PHYS1303 □□□□□□II□□□□□

□□□

TOTAL POINTS

109 / 111

QUESTION 1

1 □□□ 10 / 10

✓ - **0 pts** Correct

- **3 pts** The answer is wrong
- **6 pts** wrong

QUESTION 2

2 □□□ 10 / 10

✓ - **0 pts** Correct

- **10 pts** blank/wrong

QUESTION 3

3 □□□ 12 / 12

✓ - **0 pts** Correct

- **3 pts** □1□the answer is wrong
- **3 pts** □2□the answer is wrong
- **3 pts** □3□the answer is wrong
- **3 pts** □4□the answer is wrong

QUESTION 4

4 □□□ 9 / 9

✓ - **0 pts** Correct

- **2 pts** (1)wrong
- **2 pts** (2)wrong
- **2 pts** (3)wrong
- **9 pts** blank/wrong

QUESTION 5

5 □□□ 10 / 10

✓ - **0 pts** Correct

- **3 pts** (1)wrong
- **3 pts** (2)wrong
- **10 pts** blank/wrong

QUESTION 6

6 □□□ 10 / 10

✓ - **0 pts** Correct

- **3 pts** the answer is wrong

QUESTION 7

7 □□□ 8 / 10

- **0 pts** Correct

✓ - **2 pts** little mistake

- **5 pts** Mess
- **10 pts** No answer

QUESTION 8

8 □□□ 10 / 10

✓ - **0 pts** Correct

- **4 pts** Not□□□□□□□□□
- **10 pts** No right answer

QUESTION 9

9 □□□ 10 / 10

✓ - **0 pts** Correct

- **2 pts** calculate mistake
- **10 pts** No answer

QUESTION 10

10 □□□ 10 / 10

✓ - **0 pts** Correct

- **10 pts** No answer
- **4 pts** Not perfect/Half

QUESTION 11

11 □□□□ 10 / 10

✓ - **0 pts** Correct

- **2 pts** calculate incorrect
- **6 pts** method incorrect
- **10 pts** No answer /Not right at all

1. 设 $G(\vec{r},\vec{r}')$ 对 $\vec{r}$ 的傅立叶变换为 $\widetilde{G}(\vec{k},\vec{r}')$

则其逆变换为 $G(\vec{r},\vec{r}')=\frac{1}{(2\pi)^3}\iiint \widetilde{G}(\vec{k},\vec{r}')\exp(i\vec{k}\cdot\vec{r})d\vec{k}$

$$=\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\widetilde{G}(\vec{k},\vec{r}')\exp[i(k_x x+k_y y+k_z z)]dk_x dk_y dk_z \quad ①$$

又有结论 $\delta^3(\vec{x})=\frac{1}{(2\pi)^3}\iiint \exp(i\vec{k}\cdot\vec{x})d\vec{k}=\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp[i(k_x x+k_y y+k_z z)]dk_x dk_y dk_z \quad ②$

其中设 $k'^2=k_x^2+k_y^2+k_z^2$

将①②分别代入原方程得

$$\frac{(\nabla^2+k^2)}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\widetilde{G}(\vec{k}',\vec{r}')\exp[i(k_x x+k_y y+k_z z)]dk_x dk_y dk_z$$

$$=-\frac{1}{(2\pi)^3\varepsilon_0}\iiint \exp[i\vec{k}'\cdot(\vec{r}-\vec{r}')]d\vec{k}'$$

$$\Rightarrow \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\nabla^2+k^2)\widetilde{G}(\vec{k}',\vec{r}')\exp[i(k_x x+k_y y+k_z z)]dk_x dk_y dk_z$$

$$=-\frac{1}{(2\pi)^3\varepsilon_0}\iiint \exp[i\vec{k}'\cdot(\vec{r}-\vec{r}')]d\vec{k}'$$

$$\Rightarrow \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\{(-k_x^2-k_y^2-k_z^2)\widetilde{G}(\vec{k}',\vec{r}')\exp[i(k_x x+k_y y+k_z z)] + k^2\widetilde{G}(\vec{k}',\vec{r}')\exp[i(k_x x+k_y y+k_z z)]\}dk_x dk_y dk_z$$

$$=-\frac{1}{(2\pi)^3\varepsilon_0}\iiint \exp[i\vec{k}'\cdot(\vec{r}-\vec{r}')]d\vec{k}'$$

$$\Rightarrow \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(-k'^2+k^2)\widetilde{G}(\vec{k}',\vec{r}')\exp[i(k_x x+k_y y+k_z z)]dk_x dk_y dk_z$$

$$=-\frac{1}{(2\pi)^3\varepsilon_0}\iiint \exp[i\vec{k}'\cdot(\vec{r}-\vec{r}')]d\vec{k}'$$

两边同时对 $\vec{r}$ 作傅里叶变换得

$$(-k'^2+k^2)\widetilde{G}(\vec{k}',\vec{r}')=-\frac{\exp(-i\vec{k}\cdot\vec{r}')}{\varepsilon_0}\Longrightarrow \widetilde{G}(\vec{k}',\vec{r}')=-\frac{\exp(-i\vec{k}\cdot\vec{r}')}{\varepsilon_0}$$

利用式①逆变换

$$G(\vec{r},\vec{r}')=\frac{1}{(2\pi)^3\varepsilon_0}\iiint\frac{\exp(-i\vec{k}\cdot\vec{r}')}{k'^2-k^2}\exp(i\vec{k}\cdot\vec{r})d\vec{r}$$

以 $(\vec{r}-\vec{r}')$ 方向为 z 轴方向，转换到球坐标中

$$G(\vec{r},\vec{r}')=\frac{1}{(2\pi)^3\varepsilon_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{+\infty}\frac{\exp[ik(\vec{r}-\vec{r}')]}{k'^2-k^2}k'^2\sin\theta\, dk\, d\theta\, d\varphi$$

$$=\frac{1}{(2\pi)^3\varepsilon_0}\int_0^{2\pi}d\varphi\int_0^{\infty}\frac{k'^2}{k'^2-k^2}dk'\int_0^{\pi}\exp(ik'|\vec{r}-\vec{r}'|\cos\theta)\sin\theta\, d\theta$$

$$=\frac{1}{(2\pi)^2\varepsilon_0}\int_0^{\infty}\frac{k'^2}{k'^2-k^2}dk'\int_0^{\pi}-\exp(ik'|\vec{r}-\vec{r}'|\cos\theta)\,d\cos\theta$$

$$\overset{t=\cos\theta}{=} \frac{1}{(2\pi)^2\varepsilon_0}\int_0^{\infty}\frac{k'^2}{k'^2-k^2}dk'\int_{-1}^{1}\exp(ik'|\vec{r}-\vec{r}'|t)$$

$$=\frac{1}{(2\pi)^2\varepsilon_0}\int_0^{\infty}\frac{k'^2}{k'^2-k^2}\,\frac{\exp(ik'|\vec{r}-\vec{r}'|t)\Big|_{-1}^{1}}{ik'|\vec{r}-\vec{r}'|}dk'$$

$$=\frac{1}{i(2\pi)^2\varepsilon_0|\vec{r}-\vec{r}'|}\int_0^{\infty}\frac{k'\left[\exp(ik'|\vec{r}-\vec{r}'|)-\exp(-ik'|\vec{r}-\vec{r}'|)\right]}{k'^2-k^2}dk'$$

$$=\frac{1}{i(2\pi)^2\varepsilon_0|\vec{r}-\vec{r}'|}\int_{-\infty}^{+\infty}\frac{k'\exp(ik'|\vec{r}-\vec{r}'|)}{k'^2-k^2}dk$$

$$=\frac{1}{i(2\pi)^2\varepsilon_0|\vec{r}-\vec{r}'|}2\pi i\,\mathrm{Res}\left[\frac{k'\exp(ik'|\vec{r}-\vec{r}'|)}{k'^2-k^2},k\right]\text{(留数定理)}$$

$$=\frac{1}{i(2\pi)^2\varepsilon_0|\vec{r}-\vec{r}'|}2\pi i\lim_{k'\to k}\frac{k'\exp(ik'|\vec{r}-\vec{r}'|)}{k'^2-k^2}(k'-k)$$

$$=\frac{1}{i(2\pi)^2\varepsilon_0|\vec{r}-\vec{r}'|}2\pi i\,\frac{k\exp(ik|\vec{r}-\vec{r}'|)}{k+k}$$

$$=\frac{\exp(ik|\vec{r}-\vec{r}'|)}{4\pi\varepsilon_0|\vec{r}-\vec{r}'|}$$

1 10 / 10

✓ **- 0 pts** Correct

- 3 pts The answer is wrong

- 6 pts wrong

2. $\frac{1}{2}\pi i^{\nu+1}[J_\nu(ix)+iN_\nu(ix)]$

$$=\frac{\pi}{2}i^{\nu+1}\left[J_\nu(ix)+i\frac{J_\nu(ix)\cos\nu\pi-J_{-\nu}(ix)}{\sin\nu\pi}\right]$$

$$=\frac{\pi}{2}i^{\nu+1}\left[J_\nu(ix)+\frac{J_\nu(ix)\cos\nu\pi-J_{-\nu}(ix)}{-i\sin\nu\pi}\right]$$

$$=\frac{\pi}{2}i^{\nu+1}\frac{J_\nu(ix)\cos\nu\pi-iJ_\nu(ix)\sin\nu\pi-J_{-\nu}(ix)}{-i\sin\nu\pi}$$

$$=\frac{\pi}{2}i^{\nu+2}\frac{J_\nu(ix)e^{-i\nu\pi}-J_{-\nu}(ix)}{\sin\nu\pi}$$

$$=\frac{\pi}{2}i^{\nu+2}\frac{J_\nu(ix)[e^{-i\pi}]^\nu-J_{-\nu}(ix)}{\sin\nu\pi}$$

$$=\frac{\pi}{2}i^{\nu+2}\frac{J_\nu(ix)(-1)^\nu-J_{-\nu}(ix)}{\sin\nu\pi}.$$

$$=\frac{\pi}{2}\frac{(-i)^\nu i^2 J_\nu(ix)-i^\nu i^2 J_{-\nu}(ix)}{\sin\nu\pi}.$$

$$=\frac{\pi}{2}\frac{i^\nu J_{-\nu}(ix)-(-i)^\nu J_\nu(ix)}{\sin\nu\pi}.$$

而根据第二类虚宗量贝塞数的定义

$$K_\nu(x)\triangleq\frac{\pi}{2}\frac{I_{-\nu}(x)-I_\nu(x)}{\sin\nu\pi}$$

$$=\frac{\pi}{2}\frac{(-i)^{-\nu}J_{-\nu}(ix)-(-i)^\nu J_\nu(ix)}{\sin\nu\pi}$$

$$=\frac{\pi}{2}\frac{i^\nu J_{-\nu}(ix)-(-i)^\nu J_\nu(ix)}{\sin\nu\pi}$$

故 $\frac{1}{2}\pi i^{\nu+1}[J_\nu(ix)+iN_\nu(ix)]=K_\nu(x)$

再根据Hankel函数的定义 $H_\nu^{(1)}(x)\triangleq J_\nu(x)+iN_\nu(x)$

有 $K_\nu(x)=\frac{\pi}{2}i^{\nu+1}H_\nu^{(1)}(x)=\frac{\pi}{2}i^{\nu+1}[J_\nu(ix)+iN_\nu(ix)]$.

当ν为整数n时，取极限 $\lim_{\nu\to n}K_\nu(x)=\lim_{\nu\to n}\frac{\pi}{2}i^{\nu+1}H_\nu^{(1)}(x)=\lim_{\nu\to n}\frac{\pi}{2}i^{\nu+1}[J_\nu(ix)+iN_\nu(ix)]$

得 $K_n(x)=\frac{\pi}{2}i^{n+1}H_n^{(1)}(x)=\frac{\pi}{2}i^{n+1}[J_n(ix)+iN_n(ix)]$.

2 ▯▯▯ 10 / 10

- ✓ **- 0 pts** Correct
- **- 10 pts** blank/wrong

3. (1) $\int x J_2(x)dx$

利用递推公式 $J_2(x) = -J_0(x) + \frac{2}{x}J_1(x)$

得 $\int x J_2(x)dx = -\int x J_0(x)dx + 2\int J_1(x)dx$

$$= -xJ_1(x) - 2J_0(x)$$

(2) $\int x^4 J_1(x)dx$

$$\int x^4 J_1(x)dx = \int x^2[x^2 J_1(x)]dx = \int x^2 d[x^2 J_2(x)]$$

$$= x^4 J_2(x) - 2\int x^3 J_2(x)dx$$

$$= x^4 J_2(x) - 2x^3 J_3(x).$$

代入 $J_2(x) = \frac{2}{x}J_1(x) - J_0(x)$

和 $J_3(x) = \frac{4}{x}J_2(x) - J_1(x) = \frac{8}{x^2}J_1(x) - \frac{4}{x}J_0(x) - J_1(x)$

得 $\int x^4 J_1(x)dx = x^4[\frac{2}{x}J_1(x) - J_0(x)] - 2x^3[\frac{8}{x^2}J_1(x) - \frac{4}{x}J_0(x) - J_1(x)]$

$$= (-x^4 + 8x^2)J_0(x) + 4(x^3 - 16x)J_1(x)$$

(3) $\int_0^R J_0(x)\cos x\,dx$

$$\int_0^R J_0(x)\cos x\,dx = xJ_0(x)\cos x\Big|_0^R - \int_0^R x\,d[J_0(x)\cos x]$$

$$= xJ_0(x)\cos x\Big|_0^R - \int_0^R x[-J_1(x)\cos x - J_0(x)\sin x]dx$$

$$= xJ_0(x)\cos x\Big|_0^R + \int_0^R xJ_1(x)\cos x\,dx + \int_0^R xJ_0(x)\sin x\,dx$$

$$= xJ_0(x)\cos x\Big|_0^R + \int_0^R xJ_1(x)\cos x\,dx + \int_0^R \sin x\,d[xJ_1(x)]$$

$$= xJ_0(x)\cos x\Big|_0^R + \int_0^R xJ_1(x)\cos x\,dx + xJ_1(x)\sin x\Big|_0^R - \int_0^R xJ_1(x)\,d(\sin x)$$

$$= xJ_0(x)\cos x\Big|_0^R + \int_0^R xJ_1(x)\cos x\,dx + xJ_1(x)\sin x\Big|_0^R - \int_0^R xJ_1(x)\cos x\,dx$$

$$= RJ_0(R)\cos R + RJ_1(R)\sin R$$

(4) $3J_0'(x) + 4J_0'''(x)$

$J_0'(x) = -J_1(x)$

$J_0'''(x) = -\frac{d^2}{dx^2}J_1(x) = -\frac{1}{2}\frac{d}{dx}[J_0(x) - J_2(x)] = \frac{1}{2}J_1(x) + \frac{1}{2}\frac{d}{dx}[J_2(x)]$

$$= \frac{1}{2}J_1(x) + \frac{1}{4}J_1(x) + \frac{1}{4}J_3(x) = \frac{3}{4}J_1(x) + \frac{1}{4}J_3(x)$$

代入原式得 $3J_0'(x) + 4J_0'''(x) = J_3(x)$

3 12 / 12

✓ - 0 pts Correct

- 3 pts 1the answer is wrong
- 3 pts 2the answer is wrong
- 3 pts 3the answer is wrong
- 3 pts 4the answer is wrong

4. $J_\nu(x)$ 和 $J_{-\nu}(x)$ 是贝塞尔方程 $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\frac{x^2-\nu^2}{x^2}y=0$ 的两个线性独立的解

即
$$\begin{cases}\frac{d^2J_\nu(x)}{dx^2}+\frac{1}{x}\frac{dJ_\nu(x)}{dx}+\frac{x^2-\nu^2}{x^2}J_\nu(x)=0 & ① \\ \frac{d^2J_{-\nu}(x)}{dx^2}+\frac{1}{x}\frac{dJ_{-\nu}(x)}{dx}+\frac{x^2-\nu^2}{x^2}J_{-\nu}(x)=0 & ②\end{cases}$$

① $\cdot J_{-\nu}(x)-$ ② $\cdot J_\nu(x)$ 得

$$J_{-\nu}(x)\frac{d^2J_\nu(x)}{dx^2}+\frac{1}{x}J_{-\nu}(x)\frac{dJ_\nu(x)}{dx}-J_\nu(x)\frac{d^2J_{-\nu}(x)}{dx^2}-\frac{1}{x}J_\nu(x)\frac{dJ_{-\nu}(x)}{dx}=0$$

$$\Longrightarrow\left[J_{-\nu}(x)\frac{d^2J_\nu(x)}{dx^2}+\frac{dJ_{-\nu}(x)}{dx}\frac{dJ_\nu(x)}{dx}-J_\nu(x)\frac{d^2J_{-\nu}(x)}{dx^2}-\frac{dJ_\nu(x)}{dx}\frac{dJ_{-\nu}(x)}{dx}\right]$$
$$+\frac{1}{x}\left[J_{-\nu}(x)\frac{dJ_\nu(x)}{dx}-J_\nu(x)\frac{dJ_{-\nu}(x)}{dx}\right]=0$$

$$\Longrightarrow\frac{d}{dx}\left[J_\nu(x)\frac{dJ_{-\nu}(x)}{dx}-J_{-\nu}(x)\frac{dJ_\nu(x)}{dx}\right]+\frac{1}{x}\left[J_\nu(x)\frac{dJ_{-\nu}(x)}{dx}-J_{-\nu}(x)\frac{dJ_\nu(x)}{dx}\right]$$
$$=0$$

$$\Longrightarrow\frac{d}{dx}W[J_\nu(x),J_{-\nu}(x)]=-\frac{1}{x}W[J_\nu(x),J_{-\nu}(x)]$$

$$\Longrightarrow\ln W[J_\nu(x),J_{-\nu}(x)]=-\int\frac{1}{x}dx=-\ln x+B_1$$

$$\Longrightarrow W[J_\nu(x),J_{-\nu}(x)]=e^{-\ln x+B_1}=\frac{C_1}{x}$$

令 $x\to0$, $W[J_\nu(x),J_{-\nu}(x)]=\begin{vmatrix}\frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu & \frac{1}{\Gamma(-\nu+1)}\left(\frac{x}{2}\right)^{-\nu} \\ \frac{1}{2\Gamma(\nu)}\left(\frac{x}{2}\right)^\nu & \frac{1}{2\Gamma(-\nu)}\left(\frac{x}{2}\right)^{-\nu-1}\end{vmatrix}$

$$=\frac{1}{x}\left[\frac{1}{\Gamma(\nu+1)\Gamma(-\nu)}-\frac{1}{\Gamma(-\nu+1)\Gamma(\nu)}\right]$$
$$=\frac{1}{x}\left[\frac{1}{\nu\Gamma(\nu)\Gamma(-\nu)}+\frac{1}{\nu\Gamma(-\nu)\Gamma(\nu)}\right]$$
$$=-\frac{2}{x}\frac{1}{\Gamma(\nu)\Gamma(1-\nu)}=-\frac{2}{x}\frac{\sin\pi\nu}{\pi}\text{ (余元公式)}$$

$$\Longrightarrow C_1=-\frac{2\sin\pi\nu}{\pi}$$

$$\therefore W[J_\nu(x),J_{-\nu}(x)]=-\frac{2\sin\pi\nu}{\pi x}$$

同理，$J_\nu(x)$ 和 $Y_\nu(x)$ 也是贝塞尔方程 $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\frac{x^2-\nu^2}{x^2}y=0$ 的两个线性独立的解

即
$$\begin{cases}\dfrac{d^2J_\nu(x)}{dx^2}+\dfrac{1}{x}\dfrac{dJ_\nu(x)}{dx}+\dfrac{x^2-\nu^2}{x^2}J_\nu(x)=0 & ③\\[2ex] \dfrac{d^2Y_\nu(x)}{dx^2}+\dfrac{1}{x}\dfrac{dY_\nu(x)}{dx}+\dfrac{x^2-\nu^2}{x^2}Y_\nu(x)=0 & ④\end{cases}$$

③$\cdot Y_\nu(x)-$ ④$\cdot J_\nu(x)$ 得

$$Y_\nu(x)\frac{d^2J_\nu(x)}{dx^2}+\frac{1}{x}Y_\nu(x)\frac{dJ_\nu(x)}{dx}-J_\nu(x)\frac{d^2Y_\nu(x)}{dx^2}-\frac{1}{x}J_\nu(x)\frac{dY_\nu(x)}{dx}=0$$

$$\Longrightarrow \left[Y_\nu(x)\frac{d^2J_\nu(x)}{dx^2}+\frac{dY_\nu(x)}{dx}\frac{dJ_\nu(x)}{dx}-J_\nu(x)\frac{d^2Y_\nu(x)}{dx}-\frac{dJ_\nu(x)}{dx}\frac{dY_\nu(x)}{dx}\right]$$
$$+\frac{1}{x}\left[Y_\nu(x)\frac{dJ_\nu(x)}{dx}-J_\nu(x)\frac{dY_\nu(x)}{dx}\right]=0$$

$$\Longrightarrow \frac{d}{dx}\left[J_\nu(x)\frac{dY_\nu(x)}{dx}-Y_\nu(x)\frac{dJ_\nu(x)}{dx}\right]+\frac{1}{x}\left[J_\nu(x)\frac{dY_\nu(x)}{dx}-Y_\nu(x)\frac{dJ_\nu(x)}{dx}\right]$$
$$=0$$

$$\Longrightarrow \frac{d}{dx}W[J_\nu(x),Y_\nu(x)]=-\frac{1}{x}W[J_\nu(x),Y_\nu(x)]$$

$$\Longrightarrow \ln W[J_\nu(x),Y_\nu(x)]=-\int\frac{1}{x}dx=-\ln x+B_2$$

$$\Longrightarrow W[J_\nu(x),Y_\nu(x)]=e^{-\ln x+B_2}=\frac{C_2}{x}$$

令 $x\to+\infty$，
$$W[J_\nu(x),Y_\nu(x)]=\begin{vmatrix}\sqrt{\frac{2}{\pi x}}\cos(x-\frac{\pi}{4}-\frac{\nu\pi}{2}) & \dfrac{\sqrt{\frac{2}{\pi x}}\cos(x-\frac{\pi}{4}-\frac{\nu\pi}{2})\cos\nu\pi-\sqrt{\frac{2}{\pi x}}\cos(x-\frac{\pi}{4}+\frac{\nu\pi}{2})}{\sin\nu\pi}\\ -\sqrt{\frac{2}{\pi x}}\sin(x-\frac{\pi}{4}-\frac{\nu\pi}{2}) & \dfrac{-\sqrt{\frac{2}{\pi x}}\sin(x-\frac{\pi}{4}-\frac{\nu\pi}{2})\cos\nu\pi+\sqrt{\frac{2}{\pi x}}\sin(x-\frac{\pi}{4}+\frac{\nu\pi}{2})}{\sin\nu\pi}\end{vmatrix}$$

$$=\frac{\frac{2}{\pi x}\left[\cos(x-\frac{\pi}{4}-\frac{\nu\pi}{2})\sin(x-\frac{\pi}{4}+\frac{\nu\pi}{2})-\sin(x-\frac{\pi}{4}-\frac{\nu\pi}{2})\cos(x-\frac{\pi}{4}+\frac{\nu\pi}{2})\right]}{\sin\nu\pi}$$

$$=\frac{\frac{2}{\pi x}\sin(x-\frac{\pi}{4}+\frac{\nu\pi}{2}-x+\frac{\pi}{4}+\frac{\nu\pi}{2})}{\sin\nu\pi}=\frac{2}{\pi x}$$

$$\Longrightarrow C_2=\frac{2}{\pi}$$

$$\therefore W[J_\nu(x),Y_\nu(x)]=\frac{2}{\pi x}$$

(1) $\int \frac{dx}{xJ_\nu^2(x)}$

$$\int \frac{dx}{xJ_\nu^2(x)} = \int -\frac{\pi W[J_\nu(x), J_{-\nu}(x)]}{2\sin\pi\nu\, J_\nu^2(x)} dx$$

$$= -\frac{\pi}{2\sin\pi\nu} \int \frac{J_\nu(x)J_{-\nu}'(x) - J_\nu'(x)J_{-\nu}(x)}{J_\nu^2(x)} dx$$

$$= -\frac{\pi}{2\sin\pi\nu} \int d\left[\frac{J_{-\nu}(x)}{J_\nu(x)}\right]$$

$$= -\frac{\pi J_{-\nu}(x)}{2J_\nu(x)\sin\pi\nu} + C$$

(2) $\int \frac{dx}{xY_\nu^2(x)}$

$$\int \frac{dx}{xY_\nu^2(x)} = \int \frac{\pi W[J_\nu(x), Y_\nu(x)]}{2Y_\nu^2(x)} dx$$

$$= -\frac{\pi}{2} \int \frac{Y_\nu(x)J_\nu'(x) - Y_\nu'(x)J_\nu(x)}{Y_\nu^2(x)} dx$$

$$= -\frac{\pi}{2} \int d\left[\frac{J_\nu(x)}{Y_\nu(x)}\right]$$

$$= -\frac{\pi J_\nu(x)}{2Y_\nu(x)} + C$$

(3) $\int \frac{dx}{xJ_\nu(x)Y_\nu(x)}$

$$\int \frac{dx}{xJ_\nu(x)Y_\nu(x)} = \int \frac{\pi}{2} \frac{W[J_\nu(x), Y_\nu(x)]}{J_\nu(x)Y_\nu(x)} dx$$

$$= \int \frac{\pi}{2} \frac{J_\nu(x)Y_\nu'(x) - J_\nu'(x)Y_\nu(x)}{J_\nu(x)Y_\nu(x)} dx$$

$$= \frac{\pi}{2}\left[\int \frac{Y_\nu'(x)}{Y_\nu(x)} dx - \int \frac{J_\nu'(x)}{J_\nu(x)} dx\right]$$

$$= \frac{\pi}{2}\left[\int \frac{dY_\nu(x)}{Y_\nu(x)} - \int \frac{dJ_\nu(x)}{J_\nu(x)}\right]$$

$$= \frac{\pi}{2} \ln \frac{Y_\nu(x)}{J_\nu(x)} + C$$

4 ▯▯▯ 9 / 9

- ✓ **- 0 pts** Correct
- **- 2 pts** (1)wrong
- **- 2 pts** (2)wrong
- **- 2 pts** (3)wrong
- **- 9 pts** blank/wrong

5. 方程两边同乘 $\frac{1}{\gamma^2 z^{\alpha-2}}$ 有

$$\frac{u''}{\gamma^2 z^{\alpha-2}} + \frac{1-2\alpha}{\gamma^2 z^{\alpha-1}}u' + \left[\beta^2 z^{2\gamma-\alpha} + \frac{\alpha^2}{\gamma^2 z^{\alpha}} - \frac{\nu^2}{z^{\alpha}}\right]u = 0$$

$$\Longrightarrow \frac{1}{\gamma^2}\frac{u''}{z^{\alpha-2}} + \frac{-\gamma-2\alpha+1+\gamma}{\gamma^2}\frac{u'}{z^{\alpha-1}} + \frac{\alpha(\gamma+\alpha)-\alpha\gamma}{\gamma^2 z^{\alpha}}u + \left[\beta^2(z^\gamma)^2-\nu^2\right]\frac{u}{z^\alpha} = 0$$

$$\Longrightarrow \frac{1}{\gamma^2}\left[\frac{u''}{z^{\alpha-2}} + (-\gamma-2\alpha+1)\frac{u'}{z^{\alpha-1}} + \frac{\alpha(\gamma+\alpha)}{z^\alpha}u\right] + \frac{1}{\gamma z^{-1}}\left(\frac{u'}{z^\alpha} - \frac{\alpha u}{z^{\alpha+1}}\right) + \left[\beta^2(z^\gamma)^2-\nu^2\right]\frac{u}{z^\alpha} = 0$$

其中 $\frac{1}{\gamma^2}\left[\frac{u''}{z^{\alpha-2}} + (-\gamma-2\alpha+1)\frac{u'}{z^{\alpha-1}} + \frac{\alpha(\gamma+\alpha)}{z^\alpha}u\right]$

$$= \frac{1}{\gamma^2}\left[\frac{u''}{z^{\alpha-2}} - (\gamma+\alpha-1)\frac{u'}{z^{\alpha-1}} - \frac{\alpha u'}{z^{\alpha-1}} + \frac{\alpha(\gamma+\alpha)}{z^\alpha}u\right].$$

$$= \frac{(z^\gamma)^2}{\gamma}\frac{1}{\gamma z^{\gamma-1}}\left[\frac{u''}{z^{\gamma+\alpha-1}} - (\gamma+\alpha-1)\frac{u'}{z^{\gamma+\alpha}} - \frac{\alpha u'}{z^{\gamma+\alpha}} + \frac{\alpha(\gamma+\alpha)u}{z^{\gamma+\alpha+1}}\right].$$

$$= \frac{(z^\gamma)^2}{\gamma}\frac{1}{\frac{d(z^\gamma)}{dz}}\frac{d}{dz}\left[\frac{u'}{z^{\gamma+\alpha-1}} - \frac{\alpha u}{z^{\gamma+\alpha}}\right]$$

$$= \frac{(z^\gamma)^2}{\gamma}\frac{d}{dz^\gamma}\left[\frac{u'}{z^{\gamma+\alpha-1}} - \frac{\alpha u}{z^{\gamma+\alpha}}\right].$$

$$= (z^\gamma)^2\frac{d}{dz^\gamma}\left[\frac{1}{\gamma z^{\gamma-1}}\left(\frac{u'}{z^\alpha} - \frac{\alpha u}{z^{\alpha+1}}\right)\right]$$

$$= (z^\gamma)^2\frac{d}{dz^\gamma}\left[\frac{1}{\frac{dz^\gamma}{dz}}\frac{d}{dz}\left(\frac{u}{z^\alpha}\right)\right]$$

$$= (z^\gamma)^2\frac{d^2}{(dz^\gamma)^2}\left(\frac{u}{z^\alpha}\right) \quad ①$$

$$\frac{1}{\gamma z^{\gamma-1}}\left(\frac{u'}{z^\alpha} - \frac{\alpha u}{z^{\alpha+1}}\right) = \frac{1}{\frac{dz^\gamma}{dz}}\frac{d}{dz}\left(\frac{u}{z^\alpha}\right) = \frac{d}{dz^\gamma}\left(\frac{u}{z^\alpha}\right) \quad ②$$

将①②同代入方程中有

$$(z^\gamma)^2\frac{d^2}{(dz^\gamma)^2}\left(\frac{u}{z^\alpha}\right) + \frac{d}{dz^\gamma}\left(\frac{u}{z^\alpha}\right) + \left[\beta^2(z^\gamma)^2-\nu^2\right]\frac{u}{z^2} = 0$$

记 $x=z^\gamma$, $y=\frac{u}{z^\alpha}$，有

$$x^2\frac{d^2}{dx^2}y + \frac{d}{dx}y + \left[\beta^2x^2-\nu^2\right]y = 0$$

这是一个参数形式的贝塞尔函数，其通解为:

$$y = C_1 J_\nu(\beta x) + C_2 Y_\nu(\beta x) \quad ③$$

将 $x=z^{\gamma}$, $y=\frac{u}{z^{\alpha}}$ 同代入③中得到原方程的通解:

$$\frac{u}{z^{\alpha}}=C_1 J_{\nu}(\beta z^{\gamma})+C_2 Y_{\nu}(\beta z^{\gamma})$$

$$\Longrightarrow u=C_1 z^{\alpha} J_{\nu}(\beta z^{\gamma})+C_2 z^{\alpha} Y_{\nu}(\beta z^{\gamma})$$

1. $u''+az^{b}u=0$

在此方程中 $\begin{cases} 1-2\alpha=0 \\ (\beta\gamma)^2=1 \\ (z^{\gamma-1})^2=z^{b} \\ \alpha^2-\gamma^2\nu^2=0 \end{cases} \Longrightarrow \begin{cases} \alpha=\frac{1}{2} \\ \gamma=\frac{b+1}{2} \\ \beta=\frac{2}{b+1} \\ \nu=\frac{1}{b+1} \end{cases}$

故其通解为 $u=C_1 z^{\frac{1}{2}} J_{\frac{1}{b+1}}\left(\frac{2}{b+1} z^{\frac{b+1}{2}}\right)+C_2 z^{\frac{1}{2}} Y_{\frac{1}{b+1}}\left(\frac{2}{b+1} z^{\frac{b+1}{2}}\right)$

2. $zu''-3u'+zu=0$

$\Longrightarrow u''-\frac{3}{z}u'+u=0$

在此方程中 $\begin{cases} 1-2\alpha=-3 \\ (\beta\gamma)^2=1 \\ (z^{\gamma-1})^2=z \\ \alpha^2-\gamma^2\nu^2=0 \end{cases} \Longrightarrow \begin{cases} \alpha=2 \\ \gamma=\frac{3}{2} \\ \beta=\frac{2}{3} \\ \nu=\frac{16}{9} \end{cases}$

故其通解为 $u=C_1 z^{2} J_{\frac{16}{9}}\left(\frac{2}{3} z^{\frac{3}{2}}\right)+C_2 z^{2} Y_{\frac{16}{9}}\left(\frac{2}{3} z^{\frac{3}{2}}\right)$

5 □□□ 10 / 10

- ✓ **- 0 pts** Correct
- **- 3 pts** (1)wrong
- **- 3 pts** (2)wrong
- **- 10 pts** blank/wrong

6. 分离变量 $u(\rho,t)=R(\rho)T(t)$

代入原方程得 $\dfrac{\partial T}{kT\partial t}=\dfrac{1}{\rho R}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial R}{\partial\rho}\right)=-\lambda^2$

其中 λ 是分离常数

$$\Longrightarrow\begin{cases}T'+k\lambda^2T=0 & ①\\ \dfrac{\partial^2R}{\partial\rho^2}+\dfrac{1}{\rho}\dfrac{\partial R}{\partial\rho}+\lambda^2R=0 & ②\end{cases}$$

由①得 $T(t)=Ce^{-k\lambda^2t}$

②为0阶贝塞尔方程，其通解为

$$R(\rho)=AJ_0(\lambda\rho)+BY_0(\lambda\rho)$$

$\because u|_{\rho=0}$ 有界而 $Y_0(0)$ 无界

$\therefore B=0 \quad R(\rho)=AJ_0(\lambda\rho)$.

$\because \dfrac{\partial u}{\partial\rho}\Big|_{\rho=a}=T\dfrac{\partial R}{\partial\rho}\Big|_{\rho=a}=0\Longrightarrow\dfrac{\partial J_0(\lambda\rho)}{\partial\rho}\Big|_{\rho=a}=0$

$\therefore \lambda a=\mu'_{0m}$ 是 $J_0'(x)$ 的第 m 个零点（其实也是 $J_1(x)$ 的零点，$\mu'_{00}=0$）

综上：方程本征解为 $u_m(\rho,t)=A_mJ_0(\mu'_{0m}\frac{\rho}{a})e^{-k\lambda^2t}$

一般解为 $u(\rho,t)=\sum\limits_{m=0}^{\infty}A_mJ_0(\mu'_{0m}\frac{\rho}{a})e^{-k\lambda^2t}$

利用初始条件 $u|_{t=0}=u_0(1-\frac{\rho^2}{a^2})$ 有

$$u_0(1-\frac{\rho^2}{a^2})=\sum_{m=0}^{\infty}A_mJ_0(\mu'_{0m}\frac{\rho}{a})=A_0+\sum_{m=1}^{\infty}A_mJ_0(\mu'_{0m}\frac{\rho}{a}).$$ 上展开的系数

这表明 A_m 是 $u_0(1-\frac{\rho^2}{a^2})$ 按 $J_0(\mu'_{0m}\frac{\rho}{a})$ 在区间 $[0,a]$ 上展开的系数

$$A_0=\frac{2}{a^2J_0^2(\mu'_{00})}\int_0^a\rho u_0(1-\frac{\rho^2}{a^2})J_0(\mu'_{00}\frac{\rho}{a})d\rho$$

$$=\frac{2u_0}{a^2}\int_0^a(\rho-\frac{\rho^3}{a^2})d\rho=\frac{u_0}{2}$$

$$A_m=\frac{2}{a^2J_0^2(\mu'_{0m})}\int_0^a\rho u_0(1-\frac{\rho^2}{a^2})J_0(\mu'_{0m}\frac{\rho}{a})d\rho$$

$$=\frac{2u_0}{a^2J_0^2(\mu'_{0m})}\left[\int_0^a\rho J_0(\mu'_{0m}\frac{\rho}{a})d\rho-\int_0^a\frac{\rho^3}{a^2}J_0(\mu'_{0m}\frac{\rho}{a})d\rho\right]$$

$$=\frac{2u_0}{a^2J_0^2(\mu'_{0m})}\left\{\frac{a}{\mu'_{0m}}\int_0^a d\left[\rho J_1(\mu'_{0m}\frac{\rho}{a})\right]-\frac{1}{a\mu'_{0m}}\int_0^a\rho^2d\left[\rho J_1(\mu'_{0m}\frac{\rho}{a})\right]\right\}$$

$$=\frac{2u_0}{a^2\mu'_{0m}J_0^2(\mu'_{0m})}\left\{a\rho J_1(\mu'_{0m}\frac{\rho}{a})\Big|_0^a-\frac{\rho^3}{a}J_1(\mu'_{0m}\frac{\rho}{a})\Big|_0^a+\frac{1}{a}\int_0^a\rho J_1(\mu'_{0m}\frac{\rho}{a})d\rho^2\right\}$$

$$= \frac{4u_0}{a^3\mu'_{0m}J_0^2(\mu'_{0m})}\int_0^a \rho^2 J_1\left(\mu'_{0m}\frac{\rho}{a}\right)d\rho$$

$$= \frac{4u_0}{a^2\mu'^2_{0m}J_0^2(\mu'_{0m})}\int_0^a d\left[\rho^2 J_2\left(\mu'_{0m}\frac{\rho}{a}\right)\right]$$

$$= \frac{4u_0}{a^2\mu'^2_{0m}J_0^2(\mu'_{0m})}\rho^2 J_2\left(\mu'_{0m}\frac{\rho}{a}\right)\Big|_0^a$$

$$= \frac{4u_0 J_2(\mu'_{0m})}{\mu'^2_{0m}J_0^2(\mu'_{0m})}$$

根据递推公式 $J_0(x)+J_2(x)=\frac{2}{x}J_1(x) \Rightarrow J_2(\mu'_{0m})=\frac{2}{x}J_1(\mu'_{0m})-J_0(\mu'_{0m})$

$$= -J_0(\mu'_{0m})$$

$$\therefore A_m = -\frac{4u_0}{\mu'^2_{0m}J_0(\mu'_{0m})}$$

综上：柱体内温度分布与变化为

$$u = \frac{u_0}{2} - 4u_0\sum_{m=1}^{\infty}\frac{1}{\mu'_{0m}J_0(\mu'_{0m})}J_0\left(\mu'_{0m}\frac{\rho}{a}\right)e^{-k\left(\frac{\mu'_{0m}}{a}\right)^2 t}$$

当时间足够长时柱体达到稳定温度 $\lim\limits_{t\to+\infty}u=\frac{u_0}{2}$

6 ▯▯▯ 10 / 10

✓ **- 0 pts** Correct

- 3 pts the answer is wrong

7. $$\begin{cases} \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{\partial^2 u}{\partial z^2}=0 \\ u|_{z=0}=0,\quad u|_{z=h}=0 \\ u|_{\rho=0}\text{有界},\quad u|_{\rho=a}=u_0\sin\left(2\pi\frac{z}{h}\right) \end{cases}$$

分离变量，设 $u(\rho,z)=R(\rho)Z(z)$

代入原方程得 $-\frac{Z''}{Z}=\frac{R''+\frac{1}{\rho}R'}{R}=\lambda^2$

$$\Longrightarrow\begin{cases} Z''+\lambda^2 Z=0 & ① \\ R''+\frac{1}{\rho}R'-\lambda^2 R=0 & ② \end{cases}$$

由①得 $Z(z)=C\sin(\lambda z)+D\cos(\lambda z)$

由边界条件 $u|_{z=0}=0$ 得 $D=0$

由边界条件 $u|_{z=h}=0$ 得 $\lambda_n=\frac{n\pi}{h}$ $\Longrightarrow Z(z)=C\sin\left(\frac{n\pi}{h}z\right)$

②为0阶虚宗量贝塞尔方程，其通解为

$R(\rho)=A_0I_0(\lambda\rho)+BK_0(\lambda\rho)$

$\because u|_{\rho=0}$有界而$K_0(0)$无界

$\therefore B=0$ $\quad R(\rho)=AI_0(\lambda\rho)=A_0I_0\left(\frac{n\pi}{h}\rho\right)$

故原方程的本征解为 $u_n(\rho,z)=A_n\sin\left(\frac{n\pi}{h}z\right)I_0\left(\frac{n\pi}{h}\rho\right)$

一般解为 $u(\rho,z)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi}{h}z\right)I_0\left(\frac{n\pi}{h}\rho\right)$

利用边界条件 $u|_{\rho=a}=u_0\sin\left(2\pi\frac{z}{h}\right)$

$$\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi}{h}z\right)I_0\left(\frac{n\pi a}{h}\right)=u_0\sin\left(2\pi\frac{z}{h}\right)$$

$$\Longrightarrow A_2=\frac{u_0}{I_0\left(\frac{2\pi a}{h}\right)}$$

$$\boxed{A_n=0\quad(n\neq 2)}$$

故 $u(\rho,z)=\frac{u_0}{I\left(\frac{2\pi a}{h}\right)}\boxed{\sin\left(\frac{n\pi}{h}z\right)I_0\left(\frac{2\pi}{h}\rho\right)}$

7 ▯▯▯ 8 / 10

- **- 0 pts** Correct
- ✓ **- 2 pts** little mistake
- **- 5 pts** Mess
- **- 10 pts** No answer

8. Rodrigues公式：$P_l(x)=\frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$

$$\int_{-1}^{1}(1+x)^k P_l(x)dx=\int_{-1}^{1}(1+x)^k\frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l dx$$

$$=\frac{1}{2^l l!}\int_{-1}^{1}(1+x)^l d\left[\frac{d^{l-1}}{dx^{l-1}}(x^2-1)^l\right]$$

$$=\left[\frac{1}{2^l l!}(1+x)^k\frac{d^{l-1}}{dx^{l-1}}(x^2-1)^l\right]\Big|_{-1}^{1}-\frac{1}{2^l l!}\int_{-1}^{1}\frac{d^{l-1}}{dx^{l-1}}\left[(x^2-1)^l\right]d\left[(1+x)^k\right]$$

（由于$\frac{d^{l-1}}{dx^{l-1}}(x^2-1)^l$中含有$(x^2-1)$因子，故第一项为0）

$$=-\frac{k}{2^l l!}\int_{-1}^{1}(1+x)^{k-1}\frac{d^{l-1}}{dx^{l-1}}(x^2-1)^l dx$$

......

$$=(-1)^l\frac{k!}{2^l l!(k-l)!}\int_{-1}^{1}(1+x)^{k-l}(x^2-1)^l dx$$

$$=\frac{k!}{2^l l!(k-l)!}\int_{-1}^{1}(1+x)^k(1-x)^l dx$$

$$\xlongequal{t=\frac{x+1}{2}}\frac{k!}{2^l l!(k-l)!}\int_0^1(2t)^k(2-2t)^l d(2t-1)$$

$$=\frac{2^{k+1}k!}{l!(k-l)!}\int_0^1 t^{(k+1)-1}(1-t)^{(l+1)-1}dt$$

$$=\frac{2^{l+1}k!}{l!(k-l)!}B(k+1,l+1)$$

Beta函数

$$=\frac{2^{l+1}k!}{l!(k-l)!}\frac{\Gamma(k+1)\Gamma(l+1)}{\Gamma(k+l+2)}$$

$$=\frac{2^{l+1}k!}{l!(k-l)!}\frac{k!\,l!}{(k+l+1)!}$$

$$=\frac{2^{l+1}(k!)^2}{(k-l)!(k+l+1)!}$$

当$k<l$时 $\int_{-1}^{1}(1+x)^k P_l(x)dx=(-1)^l\frac{k!}{2^l l!(k-l)!}\int_{-1}^{1}\frac{d^l}{dx^l}\left[(1+x)^k\right](x^2-1)^l dx$

$$=0$$

8 10 / 10

✓ **- 0 pts** Correct

- 4 pts Not

- 10 pts No right answer

9. $\begin{cases} \nabla^2 u = 0, \ a<r<b \\ u|_{r=a} = u_0, \ u|_{r=b} = u_0 \cos^2\theta \end{cases}$

拉普拉斯方程在轴对称条件下的通解为

$$u(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

利用边界条件:

$$\begin{cases} u|_{r=a} = \sum_{l=0}^{\infty} \left(A_l a^l + \frac{B_l}{a^{l+1}}\right) P_l(\cos\theta) = u_0 = u_0 P_0(\cos\theta) + 0 \cdot P_2(\cos\theta) \\ u|_{r=b} = \sum_{l=0}^{\infty} \left(A_l b^l + \frac{B_l}{b^{l+1}}\right) P_l(\cos\theta) = u_0 \cos^2\theta = \frac{1}{3} u_0 P_0(\cos\theta) + \frac{2}{3} u_0 P_2(\cos\theta) \end{cases}$$

$$\xRightarrow{\text{比较系数}} \begin{cases} A_0 + \frac{B_0}{a} = u_0 \\ A_0 + \frac{B_0}{b} = \frac{1}{3} u_0 \\ A_2 a^2 + \frac{B_2}{a^3} = 0 \\ A_2 b^2 + \frac{B_2}{b^3} = \frac{2}{3} u_0 \\ A_l = 0, \ B_l = 0 \ (l \neq 0, l \neq 2) \end{cases} \Longrightarrow \begin{cases} A_0 = \frac{(-3a+b)u_0}{3(b-a)} \\ B_0 = \frac{2abu_0}{3(b-a)} \\ A_2 = \frac{2b^3 u_0}{3(b^5-a^5)} \\ B_2 = -\frac{2a^5 b^3 u_0}{3(b^5-a^5)} \\ A_l = 0, \ B_l = 0 \ (l \neq 0, l \neq 2) \end{cases}$$

$$\therefore u(r,\theta) = \frac{(-3a+b)u_0}{3(b-a)} + \frac{2abu_0}{3(b-a)r} + \frac{2a^2 b^3 u_0}{3(b^5-a^5)} \left[\left(\frac{r}{a}\right)^2 - \left(\frac{a}{r}\right)^3\right] P_2(\cos\theta)$$

9 ▯▯▯ 10 / 10

- ✓ **- 0 pts** Correct
- **- 2 pts** calculate mistake
- **- 10 pts** No answer

10. (1) $(\sin\theta-2\cos^2\theta)\cos^2\varphi$

$=\frac{1}{2}(\sin\theta-2\cos^2\theta)+\frac{1}{2}(\sin\theta-2\cos^2\theta)\cos^2\varphi$

$=\frac{1}{2}(\sin\theta-2\cos^2\theta)+\frac{1}{2}(\sin\theta-2\cos^2\theta)(e^{2i\varphi}+e^{-2i\varphi})$

设 $\sin\theta-2\cos^2\theta=\sum_{l=0}^{\infty}A_l P_l(\cos\theta)$

其中 $A_l=\frac{2l+1}{2}\int_{-1}^{1}(\sin\theta-2\cos^2\theta)P_l(\cos\theta)d\cos\theta$

当 l 为偶数, $A_l=-(2l+1)\int_{-1}^{1}\cos^2\theta P_l(\cos\theta)d\cos\theta$

$=-(2l+1)\int_{-1}^{1}x^2 P_l(x)dx$

$=-(2l+1)\int_{-1}^{1}\left[\frac{1}{3}P_0(x)+\frac{2}{3}P_2(x)\right]P_l(x)dx$

$=\begin{cases}-\frac{2}{3}, & l=0\\ -\frac{4}{3}, & l=2\\ 0 & \text{otherwise}\end{cases}$

当 l 为奇数, $A_l=\frac{2l+1}{2}\int_{-1}^{1}\sin\theta P_l(\cos\theta)d\cos\theta$

$=(2l+1)\int_{0}^{1}\sqrt{1-x^2}P_l(x)dx$

设 $\sin\theta-2\cos^2\theta=\sum_{l=0}^{\infty}B_l P_l^2(\cos\theta)$

其中 $B_l=\frac{2l+1}{2}\frac{(l+2)!}{(l-2)!}\int_{-1}^{1}(\sin\theta-2\cos^2\theta)P_l^2(\cos\theta)d\cos\theta$

当 l 为偶数, $B_l=-(2l+1)\frac{(l+2)!}{(l-2)!}\int_{-1}^{1}\cos^2\theta P_l^2(\cos\theta)d\cos\theta$

$=-(2l+1)\frac{(l+2)!}{(l-2)!}\int_{-1}^{1}x^2P_l^2(x)dx$

当 l 为奇数, $B_l=\frac{2l+1}{2}\frac{(l+2)!}{(l-2)!}\int_{-1}^{1}\sin\theta P_l^2(\cos\theta)d\cos\theta$

$=(2l+1)\frac{(l+2)!}{(l-2)!}\int_{-1}^{1}\sqrt{1-x^2}P_l^2(x)dx$

$\therefore(\sin\theta-2\cos^2\theta)\cos^2\varphi=\frac{1}{2}\sum_{l=0}^{\infty}\sqrt{\frac{4\pi}{2l+1}}A_lY_l^0(\theta,\varphi)+\frac{1}{2}\sum_{l=0}^{\infty}\sqrt{\frac{4\pi}{2l+1}\frac{(l+2)!}{(l-2)!}}B_l\left[Y_l^2(\theta,\varphi)+Y_l^{-2}(\theta,\varphi)\right]$

其中 $A_l=\begin{cases}-\frac{2}{3}, & l=0\\ -\frac{4}{3}, & l=2\\ 0, & l\text{ 为偶数}, l\neq 0,2\\ (2l+1)\int_0^1\sqrt{1-x^2}P_l(x)dx, & l\text{ 为奇数}\end{cases}$

$B_l=\begin{cases}-(2l+1)\frac{(l+2)!}{(l-2)!}\int_{-1}^{1}x^2P_l^2(x)dx, & x\text{ 为偶数}\\ (2l+1)\frac{(l+2)!}{(l-2)!}\int_{-1}^{1}\sqrt{1-x^2}P_l^2(x)dx, & x\text{ 为奇数}\end{cases}$

(2) $(1-2\sin\theta)\cos\theta\cos\varphi$

$=\frac{1}{2}(1-2\sin\theta)\cos\theta(e^{i\varphi}+e^{-i\varphi})$

设 $(1-2\sin\theta)\cos\theta=\sum_{l=0}^{\infty} C_l P_l^1(\cos\theta)$.

其中 $C_l=\frac{2l+1}{2}\frac{(l+1)!}{(l-1)!}\int_{-1}^{1}(1-2\sin\theta)\cos\theta P_l^1(\cos\theta)\, d\cos\theta$

当 l 为奇数，$C_l=-(2l+1)\frac{(l+1)!}{(l-1)!}\int_{-1}^{1}\sin\theta\cos\theta P_l^1(\cos\theta)\, d\cos\theta$

$=-2(2l+1)\frac{(l+1)!}{(l-1)!}\int_0^1\sqrt{1-x^2}\, x P_l^1(x)dx.$

当 l 为偶数，$C_l=\frac{2l+1}{2}\frac{(l+1)!}{(l-1)!}\int_{-1}^{1}\cos\theta P_l^1(\cos\theta)d\cos\theta$

$=(2l+1)\frac{(l+1)!}{(l-1)!}\int_0^1 x P_l^1(x)dx.$

$\therefore (1-2\sin\theta)\cos\theta\cos\varphi=\frac{1}{2}\sum_{l=0}^{\infty}\sqrt{\frac{4\pi}{2l+1}\frac{(l+1)!}{(l-1)!}}\, C_l\left[Y_l^1(x)+Y_l^{-1}(x)\right]$

其中 $C_l=\begin{cases}-2(2l+1)\frac{(l+1)!}{(l-1)!}\int_0^1\sqrt{1-x^2}\, xP_l^1(x)dx, & l\text{为奇数}\\(2l+1)\frac{(l+1)!}{(l-1)!}\int_0^1 xP_l^1(x)dx, & l\text{为偶数}\end{cases}$

10 □□□ 10 / 10

✓ **- 0 pts** Correct

- 10 pts No answer

- 4 pts Not perfect/Half

11. $\begin{cases} \nabla^2 u = A + Br^2\sin 2\theta\cos\varphi \\ u|_{r=a} = 0 \end{cases}$

设方程通解为 $u(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=0}^{l} P_l^m(\cos\theta)[R_{l,m}(r)\sin m\varphi + S_{l,m}(r)\cos m\varphi]$

代入方程得 $\sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{1}{r^2}[\frac{d}{dr}(r^2\frac{dR_{l,m}}{dr}) - l(l+1)R_{l,m}]P_l^m(\cos\theta)\sin m\varphi$

$+ \sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{1}{r^2}[\frac{d}{dr}(r^2\frac{dS_{l,m}}{dr}) - l(l+1)S_{l,m}]P_l^m(\cos\theta)\cos m\varphi$

$= A + Br^2\sin 2\theta\cos\varphi = AP_0^0(\cos\theta) - \frac{2}{3}Br^2P_2^1(\cos\theta)\cos\varphi.$

比较系数得 $\frac{d}{dr}(r^2\frac{dR_{l,m}}{dr}) - l(l+1)R_{l,m} = 0.$

$\frac{d}{dr}(r^2\frac{dS_{0,0}}{dr}) = Ar^2$

$\frac{d}{dr}(r^2\frac{dS_{2,1}}{dr}) - 6S_{2,1} = -\frac{2}{3}Br^4$

$\frac{d}{dr}(r^2\frac{S_{l,m}}{dr}) - l(l+1)S_{l,m} = 0$ (除 $(l,m)=(0,0)$ 和 $(l,m)=(2,1)$ 外)

由边界条件 $R_{l,m}(a)=0$, $S_{l,m}(a)=0$

由 $u|_{r=0}$ 有界, $R_{l,m}(0)$ 有界, $S_{l,m}(0)$ 有界.

$\therefore$ $R_{l,m}(r)=0.$

$S_{0,0}(r) = \frac{A}{6}(r^2-a^2)$

$S_{2,1}(r) = \frac{Br^2}{21}(a^2-r^2)$

$S_{l,m}(r) = 0$ (除 $(l,m)=(0,0)$ 和 $(l,m)=(2,1)$ 外)

$\therefore u(r,\theta,\varphi) = \frac{A}{6}(r^2-a^2) + \frac{B}{21}r^2(a^2-r^2)P_2^1(\cos\theta)\cos\varphi$

$= \frac{A}{6}(r^2-a^2) + \frac{B}{14}r^2(r^2-a^2)\sin 2\theta\cos\varphi.$

11 □□□□ 10 / 10

- ✓ **- 0 pts** Correct
- **- 2 pts** calculate incorrect
- **- 6 pts** method incorrect
- **- 10 pts** No answer /Not right at all