Quantum Mechanics
Homework Assignment 05
Fall, 2019

1. **[C-T Exercise 2-1]** $|\varphi_n\rangle$ are the eigenstates of a Hermitian operator $\hat{H}$ ($\hat{H}$ is, for example, the Hamiltonian of an arbitrary physical system). Assume that the states $|\varphi_n\rangle$ form a discrete orthonormal basis. The operator $\hat{U}(m,n)$ is defined by $\hat{U}(m,n) = |\varphi_m\rangle\langle\varphi_n|$.
   (a) Calculate the adjoint $\hat{U}^\dagger(m,n)$ of $\hat{U}(m,n)$.
   (b) Calculate the commutator $[\hat{H}, \hat{U}(m,n)]$.
   (c) Prove the relation $\hat{U}(m,n)\hat{U}^\dagger(p,q) = \delta_{nq}\hat{U}(m,p)$.
   (d) Calculate $\text{Tr}\{\hat{U}(m,n)\}$, the trace of the operator $\hat{U}(m,n)$.
   (e) Let $\hat{A}$ be an operator, with matrix elements $A_{mn} = \langle\varphi_m|\hat{A}|\varphi_n\rangle$. Prove the relation $\hat{A} = \sum_{m,n} A_{mn}\hat{U}(m,n)$.
   (f) Show that $A_{pq} = \text{Tr}\{\hat{A}U^\dagger(p,q)\}$.

2. **[C-T Exercise 2-2]** In a three-dimensional vector space, consider the operator whose matrix, in an orthonormal basis $\{|1\rangle, |2\rangle, |3\rangle\}$, is written as $\hat{L}_y = \dfrac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$.
   (a) Is $\hat{L}_y$ Hermitian? Calculate its eigenvalues and eigenvectors (giving their normalized expansion in terms of the $\{|1\rangle, |2\rangle, |3\rangle\}$ basis).
   (b) Calculate the matrices which represent the projectors onto these eigenvectors. Then verify that they satisfy the orthogonality and closure relations.

3. **[C-T Exercise 2-3]** The state space of a certain physical system is three-dimensional. Let $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ be an orthonormal basis of this space. The kets $|\psi_0\rangle$ and $|\psi_1\rangle$ are defined by

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{i}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle,$$
$$|\psi_1\rangle = \frac{1}{\sqrt{3}}|u_1\rangle + \frac{i}{\sqrt{3}}|u_3\rangle.$$

   (a) Are these kets normalized?
   (b) Calculate the matrices ρ_0 and ρ_1 representing, in the $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ basis, the projection operators onto the state $|\psi_0\rangle$ and onto the state $|\psi_1\rangle$. Verify that these matrices are Hermitian.

4. **[C-T Exercise 2-9]** Let $\hat{H}$ be the Hamiltonian operator of a physical system. Denote by $|\varphi_n\rangle$ the eigenvectors of $\hat{H}$, with eigenvalues E_n, $\hat{H}|\varphi_n\rangle = E_n|\varphi_n\rangle$.
   (a) For an arbitrary operator $\hat{A}$, prove the relation $\langle\varphi_n|[\hat{A},\hat{H}]|\varphi_n\rangle = 0$.
   (b) Consider a one-dimensional problem, where the physical system is a particle of mass m and of potential energy $\hat{V}(\hat{x})$. In this case, $\hat{H}$ is written as $\hat{H} = \dfrac{1}{2m}\hat{p}^2 + \hat{V}(\hat{x})$.
      i. In terms of $\hat{p}$, $\hat{x}$, and $\hat{V}(\hat{x})$, find the commutators: $[\hat{H},\hat{p}]$, $[\hat{H},\hat{x}]$, and $[\hat{H},\hat{x}\hat{p}]$.
      ii. Show that the matrix element $\langle\varphi_n|\hat{p}|\varphi_n\rangle$ is zero.
      iii. Establish a relation between $E_k = \langle\varphi_n|\dfrac{\hat{p}^2}{2m}|\varphi_n\rangle$ and $\langle\varphi_n|\hat{x}\dfrac{d\hat{V}(\hat{x})}{d\hat{x}}|\varphi_n\rangle$. Apply the derived relation to $\hat{V}(\hat{x}) = V_0\hat{x}^\lambda$ with $\lambda = 2, 4, 6, \cdots$ and $V_0 > 0$?

5. **[C-T Exercise 2-10]** Using the relation $\langle x|p\rangle = (2\pi\hbar)^{-1/2}e^{ipx/\hbar}$, find the expressions $\langle x|\hat{x}\hat{p}|\psi\rangle$ and $\langle x|\hat{p}\hat{x}|\psi\rangle$ in terms of $\psi(x)$. Can these results be found directly by using the fact that in the $\{|x\rangle\}$ representation, $\hat{p}$ acts like $-i\hbar\dfrac{d}{dx}$?