PHYS2202 Nonlinear Optics

Problem Set 1 Solutions

To spare ourselves having to write a factor of m^{-1} with each factor of v_a and v_b, we will rewrite the potential energy as in lecture/notes:

$$U(x) = \frac{1}{2}m\omega_0^2 x^2 + \frac{1}{3}mv_a x^3 + \frac{1}{4}mv_b x^4. \tag{1}$$

The problem was stated with the factors of m absorbed into the parameters v_a and v_b. At the end, the result derived using the form Eq. 1, we will just have to replace each v_a and v_b by v_a/m and v_b/m, respectively.

The easiest approach is to use Euler notation to represent all fields. The input electric field is

$$\begin{aligned} E(x=0,t) &= E_1\cos(\omega_1 t) + E_2\cos(\omega_2 t) + E_3\cos(\omega_3 t) \\ &= \frac{1}{2}E_1\left(e^{-i\omega_1 t} + e^{i\omega_1 t}\right) + \frac{1}{2}E_2\left(e^{-i\omega_2 t} + e^{i\omega_2 t}\right) + \frac{1}{2}E_3\left(e^{-i\omega_3 t} + e^{i\omega_3 t}\right) \end{aligned} \tag{2}$$

To save having to write more factors of 1/2 than necessary, I will write the electric field as

$$E(x=0,t) = \tilde{E}_1\cos\left(e^{-i\omega_1 t} + e^{i\omega_1 t}\right) + \tilde{E}_2\left(e^{-i\omega_2 t} + e^{i\omega_2 t}\right) + \tilde{E}\left(e^{-i\omega_3 t} + e^{i\omega_3 t}\right),$$

where $\tilde{E}_i = \frac{1}{2}E_i$. To be consistent with the original form of the electric field, it must be the case that $\tilde{E}_i = E_i/2$. We can express the final answer in terms of E_i by this substitution. If we are interested in the response (position) of the oscillator at frequency $2\omega_2 - \omega_1$, that means that we are looking at that part of $x(t)$ that goes as $e^{-i(2\omega_2-\omega_1)t}$.

We saw in lecture/notes that the third-order equation of motion is

$$\frac{d^2}{dt^2}x^{(3)}(t) + 2\Gamma\frac{d}{dt}x^{(3)}(t) + \omega_0^2 x^{(3)}(t) + 2v_a x^{(2)}(t)x^{(1)}(t) + v_b\left[x^{(1)}(t)\right]^3 = 0, \tag{3}$$

where the factor of 2 in before the v_a is due to the cascaded third order process arising from a second-order mixing of second- and first- order terms

$$v_a[x(t)]^2 = v_a\left[x^{(1)}(t) + x^{(2)}(t) + \cdots\right]^2 = v_a\left[x^{(1)}(t)\right]^2 + 2v_a x^{(1)}(t)x^{(2)}(t) + \cdots. \tag{4}$$

Writing the n^{th} order response as

$$x^{(n)}(t) = \sum_m\left(\tilde{x}^{(n)}_{\omega_m}e^{-i\omega_m t} + \tilde{x}^{*(n)}_{\omega_m}e^{i\omega_m t}\right) = \sum_m\left(\tilde{x}^{(n)}_{\omega_m}e^{-i\omega_m t} + \tilde{x}^{(n)}_{-\omega_m}e^{i\omega_m t}\right), \tag{5}$$

(where $\tilde{x}^{(n)}_{-\omega_m} = \tilde{x}^{*(n)}_{\omega_m}$) and matching terms, it should be clear that the third order response at $2\omega_2 - \omega_1$ can arise in the direct process through the $\tilde{x}^{(1)}_{\omega_2}\tilde{x}^{(1)}_{\omega_2}\tilde{x}^{*(1)}_{-\omega_1}$ term or the cascaded process through the terms associated with $\tilde{x}^{(2)}_{2\omega_2}\tilde{x}^{*(1)}_{\omega_1}$ or $\tilde{x}^{(2)}_{\omega_2-\omega_1}\tilde{x}^{(1)}_{\omega_2}$. No other unique combinations of cascaded processes produce a response at the desired frequency. For the frequency of interest, Eq. 3 then reduces to

$$\mathscr{D}(2\omega_2-\omega_1)\,\tilde{x}^{(3)}_{2\omega_2-\omega_1} = -2v_a\left[g_{a1}\tilde{x}^{(2)}_{2\omega_2}\tilde{x}^{*(1)}_{\omega_1} + g_{a2}\tilde{x}^{(2)}_{\omega_2-\omega_1}\tilde{x}^{(1)}_{\omega_2}\right] - v_b g_b\tilde{x}^{(1)}_{\omega_2}\tilde{x}^{(1)}_{\omega_2}\tilde{x}^{*(1)}_{-\omega_1}, \tag{6}$$

where the various g are numerical factors that we need to figure out.

Note that the ω_3 term in the input electric field is irrelevant for us at third order. If we are looking at a signal at $2\omega_2 - \omega_1$, the field at frequency ω_3 will only contribute to the n^{th} order response for $n \geq 5$ (we need at least 2 extra orders so that a factor of the field at frequency ω_3 is paired with its complex conjugate associated with frequency $-\omega_3$).

We have already determined $\tilde{x}^{(1)}_{\omega_i}$, $\tilde{x}^{(2)}_{2\omega_2}$, and $\tilde{x}^{(2)}_{\omega_2-\omega_1}$ in lecture:

$$\tilde{x}^{(1)}_{\omega_j} = \frac{q}{m}\frac{\tilde{E}_{\omega_j}}{\mathscr{D}(\omega_j)} = -\frac{e}{m}\frac{\tilde{E}_{\omega_j}}{\mathscr{D}(\omega_j)} \tag{7}$$

$$\tilde{x}^{(2)}_{\omega_2-\omega_1} = -2v_a\frac{e^2}{m^2}\frac{\tilde{E}_{\omega_2}\tilde{E}_{-\omega_1}}{\mathscr{D}(\omega_2-\omega_1)\mathscr{D}(\omega_2)\mathscr{D}(-\omega_1)} \tag{8}$$

$$\tilde{x}^{(2)}_{2\omega_2} = -v_a\frac{e^2}{m^2}\frac{\tilde{E}^2_{\omega_2}}{\mathscr{D}(2\omega_2)\mathscr{D}^2(\omega_2)}, \tag{9}$$

where $\mathscr{D}(\omega) = \omega_0^2 - \omega^2 - i2\Gamma\omega$. The factor of 2 in $\tilde{x}^{(2)}_{\omega_2-\omega_1}$ is just due to the factor of 2 associated with cross terms in the binomial expansion for $\left(\tilde{x}^{(1)}_{\omega_2} + \tilde{x}^{(1)}_{-\omega_1}\right)^2$.

The only thing that was not basically already done in the lecture or notes is counting how many ways we get (i.e., how many cross terms are associated with) $\tilde{x}^{(1)}_{\omega_2}\tilde{x}^{(1)}_{\omega_2}\tilde{x}^{(1)}_{-\omega_1}$ in the direct process and how many ways we get $\tilde{x}^{(2)}_{2\omega_2}\tilde{x}^{*(1)}_{\omega_1}$ and $\tilde{x}^{(2)}_{\omega_2-\omega_1}\tilde{x}^{(1)}_{\omega_2}$ in the cascaded process. This is just determined by the binomial expansion. For the direct term, we are interested in $\left(\tilde{x}^{(1)}_{\omega_2} + \tilde{x}^{(1)}_{-\omega_1}\right)^3$. By the binomial expansion there are 3 ways to get $\tilde{x}^{(1)}_{\omega_2}\tilde{x}^{(1)}_{\omega_2}\tilde{x}^{(1)}_{-\omega_1}$. Therefore, we expect a contribution to the response that looks like

$$\begin{aligned}\tilde{x}^{(3),\text{direct}}_{2\omega_2-\omega_1} &= -v_b\frac{3}{\mathscr{D}(2\omega_2-\omega_1)}\tilde{x}^{(1)}_{\omega_2}\tilde{x}^{(1)}_{\omega_2}\tilde{x}^{(1)}_{-\omega_1} = -3v_b\frac{q^3}{m^3}\frac{\tilde{E}^2_{\omega_2}\tilde{E}_{-\omega_1}}{\mathscr{D}(2\omega_2-\omega_1)\mathscr{D}^2(\omega_2)\mathscr{D}(-\omega_1)} \\ &= 3v_b\frac{e^3}{m^3}\frac{\tilde{E}^2_{\omega_2}\tilde{E}_{-\omega_1}}{\mathscr{D}(2\omega_2-\omega_1)\mathscr{D}^2(\omega_2)\mathscr{D}(-\omega_1)}.\end{aligned} \tag{10}$$

The total third order response at $2\omega_2 - \omega_1$ is then

$$\begin{aligned}\tilde{x}^{(3)}_{2\omega_2-\omega_1} &= -2v_a\left\{\left[-2v_a\frac{e^2}{m^2}\frac{\tilde{E}_{\omega_2}\tilde{E}_{-\omega_1}}{\mathscr{D}(\omega_2-\omega_1)\mathscr{D}(\omega_2)\mathscr{D}(-\omega_1)}\right]\left[-\frac{e}{m}\frac{\tilde{E}_{\omega_2}}{\mathscr{D}(\omega_2)}\right]\right. \\ &\qquad\left. + \left[-v_a\frac{e^2}{m^2}\frac{\tilde{E}^2_{\omega_2}}{\mathscr{D}(2\omega_2)\mathscr{D}^2(\omega_2)\mathscr{D}}\right]\left[-\frac{e}{m}\frac{\tilde{E}_{-\omega_1}}{\mathscr{D}(-\omega_1)}\right]\right\} \\ &\quad + 3v_b\frac{e^3}{m^3}\frac{\tilde{E}^2_{\omega_2}\tilde{E}_{-\omega_1}}{\mathscr{D}(2\omega_2-\omega_1)\mathscr{D}^2(\omega_2)\mathscr{D}(-\omega_1)} \\ &= \frac{e^3}{m^3}\frac{\tilde{E}^2_{\omega_2}\tilde{E}_{-\omega_1}}{\mathscr{D}(2\omega_2-\omega_1)\mathscr{D}^2(\omega_2)\mathscr{D}(-\omega_1)}\left\{3v_b - 2v_a^2\left[2\frac{1}{\mathscr{D}(\omega_2-\omega_1)} + \frac{1}{\mathscr{D}(2\omega_2)}\right]\right\}.\end{aligned} \tag{11}$$

If we were to use the original formulation of the potential energy, we would have to make the replacements $v_a \to v_a/m$ and $v_b \to v_b/m$ in our final result. Likewise, in terms of the electric field as written in the problem, we see that $\tilde{E}_{-\omega_n} = \tilde{E}_{\omega_n} = E_i/2$. Accounting for all of this we can write

$$\tilde{x}^{(3)}_{2\omega_2-\omega_1} = \frac{1}{8}\frac{e^3}{m^4}\frac{E_2^2E_1}{\mathscr{D}(2\omega_2-\omega_1)\mathscr{D}^2(\omega_2)\mathscr{D}(-\omega_1)}\left\{3v_b - \frac{2}{m}v_a^2\left[2\frac{1}{\mathscr{D}(\omega_2-\omega_1)} + \frac{1}{\mathscr{D}(2\omega_2)}\right]\right\}. \tag{12}$$

We could have seen this at the beginning. For a third order process, we must have that many interactions with the input field, which means we will have a common factor of $\frac{e^3}{m^3}$. For the frequency of interest, we must only see the combination of electric fields $\tilde{E}_{\omega_2}^2 \tilde{E}_{-\omega_1}$. For every input electric field and output electric field, there will be a resonant denominator. Since our input fields must be at ω_2, ω_2, and $-\omega_1$ and our output field at $2\omega_2 - \omega_1$, all terms must have a factor of $\left[\mathscr{D}\left(2\omega_2 - \omega_1\right) \mathscr{D}^2\left(\omega_2\right) \mathscr{D}\left(-\omega_1\right)\right]^{-1}$. For the direct process there must be the third order interaction parameter v_b. The only thing that we might not immediately see at the beginning for the direct process is the factor g_b, which is 3 in this case. For the indirect processes, we will have a factor of v_a^2 (in this case), since the cascade only involves two steps (the initial second-order interaction and then the single extra step in the cascade, though at higher order, such as fifth order, there would be yet more terms). Moreover, there must be a resonant denominator for each extra step in the cascade with the resonant denominator involving the frequency of the first nonlinear step: here $\omega_2 - \omega_1$ and $2\omega_2$. The only thing that might not be immediately obvious at the beginning is the value of the g_a factors.