1. $\begin{cases} \dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=6(x+y)^2,\ 0<a^2<x^2+y^2<b^2<\infty \\ u|_{x^2+y^2=a^2}=1,\ \dfrac{\partial u}{\partial n}\Big|_{x^2+y^2=b}=0 \end{cases}$

转换到极坐标系中 $\begin{cases} \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial u}{\partial\rho}\right)+\dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial\theta^2}=6\rho^2(1+\sin 2\theta) \\ u|_{\rho=a}=1 \quad ②,\quad \dfrac{\partial u}{\partial\rho}\Big|_{\rho=b}=0 \quad ③ \end{cases}$

一般解可表示为 $u(\rho,\theta)=\sum_{n=0}^{\infty}[A_n(\rho)\cos n\theta+B_n(\rho)\sin n\theta]$.

代入①中有 $\sum_{n=0}^{\infty}\left\{\left[A_n''(\rho)+\dfrac{A_n'(\rho)}{\rho}-\dfrac{n^2}{\rho^2}A_n(\rho)\right]\cos n\theta\right.$

$\left.+\left[B''(\rho)+\dfrac{B_n'(\rho)}{\rho}-\dfrac{n^2}{\rho^2}B_n(\rho)\right]\sin n\theta\right\}=6\rho^2(1+\sin 2\theta)$

$\Longrightarrow\begin{cases} A_0''(\rho)+\dfrac{A_0'(\rho)}{\rho}=6\rho^2 \quad ④ \\ A_n''(\rho)+\dfrac{A_n'(\rho)}{\rho}-\dfrac{n^2}{\rho^2}A_n(\rho)=0\Longrightarrow \rho^2A''(\rho)+\rho A_n'(\rho)-n^2A_n(\rho)=0,\ n=1,2,3\cdots \quad ⑤ \\ B_2''(\rho)+\dfrac{B_2'(\rho)}{\rho}-\dfrac{4}{\rho^2}B_2(\rho)=6\rho^2\Rightarrow\rho^2B_2''(\rho)+\rho B_2'(\rho)-4B_2(\rho)=6\rho^4 \quad ⑥ \\ B_n''(\rho)+\dfrac{B_n'(\rho)}{\rho}-\dfrac{n^2}{\rho^2}B_n(\rho)=0\Longrightarrow\rho^2B''(\rho)+\rho B_n'(\rho)-n^2B_n(\rho)=0,\ n=1,2,3\cdots \quad ⑦ \end{cases}$

由⑤，设 $A_n(\rho)=C_n\rho^n+d_n\rho^{-n}$，$n=1,2,3,\cdots$

由⑦，设 $B_n(\rho)=C_n'\rho^n+d_n'\rho^{-n}$，$n\neq 2$

由②③有 $\begin{cases} A_n(a)=0 \quad A_n'(b)=0,\ n=1,2,3 \\ B_n(a)=0 \quad B_n'(b)=0,\ n\neq 2. \end{cases}$

$\Longrightarrow\begin{cases} C_n=d_n=0 \quad (n=1,2,3,\cdots) \\ C_n'=d_n'=0 \quad (n\neq 2). \end{cases}$

由④，得 $A_0'(\rho)=e^{-\int\frac{1}{\rho}d\rho}\left[\int 6\rho^2e^{\int\frac{1}{\rho}d\rho}d\rho+C_1\right]=\dfrac{3}{2}\rho^3+\dfrac{C_0}{\rho}$.

$A_0(\rho)=\dfrac{3}{8}\rho^4+C_0\ln\rho+d_0$

由②③有 $\begin{cases} A_0(a)=1 \\ A_0'(b)=0 \end{cases}\Longrightarrow\begin{cases} C_0=-\dfrac{3}{2}b^4 \\ d_0=-\dfrac{3}{8}a^4+\dfrac{3}{2}b^4\ln a+1 \end{cases}$

$\therefore A_0(\rho)=\dfrac{3}{8}\rho^4-\dfrac{3}{2}b^4\ln\rho-\dfrac{3}{8}a^4+\dfrac{3}{2}b^4\ln a+1$

由⑥有一特解 $B_2^*(\rho)=\dfrac{1}{2}\rho^4$

故其通解为 $B_2(\rho)=C_2'\rho^2+d_2'\rho^{-2}+\dfrac{1}{2}\rho^4$

由②③得
$$\begin{cases} B_2(a)\sin(2\theta) = \left(C_2' a^2 + d_2' a^{-2} + \frac{1}{2}a^4\right)\sin(2\theta) = 0 \\ B_2'(a)\sin(2\theta) = \left(2C_2' b - 2d_2' b^{-3} + 2b^3\right)\sin(2\theta) = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} C_2' = -\dfrac{\frac{1}{2}a^6 + b^6}{a^4 + b^4} \\ d_2' = \dfrac{a^4 b^4 (b^2 - \frac{1}{2}a^2)}{a^4 + b^4} \end{cases}$$

从而 $$B_2(\rho) = -\frac{\frac{1}{2}a^6 + b^6}{a^4 + b^4}\rho^2 + \frac{a^4 b^4 (b^2 - \frac{1}{2}a^2)}{a^4 + b^4}\rho^{-2} + \frac{1}{2}\rho^4$$

综上：$$u(\rho,\theta) = \left[\frac{3}{8}(\rho^4 - a^4) - \frac{3}{2}b^4 \ln\frac{\rho}{a} + 1\right] + \left[-\frac{\frac{1}{2}a^6 + b^6}{a^4 + b^4}\rho^2 + \frac{a^4 b^4 (b^2 - \frac{1}{2}a^2)}{a^4 + b^4}\rho + \frac{1}{2}\rho^4\right]\sin 2\theta$$