PHYS1303 □□□□□□II□□□□□

□□□

TOTAL POINTS

67 / 75

QUESTION 1

1 □□□ 10 / 10

- ✓ **- 0 pts** **Correct**
- **- 5 pts** The answer is wrong
- **- 2 pts** There is a slight flaw
- **- 1 pts** Please simplify
- **- 10 pts** blank or wrong
- **- 5 pts** Late

QUESTION 2

2 □□□ 5 / 10

- **- 0 pts** Correct
- **- 10 pts** Wrong/Not given
- **- 5 pts** Handle homework late
- **- 4 pts** Error in solving process
- ✓ **- 2 pts** **Small calculation error**
- ✓ **- 3 pts** **Incorrect drawing**

QUESTION 3

3 □□□ 10 / 10

- ✓ **- 0 pts** **Correct**
- **- 10 pts** blank or wrong
- **- 3 pts** half wrong
- **- 4 pts** One situation is missing
- **- 7 pts** The answer is wrong
- **- 5 pts** late

QUESTION 4

4 □□□ 10 / 10

- ✓ **- 0 pts** **Correct**
- **- 10 pts** Wrong/Not given
- **- 5 pts** Handle late
- **- 3 pts** Incorrectly drawing
- **- 4 pts** Incorrectly processing
- **- 2 pts** Problems with simplification or computation

QUESTION 5

5 □□□ 12 / 15

- **- 0 pts** Correct
- ✓ **- 3 pts** **Please simplify**
- **- 15 pts** blank or wrong
- **- 1 pts** lack of θ
- **- 7.5 pts** late

QUESTION 6

6 □□□ 10 / 10

- ✓ **- 0 pts** **Correct**
- **- 5 pts** half right
- **- 10 pts** blank or wrong
- **- 3 pts** orthogonality
- **- 5 pts** late

QUESTION 7

7 □□□ 10 / 10

- ✓ **- 0 pts** **Correct**
- **- 10 pts** Not given/Wrong
- **- 5 pts** late
- **- 7 pts** Serious problems
- **- 5 pts** Little problems

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StudentID: 45875852

Problem 1. 解无界弦上的振动问题

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + \cos(\omega t)\cos(x), (-\infty < x < +\infty, t > 0) \\ u|_{t=0} = e^{-2x^2}, (-\infty < x < +\infty) \\ \frac{\partial u}{\partial t}|_{t=0} = \sin(x), (-\infty < x < +\infty) \end{cases}$$

Solution:

$$\phi(x) = e^{-2x^2}, \ \ \psi = \sin x, \ \ f(x,t) = \cos(\omega t)\cos x$$

利用达朗贝尔公式

$$\begin{aligned} u(x,t) =& \frac{1}{2}[\phi(x+at) + \phi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi + \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi d\tau \\ =& \frac{1}{2}[e^{-2(x+at)^2} + e^{-2(x-at)^2}] + \frac{1}{2a}\int_{x-at}^{x+at}\sin\xi d\xi + \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} \cos(\omega\tau)\cos\xi d\xi d\tau \\ =& \frac{1}{2}[e^{-2(x+at)^2} + e^{-2(x-at)^2}] + \frac{1}{2a}[-\cos(x+at) + \cos(x-at)] \\ &+ \frac{1}{2a}\int_0^t \cos(\omega\tau)\{\sin[x+a(t-\tau)] - \sin[x-a(t-\tau)]\}d\tau \\ =& \frac{1}{2}[e^{-2(x+at)^2} + e^{-2(x-at)^2}] + \frac{1}{a}\sin x\sin(at) + \frac{1}{a}\int_0^t \cos(\omega\tau)\cos x\sin[a(t-\tau)]d\tau \\ =& \frac{1}{2}[e^{-2(x+at)^2} + e^{-2(x-at)^2}] + \frac{1}{a}\sin x\sin(at) \\ &+ \frac{\cos x}{2a}\int_0^t \{\sin[\omega\tau + a(t-\tau)] - \sin[\omega\tau - a(t-\tau)]\}d\tau \\ =& \frac{1}{2}[e^{-2(x+at)^2} + e^{-2(x-at)^2}] + \frac{1}{a}\sin x\sin(at) + \frac{1}{\omega^2 - a^2}\cos x[\cos(\omega t) - \cos(at)] \end{aligned}$$

□

1 □□□ 10 / 10

- ✓ **- 0 pts** Correct
- **- 5 pts** The answer is wrong
- **- 2 pts** There is a slight flaw
- **- 1 pts** Please simplify
- **- 10 pts** blank or wrong
- **- 5 pts** Late

2. $\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} \quad (0<x<l,\ t>0) \\ \dfrac{\partial u}{\partial x}\Big|_{x=0} = 0,\ \dfrac{\partial u}{\partial x}\Big|_{x=l} = 0 \\ u|_{t=0} = e^{-x^2},\ \dfrac{\partial u}{\partial t}\Big|_{t=0} = 2axe^{-x^2} \end{cases}$

设方程有分离变量解: $u(x,t) = X(x)T(t)$

代入泛定方程有 $\dfrac{X''(x)}{X(x)} = \dfrac{T''(t)}{a^2T(t)} \triangleq -\lambda$

$$\Longrightarrow \begin{cases} X''(x) + \lambda X(x) = 0 & ① \\ T''(t) + \lambda a^2 T(t) = 0 & ② \end{cases}$$

将形式解代入边界条件有 $\begin{cases} X'(0)T(t) = 0 \\ X'(l)T(t) = 0 \end{cases}$

$$\Longrightarrow X'(0) = X'(l) = 0$$

或 $T(t) = 0$ (平庸解!)

联立① 从而有 $\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = X'(l) = 0 \end{cases}$

若 $\lambda<0$, 则方程通解为 $X(x) = Ae^{kx} + Be^{-kx}$, 其中 $k = \sqrt{-\lambda}$

由边界条件有 $\begin{cases} X'(0) = Ak - Bk = 0 \\ X'(l) = Ake^{kl} - Bke^{-kl} = 0 \end{cases}$

$$\Longrightarrow \frac{A}{B} = e^{-2kl} = 1 \Longrightarrow k = 0 \text{ 即 } \lambda = 0 \text{ (与 } \lambda<0 \text{ 矛盾)}$$

若 $\lambda=0$, 则方程通解为 $X(x) = Ax + B$

由边界条件有 $A = 0$, 从而 $X(x) = B$ (不符合初始条件)

若 $\lambda>0$, 则方程通解为 $X(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$

由边界条件有 $\begin{cases} X'(0) = B\sqrt{\lambda} = 0 \\ X'(l) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}\,l) + B\sqrt{\lambda}\cos(\sqrt{\lambda}\,l) = 0 \end{cases}$

$$\Longrightarrow B = 0 \text{ 且 } \sqrt{\lambda}\,l = n\pi$$

故本征函数 $X_n = A_n\cos\left(\frac{n\pi}{l}x\right)$ 本征值 $\lambda_n = \left(\frac{n\pi}{l}\right)^2$

将本征值代入②有 $T''(t) + \left(\frac{na\pi}{l}\right)^2 T(t) = 0$

其通解为 $T_n(t) = C_n\cos\left(\frac{na\pi}{l}t\right) + d_n\sin\left(\frac{na\pi}{l}t\right)$

从而 $u_n(x,t) = X_n(x)T_n(t) = \left[C_n\cos\left(\frac{na\pi}{l}t\right) + D_n\sin\left(\frac{na\pi}{l}t\right)\right]\cos\left(\frac{n\pi}{l}x\right)$

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

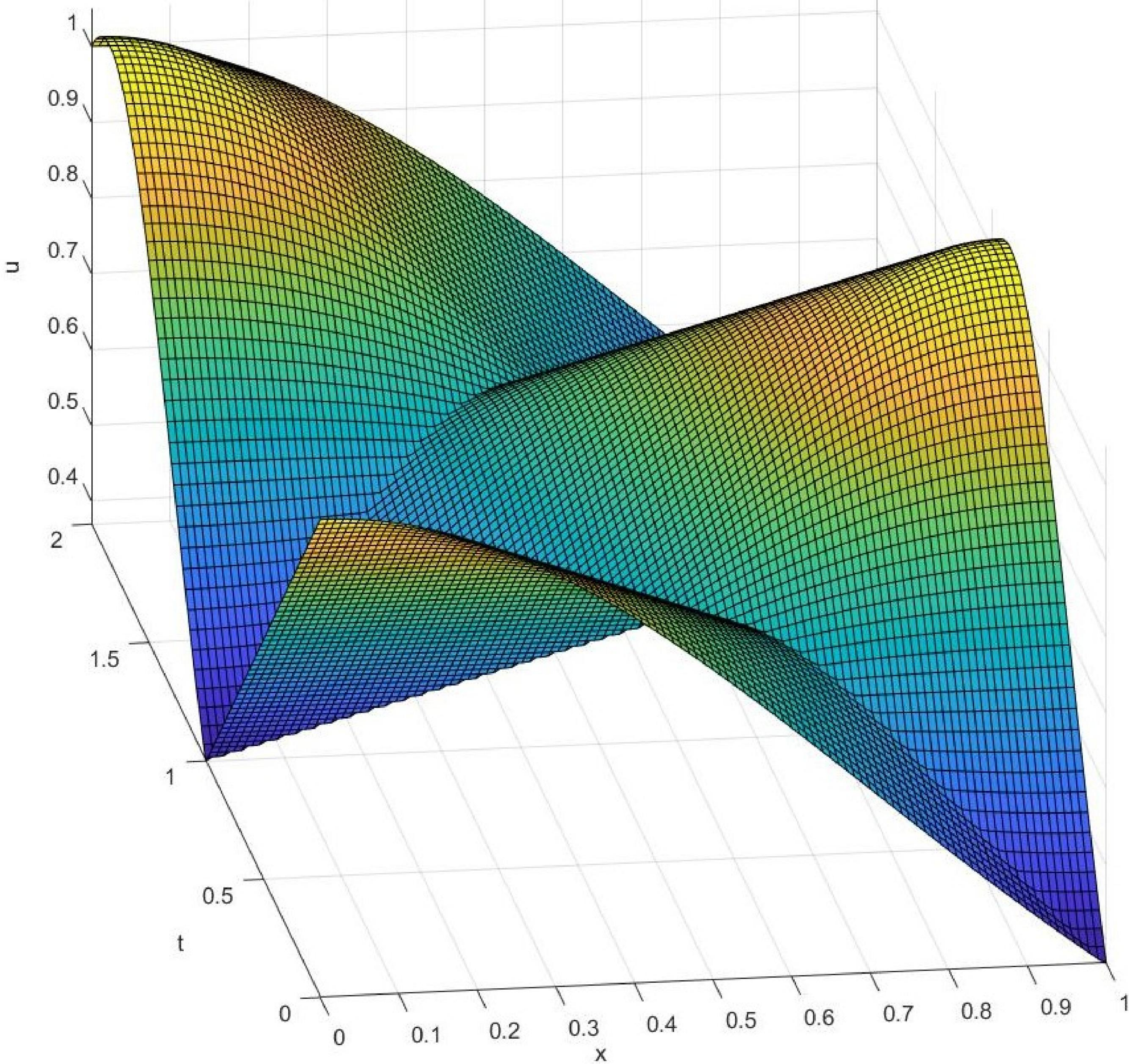
代入初始条件有 $\begin{cases} u|_{t=0} = \sum\limits_{n=0}^{\infty} C_n \cos\left(\frac{n\pi}{l}x\right) = e^{-x} \\ \frac{\partial u}{\partial t}\Big|_{t=0} = \sum\limits_{n=1}^{\infty} \frac{na\pi}{l} D_n \cos\left(\frac{n\pi}{l}x\right) = 2axe^{-x^2} \end{cases}$

$$\Longrightarrow \begin{cases} C_n = \frac{2}{l}\int_0^l e^{-x^2}\cos\left(\frac{n\pi}{l}x\right)dx, \quad n=0,1,2,\cdots \\ D_n = \frac{2}{na\pi}\int_0^l 2axe^{-x^2}\cos\left(\frac{n\pi}{l}x\right)dx, \quad n=1,2,\cdots \end{cases}$$

综上：$u(x,t) = \frac{C_0}{2} + \sum\limits_{n=1}^{\infty}\left[C_n\cos\left(\frac{na\pi}{l}t\right) + D_n\sin\left(\frac{na\pi}{l}t\right)\right]\cos\left(\frac{n\pi}{l}x\right)$

其中 $\begin{cases} C_n = \frac{2}{l}\int_0^l e^{-x^2}\cos\left(\frac{n\pi}{l}x\right)dx, \quad n=0,1,2,\cdots \\ D_n = \frac{2}{na\pi}\int_0^l 2axe^{-x^2}\cos\left(\frac{n\pi}{l}x\right)dx, \quad n=1,2,\cdots \end{cases}$

取$a=1$，$l=1, 0\leqslant t\leqslant 2$，$n=100$，作图如下页所示


u
1
0.9
0.8
0.7
0.6
0.5
0.4
t
2
1.5
1
0.5
0
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

2 ▯▯▯ 5 / 10

- **- 0 pts** Correct
- **- 10 pts** Wrong/Not given
- **- 5 pts** Handle homework late
- **- 4 pts** Error in solving process
- ✓ **- 2 pts Small calculation error**
- ✓ **- 3 pts Incorrect drawing**

3. $$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}=a^2\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}\right) \\ u|_{t=0}=\begin{cases} u_0, & x^2+y^2+z^2<R^2 \\ 0, & x^2+y^2+z^2>R^2 \end{cases}, \quad \dfrac{\partial u}{\partial t}\Big|_{t=0}=0 \end{cases}$$

转化到球坐标系中有
$$\begin{cases} \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right)+\dfrac{1}{r^2\sin^2\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial u}{\partial\theta}\right)+\dfrac{1}{r^2\sin\theta}\dfrac{\partial^2 u}{\partial\phi^2} \\ =\dfrac{1}{a^2}\dfrac{\partial^2 u}{\partial t^2} \qquad ① \\ u(r,0)=\left.\begin{cases} u_0, & r<R \\ 0, & r>R \end{cases}\right\}\overset{def}{=}\varphi(r), \quad u_t(r,0)=0 \end{cases}$$

由于 u 不依赖于 θ 和 ϕ，① 简化为 $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right)=\dfrac{1}{a^2}\dfrac{\partial^2 u}{\partial t^2}$

作函数代换 $v(r,t)=ru(r,t)$

从而有
$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2}=a^2\dfrac{\partial^2 v}{\partial r^2} & r,t>0 \\ v(r,0)=\begin{cases} ru_0, & r<R \\ 0, & r>R \end{cases} & v_t(r,0)=0 \\ v(0,t)=0 \end{cases}$$

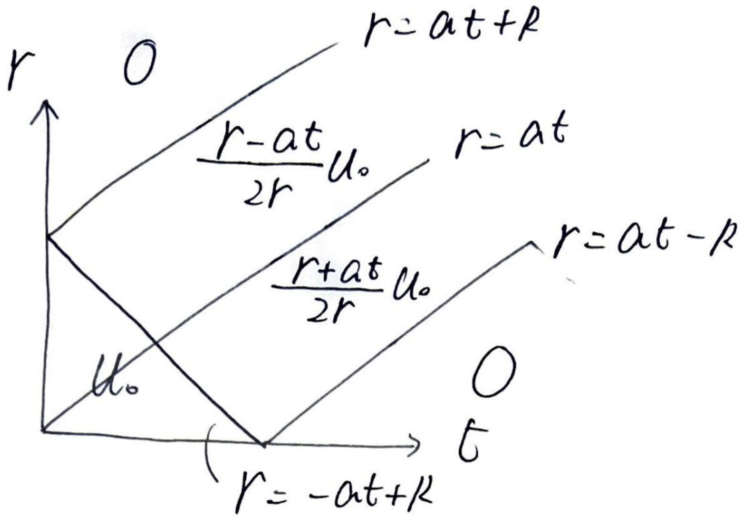
利用达朗贝尔公式

当 $r-at<0$ 时 $u(r,t)=\dfrac{1}{2r}[(r+at)\varphi(r+at)-(at-r)\varphi(at-r)]$
$$=\frac{1}{2r}(r-at)\varphi(at-r)$$
$$=\begin{cases} 0, & r+at>R,\ at-r>R \\ \dfrac{r+at}{2r}u_0, & r+at>R,\ at-r<R \\ u_0, & r+at<R,\ at-r<R \end{cases}$$

当 $r-at\geq 0$ 时 $u(r,t)=\dfrac{1}{2r}[(r+at)\varphi(r+at)+(r-at)\varphi(r-at)]$
$$=\begin{cases} 0, & r+at>R,\ r-at>R \\ \dfrac{r-at}{2r}u_0, & r+at>R,\ r-at<R \\ u_0, & r+at<R,\ r-at<R \end{cases}$$

综上：
$$u(r,t)=\begin{cases} u_0 & r+at<R \\ \dfrac{r-at}{2r}u_0 & r+at>R \text{ 且 } 0<r-at<R \\ \dfrac{r+at}{2r}u_0 & r+at>R \text{ 且 } -R<r-at<0 \\ 0, & r>at+R \text{ 或 } 0<r<at-R \end{cases}$$

回到直角坐标系
$$u(x,y,z,t)=\begin{cases} u_0, & \sqrt{x^2+y^2+z^2}+at<R \\ \dfrac{\sqrt{x^2+y^2+z^2}-at}{2\sqrt{x^2+y^2+z^2}}, & \sqrt{x^2+y^2+z^2}+at>R \text{ 且 } 0<\sqrt{x^2+y^2+z^2}-at<R \\ \dfrac{\sqrt{x^2+y^2+z^2}+at}{2\sqrt{x^2+y^2+z^2}}, & \sqrt{x^2+y^2+z^2}+at>R \text{ 且 } -R<\sqrt{x^2+y^2+z^2}-at<0 \\ 0, & \sqrt{x^2+y^2+z^2}>at+R \text{ 或 } 0<\sqrt{x^2+y^2+z^2}<at-R \end{cases}$$



r
0
$r=at+R$
$\frac{r-at}{2r}u_0$
$r=at$
$r=at-R$
$\frac{r+at}{2r}u_0$
u_0
0
t
$r=-at+R$

3 ▯▯▯ **10 / 10**

✓ **- 0 pts** Correct

- 10 pts blank or wrong

- 3 pts half wrong

- 4 pts One situation is missing

- 7 pts The answer is wrong

- 5 pts late

4.
$$\begin{cases} \dfrac{\partial u}{\partial t}=a^2\dfrac{\partial^2 u}{\partial x^2} \\ u(0,t)=u(l,t)=0 \\ u(x,0)=\dfrac{bx(l-x)}{l^2} \end{cases}$$

设方程有分离变量解 $u(x,t)=X(x)T(t)$

代入齐次方程有 $\dfrac{X''(x)}{X(x)}=\dfrac{T'(t)}{a^2T(t)}\triangleq-\lambda$

$$\Longrightarrow \begin{cases} X''(x)+\lambda X(x)=0 & ① \\ T'(t)+\lambda a^2T(t)=0 & ② \end{cases}$$

①结合边界条件得到本征函数 $X_n(x)=B_n\sin(\frac{n\pi}{l}x)$

和本征值 $\lambda_n=(\frac{n\pi}{l})^2$

将本征值代入②有 $T'(t)+(\frac{na\pi}{l})^2T(t)=0$

通解为 $T_n(t)=C_n e^{-(\frac{na\pi}{l})^2 t}$

从而 $u_n(x,t)=X_n(x)T_n(t)=C_n e^{-(\frac{na\pi}{l})^2 t}\sin(\frac{n\pi}{l}x)$

$u(x,t)=\sum\limits_{n=1}^{\infty}u_n(x,t)$

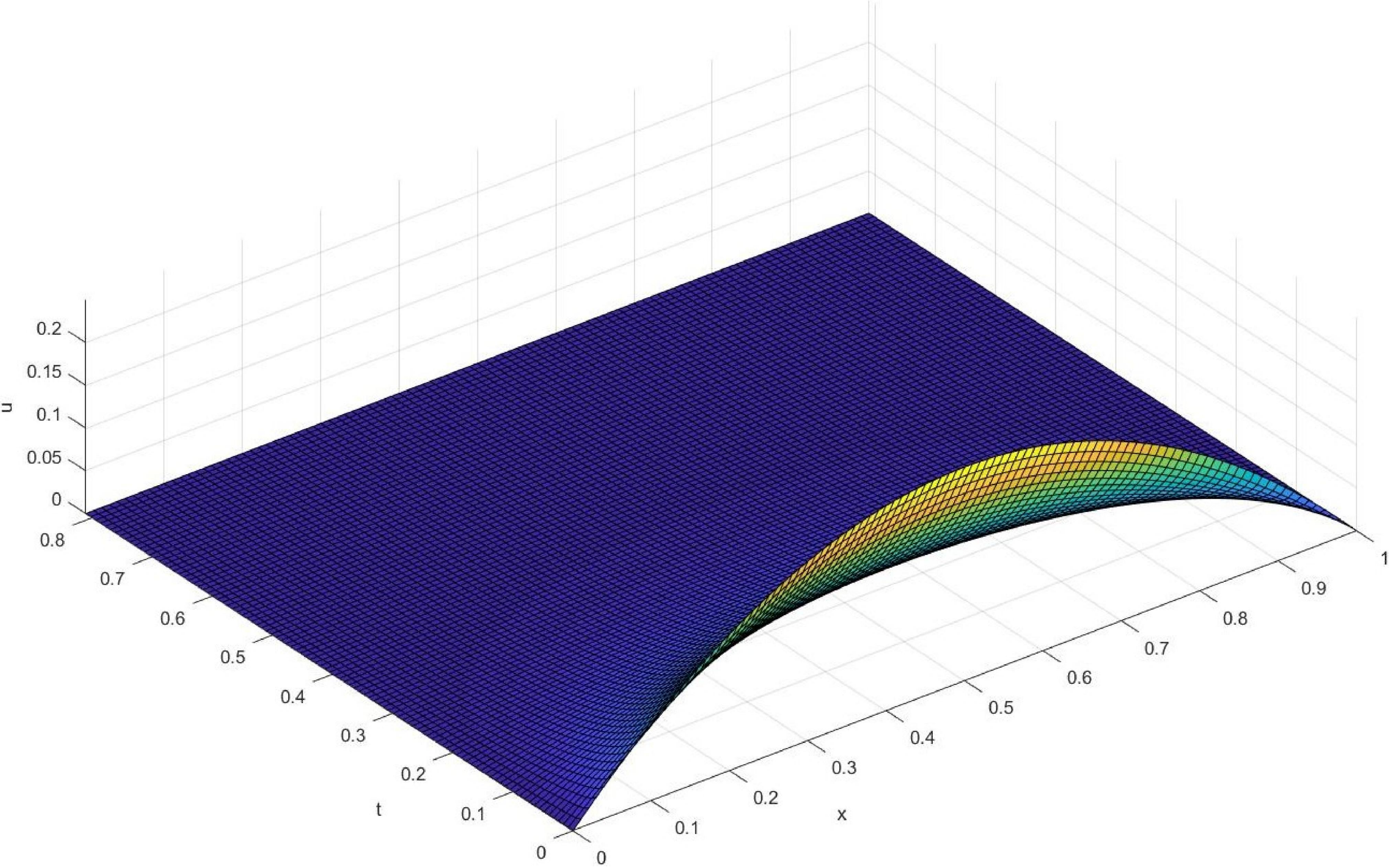
代入初始条件有 $u(x,0)=\sum\limits_{n=1}^{\infty}C_n\sin(\frac{n\pi}{l}x)=\dfrac{bx(l-x)}{l^2}$

$$\Longrightarrow C_n=\frac{2}{l}\int_0^l\frac{bx(l-x)}{l^2}\sin(\frac{n\pi}{l}x)$$
$$=\frac{4b}{n^3\pi^3}[1+(-1)^{n+1}]$$

综上: $u(x,t)=\sum\limits_{n=1}^{\infty}C_n e^{-(\frac{na\pi}{l})^2 t}\sin(\frac{n\pi}{l}x)$

其中 $C_n=\dfrac{4b}{n^3\pi^3}[1+(-1)^{n+1}]$.

取l＝1，b＝1，a＝1，n＝100, 0 ⩽ t ⩽ 8／ π ^2，作图如下页所示



u
0.2
0.15
0.1
0.05
0
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
t
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x

4 ▯▯▯ 10 / 10

- ✓ **- 0 pts** Correct
- **- 10 pts** Wrong/Not given
- **- 5 pts** Handle late
- **- 3 pts** Incorrectly drawing
- **- 4 pts** Incorrectly processing
- **- 2 pts** Problems with simplification or computation

5. (1) $a_1(x)\frac{\partial^2 u}{\partial x^2}+b_1(y)\frac{\partial^2 u}{\partial y^2}+a_2(x)\frac{\partial u}{\partial x}+b_2(y)\frac{\partial u}{\partial y}=0$

设 $u(x,y)=X(x)Y(y)$

代入原方程有 $a_1(x)X''(x)Y(y)+b_1(y)X(x)Y''(y)$

$$+a_2(x)X'(x)Y(y)+b_2(y)X(x)Y'(y)=0$$

两边同除 $X(x)Y(y)$ 有

$$\frac{a_1(x)X''(x)}{X(x)}+\frac{b_1(y)Y''(y)}{Y(y)}+\frac{a_2(x)X'(x)}{X(x)}+\frac{b_2(y)Y'(y)}{Y(y)}=0$$

$$\Longrightarrow \frac{a_1(x)X''(x)+a_2(x)X'(x)}{X(x)}=-\frac{b_1(y)Y''(y)+b_2(y)Y'(y)}{Y(y)}\triangleq\lambda$$

$$\Longrightarrow \begin{cases} a_1(x)X''(x)+a_2(x)X'(x)-\lambda X(x)=0 \\ b_1(y)Y''(y)+b_2(y)Y'(y)+\lambda Y(y)=0 \end{cases}$$

(2) $\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}=0$

设 $u(\rho,\varphi)=P(\rho)\Phi(\varphi)$

代入原方程得 $\frac{1}{\rho}\Phi(\varphi)\frac{\partial}{\partial\rho}\left(\rho\frac{\partial P(\rho)}{\partial\rho}\right)+\frac{1}{\rho^2}P(\rho)\frac{\partial^2\Phi(\varphi)}{\partial\varphi^2}=0$

两边同乘 $\frac{\rho^2}{\Phi(\varphi)P(\rho)}$ 有

$$\frac{\rho}{P(\rho)}\frac{d}{d\rho}\left(\rho\frac{dP(\rho)}{d\rho}\right)+\frac{1}{\Phi(\varphi)}\frac{d^2\Phi(\varphi)}{d\varphi^2}=0$$

$$\Longrightarrow \frac{\rho}{P(\rho)}\frac{d}{d\rho}\left(\rho\frac{dP(\rho)}{d\rho}\right)=-\frac{1}{\Phi(\varphi)}\frac{d^2\Phi(\varphi)}{d\varphi^2}\triangleq\lambda$$

$$\Longrightarrow \begin{cases} \boxed{\rho\frac{d}{d\rho}\left(\rho\frac{dP(\rho)}{d\rho}\right)}-\lambda P(\rho)=0 \\ \frac{d^2\Phi(\varphi)}{d\varphi^2}+\lambda\Phi(\varphi)=0 \end{cases}$$

(3) $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)=0$

设 $u(r,\theta)=R(r)\Theta(\theta)$

代入原方程中得. $\frac{1}{r^2}\Theta(\theta)\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)+\frac{1}{r^2\sin\theta}R(r)\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)=0$

两边同乘 $\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)+\frac{1}{\sin\theta}\frac{1}{\Theta(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)=0$

$$\Longrightarrow \frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)=-\frac{1}{\sin\theta}\frac{1}{\Theta(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)\triangleq\lambda$$

$$\Longrightarrow\begin{cases}\boxed{\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)}-\lambda R(r)=0\\ \frac{1}{\sin\theta}\boxed{\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)}+\lambda\Theta(\theta)=0\end{cases}$$

5 ▯▯▯ 12 / 15

- **- 0 pts** Correct
- ✓ **- 3 pts** **Please simplify**
- **- 15 pts** blank or wrong
- **- 1 pts** lack of θ
- **- 7.5 pts** late

6. $\begin{cases} X''+\lambda X=0 \\ \alpha_1 X(0)+\beta_1 X'(0)=0 \\ \alpha_2 X(l)+\beta_2 X'(l)=0 \end{cases}$

通解为 $X=A\cos\sqrt{\lambda}x+B\sin\sqrt{\lambda}x$

代入边界条件 $\begin{cases} \alpha_1 X(0)+\beta_1 X'(0)=\alpha_1 A+\beta_1 B\sqrt{\lambda}=0 \\ \alpha_2 X(l)+\beta_2 X'(l)=\alpha_2(A\cos\sqrt{\lambda}l+B\sin\sqrt{\lambda}l) \\ \quad +\beta_2\sqrt{\lambda}(-A\sin\sqrt{\lambda}l+B\cos\sqrt{\lambda}l)=0 \end{cases}$

$\Longrightarrow$ 本征函数 $X_n(x)=C_n\left[\sin(\sqrt{\lambda_n}x)-\frac{\beta_1}{\alpha_1}\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x)\right]$.

本征值 λ_n 是 $(\alpha_1\alpha_2+\beta_1\beta_2\lambda)\tan\sqrt{\lambda}l+(\alpha_1\beta_2-\alpha_2\beta_1)\sqrt{\lambda}=0$ 的解.

~~证明正交性~~ ~~举例说明正交性:~~

证明正交性:

$$\int_0^l\left[\sin(\sqrt{\lambda_n}x)-\frac{\beta_1}{\alpha_1}\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x)\right]\left[\sin(\sqrt{\lambda_m}x)-\frac{\beta_1}{\alpha_1}\sqrt{\lambda_m}\cos(\sqrt{\lambda_m}x)\right]$$

$$=\int_0^l \sin(\sqrt{\lambda_n}x)\sin(\sqrt{\lambda_m}x)+\frac{\beta_1^2}{\alpha_1^2}\sqrt{\lambda_m}\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x)\cos(\sqrt{\lambda_m}x)$$
$$-\frac{\beta_1}{\alpha_1}\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x)\sin(\sqrt{\lambda_m}x)-\frac{\beta_1}{\alpha_1}\sqrt{\lambda_m}\sin(\sqrt{\lambda_n}x)\cos(\sqrt{\lambda_m}x)\,dx$$

$$=\int_0^l\Big\{\frac{1}{2}\left[\cos(\sqrt{\lambda_m}-\sqrt{\lambda_n})x-\cos(\sqrt{\lambda_n}+\sqrt{\lambda_m})x\right]$$
$$+\frac{1}{2}\frac{\beta_1^2}{\alpha_1^2}\sqrt{\lambda_n}\sqrt{\lambda_m}\left[\cos(\sqrt{\lambda_n}-\sqrt{\lambda_m})x+\cos(\sqrt{\lambda_n}+\sqrt{\lambda_m})x\right]$$
$$-\frac{1}{2}\frac{\beta_1}{\alpha_1}\sqrt{\lambda_n}\left[\sin(\sqrt{\lambda_n}+\sqrt{\lambda_m})x-\sin(\sqrt{\lambda_n}-\sqrt{\lambda_m})x\right]$$
$$-\frac{1}{2}\frac{\beta_1}{\alpha_1}\sqrt{\lambda_m}\left[\sin(\sqrt{\lambda_n}+\sqrt{\lambda_m})x+\sin(\sqrt{\lambda_n}-\sqrt{\lambda_m})x\right].$$

$$=\frac{1}{2}\Big\{\frac{1+\frac{\beta_1^2}{\alpha_1^2}\sqrt{\lambda_n}\sqrt{\lambda_m}}{\sqrt{\lambda_n}-\sqrt{\lambda_m}}\sin(\sqrt{\lambda_n}-\sqrt{\lambda_m})l$$
$$-\frac{1+\frac{\beta_1^2}{\alpha_1^2}\sqrt{\lambda_n}\sqrt{\lambda_m}}{\sqrt{\lambda_n}+\sqrt{\lambda_m}}\sin(\sqrt{\lambda_n}+\sqrt{\lambda_m})l$$
$$+\frac{\beta_1}{\alpha_1}\left[\cos(\sqrt{\lambda_n}+\sqrt{\lambda_m})l-1\right]$$
$$+\frac{\beta_1}{\alpha_1}\left[\cos(\sqrt{\lambda_n}-\sqrt{\lambda_m})l-1\right]\Big\}.$$

$$=\frac{1}{2}\left\{\frac{1+\frac{\beta_1^2}{\alpha_1^2}\sqrt{\lambda_n}\sqrt{\lambda_m}}{\sqrt{\lambda_n}-\sqrt{\lambda_m}}\left[\sin\sqrt{\lambda_n}l\cos\sqrt{\lambda_m}l-\cos\sqrt{\lambda_n}l\sin\sqrt{\lambda_m}l\right]\right.$$

$$-\frac{1+\frac{\beta_1^2}{\alpha_1^2}\sqrt{\lambda_n}\sqrt{\lambda_m}}{\sqrt{\lambda_n}+\sqrt{\lambda_m}}\left[\sin\sqrt{\lambda_n}l\cos\sqrt{\lambda_m}l+\cos\sqrt{\lambda_n}l\sin\sqrt{\lambda_m}l\right]$$

$$\left.+\frac{2\beta_1}{\alpha_1}\cos\sqrt{\lambda_n}l\cos\sqrt{\lambda_m}l-\frac{2\beta_1}{\alpha_1}\right\}.$$

$$=\frac{1}{2}\left\{\frac{1+\frac{\beta_1^2}{\alpha_1^2}\sqrt{\lambda_n}\sqrt{\lambda_m}}{\sqrt{\lambda_n}-\sqrt{\lambda_m}}(\tan\sqrt{\lambda_n}l-\tan\sqrt{\lambda_m}l)\cos\sqrt{\lambda_n}l\cos\sqrt{\lambda_m}l\right.$$

$$-\frac{1+\frac{\beta_1^2}{\alpha_1^2}\sqrt{\lambda_n}\sqrt{\lambda_m}}{\sqrt{\lambda_n}+\sqrt{\lambda_m}}(\tan\sqrt{\lambda_n}l+\tan\sqrt{\lambda_m}l)\cos\sqrt{\lambda_n}l\cos\sqrt{\lambda_m}l$$

$$\left.+\frac{2\beta_1}{\alpha_1}\cos\sqrt{\lambda_n}l\cos\sqrt{\lambda_m}l-\frac{2\beta_1}{\alpha_1}\right\}$$

(代入 $\tan\sqrt{\lambda_i}l=\frac{(\alpha_1\beta_2-\alpha_2\beta_1)\sqrt{\lambda}}{\alpha_1\alpha_2+\beta_1\beta_2\lambda}$ 得)

$$=\frac{1}{2}\left\{\frac{1+\frac{\beta_1^2}{\alpha_1^2}\sqrt{\lambda_n}\sqrt{\lambda_m}}{\sqrt{\lambda_n}-\sqrt{\lambda_m}}\left[\frac{-(\alpha_1\beta_2-\alpha_2\beta_1)\sqrt{\lambda_n}}{\alpha_1\alpha_2+\beta_1\beta_2\lambda_n}+\frac{(\alpha_1\beta_2-\alpha_2\beta_1)\sqrt{\lambda_m}}{\alpha_1\alpha_2+\beta_1\beta_2\lambda_m}\right]\right.$$

$$-\frac{1+\frac{\beta_1^2}{\alpha_1^2}\sqrt{\lambda_n}\sqrt{\lambda_m}}{\sqrt{\lambda_m}+\sqrt{\lambda_m}}\left[\frac{-(\alpha_1\beta_2-\alpha_2\beta_1)\sqrt{\lambda_n}}{\alpha_1\alpha_2+\beta_1\beta_2\lambda_n}+\frac{-(\alpha_1\beta_2-\alpha_2\beta_1)\sqrt{\lambda_m}}{\alpha_1\alpha_2+\beta_1\beta_2\lambda_m}\right]$$

$$\left.+\frac{2\beta_1}{\alpha}\cos\sqrt{\lambda_n}l\cos\sqrt{\lambda_m}l-\frac{2\beta_1}{\alpha_1}\right\}.$$

$$\overset{(通分)}{=\!=}\ 0$$

由此得证本征函数正交性.

求归一化因子:

$$\int_0^l X_n^2(x)dx=C_n^2\int_0^l\sin^2(\sqrt{\lambda_n}x-\delta_n)dx\ \left(其中\tan\delta_n=\frac{\beta_1}{\alpha_1}\sqrt{\lambda_n}\right).$$

$$=C_n^2\int_0^l\frac{1-\cos(2\sqrt{\lambda_n}x-2\delta_n)}{2}dx$$

$$=C_n^2\left\{\frac{1}{2}-\frac{1}{4\sqrt{\lambda_n}}\left[\sin 2(\sqrt{\lambda_n}l-\delta_n)+\sin 2\delta_n\right]\right\}$$

$$\because\tan\sqrt{\lambda}l=-\sqrt{\lambda_n}\frac{\alpha_1\beta_2-\alpha_2\beta_1}{\alpha_1\alpha_2+\beta_1\beta_2\lambda}$$

$$\sin 2\sqrt{\lambda_n}l = \frac{2\tan\sqrt{\lambda_n}l}{1+\tan^2\sqrt{\lambda_n}l} = -2\sqrt{\lambda_n}\,\frac{\alpha_1^2\alpha_2\beta_2-\alpha_1\alpha_2^2\beta_1+\alpha_1\beta_1\beta_2^2\lambda_n-\alpha_2\beta_1^2\beta_2}{(\alpha_1^2+\beta_1^2\lambda_n)(\alpha_2^2+\beta_2^2\lambda_n)}$$

$$\cos 2\sqrt{\lambda_n}l = \frac{1-\tan^2\sqrt{\lambda_n}l}{1+\tan^2\sqrt{\lambda_n}l} = \frac{\alpha_1^2\alpha_2^2-\alpha_1^2\beta_2^2\lambda_n-\alpha_2^2\beta_1^2\lambda_n+4\alpha_1\alpha_2\beta_1\beta_2\lambda_n+\beta_1^2\beta_2^2\lambda}{(\alpha_1^2+\beta_1^2\lambda_n)(\alpha_2^2+\beta_2^2\lambda_n)}$$

$$\sin 2\delta_n = \frac{2\tan\delta_n}{1+\tan^2\delta_n} = 2\sqrt{\lambda_n}\frac{\alpha_1\beta_1}{\alpha_1^2+\beta_1^2\lambda_n} \qquad \cos 2\delta_n = \frac{1-\tan^2\delta_n}{1+\tan^2\delta_n} = \frac{\alpha_1^2-\beta_1^2\lambda_n}{\alpha_1^2+\beta_1^2\lambda_n}$$

$$\sin 2(\sqrt{\lambda_n}l-\delta_n) = \sin 2\sqrt{\lambda_n}l\cos 2\delta_n - \cos 2\sqrt{\lambda_n}l\sin 2\delta_n = -2\sqrt{\lambda_n}\frac{\alpha_2\beta_2}{\alpha_2^2+\beta_2^2\lambda_n}$$

$$\therefore \int_0^l X_n^2(x)\,dx = C_n^2\left[\frac{l}{2}+\frac{1}{2}\left(\frac{\alpha_2\beta_2}{\alpha_2^2+\beta_2^2\lambda_n}-\frac{\alpha_1\beta_1}{\alpha_1^2+\beta_1^2\lambda_n}\right)\right].$$

$$\Longrightarrow C_n = \sqrt{\frac{1}{\frac{l}{2}+\frac{1}{2}\left(\frac{\alpha_2\beta_2}{\alpha_2^2+\beta_2^2\lambda_n}-\frac{\alpha_1\beta_1}{\alpha_1^2+\beta_1^2\lambda_n}\right)}}$$

6 ▯▯▯ 10 / 10

- ✓ **- 0 pts** Correct
- **- 5 pts** half right
- **- 10 pts** blank or wrong
- **- 3 pts** orthogonality
- **- 5 pts** late

1. $\begin{cases} \dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=6(x+y)^2, & 0<a^2<x^2+y^2<b^2<\infty \\ u|_{x^2+y^2=a^2}=1, & \dfrac{\partial u}{\partial n}\Big|_{x^2+y^2=b^2}=0 \end{cases}$

转换到极坐标系中 $\begin{cases} \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\left(\rho\dfrac{\partial u}{\partial \rho}\right)+\dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial \theta^2}=6\rho^2(1+\sin 2\theta) \\ u|_{\rho=a}=1 \quad ②, \quad \dfrac{\partial u}{\partial \rho}\Big|_{\rho=b}=0 \quad ③ \end{cases}$

一般解可表示为 $u(\rho,\theta)=\sum\limits_{n=0}^{\infty}[A_n(\rho)\cos n\theta+B_n(\rho)\sin n\theta]$.

代入①中有 $\sum\limits_{n=0}^{\infty}\left\{\left[A_n''(\rho)+\dfrac{A_n'(\rho)}{\rho}-\dfrac{n^2}{\rho^2}A_n(\rho)\right]\cos n\theta\right.$

$\left.+\left[B''(\rho)+\dfrac{B_n'(\rho)}{\rho}-\dfrac{n^2}{\rho^2}B_n(\rho)\right]\sin n\theta\right\}=6\rho^2(1+\sin 2\theta)$

$\Longrightarrow \begin{cases} A_0''(\rho)+\dfrac{A_0'(\rho)}{\rho}=6\rho^2 \quad ④ \\ A_n''(\rho)+\dfrac{A_n'(\rho)}{\rho}-\dfrac{n^2}{\rho^2}A_n(\rho)=0 \Longrightarrow \rho^2A''(\rho)+\rho A_n'(\rho)-n^2A_n(\rho)=0,\ n=1,2,3,\cdots \quad ⑤ \\ B_2''(\rho)+\dfrac{B_2'(\rho)}{\rho}-\dfrac{4}{\rho^2}B_2(\rho)=6\rho^2 \Rightarrow \rho^2B_2''(\rho)+\rho B_2'(\rho)-4B_2(\rho)=6\rho^4 \quad ⑥ \\ B_n''(\rho)+\dfrac{B_n'(\rho)}{\rho}-\dfrac{n^2}{\rho^2}B_n(\rho)=0 \Longrightarrow \rho^2B''(\rho)+\rho B_n'(\rho)-n^2B_n(\rho)=0,\ n=1,2,3\cdots \quad ⑦ \end{cases}$

由⑤，设 $A_n(\rho)=C_n\rho^n+d_n\rho^{-n}$，$n=1,2,3,\cdots$

由⑦，设 $B_n(\rho)=C_n'\rho^n+d_n'\rho^{-n}$，$n\neq 2$

由②③有 $\begin{cases} A_n(a)=0 & A_n'(b)=0, & n=1,2,3 \\ B_n(a)=0 & B_n'(b)=0, & n\neq 2. \end{cases}$

$\Longrightarrow \begin{cases} C_n=d_n=0 & (n=1,2,3,\cdots) \\ C_n'=d_n'=0 & (n\neq 2). \end{cases}$

由④，有 $A_0'(\rho)=e^{-\int\frac{1}{\rho}d\rho}\left[\int 6\rho^2 e^{\int\frac{1}{\rho}d\rho}d\rho+C_1\right]=\frac{3}{2}\rho^3+\frac{C_0}{\rho}.$

$A_0(\rho)=\frac{3}{8}\rho^4+C_0\ln\rho+d_0$

由②③有 $\begin{cases} A_0(a)=1 \\ A_0'(b)=0 \end{cases} \Longrightarrow \begin{cases} C_0=-\frac{3}{2}b^4 \\ d_0=-\frac{3}{8}a^4+\frac{3}{2}b^4\ln a+1 \end{cases}$

$\therefore A_0(\rho)=\frac{3}{8}\rho^4-\frac{3}{2}b^4\ln\rho-\frac{3}{8}a^4+\frac{3}{2}b^4\ln a+1$

由⑥有一特解 $B_2^*(\rho)=\frac{1}{2}\rho^4$

故其通解为 $B_2(\rho)=C_2'\rho^2+d_2'\rho^{-2}+\frac{1}{2}\rho^4$

由②③得 $\begin{cases} B_2(a)\sin(2\theta) = (C_2' a^2 + d_2' a^{-2} + \frac{1}{2}a^4)\sin(2\theta) = 0 \\ B_2'(a)\sin(2\theta) = (2C_2' b - 2d_2' b^{-3} + 2b^3)\sin(2\theta) = 0 \end{cases}$

$$\Longrightarrow \begin{cases} C_2' = -\dfrac{\frac{1}{2}a^6 + b^6}{a^4+b^4} \\ d_2' = \dfrac{a^4 b^4 (b^2 - \frac{1}{2}a^2)}{a^4+b^4} \end{cases}$$

从而 $B_2(\rho) = -\dfrac{\frac{1}{2}a^6 + b^6}{a^4+b^4}\rho^2 + \dfrac{a^4 b^4 (b^2 - \frac{1}{2}a^2)}{a^4+b^4}\rho^{-2} + \frac{1}{2}\rho^4$

综上：$u(\rho,\theta) = \left[\frac{3}{8}(\rho^4 - a^4) - \frac{3}{2}b^4 \ln\frac{\rho}{a} + 1\right] + \left[-\dfrac{\frac{1}{2}a^6 + b^6}{a^4+b^4}\rho^2 + \dfrac{a^4 b^4 (b^2 - \frac{1}{2}a^2)}{a^4+b^4}\rho^{-2} + \frac{1}{2}\rho^4\right]\sin 2\theta$

7 □□□ 10 / 10

✓ **- 0 pts** Correct

- 10 pts Not given/Wrong

- 5 pts late

- 7 pts Serious problems

- 5 pts Little problems