SI 140
Probability and Statistics
Semester Spring 2019
Assignment 3

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Problem 1. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $P(\omega_1) = P(\omega_2) = P(\omega_3) = 1/3$, and define X, Y and Z as follows:

$$X(\omega_1) = 1, X(\omega_2) = 2, X(\omega_3) = 3;$$

$$Y(\omega_1) = 2, Y(\omega_2) = 3, Y(\omega_3) = 1;$$

$$Z(\omega_1) = 3, Z(\omega_2) = 1, Z(\omega_3) = 2.$$

Show that these three random variables have the same probability distribution. Find the probability distributions of $X+Y$, $Y+Z$, and $Z+X$.

Solution: Since

$$\begin{cases} P(X=1) = P(\omega_1) = \frac{1}{3}, P(X=2) = P(\omega_2) = \frac{1}{3}, P(X=3) = P(\omega_3) = \frac{1}{3} \\ P(Y=1) = P(\omega_3) = \frac{1}{3}, P(Y=2) = P(\omega_1) = \frac{1}{3}, P(Y=3) = P(\omega_2) = \frac{1}{3} \\ P(Z=1) = P(\omega_2) = \frac{1}{3}, P(Z=2) = P(\omega_3) = \frac{1}{3}, P(Z=3) = P(\omega_1) = \frac{1}{3} \end{cases}$$

$$\Longrightarrow P(X=i) = P(Y=i) = P(Z=i) = \frac{1}{3}, \quad i = 1, 2, 3$$

these three random variables have the same probability distribution.

$$P(X+Y) = \begin{cases} P(X=1)P(Y=2) = P(\omega_1) = \frac{1}{3}, & X+Y=3 \\ P(X=3)P(Y=1) = P(\omega_3) = \frac{1}{3}, & X+Y=4 \\ P(X=2)P(Y=3) = P(\omega_2) = \frac{1}{3}, & X+Y=5 \end{cases}$$

$$P(Y+Z) = \begin{cases} P(Y=1)P(Z=2) = P(\omega_3) = \frac{1}{3}, & Y+Z=3 \\ P(Y=3)P(Z=1) = P(\omega_2) = \frac{1}{3}, & Y+Z=4 \\ P(Y=2)P(Z=3) = P(\omega_1) = \frac{1}{3}, & Y+Z=5 \end{cases}$$

$$P(Z+X) = \begin{cases} P(Z=2)P(X=1) = P(\omega_2) = \frac{1}{3}, & Z+X=3 \\ P(Z=3)P(X=1) = P(\omega_1) = \frac{1}{3}, & Z+X=4 \\ P(Z=2)P(X=3) = P(\omega_3) = \frac{2}{9}, & Z+X=5 \end{cases}$$

□

Problem 2. In No.1 find the probability distribution of

$$X+Y-Z, \sqrt{(X^2+Y^2)Z}, \frac{Z}{|X-Y|}$$

Solution:

$$P(X+Y-Z) = \begin{cases} P(X+Y=3)P(Z=3) = P(\omega_1) = \frac{1}{3}, & X+Y-Z=0 \\ P(X+Y=4)P(Z=2) = P(\omega_3) = \frac{1}{3}, & X+Y-Z=2 \\ P(X+Y=5)P(Z=1) = P(\omega_2) = \frac{1}{3}, & X+Y-Z=4 \end{cases}$$

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$$P(\sqrt{(X^2+Y^2)Z}) = \begin{cases} P(X=1)P(Y=2)P(Z=3) = P(\omega_1) = \frac{1}{3}, & \sqrt{(X^2+Y^2)Z} = 2\sqrt{3} \\ P(X=2)P(Y=3)P(Z=1) = P(\omega_2) = \frac{1}{3}, & \sqrt{(X^2+Y^2)Z} = \sqrt{13} \\ P(X=3)P(Y=1)P(Z=2) = P(\omega_3) = \frac{1}{3}, & \sqrt{(X^2+Y^2)Z} = 2\sqrt{5} \end{cases}$$

$$P(\frac{Z}{|X-Y|}) = \begin{cases} P(X=1)P(Y=2)P(Z=3) = P(\omega_1) = \frac{1}{3}, & \frac{Z}{|X-Y|} = 3 \\ P(X=2)P(Y=3)P(Z=1) + P(X=3)P(Y=1)P(Z=2) = P(\omega_2) + P(\omega_3) = \frac{2}{3}, & \frac{Z}{|X-Y|} = 1 \end{cases}$$

□

Problem 3. Let X be integer-valued and let F be its distribution function. Show that for every x and $a < b$

$$P(X = x) = \lim_{\epsilon \downarrow 0} [F(x+\epsilon) - F(x-\epsilon)]$$

$$P(a < X < b) = \lim_{\epsilon \downarrow 0} [F(b-\epsilon) - F(a+\epsilon)]$$

[The results are true for any random variable but require more advanced proofs even when Ω is countable.]

Solution:

$$\begin{aligned} P(X=x) =& P((-\infty, x] - (-\infty, x)) \\ =& P(X \leq x) - P(X < x) \\ =& F(x) - \lim_{\epsilon \downarrow 0} F(x-\epsilon) \\ =& \lim_{\epsilon \downarrow 0} F(x+\epsilon) - \lim_{\epsilon \downarrow 0} F(x-\epsilon) \\ & \text{(because the distribution function, } F\text{, is right continuous)} \\ =& \lim_{\epsilon \downarrow 0} [F(x+\epsilon) - F(x-\epsilon)] \end{aligned}$$

$$\begin{aligned} P(a < x < b) =& F((-\infty, b] - (-\infty, a] - \{b\}) \\ =& P(x \leq b) - P(x \leq a) - P(b) \\ =& F(b) - F(a) - \lim_{\epsilon \downarrow 0} [F(b+\epsilon) - F(b-\epsilon)] \\ & \text{(because the distribution function, } F\text{, is right continuous)} \\ =& \lim_{\epsilon \downarrow 0} F(b+\epsilon) - \lim_{\epsilon \downarrow 0} F(a+\epsilon) - \lim_{\epsilon \downarrow 0} [F(b+\epsilon) - F(b-\epsilon)] \\ =& \lim_{\epsilon \downarrow 0} [F(b-\epsilon) - F(a+\epsilon)] \end{aligned}$$

□

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Problem 4. (a) Is there a discrete distribution with support 1,2,3,..., such that the value of the PMF at n is proportional to $1/n$?

(b) Is there a discrete distribution with support 1,2,3,..., such a that the value of the PMF at n is proportional to $1/n^2$?

Solution:
(a) No. Assume the PMF at n is

$$p_n = \frac{k}{n}$$

Since

$$\sum_{i=1}^{\infty} p_i = \sum_{i=1}^{\infty} \frac{k}{n} = \left\{ \begin{array}{ll} +\infty, & k > 0 \\ 0, & k = 0 \\ -\infty, & k < 0 \end{array} \right\} \neq 1$$

the assumption is incorrect, which means there is not such a discrete distribution with $1, 2, 3, ...$, that the value of PMF at n is proportional to $1/n$.
(b) Yes. Assume the PMF at n is

$$p_n = \frac{k}{n^2}$$

In this way,

$$\sum_{i=1}^{\infty} p_i = \sum_{i=1}^{\infty} \frac{k}{n^2} = \frac{\pi^2}{6} k = 1$$

so the discrete distribution

$$p_n = \frac{6}{\pi^2 n^2}, \quad n = 1, 2, 3, ...$$

satisfies that the value of the PMF at n is proportional to $1/n^2$. □

Problem 5. Let X have PMF

$$P(X = k) = cp^k/k \text{ for } k = 1, 2, ...$$

where p is a parameter with $0 < p < 1$ and c is a normalizing constant. We have $c = -1/\log(1-p)$, as seen from the Taylor series

$$-log(1-p) = p + \frac{p^2}{2} + \frac{p^3}{3} + \cdots .$$

This distribution is called the *Logarithmic* distribution (because of the log in the above Taylor series), and has often been used in ecology. Find the mean of X.

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Solution:

$$\sum_{k=1}^{\infty} P(X=k) = \sum_{k=1}^{\infty} \frac{cp^k}{k} = -c\ln(1-p) = 1 \Longrightarrow c = \frac{1}{-\ln(1-p)}$$

The mean of X is

$$E(X) = \sum_{k=1}^{\infty} kP(X=k) = \sum_{k=1}^{\infty} cp^k = \lim_{n\to\infty} \frac{1}{1-\ln(1-p)} \frac{p(1-p^n)}{1-p} = \frac{p}{[1-\ln(1-p)](1-p)}$$

□

Problem 6. Suppose F is some cumulative distribution function. Then for any real number y, the function F_y defined by $F_y(x) = F(x\text{-}y)$ is also a cumulative distribution function. In fact, F_y is just a "shifted" version of F

Solution: Since

$$F_y(X=x) = F(X=(x-y)) = P(X \leq (x-y))$$

F_y is a cumulative distribution function. □

Problem 7. Let X be a random variable, with cumulative distribution function F_X . Prove that $P(X=a)=0$ if and only if the function F_X is continuous at a.

Solution:

Sufficiency: Suppose the function F_X is continuous at a, which means

$$\lim_{y\to a^-} F_X(X=y) = F_X(X=a)$$

So

$$\begin{aligned} P(X=a) =& P((-\infty,a]-(-\infty,a)) = F(X=a) - P(X<a) \\ =& F(X=a) - \lim_{y\to a^-} F(X=y) = 0 \end{aligned}$$

Necessity: Suppose $P(X=a)=0$. Then

$$\lim_{y\to a^-} F_X(X=y) = P(X\leq a) - P(X=a) = F_X(X=a) - 0 = F_X(X=a)$$

which means F_X is left continuous at a, and

$$\lim_{y\to a^+} F_X(X=y) = P(X\leq a) + \lim_{y\to a^+} P(a<X<y) = F_X(X=a) + 0 = F_X(X=a)$$

which means F_X is right continuous at a. Therefore, the function F_X is continuous at a. Therefore, $P(X=a)=0$ if and only if the function F_X is continuous at a. □

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Problem 8. Suppose that

$$p_n = cq^{n-1}p, 0 \leq n \leq m$$

where c is a constant and m is a positive integer; cf. (4.4.8). Determine c so that $\sum_{n=1}^{m} p_n = 1$. (This scheme corresponds to the waiting time for a success when it is supposed to occur within m trials.)

Solution:

$$\sum_{n=1}^{m} p_n = \sum_{n=1}^{m} cq^{n-1}p = cp\frac{1-q^m}{1-q} = c(1-q^m) = 1 \Longrightarrow c = \frac{1}{1-q^m}$$

□

Problem 9. A perfect coin is tossed n times. Let Y_n denote the number of heads obtained minus the number of tails. Find the probability distribution of Y_n and its mean.[Hint: there is a simple relation between Y_n and the S_n in Example 9 of 4.4]

Solution: Since S_n denote the number of heads obtained in Example 9 of 4.4, $n - S_n$ denote the number of tails obtained is $n - S_n$. So the simple relation between Y_n and the S_n is

$$Y_n = S_n - (n - S_n) = 2S_n - n$$

The probability distribution of Y_n is

$$P(Y_n = k) = P(S_n = \frac{k+n}{2}) = \frac{1}{2^n}\begin{pmatrix} n \\ \frac{k+n}{2} \end{pmatrix}$$

Its mean is

$$\begin{aligned}E(Y_n) = & \sum_{Y_n=-n+2k, k=0,1,2,\ldots,n} k\frac{1}{2^n}\begin{pmatrix} n \\ \frac{k+n}{2} \end{pmatrix} \\ = & \sum_{k=0}^{n}(2k-n)\frac{1}{2^n}\begin{pmatrix} n \\ \frac{k+n}{2} \end{pmatrix} \\ = & 2\sum_{k=0}^{n} k\frac{1}{2^n}\begin{pmatrix} n \\ \frac{k+n}{2} \end{pmatrix} - n\sum_{k=0}^{n}\frac{1}{2^n}\begin{pmatrix} n \\ \frac{k+n}{2} \end{pmatrix} \\ = & 2 \times \frac{n}{2} - n \\ = & 0\end{aligned}$$

□

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Problem 10. Let

$$P(X=n)=p_n=\frac{1}{n(n+1)}, n\geq 1$$

Show that it is a probability distribution for X? Find $P(X\geq m)$ for any m and $E(X)$.

Solution: Since

$$P(X=n)=p_n=\frac{1}{n(n+1)}\geq 0$$

for any integer $n\geq 1$, and

$$\sum_{n=1}^{\infty}P(X=n)=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\sum_{n=1}^{\infty}\frac{1}{n}-\frac{1}{n+1}=\lim_{n\to\infty}(1-\frac{1}{n+1})=1$$

$P(X=n)$ is a probability distribution for X.

$$P(X\geq m)=\begin{cases}1, & m<1\\ \sum_{n=[m]}^{\infty}\frac{1}{n(n+1)}=\frac{1}{[m]}, & m\geq 1\end{cases}$$

where $[m]$ is the smallest integer that is no less than m.

$$E(X)=\sum_{n=1}^{\infty}n\frac{1}{n(n+1)}=\sum_{n=1}^{\infty}\frac{1}{n+1}=+\infty$$

Therefore, $E(X)$ does not exist. $\square$