数值试验一报告

陈稼霖 45875852

2019.3.21

1 问题的提出

正方形区域上的泊松问题

$$\begin{cases} -(\frac{\partial^2 x}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) = f(x,y), 0 < x, y < 1, \\ u(0,y) = u(1,y) = u(x,0) = u(x,1) \end{cases}$$

取$f(x,y) = 2\pi\sin(\pi x)\sin(\pi y)$，问题的精确解为$u^*(x,y) = \sin(\pi x)\sin(\pi y)$. 将正方形区域平均划分为$N\times N$个小方格，方程组的差分格式为

$$\begin{cases} -u_{i-1,j} - u_{i,j-1} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1} = h^2 f_{i,j} \\ u_{0,j} = u_{N,j} = u_{i,0} = u_{i,N} \end{cases}$$

记

$$u_j^h = \begin{pmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{N-1,j} \end{pmatrix}, f_j^h = \begin{pmatrix} f_{1,j} \\ f_{2,j} \\ \vdots \\ f_{N-1,j} \end{pmatrix}, u^h = \begin{pmatrix} u_1^h \\ u_2^h \\ \vdots \\ u_{N-1}^h \end{pmatrix}, f^h = \begin{pmatrix} f_1^h \\ f_2^h \\ \vdots \\ f_{N-1}^h \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 0 & 1 & 0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix}, L_h = \begin{pmatrix} 4I-C & -I & & & \\ -I & 4I-C & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & 4I-C & -I \\ & & & -I & 4I-C \end{pmatrix}$$

其中I为$N-1$阶单位矩阵. 将差分格式写为

$$L_h u^h = h^2 f^h \tag{1}$$

2 数值试验

1. 取$h = 0.1$，分析用不同迭代法求解方程组（1）的收敛性，并求出使$||u^{(k+1)} - u^{(k)}|| < \varepsilon$的近似解及相应的迭代次数，其中$u^{(k)}$为求解（1）的第$k$次迭代解，$\varepsilon = 10^{-6}$. 并与精确解$u^*$做比较. 考虑用（1）雅可比迭代法；（2）赛德尔迭代法；（3）超松弛迭代法（ω取1.2，1.3，1.9，0.9）.
（本次实验均以$u^{(k)} = ones((N-1)^2, 1)$为初始向量进行迭代）
（1）雅可比迭代法求出的近似解：

$u^{(k)} =$

[0.096282596713267, 0.183140370210779, 0.252071110720433, 0.296327343713280, 0.311577028014331, 0.296327343713280, 0.252071110720433, 0.183140370210779, 0.096282596713267,
0.183140370210779, 0.348353707433701, 0.479467713924059, 0.563648138734764, 0.592654687426560, 0.563648138734765, 0.479467713924059, 0.348353707433701, 0.183140370210779,
0.252071110720433, 0.479467713924059, 0.659930735448032, 0.775795057637339, 0.815719249455198, 0.775795057637339, 0.659930735448032, 0.479467713924059, 0.252071110720433,
0.296327343713280, 0.563648138734764, 0.775795057637339, 0.912001846168465, 0.958935427848119, 0.912001846168465, 0.775795057637339, 0.563648138734765, 0.296327343713280,
0.311577028014331, 0.592654687426560, 0.815719249455198, 0.958935427848119, 1.008284442881733, 0.958935427848119, 0.815719249455198, 0.592654687426560, 0.311577028014331,
0.296327343713280, 0.563648138734764, 0.775795057637339, 0.912001846168465, 0.958935427848119, 0.912001846168465, 0.775795057637339, 0.563648138734764, 0.296327343713280,
0.252071110720433, 0.479467713924059, 0.659930735448032, 0.775795057637339, 0.815719249455198, 0.775795057637339, 0.659930735448032, 0.479467713924059, 0.252071110720433,
0.183140370210779, 0.348353707433701, 0.479467713924059, 0.563648138734765, 0.592654687426560, 0.563648138734765, 0.479467713924060, 0.348353707433701, 0.183140370210779,
0.096282596713267, 0.183140370210779, 0.252071110720433, 0.296327343713280, 0.311577028014331, 0.296327343713280, 0.252071110720433, 0.183140370210779, 0.096282596713267]^T

（2）赛德尔迭代法求出的近似解：

$u^{(k)} =$

[0.096282095014263, 0.183139305299849, 0.252069468476958, 0.296325253065745, 0.311574630504295, 0.296324920581546, 0.252068902108678, 0.183138686768656, 0.096281660166459,
0.183139305299849, 0.348351437909015, 0.479464331185244, 0.563643801598307, 0.592649841163039, 0.563643200128708, 0.479463306615346, 0.348350318975516, 0.183138518652742,
0.252069468476958, 0.479464331185244, 0.659925657440196, 0.775788713795995, 0.815712102237337, 0.775787926462024, 0.659924316260711, 0.479462866482196, 0.252068438742469,
0.296325253065745, 0.563643801598307, 0.775788713795995, 0.911993855335958, 0.958926613230658, 0.911992975069768, 0.775787214311661, 0.563642164010424, 0.296324101787522,
0.311574630504295, 0.592649841163039, 0.815712102237337, 0.958926613230658, 1.008274635236227, 0.958925732964420, 0.815710602752922, 0.592648203575069, 0.311573479226012,
0.296324920581546, 0.563643200128708, 0.775787926462024, 0.911992975069768, 0.958925732964420, 0.911992178861438, 0.775786570165566, 0.563641718916405, 0.296323879240509,
0.252068902108678, 0.479463306615346, 0.659924316260711, 0.775787214311661, 0.815710602752922, 0.775786570165566, 0.659923218993784, 0.479462108289375, 0.252068059646051,
0.183138686768656, 0.348350318975516, 0.479462866482197, 0.563642164010424, 0.592648203575069, 0.563641718916405, 0.479462108289375, 0.348349490952611, 0.183138104641281,
0.096281660166459, 0.183138518652742, 0.252068438742469, 0.296324101787522, 0.311573479226012, 0.296323879240509, 0.252068059646050, 0.183138104641281, 0.096281369102766]^T

（3）超松弛迭代法求出的近似解：

当$\omega = 1.2$时，

$u^{(k)} =$

[0.096281691354415, 0.183138530015131, 0.252068396247185, 0.296323991888770, 0.311573308644520, 0.296323671559953, 0.252067849617325, 0.183137931278956, 0.096281268689381,
0.183138530015131, 0.348349956123932, 0.479462290558075, 0.563641410331563, 0.592647343119902, 0.563640846686530, 0.479461328717649, 0.348348902598140, 0.183137786301063,
0.252068396247185, 0.479462290558075, 0.659922857696348, 0.775785443815110, 0.815708696303852, 0.775784726156341, 0.659921633036897, 0.479460949160599, 0.252067449316242,
0.296323991888770, 0.563641410331563, 0.775785443815110, 0.911990047386572, 0.958922657435296, 0.911989266945657, 0.775784112020136, 0.563639951586101, 0.296322962118517,
0.311573308644520, 0.592647343119902, 0.815708696303852, 0.958922657435296, 1.008270535635037, 0.958921898321200, 0.815707400902344, 0.592645924237035, 0.311572307014415,
0.296323671559953, 0.563640846686530, 0.775784726156341, 0.911989266945657, 0.958921898321200, 0.911988599083657, 0.775783586473327, 0.563639598365687, 0.296322790334472,
0.252067849617325, 0.479461328717649, 0.659921633036897, 0.775784112020136, 0.815707400902344, 0.775783586473327, 0.659920736209750, 0.479460346402385, 0.252067156172811,
0.183137931278956, 0.348348902598140, 0.479460949160599, 0.563639951586101, 0.592645924237035, 0.563639598365687, 0.479460346402385, 0.348348242383297, 0.183137465214366,
0.096281268689381, 0.183137786301063, 0.252067449316242, 0.296322962118517, 0.311572307014415, 0.296322790334472, 0.252067156172811, 0.183137465214366, 0.096281042025073]^T

当$\omega = 1.3$时，

$u^{(k)} =$

[0.096281632281658, 0.183138393233463, 0.252068181194723, 0.296323714523445, 0.311572997474069, 0.296323362182267, 0.252067578754821, 0.183137731212908, 0.096281162818419,
0.183138393233463, 0.348349658928740, 0.479461841771555, 0.563640847418933, 0.592646724364534, 0.563640241019561, 0.479460804937937, 0.348348519553402, 0.183137585260307,
0.252068181194723, 0.479461841771555, 0.659922198626763, 0.775784633853820, 0.815707819774381, 0.775783878662965, 0.659920907386530, 0.479460422829064, 0.252067174970167,
0.296323714523445, 0.563640847418933, 0.775784633853820, 0.911989067396832, 0.958921609875873, 0.911988264121513, 0.775783260397822, 0.563639338129627, 0.296322644230624,
0.311572997474069, 0.592646724364534, 0.815707819774381, 0.958921609875873, 1.008269426939931, 0.958920845658129, 0.815706513099794, 0.592645288461249, 0.311571979221987,
0.296323362182267, 0.563640241019561, 0.775783878662965, 0.911988264121513, 0.958920845658128, 0.911987606490231, 0.775782754232017, 0.563639005383613, 0.296322485947155,
0.252067578754821, 0.479460804937937, 0.659920907386530, 0.775783260397822, 0.815706513099794, 0.775782754232017, 0.659920041934245, 0.479459853893360, 0.252066904333966,
0.183137731212908, 0.348348519553402, 0.479460422829064, 0.563639338129627, 0.592645288461249, 0.563639005383613, 0.479459853893360, 0.348347894350573, 0.183137287858479,
0.096281162818419, 0.183137585260307, 0.252067174970167, 0.296322644230624, 0.311571979221987, 0.296322485947155, 0.252066904333966, 0.183137287858479, 0.096280951919793]^T

当$\omega = 1.9$时，

$u^{(k)} =$

[0.096280584835522, 0.183136896642189, 0.252066175973691, 0.296321505822835, 0.311571541561488, 0.296321398037666, 0.252066645860813, 0.183136912720906, 0.096280744728857,
0.183136896642189, 0.348346870551354, 0.479458184617037, 0.563637578832648, 0.592643350834964, 0.563637385361955, 0.479458684582322, 0.348347059033324, 0.183137070049542,
0.252066175973690, 0.479458184617037, 0.659918444197920, 0.775780071052998, 0.815704206009224, 0.775780283367984, 0.659918237513913, 0.479458824823702, 0.252066456181786,
0.296321505822835, 0.563637578832648, 0.775780071052998, 0.911984757747605, 0.958917595801752, 0.911984510733432, 0.775780517933801, 0.563637807820625, 0.296322018990652,
0.311571541561488, 0.592643350834964, 0.815704206009224, 0.958917595801752, 1.008265913795882, 0.958917504473366, 0.815704187692265, 0.592644097866816, 0.311571472038355,
0.296321398037666, 0.563637385361955, 0.775780283367983, 0.911984510733432, 0.958917504473366, 0.911984513206622, 0.775780424012973, 0.563637667788378, 0.296321627057609,
0.252066645860813, 0.479458684582322, 0.659918237513913, 0.775780517933801, 0.815704187692265, 0.775780424012973, 0.659918299451593, 0.479458828454800, 0.252066473940547,
0.183136912720906, 0.348347059033324, 0.479458824823702, 0.563637807820625, 0.592644097866815, 0.563637667788378, 0.479458828454801, 0.348347617018303, 0.183137232782245,
0.096280744728857, 0.183137070049542, 0.252066456181786, 0.296322018990652, 0.311571472038355, 0.296321627057609, 0.252066473940547, 0.183137232782244, 0.096280630157131]^T

当$\omega = 0.9$时，

$u^{(k)} =$

[0.096282272681564, 0.183139652398577, 0.252069956791081, 0.296325837342726, 0.311575253471145, 0.296325519388657, 0.252069415406956, 0.183139061575778, 0.096281857729140,
0.183139652398577, 0.348352112742520, 0.479465276565224, 0.563644928555881, 0.592651038777218, 0.563644347898518, 0.479464287873116, 0.348351033764009, 0.183138894599879,
0.252069956791081, 0.479465276565224, 0.659926976905088, 0.775790281493278, 0.815713763305381, 0.775789514170431, 0.659925670375469, 0.479463850724592, 0.252068955380802,
0.296325837342726, 0.563644928555881, 0.775790281493278, 0.911995712466478, 0.958928575746166, 0.911994846410339, 0.775788806849324, 0.563643319248879, 0.296324707078561,
0.311575253471145, 0.592651038777218, 0.815713763305381, 0.958928575746166, 1.008276704139478, 0.958927701449238, 0.815712274629742, 0.592649414157174, 0.311574112452177,
0.296325519388657, 0.563644347898518, 0.775789514170431, 0.911994846410339, 0.958927701449238, 0.911994048076557, 0.775788154838133, 0.563642864433324, 0.296324477506940,
0.252069415406956, 0.479464287873116, 0.659925670375469, 0.775788806849324, 0.815712274629742, 0.775788154838133, 0.659924560188363, 0.479463076304811, 0.252068564486510,
0.183139061575778, 0.348351033764009, 0.479463850724592, 0.563643319248879, 0.592649414157173, 0.563642864433324, 0.479463076304811, 0.348350188624973, 0.183138468009505,
0.096281857729140, 0.183138894599879, 0.252068955380801, 0.296324707078561, 0.311574112452177, 0.296324477506940, 0.252068564486510, 0.183138468009505, 0.096281558122037]^T

精确解：

$u^* =$

[0.095491502812526, 0.181635632001340, 0.250000000000000, 0.293892626146237, 0.309016994374947, 0.293892626146237, 0.250000000000000, 0.181635632001340, 0.095491502812526,
0.181635632001340, 0.345491502812526, 0.475528258147577, 0.559016994374947, 0.587785252292473, 0.559016994374947, 0.475528258147577, 0.345491502812526, 0.181635632001340,
0.250000000000000, 0.475528258147577, 0.654508497187474, 0.769420884293813, 0.809016994374947, 0.769420884293813, 0.654508497187474, 0.475528258147577, 0.250000000000000,
0.293892626146237, 0.559016994374947, 0.769420884293813, 0.904508497187474, 0.951056516295154, 0.904508497187474, 0.769420884293813, 0.559016994374948, 0.293892626146237,
0.309016994374947, 0.587785252292473, 0.809016994374947, 0.951056516295154, 1.000000000000000, 0.951056516295154, 0.809016994374947, 0.587785252292473, 0.309016994374948,
0.293892626146237, 0.559016994374947, 0.769420884293813, 0.904508497187474, 0.951056516295154, 0.904508497187474, 0.769420884293813, 0.559016994374948, 0.293892626146237,
0.250000000000000, 0.475528258147577, 0.654508497187474, 0.769420884293813, 0.809016994374947, 0.769420884293813, 0.654508497187474, 0.475528258147577, 0.250000000000000,
0.181635632001340, 0.345491502812526, 0.475528258147577, 0.559016994374948, 0.587785252292473, 0.559016994374948, 0.475528258147577, 0.345491502812526, 0.181635632001340,
0.095491502812526, 0.181635632001340, 0.250000000000000, 0.293892626146237, 0.309016994374948, 0.293892626146237, 0.250000000000000, 0.181635632001340, 0.095491502812526]^T

表 1: 各迭代方法求解结果比较

迭代方法	雅可比迭代法	赛德尔迭代法	超松弛迭代法			
			$\omega = 1.2$	$\omega = 1.3$	$\omega = 1.9$	$\omega = 0.9$
迭代次数k	206	111	76	61	134	134
与精确解之差的1-范数$\|\|u^{(k)} - u^*\|\|_1$	0.330245247376266	0.329859823983780	0.329697874010542	0.329655405892316	0.329491803895450	0.329941467263010
与精确解之差的2-范数$\|\|u^{(k)} - u^*\|\|_2$	0.041422051869794	0.041373556852615	0.041353189819760	0.041347799507484	0.041327525203464	0.041383827166218
与精确解之差的∞-范数$\|\|u^{(k)} - u^*\|\|_\infty$	0.008284442881733	0.008274635236227	0.008270535635037	0.008269426939931	0.008265913795882	0.008276704139478

从中可以看出，收敛速度：雅可比迭代<赛德尔迭代<超松弛因子迭代法（当选取合适的松弛因子），而与精确解的差距没有明显差距。

2. 用雅可比迭代法，分析h取不同值时，求解（1）的收敛情况，求出使$||u^{(k+1)-u^{(k)}}|| < \varepsilon$的近似解及相应的迭代次数. 已知雅可比矩阵为$T = I - \frac{1}{4}L_h$，其谱半径为$\rho(T) = 1 - 2\sin^2\frac{\pi h}{2} \approx 1 - \frac{\pi^2 h^2}{2}$.

表 2: 不同h（部分）下雅可比迭代法求解结果比较

h	0.01	0.02	0.05	0.1	0.2
迭代次数k	11600	3602	722	206	55
与精确解之差的1-范数$\|\|u^{(k)} - u^*\|\|_1$	0.330245247376266	0.329859823983780	0.329697874010542	0.329655405892316	0.329491803895450
与精确解之差的2-范数$\|\|u^{(k)} - u^*\|\|_2$	0.041422051869794	0.041373556852615	0.041353189819760	0.041347799507484	0.041327525203464
与精确解之差的∞-范数$\|\|u^{(k)} - u^*\|\|_\infty$	0.008284442881733	0.008274635236227	0.008270535635037	0.008269426939931	0.008265913795882

从中可以看出，随着h的减小，迭代次数急剧上升。

3. 简单阐述上述数值结果。

（1）关于三种迭代法收敛速度：雅可比迭代公式的分量形式为

$$x_i^{(k+1)} = \frac{1}{a_{ii}}(b_i - \sum_{j\neq i}^{n} a_{ij}x_j^{(k)}),\ \ i = 1, 2, ..., n$$

可见，雅可比迭代法完全通过前一迭代得到的$x^{(k)}$的各分量进行后一次迭代运算；

赛德尔迭代公式的分量形式为

$$x_i^{(k+1)} = \frac{1}{a_{ii}}(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}),\ \ i = 1, 2, ..., n \tag{2}$$

可见，赛德尔迭代法在进行后一次迭代运算时，一部分用的是前一迭代得到的$x^{(k)}$的分量（式（2）中第三项），另一部分用的是后一次迭代中已经算出的分量（式（2）中第二项），因此赛德尔迭代法相比雅可比迭代法能更有快地逼近满足迭代终止条件的近似解；

而超松弛迭代公式的分量形式为

$$\widetilde{x}_i^{(k+1)} = \frac{1}{a_{ii}}(b_i - \sum_{i=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)})$$
$$x_i^{(k+1)} = (1-\omega)x_i^{(k)} + \omega\widetilde{x}_i^{(k+1)}$$

可见，超松弛迭代法将用赛德尔迭代法计算出来的后一次迭代计算结果和前一次迭代计算结果的加权平均作为其后一次迭代的计算结果，当选取适当的权值ω（也就是松弛因子），可以得到合适的收敛步长，从而避免“收敛过头”（比如前一步迭代得到的近似值大于精确值，但由于收敛步长过大，导致后一步迭代得到的近似值小于精确值，近似值在精确值附近来回波动，但不靠近精确值，这样一来会导致需要迭代多次才能达到终止迭代的条件），因此超松弛迭代法相比赛德尔迭代法能更有效率地逼近满足迭代终止条件的近似解。

（2）关于不同h下雅可比迭代法收敛速度：随着h的减小，解所在的正方形区域被划分得越来越细，迭代矩阵的行数和列数越来越多，迭代矩阵的谱半径越来越大

$$\rho(T) = 1 - 2\sin^2\frac{\pi h}{2}$$

收敛速度与迭代矩阵谱半径之间的关系是

$$R(T) = -\ln\rho(T)$$

从而迭代速度急剧增大。在图1中我们可以很清楚地看到收敛速度的倒数可以很好地刻画迭代次数k随着h变小而变大的趋势。

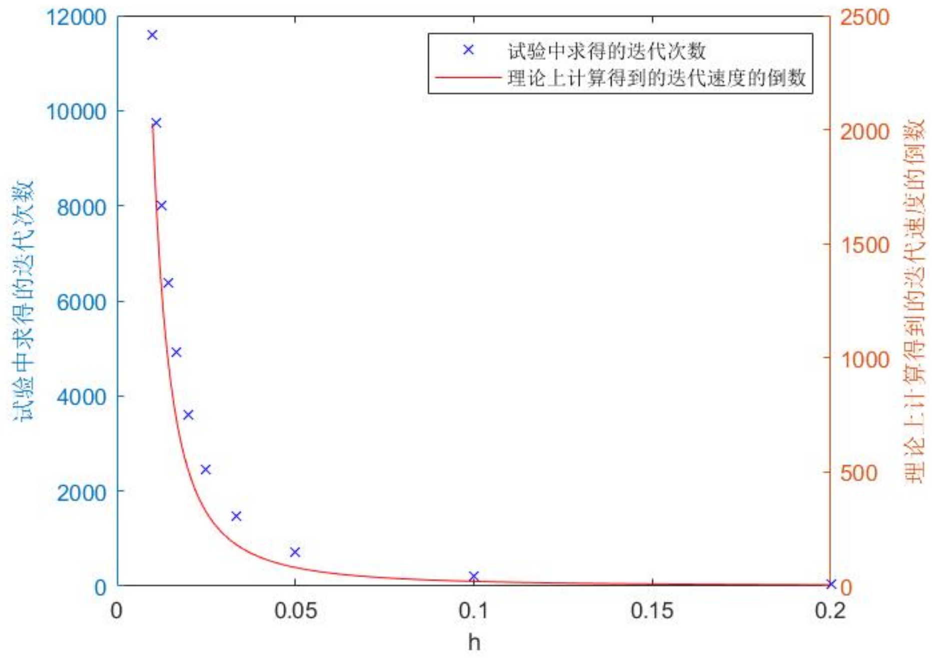

图 1: 收敛速度和迭代次数随h变化情况

3 MATLAB代码

（1）雅可比迭代法

```
clear,clc;
h = 0.1;
N = 1 / h;
epsilon = 10^(-6);
C = diag(ones(N - 2,1),1) + diag(ones(N - 2,1),-1);
L_h = zeros((N - 1) * (N - 1),(N - 1) * (N - 1));
for i = 1:N - 1
    L_h((N - 1) * (i - 1) + 1:(N - 1) * i,(N - 1) * (i - 1) + 1:(N - 1) * i) = 4 * eye(N - 1) - C;
end
for i = 1:N - 2
    L_h((N - 1) * (i - 1) + 1:(N - 1) * i,(N - 1) * i + 1:(N - 1) * (i + 1)) = -eye(N - 1);
    L_h((N - 1) * i + 1:(N - 1) * (i + 1),(N - 1) * (i - 1) + 1:(N - 1) * i) = -eye(N - 1);
end
f_h = zeros((N - 1) * (N - 1),1);
for j = 1:N - 1
    for i = 1:N - 1
        f_h((N - 1) * (j - 1) + i) = f(h * i,h * j);
    end
end
tic
D = eye((N - 1) * (N - 1)) .* L_h;% diagonal matrix of L_h
L = -(L_h - triu(L_h));% lower triangle matrix of L_h
U = -(L_h - tril(L_h));% upper triangle matrix of L_h
b = h^2 * f_h;
J = eye((N - 1) * (N - 1)) - D \ L_h;% Jacobi matrix
f1 = D \ b;
k = 0;% iteration times
u_h_k = ones((N - 1) * (N - 1),1);
u_h_past = zeros((N - 1) * (N - 1),1);
while norm(u_h_k - u_h_past,Inf) > epsilon
    k = k + 1;
    u_h_past = u_h_k;
```

```
    u_h_k = J * u_h_past + f1;
end
toc
u_star = zeros((N - 1) * (N - 1),1);% precise solution
for j = 1:N - 1
    for i = 1:N - 1
        u_star((N - 1) * (j - 1) + i) = sin(pi * h * i) * sin(pi * h * j);
    end
end
k
norm(u_h_k - u_star,1)
norm(u_h_k - u_star,2)
norm(u_h_k - u_star,Inf)
function y = f(x,y)
    y = 2 * pi^2 * sin(pi * x) * sin(pi * y);
end
```

（2）赛德尔迭代法

```
clear,clc;
h = 0.1;
N = 1 / h;
epsilon = 10^(-6);
C = diag(ones(N - 2,1),1) + diag(ones(N - 2,1),-1);
L_h = zeros((N - 1) * (N - 1),(N - 1) * (N - 1));
for i = 1:N - 1
    L_h((N - 1) * (i - 1) + 1:(N - 1) * i,(N - 1) * (i - 1) + 1:(N - 1) * i) = 4 * eye(N - 1) - C;
end
for i = 1:N - 2
    L_h((N - 1) * (i - 1) + 1:(N - 1) * i,(N - 1) * i + 1:(N - 1) * (i + 1)) = -eye(N - 1);
    L_h((N - 1) * i + 1:(N - 1) * (i + 1),(N - 1) * (i - 1) + 1:(N - 1) * i) = -eye(N - 1);
end
f_h = zeros((N - 1) * (N - 1),1);
for j = 1:N - 1
    for i = 1:N - 1
        f_h((N - 1) * (j - 1) + i) = f(h * i,h * j);
    end
end
tic
D = eye((N - 1) * (N - 1)) .* L_h;% diagonal matrix of L_h
L = -(L_h - triu(L_h));% lower triangle matrix of L_h
U = -(L_h - tril(L_h));% upper triangle matrix of L_h
b = h^2 * f_h;
G = (D - L) \ U;% Jacobi matrix
f1 = (D - L) \ b;
k = 0;% iteration times
u_h_k = ones((N - 1) * (N - 1),1);
u_h_past = zeros((N - 1) * (N - 1),1);
while norm(u_h_k - u_h_past,Inf) > epsilon
    k = k + 1;
    u_h_past = u_h_k;
    u_h_k = G * u_h_past + f1;
end
toc
u_star = zeros((N - 1) * (N - 1),1);% precise solution
for j = 1:N - 1
    for i = 1:N - 1
        u_star((N - 1) * (j - 1) + i) = sin(pi * h * i) * sin(pi * h * j);
    end
end
k
norm(u_h_k - u_star,1)
```

```
norm(u_h_k - u_star,2)
norm(u_h_k - u_star,Inf)
function y = f(x,y)
    y = 2 * pi^2 * sin(pi * x) * sin(pi * y);
end
```

（3）超松弛迭代法

```
clear,clc;
h = 0.1;
N = 1 / h;
epsilon = 10^(-6);
omega = 1.2;
C = diag(ones(N - 2,1),1) + diag(ones(N - 2,1),-1);
L_h = zeros((N - 1) * (N - 1),(N - 1) * (N - 1));
for i = 1:N - 1
    L_h((N - 1) * (i - 1) + 1:(N - 1) * i,(N - 1) * (i - 1) + 1:(N - 1) * i) = 4 * eye(N - 1) - C;
end
for i = 1:N - 2
    L_h((N - 1) * (i - 1) + 1:(N - 1) * i,(N - 1) * i + 1:(N - 1) * (i + 1)) = -eye(N - 1);
    L_h((N - 1) * i + 1:(N - 1) * (i + 1),(N - 1) * (i - 1) + 1:(N - 1) * i) = -eye(N - 1);
end
f_h = zeros((N - 1) * (N - 1),1);
for j = 1:N - 1
    for i = 1:N - 1
        f_h((N - 1) * (j - 1) + i) = f(h * i,h * j);
    end
end
tic
D = eye((N - 1) * (N - 1)) .* L_h;% diagonal matrix of L_h
L = -(L_h - triu(L_h));% lower triangle matrix of L_h
U = -(L_h - tril(L_h));% upper triangle matrix of L_h
b = h^2 * f_h;
L_omega = (D - omega * L) \ ((1 - omega) * D + omega * U);%Jacobi matrix
f1 = omega * ((D - omega * L) \ b);
k = 0;% iteration times
u_h_k = ones((N - 1) * (N - 1),1);
u_h_past = zeros((N - 1) * (N - 1),1);
while norm(u_h_k - u_h_past,Inf) > epsilon
    k = k + 1;
    u_h_past = u_h_k;
    u_h_k = L_omega * u_h_past + f1;
end
toc
u_star = zeros((N - 1) * (N - 1),1);% precise solution
for j = 1:N - 1
    for i = 1:N - 1
        u_star((N - 1) * (j - 1) + i) = sin(pi * h * i) * sin(pi * h * j);
    end
end
k
norm(u_h_k - u_star,1)
norm(u_h_k - u_star,2)
norm(u_h_k - u_star,Inf)
function y = f(x,y)
    y = 2 * pi^2 * sin(pi * x) * sin(pi * y);
end
```