Quantum Mechanics
Homework Assignment 01
Fall, 2019

Nota bene: The problems in this assignment are on two pages.

1. The scattering of a photon by an energetic electron can result in a transfer of energy from the electron to the photon. When this is the case, the scattering process is known as *the inverse Compton scattering*. Assume that, before the scattering, the wavelength of the photon is λ and the speed of the electron is v that is a relativistic speed. Also assume that the momentum vectors of the electron and photon are all in the same plane before and after the scattering.

   (a) Find an expression for the wavelength shift of the photon in the inverse Compton scattering.

   (b) What is the condition under which the inverse Compton scattering occurs.

2. The average energy of a mode in Planck's quantized theory of radiation in a cavity is given by

$$\overline{E} = \frac{\sum_{n=0}^{\infty} nh\nu e^{-nh\nu/k_B T}}{\sum_{n=0}^{\infty} e^{-nh\nu/k_B T}}.$$

   (a) Calculate the value of the sum in the denominator by rewriting it as a geometric series of the form $\sum_{n=0}^{\infty} q^n$ for some value of q (to be determined) and then evaluating the sum using the standard formula for an infinite geometric series.

   (b) Prove the identity

$$\sum_{n=0}^{\infty} ne^{-nh\nu/k_B T} = -\frac{k_B T}{h}\frac{\partial}{\partial\nu}\sum_{n=0}^{\infty} e^{-nh\nu/k_B T}.$$

   Using the answer to the previous part, further prove that

$$\sum_{n=0}^{\infty} ne^{-nh\nu/k_B T} = \frac{e^{h\nu/k_B T}}{(e^{h\nu/k_B T}-1)^2}.$$

   (c) Find an explicit expression for $\overline{E}$.

   (d) Find an expression for the spectral energy density u_ν with $u_\nu d\nu$ equal to the energy per unit volume in the frequency interval $(\nu, \nu + d\nu)$, given that the number of modes per unit volume in the frequency interval $(\nu, \nu + d\nu)$ is $D(\nu)d\nu = 8\pi\nu^2 d\nu/c^3$.

   (e) The relationship between the spectral irradiance E_ν and the spectral energy density u_ν is given by $E_\nu = cu_\nu/4$. Derive Planck's law expressed in terms of E_ν.

3. A beam of neutrons of constant velocity, mass M_n ($M_n \approx 1.67 \times 10^{-27}$ kg) and energy E, is incident on a linear chain of atomic nuclei, arranged in a regular fashion as shown in the figure (these nuclei could be, for example, those of a long linear molecule). We call ℓ the distance between two consecutive nuclei, and d, their size ($d \ll \ell$). A neutron detector D is placed far away, in a direction which makes an angle of θ with the direction of the incident neutrons.
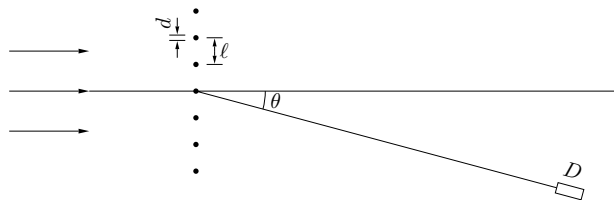
   (a) Describe qualitatively the phenomena observed at D when the energy E of the incident neutrons is varied.

(b) The counting rate, as a function of E, presents a resonance about $E = E_1$. Knowing that there are no other resonances for $E < E_1$, show that one can determine ℓ. Calculate ℓ for $\theta = 30°$ and $E_1 = 1.3 \times 10^{-20}$ J.

(c) At about what value of E must we begin to take the finite size of the nuclei into account?

4. Consider the motion of a particle of mass m in a one-dimensional harmonic potential $U(x) = m\omega^2 x^2/2$. As an approximation, the momentum p in the de Broglie relation $p = h/\lambda$ for the particle in the harmonic potential can be taken as $\sqrt{2m\overline{K}}$, where $\overline{K}$ is the average of the kinetic energy given by $\overline{K} = E/2$ with E the mechanical energy of the particle.

(a) If the mechanical energy of the particle is E, what are the two extremal points that the particle can reach according to Newtonian mechanics?

(b) From the condition that the matter wave of the particle must be fitted between the two extremal points, determine the allowed values of the energy of the particle. The numerical factor $4\sqrt{2}$ can be set approximately equal to 2π in the final result.

5. Consider two quantum states described by the following wave functions

$$\psi_1(x,0) = \left(\frac{2}{\pi a^2}\right)^{1/4} e^{-(x-a)^2/a^2},$$

$$\psi_2(x,0) = \left(\frac{2}{\pi a^2}\right)^{1/4} e^{-(x+a)^2/a^2}.$$

Let the state of a free particle in one-dimensional space be given at time $t = 0$ by

$$\psi(x,0) = A\big[\psi_1(x,0) - \psi_2(x,0)\big],$$

where A is a normalization constant.

(a) Without performing any calculation, deduce an exact value for the probability density $|\psi(0,t)|^2$ for finding the particle near $x = 0$ at an arbitrary time t.

(b) Find the value of the normalization constant A.

(c) Without solving exactly for $\psi(x,t)$, find an exact expression for the probability density $|\psi(x,t)|^2$ for finding the particle near any point x at an arbitrary time t.