SI 140
Probability and Statistics
Semester Spring 2019
Assignment 6

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Problem 1. Let X be the total from rolling 6 fair dice, and let $X_1, ..., X_6$ be the individual rolls. What is $P(X = 18)$?

Solution: The generating function of X_i $(i = 1, 2, \cdots, 6)$ is

$$g_{X_i}(z) = \frac{1}{6}z + \frac{1}{6}z^2 + \frac{1}{6}z^3 + \frac{1}{6}z^4 + \frac{1}{6}z^5 + \frac{1}{6}z^6 = \frac{z}{6}(1 + z + z^2 + z^3 + z^4 + z^5 + z^6) \ (i = 1, 2, \cdots, 6)$$

Then the generating function of $X = \sum_{i=1}^{6} X_i$ is

$$g_X(z) = \prod_{i=1}^{6} g_{X_i}(z) = \frac{z^6}{6^6}(1 + z + z^2 + z^3 + z^4 + z^5)^6 = \frac{z^6}{6^6}(\frac{1 - z^6}{1 - z})^6$$

where

$$(1 - t^6)^6 = \sum_{j=0}^{6} \binom{6}{j} (-1)^j z^{6j}$$

and

$$\frac{1}{(1 - z)^6} = (1 + z + z^2 + \cdots)^6 = \sum_{k=0}^{\infty} a_k z^k$$

Here, a_k is the number of lists ("list" means order matters) consisting of six independent non-negative integer which add to k. We can also regard a_k as the number of cases in which k balls in a line are separated by five boards, so

$$a_k = \binom{k+5}{5}$$

So

$$\frac{1}{(1 - z)^6} = (1 + z + z^2 + \cdots)^6 = \sum_{k=0}^{\infty} \binom{k+5}{5} z^k$$

and

$$g_X(z) = \frac{z^6}{6^6} \left[\sum_{j=0}^{6} \binom{6}{j} (-1)^j z^{6j} \right] \left[\sum_{k=0}^{\infty} \binom{k+5}{5} z^k \right]$$

In the generating function above, the coefficient of z^{18} item is

$$\frac{1}{6^6} \left[\binom{6}{0}\binom{17}{5} - \binom{6}{1}\binom{11}{5} + \binom{6}{2}\binom{5}{5} \right] = \frac{3431}{6^6} \approx 0.07354$$

Reference: Blitzstein, Joseph K., and Jessica Hwang. *Introduction to probability*. Chapman and Hall/CRC, 2014: p264-265. □

Problem 2. Find the MGF of $X \sim Unif(a, b)$ and $Y \sim Expo(\lambda)$.

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Solution: The PDF of X is

$$f_X(x) = \frac{1}{b-a}, \ \ x \in (a, b)$$

The MGF of X is

$$M_X(t) = \int_a^b e^{-tx} f_X(x) dx = \frac{e^{-at} - e^{-bt}}{(b-a)t}$$

The PDF of Y is

$$f_Y(y) = \lambda e^{-\lambda y}, \ \ x \geq 0$$

The MGF of Y is

$$M_Y(t) = \int_0^\infty e^{-tx} f_Y(y) dy = \frac{\lambda}{t + \lambda}$$

□

Problem 3. Find the MGF of $X \sim Bern(p)$ and $Y \sim Bin(n, p)$.

Solution: The PMF of X is

$$P_X(x) = \begin{cases} p, & \text{if } x = 1 \\ 1 - p, & \text{if } x = 0 \end{cases}$$

The MGF of X is

$$M_X(t) = (1-p) + pe^{-t}$$

The MGF of Y is

$$M_Y(t) = [M_X(t)]^n = [(1-p) + pe^{-t}]^n$$

□

Problem 4. Consider a setting where a Poisson approximation should work well: let $A_1, ..., A_n$ be independent, rare events, with n large and $p_j = P(A_j)$ small for all j. Let $X = I(A_1) + \cdots + I(A_n)$ count how many of the rare events occur, and Let $\lambda = E(X)$.

(a). Find the MGF of X.

(b). If the approximation $1 + x \approx e^x$ (this is a good approximation when x is very close to 0 but terrible when x is not close to 0) is used to write each factor in the MGF of X as e to a power.What happens to the MGF? (Hint: if $Y \sim Pois(\lambda)$, then $M_Y = e^{(e^{-t}-1)\lambda}$)

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Solution:

(a) The MGF of X is

$$\begin{aligned} M_X(t) =& \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} e^{-kt} \\ =& e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{-t})^k}{k!} \\ =& e^{-\lambda} e^{\lambda e^{-t}} = e^{\lambda(e^{-t}-1)} \end{aligned}$$

(b) If the approximation $1 + x \approx e^x$ works, the MGF of x is

$$M_X(t) \approx e^{-\lambda t} = -\lambda t + 1$$

□

Problem 5. Suppose $X \sim Bin(n, p)$. By using Binomial PGF, find the expectation $E(X)$ and variance $Var(X)$.

Solution: According to the definition, the PGF of a Bernoulli distributed random variable is

$$G_{X_i}(z) = (1-p) + pz$$

The PGF of X is

$$G_X(z) = G_{\sum_{i=1}^n X_i}(z) = \prod_{i=1}^n G_{X_i} z = [(1-p) + pz]^n$$

Then

$$\begin{aligned} G'_X(z) =& np[(1-p) + pz]^{n-1} \\ G''_X(z) =& n(n-1)p^2[(1-p) + pz]^{n-2} \end{aligned}$$

The expectation is

$$E(X) = G'(1) = np$$

The variance is

$$Var(X) = G''(1) + G'(1) - [G'(1)]^2 = n(n-1)p^2 + np - (np)^2 = np(1-p)$$

□

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Problem 6. If a random variable X has the following moment-generating function:

$$M(t) = \frac{1}{10}e^{-t} + \frac{2}{10}e^{-2t} + \frac{3}{10}e^{-3t} + \frac{4}{10}e^{-4t}$$

for all t, then what is thePMF of X?

Solution: Let $t = -\ln z$, the PGF of X is

$$G_X(z) = M_X(-\ln z) = \frac{1}{10}z^1 + \frac{2}{10}z^2 + \frac{3}{10}z^3 + \frac{4}{10}z^4$$

The PMF of X is

$$P_X(x) = \begin{cases} \frac{1}{10}, & x = 1 \\ \frac{2}{10}, & x = 2 \\ \frac{3}{10}, & x = 3 \\ \frac{4}{10}, & x = 4 \end{cases}$$

□

Problem 7. Suppose that Y has the following moment-generating function:

$$M_Y(t) = \frac{e^{-t}}{4 - 3e^{-t}}$$

I).Find E(Y)
II).Find Var(Y)

Solution:
I)

$$E(Y) = (-1)^1 E'(0) = -\frac{-4e^{-t}}{(4-3e^{-t})^2}|_{t=0} = 4$$

II)

$$E(Y^2) = (-1)^2 E''(0) = \frac{16e^{-t} + 12e^{-2t}}{(4-3e^{-t})^3}|_{t=0} = 28$$
$$Var(Y) = E(Y^2) - [E(Y)]^2 = 12$$

□

Problem 8. If a random variable X has $E[X^k] = 0.2$ k=1,2,3....,then what is thePMF of X?

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Solution: The MGF of X

$$\begin{aligned}M_X(t) =& 1+\frac{(-1)^1}{1!}E(X^1)t+\frac{(-1)^2}{2!}E(X^2)t^2+\frac{(-1)^3}{3!}E(X^3)t^3+\cdots\\ =& 0.8+0.2(1+\frac{(-1)^1}{1!}t+\frac{(-1)^2}{2!}t^2+\frac{(-1)^3}{3!}t^3+\cdots)\\ =& 0.8+0.2e^{-t}\end{aligned}$$

Let $t=-\ln z$, the PGF of X is

$$G_X(z)=M_X(-\ln z)=0.8+0.2z$$

The PMF of X is

$$P_X(x)=\begin{cases}0.2, & \text{if } x=1\\ 0.8, & \text{if } x=0\end{cases}$$

□