偏微分方程的定解问题

常微分方程 含一元未知函数$u=u(x)$, x及其若干阶导数$F(x,u,\frac{du}{dx},\dots,\frac{d^nu}{dx^n})=0$

偏微分方程 含多元未知函数$u(x_1,\dots,x_n)$及其若干阶偏导$F(x,u,\frac{\partial u}{\partial x_1},\dots,\frac{\partial^m u}{\partial x_1^{m_1}\dots\partial x_n^{m_n}})=0$，其中$m=\sum_{i=1}^n m_n$–阶数

线性偏微分方程与未知函数有关的部分仅是u及其偏导数的线性组合， 否则(如含有$u_x^2,\cos u$)为非线性偏微分方程，仅含$\boldsymbol{x}$的项(如$\cos x$)不是非线性项，非线性项或造成混沌

齐次方程 仅含对u的各种运算 **非齐次方程** 含对u运算之外的驱动项$f(x,t)$或非零自由项

波动方程(如张紧弦的机械波) 对一小段$[a,b]$，垂直方向牛二$\rho\Delta s\frac{\partial^2u(x,t)}{\partial t^2}=f_0\Delta s+T_a\sin\alpha_a-T_b\sin\alpha_b$

水平方向牛二$-T_a\cos\alpha_a+T_b\cos\alpha_b=0$ 其中$\sin\alpha=\frac{\partial u}{\partial x},\Delta s=b-a,T_a=T_b=T_0$

$$\Longrightarrow \frac{\partial^2u}{\partial t^2}=a^2\frac{\partial^2u}{\partial x^2}+f(x,t)$$

其中$a^2=\frac{T_0}{\rho}$，$f(x,t)=\frac{f_0(x,t)}{\rho}$，$u(x,t)$–时刻$t$弦上$x$处振幅，$T_0$–张力，$\rho$–线密度，$f_0$–外力线密度

$f=0$–无外力自由振动 $f=const$–常外力强迫振动 $f=-2b\frac{\partial u}{\partial t}, 2b=\frac{k}{\rho}$，$k$–阻力密度与速度之比–阻尼振动

$$\frac{\partial^2u}{\partial t^2}=\begin{cases}a^2(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2})+f(x,y,t) & \text{二维(如膜的振动)}\\ a^2\nabla^2u+f(x,y,z,t) & \text{三维(如空气中的声波)}\end{cases}$$

热传导方程描述非平衡态及其转换的弛豫，如三种输运过程：速度梯度→粘滞现象(动量传递)，温度梯度→热传导现象(能量传递)，密度梯度→扩散现象(粒子浓度传递)，微观机制：空间中分子能量、动量、浓度交换，宏观机制：傅里叶定律–场中任一点沿任一方向热流密度与该方向上温度梯度成正比，菲克定律$\boldsymbol{J}(\boldsymbol{r},t)=-\kappa\nabla u(\boldsymbol{r},t)$

(一维如均匀细杆热传导)对一小段$[x,x+\Delta x]$，$-kS\Delta t(\frac{\partial u}{\partial x}|_x-\frac{\partial u}{\partial x}|_{x+\Delta x})=c\rho S\Delta x\Delta u$

$\Longrightarrow \frac{\partial u}{\partial t}=a^2\frac{\partial^2u}{\partial x^2}$ 其中S–截面积，c–比热，ρ–体密度，$a^2=\frac{k}{c\rho}$

$$\frac{\partial u}{\partial t}=\begin{cases}a^2(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2})+f(x,y,t) & \text{二维}\\ a^2\nabla^2u+f(x,y,z,t) & \text{三维}\end{cases}\quad \text{其中}f\text{描述物体内热源}$$

稳态下$\frac{\partial u}{\partial t}=0=\frac{\partial^2u}{\partial x^2}\Longrightarrow u=Ax+B$

一般方法：对体积元，热量(t_2)–热量$(t_2)=(t_1\sim t_2)$经边界流入热量$+(t_1\sim t_2)$内部生成热量

$\int_V cu(x,y,z,t_2)\rho dV-\int_V cu(*,t_1)\rho dV=\int_{t_1}^{t_2}[\int_{\partial V}k\nabla u\cdot d\boldsymbol{S}]dt+\int_{t_1}^{t_2}[\int_V f_0(*,t)\rho dV]dt$

$\Longrightarrow\int_{t_1}^{t_2}[\int_V\frac{\partial c\rho u}{\partial t}dV]dt=\int_{t_1}^{t_2}[\int_V\nabla\cdot(k\nabla u)+f_0\rho dV]dt\Longrightarrow\frac{\partial[c\rho u]}{\partial t}=\nabla\cdot(k\nabla u)+f_0\rho$

位势方程当波动停止/温度平衡$-a^2\nabla^2u=f\begin{cases}\neq 0,\text{泊松方程}\\ =0,\text{拉普拉斯}\end{cases}$ 如静电场$\nabla^2\phi=\begin{cases}-\frac{\rho}{\epsilon_0} & \text{有源}\\ 0 & \text{无源}\end{cases}$

二阶偏微分方程标准形式变换$A(x,y)\frac{\partial^2u}{\partial x^2}+2B\frac{\partial^2u}{\partial x\partial y}+C\frac{\partial^2u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$

为使$a,c=0$，ξ和η需满足$A(\frac{\partial W}{\partial x})^2+2B\frac{\partial W}{\partial x}\frac{\partial W}{\partial y}+C(\frac{\partial W}{\partial y})^2=0$令$\frac{dy}{dx}=-\frac{\partial W}{\partial x}/\frac{\partial W}{\partial y}$得

特征方程$A(\frac{dy}{dx})^2-2B\frac{dy}{dx}+C=0$ 若$\Delta=B^2-AC>0$，解得特征线$y=y_1(x)+\gamma_1$或$y=y_2(x)+\gamma_2$

(若A不恒为0)作变换$\xi(x,y)=y-y_1(x),\eta(x,y)=y-y_2(x)$

(原方程化为$a\frac{\partial^2u}{\partial x^2}+2b\frac{\partial^2u}{\partial x\partial y}+c\frac{\partial^2u}{\partial y^2}+d\frac{\partial u}{\partial x}+e\frac{\partial u}{\partial y}+fu=g$

$$\text{其中}\begin{cases}a=A(\frac{\partial\xi}{\partial x})^2+2B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}+C(\frac{\partial\xi}{\partial y})^2\\ b=A\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+B(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x})+C\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\\ c=A(\frac{\partial\eta}{\partial x})^2+2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+C(\frac{\partial\eta}{\partial y})^2\\ d=A\frac{\partial^2\xi}{\partial x^2}+2B\frac{\partial^2\xi}{\partial x\partial y}+C\frac{\partial^2\xi}{\partial y^2}+D\frac{\partial\xi}{\partial x}+E\frac{\partial\xi}{\partial y}\\ e=A\frac{\partial^2\eta}{\partial x^2}+2B\frac{\partial^2\eta}{\partial x\partial y}+C\frac{\partial^2\eta}{\partial y^2}+D\frac{\partial\eta}{\partial x}+E\frac{\partial\eta}{\partial y}\\ f=F\qquad g=G\end{cases}$$
)

得双曲形方程$\frac{\partial^2u}{\partial\xi\partial\eta}=-\frac{1}{2b}(d\frac{\partial u}{\partial\xi}+e\frac{\partial u}{\partial\eta}+fu-g)$

若$\Delta=0$，解得特征线$\sqrt{A}\frac{\partial\xi}{\partial x}+\sqrt{C}\frac{\partial\xi}{\partial y}=0$，取$\eta$使$\frac{\partial(\xi,\eta)}{\partial(x,y)}\neq 0$

得抛物型方程$\sqrt{A}\frac{\partial\xi}{\partial x}+\sqrt{C}\frac{\partial\xi}{\partial y}=0$

若$\Delta<0$，解得特征线$y(x,y)=y_1(x,y)+iy_2(x,y)$，作变换$\xi=y_1,\eta=y_2$

得椭圆形方程$\frac{\partial^2u}{\partial\xi^2}+\frac{\partial^2u}{\partial\eta^2}=-\frac{1}{a}(d\frac{\partial u}{\partial\xi}+e\frac{\partial u}{\partial\eta}+fu-g)$

双曲型–一维弦振动方程 抛物型–一维热传导方程 椭圆型–二维拉普拉斯方程

叠加原理 对齐次线性微分方程，若u_1和u_2均为其解，则线性组合$u=au_1+bu_2$亦为其解

对非齐次线性微分方程，若u_1为其解，u_2为相应齐次微分方程解，则叠加$u=u_1+au_2$亦为其解

对自变量平面R上任一点P 为确定P点的值必须给定**依赖区**中点的值 影响区中点的值随着P点的值的变化而变化

对**双曲型方程**，影响区–P点时间后两条特征线之间的区域，依赖区–P点时间前的两特征线之间的区域

对**抛物型方程**，过P点仅一条特征线，依赖区与影响区以特征线为界

对**椭圆型方程**，特征线是虚的，与特征线线相关的数值方法不适用，信息不受影响区和依赖区的限制，可以沿各个方向传播

初始值+边界条件=定解条件 定解条件+偏微分方程(泛定方程)=定解问题

初值(Cauchy)问题 定解条件=初始条件 **边值问题** 定解条件=边界条件 **混合问题** 定解条件=初始+边界条件

第一类(Dirichlet)边界条件 已知边界位移(温度/电势)$u|_S=f_1$

第二类(Neumann)边界条件 已知边界速度(热流/电场)$\frac{\partial u}{\partial n}|_S=f_2$

第三类边界条件 已知边界弹性受力(热交换)$(u+h\frac{\partial u}{\partial n})|_S=f_3$

Chap6分离变量法 Chap7分离变量法的应用 Chap8本征函数法

分离变量法适用条件(必要而不充分)泛定方程线性齐次，且边界条件齐次

步骤 1. 对泛定方程$Lu(x,t)=0$写出形式解$u(x,t)=X(x)T(t)$ 2. 分离变量得空间函数本征值问题$\begin{cases}F_xX_n(x)=\lambda_nX_n(x)\\ \text{边界条件}\end{cases}\Longrightarrow\begin{cases}\text{本征值}\lambda_n\\ \text{本征函数}X_n(x)\end{cases}$ 3. 代入时间方程解出$T(t)$得本征解$u_n(x,t)=X_n(x)T_n(t)$ 4. 利用叠加原理得一般解$u(x,t)=\sum_n u_n(x,t)$ 5. 代入初始条件得待定系数

本征函数法基本步骤 1. 选择空间函数的本征函数集$\{X_n(x)\}$，写出泛定方程的形式解$u(x,t)=\sum_n T_n(t)X_n(x)$ 2. 将形式解代入泛定方程，直接得时间函数的常微分方程

有界弦的自由振动$\begin{cases}\frac{\partial^2u}{\partial t^2}=a^2\frac{\partial^2u}{\partial x^2} & 0<x<L,t>0\\ u|_{x=0}=0 & u|_{x=L}=0\\ u|_{t=0}=\phi(x) & u_t|_{t=0}=\psi(x)\end{cases}$ 设分离变量解$u(x,t)=X(x)T(t)$

代入原方程得$\frac{X''}{X}=\frac{T''}{aT}=-\lambda\Longrightarrow\begin{cases}X+\lambda X=0\\ T''+\lambda a^2T=0\end{cases}$ 代入边界条件$X(0)T(t)=0,X(L)T(t)=0$

因$T(t)\neq 0$，否则平庸解，故$X(0)=X(L)=0\Longrightarrow\begin{cases}X''+\lambda X=0\\ X(0)=X(L)=0\end{cases}$

当$\lambda=0$，通解为$X(x)=A+Bx$，由边界条件得$A=B=0$–平庸解

当$\lambda<0$，通解为$X(x)=A\exp(\sqrt{-\lambda}x)+B\exp(-\sqrt{-\lambda}x)$，同理亦得平庸解

当$\lambda>0$，通解为$X(x)=A\cos\sqrt{\lambda}x+B\sin\sqrt{\lambda}x$，由边界条件得

本征值$\lambda_n=(\frac{n\pi}{L})^2$，本征解$X_n(x)=B_n\sin\frac{n\pi}{L}x$

λ_n代入时间方程得$T''(t)+(\frac{na\pi}{L})^2T=0$得$T_n(t)=C_n\cos\frac{na\pi}{L}t+D_n\sin\frac{na\pi}{L}t$

原方程本征解为$u_n(t)=X_n(x)T_n(t)=(C_n\cos\frac{na\pi}{L}t+D_n\sin\frac{na\pi}{L}t)\sin\frac{n\pi}{L}x$

一般解为$u(x,t)=\sum_{n=1}^\infty u_n(x,t)=\sum_{n=1}^\infty(C_n\cos\frac{na\pi}{L}t+D_n\sin\frac{na\pi}{L}t)\sin\frac{n\pi}{L}x$

代入初始条件$\begin{cases}u|_{t=0}=\phi(x)=\sum_{n=1}^\infty C_n\sin\frac{n\pi}{L}x\\ u_t|_{t=0}=\psi(x)=\sum_{n=1}^\infty D_n\frac{na\pi}{L}\sin\frac{n\pi}{L}x\end{cases}$ 得$\begin{cases}C_n=\frac{2}{L}\int_0^L\phi(x)\sin\frac{n\pi}{L}xdx\\ D_n=\frac{2}{n\pi a}\int_0^L\psi(x)\sin\frac{n\pi}{L}xdx\end{cases}$

傅氏级数展$f(x)=\frac{a_0}{2}+\sum_{n=1}^\infty(a_n\cos\frac{2\pi n}{L}x+b\sin\frac{2\pi n}{L}x)$其中$\begin{cases}a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{2\pi n}{L}xdx\\ b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{2\pi n}{L}xdx\end{cases}$

两端固定阻尼弦振动$\begin{cases}\frac{\partial^2u}{\partial t^2}=a^2\frac{\partial^2u}{\partial x^2}+f(x,t) & 0<x<L,t>0\\ u|_{x=0}=0 & u|_{x=L}=0\\ u|_{t=0}=\phi(x) & u_t|_{t=0}=\psi(x)\end{cases}$

其中阻尼项$f(x,t)=-\frac{k}{\rho}\frac{\partial u}{\partial t}=-2b\frac{\partial u}{\partial t}$ 分离变量得$\frac{X''}{X}=\frac{T''+2bT}{a^2T}=-\lambda$

特征方程$\begin{cases}X''+\lambda X(x)=0\\ X(0)=X(L)=0\end{cases}\Longrightarrow\begin{cases}\lambda_n=(\frac{n\pi}{L})^2\\ X_n(x)=B_n\sin\frac{n\pi}{L}x\end{cases}$ 时方$T''+2bT'+(\frac{n\pi a}{L})^2T=0$

设试探解$e^{-\lambda t}$满足方程$\lambda^2-2b\lambda+(\frac{n\pi a}{L})^2=0$，判别式$\Delta=b^2-(\frac{n\pi a}{L})^2$，并记$q_n=\sqrt{|(\frac{n\pi a}{L})^2-b^2|}$

当$\Delta>0$即$n<\frac{bL}{\pi a}$，$T_n(t)=\exp(-bt)(c_n\cosh q_nt+d_n\sinh q_nt)$

当$\Delta=0$即$n=\frac{nL}{\pi a}$，$T_n(t)=c_n\exp(-bt)+d_nt\exp(-bt)$

当$\Delta<0$即$n>\frac{bL}{\pi a}$，$T_n(t)=\exp(-bt)(c_n\cos q_nt+d_n\sin q_nt)$

欠阻尼情况$n>\frac{bL}{\pi a}$下，本征解为$u_n(t)=\exp(-bt)(C_n\cos q_nt+D_n\sin q_nt)\sin\frac{n\pi}{L}x$

一般解$u(x,t)=\sum_{n=1}^\infty u_n(x,t)$代入初始条件$\begin{cases}\phi(x)=\sum_{n=1}^\infty C_n\sin\frac{n\pi}{L}x\\ \psi(x)=\sum_{n=1}^\infty(-C_nb+D_nq_n)\sin\frac{n\pi}{L}x\end{cases}$

其中$\begin{cases}C_n=\frac{2}{L}\int_0^L\phi(x)\sin\frac{n\pi}{L}xdx\\ D_n=\frac{b}{q_n}C_n+\frac{2}{Lq_n}\int_0^L\psi(x)\sin\frac{n\pi}{L}xdx\end{cases}$

两端固定弦自由振动(第二类边界条件)$\begin{cases}\frac{\partial^2u}{\partial t^2}=a^2\frac{\partial^2u}{\partial x^2} & 0<x<L,t>0\\ u|_{x=0}=0 & u_x|_{x=L}=0\\ u|_{t=0}=\phi(x) & u_t|_{t=0}=\psi(x)\end{cases}$ 分离变量$\begin{cases}X''+\lambda X=0\\ X(0)=X'(L)=0\end{cases}$

考虑边界条件本征值$\lambda_n=\frac{(2n+1)^2\pi^2}{4l^2}$，本征函数$X(x)=B_n\sin\frac{(2n+1)^2\pi}{2l}x,n=0,1,\cdots$

代入时间方程得$T_n(t)=C'_n\cos\frac{(2n+1)\pi a}{2l}t+D'_n\sin\frac{(2n+1)\pi a}{2l}t$

一般解为$u(x,t)=\sum_{n=0}^\infty(C_n\cos\frac{(2n+1)\pi a}{2l}t+D_n\sin\frac{(2n+1)\pi a}{2l}t)\sin\frac{(2n+1)\pi}{2l}x$

代入初始条件得$\begin{cases}D_n\frac{4}{(2n+1)\pi a}\int_0^L\psi(x)\sin\frac{(2n+1)\pi}{2l}xdx\\ C_n=\frac{2}{L}\int_0^L\phi(x)\sin\frac{(2n+1)\pi}{2l}xdx\end{cases}$

热传导问题(第二类边界条件)$\begin{cases}\frac{\partial u}{\partial t}=a^2\frac{\partial^2u}{\partial x^2} & 0<x<L,t>0\\ u|_{t=0}=\phi(x) & u_x|_{x=0}=0,u_x|_{x=L}=0\end{cases}$

解空间方程同前，本征值$\lambda_n=(\frac{n\pi}{L})^2$，本征函数$X_n(x)=A\cos\frac{n\pi}{L}x,n=1,2,\cdots$

代入时间方程$T_n(t)=c_n\exp[-(\frac{na\pi}{L})^2t]$一般解$u(x,t)=\frac{C_0}{2}+\sum_{n=1}^\infty C_n\exp[-(\frac{na\pi}{L})^2t]\cos\frac{n\pi}{L}x$

其中$C_n=\frac{2}{L}\int_0^L\phi(x)\cos\frac{n\pi}{L}xdx$ 当稳态，$u(x,\infty)=C_0$

热传导问题(第三类边界条件)$\begin{cases}\frac{\partial u}{\partial t}=a^2\frac{\partial^2u}{\partial x^2} & 0<x<L,t>0\\ u(0,t)=0,u(x,0)=\phi(x) & u(L,t)+hu_t(L,t)=0\end{cases}$

分离变量解空间方程$\begin{cases}X''+\lambda X=0\\ X(0)=0,X(L)+hX'(L)=0\end{cases}$ 当$\lambda=0$，$X=A+Bx$由边界条件得平庸解$A=B=0$ 当$\lambda<0$，$X=A\exp(\sqrt{-\lambda}x)+B\exp(-\sqrt{-\lambda}x)$由边界条件亦得平庸解$A=B=0$

当$\lambda>0$，$X_n(x)=A\cos\beta x+B_n\sin\beta x,\beta=\sqrt{\lambda},\tan\beta_nL=-\beta_nh$

代入时间方程得$T_n(t)=A_n\exp(-\beta_n^2a^2t)$，一般解$u(x,t)=\sum_{n=1}^\infty C_n\exp(-\beta_n^2a^2t)\sin\beta_nx$

其中$C_n=\frac{1}{L_n}\int_0^L\phi(x)\sin\beta_nxdx,L_n=\int_0^L\sin^2\beta_nxdx=\frac{L}{2}(1-\frac{\sin 2\mu_n}{2\mu_n})$

二维自由波动方程$\begin{cases}\frac{\partial^2u}{\partial t^2}=c^2(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}) & 0<x<a,0<y<b\\ u|_{t=0}=\phi(x,y),u_t|_{t=0}=\psi(x,y) & u_{x=0}=u|_{x=a}=u|_{y=0}=u|_{y=b}=0\end{cases}$

首次分离变量$u(x,y,t)=V(x,y)T(t)\Longrightarrow\begin{cases}\frac{\partial^2V}{\partial x^2}+\frac{\partial^2V}{\partial y^2}+\lambda V=0\\ T''+\lambda c^2T=0\end{cases}$ 再次$V(x,y)=X(x)Y(y)$

$\Longrightarrow\frac{X''}{X}=-\frac{Y''+\lambda Y}{Y}=-\mu\Longrightarrow\begin{cases}X''+\mu X=0 & X(0)=X(a)=0\\ Y''+\nu Y=0 & Y(0)=Y(b)=0\end{cases},\nu=\lambda-\mu$

本征值$\begin{cases}\mu_m=(\frac{m\pi}{a})^2\\ \nu_n=(\frac{n\pi}{b})^2\end{cases}$ $m,n=1,2,\cdots$本征函数$\begin{cases}X_m(x)=\sin\frac{m\pi}{a}x\\ Y_n(y)=\sin\frac{n\pi}{b}y\end{cases}$

本征值$\lambda_{mn}=(\frac{m\pi}{a})^2+(\frac{n\pi}{b})^2$ 本征函数$V_{mn}(x,y)=\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y$

一般解$u(x,y,t)=\sum_{m=1}^\infty\sum_{n=1}^\infty(C_{mn}\cos\omega_{mn}t+D_{mn}\sin\omega_{mn}t)\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y$

其中$\begin{cases}C_{mn}=\frac{4}{ab}\int_0^b\int_0^a\phi(x,y)\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}ydxdy\\ D_{mn}=\frac{4}{ab\omega_{mn}}\int_0^b\int_0^a\psi(x,y)\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}ydxdy\end{cases}$

二维热传导问题$\begin{cases}\frac{\partial u}{\partial t}=c^2(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}) & 0<x<a,0<y<b,t>0\\ u|_{t=0}=\phi(x,y) & u|_{x=0}=u|_{x=a}=u|_{y=0}=u|_{x=b}=0\end{cases}$ 两次分离变量同前

得$\omega_{mn}=c\pi\sqrt{(\frac{m}{a})^2+(\frac{n}{b})^2},u(x,y,t)=\sum_{m=1}^\infty\sum_{n=1}^\infty C_{mn}e^{-\omega_{mn}^2t}\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y$

其中$C_{mn}=\frac{4}{ab}\int_0^b\int_0^a\phi(x)\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}ydxdy$

直角坐标系下拉普拉斯方程$\begin{cases}\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=0 & 0<x<a,0<y<b\\ u(0,y)=u(a,y)=0 & u(x,0)=0,u(x,b)=f_2(x)\end{cases}$ 分离变量得

空间方程$\begin{cases}X''+\lambda X=0\\ X(0)=X(a)=0\end{cases}\Longrightarrow\begin{cases}\lambda_n=(\frac{n\pi}{a})^2\\ X_n(x)=b_n\sin\frac{n\pi}{a}x\end{cases}$

代入时间方程$Y''-\lambda Y=0$得$Y_n(y)=c_n\cosh\frac{n\pi}{a}y+d_n\sinh\frac{n\pi}{a}y$

一般解$u(x,y)=\sum_{n=1}^\infty B_n\sin\frac{n\pi}{a}x\sinh\frac{n\pi}{a}y$其中$B_n=\frac{2}{a}csch\frac{n\pi b}{a}\int_0^af_2(x)\sin\frac{n\pi}{a}xdx$

对一般性拉普拉斯边值问题$\begin{cases}u(x,0)=f_1(x) & u(x,b)=f_2(x)\\ u(0,y)=g_1(y) & u(a,y)=g_2(y)\end{cases}$ 有一般解$u(x,t)=\sum_{n=1}^\infty A_n\sin\frac{n\pi}{a}x\sin\dots$
$y)+\sum_{n=1}^\infty B_n\sin\frac{n\pi}{a}x\sinh\frac{n\pi}{a}y+\sum_{n=1}^\infty C_n\sinh\frac{n\pi}{b}(a-x)\sin\frac{n\pi}{b}y+\sum_{n=1}^\infty D_n\sinh\frac{n\pi}{b}x\sin\frac{n\pi}{b}\dots$

圆形域内二维拉普拉斯方程$\begin{cases}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial u}{\partial\rho})+\frac{1}{\rho^2}\frac{\partial^2u}{\partial\theta^2}=0\\ u(\rho_0,\theta)=f(\theta)\end{cases}$ 分离变量得$\frac{\rho^2R''+\rho R'}{R}=-\frac{\Phi''}{\Phi}=\lambda$

$\Longrightarrow\begin{cases}\text{角向方程 }\Phi''+\lambda\Phi=0\\ \text{周期性边界条件 }\Phi(\theta+2\pi)=\Phi(\theta)\end{cases}\quad\begin{cases}\text{径向方程(欧拉方程) }\rho^2R''+\rho R'-\lambda R=0\\ \text{自然边界条件 }R(0)\text{有界}\end{cases}$

当$\lambda=0$，$\Phi(\theta)=A+B\theta=A$ 当$\lambda<0$，$\Phi(\theta)=A\exp(\sqrt{-\lambda}\theta)+B\exp(-\sqrt{-\lambda}\theta)$无法满足边界条件

当$\lambda>0$，$\phi(\theta)=A\sqrt{\lambda}\theta+B\sin\sqrt{\lambda}\theta,\sqrt{\lambda}=n$

$\begin{cases}R_0(\rho)=c_0+d_0\ln\rho\\ R_n(\rho)=c_n\rho^n+d_n\rho^{-n},n=1,2,\cdots\end{cases}$ 考虑边界条件得$\begin{cases}R_0(\rho)=c_0\\ R_n(\rho)=c_n\rho^n\end{cases}$ 即$R_n(\rho)=c_n\rho^n$

一般解$u(\rho,\theta)=\frac{a_0}{2}+\sum_{n=1}^\infty(a_n\cos n\theta+b_n\sin n\theta)$

其中$a_n=\frac{1}{\pi\rho_0^n}\int_0^{2\pi}f(\theta)\cos n\theta d\theta,b_n=\frac{1}{\pi\rho_0^n}\int_0^{2\pi}f(\theta)\sin n\theta d\theta$

非齐次方程+齐次边界条件+任意初始条件，先解对应的齐次初始条件的非齐次方程，再与对应的相同边界条件的齐次方程解叠加

非齐次泛定方程+齐次初始条件(强迫弦振动)$\begin{cases}\frac{\partial^2v}{\partial t^2}=a^2\frac{\partial^2v}{\partial x^2}+f(x,t) & 0<x<L,t>0\\ v|_{x=0}=v|_{x=L}=0 & v|_{t=0}=v_t|_{t=0}=0\end{cases}$ 以对应齐次问题的解试探$v(x,t)=\sum_{n=1}^\infty T_n(t)\sin\frac{n\pi}{L}x$原方程化为$\sum_{n=1}^\infty[T''_n(t)+(\frac{na\pi}{L})^2T_n(t)]\sin\frac{n\pi}{L}x=f(x)$

将非齐次项也按本征函数展开$f(x,t)=\sum_{n=1}^\infty f_n(t)\sin\frac{n\pi}{L}x$ 其中$f_n(t)=\frac{2}{L}\int_0^Lf(x,t)\sin\frac{n\pi}{L}xdx$

得一系列常微分方程问题$\begin{cases}T''_n(t)+(\frac{na\pi}{L})^2T_n(t)-f_n(t)=0\\ T_n(0)=T'_n(0)=0\end{cases}$ 然后用拉普拉斯变换法求解

边界条件齐次化 选辅助函数$u(x,t)=v(x,t)+\Omega(x,t)$使关于$v$的问题边界条件齐次

同时齐次化(当自由项f和边界条件均不含t)$\begin{cases}\frac{\partial^2u}{\partial t^2}=a^2\frac{\partial^2u}{\partial x^2}+f(x)\\ u|_{x=0}=b,u|_{x=L}=c\\ u|_{t=0}=\phi(x),u_t|_{t=0}=\psi(x)\end{cases}$ 代入辅助函数得

$\begin{cases}\frac{\partial^2u}{\partial t^2}=a^2\frac{\partial^2u}{\partial x^2}+a^2\frac{\partial^2\Omega}{\partial x^2}+f(x)\\ v(0)+\Omega(0)=b,v(L)+\Omega(L)=c\end{cases}$ 辅助函数满足$\begin{cases}\frac{\partial^2\Omega}{\partial t^2}+f(x)=0\\ \Omega(0)=b,\Omega(L)=c\end{cases}$ 得

$\Omega(x)=b+\frac{c-b}{L}x+F(x)+\frac{F(0)+F(L)}{L}x-F(0),F(x)=-\frac{1}{a^2}\int[\int f(x)dx]dx$

Chap9施图姆-刘维尔理论及其应用

任一二阶线性偏微分方程$y''+a(x)y'+b(x)y+\lambda c(x)y=0$

均可化为**施图姆-刘维尔型方程**$\frac{d}{dx}[k(x)\frac{dy}{dx}]-q(x)y+\lambda\rho(x)y=0$

其中$k(x)=e^{\int a(x)dx},-q(x)=b(x)e^{\int a(x)dx},\lambda$–本征值$,\rho(x)=c(x)e^{\int a(x)dx}$

对$k(x),q(x),\rho(x)\geq 0$的S-L方程，若$k(x),k'(x),q(x)$在(a,b)上连续，且最多以$x=a,x=b$为一阶极点，则存在无限多个本征值，相应有无限多个本征函数，且本征值均非负

对应不同本征值的本征函数在区间$[a,b]$带权重$\rho(x)$正交$\int_a^b\rho(x)y_m(x)y_n(x)dx=0,n\neq m$

本征函数集完备，即若函数$f(x)$满足广义Dirichlet条件：(1)具有连续一阶导和分段连续二阶导;(2)满足本征函数集满足的边界条件，则必可展开为绝对且一致收敛和广义傅里叶级数$f(x)=\sum_{n=1}^{\infty}f_ny_n(x)$(对于任意函数$f(x)$，则为平均收敛)

$f_n=\frac{1}{N_n^2}\int_a^b\rho(\xi)f(\xi)y_n(\xi)d\xi,N_n^2=\int_a^b\rho(\xi)[y_n(\xi)]^2d\xi$

当Q因子$Q=k(x)[y_n(0)y_m'(0)-y_m(0)y_n'(0)]-k(L)[y_n(L)y_m'(L)-y_m(L)y_n'(L)]=0$,本征函数在区间$[0,L]$带权重$\rho(x)$正交

矢量空间对n维矢量空间，选定一组基$\{e_i,i=1,2,\cdots,n\}$，空间中任一矢量$\boldsymbol{x}$均可用这组基的线性组合表示$\boldsymbol{x}=\sum_{i=1}^n x_ie_i$，满足空间加法和数乘封闭

内积$(\boldsymbol{x},\boldsymbol{y})=\sum_{i=1}^{\infty}x_i^*y_i=(\boldsymbol{y},\boldsymbol{x})^*$　　**内积空间**：定义了内积的矢量空间，满足

1．**对称性**$(\boldsymbol{x},\boldsymbol{y})=(\boldsymbol{y},\boldsymbol{x})^*$　　2．**线性**$(a\boldsymbol{x}+b\boldsymbol{z},\boldsymbol{y})=a(\boldsymbol{x},\boldsymbol{y})+b(\boldsymbol{z},\boldsymbol{y})$

3．**正定性**$(\boldsymbol{x},\boldsymbol{x})\geq 0,(\boldsymbol{x},\boldsymbol{x})=0\Leftrightarrow\boldsymbol{x}=0$

矢量模长$(\boldsymbol{x},\boldsymbol{x})^{1/2}=||\boldsymbol{x}||$　　**归一化矢量**$\frac{\boldsymbol{x}}{||\boldsymbol{x}||}$　　**矢量正交**当$(\boldsymbol{x},\boldsymbol{y})$，两矢量正交

正交归一矢量集的完备性：有限维矢量空间中，若一正交归一矢量集不包含在另一更大的正交归一矢量集中，则称该正交归一矢量集完备

函数空间对加法$((f_1+f_2)(x)=f_1(x)+f_2(x))$和乘法$((\alpha f)(x)=\alpha f(x))$封闭的平方可积(积分$\int_a^b|f(x)|^2dx$存在)函数的集合

函数空间的内积$(f_1,f_2)=\int_a^b f_1^*(x)f_2(x)dx$　　同样满足1．**对称性** 2．**线性** 3．**正定性**

函数空间函数集的完备性对任一函数$f(x)$，展开级数$\sum_{i=1}^{\infty}c_if_i(x)$平均收敛于$f(x)$，即$\lim_{n\to\infty}\int_a^b|f(x)-\sum_{i=1}^{\infty}c_if_i(x)|^2dx=0$，称正交归一函数集$\{f_i,i=1,2,\cdots\}$完备

广义函数内积$(f_1,f_2)=\int_a^b\rho(x)f_1^*(x)f_2(x)dx$

伴算符　L和M为定义在一定函数空间的微分算符，若对$\forall u,v$恒有$(v,Lu)=(Mv,u)$即$\int_a^b v^*(Lu)dx=\int_a^b(Mv)^*udx$，则称$M$是$L$的伴算符

自伴算符：伴算符为自身的算符，即对$\forall u,v$恒有$(v,Lu)=(Lv,u)$

若L为自伴算符，则方程$Ly(x)=\lambda y(x)$称自伴算符L的本征问题

自伴算符的性质 1．**存在性**自伴算符的本征值必存在 2．且为实数

3．**正交性**自伴算符的对应不同本征值的本征函数必正交

4．**完备性**自伴算符的本征函数构成一个完备函数集，即任一在区间$[a,b]$有连续二阶导且和自伴算符边界条件相同的函数$f(x)$，均可按本征函数$\{y_n(x)\}$展开成绝对且一致收敛的级数

$f(x)=\sum_{n=1}^{\infty}c_nf_n(x)$，其中$c_n=\frac{\int_a^b f(x)y_n^*(x)dx}{\int_a^b y_n(x)y_n^*(x)dx}$

算符自伴性与本征值问题有解之间的关系：充分不必要，如L是自伴算符，iL必非自伴算符，但$i\lambda$是iL的本征值

非自伴算符的本征值不一定是实数，本征函数也不一定具有正交性

引入算符$L=-\frac{d}{dx}[k(x)\frac{d}{dx}]+q(x)$，S-L方程化为$Ly=\lambda\rho(x)y$

令$u(x)=\sqrt{\rho(x)}y(x)$，S-L方程化为$L'u(x)=\lambda u(x)$，其中$L'=-\frac{d}{dx}[\phi(x)\frac{d}{dx}]+\psi(x),\phi(x)=\frac{k(x)}{\rho(x)},\psi(x)=-\frac{1}{\sqrt{\rho(x)}}\frac{d}{dx}[k(x)\frac{d}{dx}\frac{1}{\sqrt{\rho(x)}}]+\frac{q(x)}{\rho(x)}$

在边界条件$\phi(x)(u_1^*\frac{du_2}{dx}-u_2\frac{u_1^*}{dx})|_a^b=0$下，$L'$是自伴算符

在边界条件$p(x)(y_1^*\frac{dy_2}{dx}-y_2\frac{dy_1^*}{dx})|_a^b=0$下，$L$是自伴算符

Chap10行波法

一阶线性常微分方程$\frac{dy}{dx}+P(x)y=Q(x)$的通解为$y=e^{-\int P(x)dx}[\int Q(x)e^{\int P(x)dx}dx+C]$

特征线法解含两个自变量的一阶线性偏微分方程$a(x,y)\frac{\partial u}{\partial x}+b(x,y)\frac{\partial u}{\partial y}+c(x,y)u=f(x,y)$

由特征方程$a(x,y)dy-b(x,y)dx=0$解得特征线$y=y(x)+C$

做变换$\xi=y-y(x)$，取η使$\frac{\partial(\xi,\eta)}{\partial(x,y)}\neq 0$(或直接做代换$y=y(x)$)

原方程化为仅含一个自变量的一阶常微分方程$(a\frac{\partial\eta}{\partial x}+b\frac{\partial\eta}{\partial y})\frac{\partial u}{\partial\eta}+cu=f$

求解得$u(\xi,\eta)$(积分常数中含ξ)，回代并结合初始条件得到$u(x,y)$

二阶线性偏微分方程降阶求解(波动方程)$\frac{\partial^2u}{\partial t^2}-a^2\frac{\partial^2u}{\partial x^2}=(\frac{\partial}{\partial t}+a\frac{\partial}{\partial x})(\frac{\partial}{\partial t}-a\frac{\partial}{\partial x})u=0$

$\Longrightarrow\begin{cases}\frac{\partial u}{\partial t}-\frac{\partial u}{\partial x}=v\\\frac{\partial v}{\partial t}+a\frac{\partial v}{\partial x}=0\end{cases}$　还可化成标准形式(双曲型)两次积分并结合初始条件求解

波动方程Cauchy问题(无界弦自由振动)$\begin{cases}\frac{\partial^2u}{\partial t^2}=a^2\frac{\partial^2u}{\partial x^2} & -\infty<x<+\infty,t>0\\u|_{t=0}=\phi(x) & \frac{\partial u}{\partial t}|_{t=0}=\psi(x)\end{cases}$

通解为达朗贝尔公式$u(x,t)=f_1(x+at)+f_2(x-at)$

$=[\frac{1}{2}\phi(x+at)+\frac{1}{2a}\int_0^{x+at}\psi(\xi)d\xi+\frac{C}{2}]+[\frac{1}{2}\phi(x-at)-\frac{1}{2a}\int_0^{x-at}\psi(\xi)d\xi-\frac{C}{2}]$

$=\frac{1}{2}[\phi(x+at)+\phi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi$

特解$u(x,t)$仅取决于**依赖区间**$[x-at,x+at]$上的初始条件

决定区域的边界为$x-at=X_1,x+at=X_2,t>0$

区间$[X_1,X_2]$**影响区域**的边界为$x=X_1-at,x=X_2+at,t>0$

边界的平行线为特征线$x+at=C_1(C_1\geq X_1),x-at=C_2(C_2\leq X_2)$

半无界弦振动达朗贝尔解法 端点固定$\begin{cases}\frac{\partial^2u}{\partial t^2}=a^2\frac{\partial^2u}{\partial x^2} & 0<x<\infty & t>0\\u(x,0)=\phi(x) & u_t(x,0)=\psi(x) & u(0,t)=0\end{cases}$

将$\phi(x)$和$\psi(x)$在实轴上作奇延拓化为无界弦的自由振动问题得

在端点条件影响区($x\geq 0,x-at<0$)：$u(x,t)=\frac{1}{2}[\phi(x+at)-\phi(at-x)]+\frac{1}{2a}\int_{at-x}^{x+at}\psi(\xi)d\xi$

在初始条件决定区($x\geq 0,x-at\geq 0$)：$u(x,t)=\frac{1}{2}[\phi(x+at)+\phi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi$

端点自由$\begin{cases}\frac{\partial^2u}{\partial t^2}=a^2\frac{\partial^2u}{\partial x^2} & 0<x<\infty & t>0\\u(x,0)=\phi(x) & u_t(x,0)=\psi(x) & u_t(0,t)=0\end{cases}$

对$\phi(x)$和$\psi(x)$在实轴上做偶延拓化为无界弦的自由振动问题得

在$x\geq 0,x-at<0$：$u(x,t)=\frac{1}{2}[\phi(x+at)+\phi(at-x)]+\frac{1}{2a}[\int_0^{x+at}\psi(\xi)d\xi+\int_0^{at-x}\psi(\xi)d\xi]$

在$x\geq 0,x-at\geq 0$：$u(x,t)=\frac{1}{2}[\phi(x+at)+\phi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi$

端点受迫运动$\begin{cases}\frac{\partial^2u}{\partial t^2}=a^2\frac{\partial^2u}{\partial x^2} & 0<x<\infty & t>0\\u(x,0)=\phi(x) & u_t(x,0)=\psi(x) & u(0,t)=\mu(t)\end{cases}$　先解

$\begin{cases}\frac{\partial^2w}{\partial t^2}=a^2\frac{\partial^2w}{\partial x^2}\\w(x,0)=w_t(x,0)=0 & w(0,t)=\mu(t)\end{cases}$　得$w=\begin{cases}\mu(-\frac{x-at}{a}) & x-at<0\\0 & x-at\geq 0\end{cases}$

再与对应端点固定问题解v叠加即得原端点受迫振动问题的解$u(x,t)=v(x,t)+w(x,t)$

在$x\geq 0,x-at<0$：$u(x,t)=\frac{1}{2}[\phi(x+at)-\phi(at-x)]+\frac{1}{2a}\int_{at-x}^{x+at}\psi(\xi)d\xi+u(t-\frac{x}{a})$

在$x\geq 0,x-at\geq 0$：$u(x,t)=\frac{1}{2}[\phi(x+at)+\phi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi+0$

积分微商定理 若$U(x)=\int_a^x f(x,\tau)d\tau$，则$\frac{d}{dx}U(x)=f(x,x)+\int_a^x\frac{\partial}{\partial x}f(x,\tau)d\tau$

齐次化原理$\begin{cases}\frac{\partial^2u}{\partial t^2}=a^2\frac{\partial^2u}{\partial x^2} & -\infty<x<+\infty,t>0\\u(x,0)=\phi(x) & \frac{\partial u}{\partial t}(x,0)=\psi(x)\end{cases}$　利用达朗贝尔公式解得

$u_1(x,t)=\frac{1}{2}[\phi(x+at)+\phi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi$

$\begin{cases}\frac{\partial^2w}{\partial t^2}=a^2\frac{\partial^2w}{\partial x^2} & -\infty<x<+\infty,t>\tau>0\\w|_{t-\tau=0}=0 & \frac{\partial w}{\partial t}|_{t-\tau=0}=f(x,\tau)\end{cases}$　的解为

$w(x,t;\tau)=\frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(\xi,\tau)d\xi$，由齐次化原理(积分微商定理)

$\begin{cases}\frac{\partial^2u}{\partial t^2}=a^2\frac{\partial^2u}{\partial x^2}+f(x,t) & -\infty<x<+\infty,t>0\\u(x,0)=0 & \frac{\partial u}{\partial t}(x,0)=0\end{cases}$　的解为

$u_2(x,t)=\int_0^t w(x,t;\tau)d\tau=\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(\xi,\tau)d\xi d\tau$

$\begin{cases}\frac{\partial^2u}{\partial t^2}=a^2\frac{\partial^2u}{\partial x^2}+f(x,t) & -\infty<x<+\infty,t>0\\u(x,0)=\phi(x) & \frac{\partial u}{\partial t}(x,0)=\psi(x)\end{cases}$　的解为前述两问题解的叠加$u(x,t)=u_1(x,t)+u_2(x,t)=\frac{1}{2}[\phi(x+at)+\phi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi+\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(\xi,\tau)d\xi d\tau$

高维波动方程的行波解法 中心对称的球面波$\begin{cases}\frac{\partial^2u}{\partial t^2}=a^2\nabla^2u & r,t>0\\u(r,0)=\phi(r) & u(r,0)=\psi(r)\end{cases}$

做代换$v=ru$得$\begin{cases}\frac{\partial^2v}{\partial t^2}=a^2\frac{\partial^2u}{\partial r^2} & r,t>0\\u(r,0)=\phi(r) & u(r,0)=\psi(r)\end{cases}$　利用达朗贝尔公式得

在端点条件影响区($r\geq 0,r-at<0$)：$u(r,t)=\frac{1}{2r}[(r+at)\phi(r+at)-(at-r)\phi(at-r)]+\frac{1}{2ar}\int_{at-r}^{r+at}\xi\psi(\xi)d\xi$

在初始条件决定区($r\geq 0,r-at\geq 0$)：$u(r,t)=\frac{1}{2r}[(r+at)\phi(r+at)+(r-at)\phi(r-at)]+\frac{1}{2ar}\int_{r-at}^{r+at}\xi\psi(\xi)d\xi$

一般形式的三维齐次波动方程(非球对称)$\begin{cases}\frac{\partial^2u}{\partial t^2}=a\nabla^2u & -\infty<x,y,z<+\infty\\u|_{t=0}=f(x,y,z) & u_t|_{t=0}=g(x,y,z)\end{cases}$

泊松公式为$u(x,y,z,t)=\frac{\partial}{\partial t}(\frac{t}{4\pi a^2t^2}\oint_{S_{at}^M}f(\xi,\eta,\zeta)dS)+\frac{t}{4\pi a^2t^2}\oint_{S_{at}^M}g(\xi,\eta,\zeta)dS=\frac{\partial}{\partial t}[\frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi}f(\xi,\eta,\zeta)\sin\theta d\theta d\phi]+\frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi}g(\xi,\eta,\zeta)\sin\theta d\theta d\phi$

其中S_{at}^M–以(x,y,z)为中心，$r=at$为半径的球面，$\xi=x+at\sin\theta\cos\phi,\eta=y+at\sin\sin\phi,\zeta=z+at\cos\theta$

齐次化原理$\begin{cases}u_{tt}=a\nabla^2u & \boldsymbol{x}\in R^3,t>0\\u(\boldsymbol{x},0)=f(\boldsymbol{x}) & u_t(\boldsymbol{x},0)=g(\boldsymbol{x})\end{cases}$　利用泊松公式解得

$u_1(\boldsymbol{x},t)=\frac{1}{4\pi a}\frac{\partial}{\partial t}\oint_{S_{at}^M}\frac{f(\xi,\eta,\zeta)}{r}dS+\frac{1}{4\pi a}\oint_{S_{at}^M}\frac{g(\xi,\eta,\zeta)}{r}dS$

$\begin{cases}w_{tt}=a\nabla^2w\\w(\boldsymbol{x},0)=0,w_t|_{t=\tau}=F(\boldsymbol{x},\tau)\end{cases}$　用泊松公式得$w(\boldsymbol{x},t;\tau)=\frac{1}{4\pi a}\oint_{S_{a(t-\tau)}^M}\frac{F(\xi,\eta,\zeta,\tau)}{a(t-\tau)}dS$

由齐次化原理得$\begin{cases}u_{tt}=a^2\nabla^2u+F(\boldsymbol{x},t) & x\in R^3,t>0\\u(\boldsymbol{x},0)=f(\boldsymbol{x}) & u_t(\boldsymbol{x},0)=g(\boldsymbol{x})\end{cases}$　的解为$u_2(\boldsymbol{x},t)=$

$\frac{1}{4\pi a}\int_0^t\oint_{S_{a(t-\tau)}^M}\frac{F(\xi,\eta,\zeta,\tau)}{a(t-\tau)}dS=\frac{1}{4\pi a^2}\int_0^{at}dr\oint_{S_r^M}\frac{F(\xi,\eta,\zeta,t-\frac{r}{a})}{r}dS,r=a(t-\tau)$

$\begin{cases}u_{tt}=a^2\nabla^2u+F(\boldsymbol{x},t) & x\in R^3,t>0\\u(\boldsymbol{x},0)=f(\boldsymbol{x}) & u_t(\boldsymbol{x},0)=g(\boldsymbol{x})\end{cases}$　的解为上述两问题解的叠加

$u(\boldsymbol{x},t)=u_1(\boldsymbol{x},t)+u_2(\boldsymbol{x},t)=\frac{1}{4\pi a}\frac{\partial}{\partial t}\oint_{S_{at}^M}\frac{f(\xi,\eta,\zeta)}{r}dS+\frac{1}{4\pi a}\oint_{S_{at}^M}\frac{g(\xi,\eta,\zeta)}{r}dS+\frac{1}{4\pi a^2}\iiint_{r\leq at}\frac{F(\xi,\eta,\zeta,t-\frac{r}{a})}{r}dv$

Chap11积分变换法

傅里叶变换$F(\omega)=\mathcal{F}[f(t)]=\int_{-\infty}^{+\infty}f(t)e^{-i\omega t}dt,\omega\in(-\infty,+\infty)$

傅里叶逆变换$f(t)=\mathcal{F}^{-1}[F(\omega)]=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(\omega)e^{i\omega t}d\omega$(变换存在的条件：通常要求$f(t)$绝对可积)

傅里叶变换的性质 线性$\mathcal{F}[af_1(t)+bf_2(t)]=a[f_1(t)]+b[f_2(t)]$

对称性设$\mathcal{F}[f(t)]=F(\omega)$，则$\mathcal{F}[F(t)]=2\pi f(-\omega)$

位移性$\mathcal{F}[f(t-t_0)]=e^{-i\omega t_0}F(\omega)$　　$\mathcal{F}^{-1}[F(\omega-\omega_0)]=e^{i\omega_0t}f(t)$

$\mathcal{F}[e^{i\omega_0t}f(t)]=F(\omega-\omega_0)$

相似性$\mathcal{F}[f(at)]=\frac{1}{|a|}F(\frac{\omega}{a})$　　$\mathcal{F}^{-1}[F(a\omega)]=\frac{1}{|a|}f(\frac{t}{a})(a\neq 0)$

微分性 原函数的微分性 若$\lim_{|t|\to+\infty}f(t)=0$，则$\mathcal{F}[f'(t)]=i\omega F(\omega)$

若$\lim_{|t|\to+\infty}f^{(k)}(t)=0$，则$\mathcal{F}[f^{(k)}(t)]=(i\omega)^kF(\omega)$

像函数的微分性$F'(\omega)=-i\mathcal{F}[tf(t)]$　　$\mathcal{F}[tf(t)]=iF'(\omega)$

积分性 若$\lim_{t\to+\infty}\int_{-\infty}^t f(\tau)d\tau=0$，则$\mathcal{F}[\int_{-\infty}^t f(\tau)d\tau]=\frac{1}{i\omega}F(\omega)$

卷积$f_1(t)*f_2(t)=\int_{-\infty}^{+\infty}f_1(s)f_2(t-s)ds$

卷积的性质 交换律$f*g=g*f$　　**分配律**$f*(g+h)=f*g+f*h$

结合律$f*(g*h)=(f*g)*h$　　**数乘**$A(f*g)=(Af)*g=f*(Ag),A$为常数

求导$\frac{d}{dt}(f*g(t))=f'(t)*g(t)=f(t)*g'(t)$　　$f*\delta(t)=\delta*f(t)=f(t)$

卷积定理$\mathcal{F}[f*g]=F(\omega)\cdot G(\omega)$　　$\mathcal{F}[f\cdot g]=\frac{1}{2\pi}F(\omega)*G(\omega)$

用傅里叶变换解微分方程步骤 1．方程和条件两边同对定义在$(-\infty,+\infty)$的变量(一般是x)做傅里叶变换，像函数$u(x,t)\to U(\omega,t)$，初始条件$\phi(x)\to\Phi(\omega),\psi(x)\to\Psi(\omega)$

2．解出$U(\omega,t)$　　3．$U(\omega,t)$反演得$u(x,t)$

拉普拉斯变换$F(p)=\mathcal{L}[f(t)]=\int_a^{+\infty}f(t)e^{-pt}dt$

拉普拉斯逆变换$f(t)=\mathcal{L}^{-1}[F(p)]=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}F(p)e^{pt}dp$

拉普拉斯变换的性质 线性$\mathcal{L}[af_1(t)+bf_2(t)]=aF_1(p)+bF_2(p)$

相似性设$F(p)=\mathcal{L}[f(t)]$，则$\mathcal{L}[f(at)]=\frac{1}{a}F(\frac{p}{a})$　　$\mathcal{L}^{-1}[F(ap)]=\frac{1}{a}f(\frac{t}{a})$

延迟性$\mathcal{L}[f(t-t_0)u(t-t_0)]=e^{-pt_0}F(p)$　　**平移性**$\mathcal{L}[e^{p_0t}f(t)]=F(p-p_0)$

微分性$\mathcal{L}[f'(t)]=pF(p)-f(0)$　　$F'(p)=-\mathcal{L}[tf(t)]$　　$F^{(n)}(p)=(-1)^n\mathcal{L}[t^nf(t)]$

$\mathcal{L}[f^{(n)}]=p^nF(p)-p^{n-1}f(0)-p^{n-2}f'(0)-\cdots-f^{(n-1)}(0)$

积分性$\mathcal{L}[\int_0^t f(s)ds]=\frac{F(p)}{p}$　　$\int_p^{+\infty}F(s)ds=\mathcal{L}[\frac{f(t)}{t}]$

周期性若$f(t+T)=f(t)$，$\mathcal{L}[f(t)]=\frac{\int_0^T f(t)e^{-pt}dt}{1-e^{-pT}}$

在区间$[0,+\infty]$的卷积$f_1(t)*f_2(t)=\int_0^t f_1(s)f_2(t-s)ds$

卷积定理$\mathcal{L}[f_1(t)*f_2(t)]=\mathcal{L}[f_1]\mathcal{L}[f_2]=F_1(p)F_2(p)$

用拉普拉斯变换解偏微分方程步骤与用傅里叶变换相似，针对定义在$[0,\infty)$的变量

Chap12格林函数法

格林函数$G(\boldsymbol{r},\boldsymbol{r}_0)$ 位于$\boldsymbol{r}_0$的单位点源在$\boldsymbol{r}$处产生的电势

步骤 1．建立格林函数G的定解问题 2．解定解问题得格林函数 3．解的积分即为原问题的解

无界三维泊松方程$\nabla^2 u(\boldsymbol{r})=\rho(\boldsymbol{r}), G(\boldsymbol{r};\boldsymbol{r}_0)=\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r_0}|}\Longrightarrow u(\boldsymbol{r})=\iiint\rho(\boldsymbol{r}_0)G(\boldsymbol{r},\boldsymbol{r}_0)d\boldsymbol{r}_0$无界二维泊松$u_{xx}+u_{yy}=\rho(x,y), G(x,y;\xi,\eta)=-\frac{1}{2\pi}\ln\frac{1}{|\boldsymbol{r}-\boldsymbol{r_0}|}\Longrightarrow u(x,y)=\iint\rho(\boldsymbol{r}_0)G(x,y;\xi,\eta)d\xi d\eta$

一维有界问题$\begin{cases}\frac{\partial w}{\partial t}=a^2\frac{\partial^2 w}{\partial x^2}+f(x,t)\\ w|_{t=0}=\phi(x)w|_{x=0}=w|_{x=L}=0\end{cases}$ 先求$\begin{cases}\frac{\partial w}{\partial t}=a^2\frac{\partial^2 w}{\partial x^2}\\ w|_{t=0}=\phi(x)\quad w|_{x=0}=w|_{x=L}=0\end{cases}$

$G(x,x_0,t)=\sum_{n=1}^\infty C_n\exp[-(\frac{na\pi}{L})^2t]\sin\frac{n\pi}{L}x, C_n=\frac{2}{L}\int_0^L\delta(x-x_0)\sin\frac{n\pi}{L}xdx=\frac{2}{L}\sin\frac{n\pi}{L}x_0, u(x,t)=\int_0^L\phi(x_0)G(x,x_0,t)dx_0=\frac{2}{L}\int_0^L[\sum_{n=1}^\infty e^{-(\frac{na\pi}{L})^2t}\sin\frac{n\pi}{L}x\sin\frac{n\pi}{L}x_0]\phi(x_0)dx_0$

再求$\begin{cases}\frac{\partial^2 v}{\partial t^2}=a^2\nabla^2 v+f(x,t)\\ v|_{t=0}=u|_{x=0}=u|_{x=L}=0\end{cases}$ 即求$\begin{cases}\frac{\partial^2 u}{\partial t^2}=a^2\nabla^2 u\\ u|_{t=\tau}=f(x,\tau)\quad u|_{x=0}=u|_{x=L}=0\end{cases}$ 法同前，

然后$v(x,t)=\int_0^t u(x,t,\tau)d\tau$

得$w(x,t)=\int_0^L\phi(x_0)G(x,x_0,t)dt+\int_0^t\int_0^L f(x_0,\tau)G(x_0,\tau)G(x,x_0,t-\tau)dx_0d\tau$

泊松方程边值问题基本积分公式$u(\boldsymbol{r})=\iiint_V G(\boldsymbol{r},\boldsymbol{r}_0)f(\boldsymbol{r})dV+\iint_\Sigma[G(\boldsymbol{r},\boldsymbol{r}_0)\frac{\partial u(\boldsymbol{r})}{\partial n}-u(\boldsymbol{r})\frac{\partial G(\boldsymbol{r},\boldsymbol{r}_0)}{\partial n}]dS=\iiint_V G(\boldsymbol{r},\boldsymbol{r}_0)f(\boldsymbol{r}_0)dV_0+\iint_\Sigma[G(\boldsymbol{r},\boldsymbol{r}_0)\frac{\partial u(\boldsymbol{r}_0)}{\partial n_0}-u(\boldsymbol{r})\frac{\partial G(\boldsymbol{r},\boldsymbol{r}_0)}{\partial n_0}]dS_0$

泊松方程第一类边值问题$u|_\Sigma=\phi(\boldsymbol{r}_\Sigma), u(\boldsymbol{r})=\iiint_V G(\boldsymbol{r},\boldsymbol{r}_0)f(\boldsymbol{r})dV-\iint_\Sigma u(\boldsymbol{r})\frac{\partial G(\boldsymbol{r},\boldsymbol{r}_0)}{\partial n}dS=\iiint_V G(\boldsymbol{r},\boldsymbol{r}_0)f(\boldsymbol{r}_0)dV_0-\iint_\Sigma u(\boldsymbol{r})\frac{\partial G(\boldsymbol{r},\boldsymbol{r}_0)}{\partial n_0}dS_0$

泊松方程第二类边值问题$\frac{\partial u}{\partial n}|_\Sigma=\phi(\boldsymbol{r}_\Sigma), u(\boldsymbol{r})=\iiint_V G(\boldsymbol{r},\boldsymbol{r}_0)f(\boldsymbol{r})dV+\iint_\Sigma G(\boldsymbol{r},\boldsymbol{r}_0)\frac{\partial u(\boldsymbol{r})}{\partial n}dS=\iiint_V G(\boldsymbol{r},\boldsymbol{r}_0)f(\boldsymbol{r}_0)dV_0+\iint_\Sigma G(\boldsymbol{r},\boldsymbol{r}_0)\frac{\partial u(\boldsymbol{r}_0)}{\partial n_0}dS_0$(需要推广格林函数$\nabla^2G(\boldsymbol{r},\boldsymbol{r}_0)=-\delta(\boldsymbol{r}-\boldsymbol{r}_0)+\frac{1}{V_\Omega}$)

泊松方程第三类边值问题$[\alpha u+\beta\frac{\partial u}{\partial n}]|_{Sigma}=0, u(\boldsymbol{r})=\iiint_V G(\boldsymbol{r},\boldsymbol{r}_0)f(\boldsymbol{r})dV+\frac{1}{\beta}\iint_\Sigma\phi(\boldsymbol{r})G(\boldsymbol{r},\boldsymbol{r}_0)dS=\iiint_V G(\boldsymbol{r},\boldsymbol{r}_0)f(\boldsymbol{r}_0)dV_0+\frac{1}{\beta}\iint_\Sigma\phi(\boldsymbol{r}_0)G(\boldsymbol{r},\boldsymbol{r}_0)dS_0$拉普拉斯类似

电像法真空中一半径为R_0接地导体球，距球心$a(>R_0)$处一点电荷Q，镜像电荷$Q'=-\frac{R_0}{a}Q$位于距球心$b=\frac{R_0^2}{a}$处

Chap13贝塞尔函数

从S-L型方程出发$\frac{d}{dx}[k(x)\frac{dy}{dx}]-q(x)y+\lambda\rho(x)y=0$

当$k=1,q=0,\rho=1$化为亥姆霍兹方程$\frac{d^y}{dx^2}+\lambda y=0$

当$k=x,q=m^2/x,\rho=x$化为参数形式的贝塞尔方程，当$\lambda=1$化为贝塞尔方程

当$k=x^2,q=\omega^2,\rho=1$化为球贝塞尔方程$\frac{d}{dx}[x^2\frac{dy}{dx}]-\omega^2y+\lambda x^2y=0$，当$\lambda=0$化为欧拉方程

当$k=1-x^2,q=\frac{m^2}{1-x^2},\rho=1$化为连带勒让德方程$\frac{d}{dx}[(1-x^2)\frac{dy}{dx}]-\frac{m^2}{1-x^2}y+\lambda y=0$

当$m=0$化为勒让德方程

伽马函数$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt\ (x>0)$

基本性质$\Gamma(x+1)=x\Gamma(x)\qquad\Gamma(n+1)=n!\qquad\Gamma(-n)=\infty\ (n=0,1,2,\cdots)$

$\Gamma(\frac12)=\sqrt{\pi}\qquad\Gamma(n+\frac12)=\frac{(2n)^2}{2^{2n}n!}\sqrt{\pi}\qquad\Gamma(n+\frac12+1)=\frac{(2n+1)!}{2^{2n+1}n!}\sqrt{\pi}$

对大的n有斯特林近似式$n!=\sqrt{2\pi n}n^ne^{-n}$ 余元公式$\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin\pi x}$

贝塔函数$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\frac{(x-1)!(y-1)!}{(x+y-1)!}$

贝塞尔方程的求解ν阶贝塞尔方程$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\nu^2)y=0$的解称贝塞尔函数

ν阶第一类贝氏函数$J_\nu(x)=\sum_{m=0}^\infty\frac{(-1)^m(\frac{x}{2})^{2m+\nu}}{m!\Gamma(m+\nu+1)}\qquad J_0(0)=1\ \ J_\nu(0)=0,(\nu>0)$

当ν取非负整数得整数阶贝氏函数$J_n(x)=\sum_{m=0}^\infty\frac{(-1)^m}{m!(m+n)!}(\frac{x}{2})^{2m+n}$

若$C=-\nu$，则$J_{-\nu}(x)=\sum_{m=0}^\infty\frac{(-1)^m(\frac{x}{2})^{2m-\nu}}{m!\Gamma(m-\nu+1)}$

$J_\nu(x)$和$J_{-\nu}(x)$线性独立(ν非整数) 但$J_n(x)$和$J_{-n}(x)$线性相关

为构建与J_n线性无关的解，引入诺依曼函数（第二类贝氏函数）

$Y_\nu=\begin{cases}\frac{J_\nu(x)\cos\nu x-J_{-\nu}(x)}{\sin\nu x}\\ \lim_{\alpha\to\nu}\frac{J_\alpha(x)\cos\alpha\pi-J_{-\alpha}(x)}{\sin\alpha\pi},\quad\nu非整数\end{cases}$

积分表示$Y_n(x)=\frac{1}{\pi}\int_0^\pi\sin(x\sin\theta-n\theta)d\theta-\frac{1}{\pi}\int_0^\infty[e^{nt}+(-1)^ne^{-nt}]e^{-x\sinh t}dt$

ν阶贝塞尔方程通解$y(x)=\begin{cases}AJ_\nu(x)+BY_\nu(x)\\ AJ_\nu(x)+BJ_{-\nu}(x),\quad\nu非整数\end{cases}$

生成函数级数展开式的系数是贝塞尔函数的函数$f(x,r)=\sum_n J_n(x)r^n$

整数阶贝塞尔函数的生成函数$\exp[\frac{x}{2}(r-\frac{1}{r})]=\sum_{n=-\infty}^\infty J_n(x)r^n$

递推公式$(\forall\nu)$$\begin{cases}\frac{d}{dx}[x^\nu J_\nu(x)]=x^\nu J_{\nu-1}(x)\\ \frac{d}{dx}[x^{-\nu}J_\nu(x)]=-x^{-\nu}J_{\nu+1}(x)\end{cases}$

$\frac{d}{dx}[J_0(x)]=-J_1(x)\qquad\frac{d}{dx}[xJ_1(x)]=xJ_0(x)$

$\begin{cases}J_\nu'(x)=\frac12[J_{\nu-1}(x)-J_{\nu+1}(x)]\\ J_{\nu-1}(x)+J_{\nu+1}(x)=\frac{2}{x}\nu J_\nu(x)\end{cases}\quad\begin{cases}xJ_{\nu-1}(x)=\nu J_\nu(x)+xJ_\nu'(x)\\ xJ_{\nu+1}(x)=\nu J_\nu(x)-xJ_\nu'(x)\end{cases}$

积分递推公式$\begin{cases}\int x^{\nu+1}J_\nu(x)dx=x^{\nu+1}J_{\nu+1}(x)\\ \int x^{-\nu+1}J_\nu(x)dx=-x^{-\nu+1}J_{\nu-1}(x)\end{cases}$

第二类贝氏函数递推公式与第一类相同$\begin{cases}\frac{d}{dx}[x^\nu Y_\nu(x)]=x^\nu Y_{\nu-1}(x)\\ \frac{d}{dx}[x^{-\nu}Y_\nu(x)]=-x^{-\nu}Y_{\nu+1}(x)\\ Y_\nu'(x)=\frac12[Y_{\nu-1}(x)-Y_{\nu+1}(x)]\\ Y_{\nu-1}(x)+Y_{\nu+1}(x)=\frac{2\nu}{x}Y_\nu(x)\\ xY_{\nu-1}(x)=\nu Y_\nu(x)+xY_\nu'(x)\\ xY_{\nu+1}(x)=\nu Y_\nu(x)-xY_\nu'(x)\end{cases}$

奇数阶贝氏函数$\begin{cases}J_{n+\frac12}(x)=(-1)^n\sqrt{\frac{2}{\pi}}x^{n+\frac12}(\frac1x\frac{d}{dx})^n(\frac{\sin x}{x})\\ J_{-(n+\frac12)}(x)=\sqrt{\frac{2}{\pi}}x^{n+\frac12}(\frac1x\frac{d}{dx})^n(\frac{\cos x}{x})\end{cases}$

积分表示$J_n(x)=\frac{1}{2\pi}\int_0^\pi\cos(x\sin\theta-n\theta)d\theta$

证明思路：通过整数贝塞尔函数生成函数令$r=e^{i\theta}$

比较实虚部得$\begin{cases}\cos(x\sin\theta)=\sum_{n=-\infty}^\infty J_n(x)\cos n\theta\\ \sin(x\sin\theta)=\sum_{n=-\infty}^\infty J_n(x)\sin n\theta\end{cases}$

后两边同乘$\cos m\theta$或$\sin m\theta$并在$[0,\pi]$积分即得

三角函数贝赛尔级数展开式$\begin{cases}\cos x=J_0(x)+2\sum_{n=1}^\infty(-1)^nJ_{2n}(x)\\ \sin x=2\sum_{n=0}^\infty(-1)^nJ_{2n+1}(x)\end{cases}$

渐进公式$J_n(x)\approx\sqrt{\frac{2}{\pi x}}\cos(x-\frac{x}{4}-\frac{n\pi}{2})$

参数形式的贝赛尔方程$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(\lambda^2x^2-\nu^2)y=0$

其S-L形式为$\frac{d}{dx}(x\frac{dy}{dx})-\frac{\nu^2}{x}y+\lambda^2xy=0\qquad$通解为$y=AJ_\nu(\lambda x)+BY_\nu(\lambda x)$

递推公式$\begin{cases}\frac{d}{d(\lambda x)}[(\lambda x)^{-\nu}J_\nu(\lambda x)]=-(\lambda x)^{-\nu}J_{\nu+1}(\lambda x)\\ \frac{d}{d(\lambda x)}[(\lambda x)^\nu J_\nu(\lambda x)]=(\lambda x)^\nu J_{\nu-1}(\lambda x)\end{cases}$

$\frac{d}{dx}J_0(\lambda x)=-\lambda J_1(\lambda x)\qquad\frac{d}{dx}[\frac1xJ_1(\lambda x)]=-\frac{\lambda}{x}J_2(\lambda x)$

$\frac{d}{dx}[xJ_1(\lambda x)]=\lambda xJ_0(\lambda x)\qquad\frac{d}{dx}[x^2J_2(\lambda x)]=\lambda x^2J_1(x)$

def $\lambda_{\nu m}=\frac{\mu_{\nu m}}{a}$(前已将$[0,\mu_{\nu m}]$上$J_\nu(x)$对应到$[0,1]$)将$[0,1]$上的$J_\nu(\lambda_{\nu m}x)$对应到$[0,a]$上的$J(\lambda_{\nu m})$

正交性&模值$\int_0^a xJ_\nu(\lambda_{\nu m}x)J_\nu(\lambda_{\nu k}x)dx=\frac{a^2}{2}J_{\nu+1}^2(\mu_{\nu m})\delta_{mk}$

当$a=1$化为$\int_0^1 xJ_\nu(\mu_{\nu m}x)J_\nu(\mu_{\nu k}x)dx=\frac12J_{\nu+1}^2(\mu_{\nu m})\delta_{mk}$

贝氏函数的完备性 $\left.\begin{matrix}连续点 & f(x)\\ 间断点 & \frac{f(x-0)+f(x+0)}{2}\end{matrix}\right\}=\sum_{m=1}^\infty A_mJ_\nu(\lambda_{\nu m}x)$

$\forall x\in(0,a)\qquad$其中$A_m=\frac{2}{a^2J_{\nu\pm1}^2(\mu_{\nu m})}\int_0^a xJ_\nu(\lambda_{\nu m}x)f(x)dx$

球贝塞尔函数$j_n(x)=\sqrt{\frac{\pi}{2x}}J_{n+\frac12}(x)$

$y_{nm}(x)=j_n(\lambda_{nm}x)$是球贝氏方程$x^2y''+2xy'+[kx^2-n(n+1)]y=0$解

$j_0(x)=\frac{\sin x}{x}\qquad j_1(x)=\frac{\sin x}{x^2}-\frac{\cos x}{x}\qquad j_2(x)=(\frac{3}{x^2}-\frac1x)\sin x-\frac{3}{x^2}\cos x$

递推公式$\frac{d}{dx}[x^{n+1}j_n(x)]=x^{n+1}j_{n-1}(x)\qquad\frac{d}{dx}[x^{-n}j_n(x)]=-x^{-n}j_{n+1}(x)$

正交性&模值$\int_0^a x^2j_n(\lambda_{nm}x)j_n(\lambda_{nk}x)dx=\frac{a^2}{2}j_{n+1}^2(\mu_{n+\frac12})\delta_{mk}$

Hankel函数 发散波$H_\nu^{(1)}(x)=J_\nu(x)+iN_\nu(x)\sim\sqrt{\frac{2}{\pi x}}\exp[i(x-\frac{\nu\pi}{2}-\frac{\pi}{4})]$

汇聚波$H_\nu^{(2)}(x)=J_\nu(x)-iN_\nu(x)\sim\sqrt{\frac{2}{\pi x}}\exp[-i(x-\frac{\nu\pi}{2}-\frac{\pi}{4})]$

柱形边界条件总结 第一类边界条件$R(\rho)|_{\rho=\rho_0}=0\Longrightarrow J_m(\lambda\rho_0)=0$

$\Longrightarrow\lambda_{mn}=\frac{\mu_{mn}}{\rho_0}\Longrightarrow\int_0^{\rho_0}x[J_m(\lambda_{mn}x)]^2dx=\frac{\rho_0^2}{2}[J_{m+1}(\mu_{mn})]^2$

第二类边界条件$\frac{dR}{d\rho}|_{\rho_0}=0\Longrightarrow J_m'(\lambda\rho_0)=0$(此处$\mu_{mn}$–$J_m'(x)$的第$n$个零点)

$\Longrightarrow\lambda_{mn}=\frac{\mu_{mn}}{\rho_0}\Longrightarrow\int_0^{\rho_0}x[J_m(\lambda_{mn}x)]^2dx=\frac12(\rho_0^2-\frac{m^2}{\lambda_{mn}^2})[J_m(\mu_{mn})]^2$

第三类边界条件$R(\rho_0)+HR'(\rho_0)=0\Longrightarrow J_m(\lambda\rho_0)+H\lambda J_m'(\lambda\rho_0)=0$

(此处μ_{nm}–$J_m(x)+H\lambda J_m'(x)$零点)$\Longrightarrow\lambda_{mn}=\frac{\mu_{mn}}{\rho_0}$

$\Longrightarrow\int_0^{\rho_0}x[J_m(\lambda_{mn}x)]^2dx=\frac12(\rho_0^2-\frac{m^2}{\lambda_{mn}^2}+\frac{\rho_0^2}{\lambda_{mn}^2H})[J_m(\mu_{mn})]^2$

勒让德多项式

勒让德方程$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta})=l(l+1)\Theta$

或令$x=\cos\theta,y(x)=\Theta(\theta)$化为$(1-x^2)y''-2xy'+l(l+1)y=0$

设方程有幂级数解$y=\sum_{k=0}^\infty C_kx^k$代入比较系数

得双间隔系数递推公式$C_{k+2}=\frac{(k-l)(k-+l+1)}{(k+2)(k+1)}C_k$

方程通解为$y(x)=C_0y_0(x)+C_1y_1(x)$

其中$\begin{cases}y_0(x)=&1-\frac{l(l+1)}{2!}x^2+\frac{(l-2)l(l+1)(l+3)}{4!}x^4-\frac{(l-4)(l-2)l(l+1)(l+3)(l+5)}{6!}\\ &x^6+\frac{(l-6)(l-4)(l-2)l(l+1)(l+3)(l+5)(l+7)}{8!}x^8-\cdots\\ y_1(x)=&x-\frac{(l-1)(l+2)}{3!}x^3+\frac{(l-3)(l-1)(l+2)(l+4)}{5!}x^5\\ &-\frac{(l-5)(l-3)(l-1)(l+2)(l+4)(l+6)}{7!}x^7+\cdots\end{cases}$

收敛半径$R=\lim_{k\to\infty}|\frac{C_k}{C_{k+2}}|=1$

当取l偶数，则$y_0(x)$截断为偶次幂多项式$y_0(x)=C_0+C_2x^2+\cdots+C_lx^l$

当取l奇数，则$y_1(x)$截断为奇次幂多项式$y_1(x)=C_1+C_3x^3+\cdots+C_lx^l$

取$C_l=\frac{(2l)!}{2^l(l!)^2}$得$l$阶勒让德多项式$P_l(x)=\frac{1}{2^l}\sum_{m=0}^M(-1)^m\frac{(2l-2m)!}{m!(l-m)!(l-2m)!}x^{l-2m}$

其中$M=\begin{cases}\frac{l}{2}&(l=0,2,4,\cdots)\\ \frac{l-1}{2}&(l=1,3,5,\cdots)\end{cases}$

低阶多项式$\begin{cases}P_0(x)=1 & P_1(x)=x\\ P_2(x)=\frac12(3x^2-1) & P_3(x)=\frac12(5x^3-3x)\\ P_4(x)=\frac18(35x^4-30x^2+3) & P_5(x)=\frac18(63x^5-70x^3+15x)\end{cases}$

l偶$P_l(x)$偶，l奇$P_l(x)$奇 $\qquad P_l(1)=1,\forall l$

通解为$y(x)=AP_l(x)+Q_l(x)\qquad$其中$P_l(x)$–被截断多项式$\qquad Q_l(x)$–无穷级数

勒氏多项式还可写成$P_l(x)=\frac{(2l-1)(2l-3)\cdots1}{l!}\cdot[x^l-\frac12\frac{l(l-1)}{(2l-1)}x^{l-2}+\frac{1}{2\cdot4}\frac{l(l-1)(l-2)(l-3)}{(2l-1)(2l-3)}x^{l-4}]-\cdots$

微分表示(罗德里格斯公式)$P_l(x)=\frac{1}{2^ll!}\frac{d^l}{dx^l}(x^2-1)^l$

积分表示$P_l(x)=\frac{1}{2\pi}\int_0^\pi(x+\sqrt{x^2-1}\cos\phi)^ld\phi$

对施列夫利积分$P_l(x)=\frac{1}{2^l2\pi i}\oint_C\frac{(\xi^2-1)^l}{(\xi-x)^{l+1}}d\xi$取C为$\xi=x+\sqrt{x^2-1}e^{i\phi}$

生成函数$\frac{1}{\sqrt{1-2rx+r^2}}=\begin{cases}\sum_{i=0}^\infty r^lP_l(x) & |r|<1\\ \sum_{l=0}^\infty\frac{1}{r^{l+1}}P_l(x) & r>1\end{cases}$

递推公式对$n=1,2,3,\cdots$有$(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x)$

$P_n(x)=P_{n+1}'(x)-2xP_n'(x)+P_{n-1}'(x)$

$xP_n'(x)-P_{n-1}'(x)=nP_n(x)$

$P_{n+1}'(x)-P_{n-1}'(x)=(2n+1)P_n(x)$

$nP_{n+1}'(x)+(n+1)P_{n-1}'(x)=(2n+1)xP_n'(x)$

对$n=0,1,2,\cdots$有$P_{n+1}'(x)=(n+1)P_n(x)+xP_x'(x)$

莱布尼兹乘积微分法$\frac{d^n}{dx^n}(f\cdot g)=\sum_{k=0}^n\frac{n!}{(n-k)!k!}\frac{d^kf}{dx^k}\frac{d^{n-k}g}{dx^{n-k}}$

$P_l(1)=1\quad P_l'(1)=\frac{l(l+1)}{2}\quad P_l(-1)=(-1)^l\quad P_l'(-1)=(-1)^{l-1}\frac{l(l+1)}{2}$

$P_{2n}(0)=(-1)^n\frac{(2n)!}{2^{2n}(n!)^2}\quad P_{2n+1}(0)=0$

$P_{2n}(0)-P_{2n+2}(0)=(-1)^n(\frac{4n+3}{n+1})\frac{(2n)!}{2^{2n+1}(n!)^2}$

$P_l(x)$与$k<l$次多项式正交$\int_{-1}^1f(x)P_l(x)dx=0$

正交性&模值$\int_{-1}^1P_l(x)P_m(x)=\frac{2}{2l+1}\delta_{lm}$

对$l\neq m$有$\int_x^1P_l(x)P_m(x)dx=-\int_{-1}^x=\frac{(1-x^2)[P_l(x)P_m'(x)-P_m(x)P_l'(x)]}{m(m+1)-l(l+1)}$

完备性对在$[-1,1]$上(分段)光滑的$f(x)$按勒让德多项式级数展开

$\left.\begin{matrix}连续点 & f(x)\\ 间断点 & \frac{f(x-0)+f(x+0)}{2}\end{matrix}\right\}=\sum_{l=0}^\infty C_lP_l(x)$其中$C_l=\frac{2l+1}{2}\int_{-1}^1P_l(x)f(x)dx$

连带勒让德方程$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta})+[l(l+1)-\frac{m^2}{\sin^2\theta}]\Theta=0\quad m=0,1,2,\cdots\quad l\in\mathbb{R}$

令$x=\cos\theta$化为$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+[l(l+1)-\frac{m^2}{1-x^2}]y\quad$其通解为

连带勒让德函数 $P_l^m(x)=(1-x^2)^{m/2}P_l^{(m)}(x)\quad m=0,\pm1,\pm2,\cdots,\pm l$

微分表示(罗德里格斯公式) $P_l^m(x)=\frac{(1-x^2)^{\frac{m}{2}}}{2^l l!}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l$

$P_l^{-m}(x)=(-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x)=\frac{(1-x^2)^{-m/2}}{2^l l!}\frac{d^{l-m}}{dx^{l-m}}(x^2-1)^l$

或 $\Theta(\theta)=P_l^m(\cos\theta)\quad m=0,\pm1,\pm2,\cdots,\pm l$ 当 $m>l$，$P_l^m(x)=0$

低阶连带勒氏函数

$$\begin{cases} P_l^0(x)=P_l(x) & \\ P_1^1(x)=\sqrt{1-x^2} & P_1^{-1}(x)=-\frac{1}{2}\sqrt{1-x^2}\\ P_2^1(x)=3x\sqrt{1-x^2} & P_2^{-1}(x)=-\frac{1}{2}x\sqrt{1-x^2}\\ P_2^2(x)=3(1-x^2) & P_2^{-2}(x)=\frac{1}{8}(1-x^2)\\ P_3^1(x)=\frac{3}{2}(1-x^2)^{\frac{1}{2}}(5x^2-1) & P_3^{-1}(x)=-\frac{1}{12}P_3^1(x)\\ P_3^2(x)=15(1-x^2)x & P_3^{-2}(x)=\frac{1}{8}(1-x^2)x\\ P_3^3(x)=15(1-x^2)^{\frac{3}{2}} & P_3^{-3}(x)=-\frac{1}{48}(1-x^2)^{\frac{3}{2}} \end{cases}$$

积分表示(施列夫利积分) $P_l^m(x)=\frac{(1-x^2)^{\frac{m}{2}}}{2^l}\frac{1}{2\pi i}\frac{(l+m)!}{l!}\oint_C\frac{(z^2-1)^l}{(z-x)^{l+m+1}}dz$

其中C为围绕$z=x$的任一闭合回路，取C为半径为$\sqrt{x^2-1}$的圆周得

拉普拉斯积分 $P_l^m(x)=\frac{i^m}{2\pi}\frac{(l+m)!}{l!}\int_{-\pi}^{\pi}e^{-im\psi}[\cos\theta+i\sin\theta\cos\psi]^l d\psi$

递推公式

$$\begin{cases} (2k+1)xP_k^m(x)=(k+m)P_{k-1}^m(x)+(k-m+1)P_{k+1}^m(x)\\ (2k+1)(1-x^2)^{1/2}P_k^m(x)=P_{k+1}^{m+1}(x)-P_{k-1}^{m+1}(x)\\ (2k+1)(1-x^2)^{1/2}P_k^m(x)=(k+m)(k+m-1)P_{k-1}^{m-1}(x)\\ \quad-(k-m+2)(k-m+1)P_{k+1}^{m-1}(x)\\ (2k+1)(1-x^2)\frac{dP_k^m(x)}{dx}=(k+1)(k+m)P_{k-1}^m(x)\\ \quad-k(k-m+1)P_{k+1}^m(x) \end{cases}\quad k\ge1$$

正交性&模值 $\int_{-1}^{1}P_l^m(x)P_k^m(x)=\frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{lk}$

完备性对非负整数m，区间$[-1,1]$上分段光滑函数$f(x)$可作连带勒让德级数展开

$f(x)=\sum_{l=m}^{\infty}C_lP_l^m(x)\qquad$ 其中 $C_l=\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}\int_{-1}^{1}f(x)P_l^m(x)dx$

球谐函数 $Y_l^m(\theta,\phi)=P_l^{|m|}(\cos\phi)e^{im\phi}\begin{cases} l=0,1,2,\cdots\\ m=0,\pm1,\pm2,\cdots,\pm l\end{cases}$

是方程$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial Y}{\partial\theta})+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}+l(l+1)Y=0$的通解

正交性&模值 $\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}Y_l^m(\theta,\phi)Y_k^n(\theta,\phi)\sin\theta d\theta d\phi=\frac{4\pi}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{mn}\delta_{lk}$

低阶归一化球谐函数

$$\begin{cases} Y_{0,0}(\theta,\phi)=\frac{1}{\sqrt{4\pi}} & Y_{2,\pm2}=\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm2i\phi}\\ Y_{1,0}=\sqrt{\frac{3}{4\pi}}\cos\theta & Y_{1,\pm1}=\mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}\\ Y_{2,0}=\sqrt{\frac{5}{16\pi}}(3\cos^2\theta-1) & Y_{2,\pm1}=\mp\sqrt{\frac{15}{8\pi}}\cos\theta\sin\theta e^{\pm i\phi} \end{cases}$$

完备性定义在球面上函数$f(\theta,\phi)$可展开为$f(\theta,\phi)=\sum_{l=1}^{m}\sum_{m=-l}^{l}C_l^mP_l^{|m|}(\cos\theta)e^{im\phi}=\sum_{m=0}^{\infty}\sum_{l=-m}^{m}C_l^mP_l^{|m|}(\cos\theta)e^{im\phi}$

其中$C_l^m=\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}\int_0^{\pi}\int_0^{2\pi}f(\theta,\phi)P_l^{|m|}(\cos\theta)e^{-im\phi}\sin\theta d\theta d\phi$

球谐函数另一种定义 $Y_l^m(\theta,\phi)=P_l^m(\cos\theta)\begin{Bmatrix}\sin m\phi\\ \cos m\phi\end{Bmatrix}\begin{pmatrix} m=0,1,2,\cdots,l\\ l=0,1,2,\cdots\end{pmatrix}$

正交性&模值 $\int_{-1}^{1}P_l^m(x)P_k^m(x)dx=\frac{(l+m)!}{(l-m)!}\frac{2}{2l+1}\delta_{mn}\delta lk$

完备性定义在球面上函数$f(\theta,\phi)$先对ϕ做傅里叶级数展开

$f(\theta,\phi)=\sum_{m=0}^{\infty}[A_m(\theta)\cos m\phi+B_m(\theta)\sin m\phi]$

其中 $\begin{cases} A_m(\theta)=\frac{1}{\delta_m}\int_0^{2\pi}f(\theta,\phi)\cos m\phi d\phi\\ B_m(\theta)=\frac{1}{\pi}\int_0^{2\pi}f(\theta,\phi)\sin m\phi d\phi\end{cases}$，$\delta_m=\begin{cases}2 & m=0\\ 1 & m=1,2,3,\cdots\end{cases}$

再对A_m和B_m用$P_l^m(\cos\theta)$展开得 $\begin{cases} A_m(\theta)=\sum_{l=m}^{\infty}A_l^mP_l^m(\cos\theta)\\ B_m(\theta)=\sum_{l=m}^{\infty}B_l^mP_l^m(\cos\theta)\end{cases}$

其中

$$\begin{cases} A_l^m=\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}\int_0^{\pi}A_m(\theta)P_l^m(\cos\theta)\sin\theta d\theta\\ \quad=\frac{2l+1}{2\pi\delta_m}\frac{(l-m)!}{(l+m)!}\int_0^{\pi}\int_0^{2\pi}f(\theta,\phi)P_l^m(\cos\theta)\cos m\phi\sin\theta d\theta\\ B_l^m=\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}\int_0^{\pi}B_m(\theta)P_l^m(\cos\theta)\sin\theta d\theta\\ \quad=\frac{2l+1}{2\pi}\frac{(l-m)!}{(l+m)!}\int_0^{\pi}\int_0^{2\pi}f(\theta,\phi)P_l^m(\cos\theta)\sin m\phi\sin\theta d\theta d\phi \end{cases}$$

旋度 $\nabla\times\boldsymbol{E}=[\boldsymbol{i}\,\boldsymbol{j}\,\boldsymbol{k};\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial z};E_x\,E_y\,E_x]=(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z})\boldsymbol{i}+(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x})\boldsymbol{j}+(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y})\boldsymbol{k}$

标量场的梯度无旋 $\nabla\times\nabla\phi=0\qquad$ 矢量场的旋度无源 $\nabla\times\nabla\times\boldsymbol{f}=0$

$\nabla(\phi\psi)=\phi\nabla\psi+\psi\nabla\phi\qquad\nabla\cdot(\phi\boldsymbol{f})=(\nabla\phi)\cdot\boldsymbol{f}+\phi\nabla\cdot\boldsymbol{f}$

$\nabla\times(\phi\boldsymbol{f})=(\nabla\phi)\times\boldsymbol{f}+\phi\nabla\times\boldsymbol{f}\qquad\nabla\cdot(\boldsymbol{f}\times\boldsymbol{g})=(\nabla\times\boldsymbol{f})\cdot\boldsymbol{g}-\boldsymbol{f}\cdot(\nabla\times\boldsymbol{g})$

$\nabla\times(\boldsymbol{f}\times\boldsymbol{g})=(\boldsymbol{g}\cdot\nabla)\boldsymbol{f}+(\nabla\cdot\boldsymbol{g})\boldsymbol{f}-(\boldsymbol{f}\cdot\nabla)\boldsymbol{g}-(\nabla\cdot\boldsymbol{f})\boldsymbol{g}$

$\nabla(\boldsymbol{f}\cdot\boldsymbol{g})=\boldsymbol{f}\times(\nabla\times\boldsymbol{g})+(\boldsymbol{f}\cdot\nabla)\boldsymbol{g}+\boldsymbol{g}\times(\nabla\times\boldsymbol{f})+(\boldsymbol{g}\cdot\nabla)\boldsymbol{f}$

$\nabla\times(\nabla\times\boldsymbol{f})=\nabla(\nabla\cdot\boldsymbol{f})-\nabla^2\boldsymbol{f}\qquad\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c})-\boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$

高斯定理 $\iint_{\partial V}\boldsymbol{E}\cdot d\boldsymbol{S}=\iiint\nabla\cdot\boldsymbol{E}dV\qquad$ **格林定理** $\nabla\cdot(u\nabla v)=u\nabla\cdot\nabla v+(\nabla u)\cdot(\nabla v)$

斯多克斯定理 $\oint_{\partial S}\boldsymbol{E}\cdot d\boldsymbol{l}=\iint_S(\nabla\times\boldsymbol{E})\cdot d\boldsymbol{S}$

	直角坐标系	柱坐标系	球坐标系
直角		$x=\rho\cos\phi,y=\rho\sin\phi,z$	$x=r\sin\theta\cos\phi,y=r\sin\theta\sin\phi,z=r\cos\theta$
柱	$\rho=\sqrt{x^2+y^2},\phi=\arctan(y/x),z=z$		$\rho=r\sin\theta,\phi,z=r\cos\theta$
球	$r=\sqrt{x^2+y^2+z^2},\theta=\arccos(z/r),\phi=\arctan(y/x)$	$r=\sqrt{\rho^2+z^2},\theta=\arctan(\rho/z),\phi$	
矢量$\boldsymbol{A}$	$A_x\hat{x}+A_y\hat{y}+A_z\hat{z}$	$A_\rho\hat{\rho}+A_\phi\hat{\phi}+A_z\hat{z}$	$A_r\hat{r}+A_\theta\hat{\theta}+A_\phi\hat{\phi}$
梯度∇f	$\frac{\partial f}{\partial x}\hat{x}+\frac{\partial f}{\partial y}\hat{y}+\frac{\partial f}{\partial z}\hat{z}$	$\frac{\partial f}{\partial\rho}\hat{\rho}+\frac{1}{\rho}\frac{\partial f}{\partial\phi}\hat{\phi}+\frac{\partial f}{\partial z}\hat{z}$	$\frac{\partial f}{\partial r}\hat{r}+\frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta}+\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\phi}$
散度$\nabla\cdot\boldsymbol{A}$	$\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho}+\frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi}+\frac{\partial A_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$
旋度$\nabla\times\boldsymbol{A}$	$(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z})\hat{x}+(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x})\hat{y}$ $+(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y})\hat{z}$	$(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z})\hat{\rho}+(\frac{\partial A_\rho}{\partial z}-\frac{\partial A_z}{\partial\rho})\hat{\phi}$ $+\frac{1}{\rho}(\frac{\partial(\rho A_\phi)}{\partial\rho}-\frac{\partial A_\rho}{\partial\phi})\hat{z}$	$\frac{1}{r\sin\theta}(\frac{\partial(A_\phi\sin\theta)}{\partial\theta}-\frac{\partial A_\theta}{\partial\phi})\hat{r}+\frac{1}{r}(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{\partial(rA_\phi)}{\partial r})\hat{\theta}$ $+\frac{1}{r}(\frac{\partial(rA_\theta)}{\partial r}-\frac{\partial A_r}{\partial\theta})\hat{\phi}$
拉普拉斯算子∇^2	$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial u}{\partial\rho})+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$

泛定方程$\frac{\partial^2u}{\partial t^2}=a^2\frac{\partial^2u}{\partial x^2}$和$\frac{\partial u}{\partial t}=a^2\frac{\partial^2u}{\partial x^2}$(分离变量得方程$X''(x)+\lambda X(x)=0$)的常用本征函数及其本征值

边界条件	本征值λ_n	本征函数
$u\|_{x=0}=0,u\|_{x=L}=0$	$(\frac{n\pi}{L})^2$	$B_n\sin\frac{n\pi}{L}x,n=1,2,3\cdots$
$u\|_{x=0}=0,u_x\|_{x=L}=0$	$[\frac{(2n+1)\pi}{2L}]^2$	$B_n\sin\frac{(2n+1)\pi}{2L}x,n=0,1,2,\cdots$
$u_x\|_{x=0}=0,u\|_{x=L}=0$	$[\frac{(2n+1)\pi}{2L}]^2$	$A_n\cos\frac{(2n+1)\pi}{2L}x,n=0,1,2,\cdots$
$u_x\|_{x=0}=0,u_x\|_{x=L}=0$	$[\frac{n\pi}{L}]^2$	$A_n\cos\frac{n\pi}{L}x,n=0,1,2,\cdots$
$u\|_{x=0}=0,[u+u_x]\|_{x=L}=0$	μ_n^2, s.t. $\tan\mu_nL=-\mu_n$	$\sin\mu_nx,n=1,2,\cdots$
$u_x\|_{x=0}=0,[u+u_x]\|_{x=L}=0$	μ_n^2, s.t. $\cot\mu_nL=\mu_n$	$\cos\mu_nx,n=1,2,\cdots$
$u\|_{x=0}=0,[u-u_x/2]\|_{x=L}=0$	$-\mu_0^2,\mu_n^2$, s.t. $\tanh\mu_0L=\mu_0/2,\tan\mu_nL=\mu_n/2$	$\sinh\mu_0x,\sinh\mu_nx,n=1,2,\cdots$
$u_x\|_{x=0}=0,[u-u_x/2]\|_{x=L}=0$	$-\mu_0^2,\mu_n^2$, s.t. $\coth\mu_0L=\mu_0/2,\cot\mu_nL=-\mu_n/2$	$\cosh\mu_0x,\cos\mu_nx,n=1,2,\cdots$
$[u+u_x]\|_{x=0}=0,[u+u_x]\|_{x=L}=0$	$-1,(\frac{n\pi}{L})^2$	$e^{-x},\frac{n\pi}{L}\cos\frac{n\pi}{L}x-\sin\frac{n\pi}{L}x,n=1,2,\cdots$
$[u-u_x]\|_{x=0}=0,[u-u_x]\|_{x=L}=0$	$-1,(\frac{n\pi}{L})^2$	$e^{x},\frac{n\pi}{L}\cos\frac{n\pi}{L}x+\sin\frac{n\pi}{L}x,n=1,2,\cdots$
$[u-u_x]\|_{x=0}=0,[u+u_x]\|_{x=L}=0$	μ_n^2, s.t. $\tan\mu_n=\frac{2\mu_n}{\mu_n^2-1}$	$\mu_n\cos\mu_nx+\sin\mu_nx,n=1,2,\cdots$
$[u+u_x]\|_{x=0}=0,[u-u_x]\|_{x=L}=0$	$-\mu_0^2,\mu_n^2$, s.t. $\tan\mu_0=\frac{2\mu_0}{\mu_0^2-1},\tan\mu_n=-\frac{2\mu_n}{\mu_n^2-1}$	$\mu_0\cosh\mu_0x-\sinh\mu_0x,\mu_n\cos\mu_nx-\sin\mu_nx,n=1,\cdots$

非齐次边界条件	非齐次边界条件辅助函数Ω
$u\|_{x=0}=u_1(t),u\|_{x=L}=u_2(t)$	$[u_2(t)-u_1(t)]/L+u_1(t)$
$u\|_{x=0}=u_1(t),u_x\|_{x=L}=u_2(t)$	$u_2(t)x+u_1(t)$
$u_x\|_{x=0}=u_1(t),u\|_{x=L}=u_2(t)$	$u_1(t)x+u_2(t)-Lu_1(t)$
$u_x\|_{x=0}=u_1(t),u_x\|_{x=L}=u_2(t)$	$[u_2(t)x-u_1(t)]/(2L)+u_1(t)x$

傅里叶变换表		拉普拉斯变换表	
$f(t)=\mathcal{F}^{-1}[F(\omega)]$	$F(\omega)=\mathcal{F}[f(t)]$	$f(t)=\mathcal{L}^{-1}[F(p)]$	$F(p)=\mathcal{L}[f(t)]$
矩形脉冲$\begin{cases}E & \|t\|\le\frac{\tau}{2}\\ 0 & \text{otherwise}\end{cases}$	$\begin{cases}2E\frac{\sin\frac{\omega\tau}{2}}{\omega} & \omega\ne0\\ E\tau & \omega=0\end{cases}$	单位阶跃$u(t)$	$\frac{1}{p}$
半边指数衰减$\begin{cases}0 & t<0\\ e^{-\beta t} & t\ge0\end{cases}$	$\frac{1}{\beta+i\omega}$	$\frac{t^m}{\Gamma(m+1)},m>-1$	$\frac{1}{p^{m+1}}$
双边指数衰减$Ee^{-a\|t\|}$	$\frac{2aE}{a^2+\omega^2}$	$\frac{t^n}{n!},n=0,1,2,\cdots$	$\frac{1}{p^{n+1}}$
钟形脉冲$Ae^{-\beta t^2}$	$\sqrt{\frac{\pi}{\beta}}Ae^{-\frac{\omega^2}{4\beta}}$	e^{kt}	$\frac{1}{p-k}$
傅里叶核$\frac{\sin\omega_0t}{\pi t}$	$\begin{cases}1 & \|\omega\|\le\omega_0\\ 0 & \text{otherwise}\end{cases}$	$\sin\beta t$	$\frac{\beta}{p^2+\beta^2}$
高斯分布$\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{t^2}{2\sigma^2}}$	$e^{-\frac{\sigma^2\omega^2}{2}}$	$\cos\beta t$	$\frac{p}{p^2+\beta^2}$
单位脉冲$\delta(t)$	1	$\sinh\beta t$	$\frac{\beta}{p^2-\beta^2}$
余弦$\cos\omega_0t$	$\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$	$\cosh\beta t$	$\frac{p}{p^2-\beta^2}$
正弦$\sin\omega_0t$	$i\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	单位脉冲$\delta(t)$	1
单位阶跃$u(t)$	$\frac{1}{i\omega}+\pi\delta(\omega)$	$\delta(t-a)$	e^{-ap}
直流E	$2\pi E\delta(\omega)$	$\delta'(t)$	p
指数$e^{i\omega_0t}$	$2\pi\delta(\omega-\omega_0)$	$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{p}}$
t	$2i\pi\delta'(\omega)$	$2\sqrt{\frac{t}{\pi}}$	$\frac{1}{p\sqrt{p}}$
t^n	$2i^n\pi\delta^{(n)}(\omega)$	t	$\frac{1}{p^2}$
$\frac{1}{a^2+t^2}$	$\frac{\pi}{a}e^{-a\|\omega\|}$	$a^{t/\tau}$	$\frac{1}{p-(\ln a)/\tau}$