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Problem 1. Show that if P and Q are two probability measures defined on the same (countable) sample space, then $aP + bQ$ is also a probability measure for any two nonnegative numbers a and b satisfying $a + b = 1$. Give a concrete illustration of such a mixture.

Solution: Let Ω be the sample space that P and Q are defined on and $\mathscr{F} \subset 2^{\Omega}$ be a σ-algebra. Because P and Q are two probability measures, for every set $A \in \mathscr{F}$,

$$P(A) \geq 0, \ \ Q(A) \geq 0$$

And because a and b are two nonnegative number,

$$aP(A) + bQ(A) \geq 0 \tag{1}$$

Because A and B are two probability measures, for any countable collections of disjoint sets $A_1, A_2, ... \in \mathscr{F}$,

$$P(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} P(A_j), \ \ Q(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} Q(A_j)$$

we have

$$\begin{aligned} aP(\bigcup_{j=1}^{\infty} A_j) + bQ(\bigcup_{j=1}^{\infty} A_j) =& a\sum_{j=1}^{\infty} P(A_j) + b\sum_{j=1}^{\infty} Q(A_j) \\ =& \sum_{j=1}^{\infty} [aP(A_j) + bQ(A_j)] \end{aligned} \tag{2}$$

Because P and Q are two probability measures,

$$P(\Omega) = 1, \ \ P(\Omega) = 1$$

And because $a + b = 1$,

$$aP(\Omega) + bQ(\Omega) = a \times 1 + b \times 1 = 1 \tag{3}$$

Because of (1), (2) and (3), $aP + bQ$ is also a probability measure.

A concrete illustration of such a mixture: Suppose P and Q are two probability measures defined on the same sample space $\Omega = \{0, 1\}$

$$P(A) = \begin{cases} 0, & A = \emptyset \\ 0.6, & \text{if } A = \{0\} \\ 0.4, & \text{if } A = \{1\} \\ 1, & \text{if } A = \Omega \end{cases}, \ \ Q(A) = \begin{cases} 0, & \text{if } A = \emptyset \\ 0.4, & \text{if } A = \{0\} \\ 0.6, & \text{if } A = \{1\} \\ 1, & \text{if } A = \Omega \end{cases}$$

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and

$$a = 0.5, \quad b = 0.5$$

Then we have

$$aP(A) + bQ(A) = \begin{cases} 0, & \text{if } A = \emptyset \\ 0.5, & \text{if } A = \{0\} \\ 0.5, & \text{if } A = \{1\} \\ 1, & \text{if } A = \Omega \end{cases}$$

It is obvious that $aP + bQ$ satisfies the three axioms

1. For every set $A \in \mathscr{F}$, $aP(A) + bQ(A) \geq 0$
2. For any collections of disjoint sets $A_1, A_2, ... \in \mathscr{F}$,
$$aP(\bigcup_{j=1}^{\infty} A_j) + bQ(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty}[aP(A_j) + bQ(A_j)]$$
3. $aP(\Omega) + bQ(\Omega) = 1$

Therefore, $aP + bQ$ is also a probability measure. □

Problem 2. If P is a probability measure, show that the function $P/2$ satisfies Axioms (i) and (ii) but not (iii). The function P^2 satisfies (i) and (iii) but not necessarily (ii); give a conterexample to (ii).

Solution: Let Ω be the sample space and $\mathscr{F} \in 2^{\Omega}$ be a σ-algebra. Because P is a probability measure, it satisfies the three axioms:

1. For every set $A \in \mathscr{F}$, $P(A) \geq 0$
2. For any collections of disjoint sets $A_1, A_2, ... \in \mathscr{F}$,
$$P(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} P(A_j)$$
3. $P(\Omega) = 1$

Because of 1., for every set $A \in \mathscr{F}$, $\frac{P(A)}{2} \geq 0$. And because of 2., for any collections of disjoint sets $A_1, A_2, ... \in \mathscr{F}$,

$$\frac{P(\bigcup_{j=1}^{\infty} A_j)}{2} = \frac{\sum_{j=1}^{\infty} P(A_j)}{2} = \sum_{j=1}^{\infty} \frac{P(A_j)}{2}$$

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However,

$$\frac{P(\Omega)}{2} = \frac{1}{2} \neq 1$$

Therefore, the function $P/2$ satisfies Axioms (i) and (ii) but not (iii).

It is obvious that, for every set A, $P^2(A) \geq 0$. And because of 3.,

$$P^2(\Omega) = 1^2 = 1$$

However, for a collection of disjoint sets $A_1, A_2, ... \in \mathscr{F}$,

$$P^2(\bigcup_{j=1}^{\infty} A_j) = [\sum_{j=1}^{\infty} P(A_j)]^2$$

which does not necessarily equal $\sum_{j=1}^{\infty} P^2(A_j)$.

A counterexample: sets $A_1, A_2, ...$ are disjoint, $P(A_1) = P(A_2) = \frac{1}{2}$, $P(A_j) = 0$ for $j \geq 3$,

$$P^2(\bigcup_{j=1}^{\infty} A_j) = [P(A_1) + P(A_2)]^2 = 1 \neq \frac{1}{2} = P^2(A_1) + P^2(A_2) = \sum_{j=1}^{\infty} P^2(A_j)$$

Therefore, the function P^2 satisfies (i) and (iii) but not necessarily (ii). $\square$

Problem 3. Show that if the two events (A, B) are independent, then so are (A, B^c), (A^c, B) and (A^c, B^c). Generalize this result to three independent events.

Solution: Because the two events (A, B) are independent,

$$P(A \cap B) = P(A)P(B)$$

Then we have

$$\begin{aligned}
P(A \cap B^c) =& P(A \cap (\Omega - B)) \\
=& P(A \cap \Omega - A \cap B) \\
=& P(A - A \cap B) \\
& \text{(because } (A \cap B) \subset A\text{)} \\
=& P(A) - P(A \cap B) \\
=& P(A) - P(A)P(B) \\
=& P(A)[1 - P(B)] \\
=& P(A)P(B^c)
\end{aligned}$$

$\Longrightarrow$ the two events (A, B^c) are independent. Similarly, so are (A^c, B).
And

$$\begin{aligned} P(A^c\cap B^c) =&P(A^c\cap(\Omega-B))\\ =&P(A^c-A^c\cap B)\\ =&P(A^c)-P(A^c\cap B)\\ =&P(A^c)-P(A^c)P(B)\\ =&P(A^c)[1-P(B)]\\ =&P(A^c)P(B^c) \end{aligned}$$

$\Longrightarrow$ the two events (A^c, B^c) are independent.
Generalize to this result to three independent events: If the three events (A, B, C) are independent, then

$$P(A\cap B)=P(A)P(B),\ \ P(B\cap C)=P(B)P(C),\ \ P(C\cap A)=P(C)P(A)$$
$$P(A\cap B\cap C)=P(A)P(B)P(C)$$

Then we have

$$\begin{aligned} P(A\cap B\cap C^c) =&P(A\cap B\cap(\Omega-C))\\ =&P(A\cap B-A\cap B\cap C)\\ =&P(A\cap B)-P(A\cap B\cap C)\\ =&P(A)P(B)-P(A)P(B)P(C)\\ =&P(A)P(B)[1-P(C)]\\ =&P(A)P(B)P(C^c) \end{aligned}$$

$$P(A\cap B)=P(A)P(B),\ \ P(B\cap C^c)=P(B)P(C^c),\ \ P(C^c\cap A)=P(C^c)P(A)$$

$\Longrightarrow$the three events (A, B, C^c) are independent. Similarly, so are (A, B^c, C) and (A^c, B, C).

$$\begin{aligned} P(A\cap B^c\cap C^c) =&P(A\cap B^c\cap(\Omega-C))\\ =&P(A\cap B^c-A\cap B^c\cap C)\\ =&P(A\cap B^c)-P(A\cap B^c\cap C)\\ =&P(A)P(B^c)-P(A)P(B^c)P(C)\\ =&P(A)P(B^c)[1-P(C)]\\ =&P(A)P(B^c)P(C^c) \end{aligned}$$

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$$P(A\cap B^c)=P(A)P(B^c),\ \ P(B^c\cap C^c)=P(B^c)P(C^c),\ \ P(C^c\cap A)=P(C^c)P(A)$$

$\Longrightarrow$the three events (A,B^c,C^c) are independent. Similarly, so are (A^c,B,C^c) and (A^c,B^c,C).

$$\begin{aligned}P(A^c\cap B^c\cap C^c)=&P(A^c\cap B^c\cap(\Omega-C))\\=&P(A^c\cap B^c-A^c\cap B^c\cap C)\\=&P(A^c\cap B^c)-P(A^c\cap B^c\cap C)\\=&P(A^c)P(B^c)-P(A^c)P(B^c)P(C)\\=&P(A^c)P(B^c)[1-P(C)]\\=&P(A^c)P(B^c)P(C^c)\end{aligned}$$

$$P(A^c\cap B^c)=P(A^c)P(B^c),\ \ P(B^c\cap C^c)=P(B^c)P(C^c),\ \ P(C^c\cap A^c)=P(C^c)P(A^c)$$

$\Longrightarrow$the three events (A^c,B^c,C^c) are independent. □

Problem 4. Show that if A,B,C are independent events, then A and $B\cup C$ are independent, and $A\setminus B$ and C are independent.

Solution: Because A,B,C are independent,

$$P(A\cap B)=P(A)P(B),\ \ P(B\cap C)=P(B)P(C),\ \ P(C\cap A)=P(C)P(A)$$
$$P(A\cap B\cap C)=P(A)P(B)P(C)$$

Then we have

$$\begin{aligned}P(A\cap(B\cup C))=&P((A\cap B)\cup(A\cap C))\\=&P(A\cap B)+P(A\cap C)-P((A\cap B)\cap(A\cap C))\\=&P(A\cap B)+P(A\cap C)-P(A\cap B\cap C)\\=&P(A)P(B)+P(A)P(C)-P(A)P(B)P(C)\\=&P(A)[P(B)+P(C)-P(B)P(C)]\\=&P(A)[P(B)+P(C)-P(B\cap C)]\\=&P(A)P(B\cup C)\end{aligned}$$

$\Longrightarrow$ A and $B\cup C$ are independent.

And

$$
\begin{aligned}
P((A\setminus B)\cap C) =& P(A\cap B^c\cap C)\\
& \text{(according to the conclusions of Problem 3)}\\
=& P(A)P(B^c)P(C)\\
=& P(A\cap B^c)P(C)\\
=& P(A\setminus B)P(C)
\end{aligned}
$$

$\Longrightarrow A\setminus B$ and C are independent. □

Problem 5. Let Ω be a set and $\mathscr{F}\subset 2^\Omega$ be a σ-algebra. A function P: $\mathscr{F}\to\mathbb{R}\cup\{+\infty,-\infty\}$ is called a probability measure if it satisfies the following three properties:

1. For all $A\in\mathscr{F}$, $P(A)\geq 0$
2. $P(\Omega)=1$
3. For all countable collections disjoint $A_1, A_2, \ldots$ in $\mathscr{F}$,

$$P(\bigcup_{j=1}^{\infty}A_j)=\sum_{j=1}^{\infty}P(A_j)$$

Given a nested increasing sequence of events $A_1\subset A_2\subset A_3\ldots\subset A_n\subset\ldots$ such that $\cup_{i=1}^{\infty}A_i$ is also an event, prove that

$$\lim_{n\to\infty}P(A_n)=P(\bigcup_{i=1}^{\infty}A_i)$$

Solution: Because $A_1\subset A_2\subset A_3\ldots\subset A_n\subset\ldots$, sets $A_1,(A_2-A_1),(A_3-A_2),\ldots$ are disjoint.

$$
\begin{aligned}
P(\bigcup_{i=1}^{\infty}A_i) =& P(A_1+\bigcup_{i=2}^{\infty}(A_i-A_{i-1}))\\
=& P(A_1)+\sum_{i=2}^{\infty}[P(A_i)-P(A_{i-1})]\\
=& \lim_{n\to\infty}\{P(A_1)+\sum_{i=2}^{n}[P(A_i)-P(A_{i-1})]\}\\
=& \lim_{n\to\infty}P(A_1\cap\bigcap_{i=1}^{n}(A_i-A_{i-1}))\\
=& \lim_{n\to\infty}P(A_n)
\end{aligned}
$$

□

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Problem 6. Find an example where

$$P(AB) < P(A)P(B)$$

Solution: Let $\Omega = \{0, 1\}$, $A = \{0\}$, $B = \{1\}$, $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$. Then we have

$$P(AB) = P(\emptyset) = 0 < \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = P(A)P(B)$$

□

Problem 7. What can you say about the event A if it is independent of itself? If the events A and B are disjoint and independent, what can you say of them?

Solution: If the event A is independent of itself, then

$$P(A) = P(A \cap A) = P(A)P(A)$$
$$\Longrightarrow P(A) = 0 \text{ or } P(A) = 1$$

If the events A and B are disjoint and independent, then

$$P(A)P(B) = P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0$$
$$\Longrightarrow P(A) = 0 \text{ or } P(B) = 0$$

□

Problem 8. Prove that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$$

when A, B, C are independent by considering $P(A^c B^c C^c)$

Solution:

$$\begin{aligned}
P(A \cup B \cup C) =& P(\Omega - A^c B^c C^c) \\
=& P(\Omega) - P(A^c B^c C^c) \\
& \text{(according to the conclusions of Problem 3)} \\
=& 1 - P(A^c)P(B^c)P(C^c) \\
=& 1 - [1 - P(A)][1 - P(B)][1 - P(C)] \\
=& P(A) + P(B) + P(C) - P(A)P(B) - P(A)P(C) - P(B)P(C) \\
& + P(A)P(B)P(C) \\
=& P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)
\end{aligned}$$

□

Problem 9. Let $S = (-\infty, +\infty)$, the real line. Then $\mathscr{F}$ is chosen to contain all sets of the form

$$(a, b], [a, b], [a, b), (a, b)$$

for all real numbers a and b. (Unions of these form are in $\mathscr{F}$.) Show that $\mathscr{F}$ is a Borel field.

Solution: $\mathscr{F} \subset 2^S$. For any set (a, b) s.t. a and b are real numbers, its complement

$$(-\infty, a] \cup [b, +\infty) = (\bigcup_{i=1}^{\infty}(a - i, a]) \cup (\bigcup_{i=1}^{\infty}[b, b + i)) \in \mathscr{F} \tag{4}$$

Similarly, the complements of $(a, b]$, $[a, b]$ and $[a, b)$ are also in $\mathscr{F}$.
And according to the problem description, unions of these four forms are in $\mathscr{F}$, so for $A_1, A_2, ...$ in $\mathscr{F}$,

$$\bigcup_{j=1}^{\infty} A_j \in \mathscr{F} \tag{5}$$

Because of (4) and (5), $\mathscr{F}$ is a Borel field. □

Problem 10. Suppose that the land of a square kingdom is divided into three strips A, B, C of equal area and suppose the value per unit is in the ratio of 1:3:2. For any piece of (measurable) land S in this kingdom, the relative value with respect to that of the kingdom is then given by the formula:

$$V(S) = \frac{P(SA) + 3P(SB) + 2P(SC)}{2}$$

where P is as in Example 2 of 2.1. Show that V is a probability measure.

Solution: For any piece of land S in this kingdom,

$$\begin{aligned} V(S) =& \frac{P(SA) + 3P(SB) + 2P(SC)}{2} \\ =& \frac{|SA| + 3|SB| + 2|SC|}{2|S|} \end{aligned}$$

where $|SA|, |SB|, |SC|$ and $|S|$ are area of land pieces SA, SB, SC and S, and are all positive, so

$$V(S) \geq 0 \tag{6}$$

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For any countable collections of disjoint sets $S_1, S_2, ...$ in the kingdom,

$$\begin{aligned} V(\bigcup_{j=1}^{\infty} S_j) =& \frac{P(A \cap \bigcup_{j=1}^{\infty} S_j) + 3P(B \cap \bigcup_{j=1}^{\infty} S_j) + 2P(C \cap \bigcup_{j=1}^{\infty} S_j)}{2} \\ =& \frac{P(\bigcup_{j=1}^{\infty}(S_j A)) + 3P(\bigcup_{j=1}^{\infty}(S_j B)) + 2P(\bigcup_{j=1}^{\infty}(S_j C))}{2} \\ =& \frac{\sum_{j=1}^{\infty} P(S_j A) + 3\sum_{j=1}^{\infty} P(S_j B) + 2\sum_{j=1}^{\infty} P(S_j C)}{2} \\ =& \sum_{j=1}^{\infty} \frac{P(S_j A) + 3P(S_j B) + 2P(S_j C)}{2} \\ =& \sum_{j=1}^{\infty} V(S_j) \end{aligned} \tag{7}$$

Let Ω be the total land of this kingdom

$$P(\Omega) = \frac{P(A) + 3P(B) + 2P(C)}{2} = \frac{\frac{1}{3} + 3 \times \frac{1}{3} + 2 \times \frac{1}{3}}{2} = 1 \tag{8}$$

Because of (6), (7) and (8), V is a probability measure. $\square$