PHYS2202 Nonlinear Optics Problem Set 3

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前置知识（Pre knowledge）

一.相互作用表象与含时微扰(Interaction representation and time-dependent perturbation）

考虑薛定谔方程（Consider Schrodinger equation）：

$$i\hbar\frac{\partial}{\partial t}|\varphi,t\rangle=(\hat{H}+\hat{V}(t))|\varphi,t\rangle$$

我们可以令算符承担$\hat{H}$产生的演化（We can make operators take on the evolution of the generation of $\hat{H}$）：

$$i\hbar\frac{\partial}{\partial t}|\varphi,t\rangle_I=\hat{V}_I(t)|\varphi,t\rangle_I$$

作为一个状态$|\varphi,t\rangle$，我们更关心它的演化方式（特别是$\hat{V}$导致的部分），将演化部分独立是一个很自然的决定（As a state $|\varphi,t\rangle$ we are more concerned about the way it evolves (especially the part caused by $\hat{V}$). It is a natural decision to make the evolutionary part independent）：

$$|\varphi,t\rangle=\hat{U}(t,t_0)|\varphi,t_0\rangle$$

初始状态（Initial state）：

$$\hat{U}(t_0,t_0)=1$$

则我们可以将薛定谔方程改写为(作为算符$\hat{U}$也会相似变换为相互作用表象的版本)（Then we can rewrite the Schrodinger equation as (as the operator, $\hat{U}$ will similarly transform to the version of interaction representation)）：

$$i\hbar\frac{\partial}{\partial t}\hat{U}_I(t,t_0)=\hat{V}_I(t)\hat{U}_I(t,t_0)$$

可以得到积分表达式（We can get the integral expression）：

$$\hat{U}_I(t,t_0)=1+\frac{1}{i\hbar}\int_{t_0}^{t}dt_1\hat{V}_I(t_1)\hat{U}_I(t_1,t_0)=1+\frac{1}{i\hbar}\int_{t_0}^{t}dt_1\hat{V}_I(t_1)+\frac{-1}{\hbar^2}\int_{t_0}^{t}dt_1\hat{V}_I(t_1)\int_{t_0}^{t_1}dt_2\hat{V}_I(t_2)+\cdots$$

启动时间与终止时间本质上来讲相当于引入了一个矩形的脉冲调制，所以我们通常采用$\pm\infty$来排除干扰（In essence, the start time and the end time are equivalent to the introduction of a rectangular pulse modulation, so we usually use $\pm\infty$ eliminate interference）。

通常来讲，我们能够将出现 1 个 V 解释为产生一次碰撞(Generally speaking, we can interpret the occurrence of a V as a collision)。

二．密度矩阵（density matrix）：

在讨论密度矩阵理论前，我们需要思考，我们如何描述多个态，无论是单态$|\varphi,t\rangle$还是$a|\varphi_1,t\rangle+b|\varphi_2,t\rangle$或者任何其他的描述，我们描述的都是单个态。很明显这种记法丢失了一些描述能力，$a|\varphi_1,t\rangle+b|\varphi_2,t\rangle$形式的约束太强了，我们可以放弃一些对 a，b 的约束，例如（Before we discuss the theory of density matrix, we need to think about how to describe multiple states, whether it's a single state $|\varphi,t\rangle$ or a single state $a|\varphi_1,t\rangle+b|\varphi_2,t\rangle$ It's just a single state. It's obvious that this notation has lost some descriptive power, This form（$a|\varphi_1,t\rangle+b|\varphi_2,t\rangle$）of constraint is too strong. For example, we can give up some constraints on a and b）：

$$\left(\frac{1}{2}\right)\Big|\varphi_1,t\rangle+\left(\frac{1}{2}\right)|\varphi_2,t\rangle$$

这表示各加入 50%的$|\varphi_1,t\rangle\&|\varphi_2,t\rangle$，而不记录相位信息。对于完全随机的相位信息，这个态就对应于完全退相干的态(This means that 50% $|\varphi_1,t\rangle or|\varphi_2,t\rangle$, and phase information is not recorded. For completely random phase information, this state corresponds to a completely decoherent state)。

很显然，$\left(\frac{1}{2}\right)\Big|\varphi_1,t\rangle+\left(\frac{1}{2}\right)|\varphi_2,t\rangle$这个约束又太弱了，我们没有能力提取其中的每种状态的信息了，我们不能通过计算得到某个$a|\varphi_1,t\rangle+b|\varphi_2,t\rangle$态的成分比例。我们需要一个满足"可加性"的表示方式，相加对象间没有相位，而且可加单元可以描述任意状态，即密度矩阵理论（Obviously, the constraint of $\left(\frac{1}{2}\right)\Big|\varphi_1,t\rangle+\left(\frac{1}{2}\right)|\varphi_2,t\rangle$ is too weak. We have no ability to extract the information of each state. We can't calculate the component proportion of a state of $a|\varphi_1,t\rangle+b|\varphi_2,t\rangle$. We need a representation that satisfies "additivity", There is no phase between added objects and unit can describe any state, that is, the density matrix theory）：

$$\hat{\rho}=\sum_i P_i|i\rangle\langle i|$$

这表示若干个单态直接相加，相加的各个单态的相位信息被$|i\rangle\langle i|$消除（相加的对象不包含相位信息，否则就只能产生单态），但态本身的信息则保留了下来（被加法运算的项描述的是具体的态） (This means that several singletons are added directly, and the phase information of the added singletons is eliminated by $i\rangle\langle i$ (the added object does not contain phase information, otherwise it can only generate singletons), but the information of the state itself is preserved (the added term describes the specific state).)。

密度矩阵使我们可以描述多态，而对角化后各态的比例决定状态的相关程度（完全相干$\begin{matrix}1 & 0\\ 0 & 0\end{matrix}$ / 完全退相干$\begin{matrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{matrix}$），完全相干态对其实就是单态(The density matrix enables us to describe polymorphism, and the proportion of States after diagonalization determines the degree of correlation of states (completely coherent$\begin{matrix}1 & 0\\ 0 & 0\end{matrix}$ / completely decoherent $\begin{matrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{matrix}$). A completely coherent state is actually a single state.)。

三．密度矩阵的演化(The evolution of density matrix)

密度矩阵的演化事实上就是态的演化，但与原本的单态理论不同，现在的态是一个张量。我们可以对密度算符应用含时微扰论(In fact, the evolution of density matrix is the evolution of States, but different from the original single state theory, the present state is a tensor. We can apply time-dependent perturbation theory to density operators)：

$$\hat{\rho}_I(t) = \hat{U}_I(t, t_0)\hat{\rho}_I(t_0)\hat{U}_I^{-1}(t, t_0)$$

直接计算积分参数较多，可以考虑无穷小时间平移算符(There are many integral parameters in direct calculation, so the infinitesimal time translation operator can be considered)：

$$\hat{U}_I(t+dt, t) = 1 - i\frac{\hat{V}_I(t)}{\hbar}dt$$

得到微分形式的演化方程(Get the evolution equation of differential form)：

$$\frac{\partial}{\partial t}\hat{\rho}_I(t) = -i\frac{\hat{V}_I(t)}{\hbar}\hat{\rho}_I(t) + \hat{\rho}_I(t)i\frac{\hat{V}_I(t)}{\hbar} = -\frac{i}{\hbar}[\hat{V}_I(t),\ \hat{\rho}_I(t)]$$

可以看到，对易子和相似变换间有紧密关系，基于这个关系，我们通常称$\hat{V}_I(t)$为生成元。和态矢量不同，密度矩阵的演化是对易子形式的，我们可以作这样的解读：算符是一对共轭空间两矢量的并矢，它们要做共轭的演化才能让算符达到最终的状态（It can be seen that there is a close relationship between commutators and similar transformations. Based on this relationship, we usually call $\hat{V}_I(t)$ as the generator. Different from the state vector, the evolution of the density matrix is in the form of commutator. We can interpret it as follows: the operator is the dyadic of two vectors in a pair of conjugate spaces. They need to do the evolution of conjugate to make the operator reach the final state）。

如果将密度矩阵展开为 V 的级数，可以得到(If we expand the density matrix to a series of V, we can get)：

$$\frac{\partial}{\partial t}\hat{\rho}_I^{(n)}(t) = -\frac{i}{\hbar}[\hat{V}_I(t),\ \hat{\rho}_I^{(n-1)}(t)]$$

这时候我们就可以轻松的转换为积分形式了，选择久远过去为起点(At this time, we can easily convert it into integral form and choose the long past as the starting time)：

$$\hat{\rho}_I^{(n)}(t) = -\frac{i}{\hbar}\int_{-\infty}^{t}[\hat{V}_I(\tau_1),\ \hat{\rho}_I^{(n-1)}(\tau_1)]d\tau_1$$

更进一步会使得问题更为清晰，我们可以得到我们所熟悉的薛定谔绘景的版本(Further, it will make the problem clearer. We can get the familiar version of Schrodinger picture)：

$$\hat{\rho}^{(n)}(t) = -\frac{i}{\hbar}\left[\int_{-\infty}^{t}\hat{U}_0(t-\tau_1)\hat{V}(\tau_1)\hat{\rho}^{(n-1)}(\tau_1)\hat{U}_0(-(t-\tau_1))d\tau_1 - \int_{-\infty}^{t}\hat{U}_0(t-\tau_1)\hat{\rho}^{(n-1)}(\tau_1)\hat{V}(\tau_1)\hat{U}_0(-(t-\tau_1))d\tau_1\right]$$

$$= (-\frac{i}{\hbar})^2\left[\int_{-\infty}^{t}\hat{U}_0(t-\tau_1)\hat{V}(\tau_1)\int_{-\infty}^{\tau_1}\hat{U}_0(\tau_1-\tau_2)\hat{V}(\tau_2)\hat{\rho}^{(n-2)}(\tau_2)\hat{U}_0(-(\tau_1-\tau_2))d\tau_2\,\hat{U}_0(-(t-\tau_1))d\tau_1\right.$$

$$-\int_{-\infty}^{t}\hat{U}_0(t-\tau_1)\hat{V}(\tau_1)\int_{-\infty}^{\tau_1}\hat{U}_0(\tau_1-\tau_2)\hat{\rho}^{(n-2)}(\tau_2)\hat{V}(\tau_2)\hat{U}_0(-(\tau_1-\tau_2))d\tau_2\,\hat{U}_0(-(t-\tau_1))d\tau_1$$

$$-\int_{-\infty}^{t}\hat{U}_0(t-\tau_1)\int_{-\infty}^{\tau_1}\hat{U}_0(\tau_1-\tau_2)\hat{V}(\tau_2)\hat{\rho}^{(n-2)}(\tau_2)\hat{U}_0(-(\tau_1-\tau_2))d\tau_2\hat{V}(\tau_1)\,\hat{U}_0(-(t-\tau_1))d\tau_1$$

$$\left.+\int_{-\infty}^{t}\hat{U}_0(t-\tau_1)\int_{-\infty}^{\tau_1}\hat{U}_0(\tau_1-\tau_2)\hat{\rho}^{(n-2)}(\tau_2)\hat{V}(\tau_2)\hat{U}_0(-(\tau_1-\tau_2))d\tau_2\,\hat{V}(\tau_1)\hat{U}_0(-(t-\tau_1))d\tau_1\right]$$

这种写法会有明显的碰撞的含义，$\hat{V}(\tau_2)\hat{\rho}^{(n-2)}(\tau_2)$ 代表产生了作用，一对$\hat{U}_0(\tau_1-\tau_2)$ 则描述时间演化。(This kind of writing will have the obvious meaning of collision，$\hat{V}(\tau_2)\hat{\rho}^{(n-2)}$ represents an effect, and a pair of $\hat{U}_0(\tau_1-\tau_2)$ describes the time evolution.)

三⁺,哈密顿本征态表象下的演化（Evolution under Hamiltonian eigenstate representation）

在$\hat{\mathrm{H}}$的本征态下，时间演化算符$\hat{\mathrm{U}}(\mathrm{t},\mathrm{t}_0)$将成为数字，我们也就得到了‘前置知识一,状态的时间演化’的张量版本(In the eigenstate of $\hat{\mathrm{H}}$, the time evolution operator $\hat{\mathrm{U}}(\mathrm{t},\mathrm{t}_0)$ will become a number.We also get the tensor version of ‘pre knowledge one-Time evolution of States’.)

让我们从零级项开始(Let's start with a zero level term)：

值得注意的是，为了形式简单，我直接约定重复指标求和，时间项的连续指标则没有做类似的处理(It is worth noting that, for Simple form, I directly agreed to sum the repeated indicators, while the continuous indicators of time items did not do similar processing)。

$\hat{\rho}^{(0)}(t) = \hat{\rho}(-\infty)$ 在散射之前，密度矩阵是常数(Before scattering, the density matrix is constant)。

$\hat{\rho}^{(1)}(t) = -\frac{\mathrm{i}}{\hbar}\left[\int_{-\infty}^{\mathrm{t}} e^{-i\omega_{mn}(\mathrm{t}-\tau_1)}(\mathrm{V}_{ms}(\tau_1)\rho_{sn} - \rho_{ms}\mathrm{V}_{sn}(\tau_1))d\tau_1\right]|m\rangle\langle n|$ 反对称的形式保证概率守恒(The form of antisymmetry guarantees the conservation of probability)

$$\hat{\rho}^{(2)}(t) = (-\frac{\mathrm{i}}{\hbar})^2\left[\int_{-\infty}^{\mathrm{t}}\int_{-\infty}^{\tau_1} e^{-i\omega_{mn}(t-\tau_1)}\left(e^{-i\omega_{sn}(\tau_1-\tau_2)}\mathrm{V}_{ms}(\tau_1)\mathrm{V}_{sp}(\tau_2)\rho_{pn} - e^{-i\omega_{sp}(\tau_1-\tau_2)}\mathrm{V}_{ms}(\tau_1)\rho_{sp}\mathrm{V}_{pn}(\tau_2) - e^{-i\omega_{mp}(\tau_1-\tau_2)}\mathrm{V}_{ms}(\tau_2)\rho_{sp}\mathrm{V}_{pn}(\tau_1)\right.\right.$$

$$\left.\left. + e^{-i\omega_{mp}(\tau_1-\tau_2)}\rho_{ms}\mathrm{V}_{sp}(\tau_1)\mathrm{V}_{pn}(\tau_2)\right) d\tau_2\, d\tau_1\right]|m\rangle\langle n|$$

我们可以看出，每次发生作用，$\mathrm{V}_{sp}(\tau_2)\rho_{pn}$外围指标 sn 就是时间演化的指标，时间演化范围此次作用到下一次作用的范围(We can see that every time an action takes place, the $\mathrm{V}_{sp}(\tau_2)\rho_{pn}$ peripheral index sn is the index of time evolution, and the time evolution range is the range of this action to the next action)。

作业正文(Homework text)

1.

(a) δ 函数意味着仅允许特定时刻的碰撞，则对于这个势场，至多发生两次碰撞(The δ function means that only collisions at a certain time are allowed, so for this potential field, at most two collisions will occur).

首先要做的是将本题相互作用势场写出来，假定$\tau_a < \tau_b$(The first thing to do is to write out the interaction potential field of this topic, assuming that$\tau_a < \tau_b$)：

$$\mathrm{V}_{mn}(t) = \sum_i q^i \vec{x}_{mn}^i \cdot E(t) = \sum_i q^i z_{mn}^i\, \mathcal{E}[\delta(t-\tau_a) + \delta(t-\tau_b)] = \mathbb{v}_{mn}[\delta(t-\tau_a) + \delta(t-\tau_b)]$$

这里我让τ_a吸收了空间相位（$\tau_i = \tau_i' + \frac{\vec{k}\cdot\vec{x}}{w_{21}}$），来让形式简单一些(Here I let τ_a absorb the spatial phase to make the form simpler)。

初始密度矩阵(Initial density matrix)：

$\hat{\rho}^{(0)}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

一级密度矩阵(First order density matrix)：

$$\hat{\rho}^{(1)}(t) = -\frac{\mathrm{i}}{\hbar}\left[\int_{-\infty}^{\mathrm{t}} e^{-i\omega_{mn}(\mathrm{t}-\tau_1)}(\mathrm{V}_{ms}(\tau_1)\rho_{sn} - \rho_{ms}\mathrm{V}_{sn}(\tau_1))d\tau_1\right]|m\rangle\langle n| = \begin{cases} 0 \quad (t \le \tau_a) \\ -\frac{\mathrm{i}}{\hbar}[\mathbb{v}, \hat{\rho}^{(0)}]_{mn} e^{-i\omega_{mn}(\mathrm{t}-\tau_a)} \quad (\tau_a < t \le \tau_b) \\ -\frac{\mathrm{i}}{\hbar}[\mathbb{v}, \hat{\rho}^{(0)}]_{mn} e^{-i\omega_{mn}(\mathrm{t}-\tau_a)} + -\frac{\mathrm{i}}{\hbar}[\mathbb{v}, \hat{\rho}^{(0)}]_{mn} e^{-i\omega_{mn}(\mathrm{t}-\tau_b)} (\tau_b < t) \end{cases}$$

对于单个偶极子的情况，可以更进一步的写出具体的答案。这个双态系统 z 的表示依据具体问题来确定,可以计 z 对应的厄密矩阵为：$\begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix}$(For the case of a single dipole, we can further write the specific answer. The representation of the two-state system Z is determined according to the specific problems. The Hermite matrix corresponding to Z can be calculated as$\begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix}$)。

并且，如果我们考虑积分$\int_{-\infty}^{\mathrm{T}} \sum_i \delta(t-\tau_i) f(t)\, dt = \sum_i \theta(t-\tau_i) f(\tau_i)$（对于连续的信号我们需要统计所有时间可能，这里的求和$\sum_i \delta(t-\tau_i) f(t)$，会变成积分），形式将更为简单(And, if we think about integration$\int_{-\infty}^{\mathrm{T}} \sum_i \delta(t-\tau_i) f(t)\, dt = \sum_i \theta(t-\tau_i) f(\tau_i)$（For the continuous signal, we need to count all the time possibilities, and the sum$\sum_i \delta(t-\tau_i) f(t)$ here will become integral）,The form will be simpler)：

$$\hat{\rho}^{(1)}(t) = -\frac{\mathrm{i}}{\hbar} q\mathcal{E} \sum_i \theta(t-\tau_i) \begin{pmatrix} 0 & -z_{12} e^{-i\omega_{12}(\mathrm{t}-\tau_i)} \\ z_{12} e^{i\omega_{12}(\mathrm{t}-\tau_i)} & 0 \end{pmatrix}$$

其中，$\theta(\mathrm{t}) = \begin{cases} 0 & \mathrm{t} \le 0 \\ 1 & \mathrm{t} > 0 \end{cases}$，零点设为 0 意味着不考虑同时发生两次碰撞的情况，对于常规的函数这个定义无所谓，某时间点的概率总是 0。但对于$\delta(t-\tau_i)$，我们必须明确的定义避免所谓同时碰撞情况的发生。这个处理是纯数学的，正因为我们在处理时间点，我们需要保证这种处理本身不造成新的问题(in there，$\theta(\mathrm{t}) = \begin{cases} 0 & \mathrm{t} \le 0 \\ 1 & \mathrm{t} > 0 \end{cases}$Setting zero to zero means not considering two collisions at the same time. It doesn't matter for the definition of normal function. The probability of a certain time point is always zero. But for $\delta(t-\tau_i)$, we must make clear the definition to avoid the so-called simultaneous collision. This process is purely mathematical, because we are at the point of processing time, we need to ensure that this process itself does not cause new problems)。

这个式子可以解读为，单次碰撞发生了两/n 次(这个读法太糟糕了，可以考虑一般的函数，我们积分是在求解不同时刻单次碰撞的总影响，或者叫碰撞不是相继发生的)(This formula can be interpreted as that a single collision occurs twice / N times (this reading method is too bad, we can consider the general function, we integrate to solve the total impact of a single collision at different times, or the collision does not happen in succession))。

我们可以进一步的求出二阶的结果(We can get the second order result further)：

$$\hat{\rho}^{(2)}(t) = -\frac{\mathrm{i}}{\hbar}\left[\int_{-\infty}^{\mathrm{t}} \mathrm{e}^{-\mathrm{i}\omega_{\mathrm{mn}}(\mathrm{t}-\tau_1)} \left(\mathrm{V}_{\mathrm{ms}}(\tau_1)\rho_{\mathrm{sn}}^{(1)}(\tau_1) - \rho_{\mathrm{ms}}^{(1)}(\tau_1)\mathrm{V}_{\mathrm{sn}}(\tau_1)\right) \mathrm{d}\tau_1\right] |\mathrm{m}\rangle\langle\mathrm{n}|$$

$$= -\frac{\mathrm{i}}{\hbar}\left[\sum_i \theta(t-\tau_i)\, e^{-i\omega_{mn}(\mathrm{t}-\tau_i)} \left(\mathrm{V}_{\mathrm{ms}}(\tau_i)\rho_{\mathrm{sn}}^{(1)}(\tau_i) - \rho_{\mathrm{ms}}^{(1)}(\tau_i)\mathrm{V}_{\mathrm{sn}}(\tau_i)\right)\right] |\mathrm{m}\rangle\langle\mathrm{n}|$$

$$= \left(-\frac{\mathrm{i}}{\hbar} q\mathcal{E}\right)^2 \left[\theta(t-\tau_b) e^{-i\omega_{mn}(\mathrm{t}-\tau_b)} \left[\begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix}, \begin{pmatrix} 0 & -z_{12} e^{-i\omega_{12}(\tau_b-\tau_a)} \\ z_{12} e^{i\omega_{12}(\tau_b-\tau_a)} & 0 \end{pmatrix}\right]_{mn}\right] |\mathrm{m}\rangle\langle\mathrm{n}|$$

$$= \left(-\frac{\mathrm{i}}{\hbar} q\mathcal{E}\right)^2 \theta(t-\tau_b) \begin{bmatrix} 2{z_{12}}^2 cos(\omega_{12}(\tau_b-\tau_a)) & (z_{22}-z_{11}) z_{12} e^{-i\omega_{12}(t-\tau_a)} \\ (z_{22}-z_{11}) z_{12} e^{i\omega_{12}(t-\tau_a)} & -2{z_{12}}^2 cos(\omega_{12}(\tau_b-\tau_a)) \end{bmatrix} d$$

说实话这有点难算，不过θ总可以采用这样的解读，即给碰撞排序，仅有正确的时间顺序可以被保留下来，我们可以快速确定正确的结果(比如对于这个问题，‘二次碰撞’仅允许一种可能性‘τ_a被散射的粒子在τ_b时刻被二次散射’)。对于连续的情况，我们就没有办法在连续的积分中简化$\theta(t-\tau_b)$

了，它将被一个完整的时序的积分代替（To be honest, it's a bit difficult to calculate, but θ can always adopt the interpretation that only the correct time sequence can be preserved, and we can quickly determine the correct results (for example, for this problem, "second collision" only allows one possibility, and the first scattered at the time τ_a , second scattered at the time τ_b) For continuous cases, there is no way to simplify $\theta(t-\tau_b)$ in continuous integrals, which will be replaced by integral of a complete sequence)。

之前的叙述可能会让人困惑，我应当这么说：对于任意脉冲 f(t)，总可以写成$\int_{-\infty}^{\infty}\delta(t-\tau_i)f(\tau_i)d\tau_i$的形式。这让我们可以很明确的将任意级数的微扰写成如下形式(I should say that: for any pulse f(t), it can always be written as $\int_{-\infty}^{\infty}\delta(t-\tau_i)f(\tau_i)d\tau_i$. This allows us to explicitly write the perturbation of any series as follows:):

$$\hat{\rho}^{(1)}(t)=-\frac{\mathrm{i}}{\hbar}q\mathcal{E}\int\theta(t-\tau_i)\begin{pmatrix}0 & -z_{12}e^{-i\omega_{12}(\mathrm{t}-\tau_i)}\\ z_{12}e^{i\omega_{12}(\mathrm{t}-\tau_i)} & 0\end{pmatrix}d\tau_i$$

这是一个时序积分，我们只允许过去的信号对现在产生影响，$\theta(t-\tau_i)$保证了它成立。我们可以借此得到 n 级的形式(This is a time series integral. We only allow the signals of the past to have an impact on the present. $\theta(t-\tau_i)$ ensures that it works. We can get the form of level n:)：

$$\hat{\rho}^{(n)}(t)=-\frac{\mathrm{i}}{\hbar}q\mathcal{E}\int T(t,\tau_1,\tau_2\cdots\tau_i)[\cdots]d\tau_1 d\tau_2\cdots d\tau_i$$

[⋯]通常十分复杂，但是如果对离散的体系，在动量守恒约束下，问题就会变得简单多了([⋯] is usually very complex, but if the discrete system is constrained by the conservation of momentum, the problem will become much simpler)

(b)

$$\langle \mathrm{z}\rangle=\mathrm{tr}\big(\mathrm{z}\hat{\rho}(\mathrm{t})\big)=\mathrm{tr}\left(\mathrm{z}\hat{\rho}^{(0)}(\mathrm{t})+\mathrm{z}\hat{\rho}^{(1)}(\mathrm{t})+\mathrm{z}\hat{\rho}^{(2)}(\mathrm{t})\right)$$

$$=z_{11}+\left(-\frac{\mathrm{i}}{\hbar}q\mathcal{E}\right)\sum_{i=a,b}\theta(t-\tau_i)2i\,z_{12}{}^2 sin(\omega_{12}(t-\tau_i))+\left(-\frac{\mathrm{i}}{\hbar}q\mathcal{E}\right)^2\theta(t-\tau_b)2z_{12}{}^2(z_{11}-z_{22})[cos(\omega_{12}(\tau_b-\tau_a))$$

$$-cos(\omega_{12}(\mathrm{t}-\tau_a))])$$

如果考虑空间相位带来的影响，对于足够大的介质，只有不含空间相位的反应被允许发生,而这一项会出现在二阶微扰里面，就是脉冲间隔所对应的项（这里我释放了被时间吸收的项）(If the influence of spatial phase is considered, only the reaction without spatial phase is allowed to occur for a medium large enough, and this term will appear in the second-order perturbation, which is the term corresponding to the pulse interval (here I release the term absorbed by time))：

$$\boldsymbol{avg}(\hat{\boldsymbol{\rho}}^{(2)}(\mathbf{t}))=\left(-\frac{\mathbf{i}}{\hbar}\boldsymbol{q}\boldsymbol{\mathcal{E}}\right)^2\boldsymbol{\theta}(\boldsymbol{t}-\boldsymbol{\tau_b})\begin{bmatrix}\mathbf{2}\boldsymbol{z_{12}}^2\boldsymbol{cos}(\boldsymbol{\omega_{12}}(\boldsymbol{\tau_b}-\boldsymbol{\tau_a})) & \mathbf{0}\\ \mathbf{0} & -\mathbf{2}\boldsymbol{z_{12}}^2\boldsymbol{cos}(\boldsymbol{\omega_{12}}(\boldsymbol{\tau_b}-\boldsymbol{\tau_a}))\end{bmatrix}$$

$$avg(\langle \mathrm{z}\rangle)=z_{11}+\left(-\frac{\mathrm{i}}{\hbar}q\mathcal{E}\right)^2\theta(t-\tau_b)2z_{12}{}^2(z_{11}-z_{22})avg\left(cos\left(\omega_{12}\left(\tau_b-\tau_a+\frac{\Delta\vec{k}\cdot\vec{x}}{w_{21}}\right)\right)\right)$$

$$=\begin{cases}z_{11}+\left(-\frac{\mathrm{i}}{\hbar}q\mathcal{E}\right)^2\theta(t-\tau_b)2z_{12}{}^2(z_{11}-z_{22})cos\big(\omega_{12}(\tau_b-\tau_a)\big) & if\,\Delta\vec{k}=0\\ \sim z_{11} & if\,\Delta\vec{k}\neq 0\end{cases}$$

在$\Delta\vec{k}$零附近才会有比较明显的信号,在这个问题里面问题还不明显，三维空间，在正确的方向我们才可以看到强信号(Only when △ K is near zero can there be a more obvious signal. In this problem, the problem is not obvious. In three-dimensional space, we can see a strong signal in the right direction.)。

2.

(a)对于非共线的碰撞，让τ_i吸收相位或许是一个明智的决定，但对于坐标的吸收使得在不同的空间位置，三者的大小关系会发生变化。在动量守恒的方向，额外的时间项会进行抵消，不会附加一个空间相位(For non collinear collisions, it may be a wise decision to let the τ_i absorption phase change, but for the absorption of coordinates, the size relationship of the three will change in different spatial positions. In the direction of conservation of momentum, the extra time term will be cancelled, and a space phase will not be added)。

任意选取一个区域，我们很快（观察结果，我们）可以得到(If we choose any region, we can get it soon)：

$\hat{\rho}^{(0)}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\hat{\rho}^{(1)}(t) = -\frac{\mathrm{i}}{\hbar} q\mathcal{E} \sum_i \theta(t-\tau_i) \begin{pmatrix} 0 & -z_{12} e^{-i\omega_{12}(t-\tau_i)} \\ z_{12} e^{i\omega_{12}(t-\tau_i)} & 0 \end{pmatrix}$

$\hat{\rho}^{(2)}(t) = \left(-\frac{\mathrm{i}}{\hbar} q\mathcal{E}\right)^2 \sum_{ij=ab,ac,bc} \theta(t-\tau_j) \begin{bmatrix} 2{z_{12}}^2 cos(\omega_{12}(\tau_j-\tau_i)) & (z_{22}-z_{11}) z_{12} e^{-i\omega_{12}(t-\tau_i)} \\ (z_{22}-z_{11}) z_{12} e^{i\omega_{12}(t-\tau_i)} & -2{z_{12}}^2 cos(\omega_{12}(\tau_j-\tau_i)) \end{bmatrix}$

$\hat{\rho}^{(3)}(t) = \left(-\frac{\mathrm{i}}{\hbar} q\mathcal{E}\right)^3 \theta(t-\tau_c) \left[\begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix}, \begin{bmatrix} 2{z_{12}}^2 cos(\omega_{12}(\tau_b-\tau_a)) & (z_{22}-z_{11}) z_{12} e^{-i\omega_{12}(t-\tau_a)} \\ (z_{22}-z_{11}) z_{12} e^{i\omega_{12}(t-\tau_a)} & -2{z_{12}}^2 cos(\omega_{12}(\tau_b-\tau_a)) \end{bmatrix} \right]$

$= \left(-\frac{\mathrm{i}}{\hbar} q\mathcal{E}\right)^3 \theta(t$

$-\tau_c) \begin{pmatrix} 2i(z_{22}-z_{11}) {z_{12}}^2 sin(\omega_{12}(\tau_c-\tau_a)) & -4{z_{12}}^3 cos(\omega_{12}(\tau_b-\tau_a)) e^{-i\omega_{12}(t-\tau_c)} + z_{12}(z_{11}-z_{22})^2 e^{i\omega_{12}(t-\tau_a)} \\ 4{z_{12}}^3 cos(\omega_{12}(\tau_b-\tau_a)) e^{i\omega_{12}(t-\tau_c)} + z_{12}(z_{11}-z_{22})^2 e^{i\omega_{12}(t-\tau_a)} & -2i(z_{22}-z_{11}) {z_{12}}^2 sin(\omega_{12}(\tau_c-\tau_a)) \end{pmatrix}$

如果对于连续的信号，我们这里附加的时序因子将会遍历所有可能，不会展开成有限的项。正是因为它离散，我们可以快速的化简形式(For continuous signals, the additional timing factor here will traverse all possibilities and will not expand into a finite term. Because it is discrete, we can quickly simplify the form)。

t 大于τ_c（吸收了空间相位），$\theta(t-\tau_c)=1$，演化矩阵仅同时间相关而空间无关的部分只有$-4{z_{12}}^3 cos(\omega_{12}(\tau_b-\tau_a)) e^{i\omega_{12}(t-\tau_c)}$，如果只需要采用分量的记法或许可以更快的得到(T is greater than τ_c (absorbing the spatial phase), $\theta(t-\tau_c)=1$, and the time-dependent&Space -independence part of the evolution matrix is only $-4{z_{12}}^3 cos(\omega_{12}(\tau_b-\tau_a)) e^{i\omega_{12}(t-\tau_c)}$). If only the quantizing notation is needed, it may be faster)。

在所给的方向，只有对 x 稳定的信号会被保留。注意我们的τ_i吸收了空间相位，只有一种组合可以消除空间相位，我们可以得到(In the given direction,only the signal stable for X is retained. Note that our τ_i absorbs the spatial phase, only one combination can eliminate the spatial phase, and we can get:)：

$$\boldsymbol{avg(\hat{\rho}^{(3)}(t)) = \left(-\frac{\mathrm{i}}{\hbar} q\mathcal{E}\right)^3 \theta(t-\tau_c) \begin{pmatrix} 0 & -4{z_{12}}^3 e^{-i\omega_{12}(t-(\tau_c+\tau_b-\tau_a))} \\ 4{z_{12}}^3 e^{i\omega_{12}(t-(\tau_c+\tau_b-\tau_a))} & 0 \end{pmatrix}}$$

我们在$\vec{k}_c+\vec{k}_b-\vec{k}_a$的这个方向将几乎不会看到一阶和二阶微扰的信号,除非偶然简并。如果一开始就以这个为目标，计算会简单不少，很多项可以直接排除，比如二阶微扰的非对角项(We will hardly see the first and second order perturbations in this direction of the topic，Except by chance. If we take this as the goal at the beginning, the calculation will be simpler and many items can be excluded directly，For example, the off diagonal term of the second order perturbation)。

$$\boldsymbol{avg(\hat{\rho}^{(0)}(t)) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}$$

$$\boldsymbol{avg(\hat{\rho}^{(1)}(t)) = -\frac{\mathrm{i}}{\hbar} q\mathcal{E} \sum_i \theta(t-\tau_i) z_{12} \begin{pmatrix} 0 & -e^{-i\omega_{12}(t-\tau_i)} \\ e^{i\omega_{12}(t-\tau_i)} & 0 \end{pmatrix}}$$

$$\boldsymbol{avg(\hat{\rho}^{(2)}(t)) = \left(-\frac{\mathrm{i}}{\hbar} q\mathcal{E}\right)^2 \sum_{ij=ab,ac,bc} \theta(t-\tau_j) 2{z_{12}}^2 cos(\omega_{12}(\tau_j-\tau_i)) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$

$$\boldsymbol{avg(\hat{\rho}^{(3)}(t)) = \left(-\frac{\mathrm{i}}{\hbar} q\mathcal{E}\right)^3 \theta(t-\tau_c) 4{z_{12}}^3 cos(\omega_{12}(\tau_j-\tau_i)) \begin{pmatrix} 0 & -e^{-i\omega_{12}(t-\tau_c)} \\ e^{i\omega_{12}(t-\tau_c)} & 0 \end{pmatrix}}$$

可以看到形式是十分简明的，使用这个序列递推会快很多，emmm 我一直是全部推导(It can be seen that form is very concise. It will be much faster to use this sequence to recurse. Emmm is always deduce all).

关于这里的处理，我有一些是要说明的，之前我们处理的是单个偶极子，如果我们处理一个近似连续的体系，空间上相位的积累将导致强烈的干涉

相消，使得我们得到‘动量守恒’的信号最强。这并不是动量守恒被破坏了，对于一个场，我们倾向于将他视为粒子云（这个处理事实上是在改变可加的对象，单粒子同样），动量守恒约束的结果同样是“云”(As for the processing here, I have some points to explain. Before, we dealt with a single dipole. If we deal with an approximately continuous system, the accumulation of phases in space will lead to strong interference cancellation, making us obtain the strongest signal of "momentum conservation". It's not that momentum conservation is broken. For a field, we tend to think of it as a particle cloud (this treatment is actually changing the additive object, the same for a single particle). The result of momentum conservation constraint is also a "cloud".)。

(b)

在所给的时间 t=$\tau_c+\tau_b-\tau_a$(At the given time t=$\tau_c+\tau_b-\tau_a$):

$$avg(\hat{\rho})=\hat{\rho}^{(0)}+avg(\hat{\rho}^{(3)}(t))=\begin{pmatrix}1&0\\0&0\end{pmatrix}+\left(-\frac{\mathrm{i}}{\hbar}q\mathcal{E}\right)^3 4{z_{12}}^3\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

(c)对于能量是个分布的情况，我们应当对频率做积分(If the energy is a distribution, we should integrate the frequency):

$$avg(\hat{\rho}^{(3)}(t))=\int dw_{21}g(w_{21})\left(-\frac{\mathrm{i}}{\hbar}q\mathcal{E}\right)^3\theta(t-\tau_c)4{z_{12}}^3\begin{pmatrix}0&-e^{-i\omega_{12}(t-(\tau_c+\tau_b-\tau_a))}\\e^{i\omega_{12}(t-(\tau_c+\tau_b-\tau_a))}&0\end{pmatrix}$$

这个式子对应于傅里叶变换(This formula corresponds to the Fourier transform):

$$avg(\hat{\rho}^{(3)}(t))=\left(-\frac{\mathrm{i}}{\hbar}q\mathcal{E}\right)^3\theta(t-\tau_c)\begin{pmatrix}0&-4{z_{12}}^3G((\tau_c+\tau_b-\tau_a)-t)\\4{z_{12}}^3G(t-(\tau_c+\tau_b-\tau_a)&0\end{pmatrix}$$

i.如果 t 很大(至少远大于$\frac{1}{\Delta w}$,对于标记谱特征宽带的量 Δw，它和时间特征宽带Δt的乘积总是大于一个常数)，G 通常会变得很小(高斯波包的傅里叶变换仍然是高斯波包)(If t is large((at least far greater than 1/ Δ W, for the amount of the marked spectrum characteristic broadband, Δ W, the product of it and the time characteristic broadband Δ t is always greater than a constant)), G usually becomes very small (the Fourier transform of Gaussian wave packet is still Gaussian wave packet))

$$avg(\hat{\rho}^{(3)}(t))=0$$

$$avg(\hat{\rho})=\hat{\rho}^{(0)}$$

即不产生信号，我们之前没有出现这样的情况是在于我们使用狄拉克函数(no signal is generated,We haven't seen this before because we use Dirac functions)。

ii.如果 t 满足 t=$\tau_c+\tau_b-\tau_a$，则我们得到 (If t satisfies t=$\tau_c+\tau_b-\tau_a$, then we get):

$$avg(\hat{\rho}^{(3)}(t))=\left(-\frac{\mathrm{i}}{\hbar}q\mathcal{E}\right)^3\theta(t-\tau_c)\begin{pmatrix}0&-4{z_{12}}^3G(0)\\4{z_{12}}^3G(0)&0\end{pmatrix}$$

高斯分布这对应着反映的峰值信号(The Gaussian distribution corresponds to the reflected peak signal)。

总结(Summary):

这次作业在具体的运算时，我发现我们需要一个时序算符，而对于离散的情况，时序算符可以快速的合并起来，极端的情况会合并为 1；空间相位我常常将它储存在 t 里面，在牵扯空间相位的情况，与空间坐标无关的项总能贡献最强的信号，这对应着动量守恒；(In this operation, I found that we need a time sequence operator. For the discrete case, the time sequence operator can be combined quickly, and in the extreme case, it can be combined into 1. I often store the spatial phase in T. in the case of involving the spatial phase, the term irrelevant to the spatial coordinate can always contribute the strongest signal, which corresponds to the conservation of momentum;)

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PS.使用这样的 matlab 语句会省心一些，我一开始推导所有的项，浪费了许多时间(Using MATLAB statements like this will save you some time，I wasted a lot of time deducing all the items at the beginning)：

```
%a=[1,0;0,0];i=1;

syms z11 z12 z22 p1 p2 p3 d1 d2 d3;

P=[p1,p2,p3];

D=[d1,d2,d3];

Z=[z11,z12;z12,z22];

b=Z*a-a*Z;

a=[b(1),b(3)*P(i);b(2)*D(i),b(4)]
```

```
i=i+1;
```

运行两次的结果（虽然可读性差，不过用来验算是没有问题的，可以更改指数项的指标比如用d、p代替）(The result of two runs (although the readability is poor, there is no problem in checking the calculation. You can change the index of index items, such as D and P instead))：

[d1*z12^2 + p1*z12^2, -p2*(p1*z11*z12 - p1*z12*z22)]

[-d2*(d1*z11*z12 - d1*z12*z22), - d1*z12^2 - p1*z12^2]

三次(The three time)：

[p2*z12*(p1*z11*z12 - p1*z12*z22) - d2*z12*(d1*z11*z12 - d1*z12*z22), -p3*(2*z12*(d1*z12^2 + p1*z12^2) + p2*z11*(p1*z11*z12 - p1*z12*z22) - p2*z22*(p1*z11*z12 - p1*z12*z22))]

[d3*(2*z12*(d1*z12^2 + p1*z12^2) + d2*z11*(d1*z11*z12 - d1*z12*z22) - d2*z22*(d1*z11*z12 - d1*z12*z22)), d2*z12*(d1*z11*z12 - d1*z12*z22) - p2*z12*(p1*z11*z12 - p1*z12*z22)]

‘1’对应时间段‘a-b’；‘2’对应时间段‘b-c’；‘3’对应时间段‘c-end’；('1' corresponds to time period 'A-B'; '2' corresponds to time period 'B-C'; '3' corresponds to time period 'C-end';)

//