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SI 140
Probability and Statistics
Semester Spring 2019
Assignment 6

Problem 1. Let $U_1, U_2, ..., U_{60}$ be i.i.d. Unif(0,1) and $X = U_1 + U_2 + \cdots + U_{60}$

1. which important distribution is the distribution of X very close to? Specify what the parameters are, and state which theorem justifies your choice.

2. Give a simple but accurate approximation for $P(X > 17)$. Justify briefly.

Solution:
1. The distribution of X is very close to Normal Distribution, with expectation $\mu = 60E(U_i) = 60 \times \frac{1}{2} = 30$ and variance $\sigma^2 = 60Var(U_i) = 60\int_0^1 (x - \frac{1}{2})^2 dx = 5$.
Central Limit Theorem justifies my choice.
2. Approximation: $P(X > 17) \approx 1$. Justification:

$$P(X > 17) = P(\frac{X - 30}{\sqrt{5/60}} > \frac{17 - 30}{\sqrt{5/60}}) \approx \Phi(26\sqrt{3}) \approx 1$$

□

Problem 2. Let X and Y be Pois(λ) r.v.s. and $T = X + Y$. Suppose that X and Y are not independent, and in fact $X = Y$. Prove or disprove the claim that $T \sim Pois(2\lambda)$ in this scenario.

Solution: Disprove:
The p.g.f. of X (or Y) is

$$G_X(z) = \exp[\lambda(z - 1)]$$

Since $T = X + Y$ and $X = Y$,

$$T = 2X$$

The p.g.f. of T is

$$G_T(z) = G_{2X}(z) = G_X(z^2) = \exp[\lambda(z^2 - 1)]$$

However, the p.g.f. of $Pois(2\lambda)$ should be

$$G_{Pois(2\lambda)}(z) = \exp[2\lambda(z - 1)]$$

Therefore, $T \sim Pois(2\lambda)$ is incorrect. □

Problem 3. Let X, Y, Z be r.v.s such that $X \sim \mathcal{N}(0, 1)$ and conditional on $X = x$, Y and Z are i.i.d. $\mathcal{N}(x, 1)$.

1. Find the joint PDF of X, Y, Z.

2. By definition, Y and Z are conditionally independent given X. Discuss intuitively whether or not Y and Z are also unconditianally independent.

3. Find the joint PDF of Y and Z. You can leave your answer as an integral, though the integral can be done with some algebra (such as completing the square) and facts about the Normal distribution.

Solution:

1. The PDF of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})$$

The PDF of $Y|X$ and $Z|X$ are

$$\begin{aligned} f_{Y|X}(y|x) =& \frac{1}{\sqrt{2\pi}} \exp[-\frac{(y-x)^2}{2}] \\ f_{Z|X}(z|x) =& \frac{1}{\sqrt{2\pi}} \exp[-\frac{(z-x)^2}{2}] \end{aligned}$$

The joint PDF of X, Y, Z is

$$\begin{aligned} f_{X,Y,Z}(x,y,z) =& f_X(x) f_{Y,Z|X}(y,z|x) = f_X(x) f_{Y|X}(y|x) f_{Z|X}(z|x) \\ =& \frac{1}{(2\pi)^{3/2}} \exp[-\frac{x^2 + (y-x)^2 + (z-x)^2}{2}] \end{aligned}$$

2. Intuitively speaking, Y and Z are <u>not</u> unconditionally independent.

3. The joint PDF of (Y,Z) is

$$\begin{aligned} f_{Y,Z}(y,z) =& \int_{-\infty}^{+\infty} f_{X,Y,Z}(x,y,z) dx \\ =& \int_{-\infty}^{+\infty} \frac{1}{(2\pi)^{3/2}} \exp[-\frac{3x^2 - 2(y+z)x + y^2 + z^2}{2}] dx \\ =& \frac{1}{(2\pi)^{3/2}} \exp[-\frac{y^2+z^2}{2}] \int_{-\infty}^{+\infty} \exp[-\frac{3x^2 - 2(y+z)x}{2}] dx \\ =& \frac{1}{(2\pi)^{3/2}} \exp[-\frac{y^2+z^2}{2}] \int_{-\infty}^{+\infty} \exp[-\frac{[\sqrt{3}x - \frac{(y+z)}{\sqrt{3}}]^2}{2} + \frac{(y+z)^2}{6}] dx \\ =& \frac{1}{2\sqrt{3}\pi} \exp[-\frac{y^2+z^2-xy}{3}] \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp[-\frac{[\sqrt{3}x - \frac{(y+z)}{\sqrt{3}}]^2}{2}] d(\sqrt{3}x) \\ =& \frac{1}{2\sqrt{3}\pi} \exp[-\frac{y^2+z^2-xy}{3}] \end{aligned}$$

$\square$

Problem 4. A large freight elevator can transport a maximum of 9800 pounds. Suppose a load of cargo containing 49 boxes must be transported via the elevator. Experience has shown that the weight of boxes of this type of cargo follows a distribution with mean $\mu = 205$ pounds and standard deviation $\sigma = 15$ pounds. Based on this information, what is the probability that all 49 boxes can be safely loaded onto the freight elevator and transported?

Solution: Let weight of i-th box be X_i and the total wight of the 49 boxes be S_n.

$$S_n = \sum_{i=1}^{49} X_i$$

We can approximately regard S_n as normal distribution with expectation of $49\mu = 10045$ and variance of $49\sigma^2 = 11025$, so the probability that all 49 boxes can be safely loaded onto the freight elevator and transported is

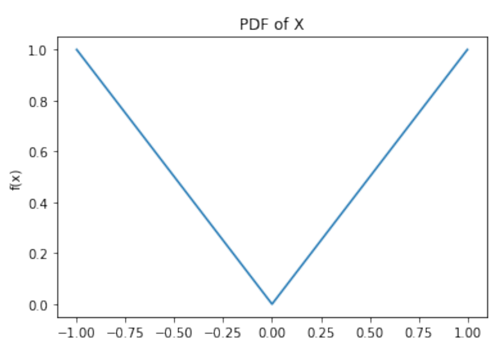
$$P(S_n \leq 9800) = P(\frac{S_n - 10045}{\sqrt{11025}} \leq \frac{9800 - 10045}{\sqrt{11025}}) = \Phi(-\frac{7}{3}) = 0.009815$$

□

Problem 5. In this experiment we will sample 500 samples from probability distribute function $f(x)$. Using the 500 outcomes we will compute the sample mean . We will repeat until we obtain 1000 values of $\bar{X}$.

1).Plot the histogram of X.

2).Plot the histogram of $\bar{X}$.



Solution:

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1)

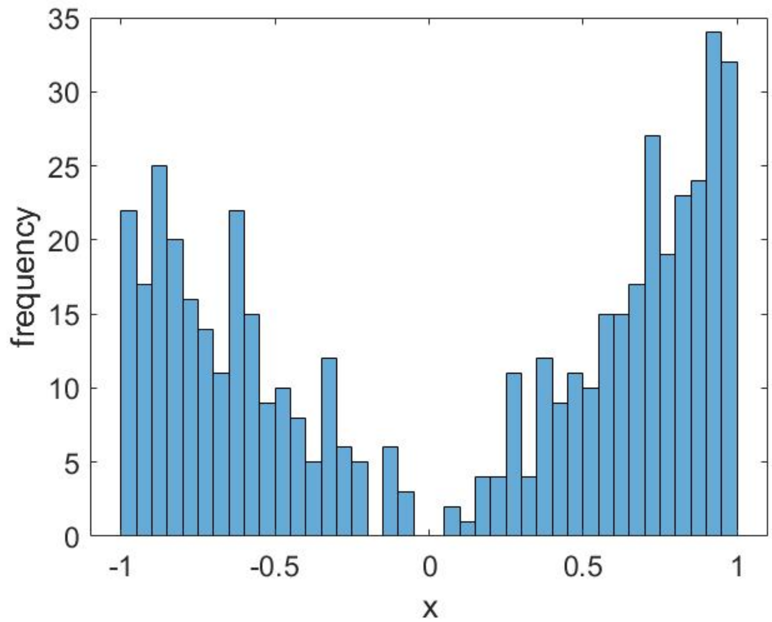

图 1: The histogram of x

2)

$\square$

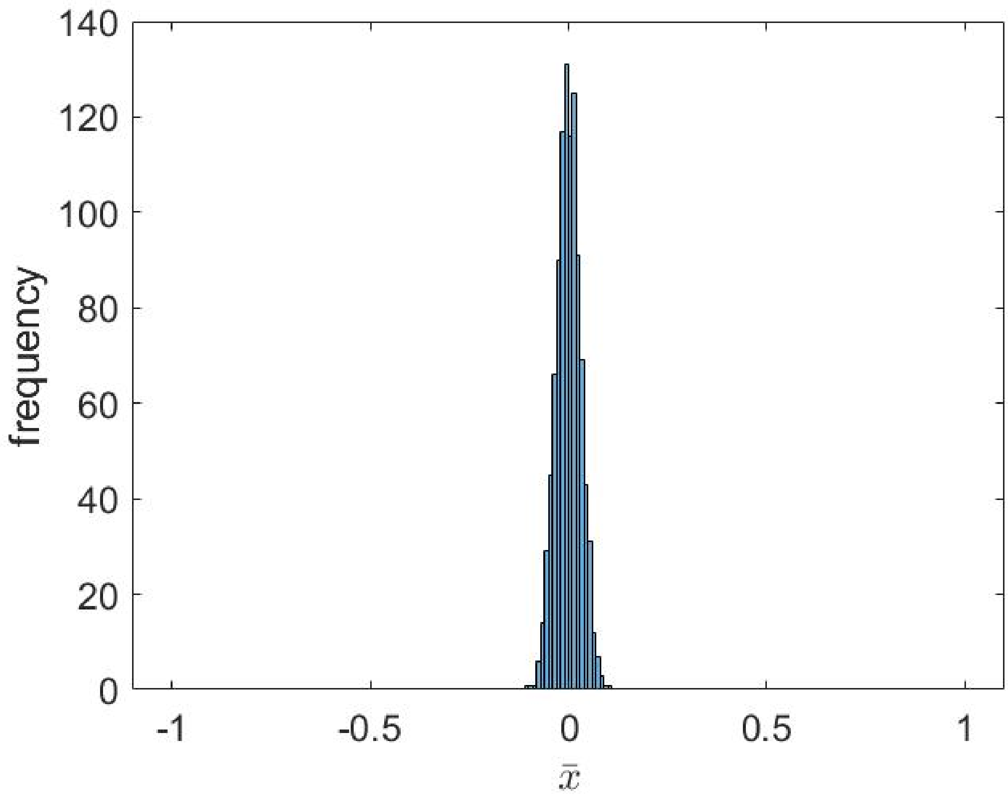

图 2: The histogram of $\bar{x}$

MATLAB代码

```
close all;clear;clc;
x = zeros(500,1);
x_bar = zeros(1000,1);
for i = 1:1000
    for j = 1:500
        x(j) = f5(rand);
    end
    x_bar(i) = mean(x);
end
figure(1);
histogram(x,'BinEdges',-1:0.05:1);
xlabel('x');
ylabel('frequency');
figure(2);
histogram(x_bar,'BinEdges',-1:0.01:1);
```

```
xlabel('$\bar{x}$','Interpreter','LaTeX');
ylabel('frequency');
function y = f5(x)
    % mapping Unif(0,1) to f(x)
    y = sign(x - 1 / 2) .* sqrt(abs(2 * x - 1));
end
```