2. $\begin{cases} \dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2} \quad (0<x<l,\ t>0) \\ \dfrac{\partial u}{\partial x}\Big|_{x=0}=0,\ \dfrac{\partial u}{\partial x}\Big|_{x=l}=0 \\ u|_{t=0}=e^{-x^2},\ \dfrac{\partial u}{\partial t}\Big|_{t=0}=2axe^{-x^2} \end{cases}$

设方程有分离变量解：$u(x,t)=X(x)T(t)$

代入泛定方程有 $\dfrac{X''(x)}{X(x)}=\dfrac{T''(t)}{a^2T(t)}\triangleq-\lambda$

$$\Longrightarrow \begin{cases} X''(x)+\lambda X(x)=0 & ① \\ T''(t)+\lambda a^2T(t)=0 & ② \end{cases}$$

将形式解代入边界条件有 $\begin{cases} X'(0)T(t)=0 \\ X'(l)T(t)=0 \end{cases}$

$$\Longrightarrow X'(0)=X'(l)=0$$

或 $T(t)=0$ (平庸解!)

联立①从而有 $\begin{cases} X''(x)+\lambda X(x)=0 \\ X'(0)=X'(l)=0 \end{cases}$

若 $\lambda<0$，则方程通解为 $X(x)=Ae^{kx}+Be^{-kx}$，其中 $k=\sqrt{-\lambda}$

由边界条件有 $\begin{cases} X'(0)=Ak-Bk=0 \\ X'(l)=Ake^{kl}-Bke^{-kl}=0 \end{cases}$

$$\Longrightarrow \frac{A}{B}=e^{-2kl}=1 \Longrightarrow k=0 \text{ 即 } \lambda=0 \text{ (与 } \lambda<0 \text{ 矛盾)}$$

若 $\lambda=0$，则方程通解为 $X(x)=Ax+B$

由边界条件有 $A=0$，从而 $X(x)=B$ (不符合初始条件)

若 $\lambda>0$，则方程通解为 $X(x)=A\cos(\sqrt{\lambda}x)+B\sin(\sqrt{\lambda}x)$

由边界条件有 $\begin{cases} X'(0)=B\sqrt{\lambda}=0 \\ X'(l)=-A\sqrt{\lambda}\sin(\sqrt{\lambda}l)+B\sqrt{\lambda}\cos(\sqrt{\lambda}l)=0 \end{cases}$

$$\Longrightarrow B=0 \text{ 且 } \sqrt{\lambda}l=n\pi$$

故本征函数 $X_n=A_n\cos(\frac{n\pi}{l}x)$ 本征值 $\lambda_n=(\frac{n\pi}{l})^2$

将本征值代入②有 $T''(t)+(\frac{na\pi}{l})^2T(t)=0$

其通解为 $T_n(t)=C_n\cos(\frac{na\pi}{l}t)+d_n\sin(\frac{na\pi}{l}t)$

从而 $u_n(x,t)=X_n(x)T_n(t)=\left[C_n\cos(\frac{na\pi}{l}t)+D_n\sin(\frac{na\pi}{l}t)\right]\cos(\frac{n\pi}{l}x)$

$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)$

代入初始条件有
$$\begin{cases} u|_{t=0} = \sum_{n=0}^{\infty} C_n \cos\left(\frac{n\pi}{l}x\right) = e^{-x^2} \\ \frac{\partial u}{\partial t}\Big|_{t=0} = \sum_{n=1}^{\infty} \frac{na\pi}{l} D_n \cos\left(\frac{n\pi}{l}x\right) = 2axe^{-x^2} \end{cases}$$
$$\Longrightarrow \begin{cases} C_n = \frac{2}{l}\int_0^l e^{-x^2}\cos\left(\frac{n\pi}{l}x\right)dx, \quad n=0,1,2,\cdots \\ D_n = \frac{2}{na\pi}\int_0^l 2axe^{-x^2}\cos\left(\frac{n\pi}{l}x\right)dx, \quad n=1,2,\cdots \end{cases}$$

综上：
$$u(x,t) = \frac{C_0}{2} + \sum_{n=1}^{\infty}\left[C_n\cos\left(\frac{na\pi}{l}t\right) + D_n\sin\left(\frac{na\pi}{l}t\right)\right]\cos\left(\frac{n\pi}{l}x\right)$$

其中
$$\begin{cases} C_n = \frac{2}{l}\int_0^l e^{-x^2}\cos\left(\frac{n\pi}{l}x\right)dx, \quad n=0,1,2,\cdots \\ D_n = \frac{2}{na\pi}\int_0^l 2axe^{-x^2}\cos\left(\frac{n\pi}{l}x\right)dx, \quad n=1,2,\cdots \end{cases}$$

3. $$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right) \\ u|_{t=0} = \begin{cases} u_0, & x^2+y^2+z^2<R^2 \\ 0, & x^2+y^2+z^2>R^2 \end{cases}, \quad \dfrac{\partial u}{\partial t}\Big|_{t=0}=0 \end{cases}$$

转化到球坐标系中有 $$\begin{cases} \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta\dfrac{\partial u}{\partial \theta}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial^2 u}{\partial \phi^2} \\ = \dfrac{1}{a^2}\dfrac{\partial^2 u}{\partial t^2} \qquad ① \\ u(r,0) = \left.\begin{cases} u_0, & r<R \\ 0, & r>R \end{cases}\right\} \overset{def}{=} \varphi(r), \quad u_t(r,0)=0 \end{cases}$$

由于 u 不依赖于 θ 和 ϕ, ① 简化为 $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right) = \dfrac{1}{a^2}\dfrac{\partial^2 u}{\partial t^2}$

作函数代换 $v(r,t) = r u(r,t)$

从而有 $$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} = a^2\dfrac{\partial^2 v}{\partial r^2} & r, t>0 \\ v(r,0) = \begin{cases} r u_0, & r<R \\ 0, & r>R \end{cases} & v_t(r,0)=0 \\ v(0,t)=0 \end{cases}$$

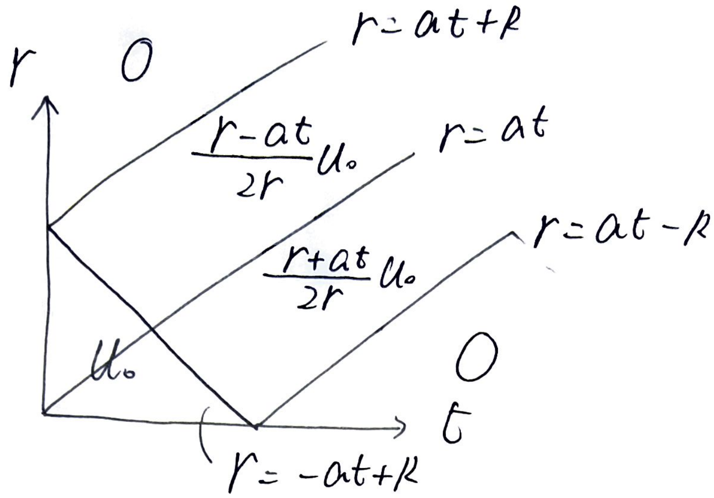
利用达朗贝尔公式

当 $r-at<0$ 时 $$u(r,t) = \frac{1}{2r}\left[(r+at)\varphi(r+at) - (at-r)\varphi(at-r)\right]$$
$$= \frac{1}{2r}(r-at)\varphi(at-r)$$
$$= \begin{cases} 0, & r+at>R,\ at-r>R \\ \dfrac{r+at}{2r}u_0, & r+at>R,\ at-r<R \\ u_0, & r+at<R,\ at-r<R \end{cases}$$

当 $r-at\ge 0$ 时 $$u(r,t) = \frac{1}{2r}\left[(r+at)\varphi(r+at) + (r-at)\varphi(r-at)\right]$$
$$= \begin{cases} 0, & r+at>R,\ r-at>R \\ \dfrac{r-at}{2r}u_0, & r+at>R,\ r-at<R \\ u_0, & r+at<R,\ r-at<R \end{cases}$$

综上: $$u(r,t) = \begin{cases} u_0, & r+at<R \\ \dfrac{r-at}{2r}u_0, & r+at>R \text{ 且 } 0<r-at<R \\ \dfrac{r+at}{2r}u_0, & r+at>R \text{ 且 } -R<r-at<0 \\ 0, & r>at+R \text{ 或 } 0<r<at-R \end{cases}$$

回到直角坐标系 $$u(x,y,z,t) = \begin{cases} u_0, & \sqrt{x^2+y^2+z^2}+at<R \\ \dfrac{\sqrt{x^2+y^2+z^2}-at}{2\sqrt{x^2+y^2+z^2}}, & \sqrt{x^2+y^2+z^2}+at>R \text{ 且 } 0<\sqrt{x^2+y^2+z^2}-at<R \\ \dfrac{\sqrt{x^2+y^2+z^2}+at}{2\sqrt{x^2+y^2+z^2}}, & \sqrt{x^2+y^2+z^2}+at>R \text{ 且 } -R<\sqrt{x^2+y^2+z^2}-at<0 \\ 0, & \sqrt{x^2+y^2+z^2}>at+R \text{ 或 } 0<\sqrt{x^2+y^2+z^2}<at-R \end{cases}$$


$r = at + R$
r
0
$\frac{r-at}{2r}u_0$
$r = at$
$r = at - R$
$\frac{r+at}{2r}u_0$
u_0
0
t
$r = -at + R$

4.
$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = \dfrac{bx(l-x)}{l^2} \end{cases}$$

设方程有分离变量解 $u(x,t) = X(x)T(t)$

代入泛定方程有 $\dfrac{X''(x)}{X(x)} = \dfrac{T''(t)}{a^2T(t)} \stackrel{\Delta}{=} -\lambda$

$$\Longrightarrow \begin{cases} X''(x) + \lambda X(t) = 0 & ① \\ T'(t) + \lambda a^2 T(t) = 0 & ② \end{cases}$$

①结合边界条件得到本征函数 $X_n(x) = B_n \sin(\frac{n\pi}{l}x)$

和本征值 $\lambda_n = (\frac{n\pi}{l})^2$

将本征值代入②有 $T'(t) + (\frac{na\pi}{l})^2 T(t) = 0$

通解为 $T_n(t) = C_n e^{-(\frac{na\pi}{l})^2 t}$

从而 $u_n(x,t) = X_n(x)T_n(t) = C_n e^{-(\frac{na\pi}{l})^2 t} \sin(\frac{n\pi}{l}x)$

$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$

代入初始条件有 $u(x,0) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi}{l}x) = \dfrac{bx(l-x)}{l^2}$

$$\Longrightarrow C_n = \frac{2}{l}\int_0^l \frac{bx(l-x)}{l^2} \sin(\frac{n\pi}{l}x)$$

$$= \frac{4b}{n^3\pi^3}[1+(-1)^{n+1}]$$

综上: $u(x,t) = \sum_{n=1}^{\infty} C_n e^{-(\frac{na\pi}{l})^2 t} \sin(\frac{n\pi}{l}x)$

其中 $C_n = \dfrac{4b}{n^3\pi^3}[1+(-1)^{n+1}]$.

5. (1) $a_1(x)\frac{\partial^2 u}{\partial x^2}+b_1(y)\frac{\partial^2 u}{\partial y^2}+a_2(x)\frac{\partial u}{\partial x}+b_2(y)\frac{\partial u}{\partial y}=0$

设 $u(x,y)=X(x)Y(y)$

代入原方程有 $a_1(x)X''(x)Y(y)+b_1(y)X(x)Y''(y)$
$+a_2(x)X'(x)Y(y)+b_2(y)X(x)Y'(y)=0$

两边同除 $X(x)Y(y)$ 有

$$\frac{a_1(x)X''(x)}{X(x)}+\frac{b_1(y)Y''(y)}{Y(y)}+\frac{a_2(x)X'(x)}{X(x)}+\frac{b_2(y)Y'(y)}{Y(y)}=0$$

$$\Longrightarrow \frac{a_1(x)X''(x)+a_2(x)X'(x)}{X(x)}=-\frac{b_1(y)Y''(y)+b_2(y)Y'(y)}{Y(y)}\triangleq\lambda$$

$$\Longrightarrow \begin{cases} a_1(x)X''(x)+a_2(x)X'(x)-\lambda X(x)=0 \\ b_1(y)Y''(y)+b_2(y)Y'(y)+\lambda Y(y)=0 \end{cases}$$

(2) $\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial u}{\partial\rho})+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}=0$

设 $u(\rho,\varphi)=P(\rho)\Phi(\varphi)$

代入原方程得 $\frac{1}{\rho}\Phi(\varphi)\frac{\partial}{\partial\rho}(\rho\frac{\partial P(\rho)}{\partial\rho})+\frac{1}{\rho^2}P(\rho)\frac{\partial^2\Phi(\varphi)}{\partial\varphi^2}=0$

两边同乘 $\frac{\rho^2}{\Phi(\varphi)P(\rho)}$ 有

$$\frac{\rho}{P(\rho)}\frac{d}{d\rho}(\rho\frac{dP(\rho)}{d\rho})+\frac{1}{\Phi(\varphi)}\frac{d^2\Phi(\varphi)}{d\varphi^2}=0$$

$$\Longrightarrow \frac{\rho}{P(\rho)}\frac{d}{d\rho}(\rho\frac{dP(\rho)}{d\rho})=-\frac{1}{\Phi(\varphi)}\frac{d^2\Phi(\varphi)}{d\varphi^2}\triangleq\lambda$$

$$\Longrightarrow \begin{cases} \rho\frac{d}{d\rho}(\rho\frac{dP(\rho)}{d\rho})-\lambda P(\rho)=0 \\ \frac{d^2\Phi(\varphi)}{d\varphi^2}+\lambda\Phi(\varphi)=0 \end{cases}$$

(3) $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)=0$

设 $u(r,\theta)=R(r)\Theta(\theta)$

代入原方程中得. $\frac{1}{r^2}\Theta(\theta)\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)+\frac{1}{r^2\sin\theta}R(r)\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)=0$

两边同乘 $\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)+\frac{1}{\sin\theta}\frac{1}{\Theta(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)=0$

$$\Longrightarrow \frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)=-\frac{1}{\sin\theta}\frac{1}{\Theta(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)\triangleq\lambda$$

$$\Longrightarrow \begin{cases}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)-\lambda R(r)=0\\ \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)+\lambda\Theta(\theta)=0\end{cases}$$

6. $\begin{cases} X''+\lambda X=0 \\ \alpha_1 X(0)+\beta_1 X'(0)=0 \\ \alpha_2 X(l)+\beta_2 X'(l)=0 \end{cases}$

通解为 $X=A\cos\sqrt{\lambda}x+B\sin\sqrt{\lambda}x$

代入边界条件 $\begin{cases} \alpha_1 X(0)+\beta_1 X'(0)=\alpha_1 A+\beta_1 B\sqrt{\lambda}=0 \\ \alpha_2 X(l)+\beta_2 X'(l)=\alpha_2(A\cos\sqrt{\lambda}l+B\sin\sqrt{\lambda}l) \\ \qquad +\beta_2\sqrt{\lambda}(-A\sin\sqrt{\lambda}l+B\cos\sqrt{\lambda}l)=0 \end{cases}$

$\Longrightarrow$ 本征函数 $X_n(x)=C_n\left[\sin(\sqrt{\lambda_n}x)-\frac{\beta_1}{\alpha_1}\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x)\right]$.

本征值 λ_n 是 $(\alpha_1\alpha_2+\beta_1\beta_2\lambda)\tan\sqrt{\lambda}l+(\alpha_1\beta_2-\alpha_2\beta_1)\sqrt{\lambda}=0$ 的解.

~~证明正交性~~ ~~举例说明正交性:~~

证明正交性:

$$\int_0^l\left[\sin(\sqrt{\lambda_n}x)-\frac{\beta_1}{\alpha_1}\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x)\right]\left[\sin(\sqrt{\lambda_m}x)-\frac{\beta_1}{\alpha_1}\sqrt{\lambda_m}\cos(\sqrt{\lambda_m}x)\right]$$

$$=\int_0^l \sin(\sqrt{\lambda_n}x)\sin(\sqrt{\lambda_m}x)+\frac{\beta_1^2}{\alpha_1^2}\sqrt{\lambda_m}\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x)\cos(\sqrt{\lambda_m}x)$$
$$-\frac{\beta_1}{\alpha_1}\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x)\sin(\sqrt{\lambda_m}x)-\frac{\beta_1}{\alpha_1}\sqrt{\lambda_m}\sin(\sqrt{\lambda_n}x)\cos(\sqrt{\lambda_m}x)\,dx$$

$$=\int_0^l\Big\{\frac{1}{2}\left[\cos(\sqrt{\lambda_m}-\sqrt{\lambda_n})x-\cos(\sqrt{\lambda_n}+\sqrt{\lambda_m})x\right]$$
$$+\frac{1}{2}\frac{\beta_1^2}{\alpha_1^2}\sqrt{\lambda_n}\sqrt{\lambda_m}\left[\cos(\sqrt{\lambda_n}-\sqrt{\lambda_m})x+\cos(\sqrt{\lambda_n}+\sqrt{\lambda_m})x\right]$$
$$-\frac{1}{2}\frac{\beta_1}{\alpha_1}\sqrt{\lambda_n}\left[\sin(\sqrt{\lambda_n}+\sqrt{\lambda_m})x-\sin(\sqrt{\lambda_n}-\sqrt{\lambda_m})x\right]$$
$$-\frac{1}{2}\frac{\beta_1}{\alpha_1}\sqrt{\lambda_m}\left[\sin(\sqrt{\lambda_n}+\sqrt{\lambda_m})x+\sin(\sqrt{\lambda_n}-\sqrt{\lambda_m})x\right].$$

$$=\frac{1}{2}\Big\{\frac{1+\frac{\beta_1^2}{\alpha_1^2}\sqrt{\lambda_n}\sqrt{\lambda_m}}{\sqrt{\lambda_n}-\sqrt{\lambda_m}}\sin(\sqrt{\lambda_n}-\sqrt{\lambda_m})l$$
$$-\frac{1+\frac{\beta_1^2}{\alpha_1^2}\sqrt{\lambda_n}\sqrt{\lambda_m}}{\sqrt{\lambda_n}+\sqrt{\lambda_m}}\sin(\sqrt{\lambda_n}+\sqrt{\lambda_m})l$$
$$+\frac{\beta_1}{\alpha_1}\left[\cos(\sqrt{\lambda_n}+\sqrt{\lambda_m})l-1\right]$$
$$+\frac{\beta_1}{\alpha_1}\left[\cos(\sqrt{\lambda_n}-\sqrt{\lambda_m})l-1\right]\Big\}.$$

$$=\frac{1}{2}\left\{\frac{1+\frac{\beta_1^2}{\alpha_1^2}\sqrt{\lambda_n}\sqrt{\lambda_m}}{\sqrt{\lambda_n}-\sqrt{\lambda_m}}\left[\sin\sqrt{\lambda_n}l\cos\sqrt{\lambda_m}l-\cos\sqrt{\lambda_n}l\sin\sqrt{\lambda_m}l\right]\right.$$

$$-\frac{1+\frac{\beta_1^2}{\alpha_1^2}\sqrt{\lambda_n}\sqrt{\lambda_m}}{\sqrt{\lambda_n}+\sqrt{\lambda_m}}\left[\sin\sqrt{\lambda_n}l\cos\sqrt{\lambda_m}l+\cos\sqrt{\lambda_n}l\sin\sqrt{\lambda_m}l\right]$$

$$\left.+\frac{2\beta_1}{\alpha_1}\cos\sqrt{\lambda_n}l\cos\sqrt{\lambda_m}l-\frac{2\beta_1}{\alpha_1}\right\}.$$

$$=\frac{1}{2}\left\{\frac{1+\frac{\beta_1^2}{\alpha_1^2}\sqrt{\lambda_n}\sqrt{\lambda_m}}{\sqrt{\lambda_n}-\sqrt{\lambda_m}}(\tan\sqrt{\lambda_n}l-\tan\sqrt{\lambda_m}l)\cos\sqrt{\lambda_n}l\cos\sqrt{\lambda_m}l\right.$$

$$-\frac{1+\frac{\beta_1^2}{\alpha_1^2}\sqrt{\lambda_n}\sqrt{\lambda_m}}{\sqrt{\lambda_n}+\sqrt{\lambda_m}}(\tan\sqrt{\lambda_n}l+\tan\sqrt{\lambda_m}l)\cos\sqrt{\lambda_n}l\cos\sqrt{\lambda_m}l$$

$$\left.+\frac{2\beta_1}{\alpha_1}\cos\sqrt{\lambda_n}l\cos\sqrt{\lambda_m}l-\frac{2\beta_1}{\alpha_1}\right\}$$

(代入 $\tan\sqrt{\lambda_i}l=\frac{(\alpha_1\beta_2-\alpha_2\beta_1)\sqrt{\lambda}}{\alpha_1\alpha_2+\beta_1\beta_2\lambda}$ 得)

$$=\frac{1}{2}\left\{\frac{1+\frac{\beta_1^2}{\alpha_1^2}\sqrt{\lambda_n}\sqrt{\lambda_m}}{\sqrt{\lambda_n}-\sqrt{\lambda_m}}\left[\frac{-(\alpha_1\beta_2-\alpha_2\beta_1)\sqrt{\lambda_n}}{\alpha_1\alpha_2+\beta_1\beta_2\lambda_n}+\frac{(\alpha_1\beta_2-\alpha_2\beta_1)\sqrt{\lambda_m}}{\alpha_1\alpha_2+\beta_1\beta_2\lambda_m}\right]\right.$$

$$-\frac{1+\frac{\beta_1^2}{\alpha_1^2}\sqrt{\lambda_n}\sqrt{\lambda_m}}{\sqrt{\lambda_m}+\sqrt{\lambda_m}}\left[\frac{-(\alpha_1\beta_2-\alpha_2\beta_1)\sqrt{\lambda_n}}{\alpha_1\alpha_2+\beta_1\beta_2\lambda_n}+\frac{-(\alpha_1\beta_2-\alpha_2\beta_1)\sqrt{\lambda_m}}{\alpha_1\alpha_2+\beta_1\beta_2\lambda_m}\right]$$

$$\left.+\frac{2\beta_1}{\alpha}\cos\sqrt{\lambda_n}l\cos\sqrt{\lambda_m}l-\frac{2\beta_1}{\alpha_1}\right\}.$$

$$\underset{(通分)}{=\!=}0$$

由此得证本征函数正交性.

求归一化因子:

$$\int_0^l X_n^2(x)dx=C_n^2\int_0^l\sin^2(\sqrt{\lambda_n}x-\delta_n)dx\quad(其中\ \tan\delta_n=\frac{\beta_1}{\alpha_1}\sqrt{\lambda_n}).$$

$$=C_n^2\int_0^l\frac{1-\cos(2\sqrt{\lambda_n}x-2\delta_n)}{2}dx$$

$$=C_n^2\left\{\frac{l}{2}-\frac{1}{4\sqrt{\lambda_n}}\left[\sin 2(\sqrt{\lambda_n}l-\delta_n)+\sin 2\delta_n\right]\right\}$$

$$\because\tan\sqrt{\lambda}l=-\sqrt{\lambda_n}\frac{\alpha_1\beta_2-\alpha_2\beta_1}{\alpha_1\alpha_2+\beta_1\beta_2\lambda}$$

$$\sin 2\sqrt{\lambda_n}\,l = \frac{2\tan\sqrt{\lambda_n}\,l}{1+\tan^2\sqrt{\lambda_n}\,l} = -2\sqrt{\lambda_n}\,\frac{\alpha_1^2\alpha_2\beta_2 - \alpha_1\alpha_2^2\beta_1 + \alpha_1\beta_1\beta_2^2\lambda_n - \alpha_2\beta_1^2\beta_2}{(\alpha_1^2+\beta_1^2\lambda_n)(\alpha_2^2+\beta_2^2\lambda_n)}$$

$$\cos 2\sqrt{\lambda_n}\,l = \frac{1-\tan^2\sqrt{\lambda_n}\,l}{1+\tan^2\sqrt{\lambda_n}\,l} = \frac{\alpha_1^2\alpha_2^2 - \alpha_1^2\beta_2^2\lambda_n - \alpha_2^2\beta_1^2\lambda_n + 4\alpha_1\alpha_2\beta_1\beta_2\lambda_n + \beta_1^2\beta_2^2\lambda}{(\alpha_1^2+\beta_1^2\lambda_n)(\alpha_2^2+\beta_2^2\lambda_n)}$$

$$\sin 2\delta_n = \frac{2\tan\delta_n}{1+\tan^2\delta_n} = 2\sqrt{\lambda_n}\,\frac{\alpha_1\beta_1}{\alpha_1^2+\beta_1^2\lambda_n} \qquad \cos 2\delta_n = \frac{1-\tan^2\delta_n}{1+\tan^2\delta_n} = \frac{\alpha_1^2-\beta_1^2\lambda_n}{\alpha_1^2+\beta_1^2\lambda_n}$$

$$\sin 2(\sqrt{\lambda_n}\,l-\delta_n) = \sin 2\sqrt{\lambda_n}\,l\cos 2\delta_n - \cos 2\sqrt{\lambda_n}\,l\sin 2\delta_n = -2\sqrt{\lambda_n}\,\frac{\alpha_2\beta_2}{\alpha_2^2+\beta_2^2\lambda_n}$$

$$\therefore \int_0^l X_n^2(x)\,dx = C_n^2\left[\frac{l}{2}+\frac{1}{2}\left(\frac{\alpha_2\beta_2}{\alpha_2^2+\beta_2^2\lambda_n} - \frac{\alpha_1\beta_1}{\alpha_1^2+\beta_1^2\lambda_n}\right)\right].$$

$$\Longrightarrow C_n = \sqrt{\frac{1}{\frac{l}{2}+\frac{1}{2}\left(\frac{\alpha_2\beta_2}{\alpha_2^2+\beta_2^2\lambda_n} - \frac{\alpha_1\beta_1}{\alpha_1^2+\beta_1^2\lambda_n}\right)}}$$

7. $$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=6(x+y)^2,\ 0<a^2<x^2+y^2<b^2<\infty \\ u\big|_{x^2+y^2=a^2}=1,\ \dfrac{\partial u}{\partial n}\Big|_{x^2+y^2=b^2}=0 \end{cases}$$

其中 n 为边界 $x^2+y^2=b^2$ 的外法向

转换到极坐标系中有 $$\begin{cases} \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial u}{\partial\rho}\right)+\dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial\theta^2}=6\rho^2 \\ u\big|_{\rho=a}=1,\ \dfrac{\partial u}{\partial\rho}\Big|_{\rho=b}=0 \end{cases}$$

由问题的对称性，u 与 θ 无关而仅与 ρ 有关

从而简化为 $$\begin{cases} \dfrac{1}{\rho}\dfrac{d}{d\rho}\left(\rho\dfrac{du}{d\rho}\right)=\dfrac{d^2u}{d\rho^2}+\dfrac{1}{\rho}\dfrac{du}{d\rho}=6\rho^2 & ① \\ u\big|_{\rho=a}=1\ ②,\ \dfrac{du}{d\rho}\Big|_{\rho=b}=0 & ③ \end{cases}$$

由①得 $$\frac{du}{d\rho}=e^{-\int\frac{1}{\rho}d\rho}\left[\int 6\rho^2 e^{\int\frac{1}{\rho}d\rho}d\rho+C_1\right]$$
$$=\frac{3}{2}\rho^3+\frac{C_1}{\rho}$$

代入③中得 $\dfrac{du}{d\rho}\Big|_{\rho=b}=\dfrac{3}{2}b^3+\dfrac{C_1}{b}=0\Longrightarrow C_1=-\dfrac{3}{2}b^4$

从而 $\dfrac{\partial u}{\partial\rho}=\dfrac{3}{2}\left(\rho^3-\dfrac{b^4}{\rho}\right)$

$\Longrightarrow u=\displaystyle\int\frac{\partial u}{\partial\rho}d\rho=\frac{3}{2}\left(\frac{1}{4}\rho^4-b^4\ln\rho\right)+C_2$

代入②中得 $u\big|_{\rho=a}=\dfrac{3}{2}\left(\dfrac{1}{4}a^4-b^4\ln a\right)+C_2=0.\Longrightarrow C_2=\dfrac{3}{2}\left(b^4\ln a-\dfrac{1}{4}a^4\right)$

从而 $u=\dfrac{3}{8}(\rho^4-a^4)-\dfrac{3}{2}b^4\ln\dfrac{\rho}{a}$

回到直角坐标系有 $u(x,y)=\dfrac{3}{8}\left[(x^2+y^2)^2-a^4\right]-\dfrac{3}{2}b^4\ln\dfrac{\sqrt{x^2+y^2}}{a}$