SI 140
Probability and Statistics
Semester Spring 2019
Assignment 4

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Problem 1. Let $\lambda > 0$ and define f as follows:

$$f(u) = \begin{cases} \frac{1}{2}\lambda e^{-\lambda u} & \text{if } u \geq 0; \\ \frac{1}{2}\lambda e^{+\lambda u} & \text{if } u < 0 \end{cases} \tag{1}$$

This f is called bilateral exponential. If X has density f, find the density of $|X|$.

Solution: When $x > 0$,

$$\begin{aligned} F_{|X|}(x) = P(|X| \leq x) &= P(-x \leq X \leq x) \\ &= \int_{-x}^{0} f(u)du + \int_{0}^{\infty} f(u)du \\ &= 1 - e^{-\lambda x} \end{aligned}$$

$$f(x) = F'_{|X|}(x) = \lambda e^{-\lambda x}, \quad x > 0$$

□

Problem 2. If X is a positive random variable with density f, find the density of $+\sqrt{X}$. Apply this to the distribution of the side length of a square when its area is uniformly distributed in $[a, b]$.

Solution: When $x > 0$

$$F_{\sqrt{X}}(x) = P(\sqrt{X} \leq x) = P(X \leq x^2) = \int_{-\infty}^{x^2} f(x)dx$$

$$f_{\sqrt{X}}(x) = F'_{\sqrt{X}}(x) = 2xf(x^2), \quad x > 0$$

If $f(x) = \frac{1}{b-a}, \quad a < x < b$, then

$$f_{\sqrt{X}}(x) = \frac{2x}{b-a}, \quad \sqrt{a} < x < \sqrt{b}$$

□

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Problem 3. If X has density f, find the density of (i)$aX+b$ where a and b are constants; (ii) X^2.

Solution: (i). If $a=0$, then $f(x)=0$

When $a \neq 0$,

$$F_{aX+b}(x) = P(aX+b \leq x) = \begin{cases} P(X \leq \frac{x-b}{a}), & a>0 \\ P(X \geq \frac{x-b}{a}), & a<0 \end{cases} = \begin{cases} \int_{-\infty}^{\frac{x-b}{a}} f(x)dx, & a>0 \\ \int_{\frac{x-b}{a}}^{\infty} f(x)dx, & a<0 \end{cases}$$

$$f_{aX+b}(x) = F'_{aX+b}(x) = \begin{cases} \frac{1}{a}f(\frac{x-b}{a}), & a>0 \\ -\frac{1}{a}f(\frac{x-b}{a}), & a<0 \end{cases} = \frac{1}{|a|}f(\frac{x-b}{a})$$

(ii). When $x>0$,

$$F_{X^2}(x) = P(X^2 \leq x) = P(-\sqrt{x} \leq X \leq \sqrt{x}) = F(\sqrt{x}) - F(-\sqrt{x})$$

Thus,

$$f_{X^2}(x) = \frac{1}{2\sqrt{x}}\big(f(\sqrt{x}) + f(-\sqrt{x})\big), \quad x>0$$

□

Problem 4. If f and g are two density functions, show that $\lambda f + \mu g$ is also a density function, where $\lambda + \mu = 1, \lambda \geq 0, \mu \geq 0$.

Solution:

$$\lambda f(x) + \mu g(x) \geq 0, \qquad \forall x$$

$$\int_{-\infty}^{\infty} \lambda f(x) + \mu g(x) dx = \lambda + \mu = 1$$

□

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Problem 5. Let

$$f(u) = ue^{-u}, \quad u \geq 0$$

Show that f is a density function. Find $\int_0^\infty uf(u)du$.

Solution: Because $u > 0$, $f(u) = \mu e^{-u} \geq 0$ and $\int_{-\infty}^{\infty} ue^{-u}du = \int_0^\infty ue^{-u}du = 1$, thus f is a density function.

$$\int_0^\infty uf(u)du = 2$$

□

Problem 6. A number of μ is called the median of the random variable X iff $P(X \geq \mu) \geq 1/2$ and $P(X \leq \mu) \geq 1/2$. Show that such a number always exists but need not be unique. Here is a practical example. After n examination papers have been graded, they are arranged in descending order. There is one in the middle if n is odd, two if n is even, corresponding to the median(s). Explain the probability model used.

Solution: For continuous random variables, we can find the point μ such that $P(X \leq \mu) = \frac{1}{2}$ because of the continuity of CDF.

For discrete random variables,

$$P(X \geq \mu) = 1 - P(X < \mu) \geq \frac{1}{2}$$

which means that μ is a median if it satisfies

$$P(X < \mu) \leq 1/2 \quad \text{and} \quad P(X \leq \mu) \geq 1/2$$

Thus, median $= \min\{\mu \in \mathcal{S} | P(X \leq \mu) \geq 1/2\}$, where $\mathcal{S}$ is sample space.
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Problem 7. Suppose X_1, X_2, X_3 are independent identically distributed (i.i.d.) Unif$(0,1)$ random variables and let $Y = X_1 + X_2 + X_3$. (i). Find PDF of Y; (ii). Find $E(Y)$.

Solution: Suppose $Z = X + Y$,

$$\begin{aligned} F_Z(z) &= P(Z \leq z) \\ &= P(X + Y \leq z) \\ &= \iint_{x+y\leq z} f(x,y)dxdy \\ &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{z-x} f(x,y)dy \\ &= \int_{-\infty}^{z-x} \left(\int_{-\infty}^{\infty} f(x,y)dx \right) dy \end{aligned}$$

$$f(z) = F'_Z(z) = \int_{-\infty}^{\infty} f(x, z-x)dx$$

If X and Y are independent, then

$$f(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx$$

(i).
Let $Z = X_1 + X_2$,

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X_1}(x_1) f_{X_2}(z-x_1)dx_1 = \begin{cases} \int_0^z dx_1, & 0 \leq z < 1 \\ \int_{z-1}^1 dx_1, & 1 \leq z \leq 2 \\ 0, & \text{o.w} \end{cases} = \begin{cases} z, & 0 \leq z < 1 \\ 2-z, & 1 \leq z \leq 2 \\ 0, & \text{o.w.} \end{cases}$$

Then, $Y = Z + X_3$

$$f_Y(y) = \int_{-\infty}^{\infty} f_Z(z) f_{X_3}(y-z)dz = \begin{cases} \int_0^y zdz, & 0 \leq y < 1 \\ \int_{y-1}^1 zdz + \int_1^y 2-zdz, & 1 \leq y < 2 \\ \int_{y-1}^2 2-zdz, & 2 \leq y \leq 3 \\ 0, & \text{o.w.} \end{cases}$$

$$= \begin{cases} \frac{1}{2}y^2, & 0 \leq y < 1 \\ -y^2 + 3y - \frac{3}{2}, & 1 \leq y < 2 \\ \frac{1}{2}y^2 - 3y + \frac{9}{2}, & 2 \leq y \leq 3 \\ 0, & \text{o.w.} \end{cases}$$

(ii).

$$E(Y) = 3E(X_1) = \frac{3}{2}$$

□

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Problem 8. There are 40 people in a room. Assume each person's birthday is equally likely to be any of the 365 days of the year (we exclude February 29), and that peoples birthdays are independent (we assume there are no twins in the room). What is the probability that two or more people in the group have the same birthday?

Solution:

$$\begin{aligned}
&P(\text{"two or more have same birthday"}) \\
&= 1 - P(\text{"no two people share a birthday"}) \\
&= 1 - \frac{\text{\# of ways to sample the 365 days without replacement}}{\text{\# of ways to sample the 365 days with replacement}} \\
&= 1 - \frac{365 \times 364 \times \cdots \times (365 - 40 + 1)}{365^{40}} \\
&\approx 0.8912
\end{aligned}$$

$\square$

Problem 9. Let X_1,...,X_n be independent, with $X_j \sim \text{Expo}(\lambda_j)$. Let $L = \min\{X_1, ..., X_n\}$. Show that $L \sim \text{Expo}(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$ and find $E(L)$.

Solution:

$$\begin{aligned}
F_L(t) = P(L \leq t) &= 1 - P(L > t) \\
&= 1 - P(X_1 > t, ..., X_n > t) \\
&= 1 - e^{-(\lambda_1 + \cdots + \lambda_n)t}
\end{aligned}$$

Thus,

$$f(t) = F_L'(t) = (\lambda_1 + \cdots + \lambda_n)e^{-(\lambda_1 + \cdots + \lambda_n)t}$$

$$E(L) = \frac{1}{\lambda_1 + \cdots + \lambda_n}$$

$\square$

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Problem 10. (Expectation via Survival Function) Let X be a nonnegative random variable. Let F be the CDF of X, and $G(x) = 1 - F(x) = P(X > x)$. The function G is called the survival function of X. Show that
(i). The expectation of a nonnegative integer-valued discrete random variable X is

$$E(X) = \sum_{n=0}^{\infty} G(n)$$

(ii). The expectation of a nonnegative continuous random variable X is

$$E(X) = \int_0^{\infty} G(x)dx$$

Solution: (i).

$$\begin{aligned}
\sum_{n=0}^{\infty} G(n) &= \sum_{n=0}^{\infty} P(X > n) \\
&= \sum_{n=0}^{\infty} P(X \geq n+1) \\
&= \sum_{n=0}^{\infty} \sum_{m=n+1}^{\infty} P(X = m) \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{m} P(X = m) \\
&= \sum_{m=1}^{\infty} mP(X = m) \\
&= E(X)
\end{aligned}$$

(ii).

$$G(x) = P(X > x) = \int_x^{\infty} f(y)dy$$

$$\begin{aligned}
\int_0^{\infty} G(x)dx &= \int_0^{\infty} \int_x^{\infty} f(y)dydx \\
&= \int_0^{\infty} dy \int_0^{y} f(y)dx \\
&= \int_0^{\infty} yf(y)dy \\
&= E(X)
\end{aligned}$$

□