Quantum Mechanics

Solutions to the Problems in Homework Assignment 07

Fall, 2018

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1. Starting from the time-dependent Schrödinger equation in the Dirac notation, $i\hbar\dfrac{d\,|\psi(t)\rangle}{dt} = \left[\dfrac{\hat{\vec{p}}^{\,2}}{2m} + \hat{V}(\hat{\vec{r}})\right]|\psi(t)\rangle$, derive the time-dependent Schrödinger equation in the $\{|\vec{p}\rangle\}$ representation,

$$i\hbar\frac{\partial}{\partial t}\overline{\psi}(\vec{p},t) = \left[\,\frac{\vec{p}^{\,2}}{2m} + \hat{V}(i\hbar\vec{\nabla}_{\vec{p}})\,\right]\overline{\psi}(\vec{p},t).$$

To obtain the time-dependent Schrödinger equation in the $\{|\vec{p}\rangle\}$ representation, we project the time-dependent Schrödinger equation onto the $\{|\vec{p}\rangle\}$ basis. We have

$$i\hbar\,\langle\vec{p}|\frac{d}{dt}|\psi(t)\rangle = \langle\vec{p}|\left[\frac{\hat{\vec{p}}^{\,2}}{2m} + \hat{V}(\hat{\vec{r}})\right]|\psi(t)\rangle\,.$$

Making use of

$$\begin{aligned}
\langle\vec{p}|\frac{d}{dt}|\psi(t)\rangle &= \frac{\partial}{\partial t}\,\langle\vec{p}|\psi(t)\rangle = \frac{\partial}{\partial t}\overline{\psi}(\vec{p},t),\\
\langle\vec{p}|\frac{\hat{\vec{p}}^{\,2}}{2m}|\psi(t)\rangle &= \frac{\vec{p}^{\,2}}{2m}\,\langle\vec{p}|\psi(t)\rangle = \frac{\vec{p}^{\,2}}{2m}\overline{\psi}(\vec{p},t),\\
\langle\vec{p}|\hat{V}(\hat{\vec{r}})|\psi(t)\rangle &= \hat{V}(i\hbar\vec{\nabla}_{\vec{p}})\,\langle\vec{p}|\psi(t)\rangle = \hat{V}(i\hbar\vec{\nabla}_{\vec{p}})\overline{\psi}(\vec{p},t),
\end{aligned}$$

we have the following time-dependent Schrödinger equation in the $\{|\vec{p}\rangle\}$ representation

$$i\hbar\frac{\partial}{\partial t}\overline{\psi}(\vec{p},t) = \left[\,\frac{\vec{p}^{\,2}}{2m} + \hat{V}(i\hbar\vec{\nabla}_{\vec{p}})\,\right]\overline{\psi}(\vec{p},t).$$

2. Introducing the Fourier transform of the potential energy $V(\vec{r})$ in the $\{|\vec{r}\rangle\}$ representation, $\overline{V}(\vec{p}) = \dfrac{1}{(2\pi\hbar)^{3/2}}\displaystyle\int d^3r\; e^{-i\vec{p}\cdot\vec{r}/\hbar}V(\vec{r})$, show that the time-dependent Schrödinger equation in the $\{|\vec{p}\rangle\}$ representation can be also written as

$$i\hbar\frac{\partial}{\partial t}\overline{\psi}(\vec{p},t) = \frac{\vec{p}^{\,2}}{2m}\overline{\psi}(\vec{p},t) + \frac{1}{(2\pi\hbar)^{3/2}}\int d^3p'\;\overline{V}(\vec{p}-\vec{p}\,')\overline{\psi}(\vec{p}\,',t).$$

We now rewrite the potential energy term in the above-derived time-dependent Schrödinger equation in the $\{|\vec{p}\rangle\}$ representation,

$$i\hbar\frac{\partial}{\partial t}\overline{\psi}(\vec{p},t) = \left[\,\frac{\vec{p}^{\,2}}{2m} + \hat{V}(i\hbar\vec{\nabla}_{\vec{p}})\,\right]\overline{\psi}(\vec{p},t).$$

We first put the potential energy term into the following form

$$\hat{V}(i\hbar\vec{\nabla}_{\vec{p}})\overline{\psi}(\vec{p},t) = \langle\vec{p}|\hat{V}(\hat{\vec{r}})|\psi(t)\rangle\,.$$

Inserting the magic one

$$\int d^3r\;\;|\vec{r}\rangle\langle\vec{r}| = 1$$

between $\langle\vec{p}|$ and $\hat{V}(\hat{\vec{r}})$ and inserting the magic one

$$\int d^3p'\;\;|\vec{p}\,'\rangle\langle\vec{p}\,'| = 1$$

between $\hat{V}(\hat{\vec{r}})$ and $|\psi(t)\rangle$, we have

$$\hat{V}(i\hbar\vec{\nabla}_{\vec{p}})\overline{\psi}(\vec{p},t) = \int d^3p' \int d^3r \ \langle\vec{p}|\vec{r}\rangle\langle\vec{r}|\hat{V}(\hat{\vec{r}})|\vec{p}'\rangle\langle\vec{p}'|\psi(t)\rangle\,.$$

Making use of

$$\langle\vec{r}|\hat{V}(\hat{\vec{r}})|\vec{p}'\rangle = V(\vec{r})\langle\vec{r}|\vec{p}'\rangle,$$

we have

$$\hat{V}(i\hbar\vec{\nabla}_{\vec{p}})\overline{\psi}(\vec{p},t) = \int d^3p' \int d^3r \ \langle\vec{p}|\vec{r}\rangle V(\vec{r})\langle\vec{r}|\vec{p}'\rangle\langle\vec{p}'|\psi(t)\rangle\,.$$

Utilizing

$$\langle\vec{p}|\vec{r}\rangle = \frac{1}{(2\pi)^{3/2}}e^{-i\vec{p}\cdot\vec{r}/\hbar},\ \langle\vec{r}|\vec{p}'\rangle = \frac{1}{(2\pi)^{3/2}}e^{i\vec{p}'\cdot\vec{r}/\hbar},\ \langle\vec{p}'|\psi(t)\rangle = \overline{\psi}(\vec{p}',t),$$

we have

$$\hat{V}(i\hbar\vec{\nabla}_{\vec{p}})\overline{\psi}(\vec{p},t) = \frac{1}{(2\pi)^3}\int d^3p' \int d^3r \ e^{-i(\vec{p}-\vec{p}')\cdot\vec{r}/\hbar}V(\vec{r})\overline{\psi}(\vec{p}',t).$$

Introducing

$$\overline{V}(\vec{p}) = \frac{1}{(2\pi\hbar)^{3/2}}\int d^3r \ e^{-i\vec{p}\cdot\vec{r}/\hbar}V(\vec{r}),$$

we have

$$\hat{V}(i\hbar\vec{\nabla}_{\vec{p}})\overline{\psi}(\vec{p},t) = \frac{1}{(2\pi)^{3/2}}\int d^3p' \ V(\vec{p}-\vec{p}')\overline{\psi}(\vec{p}',t).$$

Inserting the above expression of $\hat{V}(i\hbar\vec{\nabla}_{\vec{p}})\overline{\psi}(\vec{p},t)$ into

$$i\hbar\frac{\partial}{\partial t}\overline{\psi}(\vec{p},t) = \left[\ \frac{\vec{p}^{\,2}}{2m} + \hat{V}(i\hbar\vec{\nabla}_{\vec{p}})\ \right]\overline{\psi}(\vec{p},t),$$

we obtain

$$i\hbar\frac{\partial}{\partial t}\overline{\psi}(\vec{p},t) = \frac{\vec{p}^{\,2}}{2m}\overline{\psi}(\vec{p},t) + \frac{1}{(2\pi\hbar)^{3/2}}\int d^3p' \ \overline{V}(\vec{p}-\vec{p}^{\,\prime})\overline{\psi}(\vec{p}^{\,\prime},t).$$

3. In the $\{|p_x\rangle\}$ representation, find the energy eigenvalue and eigenfunction of a particle of mass m in the one-dimensional δ-function potential well

$$V(x) = -\lambda\delta(x),\ \lambda > 0.$$

From the time-dependent Schrödinger equation in the $\{|\vec{p}\rangle\}$ representation

$$i\hbar\frac{\partial}{\partial t}\overline{\psi}(\vec{p},t) = \frac{\vec{p}^{\,2}}{2m}\overline{\psi}(\vec{p},t) + \frac{1}{(2\pi\hbar)^{3/2}}\int d^3p' \ \overline{V}(\vec{p}-\vec{p}^{\,\prime})\overline{\psi}(\vec{p}^{\,\prime},t),$$

we have the following stationary Schrödinger equation in the $\{|\vec{p}\rangle\}$ representation

$$\frac{\vec{p}^{\,2}}{2m}\overline{\varphi}(\vec{p}) + \frac{1}{(2\pi\hbar)^{3/2}}\int d^3p' \ \overline{V}(\vec{p}-\vec{p}^{\,\prime})\overline{\varphi}(\vec{p}^{\,\prime}) = E\,\overline{\varphi}(\vec{p}).$$

In one dimension, we have

$$\frac{p_x^2}{2m}\overline{\varphi}(p_x) + \frac{1}{(2\pi\hbar)^{1/2}}\int_{-\infty}^{\infty} dp_x' \ \overline{V}(p_x-p_x')\overline{\varphi}(p_x') = E\,\overline{\varphi}(p_x)$$

with

$$\overline{V}(p_x) = \frac{1}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} dx\, e^{-ip_x x/\hbar} V(x).$$

For $V(x) = -\lambda\delta(x)$, we have

$$\overline{V}(p_x) = -\frac{\lambda}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} dx\, e^{-ip_x x/\hbar} \delta(x) = -\frac{\lambda}{(2\pi\hbar)^{1/2}}$$

which is a constant. The stationary Schrödinger equation in the $\{|p_x\rangle\}$ representation for a particle in the δ-function potential $V(x) = -\lambda\delta(x)$ becomes

$$\frac{p_x^2}{2m}\overline{\varphi}(p_x) - \frac{\lambda}{2\pi\hbar} \int_{-\infty}^{\infty} dp_x'\, \overline{\varphi}(p_x') = E\,\overline{\varphi}(p_x).$$

The above equation is an integral equation for $\overline{\varphi}(p_x)$. A trick for solving the above equation is to first differentiate it with respect to p_x so that the term containing the integral of $\overline{\varphi}(p_x)$ disappears and an expression of $\overline{\varphi}(p_x)$ in terms of E and p_x can be obtained. We then substitute the obtained expression of $\overline{\varphi}(p_x)$ back into the equation so that the value of E can be determined. Differentiating the above equation with respect to p_x yields

$$\frac{p_x}{m}\overline{\varphi}(p_x) = \left(E - \frac{p_x^2}{2m}\right)\frac{d\overline{\varphi}(p_x)}{dp_x}.$$

Making a rearrangement of the above equation, we have

$$\frac{d\overline{\varphi}(p_x)}{\overline{\varphi}(p_x)} = -\frac{d(p_x^2)}{p_x^2 - 2mE}.$$

Integrating, we obtain

$$\overline{\varphi}(p_x) = \frac{A}{p_x^2 - 2mE},$$

where A is the normalization constant. From the normalization condition, we have

$$1 = \int_{-\infty}^{\infty} dp_x \big|\overline{\varphi}(p_x)\big|^2 = |A|^2 \int_{-\infty}^{\infty} dp_x\, \frac{1}{(p_x^2 - 2mE)^2}.$$

Noting that $E < 0$ for a bound state in the δ-function potential $V(x) = -\lambda\delta(x)$, we can rewrite the above equation as

$$1 = |A|^2 \int_{-\infty}^{\infty} dp_x\, \frac{1}{(p_x^2 + 2m|E|)^2}.$$

The integrand of the above integral has two second-order poles at $p_x = \pm\sqrt{2m|E|}$ in the complex plane. Closing the contour in the upper-half complex plane, we have

$$\begin{aligned} 1 &= 2\pi i|A|^2 \lim_{z\to i\sqrt{2m|E|}} \frac{d}{dz}\frac{(z - i\sqrt{2m|E|})^2}{(z^2 + 2m|E|)^2} = 2\pi i|A|^2 \lim_{z\to i\sqrt{2m|E|}} \frac{d}{dz}\frac{1}{(z + i\sqrt{2m|E|})^2} \\ &= 2\pi i|A|^2 \lim_{z\to i\sqrt{2m|E|}} \left[-\frac{2}{(z + i\sqrt{2m|E|})^3}\right] = \frac{\pi|A|^2}{2(2m|E|)^{3/2}}. \end{aligned}$$

Solving for $|A|$ from the above equation yields

$$|A| = \left(\frac{2}{\pi}\right)^{1/2} (2m|E|)^{3/4}.$$

We choose

$$A = \left(\frac{2}{\pi}\right)^{1/2} (2m|E|)^{3/4}.$$

The normalized wave function of the bound state is then given by

$$\overline{\varphi}(p_x) = \left(\frac{2}{\pi}\right)^{1/2} \frac{(2m|E|)^{3/4}}{p_x^2 - 2mE} = \left(\frac{2}{\pi}\right)^{1/2} \frac{(2m|E|)^{3/4}}{p_x^2 + 2m|E|}.$$

To find the energy of the bound state, we insert the above-obtained wave function into the stationary Schrödinger equation

$$\frac{p_x^2}{2m}\overline{\varphi}(p_x) - \frac{\lambda}{2\pi\hbar}\int_{-\infty}^{\infty} dp_x'\, \overline{\varphi}(p_x') = E\,\overline{\varphi}(p_x).$$

We first evaluate the integral in the above equation. We have

$$\begin{aligned}\int_{-\infty}^{\infty} dp_x'\, \overline{\varphi}(p_x') &= \left(\frac{2}{\pi}\right)^{1/2} (2m|E|)^{3/4} \int_{-\infty}^{\infty} dp_x' \frac{1}{p_x'^2 + 2m|E|} \\ &= 2\pi i\left(\frac{2}{\pi}\right)^{1/2} (2m|E|)^{3/4} \lim_{z\to i\sqrt{2m|E|}} \frac{z - i\sqrt{2m|E|}}{z^2 + 2m|E|} \\ &= (2\pi)^{1/2}\big(2m|E|\big)^{1/4}.\end{aligned}$$

We then have from the stationary Schrödinger equation

$$\left(\frac{p_x^2}{2m} - E\right)\left(\frac{2}{\pi}\right)^{1/2} \frac{(2m|E|)^{3/4}}{p_x^2 + 2m|E|} = \frac{\lambda}{2\pi\hbar}(2\pi)^{1/2}\big(2m|E|\big)^{1/4}$$

from which we obtain

$$E = -\frac{m\lambda^2}{2\hbar^2}.$$

4. In the $\{|\vec{p}\rangle\}$ representation, the wave function of a particle at a given time is given by $\overline{\psi}(\vec{p}) = Ne^{-\alpha|\vec{p}|/\hbar}$ with $\alpha > 0$. Find the value of the normalization constant N and the wave function $\psi(\vec{r})$ in the $\{|\vec{r}\rangle\}$ representation.

From the normalization condition, we have

$$\frac{1}{|N|^2} = \int d^3p\, e^{-2\alpha|\vec{p}|/\hbar} = \int_0^{\infty} dp\, p^2 \int_0^{\pi} d\theta\, \sin\theta \int_0^{2\pi} d\phi\, e^{-2\alpha p/\hbar} = 4\pi \cdot \frac{2!}{(2\alpha/\hbar)^3} = \pi\left(\frac{\hbar}{\alpha}\right)^3.$$

We thus have

$$|N| = \frac{1}{\pi^{1/2}}\left(\frac{\alpha}{\hbar}\right)^{3/2}.$$

We choose

$$N = \frac{1}{\pi^{1/2}}\left(\frac{\alpha}{\hbar}\right)^{3/2}.$$

The normalized wave function is then given by

$$\overline{\psi}(\vec{p}) = \frac{1}{\pi^{1/2}}\left(\frac{\alpha}{\hbar}\right)^{3/2} e^{-\alpha|\vec{p}|/\hbar}.$$

The wave function $\psi(\vec{r})$ in the $\{|\vec{r}\rangle\}$ representation is given by

$$\begin{aligned}\psi(\vec{r}) &= \frac{1}{(2\pi\hbar)^{3/2}}\int d^3p\, e^{i\vec{p}\cdot\vec{r}/\hbar}\overline{\psi}(\vec{p}) = \frac{1}{(2\pi\hbar)^{3/2}}\frac{1}{\pi^{1/2}}\left(\frac{\alpha}{\hbar}\right)^{3/2}\int d^3p\, e^{i\vec{p}\cdot\vec{r}/\hbar}e^{-\alpha|\vec{p}|/\hbar} \\ &= \frac{1}{(2\pi\hbar)^{3/2}}\frac{1}{\pi^{1/2}}\left(\frac{\alpha}{\hbar}\right)^{3/2}\int_0^{\infty} dp\, p^2 \int_0^{\pi} d\theta\, \sin\theta \int_0^{2\pi} d\phi\, e^{ipr\cos\theta/\hbar}e^{-\alpha p/\hbar}.\end{aligned}$$

Performing the integrals over the angles, we have

$$\psi(\vec{r}) = \frac{2\pi\hbar}{ir}\frac{1}{(2\pi\hbar)^{3/2}}\frac{1}{\pi^{1/2}}\left(\frac{\alpha}{\hbar}\right)^{3/2}\int_0^\infty dp\, p\left(e^{ipr/\hbar} - e^{-ipr/\hbar}\right)e^{-\alpha p/\hbar}$$
$$= \frac{\alpha^{3/2}}{2^{1/2}\pi\hbar^2 ir}\int_0^\infty dp\, p\left[e^{-(\alpha-ir)p/\hbar} - e^{-(\alpha+ir)p/\hbar}\right].$$

Performing the integral over p, we have

$$\psi(\vec{r}) = \frac{\alpha^{3/2}}{2^{1/2}\pi\hbar^2 ir}\left[\frac{\hbar^2}{(\alpha-ir)^2} - \frac{\hbar^2}{(\alpha+ir)^2}\right] = \left(\frac{8\alpha^5}{\pi^2}\right)^{1/2}\frac{1}{(\alpha^2+r^2)^2}.$$

5. For a particle in one-dimensional space, find the expression of the operator $\hat{x}^{-1} = \dfrac{1}{\hat{x}}$ in the $\{|p_x\rangle\}$ representation and the expression of the operator $\hat{p}_x^{-1} = \dfrac{1}{\hat{p}_x}$ in the $\{|x\rangle\}$ representation.
Note that $\hat{x}^{-1}$ is the inverse of $\hat{x}$ and that $\hat{p}_x^{-1}$ is the inverse of $\hat{p}_x$.

To find the inverse $\hat{A}^{-1}$ of an operator $\hat{A}$, we use

$$\hat{A}\hat{A}^{-1} = \hat{A}^{-1}\hat{A} = 1.$$

Inverse $\hat{x}^{-1}$ of $\hat{x}$ in the $\{|p_x\rangle\}$ representation. The expression of $\hat{x}$ in the $\{|p_x\rangle\}$ representation is given by

$$\hat{x} = i\hbar\frac{d}{dp_x}.$$

Let $\overline{\psi}(p_x)$ be an arbitrary wave function in the $\{|p_x\rangle\}$ representation. Acting $\hat{x}\hat{x}^{-1} = 1$ on $\overline{\psi}(p_x)$, we have

$$i\hbar\frac{d}{dp_x}\left[\hat{x}^{-1}\overline{\psi}(p_x)\right] = \overline{\psi}(p_x).$$

Multiplying both sides of the above equation with $dp_x/i\hbar$, we have

$$d\left[\hat{x}^{-1}\overline{\psi}(p_x)\right] = \frac{1}{i\hbar}\overline{\psi}(p_x)dp_x.$$

Integrating the above equation and utilizing $\lim_{p_x\to-\infty}\hat{x}^{-1}\overline{\psi}(p_x) = 0$, we have

$$\hat{x}^{-1}\overline{\psi}(p_x) = \frac{1}{i\hbar}\int_{-\infty}^{p_x} dp_x\,\overline{\psi}(p_x).$$

Since $\overline{\psi}(p_x)$ is arbitrary, we have

$$\hat{x}^{-1} = \frac{1}{i\hbar}\int_{-\infty}^{p_x} dp_x.$$

The above expression is the expression of the operator $\hat{x}^{-1} = \dfrac{1}{\hat{x}}$ in the $\{|p_x\rangle\}$ representation. With $\hat{x}^{-1}$ given in the above, the action of the product $\hat{x}^{-1}\hat{x}$ on an arbitrary wave function $\overline{\psi}(p_x)$ in the $\{|p_x\rangle\}$ representation is given by

$$\hat{x}^{-1}\hat{x}\overline{\psi}(p_x) = \frac{1}{i\hbar}\int_{-\infty}^{p_x} dp_x\, i\hbar\frac{d\overline{\psi}(p_x)}{dp_x} = \int_{-\infty}^{p_x} dp_x\,\frac{d\overline{\psi}(p_x)}{dp_x} = \overline{\psi}(p_x)\Big|_{p_x=-\infty}^{p_x} = \overline{\psi}(p_x),$$

where we have made use of $\lim_{p_x\to-\infty}\overline{\psi}(p_x) = 0$. Thus, $\hat{x}^{-1}\hat{x} = 1$ holds.

Inverse $\hat{p}_x^{-1}$ of $\hat{p}_x$ in the $\{|x\rangle\}$ representation. The expression of $\hat{p}_x$ in the $\{|x\rangle\}$ representation is given by

$$\hat{p}_x = -i\hbar\frac{d}{dx}.$$

Let $\psi(x)$ be an arbitrary wave function in the $\{|x\rangle\}$ representation. Acting $\hat{p}_x\hat{p}_x^{-1} = 1$ on $\psi(x)$, we have

$$-i\hbar\frac{d}{dx}\left[\hat{p}_x^{-1}\psi(x)\right] = \psi(x).$$

Multiplying both sides of the above equation with $-dx/i\hbar$, we have

$$d\left[\hat{p}_x^{-1}\psi(x)\right] = -\frac{1}{i\hbar}\psi(x)dx.$$

Integrating the above equation and utilizing $\lim_{x\to-\infty}\hat{p}_x^{-1}\psi(x) = 0$, we have

$$\hat{p}_x^{-1}\psi(x) = -\frac{1}{i\hbar}\int_{-\infty}^{x} dx\ \psi(x).$$

Since $\psi(x)$ is arbitrary, we have

$$\hat{p}_x^{-1} = -\frac{1}{i\hbar}\int_{-\infty}^{x} dx.$$

The above expression is the expression of the operator $\hat{p}_x^{-1} = \dfrac{1}{\hat{p}_x}$ in the $\{|x\rangle\}$ representation. With $\hat{p}_x^{-1}$ given in the above, the action of the product $\hat{p}_x^{-1}\hat{p}_x$ on an arbitrary wave function $\psi(x)$ in the $\{|x\rangle\}$ representation is given by

$$\hat{p}_x^{-1}\hat{p}_x\psi(x) = -\frac{1}{i\hbar}\int_{-\infty}^{x} dx\ (-i\hbar)\frac{d\psi(x)}{dx} = \int_{-\infty}^{x} dx\ \frac{d\psi(x)}{dx} = \psi(x)\Big|_{x=-\infty}^{x} = \psi(x),$$

where we have made use of $\lim_{x\to-\infty}\psi(x) = 0$. Thus, $\hat{p}_x^{-1}\hat{p}_x = 1$ holds.