Quantum Mechanics
Homework Assignment 15
Fall, 2019

1. **[C-T Exercise 13-1]** Consider a one-dimensional harmonic oscillator of mass m, angular frequency ω_0 and charge q. Let $|\varphi_n\rangle$ and $E_n = (n + 1/2)\hbar\omega_0$ be the eigenstates and eigenvalues of its Hamiltonian $\hat{H}_0$.
   For $t < 0$, the oscillator is in the ground state $|\varphi_0\rangle$. At $t = 0$, it is subjected to an electric field "pulse" of duration τ. The corresponding perturbation can be written $\hat{W}(t) = \begin{cases} -q\mathscr{E}\hat{x}, & 0 \le t \le \tau, \\ 0, & t < 0, t > \tau. \end{cases}$ Here $\mathscr{E}$ is the field amplitude and $\hat{x}$ is the position observable. Let $\mathscr{P}_{0n}$ be the probability of finding the oscillator in the state $|\varphi_n\rangle$ after the pulse.
   (a) Calculate $\mathscr{P}_{01}$ by using first-order time-dependent perturbation theory. How does $\mathscr{P}_{01}$ vary with τ, for fixed ω_0?
   (b) Show that, to obtain $\mathscr{P}_{02}$, the time-dependent perturbation theory calculation must be pursued at least to second order. Calculate $\mathscr{P}_{02}$ to this perturbation order.

2. **[C-T Exercise 13-2]** Consider two spin 1/2's, $\hat{\vec{S}}_1$ and $\hat{\vec{S}}_2$, coupled by an interaction of the form $a(t)\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2$; $a(t)$ is a function of time which approaches zero when $|t|$ approaches infinity, and takes on non-negligible values (on the order of a_0) only inside an interval, whose width is of the order of τ, about $t = 0$.
   (a) At $t = -\infty$, the system is in the state $|+-\rangle$ (an eigenstate of $\hat{\vec{S}}_1$ and $\hat{\vec{S}}_2$ with the eigenvalues $+\hbar/2$ and $-\hbar/2$). Calculate, without approximations, the state of the system at $t = +\infty$. Show that the probability $\mathscr{P}(+- \to -+)$ of finding, at $t = +\infty$, the system in the state $|-+\rangle$ depends only on the integral $\displaystyle\int_{-\infty}^{+\infty} dt\, a(t)$.
   (b) Calculate $\mathscr{P}(+- \to -+)$ by using first-order time-dependent perturbation theory. Discuss the validity conditions for such an approximation by comparing the results obtained with those of the preceding question.
   (c) Now assume that the two spins are also interacting with a static magnetic field $\vec{B}_0$ parallel to Oz. The corresponding Zeeman Hamiltonian can be written $\hat{H}_0 = -B_0(\gamma_1\hat{S}_{1z} + \gamma_2\hat{S}_{2z})$, where γ_1 and γ_2 are the gyromagnetic ratios of the two spins, assumed to be different.
   Assume that $a(t) = a_0 e^{-t^2/\tau^2}$. Calculate $\mathscr{P}(+- \to -+)$ by first-order time-dependent perturbation theory. With fixed a_0 and τ, discuss the variation of $\mathscr{P}(+- \to -+)$ with respect to B_0.

3. A particle of mass μ is scattered by the central field $V(r) = \dfrac{\alpha}{r^2}$ with $\alpha > 0$. Find the differential and total scattering cross sections under the first-order Born approximation.

4. **[C-T Complement $\mathbf{C_{VIII}}$-3 Exercise b]** Consider a central potential $V(r)$ such that $V(r) = \begin{cases} -V_0, & r < r_0, \\ 0, & r > r_0. \end{cases}$
   Here V_0 is a positive constant. Set $k_0 = \sqrt{2\mu V_0/\hbar^2}$ with μ the mass of the particle subject to the potential. We shall confine ourselves to the study of the s wave ($\ell = 0$).
   (a) **Bound states ($E < 0$)]**
      i. Write the radial equation in the two regions $r > r_0$ and $r < r_0$, as well as the condition at the origin. Show that, if one sets $\rho = \sqrt{-2\mu E/\hbar^2}$ and $K = \sqrt{k_0^2 - \rho^2}$, the function $u_0(r)$ is necessarily of the form $u_0(r) = \begin{cases} Ae^{-\rho r}, & r > r_0, \\ B\sin(Kr), & r < r_0. \end{cases}$
      ii. Write the matching conditions at $r = R_0$. Deduce from them that the only possible values for ρ are those which satisfy the equation $\tan(Kr_0) = -K/\rho$.
      iii. Discuss the equation $\tan(Kr_0) = -K/\rho$. Indicate the number of s bound states as a function of the depth of the well (for fixed r_0) and show, in particular, that there are no bound states if this depth is too small.
   (b) **Scattering resonances ($E > 0$)]**

i. Again write the radial equation, this time setting $k = \sqrt{2\mu E}/\hbar$ and $K' = \sqrt{k_0^2 + k^2}$. Show that $u_{k,0}(r)$ is of the form $u_{k,0}(r) = \begin{cases} A\sin(kr + \delta_0), & r > r_0, \\ B\sin(K'r), & r < r_0. \end{cases}$

ii. Choosing $A = 1$. Show, using the continuity conditions at $r = r_0$, that the constant B and the phase shift δ_0 are given by $B^2 = k^2/\left[k^2 + k_0^2\cos^2(K'r_0)\right]$ and $\delta_0 = -kr_0 + \alpha(k)$ with $\tan\alpha(k) = (k/K')\tan(K'r_0)$.

iii. Trace the curve representing B^2 as a function of k. This curve clearly shows resonances, for which B^2 is maximum. What are the values of k associated with these resonances? What is then the value of $\alpha(k)$? Show that, if there exists such a resonance for a small energy ($kr_0 \ll 1$), the corresponding contribution of the s wave to the total cross section is practically maximal.

(c) **Relation between bound states and scattering resonances**

Assume that $k_0 r_0$ is very close to $(2n+1)\pi/2$, where n is an integer, and set $k_0 r_0 = (2n+1)\pi/2 + \varepsilon$ with $|\varepsilon| \ll 1$.

i. Show that, if ε is positive, there exists a bound state whose binding energy $E = -\hbar^2\rho^2/2\mu$ is given by $\rho \simeq \varepsilon k_0$.

ii. Show that if, on the other hand, ε is negative, there exists a scattering resonance at energy $E = \hbar^2 k^2/2\mu$ such that $k^2 \simeq -2k_0\varepsilon/r_0$.

iii. Deduce from this that if the depth of the well is gradually decreased (for fixed r_0), the bound state which disappears when $k_0 r_0$ passes through an odd multiple of $\pi/2$ gives rise to a low energy scattering resonance.

5. **[C-T Exercise 14-3]** Consider the state space of an electron, spanned by the two vectors $|\varphi_{p_x}\rangle$ and $|\varphi_{p_y}\rangle$ which represent two atomic orbitals, p_x and p_y, of wave functions $\varphi_{p_x}(\vec{r})$ and $\varphi_{p_y}(\vec{r})$, $\varphi_{p_x}(\vec{r}) = xf(r) = rf(r)\sin\theta\cos\phi$, $\varphi_{p_y}(\vec{r}) = yf(r) = rf(r)\sin\theta\sin\phi$.

(a) Write, in terms of $|\varphi_{p_x}\rangle$ and $|\varphi_{p_y}\rangle$, the state $|\varphi_{p_\alpha}\rangle$ which represents the p_α orbital pointing in the direction of the xOy plane which makes an angle α with Ox.

(b) Consider two electrons whose spins are both in the $|+\rangle$ state, the eigenstate of $\hat{S}_z$ of eigenvalue $+\hbar/2$. Write the normalized state vector $|\psi\rangle$ which represents the system of two electrons, one of which is in the state $|\varphi_{p_x}\rangle$ and the other in the state $|\varphi_{p_y}\rangle$.

(c) Same question, with one of the electrons in the state $|\varphi_{p_\alpha}\rangle$ and the other one in the state $|\varphi_{p_\beta}\rangle$, where α and β are two arbitrary angles. Show that the state vector $|\psi\rangle$ obtained is the same.

(d) The system is in the state $|\psi\rangle$ of question (b). Calculate the probability density $\mathscr{P}(r,\theta,\phi;r',\theta',\phi')$ of finding one electron at (r,θ,ϕ) and the other one at (r',θ',ϕ'). Show that the electronic density $\rho(r,\theta,\phi)$ [the probability density of finding any electron at (r,θ,ϕ)] is symmetrical with respect to revolution about the Oz axis. Determine the probability density of having $\phi - \phi' = \phi_0$, where ϕ_0 is given. Discuss the variation of this probability density with respect to ϕ_0.