\nabla\frac{1}{|\vec{r}-\vec{r}_0|}\cdot d\vec{S}$$

$$=\oiint_{|\vec{r}-\vec{r}_0|=R}-\frac{(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3}\cdot d\vec{S}$$

$$= -\iint_{|\vec{r}-\vec{r}_0|=R} \frac{ds}{|\vec{r}-\vec{r}_0|^2} = -\iint_{|\vec{r}-\vec{r}_0|=R} \frac{ds}{R^2}$$

$$= -\frac{4\pi R^2}{R^2} = -4\pi = \iiint_{\mathbb{R}^3} -4\pi\delta(\vec{r}-\vec{r}_0)\,dV$$

综上可见：$\nabla^2 \frac{1}{|\vec{r}-\vec{r}_0|} = -4\pi\delta(\vec{r}-\vec{r}_0)$

4. (1) $\int_{-\infty}^{\infty}\frac{\cos\omega t}{\omega^2+a^2}d\omega=\frac{\pi}{a}e^{-a|t|}\quad (a>0)$

$$\int_{-\infty}^{\infty}\frac{\cos\omega t}{\omega^2+a^2}d\omega=Re\left(\int_{-\infty}^{+\infty}\frac{e^{i\omega t}}{\omega^2+a^2}d\omega\right)$$

当 $t\geq 0$ 时 $\frac{e^{i\omega|t|}}{\omega^2+a^2}$ 在上半复平面有一个奇点 $\omega=ai$，且为一阶极点

$$\int_{-\infty}^{+\infty}\frac{e^{i\omega t}}{\omega^2+a^2}d\omega=2\pi i\,Re\left[\frac{e^{i\omega t}}{\omega^2+a^2},ai\right]$$

$$=2\pi i\lim_{\omega\to ai}\frac{e^{i\omega t}}{\omega+ai}$$

$$=\frac{\pi}{a}e^{-at}\quad \int_{-\infty}^{+\infty}\frac{\cos\omega t}{\omega^2+a^2}d\omega=Re\left(\int_{-\infty}^{+\infty}\frac{e^{i\omega t}}{\omega^2+a^2}d\omega\right)=\frac{\pi}{a}e^{-at}$$

当 $t<0$ 时 $\int_{-\infty}^{\infty}\frac{\cos\omega t}{\omega^2+a^2}d\omega=\int_{-\infty}^{\infty}\frac{\cos\omega(-t)}{\omega^2+a^2}d\omega=Re\left(\int_{-\infty}^{+\infty}\frac{e^{i\omega(-t)}}{\omega^2+a^2}d\omega\right)$

$$=\frac{\pi}{a}e^{-a(-t)}$$

综上：$\int_{-\infty}^{\infty}\frac{\cos\omega t}{\omega^2+a^2}d\omega=\frac{\pi}{a}e^{-a|t|}$

(2) 设 $|\vec{r}|=\sqrt{x^2+y^2+z^2}$, $|\vec{k}|=\sqrt{k_1^2+k_2^2+k_3^2}$

(i) $\frac{1}{r}\leftrightarrow\sqrt{\frac{2}{\pi}}\frac{1}{k^2}$

我们先做 $\frac{e^{-ar}}{r}$ 的傅立叶变换

$$\mathcal{F}\left[\frac{e^{-ar}}{r}\right]=\frac{1}{\sqrt{2\pi}}\iiint_{\mathbb{R}^3}\frac{e^{-ar}}{r}e^{-i\vec{k}\cdot\vec{r}}d^3r$$

$$=\frac{1}{\sqrt{2\pi}}\int_0^{+\infty}\frac{e^{-ar}}{r}r^2dr\int_0^{\pi}e^{-ik\cdot r\cdot\cos\theta}\sin\theta\, d\theta\int_0^{2\pi}d\varphi$$

$$=\sqrt{2\pi}\int_0^{+\infty}\frac{e^{-ar}}{r}r^2dr\int_{-1}^{1}e^{ikr(-\cos\theta)}d(-\cos\theta)$$

$$=\sqrt{2\pi}\int_0^{+\infty}\frac{e^{-ar}}{r}r^2\left.\frac{e^{ikr(-\cos\theta)}}{ikr}\right|_{-\cos\theta=-1}^{1}dr$$

$$=\frac{2\sqrt{2\pi}}{k}\int_0^{+\infty}e^{-ar}\sin kr\,dr$$

$$=\frac{2\sqrt{2\pi}}{k}Im\left[\int_0^{+\infty}e^{-ar}e^{ikr}dr\right]$$

$$=\frac{2\sqrt{2\pi}}{k}Im\left[\int_0^{+\infty}e^{(-a+ik)r}dr\right]$$

$$=\frac{2\sqrt{2\pi}}{k}Im\left[\left.\frac{e^{(-a+ik)r}}{-a+ik}\right|_0^{+\infty}\right]$$

$$=\frac{2\sqrt{2\pi}}{k}\left.\frac{e^{-ar}[k\cos kr+a\sin kr]}{q^2+a^2}\right|_0^{+\infty}$$

$$=\frac{2\sqrt{2\pi}}{q^2+a^2}$$

取 $a\to 0$，得到 $\mathcal{F}\left[\frac{1}{r}\right]=2\sqrt{2\pi}\frac{1}{q^2}$

(ii) $$\mathcal{F}^{-1}\left[\frac{\sin ak}{k}\right]=\frac{1}{\sqrt{2\pi}}\iiint_{\mathbb{R}^3}\frac{\sin ak}{k}e^{i\vec{k}\cdot\vec{r}}d^3k$$

$$=\frac{1}{\sqrt{2\pi}}\int_0^{+\infty}\frac{\sin ak}{k}k^2dk\int_0^{\pi}e^{ikr\cos\theta}\sin\theta d\theta\int_0^{2\pi}d\varphi$$

$$=\sqrt{2\pi}\int_0^{+\infty}\frac{\sin ak}{k}k^2dk\int_{-1}^{1}e^{-ikr(-\cos\theta)}d(-\cos\theta)$$

$$=\sqrt{2\pi}\int_0^{+\infty}\frac{\sin ak}{k}k^2\left.\frac{e^{-ikr(-\cos\theta)}}{-ikr}\right|_{-\cos\theta=-1}^{1}dk$$

$$=\frac{2\sqrt{2\pi}}{r}\int_0^{+\infty}\sin ak\sin kr\,dk$$

(由三角函数正交性) $$2\sqrt{2\pi}\,\frac{\delta(r-a)}{r}$$

5. $F(p) = \int_0^{+\infty} f(t) e^{-pt} dt$

$$= \int_0^a f(t) e^{-pt} dt + \int_a^{2a} f(t) e^{-pt} dt + \int_{2a}^{3a} f(t) e^{-pt} dt + \cdots$$

$$= \int_0^a f(t) e^{-pt} dt + \int_0^a f(t_1+a) e^{-p(t_1+a)} d(t_1+a) + \int_0^a f(t_2+2a) e^{-p(t_2+a)} d(t_2+a) + \cdots$$

$$= \int_0^a f(t) e^{-pt} dt + e^{-pa} \int_0^a f(t_1) e^{-pt_1} dt_1 + e^{-2pa} \int_0^a f(t_2) e^{-pt_2} dt_2$$

$$= (1 + e^{-pa} + e^{-2pa} + \cdots) \int_0^a f(t) e^{-pt} dt$$

$$= \frac{1}{1 - e^{-pa}} \int_0^a f(t) e^{-pt} dt$$

6 $\begin{cases}\dfrac{\partial u}{\partial t}=a^2\dfrac{\partial^2 u}{\partial x^2}+f(x,t), -\infty<x<+\infty\\ u|_{t=0}=\varphi(x)\end{cases}$

(1) 将 $u(x,t)$ 和 $f(x,t)$ 关于 $t$ 的拉普拉斯变换分别记作 $U(x,p)$ 和 $F(x,p)$

方程两边同关于 $t$ 做拉普拉斯变换得

$$pU(x,p)-u(x,0)=a^2\frac{d^2U(x,p)}{dx^2}+F(x,p)$$

即 $$\frac{d^2U(x,p)}{dx^2}-\frac{p}{a^2}U(x,p)=-\frac{F(x,p)+\varphi(x)}{a^2}$$

对应齐次方程 $\dfrac{d^2U(x,p)}{dx^2}-\dfrac{p}{a^2}U(x,p)=0$ 的两特征解为

$$U_1(x,p)=e^{\frac{\sqrt{p}}{a}x} \text{ 和 } U_2(x,p)=e^{-\frac{\sqrt{p}}{a}x}$$

利用常数变易法得原方程通解

$$C_1(x)=-\int_{-\infty}^{x}\frac{U_2\cdot\left[-\frac{F(\xi,p)+\varphi(\xi)}{a^2}\right]}{\begin{vmatrix}U_1 & U_2\\ U_1' & U_2'\end{vmatrix}}d\xi=-\frac{1}{2\sqrt{p}a}\int_{-\infty}^{x}[F(\xi,p)+\varphi(\xi)]e^{-\frac{\sqrt{p}}{a}\xi}d\xi$$

$$C_2(x)=\int_{-\infty}^{x}\frac{U_1\cdot\left[-\frac{F(\xi,p)+\varphi(\xi)}{a^2}\right]}{\begin{vmatrix}U_1 & U_2\\ U_1' & U_2'\end{vmatrix}}d\xi=\frac{1}{2\sqrt{p}a}\int_{-\infty}^{x}[F(\xi,p)+\varphi(\xi)]e^{\frac{\sqrt{p}}{a}\xi}d\xi$$

从而原方程的解为

$$\begin{aligned}U(x,p)&=C_1(x)U_1(x,p)+C_2(x)U_2(x,p)\\&=\frac{1}{2\sqrt{p}a}\Big[-\int_{-\infty}^{x}\varphi(\xi)e^{\frac{\sqrt{p}}{a}(x-\xi)}d\xi+\int_{-\infty}^{x}\varphi(\xi)e^{\frac{\sqrt{p}}{a}(\xi-x)}d\xi\\&\quad-\int_{-\infty}^{x}F(\xi,p)e^{\frac{\sqrt{p}}{a}(x-\xi)}d\xi+\int_{-\infty}^{x}F(\xi,p)e^{\frac{\sqrt{p}}{a}(\xi-x)}d\xi\Big]\\&=\frac{1}{2\sqrt{p}a}\Big[\int_{x}^{+\infty}\varphi(\xi)e^{\frac{\sqrt{p}}{a}(x-\xi)}d\xi+\int_{-\infty}^{x}\varphi(\xi)e^{\frac{\sqrt{p}}{a}(\xi-x)}d\xi\\&\quad+\int_{x}^{+\infty}F(\xi,p)e^{\frac{\sqrt{p}}{a}(x-\xi)}d\xi+\int_{-\infty}^{x}F(\xi,p)e^{\frac{\sqrt{p}}{a}(\xi-x)}d\xi\Big]\end{aligned}$$

利用拉普拉斯逆变换

$$\mathcal{L}^{-1}\left[\frac{1}{2\sqrt{p}a}e^{\frac{\sqrt{p}(x-\xi)}{a}}\right]=\mathcal{L}^{-1}\left[\frac{1}{2\sqrt{p}a}e^{\frac{\sqrt{p}(\xi-x)}{a}}\right]=\frac{e^{-\frac{(x-\xi)^2}{4a^2t}}}{2a\sqrt{\pi t}}$$

$$\mathcal{L}^{-1}\left[\frac{1}{2\sqrt{p}a}F(\xi,p)e^{\frac{\sqrt{p}(x-\xi)}{a}}\right]=\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{f(\xi,\tau)\cdot e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}}{\sqrt{t-\tau}}d\tau \quad \text{(卷积定理)}$$

$$=\mathcal{L}^{-1}\left[\frac{1}{2\sqrt{p}a}F(x,p)e^{\frac{\sqrt{p}(\xi-x)}{a}}\right]$$

得到 $u(x,t)=\mathcal{L}[U(x,p)]=\frac{1}{2a\sqrt{\pi t}}\left[\int_x^{+\infty}\varphi(\xi)e^{-\frac{(x-\xi)^2}{4at^2}}d\xi+\int_{-\infty}^{x}\varphi(\xi)e^{-\frac{(x-\xi)^2}{4at^2}}d\xi\right]$

$$+\frac{1}{2a\sqrt{\pi}}\left[\int_x^{+\infty}\int_0^t\frac{f(\xi,\tau)e^{-\frac{(x-\xi)^2}{4a(t-\tau)}}}{\sqrt{t-\tau}}d\tau d\xi+\int_{-\infty}^{x}\int_0^t\frac{f(\xi,\tau)e^{-\frac{(x-\xi)^2}{4a(t-\tau)}}}{\sqrt{t-\tau}}d\tau d\xi\right]$$

$$=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}\varphi(\xi)e^{-\frac{(x-\xi)^2}{4at^2}}d\xi+\frac{1}{2a\sqrt{\pi}}\int_{-\infty}^{+\infty}\int_0^t\frac{f(\xi,\tau)e^{-\frac{(x-\xi)^2}{4a(t-\tau)}}}{\sqrt{t-\tau}}d\tau d\xi$$

(2) 记 $u(x,t)$, $f(x,t)$ 和 $\varphi(x)$ 关于 $x$ 的傅立叶变换分别为 $U(\omega,t)$, $F(\omega,t)$ 和 $\phi(\omega)$

原方程组化为

$$\begin{cases}\frac{dU(\omega,t)}{dt}+\omega^2a^2U(\omega,t)=F(\omega,t) & (1)\\ U|_{t=0}=\phi(\omega) & (2)\end{cases}$$

利用常数变易法，得到(1)的通解

$$U(\omega,t)=e^{-\int_0^t\omega^2a^2d\tau}\left[\int_0^tF(\omega,\ )e^{\int_0^\tau\omega^2a^2d\tau_1}d\tau+C\right]$$

$$=e^{-\omega^2a^2t}\left[\int_0^tF(\omega,\tau)e^{\omega^2a^2\tau}d\tau+C\right]$$

代入(2)得 $C=\phi(\omega)$

从而 $U(\omega,t)=e^{-\omega^2a^2t}\left[\int_0^tF(\omega,\tau)e^{\omega^2a^2\tau}d\tau+\phi(\omega)\right]$

利用傅里叶逆变换

$$\mathcal{F}^{-1}\left[F(\omega,\tau)e^{\omega^2a^2(\tau-t)}\right]=\frac{1}{2a\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{f(\xi,\tau)e^{-\frac{(x-\xi)^2}{4a(t-\tau)}}}{\sqrt{t-\tau}}d\xi \quad (卷积定理)$$

$$\mathcal{F}^{-1}\left[e^{-\omega^2a^2t}\phi(\omega)\right]=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}\varphi(\xi)e^{-\frac{(x-\xi)^2}{4at^2}}d\xi \quad (卷积定理)$$

得到 $u(x,t)=\mathcal{F}^{-1}[U(\omega,t)]=\frac{1}{2a\sqrt{\pi}}\int_0^t\int_{-\infty}^{+\infty}\frac{f(\xi,\tau)e^{-\frac{(x-\xi)^2}{4a(t-\tau)}}}{\sqrt{t-\tau}}d\xi d\tau$

$$+\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}\varphi(\xi)e^{-\frac{(x-\xi)^2}{4at^2}}d\xi$$

7. $$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2\dfrac{\partial^2 u}{\partial x^2} = 0 \\ u|_{t=0} = \varphi(x),\ \dfrac{\partial u}{\partial t}\Big|_{t=0} = \psi(x) \end{cases}$$

对t作拉普拉斯变换得

$$p^2 U(x,p) - p u(x,0) - u_t(x,0) - c^2\frac{d^2}{dx^2}U(x,p) = 0$$

代入 $u|_{t=0}$ 和 $\frac{\partial u}{\partial t}\big|_{t=0}$ 得

$$c^2\frac{d^2 U(x,p)}{dx^2} - p^2 U(x,p) + p\varphi(x) + \psi(x) = 0$$

对x作傅立叶变换得

$$-\omega^2 c^2 U(\omega,p) - p^2 U(\omega,p) + p\Phi(\omega) + \Psi(\omega) = 0$$

$$\Longrightarrow U(\omega,p) = \frac{p\Phi(\omega) + \Psi(\omega)}{\omega^2 c^2 + p^2}$$

作拉普拉斯反变换得

$$U(\omega,t) = \Phi(\omega)\cos\omega ct + \frac{\Psi(\omega)}{\omega c}\sin\omega ct$$

作傅立叶反变换得

$$u(x,t) = \int_{-\infty}^{+\infty}\varphi(\xi)\frac{\delta(x-\xi+ct)+\delta(x-\xi-ct)}{2}d\xi$$

$$+\int_{x-ct}^{x+ct}\psi(\xi)\frac{1}{2c}d\xi$$

$$= \frac{\varphi(x+ct)+\varphi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(\xi)d\xi$$