

**Problem 1.** 解无界弦上的振动问题

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + \cos(\omega t) \cos(x), (-\infty < x < +\infty, t > 0) \\ u|_{t=0} = e^{-2x^2}, (-\infty < x < +\infty) \\ \frac{\partial u}{\partial t}|_{t=0} = \sin(x), (-\infty < x < +\infty) \end{cases}$$

*Solution:*

$$\phi(x) = e^{-2x^2}, \quad \psi = \sin x, \quad f(x, t) = \cos(\omega t) \cos x$$

利用达朗贝尔公式

$$\begin{aligned} u(x, t) &= \frac{1}{2}[\phi(x + at) + \phi(x - at)] + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\xi) d\xi + \frac{1}{2a} \int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi, \tau) d\xi d\tau \\ &= \frac{1}{2}[e^{-2(x+at)^2} + e^{-2(x-at)^2}] + \frac{1}{2a} \int_{x-at}^{x+at} \sin \xi d\xi + \frac{1}{2a} \int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} \cos(\omega\tau) \cos \xi d\xi d\tau \\ &= \frac{1}{2}[e^{-2(x+at)^2} + e^{-2(x-at)^2}] + \frac{1}{2a}[-\cos(x + at) + \cos(x - at)] \\ &\quad + \frac{1}{2a} \int_0^t \cos(\omega\tau) \{\sin[x + a(t - \tau)] - \sin[x - a(t - \tau)]\} d\tau \\ &= \frac{1}{2}[e^{-2(x+at)^2} + e^{-2(x-at)^2}] + \frac{1}{a} \sin x \sin(at) + \frac{1}{a} \int_0^t \cos(\omega\tau) \cos x \sin[a(t - \tau)] d\tau \\ &= \frac{1}{2}[e^{-2(x+at)^2} + e^{-2(x-at)^2}] + \frac{1}{a} \sin x \sin(at) \\ &\quad + \frac{\cos x}{2a} \int_0^t \{\sin[\omega\tau + a(t - \tau)] - \sin[\omega\tau - a(t - \tau)]\} d\tau \\ &= \frac{1}{2}[e^{-2(x+at)^2} + e^{-2(x-at)^2}] + \frac{1}{a} \sin x \sin(at) + \frac{1}{\omega^2 - a^2} \cos x [\cos(\omega t) - \cos(at)] \end{aligned}$$

□

$$2. \begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, & (0 < x < l, t > 0) \\ \end{cases}$$

$$\begin{cases} \frac{\partial u}{\partial x} \Big|_{x=0} = 0, \quad \frac{\partial u}{\partial x} \Big|_{x=l} = 0 \\ u \Big|_{t=0} = e^{-x}, \quad \frac{\partial u}{\partial t} \Big|_{t=0} = 2axe^{-x}. \end{cases}$$

设方程分离变量解:  $u(x,t) = X(x)T(t)$

$$\text{代入范德方程有 } \frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2 T(t)} \triangleq -\lambda$$

$$\Rightarrow \begin{cases} X''(x) + \lambda X(x) = 0 & ① \\ T''(t) + \lambda a^2 T(t) = 0 & ② \end{cases}$$

$$\text{将形式解代入边界条件有 } \begin{cases} X(0)T(t) = 0 \\ X(l)T(t) = 0 \end{cases}$$

$$\Rightarrow X(0) = X(l) = 0$$

或  $T(t) = 0$  (矛盾解!)

$$\text{联立 } ① \text{ 从而有 } \begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = X'(l) = 0 \end{cases}$$

若  $\lambda < 0$ , 则方程通解为  $X(x) = Ae^{kx} + Be^{-kx}$ , 其中  $k = \sqrt{-\lambda}$

$$\text{由边界条件有 } \begin{cases} X(0) = Ak - Bk = 0 \\ X(l) = Ake^{kl} - Bke^{-kl} = 0 \end{cases}$$

$$\Rightarrow \frac{A}{B} = e^{-2kl} = 1 \Rightarrow k = 0 \text{ if } \lambda = 0 \text{ (与 } \lambda < 0 \text{ 矛盾)}$$

若  $\lambda = 0$ , 则方程通解为  $X(x) = Ax + B$

由边界条件有  $A = 0$ , 从而  $X(x) = B$  (不满足初始条件)

$$\text{若 } \lambda > 0, \text{ 则方程通解为 } X(x) = A \cos(\sqrt{\lambda} x) + B \sin(\sqrt{\lambda} x)$$

$$\text{由边界条件有 } \begin{cases} X'(0) = B\sqrt{\lambda} = 0 \\ X'(l) = A\sqrt{\lambda} \sin(\sqrt{\lambda} l) + B\sqrt{\lambda} \cos(\sqrt{\lambda} l) = 0 \end{cases}$$

$$\Rightarrow B = 0 \text{ 且 } \sqrt{\lambda} l = n\pi$$

故特征函数  $X_n = A_n \cos\left(\frac{n\pi}{l} x\right)$  特征值  $\lambda_n = \left(\frac{n\pi}{l}\right)^2$

$$\text{将特征值代入 } ② \text{ 有 } T''(t) + \left(\frac{n\pi}{l}\right)^2 T(t) = 0$$

其通解为  $T_n(t) = C_n \cos\left(\frac{n\pi}{l} t\right) + D_n \sin\left(\frac{n\pi}{l} t\right)$

$$\text{从而 } u_n(x, t) = X_n(x)T_n(t) = \left[C_n \cos\left(\frac{n\pi}{l} t\right) + D_n \sin\left(\frac{n\pi}{l} t\right)\right] \cos\left(\frac{n\pi}{l} x\right)$$

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t)$$

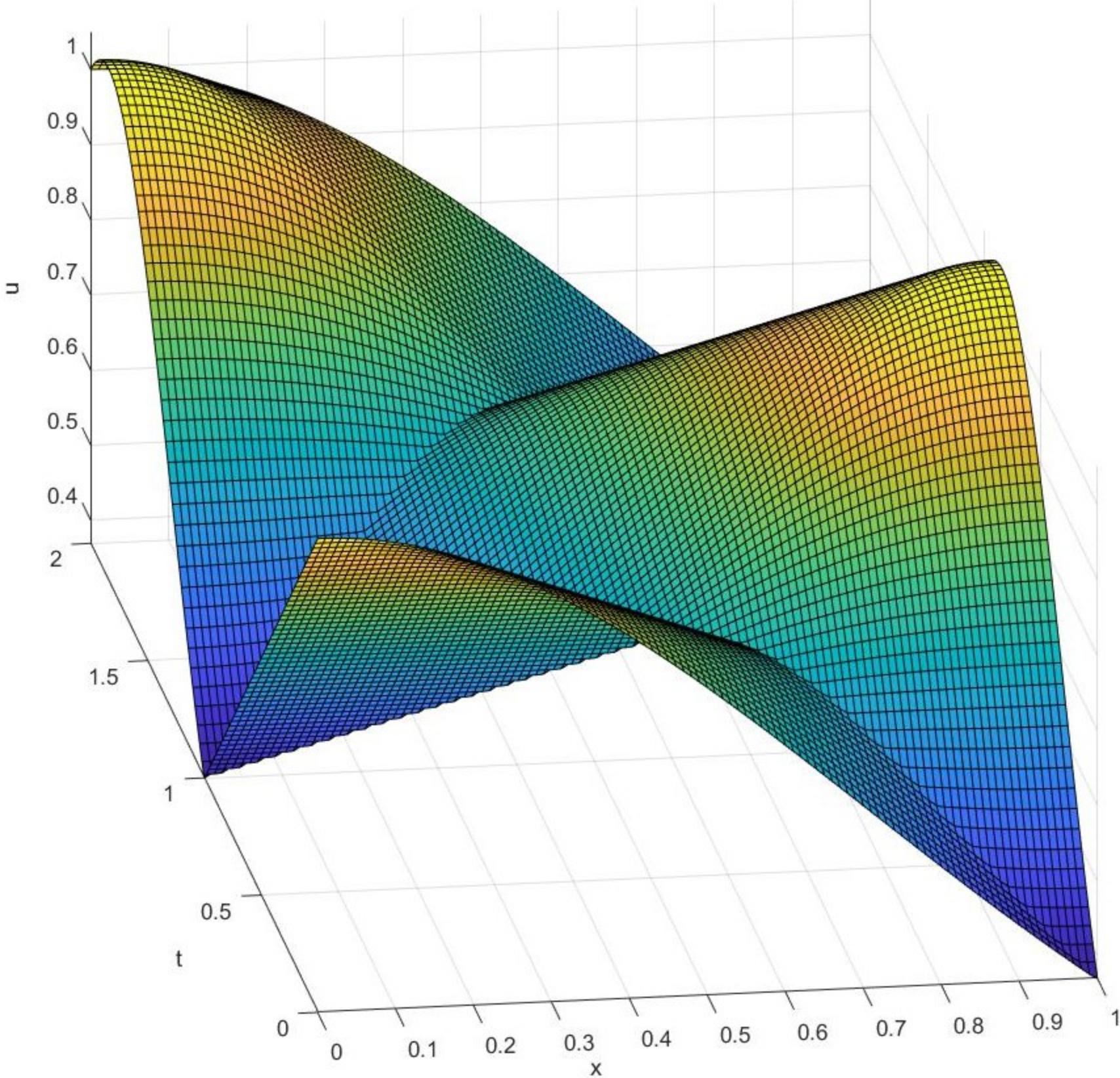
$$\text{代入初始条件得} \begin{cases} u|_{t=0} = \sum_{n=0}^{\infty} C_n \cos\left(\frac{n\pi}{l}x\right) = e^{-x} \\ \left|\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} \frac{n\alpha\pi}{l} D_n \cos\left(\frac{n\pi}{l}x\right) = 2ax e^{-x^2} \end{cases}$$

$$\Rightarrow \begin{cases} C_n = \frac{2}{l} \int_0^l e^{-x^2} \cos\left(\frac{n\pi}{l}x\right) dx, \quad n=0,1,2,\dots \\ D_n = \frac{2}{n\alpha\pi} \int_0^l 2ax e^{-x^2} \cos\left(\frac{n\pi}{l}x\right) dx, \quad n=1,2,\dots \end{cases}$$

练习 b:  $u(x,t) = \frac{C_0}{2} + \sum_{n=1}^{\infty} [C_n \cos\left(\frac{n\alpha\pi}{l}t\right) + D_n \sin\left(\frac{n\alpha\pi}{l}t\right)] \cos\left(\frac{n\pi}{l}x\right)$

其中  $\begin{cases} C_n = \frac{2}{l} \int_0^l e^{-x^2} \cos\left(\frac{n\pi}{l}x\right) dx, \quad n=0,1,2,\dots \\ D_n = \frac{2}{n\alpha\pi} \int_0^l 2ax e^{-x^2} \cos\left(\frac{n\pi}{l}x\right) dx, \quad n=1,2,\dots \end{cases}$

取  $a = 1, l = 1, 0 \leq t \leq 2, n = 100$ , 作图如下页所示



$$3. \int \frac{\partial^2 u}{\partial t^2} = a^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

$$\left\{ \begin{array}{l} u|_{t=0} = \begin{cases} u_0, & x^2 + y^2 + z^2 < R^2 \\ 0, & x^2 + y^2 + z^2 > R^2 \end{cases} \\ \frac{\partial u}{\partial t}|_{t=0} = 0 \end{array} \right.$$

转到球坐标系中得  $\int \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial u}{\partial r}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial u}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2}$

$$\left. \begin{aligned} &= \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2} \end{aligned} \right) \quad \textcircled{1}$$

$$u(r, 0) = \begin{cases} u_0, & r < R \\ 0, & r > R \end{cases} \stackrel{\text{def}}{=} \varphi(r), \quad u(r, 0) = 0$$

由于  $u$  不依赖于  $\theta$  和  $\phi$ , \textcircled{1} 可化为  $\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial u}{\partial r}) = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2}$

作函数代换  $v(r, t) = r u(r, t)$

从而有  $\frac{\partial^2 v}{\partial t^2} = a^2 \frac{\partial^2 v}{\partial r^2}, \quad r, t > 0$

$$\left\{ \begin{array}{l} v(r, 0) = \begin{cases} r u_0, & r < R \\ 0, & r > R \end{cases} \quad v_t(r, 0) = 0 \\ v(0, t) = 0 \end{array} \right.$$

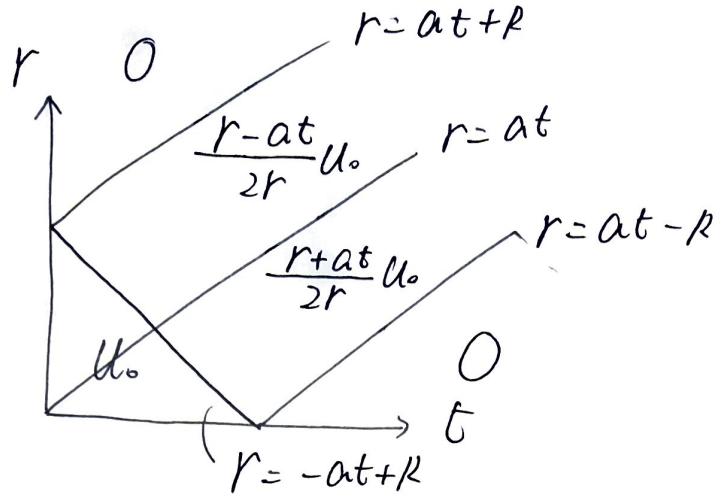
利用达朗贝尔公式

$$\begin{aligned} \text{当 } r - at < 0 \text{ 时} \quad u(r, t) &= \frac{1}{2r} [(r+at)\varphi(r+at) - (at-r)\varphi(at-r)] \\ &= \frac{1}{2r} (r-at)\varphi(at-r) \\ &= \begin{cases} 0, & r+at > R, at-r > R \\ \frac{r+at}{2r} u_0, & r+at > R, at-r < R \\ u_0, & r+at < R, at-r < R \end{cases} \end{aligned}$$

$$\begin{aligned} \text{当 } r - at \geq 0 \text{ 时} \quad u(r, t) &= \frac{1}{2r} [(r+at)\varphi(r+at) + (r-at)\varphi(r-at)] \\ &= \begin{cases} 0, & r+at > R, r-at > R \\ \frac{r+at}{2r} u_0, & r+at > R, r-at < R \\ u_0, & r+at < R, r-at < R \end{cases} \end{aligned}$$

综上:  $u(r, t) = \begin{cases} \frac{u_0}{2r} u_0, & r+at < R \\ \frac{r+at}{2r} u_0, & r+at > R \text{ 且 } 0 < r-at < R \\ \frac{r+at}{2r} u_0, & r+at > R \text{ 且 } -R < r-at < 0 \\ u_0, & r+at < R, r-at < R \end{cases}$

回到直角坐标系  $u(x, y, z, t) = \begin{cases} u_0, & \sqrt{x^2 + y^2 + z^2} + at < R \\ \frac{\sqrt{x^2 + y^2 + z^2} - at}{2\sqrt{x^2 + y^2 + z^2}}, & \sqrt{x^2 + y^2 + z^2} + at > R \text{ 且 } 0 < \sqrt{x^2 + y^2 + z^2} - at < R \\ \frac{\sqrt{x^2 + y^2 + z^2} + at}{2\sqrt{x^2 + y^2 + z^2}}, & \sqrt{x^2 + y^2 + z^2} + at > R \text{ 且 } -R < \sqrt{x^2 + y^2 + z^2} - at < 0 \\ 0, & \sqrt{x^2 + y^2 + z^2} > at + R \text{ 或 } 0 < \sqrt{x^2 + y^2 + z^2} < at - R \end{cases}$



$$4. \begin{cases} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \\ u(0, t) = u(l, t) = 0 \\ u(x, 0) = \frac{6x(l-x)}{l^2} \end{cases}$$

设方程分离变量解  $u(x, t) = X(x)T(t)$

$$\text{代入范定方程有 } \frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2 T(t)} \stackrel{!}{=} -\lambda$$

$$\Rightarrow \begin{cases} X''(x) + \lambda X(x) = 0 & \textcircled{1} \\ T'(t) + \lambda a^2 T(t) = 0 & \textcircled{2} \end{cases}$$

①结合边界条件得到特征函数  $X_n(x) = B_n \sin(\frac{n\pi}{l}x)$   
和本征值  $\lambda_n = (\frac{n\pi}{l})^2$

$$\text{将本征值代入②有 } T'(t) + (\frac{n\pi}{l})^2 T(t) = 0$$

$$\text{通解为 } T_n(t) = C_n e^{-\frac{(n\pi/l)^2 t}{2}}$$

$$\text{从而 } u_n(x, t) = X_n(x)T_n(t) = C_n e^{-\frac{(n\pi/l)^2 t}{2}} \sin(\frac{n\pi}{l}x)$$

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t)$$

$$\text{代入初值条件有 } u(x, 0) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi}{l}x) = \frac{6x(l-x)}{l^2}$$

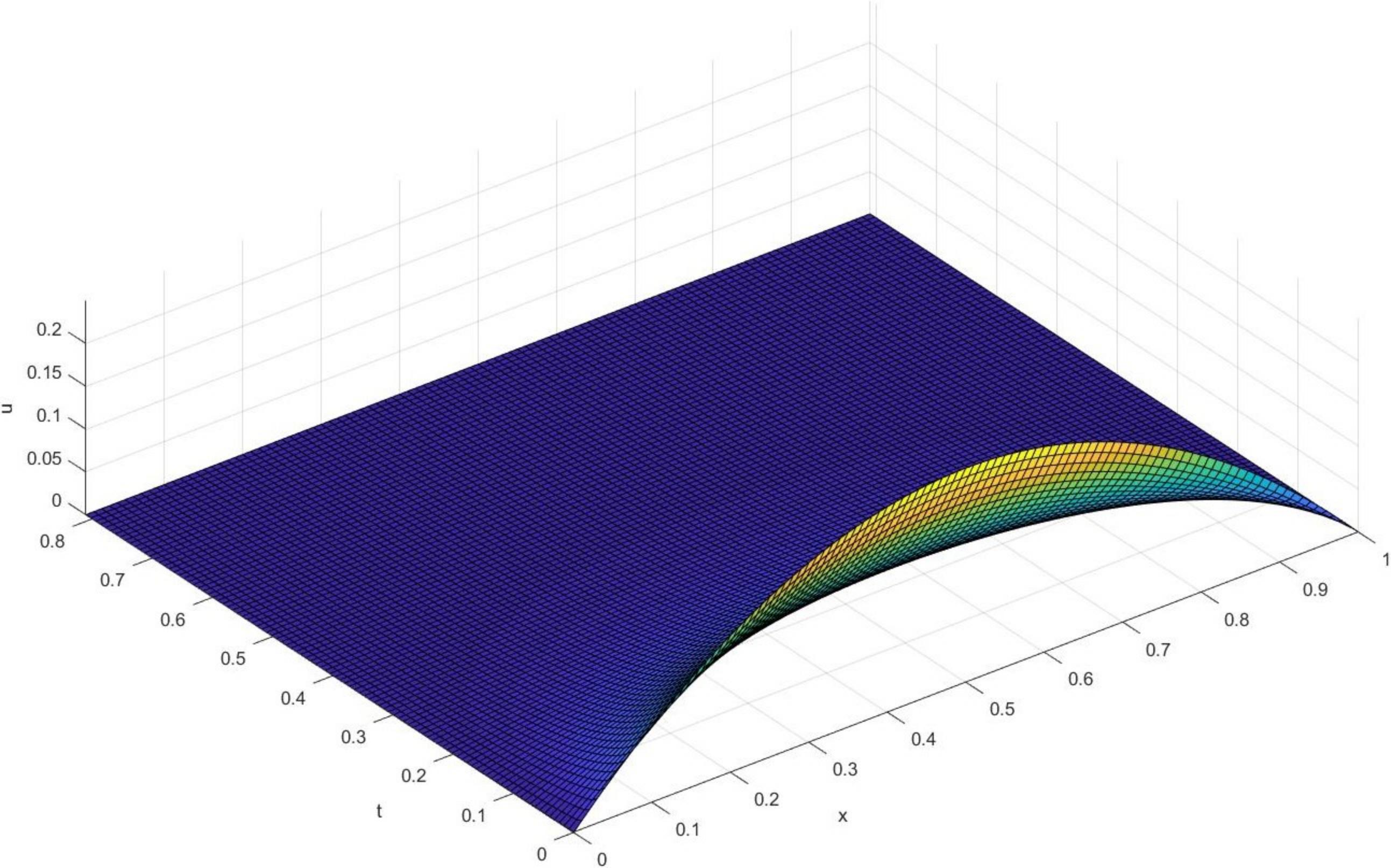
$$\Rightarrow C_n = \frac{2}{l} \int_0^l \frac{6x(l-x)}{l^2} \sin(\frac{n\pi}{l}x) dx$$

$$= \frac{4b}{n^3 \pi^3} [1 + (-1)^{n+1}]$$

$$\text{综上: } u(x, t) = \sum_{n=1}^{\infty} C_n e^{-\frac{(n\pi/l)^2 t}{2}} \sin(\frac{n\pi}{l}x)$$

$$\text{其中 } C_n = \frac{4b}{n^3 \pi^3} [1 + (-1)^{n+1}].$$

取  $l = 1$ ,  $b = 1$ ,  $a = 1$ ,  $n = 100$ ,  $0 \leq t \leq 8/\pi^2$ , 作图如下页所示



$$5. (1) a_1(x) \frac{\partial^2 u}{\partial x^2} + b_1(y) \frac{\partial^2 u}{\partial y^2} + a_2(x) \frac{\partial u}{\partial x} + b_2(y) \frac{\partial u}{\partial y} = 0$$

$$\text{设 } u(x, y) = X(x) Y(y)$$

$$\text{代入原方程得 } a_1(x) X''(x) Y(y) + b_1(y) X(x) Y''(y) \\ + a_2(x) X'(x) Y(y) + b_2(y) X(x) Y'(y) = 0$$

两边同除  $X(x) Y(y)$

$$\frac{a_1(x) X''(x)}{X(x)} + \frac{b_1(y) Y''(y)}{Y(y)} \\ + \frac{a_2(x) X'(x)}{X(x)} + \frac{b_2(y) Y'(y)}{Y(y)} = 0$$

$$\Rightarrow \frac{a_1(x) X''(x) + a_2(x) X'(x)}{X(x)} \\ = - \frac{b_1(y) Y''(y) + b_2(y) Y'(y)}{Y(y)} \triangleq \lambda$$

$$\Rightarrow \begin{cases} a_1(x) X''(x) + a_2(x) X'(x) - \lambda X(x) = 0 \\ b_1(y) Y''(y) + b_2(y) Y'(y) + \lambda Y(y) = 0 \end{cases}$$

$$(2) \frac{1}{P} \frac{\partial}{\partial P} (P \frac{\partial u}{\partial P}) + \frac{1}{P^2} \frac{\partial^2 u}{\partial \varphi^2} = 0$$

$$\text{设 } u(P, \varphi) = P(\varphi) \Phi(\varphi)$$

$$\text{代入原方程得 } \frac{1}{P} \Phi(\varphi) \frac{\partial}{\partial P} (P \frac{\partial P(\varphi)}{\partial P}) + \frac{1}{P^2} P(\varphi) \frac{\partial^2 \Phi(\varphi)}{\partial \varphi^2} = 0$$

$$\text{两边同乘 } \frac{P^2}{\Phi(\varphi) P(\varphi)}$$

$$\frac{P}{P(\varphi)} \frac{d}{dP} (P \frac{dP(\varphi)}{dP}) + \frac{1}{\Phi(\varphi)} \frac{d^2 \Phi(\varphi)}{d\varphi^2} = 0$$

$$\Rightarrow \frac{P}{P(\varphi)} \frac{d}{dP} (P \frac{dP(\varphi)}{dP}) = - \frac{1}{\Phi(\varphi)} \frac{d^2 \Phi(\varphi)}{d\varphi^2} \triangleq \lambda$$

$$\Rightarrow \begin{cases} P \frac{d}{dP} (P \frac{dP(\varphi)}{dP}) - \lambda P(\varphi) = 0 \\ \frac{d^2 \Phi(\varphi)}{d\varphi^2} + \lambda \Phi(\varphi) = 0 \end{cases}$$

$$(3) \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial u}{\partial r}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial u}{\partial \theta}) = 0$$

設  $u(r, \theta) = R(r)\Theta(\theta)$

代入原方程中得.  $\frac{1}{r^2} \Theta(\theta) \frac{d}{dr} (r^2 \frac{dR(r)}{dr}) + \frac{1}{r^2 \sin \theta} R(r) \frac{d}{d\theta} (\sin \theta \frac{d\Theta(\theta)}{d\theta}) = 0$   
 两边同乘  $\frac{1}{R(r)} \frac{d}{dr} (r^2 \frac{dR(r)}{dr}) + \frac{1}{\sin \theta} \frac{1}{\Theta(\theta)} \frac{d}{d\theta} (\sin \theta \frac{d\Theta(\theta)}{d\theta}) = 0$

$$\Rightarrow \frac{1}{R(r)} \frac{d}{dr} (r^2 \frac{dR(r)}{dr}) = - \frac{1}{\sin \theta} \frac{1}{\Theta(\theta)} \frac{d}{d\theta} (\sin \theta \frac{d\Theta(\theta)}{d\theta}) \triangleq \lambda$$

$$\Rightarrow \begin{cases} \frac{d}{dr} (r^2 \frac{dR(r)}{dr}) - \lambda R(r) = 0 \\ \frac{1}{\sin \theta} \frac{d}{d\theta} (\sin \theta \frac{d\Theta(\theta)}{d\theta}) + \lambda \Theta(\theta) = 0 \end{cases}$$

$$6. \begin{cases} X'' + \lambda X = 0 \\ \alpha_1 X(0) + \beta_1 X'(0) = 0 \\ \alpha_2 X(l) + \beta_2 X'(l) = 0 \end{cases}$$

通解为  $X = A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x$

代入边界条件  $\begin{cases} \alpha_1 X(0) + \beta_1 X'(0) = \alpha_1 A + \beta_1 B \sqrt{\lambda} = 0 \\ \alpha_2 X(l) + \beta_2 X'(l) = \alpha_2 (A \cos \sqrt{\lambda} l + B \sin \sqrt{\lambda} l) + \beta_2 \sqrt{\lambda} (-A \sin \sqrt{\lambda} l + B \cos \sqrt{\lambda} l) = 0 \end{cases}$

$$\Rightarrow \text{特征函数 } X_n(x) = C_n \left[ \sin(\sqrt{\lambda_n} x) - \frac{\beta_1}{\alpha_1} \sqrt{\lambda_n} \cos(\sqrt{\lambda_n} x) \right].$$

特征值  $\lambda_n$  是  $(\alpha_1 \alpha_2 + \beta_1 \beta_2) \tan \sqrt{\lambda} l + (\alpha_1 \beta_2 - \alpha_2 \beta_1) \sqrt{\lambda} = 0$  的解

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证明已知：

$$\begin{aligned} & \int_0^l \left[ \sin(\sqrt{\lambda_n} x) - \frac{\beta_1}{\alpha_1} \sqrt{\lambda_n} \cos(\sqrt{\lambda_n} x) \right] \left[ \sin(\sqrt{\lambda_m} x) - \frac{\beta_1}{\alpha_1} \sqrt{\lambda_m} \cos(\sqrt{\lambda_m} x) \right] \\ &= \int_0^l \sin(\sqrt{\lambda_n} x) \sin(\sqrt{\lambda_m} x) + \frac{\beta_1^2}{\alpha_1^2} \sqrt{\lambda_m} \sqrt{\lambda_n} \cos(\sqrt{\lambda_n} x) \cos(\sqrt{\lambda_m} x) \\ & \quad - \frac{\beta_1}{\alpha_1} \sqrt{\lambda_n} \cos(\sqrt{\lambda_n} x) \sin(\sqrt{\lambda_m} x) - \frac{\beta_1}{\alpha_1} \sqrt{\lambda_m} \sin(\sqrt{\lambda_n} x) \cos(\sqrt{\lambda_m} x) dx \\ &= \int_0^l \left[ \frac{1}{2} [\cos(\sqrt{\lambda_n} - \sqrt{\lambda_m})x - \cos(\sqrt{\lambda_n} + \sqrt{\lambda_m})x] \right. \\ & \quad \left. + \frac{1}{2} \frac{\beta_1^2}{\alpha_1^2} \sqrt{\lambda_n} \sqrt{\lambda_m} [\cos(\sqrt{\lambda_n} - \sqrt{\lambda_m})x + \cos(\sqrt{\lambda_n} + \sqrt{\lambda_m})x] \right. \\ & \quad \left. - \frac{1}{2} \frac{\beta_1}{\alpha_1} \sqrt{\lambda_n} [\sin(\sqrt{\lambda_n} + \sqrt{\lambda_m})x - \sin(\sqrt{\lambda_n} - \sqrt{\lambda_m})x] \right. \\ & \quad \left. - \frac{1}{2} \frac{\beta_1}{\alpha_1} \sqrt{\lambda_m} [\sin(\sqrt{\lambda_n} + \sqrt{\lambda_m})x + \sin(\sqrt{\lambda_n} - \sqrt{\lambda_m})x] \right] \\ &= \frac{1}{2} \left\{ \frac{1 + \frac{\beta_1^2}{\alpha_1^2} \sqrt{\lambda_n} \sqrt{\lambda_m}}{\sqrt{\lambda_n} - \sqrt{\lambda_m}} \sin(\sqrt{\lambda_n} - \sqrt{\lambda_m})l \right. \\ & \quad \left. - \frac{1 + \frac{\beta_1^2}{\alpha_1^2} \sqrt{\lambda_n} \sqrt{\lambda_m}}{\sqrt{\lambda_n} + \sqrt{\lambda_m}} \sin(\sqrt{\lambda_n} + \sqrt{\lambda_m})l \right. \\ & \quad \left. + \frac{\beta_1}{\alpha_1} [\cos(\sqrt{\lambda_n} + \sqrt{\lambda_m})l - 1] \right. \\ & \quad \left. + \frac{\beta_1}{\alpha_1} [\cos(\sqrt{\lambda_n} - \sqrt{\lambda_m})l - 1] \right\}. \end{aligned}$$

$$= \frac{1}{2} \left\{ \frac{1 + \frac{\beta_1^2}{\alpha_1^2} \sqrt{\lambda_n} \sqrt{\lambda_m}}{\sqrt{\lambda_n} - \sqrt{\lambda_m}} [\sin \sqrt{\lambda_n} l \cos \sqrt{\lambda_m} l - \cos \sqrt{\lambda_n} l \sin \sqrt{\lambda_m} l] \right. \\ - \frac{1 + \frac{\beta_1^2}{\alpha_1^2} \sqrt{\lambda_n} \sqrt{\lambda_m}}{\sqrt{\lambda_n} + \sqrt{\lambda_m}} [\sin \sqrt{\lambda_n} l \cos \sqrt{\lambda_m} l + \cos \sqrt{\lambda_n} l \sin \sqrt{\lambda_m} l] \\ \left. + \frac{2\beta_1}{\alpha_1} \cos \sqrt{\lambda_n} l \cos \sqrt{\lambda_m} l - \frac{2\beta_1}{\alpha_1} \right\}.$$

$$= \frac{1}{2} \left\{ \frac{1 + \frac{\beta_1^2}{\alpha_1^2} \sqrt{\lambda_n} \sqrt{\lambda_m}}{\sqrt{\lambda_n} - \sqrt{\lambda_m}} (\tan \sqrt{\lambda_n} l - \tan \sqrt{\lambda_m} l) \cos \sqrt{\lambda_n} l \cos \sqrt{\lambda_m} l \right. \\ - \frac{1 + \frac{\beta_1^2}{\alpha_1^2} \sqrt{\lambda_n} \sqrt{\lambda_m}}{\sqrt{\lambda_n} + \sqrt{\lambda_m}} (\tan \sqrt{\lambda_n} l + \tan \sqrt{\lambda_m} l) \cos \sqrt{\lambda_n} l \cos \sqrt{\lambda_m} l \\ \left. + \frac{2\beta_1}{\alpha_1} \cos \sqrt{\lambda_n} l \cos \sqrt{\lambda_m} l - \frac{2\beta_1}{\alpha_1} \right\}$$

$$(4) \lambda \tan \sqrt{\lambda_i} l = \frac{(\alpha_1 \beta_2 - \alpha_2 \beta_1) \sqrt{\lambda}}{\alpha_1 \alpha_2 + \beta_1 \beta_2 \lambda} \quad (\text{A})$$

$$= \sum \left\{ \frac{1 + \frac{\beta_1^2}{\alpha_1^2} \sqrt{\lambda_n} \sqrt{\lambda_m}}{\sqrt{\lambda_n} - \sqrt{\lambda_m}} \left[ \frac{-(\alpha_1 \beta_2 - \alpha_2 \beta_1) \sqrt{\lambda_n}}{\alpha_1 \alpha_2 + \beta_1 \beta_2 \lambda_n} + \frac{(\alpha_1 \beta_2 - \alpha_2 \beta_1) \sqrt{\lambda_m}}{\alpha_1 \alpha_2 + \beta_1 \beta_2 \lambda_m} \right] \right. \\ - \frac{1 + \frac{\beta_1^2}{\alpha_1^2} \sqrt{\lambda_n} \sqrt{\lambda_m}}{\sqrt{\lambda_m} + \sqrt{\lambda_m}} \left[ \frac{-(\alpha_1 \beta_2 - \alpha_2 \beta_1) \sqrt{\lambda_n}}{\alpha_1 \alpha_2 + \beta_1 \beta_2 \lambda_n} + \frac{-(\alpha_1 \beta_2 - \alpha_2 \beta_1) \sqrt{\lambda_m}}{\alpha_1 \alpha_2 + \beta_1 \beta_2 \lambda_m} \right] \\ \left. + \frac{2\beta_1}{\alpha_1} \cos \sqrt{\lambda_n} l \cos \sqrt{\lambda_m} l - \frac{2\beta_1}{\alpha_1} \right\}.$$

(通分)

由此得证本征函数正交性.

本节一练习:

$$\int_0^l X_n(x) dx = C_n^2 \int_0^l \sin^2(\sqrt{\lambda_n} x - \delta_n) dx \quad (\text{其中 } \tan \delta_n = \frac{\beta_1}{\alpha_1} \sqrt{\lambda_n})$$

$$= C_n^2 \int_0^l \frac{1 - \cos(2\sqrt{\lambda_n} x - 2\delta_n)}{2} dx$$

$$= C_n^2 \left\{ \frac{1}{2} - \frac{1}{4\sqrt{\lambda_n}} [\sin 2(\sqrt{\lambda_n} l - \delta_n) + \sin 2\delta_n] \right\}$$

$$\therefore \tan \sqrt{\lambda} l = -\sqrt{\lambda_n} \frac{\alpha_1 \beta_2 - \alpha_2 \beta_1}{\alpha_1 \alpha_2 + \beta_1 \beta_2 \lambda}$$

$$\sin 2\sqrt{\lambda_n}l = \frac{2\tan \sqrt{\lambda_n}l}{1+\tan^2 \sqrt{\lambda_n}l} = -2\sqrt{\lambda_n} \frac{\alpha_1^2 \alpha_2 \beta_2 - \alpha_1 \alpha_2 \beta_1 + \alpha_1 \beta_1 \beta_2^2 \lambda_n - \alpha_2 \beta_1^2 \beta_2}{(\alpha_1^2 + \beta_1^2 \lambda_n)(\alpha_2^2 + \beta_2^2 \lambda_n)}$$

$$\cos 2\sqrt{\lambda_n}l = \frac{1-\tan^2 \sqrt{\lambda_n}l}{1+\tan^2 \sqrt{\lambda_n}l} = \frac{\alpha_1^2 \alpha_2^2 - \alpha_1^2 \beta_2^2 \lambda_n - \alpha_2^2 \beta_1^2 \lambda_n + 4\alpha_1 \alpha_2 \beta_1 \beta_2 \lambda_n + \beta_1^2 \beta_2^2 \lambda_n}{(\alpha_1^2 + \beta_1^2 \lambda_n)(\alpha_2^2 + \beta_2^2 \lambda_n)}$$

$$\sin 2\delta_n = \frac{2\tan \delta_n}{1+\tan^2 \delta_n} = 2\sqrt{\lambda_n} \frac{\alpha_1 \beta_1}{\alpha_1^2 + \beta_1^2 \lambda_n} \quad \cos 2\delta_n = \frac{1-\tan^2 \delta_n}{1+\tan^2 \delta_n} = \frac{\alpha_1^2 - \beta_1^2 \lambda_n}{\alpha_1^2 + \beta_1^2 \lambda_n}$$

$$\sin 2(\sqrt{\lambda_n}l - \delta_n) = \sin 2\sqrt{\lambda_n}l \cos 2\delta_n - \cos 2\sqrt{\lambda_n}l \sin 2\delta_n = 2\sqrt{\lambda_n} \frac{\alpha_2 \beta_2}{\alpha_2^2 + \beta_2^2 \lambda_n}$$

$$\therefore \int_0^L X_n(x) dx = C_n^2 \left[ \frac{l}{2} + \frac{1}{2} \left( \frac{\alpha_2 \beta_2}{\alpha_2^2 + \beta_2^2 \lambda_n} - \frac{\alpha_1 \beta_1}{\alpha_1^2 + \beta_1^2 \lambda_n} \right) \right].$$

$$\Rightarrow C_n = \sqrt{\frac{1}{\frac{l}{2} + \frac{1}{2} \left( \frac{\alpha_2 \beta_2}{\alpha_2^2 + \beta_2^2 \lambda_n} - \frac{\alpha_1 \beta_1}{\alpha_1^2 + \beta_1^2 \lambda_n} \right)}}$$

$$1. \begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6(x+y)^2, & 0 < x^2 + y^2 < b^2 < \infty \\ u|_{x^2 + y^2 = a^2} = 1, \quad \frac{\partial u}{\partial n}|_{x^2 + y^2 = b^2} = 0 \end{cases}$$

转换到极坐标系中  $\begin{cases} \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \frac{\partial u}{\partial \rho}) + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \theta^2} = 6\rho^2(1 + \sin 2\theta) \\ u|_{\rho=a} = 1 \quad ②, \quad \frac{\partial u}{\partial \rho}|_{\rho=b} = 0 \quad ③ \end{cases}$

一般解可表示为  $u(\rho, \theta) = \sum_{n=0}^{\infty} [A_n(\rho) \cos n\theta + B_n(\rho) \sin n\theta]$ .

代入①中有  $\sum_{n=0}^{\infty} \left\{ \left[ A_n''(\rho) + \frac{A_n'(\rho)}{\rho} - \frac{n^2}{\rho^2} A_n(\rho) \right] \cos n\theta + \left[ B_n''(\rho) + \frac{B_n'(\rho)}{\rho} - \frac{n^2}{\rho^2} B_n(\rho) \right] \sin n\theta \right\} = 6\rho^2(1 + \sin 2\theta)$

$$\Rightarrow \begin{cases} A_0''(\rho) + \frac{A_0'(\rho)}{\rho} = 6\rho^2 \quad ④ \\ A_n''(\rho) + \frac{A_n'(\rho)}{\rho} - \frac{n^2}{\rho^2} A_n(\rho) = 0 \Rightarrow \rho^2 A_n''(\rho) + \rho A_n'(\rho) - n^2 A_n(\rho) = 0, \quad n=1, 2, 3, \dots \\ B_2''(\rho) + \frac{B_2'(\rho)}{\rho} - \frac{4}{\rho^2} B_2(\rho) = 6\rho^2 \Rightarrow \rho^2 B_2''(\rho) + \rho B_2'(\rho) - 4B_2(\rho) = 6\rho^4 \quad ⑤ \\ B_n''(\rho) + \frac{B_n'(\rho)}{\rho} - \frac{n^2}{\rho^2} B_n(\rho) = 0 \Rightarrow \rho^2 B_n''(\rho) + \rho B_n'(\rho) - n^2 B_n(\rho) = 0, \quad n=1, 2, 3, \dots \end{cases}$$

由④, 设  $A_n(\rho) = C_n \rho^n + d_n \rho^{-n}, \quad n=1, 2, 3, \dots$

由⑤, 设  $B_n(\rho) = C'_n \rho^n + d'_n \rho^{-n}, \quad n \neq 2$

由②③有  $\begin{cases} A_n(a) = 0 \quad A_n(b) = 0, \quad n=1, 2, 3 \\ B_n(a) = 0 \quad B_n(b) = 0, \quad n \neq 2 \end{cases}$

$$\Rightarrow \begin{cases} C_n = d_n = 0 \quad (n=1, 2, 3, \dots) \\ C'_n = d'_n = 0 \quad (n \neq 2). \end{cases}$$

$$\text{由④, 得 } A_0'(\rho) = e^{-\int \frac{1}{\rho} d\rho} \left[ \int 6\rho^2 e^{\int \frac{1}{\rho} d\rho} d\rho + C_1 \right] = \frac{3}{2} \rho^3 + \frac{C_0}{\rho}.$$

$$A_0(\rho) = \frac{3}{8} \rho^4 + C_0 \ln \rho + d_0$$

$$\text{由②③得 } \begin{cases} A_0(a) = 1 \\ A_0'(b) = 0 \end{cases} \Rightarrow \begin{cases} C_0 = -\frac{3}{2} b^4 \\ d_0 = -\frac{3}{8} a^4 + \frac{3}{2} b^4 / \ln a + 1 \end{cases}$$

$$\therefore A_0(\rho) = \frac{3}{8} \rho^4 - \frac{3}{2} b^4 / \ln \rho - \frac{3}{8} a^4 + \frac{3}{2} b^4 / \ln a + 1$$

$$\text{由⑤得一特解 } B_2(\rho) = \frac{1}{2} \rho^4$$

$$\text{故其通解为 } B_2(\rho) = C'_2 \rho^2 + d'_2 \rho^2 + \frac{1}{2} \rho^4$$

$$\text{由} \textcircled{2} \text{ 得 } \begin{cases} B_2(a) \sin(2\theta) = (c_2 a^2 + d_2 a^{-2} + \frac{1}{2} a^4) \sin(\theta) = 0 \\ B_2'(a) \sin(2\theta) = (2c_2 b - 2d_2 b^{-3} + 2b^3) \sin(2\theta) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} c_2' = -\frac{\frac{1}{2}a^4 + b^6}{a^4 + b^4} \\ d_2' = \frac{a^4 b^4 (b^2 - \frac{1}{2}a^4)}{a^4 + b^4} \end{cases}$$

从而  $B_2(p) = -\frac{\frac{1}{2}a^4 + b^6}{a^4 + b^4} p^2 + \frac{a^4 b^4 (b^2 - \frac{1}{2}a^4)}{a^4 + b^4} p^{-2} + \frac{1}{2} p^4$

续 b:  $u(p, \theta) = \left[ \frac{3}{8}(p^4 - a^4) - \frac{3}{2}b^4 \ln \frac{p}{a} + 1 \right] + \left[ -\frac{\frac{1}{2}a^4 + b^6}{a^4 + b^4} p^2 + \frac{a^4 b^4 (b^2 - \frac{1}{2}a^4)}{a^4 + b^4} p^{-2} + \frac{1}{2} p^4 \right] \sin 2\theta$