Name: 陈稼霖
StudentID: 45875852

SI 140
Probability and Statistics
Semester Spring 2019
Assignment 1

Problem 1. If two sets have identical complements, then they are themselves identical. Show this in two ways:(i) by verbal definition, (ii) by using formula$(A^c)^c$

Solution:

(i) We mark these two sets as A and B, respectively. For any member of A, marked as ω, it does not belong to complement of A. Since A and B have identical complements, ω does not belong to complement of B and this means that ω belongs to B. Therefore, any member ω of A belongs to B, $A \subset B$......①.

Similarly, we have $B \subset A$......②. Because of ① and ②, A and B are identical.

(ii) Because $A^c = B^c$, $(A^c)^c = (B^c)^c$. According to the formula, $(A^c)^c = A$ and $(B^c)^c = B$. Therefore, $A = B$. □

Problem 2. Show that

$$(A \cup B) \cap C \neq A \cup (B \cap C)$$

but also give some special cases where there is equality.

Solution:

Suppose

$$A = \{1, 2\}, B = \{2, 3\}, C = \{3, 1\}$$

Then

$$(A \cup B) \cap C = \{1, 3\}, A \cup (B \cap C) = \{1, 2, 3\}$$

Therefore,

$$(A \cup B) \cap C \neq A \cup (B \cap C)$$

Some special cases where there is equality:

Case 1: Suppose

$$A = B = C = \emptyset$$

In this case,

$$(A \cup B) \cap C = \emptyset = A \cup (B \cap C)$$

Case 2: Suppose

$$A = \{1\}, B = C = \{1, 2\}$$

In this case,

$$(A \cup B) \cap C = \{1, 2\} = A \cup (B \cap C)$$

□

Problem 3. Show that A $\subset$ B if and only if AB = A; or A$\cup$ B = B. (So the relation of inclusion can be defined through identity and the operations.)

Solution:

(i) Proof that $A \subset B$ if and only if $A \cap B = A$:

Necessity: If $AB = A$, then any member of A, marked as ω_1, belongs to $A \cap B$. According to the definition of intersection, any member of $A \cap B$ belongs to B, so ω_1 belongs to B. Therefore, any member of A belongs to B, $A \subset B$.

Sufficiency: If $A \subset B$, then any member of A belongs to A and B at the same time. Therefore, according to the definition of intersection, any member of A belongs to $A \cap B$, $A \subset (A \cap B)$......①.

According to the definition of intersection, any member of $A \cap B$ belongs to A, $(A \cap B) \subset A$......②.

Because of ① and ②, $A \cap B = A$.

(ii) Proof that $A \subset B$ if and only if $A \cup B = B$:

Necessity: According to the definition of union, any member of A, marked as ω_2, belongs to $A \cup B$. If $A \cup B = B$, then ω_2 belongs to B. Therefore, any member of A belongs to B, $A \subset B$.

Sufficiency: According to the definition of union, any member of $A \cup B$, marked as ω_3 belongs to A or B. If ω_3 belongs to A, since $A \subset B$, then ω_3 belongs to B. If ω_3 belongs to B, needless to say it belongs to B. Therefore, any member of $A \cup B$ belongs to B, $A \subset (A \cup B)$......③.

According to the definition of union, any member of B belongs to $A \cup B$, $B \subset (A \cup B)$......④.

Because of ③ and ④, $A \cup B = B$. $\square$

Problem 4. Show that there is a distributive law also for difference:

$$(A \backslash B) \cap C = (A \cap C) \backslash (B \cap C).$$

Is the dual

$$(A \cap B) \backslash C = (A \backslash C) \cap (B \backslash C)$$

also true?

Solution:

(i) Proof that $(A \backslash B) \cap C = (A \cap C) \backslash (B \cap C)$:

The left side of the equation is

$$(A\backslash B) \cap C = A \cap B^c \cap C$$

The right side of the equation is

$$\begin{aligned}(A \cap C)\backslash(B \cap C) =&(A \cap C) \cap (B \cap C)^c\\ =&(A \cap C) \cap (B^c \cup C^c)\\ =&((A \cap C) \cap B^c) \cup ((A \cap C) \cup C^c)\\ =&(A \cap B^c \cap C) \cup \emptyset\\ =&A \cap B^c \cap C\end{aligned}$$

Therefore,

$$(A\backslash B) \cap C = (A \cap C)\backslash(B \cap C)$$

(ii) The second equation, $(A \cap B)\backslash C = (A\backslash C) \cap (B\backslash C)$, is also true. Proof:

The left side of the equation is

$$(A \cap B)\backslash C = A \cap B \cap C^c$$

The right side of the equation is

$$\begin{aligned}(A\backslash C) \cap (B\backslash C) =&(A \cap C^c) \cap (B \cap C^c)\\ =&A \cap B \cap C^c\end{aligned}$$

Therefore,

$$(A \cap B)\backslash C = (A\backslash C) \cap (B\backslash C)$$

□

Problem 5. Show that A ⊂ B if and only if $I_A \leq I_B$; and A ∩ B = ∅ if and only if $I_A I_B$ =0.

Solution:

Proof that $A \subset B$ if and only if $I_A \leq I_B$:

Necessity: Suppose $I_A \leq I_B$, then there are three cases for I_A and I_B. ①If $I_A = I_B = 0$, then ω belongs to neither A nor B; ②If $I_A = 0, I_B = 1$, then ω belongs to B but does not belongs to A; ③If $I_A = I_B = 1$, then ω belongs to both A and B. Therefore, if ω belongs to A, it must belongs to B, which means $A \subset B$.

Sufficiency: Suppose $A \subset B$, then there are three cases for ω. ①If ω belongs to A, then it also belongs to B and $I_A = I_B = 1$; ②If ω does not belong to A but belongs to B, then $I_A = 0, I_B = 1$; ③If ω belongs to neither A nor B, then $I_A = I_B = 0$. Therefore, $I_A \leq I_B$.

Proof that $A \cap B = \emptyset$ if and only if $I_A I_B = 0$:

Necessity: Suppose $I_A I_B = 0$. For any ω belongs to A, $I_A(\omega) = 1$. This make $I_B(\omega) = 0$, which means ω does not belongs to B. Therefore, there is no ω that belongs to both A and B at the same time, $A \cap B = 0$.

Sufficiency: If $A \cap B = 0$, then there are three case for ω. ①If ω belongs to A, then it can not belong to B (or $A \cap B \neq \emptyset$). In this case, $I_A I_B = 1 \cdot 0 = 0$; ②If ω belongs to B, then it can not belong to A (for the same reason in ①) and $I_A I_B = 0 \cdot 1 = 0$, ③If ω belongs to neither A nor B, then $I_A = I_B = 0 \cdot 0 = 0$. Therefore, $I_A I_B = 0$. □

Problem 6. Given n events $A_1, A_2, ..., A_n$ and indicators $I_j, j = 1, ..., n$ ($I_j = 1$ if A_j occur, else $I_j = 0$). Let $X = \sum_{j=1}^{n} I_j$ be the number of events that occur. You need to find the number of pairs of distinct events that occur: (i) Write your answer in terms of X. (ii) Write your answer in terms of indicators.

Solution:

(i) The number of pairs of distinct events that occur in terms of X is $\frac{X(X-1)}{2}$.

(ii) The number of pairs of distinct events that occur in terms of indicators is

$$\frac{\sum_{k=1}^{n} I_k(\sum_{k=1}^{n} I_k - 1)}{2} \text{ or } \sum_{i=1}^{n}\sum_{j=i+1}^{n} I_i I_j$$

□

Problem 7. Express $I_{A\cup B\cup C}$ as a polynomial of I_A, I_B, I_C.

Solution:

$$\begin{aligned}
I_{A\cup B\cup C} =& 1 - I_{(A\cup B)^c C^c} \\
=& 1 - I_{(A\cup B)^c} I_{C^c} \\
=& 1 - I_{A^c B^c}(1 - I_C) \\
=& 1 - I_{A^c} I_{B^c}(1 - I_C) \\
=& 1 - (1 - I_A)(1 - I_B)(1 - I_C) \\
=& I_A I_B I_C - I_A I_B - I_B I_C - I_C I_A + I_A + I_B + I_C
\end{aligned}$$

□

Problem 8. Show that

$$I_{ABC} = I_A + I_B + I_C - I_{A\cup B} - I_{A\cup C} - I_{B\cup C} + I_{A\cup B\cup C}$$

You can verify this directly, but it is nicer to derive it from problem 7 by duality.

Solution:

$$\begin{aligned}
I_{ABC} =& I_A I_B I_C \\
& \text{(according to the conclusion of problem 7)} \\
=& I_A I_B + I_B I_C + I_C I_A - I_A - I_B - I_C + I_{A\cup B\cup C} \\
=& I_A + I_B + I_C + (I_A I_B - I_A - I_B) + (I_B I_C - I_B - I_C) + (I_C I_A - I_C - I_A) + I_{A\cup B\cup C} \\
& \text{(according to equation (1.4.8) in textbook)} \\
=& I_A + I_B + I_C - I_{A\cup B} - I_{A\cup C} - I_{B\cup C} + I_{A\cup B\cup C}
\end{aligned}$$

□

Problem 9. Prove that the set of all rational numbers is countable.

Solution: Any nonzero rational number can be written as the following format

$$\pm\frac{p}{q}$$

where p and q are two non-negative integers.

Therefore, in principle, we can list all the nonzero rational numbers in the following table (Figure 1), where the number at row i and column j of the table is

$$(-1)^{(j+1)}\frac{i}{u(j)}$$

where

$$u(j) = \begin{cases} \frac{j+1}{2}, & \text{if } j \text{ is odd} \\ \frac{j}{2}, & \text{if } j \text{ is even} \end{cases}$$

We construct the mapping that associates natural number 0 to rational number 0 and associates other nonzero natural numbers greater than to nonzero rational numbers in the form showed in Figure 1.

In this way, we can associate every rational number with a unique natural number. Therefore, the set of all rational numbers is countable. □

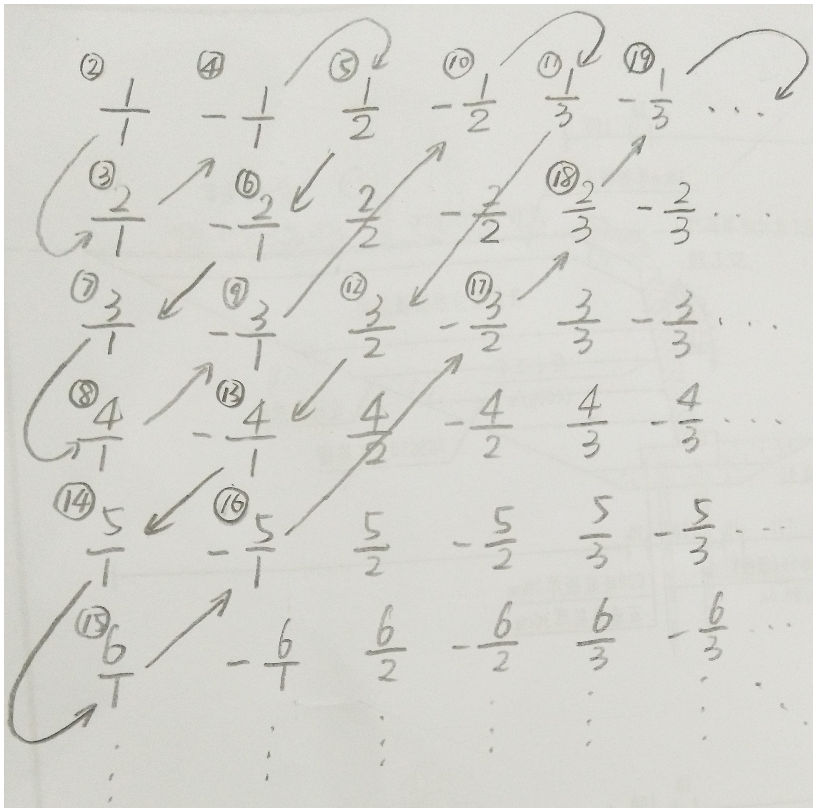

图 1: Problem 9

Problem 10. Let A be the set of all sequences whose elements are the digits 0 and 1. For example, the following sequence is a element of A.

$$1, 0, 1, 0, 0, 0, 1, 1, ...$$

Prove that set A is uncountable. (Hint: You can prove it by using Cantor's diagonal process.)

Solution: Suppose we associate every zero-one sequence in set A with a unique natural number like the following list does:

$$
\begin{aligned}
0 \rightarrow & a_{01}, a_{02}, a_{03}, a_{04}, ...\\
1 \rightarrow & a_{11}, a_{12}, a_{13}, a_{14}, ...\\
2 \rightarrow & a_{21}, a_{22}, a_{23}, a_{24}, ...\\
3 \rightarrow & a_{31}, a_{32}, a_{33}, a_{34}, ...\\
& ......
\end{aligned}
$$

where $a_{ij} = 0$ or 1.

Name: 陈稼霖
StudentID: 45875852

Then we construct the following zero-one sequence through Cantor's diagonal process:

$$b_0, b_1, b_2, b_3, ...$$

where

$$b_i = \begin{cases} 1, & \text{if } a_{i\ i+1} = 0 \\ 0, & \text{if } a_{i\ i+1} = 1 \end{cases}$$

Obviously, sequence $b_0, b_1, b_2, b_3, ...$ belong to set A. However, according to the definition of the b_i, there will always be at least one digit (the ith digit) different for sequence $b_0, b_1, b_2, b_3, ...$ and sequence $a_{i1}, a_{i2}, a_{i3}, a_{i4}, ...$ Here comes a contradiction. Therefore, we can not associate every zero-one sequence in set A with a unique natural number, which means that set A is uncountable. $\square$