Name: 陈稼霖
StudentID: 45875852

Problem 1. [C-T Exercise 3-1] In a one-dimensional problem, consider a particle whose wave function is

$$\psi(x) = N\frac{e^{ip_0x/\hbar}}{\sqrt{x^2+a^2}}$$

where a and p_0 are real constants and N is a normalization coefficient.

(a) Determine N so that $\psi(x)$ is normalized.

(b) The position of the particle is measured. What is the probability of finding a result between $-a/\sqrt{3}$ and $+a/\sqrt{3}$.

(c) Calculate the mean value of the momentum of a particle which has $\psi(x)$ for its wave function.

Solution:

(a) The normalization condition is

$$\int_{-\infty}^{+\infty} dx\psi^*(x)\psi(x) = N^2\int_{-\infty}^{+\infty}\frac{dx}{x^2+a^2} = N^2\left.\frac{\arctan\frac{x}{a}}{a}\right|_{-\infty}^{+\infty} = N^2\frac{\pi}{a} = 1 \tag{1}$$

Therefore, the normalization coefficient is

$$N = \sqrt{\frac{a}{\pi}} \tag{2}$$

so the normalized wave function is

$$\psi(x) = \sqrt{\frac{a}{\pi}}\frac{e^{ip_0x/\hbar}}{\sqrt{x^2+a^2}} \tag{3}$$

(b) The probability of finding the position of the particle between $-a/\sqrt{3}$ and $+a/\sqrt{3}$ is

$$\begin{aligned}P(-a/\sqrt{3} < x < +a/\sqrt{3}) = &\int_{-a/\sqrt{3}}^{+a/\sqrt{3}} dx\psi^*(x)\psi(x) = \frac{a}{\pi}\int_{-a/\sqrt{3}}^{+a/\sqrt{3}}\frac{dx}{x^2+a^2}\\ =&\frac{a}{\pi}\left.\frac{\arctan\frac{x}{a}}{a}\right|_{-a/\sqrt{3}}^{+a/\sqrt{3}} = \frac{2}{3}\end{aligned} \tag{4}$$

(c) The mean value of the momentum of the particle is

$$\begin{aligned}\langle p_x\rangle =& \int_{-\infty}^{+\infty} dx\psi^*(x)\hat{p}_x\psi(x) = \frac{a}{\pi}\int_{-\infty}^{+\infty} dx\frac{e^{-ip_0x/\hbar}}{\sqrt{x^2+a^2}}(-i\hbar\frac{d}{dx})\frac{e^{ip_0x/\hbar}}{\sqrt{x^2+a^2}}\\ =&-\frac{ia\hbar}{\pi}\int_{-\infty}^{+\infty} dx\left[\frac{ip_0}{\hbar(x^2+a^2)} - \frac{x}{(x^2+a^2)^2}\right]\\ =&-\frac{ia\hbar}{\pi}\left\{\frac{ip_0}{\hbar}\left.\frac{\arctan\frac{x}{a}}{a}\right|_{-\infty}^{+\infty} - 2\pi i\text{Res}\left[\frac{x}{(x^2+a^2)^2}, i|a|\right]\right\}\\ =&-\frac{ia\hbar}{\pi}\left\{\frac{ip_0}{\hbar}\frac{\pi}{a} - 2\pi i\lim_{x\to i|a|}\frac{d}{dx}\frac{x}{(x+i|a|)^2}\right\}\\ =&-\frac{ia\hbar}{\pi}\left\{\frac{ip_0}{\hbar}\frac{\pi}{a} - 2\pi i\lim_{x\to i|a|}\frac{i|a|-x}{(x+i|a|)^3}\right\} = p_0\end{aligned} \tag{5}$$

□

Problem 2. [C-T Exercise 3-12] Consider a particle of mass m submitted to the potential

$$V(x) = \begin{cases} 0, & 0 \leq x \leq a, \\ +\infty, & x < 0, x > a. \end{cases}$$

$|\varphi\rangle$'s are the eigenstates of the Hamiltonian $\hat{H}$ of the system, and their eigenvalues are $E_n = n^2\pi^2\hbar^2/2ma^2$. The state of the particle at the instant $t = 0$ is

$$|\psi(0)\rangle = a_1|\varphi_1\rangle + a_2|\varphi_2\rangle + a_3|\varphi_3\rangle + a_4|\varphi_4\rangle$$

(a) What is the probability, when the energy of the particle in the state $|\psi(0)\rangle$ is measured, of finding a value smaller than $3\pi^2\hbar^2/ma^2$?

(b) What is the mean value and what is the root-mean-square deviation of the energy of the particle in the state $|\psi(0)\rangle$?

(c) Calculate the state vector $|\psi(t)\rangle$ at the instant t. Do the results found in the previous two parts at the instant $t = 0$ remain valid at an arbitrary time t?

(d) When the energy is measured, the result $8\pi^2\hbar^2/ma^2$ is found. After the measurement, what is the state of the system? What is the result if the energy is measured again?

Solution:

(a) The normalizaed state vector is

$$|\psi(0)\rangle = N[a_1|\varphi_1\rangle + a_2|\varphi_2\rangle + a_3|\varphi_3\rangle + a_4|\varphi_4\rangle] \tag{6}$$

The normalization condition is

$$\langle\psi(0)|\psi(0)\rangle = N^2(|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2) = 1 \tag{7}$$

$$\Longrightarrow N = \frac{1}{\sqrt{|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2}} \tag{8}$$

The Hamiltonian eigenvalues are

$$E_1 = \frac{\pi^2\hbar^2}{2ma^2}, \quad E_2 = \frac{2\pi^2\hbar^2}{ma^2}, \quad E_3 = \frac{9\pi^2\hbar^2}{2ma^2}, \quad E_4 = \frac{8\pi^2\hbar^2}{ma^2} \tag{9}$$

The probability that measured energy of the particle is smaller than $3\pi^2\hbar^2/ma^2$ is

$$P(E < 3\pi^2\hbar^2/ma^2) = (\langle\psi_1|\psi(0)\rangle)^2 + (\langle\psi_2|\psi(0)\rangle)^2 = \frac{|a_1|^2 + |a_2|^2}{|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2} \tag{10}$$

(b) The mean value of the energy of the particle is

$$\begin{aligned}\langle E\rangle =&E_1(\langle\psi_1|\psi(0)\rangle)^2+E_2(\langle\psi_2|\psi(0)\rangle)^2+E_3(\langle\psi_3|\psi(0)\rangle)^2+E_4(\langle\psi_4|\psi(0)\rangle)^2\\ =&\frac{|a_1|^2+4|a_2|^2+9|a_3|^2+16|a_4|^2}{|a_1|^2+|a_2|^2+|a_3|^2+|a_4|^2}\frac{\pi^2\hbar^2}{2ma^2}\end{aligned}\tag{11}$$

The mean value of the square of the energy of the particle is

$$\begin{aligned}\langle E^2\rangle =&E_1^2(\langle\psi_1|\psi(0)\rangle)^2+E_2^2(\langle\psi_2|\psi(0)\rangle)^2+E_3^2(\langle\psi_3|\psi(0)\rangle)^2+E_4^2(\langle\psi_4|\psi(0)\rangle)^2\\ =&\frac{|a_1|^2+16|a_2|^2+81|a_3|^2+256|a_4|^2}{|a_1|^2+|a_2|^2+|a_3|^2+|a_4|^2}\left(\frac{\pi^2\hbar^2}{2ma^2}\right)^2\end{aligned}\tag{12}$$

The root-mean-square deviation of the energy of the particle is

$$\begin{aligned}\Delta E =&\sqrt{\langle E^2\rangle-\langle E\rangle^2}\\ =&\frac{\sqrt{9|a_1|^2|a_2|^2+64|a_1|^2|a_3|^2+225|a_1|^2|a_4|^2+25|a_2|^2|a_3|^2+144|a_2|^2|a_4|^2+49|a_3|^2|a_4|^2}}{|a_1|^2+|a_2|^2+|a_3|^2+|a_4|^2}\frac{\pi^2\hbar^2}{2ma^2}\end{aligned}\tag{13}$$

(c) At instant t, the state vector is

$$\begin{aligned}|\psi(t)\rangle =&\frac{a_1|\psi_1\rangle e^{-iE_1t/\hbar}+a_2|\psi_2\rangle e^{-iE_2t/\hbar}+a_3|\psi_3\rangle e^{-iE_3t/\hbar}+a_4|\psi_4\rangle e^{-iE_4t/\hbar}}{|a_1|^2+|a_2|^2+|a_3|^2+|a_4|^2}\\ =&\frac{a_1|\psi_1\rangle e^{-i\frac{\pi^2\hbar^2}{2ma^2}t/\hbar}+a_2|\psi_2\rangle e^{-i\frac{2\pi^2\hbar^2}{ma^2}t/\hbar}+a_3|\psi_3\rangle e^{-i\frac{9\pi^2\hbar^2}{2ma^2}t/\hbar}+a_4|\psi_4\rangle e^{-i\frac{8\pi^2\hbar^2}{ma^2}t/\hbar}}{|a_1|^2+|a_2|^2+|a_3|^2+|a_4|^2}\end{aligned}\tag{14}$$

Because the probabilities of finding each eigenstate remains the same as at instant $t=0$, the result found in the previous two parts remains valid at an arbitrary time t.

(d) After the measurement, the state of the system is $|\psi_4\rangle$.
The result $8\pi^2\hbar^2/ma^2$ will be found again if the energy is measured again.

□

Problem 3. [C-T Exercise 3-13] In a two-dimensional problem, consider a particle of mass m; its Hamiltonian $\hat{H}$ is written as $\hat{H}=\hat{H}_x+\hat{H}_y$ with

$$\hat{H}_x=\frac{\hat{p}_x^2}{2m}+\hat{V}(\hat{x}),\hat{H}_y=\frac{\hat{p}_y^2}{2m}+\hat{V}(y)$$

The potential energy $V(x)$ [or $V(y)$] is zero when x (or y) is included in the interval $[0,a]$ and is infinite everywhere else.

(a) Of the following sets of operators, which form a CSCO?

$$\{\hat{H}\},\{\hat{H}_x\},\{\hat{H}_x,\hat{H}_y\},\{\hat{H},\hat{H}_x\}$$

(b) Consider a particle whose wave function is

$$\psi(x,y)=N\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{2\pi y}{a}\right)$$

if $0\leq x\leq a$ and $0\leq y\leq a$, and is zero everywhere else (where N is a constant).

i. What is the mean value $\langle\hat{H}\rangle$ of the energy of the particle? If the energy $\hat{H}$ is measured, what results can be found, and with what probabilities?

ii. The observable $\hat{H}_x$ is measured; what results can be found, and with what probabilities? If this measurement yields the result $\pi^2\hbar^2/2ma^2$, what will be the results of a subsequent measurement of $\hat{H}_y$, and with what probabilities?

iii. Instead of performing the preceding measurements, one now performs a simultaneous measurement of $\hat{H}_x$ and $\hat{p}_y$. What are the probabilities of finding the following results?

$$E_x=\frac{9\pi^2\hbar^2}{2ma^2}\text{ and }p_0\leq p_y\leq p_0+dp$$

Solution:

(a) The eigenstates and the eigenvalue of $\hat{H}$, $\hat{H}_x$ and $\hat{H}_y$ are list below

表 1: The eitenstates and the eigenvalue of $\hat{H}$, $\hat{H}_x$ and $\hat{H}_y$

Eigenstates	Eigenvalues of:		
	$\hat{H}$	$\hat{H}$	$\hat{H}_y$
$\psi_{11}(x,y)=\frac{2}{a}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)$	$\frac{\pi^2\hbar^2}{ma^2}$	$\frac{\pi^2\hbar^2}{2ma^2}$	$\frac{\pi^2\hbar^2}{2ma^2}$
$\psi_{12}(x,y)=\frac{2}{a}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)$	$\frac{5\pi^2\hbar^2}{2ma^2}$	$\frac{\pi^2\hbar^2}{2ma^2}$	$\frac{2\pi^2\hbar^2}{ma^2}$
$\psi_{21}(x,y)=\frac{2}{a}\sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)$	$\frac{5\pi^2\hbar^2}{2ma^2}$	$\frac{2\pi^2\hbar^2}{ma^2}$	$\frac{\pi^2\hbar^2}{2ma^2}$
$\psi_{22}(x,y)=\frac{2}{a}\sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)$	$\frac{4\pi^2\hbar^2}{ma^2}$	$\frac{2\pi^2\hbar^2}{ma^2}$	$\frac{2\pi^2\hbar^2}{ma^2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\psi_{mn}(x,y)=\frac{2}{a}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right)$	$\frac{\pi^2\hbar^2}{2ma^2}(m^2+n^2)$	$\frac{m^2\pi^2\hbar^2}{2ma^2}$	$\frac{n^2\pi^2\hbar^2}{2ma^2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Therefore, $\{\hat{H}\}$ cannot form a CSCO;
$\{\hat{H}_x\}$ cannot form a CSCO;
$\{\hat{H}_x,\hat{H}_y\}$ can form a CSCO;
$\{\hat{H},\hat{H}_x\}$ can form a CSCO.

(b) i. The normalization condition is

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dxdy\psi^*(x,y)\psi(x,y)\\
=&N^2\int_0^a dx\cos^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{2\pi x}{a}\right)\int_0^a dy\cos^2\left(\frac{\pi y}{a}\right)\sin^2\left(\frac{2\pi y}{a}\right)\\
=&16N^2\int_0^a dx\cos^4\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{2\pi x}{a}\right)\int_0^a dy\cos^4\left(\frac{\pi y}{a}\right)\sin^2\left(\frac{\pi y}{a}\right)\\
=&16N^2\int_0^a dx\cos^4\left(\frac{\pi x}{a}\right)\left[1-\cos^2\left(\frac{2\pi x}{a}\right)\right]\int_0^a dy\cos^4\left(\frac{\pi y}{a}\right)\left[1-\cos^2\left(\frac{\pi y}{a}\right)\right]\\
=&16N^2\cdot 2\frac{a}{\pi}\left(\frac{3!!}{4!!}\frac{\pi}{2}-\frac{5!!}{6!!}\frac{\pi}{2}\right)\cdot 2\frac{a}{\pi}\left(\frac{3!!}{4!!}\frac{\pi}{2}-\frac{5!!}{6!!}\frac{\pi}{2}\right)\\
=&\frac{a^2}{16}N^2=1
\end{aligned}\tag{15}$$

$$\Longrightarrow N=\frac{4}{a}\tag{16}$$

so the normalized wave function is

$$\psi(x,y)=\frac{4}{a}\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{2\pi y}{a}\right)\tag{17}$$

The wave function can be written as

$$\begin{aligned}
\psi(x,y)=&\sum_{m=1}^{\infty}a_m\sin\left(\frac{m\pi x}{a}\right)\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{a}\right)\\
=&\frac{1}{a}\left[\sin\left(\frac{3\pi x}{a}\right)+\sin\left(\frac{\pi x}{a}\right)\right]\left[\sin\left(\frac{3\pi y}{a}\right)+\sin\left(\frac{\pi y}{a}\right)\right]\\
=&\frac{1}{2}[\psi_{11}(x,y)+\psi_{13}(x,y)+\psi_{31}(x,y)+\psi_{33}(x,y)]
\end{aligned}\tag{18}$$

The mean value of the energy of the particle is

$$\begin{aligned}
\langle\hat{H}\rangle=&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dxdy\psi^*(x,y)\hat{H}\psi(x,y)\\
=&\frac{1}{4}\frac{\pi^2\hbar^2}{2ma^2}[(1^2+1^2)+(1^2+3^2)+(3^2+1^2)+(3^2+3^2)]=\frac{5\pi^2\hbar^2}{ma^2}
\end{aligned}\tag{19}$$

Since

$$
\begin{aligned}
&P\left(E=\frac{\pi^2\hbar^2}{ma^2}\right)=\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dxdy\psi_{11}^*(x,y)\psi(x,y)\right]^2\\
=&\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dxdy\psi_{11}^*(x,y)\cdot\frac{1}{2}[\psi_{11}(x,y)+\psi_{13}(x,y)+\psi_{31}(x,y)+\psi_{33}(x,y)]\right]^2\\
=&\frac{1}{4}
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&P\left(E=\frac{5\pi^2\hbar^2}{2ma^2}\right)\\
=&\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dxdy\psi_{13}^*(x,y)\psi(x,y)\right]^2+\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dxdy\psi_{31}^*(x,y)\psi(x,y)\right]^2\\
=&\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dxdy\psi_{13}^*(x,y)\cdot\frac{1}{2}[\psi_{11}(x,y)+\psi_{13}(x,y)+\psi_{31}(x,y)+\psi_{33}(x,y)]\right]^2\\
&+\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dxdy\psi_{31}^*(x,y)\cdot\frac{1}{2}[\psi_{11}(x,y)+\psi_{13}(x,y)+\psi_{31}(x,y)+\psi_{33}(x,y)]\right]^2\\
=&\frac{1}{4}+\frac{1}{4}=\frac{1}{2}
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
&P\left(E=\frac{9\pi^2\hbar^2}{2ma^2}\right)=\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dxdy\psi_{33}^*(x,y)\psi(x,y)\right]^2\\
=&\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dxdy\psi_{11}^*(x,y)\cdot\frac{1}{2}[\psi_{33}(x,y)+\psi_{13}(x,y)+\psi_{31}(x,y)+\psi_{33}(x,y)]\right]^2\\
=&\frac{1}{4}
\end{aligned}
\tag{22}
$$

If the energy $\hat{H}$ is measured, result $\frac{\pi^2\hbar^2}{ma^2}$ can be found with probability $\frac{1}{4}$;
result $\frac{5\pi^2\hbar^2}{2ma^2}$ can be found with probability $\frac{1}{2}$;
result $\frac{9\pi^2\hbar^2}{2ma^2}$ can be found with probability $\frac{1}{2}$.

ii. Since

$$\begin{aligned}
&P\left(E_x=\frac{\pi^2\hbar^2}{2ma^2}\right)=\sum_n\left[\int_{-\infty}^{+\infty}\psi_{1n}^*(x,y)\psi(x,y)\right]^2\\
=&\left[\int_{-\infty}^{+\infty}\psi_{1,1}^*(x,y)\cdot\frac{1}{2}[\psi_{11}(x,y)+\psi_{13}(x,y)+\psi_{31}(x,y)+\psi_{33}(x,y)]\right]^2\\
&+\left[\int_{-\infty}^{+\infty}\psi_{1,3}^*(x,y)\cdot\frac{1}{2}[\psi_{11}(x,y)+\psi_{13}(x,y)+\psi_{31}(x,y)+\psi_{33}(x,y)]\right]^2\\
=&\frac{1}{4}+\frac{1}{4}=\frac{1}{2}
\end{aligned}\tag{23}$$

$$\begin{aligned}
&P\left(E_x=\frac{9\pi^2\hbar^2}{2ma^2}\right)=\sum_n\left[\int_{-\infty}^{+\infty}\psi_{3n}^*(x,y)\psi(x,y)\right]^2\\
=&\left[\int_{-\infty}^{+\infty}\psi_{3,1}^*(x,y)\cdot\frac{1}{2}[\psi_{11}(x,y)+\psi_{13}(x,y)+\psi_{31}(x,y)+\psi_{33}(x,y)]\right]^2\\
&+\left[\int_{-\infty}^{+\infty}\psi_{3,3}^*(x,y)\cdot\frac{1}{2}[\psi_{11}(x,y)+\psi_{13}(x,y)+\psi_{31}(x,y)+\psi_{33}(x,y)]\right]^2\\
=&\frac{1}{4}+\frac{1}{4}=\frac{1}{2}
\end{aligned}\tag{24}$$

If observable $\hat{H}_x$ is measured, result $\frac{\pi^2\hbar^2}{2ma^2}$ will be find with probability $\frac{1}{2}$; result $\frac{9\pi^2\hbar^2}{2ma^2}$ will be find with probability $\frac{1}{2}$.
If this measurement yields the result $\pi^2\hbar^2/2ma^2$, then the subsequent state of the partial is

$$\begin{aligned}
\psi_1(x,y)=&\frac{\sqrt{2}}{a}\sin\left(\frac{\pi x}{a}\right)\left[\sin\left(\frac{3\pi y}{a}\right)+\sin\left(\frac{\pi y}{a}\right)\right]\\
=&\frac{1}{\sqrt{2}}[\psi_{13}(x,y)+\psi_{33}(x,y)]
\end{aligned}\tag{25}$$

Then

$$\begin{aligned}
P(E_y=\frac{\pi^2\hbar^2}{2ma^2})=&\sum_m\left[\int_{-\infty}^{+\infty}dy\psi_{m1}^*(y)\psi_1(x,y)\right]^2\\
=&\left[\int_{-\infty}^{+\infty}dy\psi_{11}(y)\cdot\frac{1}{\sqrt{2}}[\psi_{13}(x,y)+\psi_{33}(x,y)]\right]^2=\frac{1}{2}
\end{aligned}\tag{26}$$

$$\begin{aligned}
P(E_y=\frac{9\pi^2\hbar^2}{2ma^2})=&\sum_m\left[\int_{-\infty}^{+\infty}dy\psi_{m3}^*(y)\psi_1(x,y)\right]^2\\
=&\left[\int_{-\infty}^{+\infty}dy\psi_{13}(y)\cdot\frac{1}{\sqrt{2}}[\psi_{13}(x,y)+\psi_{33}(x,y)]\right]^2=\frac{1}{2}
\end{aligned}\tag{27}$$

Therefore, the subsequent measurement of $\hat{H}$ will have result $\frac{\pi^2\hbar^2}{2ma^2}$ with probability $\frac{1}{2}$ and result $\frac{9\pi^2\hbar^2}{2ma^2}$ with probability $\frac{1}{2}$.

iii. The probability of finding result $E_x=\frac{9\pi^2\hbar^2}{2ma^2}$ when measuring $\hat{H}_x$ is

$$\begin{aligned}&P\left(E_x=\frac{9\pi^2\hbar^2}{2ma^2}\right)=\sum_n\left[\int_{-\infty}^{+\infty}\psi_{3n}(x,y)\right]^2\\=&\left[\int_{-\infty}^{+\infty}\psi_{31}(x,y)\cdot\frac{1}{2}[\psi_{11}(x,y)+\psi_{13}(x,y)+\psi_{31}(x,y)+\psi_{33}(x,y)]\right]^2\\&+\left[\int_{-\infty}^{+\infty}\psi_{33}(x,y)\cdot\frac{1}{2}[\psi_{11}(x,y)+\psi_{13}(x,y)+\psi_{31}(x,y)+\psi_{33}(x,y)]\right]^2\\=&\frac{1}{4}+\frac{1}{4}=\frac{1}{2}\end{aligned}\tag{28}$$

The probability of finding result $p_0\leq p_y\leq p_0+dp$ when measuring $\hat{p}_y$ is

$$P(p_0\leq p_y\leq P_0+dp)=\frac{1}{2}[\langle p_0|(|\psi_{y,1}\rangle+|\psi_{y,3}\rangle)]^2dp\tag{29}$$

where

$$\begin{aligned}&\langle p_0|\psi_{y,n}\rangle=\frac{1}{\sqrt{2\pi\hbar}}\int_0^a dye^{-ip_0y/\hbar}\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi y}{a}\right)\\=&\frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{2}{a}}\frac{\hbar}{-ip_0}\int_0^a\sin\left(\frac{n\pi y}{a}\right)de^{-ip_0y/\hbar}\\=&\frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{2}{a}}\frac{\hbar}{-ip_0}\left\{\left[e^{-ip_0y/\hbar}\sin\left(\frac{n\pi y}{a}\right)\right]\Big|_0^a-\int_0^a e^{-ip_0y/\hbar}d\sin\left(\frac{n\pi y}{a}\right)\right\}\\=&-\frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{2}{a}}\frac{\hbar}{-ip_0}\frac{n\pi}{a}\int_0^a e^{-ip_0y/\hbar}\cos\left(\frac{n\pi y}{a}\right)dy\\=&-\frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{2}{a}}\left(\frac{\hbar}{-ip_0}\right)^2\frac{n\pi}{a}\int_0^a\cos\left(\frac{n\pi y}{a}\right)de^{-ip_0y/\hbar}\\=&-\frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{2}{a}}\left(\frac{\hbar}{-ip_0}\right)^2\frac{n\pi}{a}\left\{\left[e^{-ip_0y/\hbar}\cos\left(\frac{n\pi y}{a}\right)\right]\Big|_0^a-\int_0^a e^{-ip_0y/\hbar}d\cos\left(\frac{n\pi y}{a}\right)\right\}\\=&-\frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{2}{a}}\left(\frac{\hbar}{-ip_0}\right)^2\frac{n\pi}{a}\left\{\left[e^{-ip_0a/\hbar}(-1)^n-1\right]+\frac{n\pi}{a}\int_0^a e^{-ip_0y/\hbar}\sin\left(\frac{n\pi y}{a}\right)dy\right\}\end{aligned}\tag{30}$$

$$\Longrightarrow\langle p_0|\psi_{y,n}\rangle=\frac{\sqrt{\pi a}\hbar^{3/2}n}{(p_ya)^2-(n\pi\hbar)^2}\left[1-(-1)^ne^{-ip_ya/\hbar}\right]\tag{31}$$

so the probability of finding result $p_0\leq p_y\leq p_0+dp$ when measuring $\hat{p}_y$ is

$$\begin{aligned}&P(p_0\leq p_y\leq p_0+dp)\\=&\frac{1}{2}\left\{\frac{\sqrt{\pi a}\hbar^{3/2}}{(p_0a)^2-(\pi\hbar)^2}\left[1+e^{-ip_0a/\hbar}\right]+\frac{\sqrt{\pi a}\hbar^{3/2}3}{(p_0a)^2-(3\pi\hbar)^2}\left[1+e^{-ip_0a/\hbar}\right]\right\}^2dp\\=&32\pi a\hbar^3\left\{\frac{(pa)^2-3(\pi\hbar)^2}{(p_0a)^2-(\pi\hbar)^2}\cos\left(\frac{p_0a}{2\hbar}\right)\right\}^2dp\end{aligned}\tag{32}$$

Therefore, the probability of finding the following results $E_x=\frac{9\pi^2\hbar^2}{2ma^2}$ and $p_0\leq p_y\leq p_0+dp$ is

$$\begin{aligned}P\left(E_x=\frac{9\pi^2\hbar^2}{2ma^2},p_0\leq p_y\leq P_0+dp\right)=&P\left(E_x=\frac{9\pi^2\hbar^2}{2ma^2}\right)P(p_0\leq p_y\leq P_0+dp)\\=&16\pi a\hbar^3\left\{\frac{(pa)^2-3(\pi\hbar)^2}{(p_0a)^2-(\pi\hbar)^2}\cos\left(\frac{p_0a}{2\hbar}\right)\right\}^2dp\end{aligned}\tag{33}$$

□

Problem 4. [C-T Exercise 3-14] Consider a physical system whose state space, which is three-dimensional, is spanned by the orthonormal basis formed by the three kets $|u_1\rangle$, $|u_2\rangle$, and $|u_3\rangle$. In this basis, the Hamiltonian operator $\hat{H}$ of the system and the two observables $\hat{A}$ and $\hat{B}$ are written as

$$H = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad A = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = b \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where ω_0, a, and b are positive real constants. The physical system at time $t = 0$ is in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle$$

(a) At time $t = 0$, the energy of the system is measured. What values can be found, and with what probabilities? Calculate, for the system in the state $|\psi(0)\rangle$, the mean value $\langle\hat{H}\rangle$ and the root-mean-square deviation ΔH.

(b) Instead of measuring $\hat{H}$ at time $t = 0$, one measures $\hat{A}$; what results can be found, and with what probabilities? What is the state vector immediately after the measurement?

(c) Calculate the state vector $|\psi(t)\rangle$ of the system at time t.

(d) Calculate the mean values $\langle\hat{A}\rangle(t)$ and $\langle\hat{B}\rangle(t)$ of $\hat{A}$ and $\hat{B}$ at time t. What comments can be made?

(e) What results are obtained if the observable $\hat{A}$ is measured at time t? Same question for the observable $\hat{B}$. Interpret.

Solution:

(a) The energy eigenvalue of the three kets forming the orthonormal basis are

$$\hat{H}|u_1\rangle = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \hbar\omega \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \hbar\omega|u_1\rangle \Longrightarrow E_1 = \hbar\omega \tag{34}$$

$$\hat{H}|u_1\rangle = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 2\hbar\omega \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \hbar\omega|u_1\rangle \Longrightarrow E_2 = 2\hbar\omega \tag{35}$$

$$\hat{H}|u_1\rangle = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 2\hbar\omega \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \hbar\omega|u_1\rangle \Longrightarrow E_3 = 2\hbar\omega \tag{36}$$

Since

$$P(E_1) = |\langle u_1|\psi(0)\rangle|^2 = \frac{1}{2} \tag{37}$$
$$P(E_2) = |\langle u_2|\psi(0)\rangle|^2 = \frac{1}{4} \tag{38}$$
$$P(E_3) = |\langle u_3|\psi(0)\rangle|^2 = \frac{1}{4} \tag{39}$$
$$\tag{40}$$

Value $\hbar\omega$ can be found with probability $\frac{1}{2}$;
value $2\hbar\omega$ can be found with probability $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.
The mean value of of the energy is

$$\langle\hat{H}\rangle = \langle\psi(0)|\hat{H}|\psi(0)\rangle = \hbar\omega \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \frac{3}{2}\hbar\omega \tag{41}$$

The mean value of square of the energy is

$$\begin{aligned} \langle\hat{H}^2\rangle =& \langle\psi(0)|\hat{H}^2|\psi(0)\rangle = \hbar^2\omega^2 \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \\ =& \frac{5}{2}\hbar^2\omega^2 \end{aligned} \tag{42}$$

The root-mean-square deviation of the energy is

$$\Delta H = \sqrt{\langle H^2\rangle - \langle H\rangle^2} = \frac{1}{2}\hbar\omega \tag{43}$$

(b) The characteristic equation of $\hat{A}$

$$|A - A_m I| = a \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = -a(\lambda+1)(\lambda-1)^2 = 0 \tag{44}$$

gives the eigenvalues of $\hat{A}$

$$A_1 = A_2 = a\lambda_{1,2} = a, \quad A_3 = a\lambda_3 = -a \tag{45}$$

and the corresponding eigenvectors of $\hat{A}$

$$|u_1\rangle, \quad \frac{1}{\sqrt{2}}(|u_2\rangle + |u_3\rangle), \quad \frac{1}{\sqrt{2}}(|u_2\rangle - |u_3\rangle) \tag{46}$$

The system state at time $t = 0$ can be written as

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|u_2\rangle + \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(|u_2\rangle + |u_3\rangle)\right] \tag{47}$$

Since

$$P(A_1) = |\langle u_1|\psi(0)\rangle|^2 = \frac{1}{2} \tag{48}$$

$$P(A_2) = |\frac{1}{\sqrt{2}}(\langle u_1| + \langle u_2|)\psi(0)\rangle|^2 = \frac{1}{2} \tag{49}$$

Result $A_{1,2} = a$ can be found with probability $\frac{1}{2} + \frac{1}{2} = 1$;
result $A_3 = -a$ can be found with probability $\frac{1}{2}$.
After the measurement, the state vector remains $\frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(|u_2\rangle + |u_3\rangle)\right]$

(c) The state vector of the system at time t is

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}|u_1\rangle e^{-i\omega t} + \frac{1}{2}|u_2\rangle e^{-2i\omega t} + \frac{1}{2}|u_3\rangle e^{-2i\omega t} \tag{50}$$

(c) The mean value of $\hat{A}$ at time t is

$$\begin{aligned}\langle\hat{A}\rangle =&\langle\psi(t)|\hat{A}|\psi(t)\rangle = a\begin{pmatrix}\frac{1}{\sqrt{2}}e^{i\omega t}\\ \frac{1}{2}e^{2i\omega t}\\ \frac{1}{2}e^{2i\omega t}\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}e^{-i\omega t} & \frac{1}{2}e^{-2i\omega t} & \frac{1}{2}e^{-2i\omega t}\end{pmatrix}\\ =&a\end{aligned} \tag{51}$$

The mean value of $\hat{A}$ at time t is

$$\begin{aligned}\langle\hat{B}\rangle =&\langle\psi(t)|\hat{B}|\psi(t)\rangle = b\begin{pmatrix}\frac{1}{\sqrt{2}}e^{i\omega t}\\ \frac{1}{2}e^{2i\omega t}\\ \frac{1}{2}e^{2i\omega t}\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}e^{-i\omega t} & \frac{1}{2}e^{-2i\omega t} & \frac{1}{2}e^{-2i\omega t}\end{pmatrix}\\ =&b\left(\frac{1}{2\sqrt{2}}e^{-i\omega t} + \frac{1}{2\sqrt{2}}e^{i\omega t} + \frac{1}{4}\right) = \left(\frac{1}{\sqrt{2}}\cos\omega t + \frac{1}{4}\right)b\end{aligned} \tag{52}$$

Comment: $\hat{A}$ is a constant of motion while $\hat{B}$ not.

(e) If the observable $\hat{A}$ is measured at time t, result a is obtained, because $\hat{A}$ is a constant of the motion and the probability of finding an eigenvalue of a constant of the motion is not time-independent.
The characteristic equation of $\hat{B}$

$$|B - B_n I| = b\begin{vmatrix}-\lambda & 1 & 0\\ 1 & -\lambda & 0\\ 0 & 0 & 1-\lambda\end{vmatrix} = -b(\lambda+1)(\lambda-1)^2 \tag{53}$$

gives the eigenvalue of $\hat{B}$

$$B_1 = b\lambda_1 = -b, \quad B_2 = B_3 = b\lambda_{2,3} = -b \tag{54}$$

and the corresponding eigenvectors of $\hat{B}$

$$\frac{1}{\sqrt{2}}(|u_1\rangle - |u_2\rangle), \quad \frac{1}{\sqrt{2}}(|u_1\rangle + |u_2\rangle), \quad |u_3\rangle \tag{55}$$

The state vector of the system at time t can be written as

$$\begin{aligned}|\psi(t)\rangle =& \frac{1}{2}\left(\frac{1}{\sqrt{2}}e^{-i\omega t} - \frac{1}{2}e^{-2i\omega t}\right)\left[\frac{1}{\sqrt{2}}(|u_1\rangle - |u_2\rangle)\right] \\ &+ \frac{1}{2}\left(\frac{1}{\sqrt{2}}e^{-i\omega t} + \frac{1}{2}e^{-2i\omega t}\right)\left[\frac{1}{\sqrt{2}}(|u_1\rangle + |u_2\rangle)\right] + \frac{1}{2}|u_3\rangle\end{aligned} \tag{56}$$

Since

$$P(B_1) = \left|\frac{1}{\sqrt{2}}(\langle u_1| - \langle u_2|)|\psi(t)\rangle\right|^2 = \left|\frac{1}{2}\left(\frac{1}{\sqrt{2}}e^{-i\omega t} - \frac{1}{2}e^{-2i\omega t}\right)\right|^2 = \frac{3 - 2\sqrt{2}\cos\omega t}{8} \tag{57}$$

$$P(B_2) = \left|\frac{1}{\sqrt{2}}(\langle u_1| + \langle u_2|)|\psi(t)\rangle\right|^2 = \left|\frac{1}{2}\left(\frac{1}{\sqrt{2}}e^{-i\omega t} + \frac{1}{2}e^{-2i\omega t}\right)\right|^2 = \frac{3 + 2\sqrt{2}\cos\omega t}{8} \tag{58}$$

$$P(B_3) = |\langle u_3|\psi(t)\rangle|^2 = \frac{1}{4} \tag{59}$$

Result $B_1 = -b$ is obtained with probability $\frac{3-2\sqrt{2}\cos\omega t}{8}$;
result $B_{2,3} = b$ is obtained with probability $\frac{3+2\sqrt{2}\cos\omega t}{8} + \frac{1}{4} = \frac{5+2\sqrt{2}\cos\omega t}{8}$.

□

Problem 5. [C-T Exercise 3-8] Let $\vec{j}(\vec{r})$ be the probability current density associated with a wave function $\psi(\vec{r})$ describing the state of a particle of mass m.

(a) Show that

$$m\int d^3r\vec{j}(\vec{r}) = \langle\hat{\vec{p}}\rangle$$

where $\langle\hat{\vec{p}}\rangle$ is the mean value of the momentum.

(b) Consider the operator $\hat{\vec{L}}$ (orbital angular momentum) defined by $\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$. Are the three components of $\hat{\vec{L}}$ Hermitian operators? Establish the relation

$$m\int d^3r[\vec{r} \times \vec{j}(\vec{r})] = \langle\hat{\vec{L}}\rangle$$

Solution:

(a) The probability current density is

$$\vec{j} = \frac{\hbar}{2im}[\psi^*(\vec{r})\vec{\nabla}\psi(\vec{r}) - \psi(\vec{r})\vec{\nabla}\psi^*(\vec{r})] \tag{60}$$

so

$$\begin{aligned} m\int d^3r\vec{j}(\vec{r}) =& \frac{1}{2}\int d^3r[\psi^*(\vec{r})(-i\hbar\vec{\nabla})\psi(\vec{r})+\psi(\vec{r})(-i\hbar\vec{\nabla})\psi^*(\vec{r})] \\ =& \frac{1}{2}\int d^3r[\psi^*(\vec{r})\hat{\vec{p}}\psi(\vec{r})+\psi(\vec{r})\hat{\vec{p}}\psi^*(\vec{r})] \\ =& \frac{1}{2}(\langle\hat{\vec{p}}\rangle+\langle\hat{\vec{p}}\rangle)=\langle\hat{\vec{p}}\rangle \end{aligned} \tag{61}$$

(b) The three components of $\hat{\vec{L}}$ are

$$\hat{\vec{L}}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \tag{62}$$

$$\hat{\vec{L}}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \tag{63}$$

$$\hat{\vec{L}}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \tag{64}$$

The Hermitian conjugation of the three components above are

$$\hat{\vec{L}}_x^\dagger = \hat{p}_z^\dagger\hat{y}^\dagger - \hat{p}_y^\dagger\hat{z}^\dagger = \hat{p}_z\hat{y} - \hat{p}_y\hat{z} = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = \hat{L}_x \tag{65}$$

$$\hat{\vec{L}}_y^\dagger = \hat{p}_x^\dagger\hat{z}^\dagger - \hat{p}_z^\dagger\hat{x}^\dagger = \hat{p}_x\hat{z} - \hat{p}_z\hat{x} = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z = \hat{L}_y \tag{66}$$

$$\hat{\vec{L}}_z^\dagger = \hat{p}_y^\dagger\hat{x}^\dagger - \hat{p}_x^\dagger\hat{y}^\dagger = \hat{p}_y\hat{x} - \hat{p}_x\hat{y} = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = \hat{L}_z \tag{67}$$

Therefore, the three components of $\hat{\vec{L}}$ Hermitian operator are Hermitian operators.

$$\begin{aligned} m\int d^3r[\vec{r}\times\vec{j}(\vec{r})] =& \frac{1}{2}\int d^3r[\vec{r}\times\psi^*(\vec{r})(-i\hbar\vec{\nabla})\psi(\vec{r})+\vec{r}\times\psi(\vec{r})(-i\hbar\vec{\nabla})\psi^*(\vec{r})] \\ =& \frac{1}{2}\int d^3r[\vec{r}\times\psi^*(\vec{r})\hat{\vec{p}}\psi(\vec{r})+\vec{r}\times\psi(\vec{r})\hat{\vec{p}}\psi^*(\vec{r})] \\ =& \frac{1}{2}\int d^3r[\psi^*(\vec{r})\hat{\vec{L}}\psi(\vec{r})+\psi(\vec{r})\hat{\vec{L}}\psi^*(\vec{r})] \\ =& \frac{1}{2}(\langle\hat{\vec{L}}\rangle+\langle\hat{\vec{L}}\rangle)=\langle\hat{\vec{L}}\rangle \end{aligned} \tag{68}$$

$\square$