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作业
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目录

Chapter 6

光束在均匀介质和类透镜介质中的传播

习题 6.1: 推导 (6.2-1) 至 (6.2-4) 各式. □

证: 对相同透镜构成的波导, (在 $n+1$ 处) 出射和 (在 n 处) 出射光线之间的关系为

$$\begin{bmatrix} r_{n+1} \\ r'_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & d \\ -1/f & (1-d/f) \end{bmatrix} \begin{bmatrix} r_n \\ r'_n \end{bmatrix}, \tag{6.1}$$

或将其写成方程形式为

$$r_{n+1} = r_n + d r'_n, \tag{6.2}$$

$$r'_{n+1} = -\frac{1}{f} r_n + \left(1 - \frac{d}{f}\right) r'_n. \tag{6.3}$$

由式 (6.2) 得

$$r'_n = \frac{1}{d}(r_{n+1} - r_n), \tag{6.4}$$

因而有

$$r'_{n+1} = \frac{1}{d}(r_{n+2} - r_{n+1}), \tag{6.5}$$

将以上两式代入式 (6.3) 中得

$$r_{n+2} - \left(2 - \frac{d}{f}\right) r_{n+1} + r_n = 0. \tag{6.6}$$

上式为微分方程

$$r'' + \frac{d}{f} r = 0 \tag{6.7}$$

的差分方程, 故令试探解为

$$r_n = r_0 e^{in\theta}, \tag{6.8}$$

将其代入式 (6.6) 中可得

$$e^{2i\theta} - \left(2 - \frac{d}{f}\right) e^{i\theta} + 1 = 0, \tag{6.9}$$

解得

$$e^{i\theta} = \left(1 - \frac{d}{f}\right) \pm i\sqrt{\frac{d}{f} - \frac{d^2}{4f^2}}, \tag{6.10}$$

于是得到

$$r_n = r_{最大} \sin\theta(n\theta + \delta), \tag{6.11}$$

其中

$$\cos\theta = \mathrm{Re}\left[e^{i\theta}\right] = \left(1 - \frac{d}{2f}\right), \tag{6.12}$$

此即式 (6.2-2),

$$r_{最大} = \frac{r_0}{\sin\delta}. \tag{6.13}$$

利用 (6.2) 得

$$r_{最大} \sin(\theta + \delta) = r_0 + dr_0', \tag{6.14}$$

$$\Longrightarrow \frac{r_0}{\sin\delta}(\sin\theta\cos\delta + \cos\theta\sin\delta) = r_0 + dr_0', \tag{6.15}$$

$$\Longrightarrow \tan\delta = \frac{\sin\delta}{\cos\delta} = \frac{r_0 \sin\theta}{(r_0 + dr_0') - r_0\cos\theta} = \frac{\sqrt{\frac{4f}{d} - 1}}{1 + 2f\frac{r_0'}{r_0}}, \tag{6.16}$$

此即式 (6.2-4), 从而

$$(r_{最大})^2 = \frac{r_0^2}{\sin^2\delta} = \frac{r_0^2(1 + \tan^2\delta)}{\tan^2\delta} = \frac{4f}{4f - d}(r_0^2 + dr_0r_0' + dfr_0'^2), \tag{6.17}$$

此即式 (6.2-3). 光线稳定的条件为

$$\frac{d}{f} - \frac{d^2}{4f^2} \geq 0, \tag{6.18}$$

$$\Longrightarrow 0 \leq d \leq 4f, \tag{6.19}$$

此即式 (6.2-1). □

习题 6.2: 证明方程

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_s \\ r_s' \end{bmatrix} = \lambda \begin{bmatrix} r_s \\ r_s' \end{bmatrix}$$

的本征值是 $\lambda = e^{\pm i\theta}$, 其中 $\exp(\pm i\theta)$ 由式 (6.1-13) 给出. 注意, 按照式 (6.1-5), 上述矩阵方程也可以写为

$$\begin{bmatrix} r_{s+1} \\ r_{s+1}' \end{bmatrix} = \lambda \begin{bmatrix} r_s \\ r_s' \end{bmatrix}. \tag{6.20}$$

□

证: 上述矩阵的特征方程为

$$\det\left(\begin{vmatrix} A & B \\ C & D \end{vmatrix} - \lambda I\right) = \begin{vmatrix} A - \lambda & B \\ C & D - \lambda \end{vmatrix} = \lambda^2 - (A + D)\lambda + (AD - BC) = \lambda^2 - 2\lambda + 1 = 0, \tag{6.21}$$

从而解得特征值为

$$\lambda = e^{\pm i\theta}, \tag{6.22}$$

其中 $\exp(\pm i\theta)$ 由式 (6.1-13) 给出:

$$e^{\pm i\theta} = b \pm \sqrt{1-b^2}. \tag{6.23}$$

□

习题 6.3: 对于一个平面波入射到透镜上的情况证明式 (6.4-1) 成立, 由此合理地论证式 (6.4-1) 是正确的. □

证: 设透镜的折射率为 n, (x,y) 处厚度为 $t(\sqrt{x^2+y^2})$. 当平面波入射到透镜上, 沿光轴入射透镜和平行光轴于 (x,y) 入射透镜的光线汇聚于焦点, 根据费马原理, 两者自入射到汇聚走过的光程相等:

$$nt(0)+f = nt(\sqrt{x^2+y^2}) + (t(\sqrt{x^2+y^2})-t(0)) + \sqrt{f^2+x^2+y^2}. \tag{6.24}$$

傍轴近似下, 有

$$nt(0)+f \approx nt(\sqrt{x^2+y^2}) + [t(\sqrt{x^2+y^2})-t(0)] + f\left(1+\frac{x^2+y^2}{2f^2}\right), \tag{6.25}$$

$$\Longrightarrow nt(\sqrt{x^2+y^2}) + [t(\sqrt{x^2+y^2})-t(0)] = nt(0) - \frac{x^2+y^2}{2f}. \tag{6.26}$$

透镜对平面波的作用是引起相移

$$\phi(\sqrt{x^2+y^2}) = k\{nt(\sqrt{x^2+y^2}) + [t(\sqrt{x^2+y^2})-t(0)]\} = knt(0) - k\frac{x^2+y^2}{2f}, \tag{6.27}$$

而不影响振幅, 因此 (忽略整体相因子) 有

$$E_R(x,y) = E_L(x,y)\exp\left(-ik\frac{x^2+y^2}{2f}\right), \tag{6.28}$$

此即式 (6.4-1). □

习题 6.4: 推导方程式 (6.4-7). □

证: l 处的光线可表为

$$r(l^-) = r_0\cos\left(\sqrt{\frac{k_2}{k}}l\right), \tag{6.29}$$

$$r'(l^-) = -\sqrt{\frac{k_2}{k}}r_0\sin\left(\sqrt{\frac{k_2}{k}}l\right). \tag{6.30}$$

边界处出射可表为

$$r(l^+) = r(l^-) = r_0\cos\left(\sqrt{\frac{k_2}{k}}l\right), \tag{6.31}$$

$$r'(l^+) = n_0 r'(l^-) = -n_0\sqrt{\frac{k_2}{k}}r_0\sin\left(\sqrt{\frac{k_2}{k}}l\right). \tag{6.32}$$

公共焦点与出射面之间的距离为

$$h = \left|\frac{r_{\rm out}}{r'_{\rm out}}\right| = \frac{1}{n_0}\sqrt{\frac{k_2}{k}}\cot\left(\sqrt{\frac{k_2}{k}}l\right). \tag{6.33}$$

□

习题 6.5: 证明占据 $0 \le z \le l$ 区域的类透镜介质把位于 $z<0$ 的轴上点成像为单点. (若成像点位于 $z<l$ 处, 则是虚像.) $\square$

证: 设轴上物点坐标为 $-z_0<0$, 则由该点发出的某条光线可表为 $(0, r_0')$, 经 $-z_0<z<0$ 区域的直线传播后在入射面的入射光线可表为

$$\begin{bmatrix} r(0^-) \\ r'(0^-) \end{bmatrix} = \begin{bmatrix} 1 & z_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ r_0' \end{bmatrix} = \begin{bmatrix} z_0 r' \\ r_0' \end{bmatrix}. \tag{6.34}$$

经入射面折射后, 光线可表为

$$\begin{bmatrix} r(0^+) \\ r'(0^+) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{n_0} \end{bmatrix} \begin{bmatrix} r(0^-) \\ r'(0^-) \end{bmatrix} = \begin{bmatrix} z_0 r' \\ \frac{r_0'}{n_0} \end{bmatrix}. \tag{6.35}$$

经类透镜介质中传播后, 在出射面处出射前的光线可表为

$$\begin{bmatrix} r(l^-) \\ r'(l^-) \end{bmatrix} = \begin{bmatrix} \cos\left(\sqrt{\frac{k_2}{k}}l\right) & \sqrt{\frac{k}{k_2}}\sin\left(\sqrt{\frac{k_2}{k}}l\right) \\ -\sqrt{\frac{k_2}{k}}\sin\left(\sqrt{\frac{k_2}{k}}l\right) & \cos\left(\sqrt{\frac{k_2}{k}}l\right) \end{bmatrix} \begin{bmatrix} r(0^+) \\ r'(0^+) \end{bmatrix} = \begin{bmatrix} z_0 r_0' \cos\left(\sqrt{\frac{k_2}{k}}l\right) + \sqrt{\frac{k}{k_2}}\frac{r_0'}{n_0}\sin\left(\sqrt{\frac{k_2}{k}}l\right) \\ -\sqrt{\frac{k_2}{k}} z_0 r_0' \sin\left(\sqrt{\frac{k_2}{k}}l\right) + \frac{r_0'}{n_0}\cos\left(\sqrt{\frac{k_2}{k}}l\right) \end{bmatrix}. \tag{6.36}$$

经出射面折射后, 光线可表为

$$\begin{bmatrix} r(l^+) \\ r'(l^+) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & n_0 \end{bmatrix} \begin{bmatrix} r(l^-) \\ r'(l^-) \end{bmatrix} = \begin{bmatrix} z_0 r_0' \cos\left(\sqrt{\frac{k_2}{k}}l\right) + \sqrt{\frac{k}{k_2}}\frac{r_0'}{n_0}\sin\left(\sqrt{\frac{k_2}{k}}l\right) \\ -\sqrt{\frac{k_2}{k}} n_0 z_0 r_0' \sin\left(\sqrt{\frac{k_2}{k}}l\right) + r_0'\cos\left(\sqrt{\frac{k_2}{k}}l\right) \end{bmatrix}. \tag{6.37}$$

在 $z=l+d$ 处,

$$r(l+d) = r(l^+) + d r'(l^+) = z_0 r_0' \cos\left(\sqrt{\frac{k_2}{k}}l\right) + \sqrt{\frac{k}{k_2}}\frac{r_0'}{n_0}\sin\left(\sqrt{\frac{k_2}{k}}l\right) + \left[-\sqrt{\frac{k_2}{k}} n_0 z_0 r_0' \sin\left(\sqrt{\frac{k_2}{k}}l\right) + r_0'\cos\left(\sqrt{\frac{k_2}{k}}l\right)\right] d = \tag{6.38}$$

其中

$$d = \frac{n_0 z_0 \cos\left(\sqrt{\frac{k_2}{k}}l\right) + \sqrt{\frac{k}{k_2}}\sin\left(\sqrt{\frac{k_2}{k}}l\right)}{\sqrt{\frac{k_2}{k}} n_0^2 z_0 \sin\left(\sqrt{\frac{k_2}{k}}l\right) - n_0\cos\left(\sqrt{\frac{k_2}{k}}l\right)}, \tag{6.39}$$

注意到 $r(l+d)=0$ 不依赖于 r_0', 故类透镜介质把轴上物点成像为单点. $\square$

习题 6.6: 推导表 6.1 列出的光线矩阵. $\square$

证: (1) **长度为 d 的直线段**: 光线 (r_i, r_i') 在均匀介质中传播 d, 则其与光轴的距离变为

$$r_o = r_i + d r_i', \tag{6.40}$$

斜率不变

$$r_o' = r_i', \tag{6.41}$$

故

$$\begin{bmatrix} r_o \\ r_o' \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_i \\ r_i' \end{bmatrix}. \tag{6.42}$$

(2) **薄透镜 (焦距 f)**: 光线 (r_i, r_i') 经过薄透镜, 则其与光轴的距离不变

$$r_o = r_i, \tag{6.43}$$

傍轴近似下, 斜率 r_o' 满足

$$r_i' - r_o' = \frac{r_o}{f}, \tag{6.44}$$

$$\Longrightarrow r_o =' = -\frac{r_o}{f} + r_i', \tag{6.45}$$

故

$$\begin{bmatrix} r_o \\ r_o' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 0 \end{bmatrix} \begin{bmatrix} r_i \\ r_i' \end{bmatrix}. \tag{6.46}$$

(3) **电介质界面 (折射率 n_1, n_2)**: 光线 (r_i, r_i') 经过电介质界面折射, 则其与光轴的距离不变

$$r_o = r_i, \tag{6.47}$$

傍轴近似下, 斜率变为

$$r_o' = \frac{n_1}{n_2} r_i', \tag{6.48}$$

故

$$\begin{bmatrix} r_o \\ r_o' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix} \begin{bmatrix} r_i \\ r_i' \end{bmatrix}. \tag{6.49}$$

(4) **球面电介质界面 (半径 R)**: 光线 (r_i, r_i') 经过球面电介质界面折射, 则其与光轴的距离不变

$$r_o = r_i, \tag{6.50}$$

傍轴近似下, 斜率 r_o' 满足

$$n_1 \left(\frac{r_i}{R} - r_i' \right) = n_2 \left(\frac{r_i}{R} - r_o' \right), \tag{6.51}$$

$$\Longrightarrow r_o' = \frac{n_2 - n_1}{n_2} \frac{r_i}{R} + \frac{n_1}{n_2} r_i', \tag{6.52}$$

故

$$\begin{bmatrix} r_o \\ r_o' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n_2 - n_1}{n_2} \frac{1}{R} & \frac{n_1}{n_2} \end{bmatrix} \begin{bmatrix} r_i \\ r_i' \end{bmatrix}. \tag{6.53}$$

(5) **球面反射镜 (曲率半径 R)**: 光线 (r_i, r_i') 经过球面反射镜反射, 则其与光轴的距离不变

$$r_o = r_i, \tag{6.54}$$

傍轴近似下, 斜率 r_o' 满足

$$\frac{r_i}{R} - r_i' = -r_o' - \frac{r_i'}{R}, \tag{6.55}$$

$$\Longrightarrow r_o' = -\frac{2}{R} r_i + r_i', \tag{6.56}$$

故

$$\begin{bmatrix} r_o \\ r_o' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix} \begin{bmatrix} r_i \\ r_i' \end{bmatrix}. \tag{6.57}$$

(6) **有二次型折射率变化曲线的介质**: 介质的折射率分布为

$$n(x,y)=n_0\left[1-\frac{k_2}{2k}(x^2+y^2)\right], \tag{6.58}$$

其中 k_2 为常数. 光线在非均匀介质中传播的微分方程为

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s}\right)=\nabla, \tag{6.59}$$

其中 s 为沿光线的切向距离, $\boldsymbol{r}$ 为光线的位矢. 傍轴近似下, 可用 $\frac{\mathrm{d}}{\mathrm{d}z}$ 代替 $\frac{\mathrm{d}}{\mathrm{d}s}$, 并将式 (6.58) 代入可得

$$\frac{\mathrm{d}^2r}{\mathrm{d}z^2}+\left(\frac{k_2}{k}\right)r=0, \tag{6.60}$$

对入射光线 $\begin{bmatrix}r(0)\\r'(0)\end{bmatrix}=\begin{bmatrix}r_i\\r_i'\end{bmatrix}$, 解得

$$r(z)=r_i\cos\left(\sqrt{\frac{k_2}{k}}z\right)+\sqrt{\frac{k}{k_2}}r_i'\sin\left(\sqrt{\frac{k_2}{k}}z\right), \tag{6.61}$$

$$r'(z)=-\sqrt{\frac{k_2}{k}}r_i\sin\left(\sqrt{\frac{k_2}{k}}z\right)+r_i'\cos\left(\sqrt{\frac{k_2}{k}}z\right), \tag{6.62}$$

故

$$\begin{matrix}r(l)\\r'(l)\end{matrix}=\begin{bmatrix}\cos\left(\sqrt{\frac{k_2}{k}}l\right) & \sqrt{\frac{k}{k_2}}\sin\left(\sqrt{\frac{k_2}{k}}l\right)\\-\sqrt{\frac{k_2}{k}}\sin\left(\sqrt{\frac{k_2}{k}}l\right) & \cos\left(\sqrt{\frac{k_2}{k}}l\right)\end{bmatrix}\begin{bmatrix}r_i\\r_i'\end{bmatrix}. \tag{6.63}$$

□

习题 6.7: 若透镜相对于入射光束置于任意位置处 (即不置于腰部), 对于这种情况求解导出式 (6.7-11) 和 (6.7-12) 的问题. □

证: 设入射高斯光束的束腰半径为 ω_{01}, 透镜位于距入射光束束腰 z_1 处, 则透镜前表面处光斑半径的平方 $\omega_1^2=[\omega_1(z_1)]^2=\omega_{01}^2\left[1+\left(\frac{z_1}{z_{01}}\right)^2\right]$, 等相位面曲率半径 $R_1=R_1(z_1)=z_1\left[1+\left(\frac{z_{01}}{z_1}\right)^2\right]$, 其中 $z_{01}=\frac{\pi\omega_{01}^2n}{\lambda}$, 因而

$$\frac{1}{q_1}=\frac{1}{R_1}-i\frac{\lambda}{\pi n\omega_1^2}=\frac{1}{z_1+iz_{01}}. \tag{6.64}$$

光束经过透镜折射, 在透镜后表面的参量为

$$\frac{1}{q_2}=\frac{1}{q_1}-\frac{1}{f}, \tag{6.65}$$

$$\Longrightarrow q_2=\frac{f(z_1+z_{01})}{f-(z_1+iz_{01})}. \tag{6.66}$$

光束经过距离 l 的传播, 在平面 (3) 处的参量为

$$q_3=q_2+l=\frac{(f-l)(z_1+iz_{01})+fl}{f-(z_1+iz_{01})}, \tag{6.67}$$

$$\Longrightarrow \frac{1}{q_3}=\frac{1}{R_3}-i\frac{\lambda}{\pi\omega_3^2n}=\frac{\{(f-z_1)[(f-l)z_1+fl]-z_{01}^2(f-l)\}-if^2z_{01}}{[(f-l)z_1+fl]^2+[(f-l)z_{01}]^2}. \tag{6.68}$$

在新束腰处 $R_3=\infty$, 由此得新束腰位置为

$$l=\frac{f}{1+\frac{f(f-z_1)}{z_{01}^2-(f-z_1)z_1}}, \tag{6.69}$$

且新束腰半径与原束腰半径之比为

$$\frac{\omega_3}{\omega_{01}} = \frac{\frac{f}{z_{01}}}{\sqrt{1+\left(\frac{f-z_1}{z_{01}}\right)^2}}. \tag{6.70}$$

□

习题 6.8: (a) 若高斯光束垂直入射到折射率为 n 的固体棱镜上, 如下图所示. 试求出射光束的远场衍射角.

(b) 若棱镜向左移动一直到它的入射面位于 $s=-l_1$ 处. 试求出新的出射光束的腰部大小及腰部位置. (假设晶体足够长, 以致光束的腰部位于晶体内.)

□

解: (a) $z=l_1$ 处入射固体棱镜前, 高斯光束光斑半径的平方为 $[\omega(l_1^-)]^2=\omega_0^2\left[1+\left(\frac{l_1}{z_0}\right)^2\right]$, 等相位面曲率半径为 $R(l_1^-)=z_1\left[1+\left(\frac{z_0}{l_1}\right)^2\right]$, 其中 $z_0=\frac{\pi\omega_0^2 n}{\lambda}$, 因而

$$\frac{1}{q(l_1^-)} = \frac{1}{R(l_1^-)} - i\frac{\lambda}{\pi n[\omega(l_1^-)]^2} = \frac{1}{l_1+iz_0}, \tag{6.71}$$

$$\Longrightarrow q(l_1^-) = l_1+iz_0. \tag{6.72}$$

$z=l_1$ 处, 光束入射后的参量为

$$q(l_1^+) = \frac{q(l_1^-)}{\frac{1}{n}} = n(l_1+iz_0). \tag{6.73}$$

$z=l_2$ 出射前光束的参量为

$$q(l_2^-) = q(l_1^+) + (l_2-l_1) = n(l_1+iz_0)+(l_2-l_1). \tag{6.74}$$

$z=l_2$ 光束出射后的参量为

$$q(l_2^+) = \frac{q(l_2^-)}{n} = (l_1+iz_0)+\frac{l_2-l_1}{n}. \tag{6.75}$$

$z=l_3$ 处光束的参量为

$$q(l_3) = q(l_2^+) + l = (l_1+iz_0)+\frac{l_2-l_1}{n}+(l_3-l_2). \tag{6.76}$$

由

$$\frac{1}{q(l_3)} = \frac{1}{R(l_3)} - i\frac{\lambda}{\pi n[\omega(l_3)]^2}, \tag{6.77}$$

得光束等相位面曲率半径

$$\frac{1}{R_3} = \frac{l_1+\frac{l_2-l_1}{n}+(l_3-l_2)}{\left[l_1+\frac{l_2-l_1}{n}+(l_3-l_2)\right]^2+z_0^2}, \tag{6.78}$$

及束腰半径

$$\omega(l_3) = \omega_0\sqrt{1+\left[\frac{l_1+\frac{l_2-l_1}{n}+(l_3-l_2)}{z_0}\right]^2} \tag{6.79}$$

在新束腰处 $R(l_3) = \infty$, 由此得

$$l_3 = l_2 - \left(l_1 + \frac{l_2 - l_1}{n}\right), \tag{6.80}$$

及新束腰半径

$$\omega_0' = \omega_0. \tag{6.81}$$

此时出射光束的远场衍射角为

$$\theta \approx \frac{\lambda}{\pi\omega_0' n} = \frac{\lambda}{\pi\omega_0 n}. \tag{6.82}$$

(b) 当棱镜入射面位于 $z = -l_1$ 处, 则新的出射光束的腰部半径为

$$\omega_0' = \omega_0, \tag{6.83}$$

位置为

$$l_3 = (l_1 + l_2) - \frac{l_1 + l_2}{n}. \tag{6.84}$$

□

习题 6.9: 波长为 λ 的高斯光束入射到置于 $Z = l$ 的透镜上, 如下图所示. 要使出射光束的腰位于晶体样品的前表面上, 试计算透镜的焦距 f. 证明 (对给定的 l 和 L) 可能存在两个解. 对每个解画出光束的传播情况. □

证: 利用习题 6.7 的结论, 新束腰距离透镜

$$\frac{f}{1 + \frac{f(f-l)}{z_0^2 - (f-l)l}} = L, \tag{6.85}$$

$$\Longrightarrow (l + L)f^2 - (l^2 + 2lL + z_0^2)f + (l^2 + z_0^2)L = 0, \tag{6.86}$$

解得当 $(l^2 + z_0^2)^2 - 4L^2 z_0^2 \geq 0$ 时,

$$f = \frac{l^2 + 2lL + z_0^2 \pm \sqrt{(l^2 + z_0^2)^2 - 4L^2 z_0^2}}{l + L}. \tag{6.87}$$

□

习题 6.10: 补全 6.12 节推导过程中所有略去的步骤. □

证: 仿照 6.5 节, 取 $E = \psi(x, y, z)e^{-ikz}$, 将亥姆霍兹方程化为

$$\nabla_t^2\psi - 2ik\psi' - k(k_{2x}x^2 + k_{2y}y^2)\psi = 0, \tag{6.88}$$

其中 $\psi' = \frac{\partial\psi}{\partial z}$. 假设波动方程的一个解为

$$\psi = \exp\left\{-i\left[P(z) + \frac{Q(z)x^2}{2} + \frac{Q_y(z)y^2}{2}\right]\right\}, \tag{6.89}$$

将其代入式 (6.88) 得

$$Q_x^2 x^2 + Q_y y^2 + iQ_x + iQ_y + 2kP' + k(Q_x' x + Q_y' y) + kk_2(x^2 + y^2) = 0. \tag{6.90}$$

上式对任一 x,y 均成立, 故 x,y 的各次幂的系数均等于零, 从而导出

$$Q_x^2+k\frac{\mathrm{d}Q_x}{\mathrm{d}z}+kk_{2x}=0,\quad Q_y^2+k\frac{\mathrm{d}Q_y}{\mathrm{d}z}+kk_{2y}=0 \tag{6.91}$$

和

$$\frac{\mathrm{d}P}{\mathrm{d}z}=-i\left(\frac{Q_x+Q_y}{2k}\right). \tag{6.92}$$

假设介质均匀, 则 $k_{2x}=k_{2y}=0$, 式 (6.91) 可化为

$$Q_x^2+k\frac{\mathrm{d}Q_x}{\mathrm{d}z}=0,\quad Q_y^2+k\frac{\mathrm{d}Q_y}{\mathrm{d}z}=0. \tag{6.93}$$

引入函数 $s_x(z)$ 和 $s_y(z)$, 满足

$$Q_x=k\frac{s_x'}{s_x},\quad Q_y=k\frac{s_y'}{s_y}. \tag{6.94}$$

由式 (6.93) 可得

$$s_x''=0,\quad s_y''=0, \tag{6.95}$$

因而

$$s_x=a_xz+b_x,\quad s_y=a_yz+b_y, \tag{6.96}$$

或

$$Q_x(z)=k\frac{a_x}{a_xz+b_z},\quad Q_y(z)=k\frac{a_y}{a_yz+b_y}, \tag{6.97}$$

其中 a_x,b_x,a_y,b_y 为任意常数. 定义

$$q_x(z)=\frac{k}{Q_x(z)},\quad q_y(z)=\frac{k}{Q_y(z)}, \tag{6.98}$$

从而将式 (6.97) 改写为

$$q_x=z+C_x,\quad q_y=z+C_y, \tag{6.99}$$

其中 C_x 和 C_y 为任意积分常数. 将 C_x 和 C_y 写成如下形式:

$$C_x=-z_x+q_{0x},\quad C_y=-z_y+q_{0y}, \tag{6.100}$$

其中 z_x 和 z_y 为实数, q_{0x} 和 q_{0y} 为虚数. 将 $q_x(z)$ 和 $q_y(z)$ 代入式 (6.92) 可得

$$P'=-\frac{i}{2}\left[\frac{1}{z+C_x}+\frac{1}{z+C_y}\right], \tag{6.101}$$

取积分常数为零得

$$P=-\frac{i}{2}\left[\ln\left(1+\frac{z-z_x}{q_{0x}}\right)+\ln\left(1+\frac{z-z_y}{q_{0y}}\right)\right]. \tag{6.102}$$

将上式代入 ψ 的假设解中得

$$\psi=\exp\left\{-i\left[-\frac{i}{2}\ln\left(1+\frac{z-z_x}{q_0}\right)-\frac{i}{2}\ln\left(1+\frac{z-z_y}{q_{0y}}\right)+\frac{kx^2}{2(q_{0x}+z-z_x)}+\frac{ky^2}{2(q_{0y}+z-z_y)}\right]\right\}. \tag{6.103}$$

取

$$q_{0x} = i\frac{\pi\omega_{ox}^2 n}{\lambda}, \quad q_{0y} = i\frac{\pi\omega_{0y}^2 n}{\lambda}, \tag{6.104}$$

则式 (6.103) 中的

$$\begin{aligned}
&\exp\left[-\frac{1}{2}\ln\left(1+\frac{z-z_x}{q_{0x}}\right)-\frac{1}{2}\ln\left(1+\frac{z-z_y}{q_{0y}}\right)\right] = \frac{1}{\left[1+\frac{z-z_x}{q_{0x}}\right]^{1/2}\left[1+\frac{z-z_y}{q_{0y}}\right]^{1/2}} \\
=&\frac{1}{\left[\sqrt{1+\left|\frac{z-z_x}{q_{0x}}\right|^2}\exp\left(i\arctan\frac{z-z_x}{iq_{0y}}\right)\right]^{1/2}\left[\sqrt{1+\left|\frac{z-z_y}{q_{0y}}\right|^2}\exp\left(i\arctan\frac{z-z_y}{iq_{0y}}\right)\right]^{1/2}} \\
=&\frac{\sqrt{\omega_{0x}\omega_{0y}}}{\sqrt{\omega_x(z)\omega_y(z)}}\exp[i\eta(z)],
\end{aligned} \tag{6.105}$$

其中

$$\omega_x^2(z) = \omega_{0x}^2\left[1+\left(\frac{\lambda(z-z_x)}{\pi\omega_{0x}^2 n}\right)^2\right], \quad \omega_y^2(z) = \omega_{0y}^2\left[1+\left(\frac{\lambda(z-z_y)}{\pi\omega_{0y}^2 n}\right)^2\right], \tag{6.106}$$

$$\eta = \frac{1}{2}\tan^{-1}\left(\frac{\lambda(z-z_x)}{\pi\omega_{0x}^2 n}\right)+\frac{1}{2}\tan^{-1}\left(\frac{\lambda(z-z_y)}{\pi\omega_{0y}^2 n}\right). \tag{6.107}$$

式 (6.103) 中的

$$\begin{aligned}
&\exp\left[\frac{-ikx^2}{2(q_{0x}+z-z_x)}+\frac{-iky^2}{2(q_0+z-z_y)}\right] = \exp\left\{\frac{-i\frac{k}{q_{0x}}x^2}{2\left(1+\frac{z-z_{0x}}{q_{0x}}\right)}+\frac{-i\frac{k}{q_{0y}}y^2}{2\left(1+\frac{z-z_{0y}}{q_{0y}}\right)}\right\} \\
=&\exp\left\{-x^2\frac{1}{\omega_{0x}^2\left(1+\frac{z-z_x}{q_{0x}}\right)}-y^2\frac{1}{\omega_{0y}^2\left(1+\frac{z-z_y}{q_{0y}}\right)}\right\} \\
=&\exp\left\{-x^2\frac{1-\frac{z-z_x}{q_{0x}}}{\omega_{0x}^2\left(1+\left|\frac{z-z_x}{q_{0x}}\right|^2\right)}-y^2\frac{1-\frac{z-z_y}{q_{0y}}}{\omega_{0y}^2\left(1+\left|\frac{z-z_y}{q_{0y}}\right|^2\right)}\right\} \\
=&\exp\left\{-x^2\left(\frac{1}{\omega_x^2(x)}+\frac{ik}{2R_x(z)}\right)-y^2\left(\frac{1}{\omega_y^2(z)}+\frac{ik}{2R_y(z)}\right)\right\},
\end{aligned} \tag{6.108}$$

其中

$$R_x(z) = (z-z_x)\left[1+\left(\frac{\pi\omega_{0x}^2 n}{\lambda(z-z_x)}\right)^2\right], \quad R_y(z) = (z-z_y)\left[1+\left(\frac{\pi\omega_{0y}^2 n}{\lambda(z-z_y)}\right)\right]. \tag{6.109}$$

综上, 在均匀介质中,

$$\begin{aligned}
E(x,y,z) =&E_0\frac{\sqrt{\omega_{0x}\omega_{0y}}}{\sqrt{\omega_x(z)\omega_y(z)}}\exp\left\{-i[kz-\eta(z)]-\frac{ikx^2}{2q_x(z)}-\frac{iky^2}{2q_y(z)}\right\} \\
=&E_0\frac{\sqrt{\omega_{0x}\omega_{0y}}}{\sqrt{\omega_x(z)\omega_y(z)}}\exp\left\{-i[kz-\eta(z)]-x^2\left(\frac{1}{\omega_x^2(z)}+\frac{ik}{2R_x(z)}\right)-y^2\left(\frac{1}{\omega_y^2(z)}+\frac{ik}{2R_y(z)}\right)\right\}.
\end{aligned} \tag{6.110}$$

对比 ψ 的假设解和上式可得

$$\frac{1}{q_x(z)} =\frac{1}{R_x(z)}-i\frac{\lambda}{\pi n\omega_x^2(z)}, \tag{6.111}$$

$$\frac{1}{q_y(z)} = \frac{1}{R_y(z)} - i\frac{\lambda}{\pi n \omega_y^2(z)}. \tag{6.112}$$

在折射率分布为

$$n^2(\boldsymbol{r}) = n^2\left(1 - \frac{n_{2x}}{n}x^2 - \frac{n_{2y}}{n}y^2\right) \tag{6.113}$$

的二次型类透镜介质中, 亥姆霍兹方程可化为

$$\nabla^2 \boldsymbol{E} + k^2\left(1 - \frac{n_{2x}}{n}x^2 - \frac{n_{2y}}{n}y^2\right). \tag{6.114}$$

假设标量场的形式为 $E(\boldsymbol{r}) = \psi(\boldsymbol{r})\exp(-i\beta z)$, 则上述方程可化为

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \left[k^2\left(1 - \frac{n_{2x}}{n}x^2 - \frac{n_{2y}}{n}y^2\right) - \beta^2\right]\psi = 0. \tag{6.115}$$

取 $\psi = f(x)g(y)$, 并将上式除以 ψ 得

$$\frac{1}{f}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \left(\lambda_1 - k^2\frac{n_{2x}}{n}x^2\right) = -\frac{1}{g}\frac{\mathrm{d}^2 g}{\mathrm{d}y^2} + \left[(k^2 - \beta^2 - \lambda_1) - k^2\frac{n_2}{n}y^2\right], \tag{6.116}$$

分离变量得两个微分方程:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \left(\lambda_1 - k^2\frac{n_2}{n}x^2\right)f = 0, \tag{6.117}$$

$$\frac{\mathrm{d}^2 g}{\mathrm{d}y^2} + \left[(k - \beta^2 - \lambda_1) - k^2\frac{n_2}{n}y^2\right]g = 0. \tag{6.118}$$

做变量代换

$$\xi = \frac{\sqrt{2}x}{\omega_x}, \quad \omega_x = \left(\frac{\lambda_1}{\pi}\right)^{1/2}\left(\frac{1}{nn_{2x}}\right)^{1/4}, \tag{6.119}$$

从而将上面的微分方程化为

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} + \left(\frac{\omega_x^2\lambda_1}{2} - \xi^2\right)f = 0. \tag{6.120}$$

假设上述方程的解具有如下形式:

$$f\left(\frac{\omega_x\xi}{\sqrt{2}}\right) = H(\xi)e^{-\xi^2/2}, \tag{6.121}$$

其中 $H(\xi)$ 是一个有限级数的多项式. 将该假设解代入微分方程得

$$\frac{\mathrm{d}^2 H}{\mathrm{d}\xi^2} - 2\xi\frac{\mathrm{d}H}{\mathrm{d}\xi} + \left(\frac{\omega_x^2\lambda_1}{2} - 1\right)H = 0, \tag{6.122}$$

从而解得 $H(\xi)$ 为厄米多项式 $H_l(\xi)$, 且

$$\frac{\omega_x^2\lambda_1}{2} = 2l + 1. \tag{6.123}$$

对关于 y 的微分方程也可同理求解, 故

$$E_{l,m}(\boldsymbol{r}) = E_0 e^{-i\beta_{l,m}z} H_l\left(\sqrt{2}\frac{x}{\omega_x}\right) H_m\left(\sqrt{2}\frac{y}{\omega_y}\right)\exp\left(-\frac{x^2}{\omega_x^2} - \frac{y^2}{\omega_y^2}\right), \tag{6.124}$$

且

$$k^2 - \beta^2 = \lambda_1^2 + (k^2 - \beta^2 - \lambda_1) = \frac{2}{\omega_x^2}[2(l+1)] + \frac{2}{\omega_y^2}[2(m+1)], \tag{6.125}$$

$$\Longrightarrow \beta_{l,m} = k\left\{1 - \frac{2}{k}\left[\sqrt{\frac{n_{2x}}{n}}\left(l + \frac{1}{2}\right) + \sqrt{\frac{n_{2y}}{n}}\left(m + \frac{1}{2}\right)\right]\right\}^{1/2}. \tag{6.126}$$

□

习题 6.11: 证明式 (6.10-10).
提示: 把光脉冲场看作载波和包络函数的乘积

$$E(z,t) = E_0 e^{i(\omega_0 t - k_0 z)} \int_{-\infty}^{+\infty} G(\Delta\omega) e^{i(\Delta\omega t - \Delta k z)}\, \mathrm{d}(\Delta\omega)$$

式中 $\Delta\omega \equiv \omega - \omega_0$, $\Delta k = k(\omega) - k_0$. □

证: 由傅里叶变换得光场频谱为

$$G(\Delta\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{E(z,t)}{E_0} e^{-i[(\omega_0+\Delta\omega)\tau - (k_0+\Delta k)z]}\, \mathrm{d}\tau. \tag{6.127}$$

对一个持续时间为 τ 的光脉冲, 该脉冲的谱宽满足

$$\Delta\omega \frac{\tau}{2} = \pi, \tag{6.128}$$

$$\Longrightarrow \Delta\omega = \frac{2\pi}{\tau}. \tag{6.129}$$

单频光传播距离 L 所用的时间为

$$t = \frac{L}{v_g}. \tag{6.130}$$

该脉冲传输距离 L 后, 增宽为

$$\Delta\tau \approx \left|\frac{\mathrm{d}\tau}{\mathrm{d}t}\right| \Delta\omega = \left|\frac{\mathrm{d}\left(\frac{L}{v_g}\right)}{\mathrm{d}\omega}\right| \Delta\omega = \frac{L}{v_g^2} \frac{\mathrm{d}v_g}{\mathrm{d}\omega} \frac{2\pi}{\tau}. \tag{6.131}$$

□

习题 6.12: 一根二次型折射率变化的玻璃纤维长 1000 米, $n = 1.5$, $n_2 = 5 \times 10^2$ 厘米$^{-2}$. 波长 $\lambda = 1$ 微米的光束在此纤维中传播, 试求在 (a) 单模 $l = m = 0$ 激发情况下, (b) $l = m = 5$ 情况下此载波的光斑尺寸及最大的脉冲重复频率. □

解: (a) 单模 $l = m = 0$ 激发情况下, 光斑半径为

$$w_0 = \left(\frac{\lambda}{\pi}\right)^{1/2} \left(\frac{1}{nn_2}\right)^{1/4} = 1.08 \times 10^{-5}\,\mathrm{m} = 10.8\,\mu\mathrm{m}, \tag{6.132}$$

最大脉冲重复频率为

$$f_{\max} \approx \frac{1}{\Delta\tau} = \frac{1}{\frac{Lnn_2\Delta\omega}{c^2k^3}\left[1 + \frac{n_2/n}{2k^2}(0+0+1)^2\right]^2 (0+0+1)^2} = \frac{1.00 \times 10^{28}\,\mathrm{Hz}^2}{\Delta\omega}, \tag{6.133}$$

其中 $\Delta\omega$ 为载波的光谱宽度.

(b) $l = m = 5$ 情况下, 光斑半径为

$$w_5 = \sqrt{2*5+1}\,w_0 = 3.58 \times 10^{-5}\,\mathrm{m} = 35.8\,\mu\mathrm{m}, \tag{6.134}$$

最大脉冲重复频率为

$$f_{\max} \approx \frac{1}{\Delta\tau} = \frac{1}{\frac{Lnn_2\Delta\omega}{c^2k^3}\left[1 + \frac{n_2/n}{2k^2}(5+5+1)^2\right]^2 (5+5+1)^2} = \frac{8.30 \times 10^{25}\,\mathrm{Hz}^2}{\Delta\omega}. \tag{6.135}$$

□

习题 6.13: 推导式 (6.10-4) 和 (6.10-5). □

证: 二次型折射率变化的介质中矢量波动方程为

$$\nabla^2 \boldsymbol{E} + k^2 \left(1 - \frac{n_2}{n} r^2\right) \boldsymbol{E} = 0, \tag{6.136}$$

其中光束在折射率为 n 的均匀介质中的传播常数 $k = \frac{2\pi n}{\lambda}$. 设标量场的形式为 $E(\boldsymbol{r}) = \psi(\boldsymbol{r}) \exp(i\beta z)$, 从而上述方程化为

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \left[k^2 \left(1 - \frac{n_2}{n} r^2\right) - \beta^2\right] \psi = 0. \tag{6.137}$$

取 $\psi = f(x)g(y)$, 并将上式除以 ψ 得

$$\frac{1}{f}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \left(\lambda_1 - k^2 \frac{n_2}{n} x^2\right) = -\frac{1}{g}\frac{\mathrm{d}^2 g}{\mathrm{d}y^2} - \left[(k^2 - \beta^2 - \lambda_1) - k^2 \frac{n_2}{n} y^2\right], \tag{6.138}$$

分离变量得两个微分方程:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \left(\lambda_1 - k^2 \frac{n_2}{n} x^2\right) f = 0, \tag{6.139}$$

$$\frac{\mathrm{d}^2 g}{\mathrm{d}y^2} + \left[(k^2 - \beta^2 - \lambda_1) - k^2 \frac{n_2}{n} y^2\right] g = 0. \tag{6.140}$$

作变量代换

$$\xi = \frac{\sqrt{2}x}{\omega}, \quad \omega = \left(\frac{\lambda_1}{\pi}\right)^{1/2} \left(\frac{1}{nn_2}\right)^{1/4}, \tag{6.141}$$

从而将上面的微分方程化为

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} + \left(\frac{\omega^2 \lambda_1}{2} - \xi^2\right) f = 0. \tag{6.142}$$

假设上述方程的解具有如下形式:

$$f\left(\frac{\omega\xi}{\sqrt{2}}\right) = H(\xi) e^{-\xi^2/2}, \tag{6.143}$$

其中 $H(\xi)$ 是一个有限级数的多项式. 将该假设解代入微分方程得

$$\frac{\mathrm{d}^2 H}{\mathrm{d}\xi^2} - 2\xi \frac{\mathrm{d}H}{\mathrm{d}\xi} + \left(\frac{\omega^2 \lambda_1}{2} - 1\right) H = 0, \tag{6.144}$$

从而解得 $H(\xi)$ 为厄米多项式 $H_l(\xi)$, 且

$$\frac{\omega^2 \lambda_1}{2} = 2l + 1. \tag{6.145}$$

对关于 y 的微分方程也可同理求解, 故

$$\psi_{l,m}(x,y) = f_l(x) g_m(y) = E_0 H_l\left(\sqrt{2}\frac{x}{\omega}\right) H_m\left(\sqrt{2}\frac{y}{\omega}\right) \exp\left(-\frac{x^2 + y^2}{\omega^2}\right), \tag{6.146}$$

且

$$k^2 - \beta^2 = \lambda_1 + (k^2 - \beta^2 - \lambda_1) = \frac{2}{\omega^2}[(2l+1) + (2m+1)], \tag{6.147}$$

$$\Longrightarrow \beta_{l,m} = k\left[1 - \frac{2}{k}\sqrt{\frac{n_2}{n}}(l + m + 1)\right]^{1/2}. \tag{6.148}$$

□