1. **[C-T Exercise 3-1]** In a one-dimensional problem, consider a particle whose wave function is

$$\psi(x) = N \frac{e^{ip_0 x/\hbar}}{\sqrt{x^2+a^2}},$$

where $a$ and $p_0$ are real constants and $N$ is a normalization coefficient.

(a) Determine $N$ so that $\psi(x)$ is normalized.

(b) The position of the particle is measured. What is the probability of finding a result between $-a/\sqrt{3}$ and $+a/\sqrt{3}$?

(c) Calculate the mean value of the momentum of a particle which has $\psi(x)$ for its wave function.

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(a) We assume that $a > 0$. From the normalization condition, we have

$$1 = \int_{-\infty}^{\infty} dx\, |\psi(x)|^2 = |N|^2 \int_{-\infty}^{\infty} dx\, \frac{1}{x^2+a^2}.$$

Extending the integrand onto the entire complex plane. We then see that the integrand possesses two first-order poles at $z = \pm ia$. Closing the contour in the upper-half complex plane and making use of the residue theorem, we have

$$1 = |N|^2 \cdot 2\pi i \lim_{z\to ia} \frac{(z-ia)}{(z-ia)(z+ia)} = |N|^2 \frac{\pi}{a}.$$

We thus have $|N| = \sqrt{a/\pi}$. We choose $N = \sqrt{a/\pi}$. The normalized wave function is then given by

$$\psi(x) = \sqrt{\frac{a}{\pi}} \frac{e^{ip_0 x/\hbar}}{\sqrt{x^2+a^2}}.$$

(b) The probability of finding a result between $-a/\sqrt{3}$ and $+a/\sqrt{3}$ is given by

$$\mathscr{P}_{\hat{x}}(-a/\sqrt{3}, +a/\sqrt{3}) = \int_{-a/\sqrt{3}}^{+a/\sqrt{3}} dx\, |\psi(x)|^2 = \frac{a}{\pi} \int_{-a/\sqrt{3}}^{+a/\sqrt{3}} dx\, \frac{1}{x^2+a^2} = \frac{1}{\pi} \int_{-\pi/6}^{+\pi/6} d\theta = \frac{1}{3}.$$

(c) The mean value of the momentum of a particle which has $\psi(x)$ for its wave function can be evaluated as follows

$$\begin{aligned}
\langle \hat{p}_x \rangle_\psi &= \int_{-\infty}^{\infty} dx\, \psi^*(x) \hat{p}_x \psi(x) = -i\hbar \int_{-\infty}^{\infty} dx\, \psi^*(x) \frac{d\psi(x)}{dx} \\
&= \frac{a}{\pi} \int_{-\infty}^{\infty} dx \left( p_0 + \frac{i\hbar x}{x^2+a^2} \right) \frac{1}{x^2+a^2} \\
&= \frac{p_0 a}{\pi} \int_{-\infty}^{\infty} dx\, \frac{1}{x^2+a^2} = p_0.
\end{aligned}$$

Thus, the mean value of the momentum of the particle is given by $\langle \hat{p}_x \rangle_\psi = p_0$.

2. **[C-T Exercise 3-12]** Consider a particle of mass $m$ submitted to the potential

$$V(x) = \begin{cases} 0, & 0 \le x \le a, \\ +\infty, & x < 0,\ x > a. \end{cases}$$

$|\varphi_n\rangle$'s are the eigenstates of the Hamiltonian $\hat{H}$ of the system, and their eigenvalues are $E_n = n^2\pi^2\hbar^2/2ma^2$. The state of the particle at the instant $t = 0$ is

$$|\psi(0)\rangle = a_1 |\varphi_1\rangle + a_2 |\varphi_2\rangle + a_3 |\varphi_3\rangle + a_4 |\varphi_4\rangle.$$

(a) What is the probability, when the energy of the particle in the state $|\psi(0)\rangle$ is measured, of finding a value smaller than $3\pi^2\hbar^2/ma^2$?

(b) What is the mean value and what is the root-mean-square deviation of the energy of the particle in the state $|\psi(0)\rangle$?

(c) Calculate the state vector $|\psi(t)\rangle$ at the instant $t$. Do the results found in the previous two parts at the instant $t = 0$ remain valid at an arbitrary time $t$?

(d) When the energy is measured, the result $8\pi^2\hbar^2/ma^2$ is found. After the measurement, what is the state of the system? What is the result if the energy is measured again?

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(a) We first normalize the state vector $|\psi(0)\rangle$ of the particle. Let the normalized state vector $|\psi(0)\rangle$ be

$$|\psi(0)\rangle = N\big[a_1\,|\varphi_1\rangle + a_2\,|\varphi_2\rangle + a_3\,|\varphi_3\rangle + a_4\,|\varphi_4\rangle\big].$$

From the normalization $\langle\psi(0)|\psi(0)\rangle = 1$, we have

$$1 = \langle\psi(0)|\psi(0)\rangle = |N|^2\big(|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2\big).$$

We thus have

$$|N| = \frac{1}{\sqrt{|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2}}.$$

We choose $N$ to be given by

$$N = \frac{1}{\sqrt{|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2}}.$$

The normalized state vector is then given by

$$|\psi(0)\rangle = \frac{1}{\sqrt{|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2}}\big[a_1\,|\varphi_1\rangle + a_2\,|\varphi_2\rangle + a_3\,|\varphi_3\rangle + a_4\,|\varphi_4\rangle\big].$$

The first four energy eigenvalues are

$$E_1 = \frac{\pi^2\hbar^2}{2ma^2},\ E_2 = \frac{2\pi^2\hbar^2}{ma^2},\ E_3 = \frac{9\pi^2\hbar^2}{2ma^2},\ E_4 = \frac{8\pi^2\hbar^2}{ma^2}.$$

From the above first four energy eigenvalues, we see that

$$E_1, E_2 < \frac{3\pi^2\hbar^2}{ma^2}.$$

Thus, when the energy of the particle in the state $|\psi(0)\rangle$ is measured, the probability of finding a value smaller than $3\pi^2\hbar^2/ma^2$ is given by

$$\mathscr{P}_{\hat{H}}(E < 3\pi^2\hbar^2/ma^2) = |\,\langle\varphi_1|\psi(0)\rangle\,|^2 + |\,\langle\varphi_2|\psi(0)\rangle\,|^2 = \frac{|a_1|^2 + |a_2|^2}{|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2}.$$

(b) The mean value of the energy of the particle in the state $|\psi(0)\rangle$ is given by

$$\overline{E} = \frac{|a_1|^2 E_1 + |a_2|^2 E_2 + |a_3|^2 E_3 + |a_4|^2 E_4}{|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2} = \frac{|a_1|^2 + 4|a_2|^2 + 9|a_3|^2 + 16|a_4|^2}{|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2}\,\frac{\pi^2\hbar^2}{2ma^2}.$$

The mean value of the square of the energy of the particle in the state $|\psi(0)\rangle$ is given by

$$\overline{E^2} = \frac{|a_1|^2 E_1^2 + |a_2|^2 E_2^2 + |a_3|^2 E_3^2 + |a_4|^2 E_4^2}{|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2} = \frac{|a_1|^2 + 16|a_2|^2 + 81|a_3|^2 + 256|a_4|^2}{|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2}\left(\frac{\pi^2\hbar^2}{2ma^2}\right)^2.$$

The variance of the energy of the particle in the state $|\psi(0)\rangle$ is given by

$$\begin{aligned}
\mathrm{var}(E) &= \overline{E^2} - (\overline{E}\,)^2 \\
&= \left[ \frac{|a_1|^2 + 16|a_2|^2 + 81|a_3|^2 + 256|a_4|^2}{|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2} - \left( \frac{|a_1|^2 + 4|a_2|^2 + 9|a_3|^2 + 16|a_4|^2}{|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2} \right)^2 \right] \left( \frac{\pi^2\hbar^2}{2ma^2} \right)^2 \\
&= \frac{1}{(|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2)^2} \left( \frac{\pi^2\hbar^2}{2ma^2} \right)^2 \\
&\quad \times \big[ (|a_1|^2 + 16|a_2|^2 + 81|a_3|^2 + 256|a_4|^2)(|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2) \\
&\qquad - (|a_1|^2 + 4|a_2|^2 + 9|a_3|^2 + 16|a_4|^2)^2 \big] \\
&= \frac{1}{(|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2)^2} \left( \frac{\pi^2\hbar^2}{2ma^2} \right)^2 \\
&\quad \times \big[ \, |a_1|^4 + 16|a_1|^2|a_2|^2 + 81|a_1|^2|a_3|^2 + 256|a_1|^2|a_4|^2 \\
&\qquad + |a_1|^2|a_2|^2 + 16|a_2|^4 + 81|a_2|^2|a_3|^2 + 256|a_2|^2|a_4|^2 \\
&\qquad + |a_1|^2|a_3|^2 + 16|a_2|^2|a_3|^2 + 81|a_3|^4 + 256|a_3|^2|a_4|^2 \\
&\qquad + |a_1|^2|a_4|^2 + 16|a_2|^2|a_4|^2 + 81|a_3|^2|a_4|^2 + 256|a_4|^4 \\
&\qquad - |a_1|^4 - 4|a_1|^2|a_2|^2 - 9|a_1|^2|a_3|^2 - 16|a_1|^2|a_4|^2 \\
&\qquad - 4|a_1|^2|a_2|^2 - 16|a_2|^4 - 36|a_2|^2|a_3|^2 - 64|a_2|^2|a_4|^2 \\
&\qquad - 9|a_1|^2|a_3|^2 - 36|a_2|^2|a_3|^2 - 81|a_3|^4 - 144|a_3|^2|a_4|^2 \\
&\qquad - 16|a_1|^2|a_4|^2 - 64|a_2|^2|a_4|^2 - 144|a_3|^2|a_4|^2 - 256|a_4|^4 \, \big] \\
&= \frac{1}{(|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2)^2} \left( \frac{\pi^2\hbar^2}{2ma^2} \right)^2 \\
&\quad \times \big[ \, 12|a_1|^2|a_2|^2 + 72|a_1|^2|a_3|^2 + 240|a_1|^2|a_4|^2 \\
&\qquad - 3|a_1|^2|a_2|^2 + 45|a_2|^2|a_3|^2 + 192|a_2|^2|a_4|^2 \\
&\qquad - 8|a_1|^2|a_3|^2 - 20|a_2|^2|a_3|^2 + 112|a_3|^2|a_4|^2 \\
&\qquad - 15|a_1|^2|a_4|^2 - 48|a_2|^2|a_4|^2 - 63|a_3|^2|a_4|^2 \, \big] \\
&= \frac{1}{(|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2)^2} \left( \frac{\pi^2\hbar^2}{2ma^2} \right)^2 \\
&\quad \times \big[ \, 9|a_1|^2|a_2|^2 + 64|a_1|^2|a_3|^2 + 225|a_1|^2|a_4|^2 + 25|a_2|^2|a_3|^2 + 144|a_2|^2|a_4|^2 + 49|a_3|^2|a_4|^2 \, \big].
\end{aligned}$$

The root-mean-square deviation of the energy of the particle in the state $|\psi(0)\rangle$ is then given by

$$\begin{aligned}
\Delta E &= \sqrt{\mathrm{var}(E)} \\
&= \frac{1}{|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2} \frac{\pi^2\hbar^2}{2ma^2} \\
&\quad \times \big[ \, 9|a_1 a_2|^2 + 64|a_1 a_3|^2 + 255|a_1|^2|a_4|^2 + 25|a_2 a_3|^2 + 144|a_2 a_4|^2 + 49|a_3 a_4|^2 \, \big]^{1/2}.
\end{aligned}$$

(c) At the instant $t$, the state vector $|\psi(t)\rangle$ is given by

$$|\psi(t)\rangle = \frac{1}{\sqrt{|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2}} \big[ a_1 e^{-iE_1 t/\hbar} \, |\varphi_1\rangle + a_2 \, e^{-iE_2 t/\hbar} \, |\varphi_2\rangle + a_3 e^{-iE_3 t/\hbar} \, |\varphi_3\rangle + a_4 e^{-iE_4 t/\hbar} \, |\varphi_4\rangle \big].$$

Since the probabilities of finding the particle in the energy eigenstates are independent of time $t$, the results found in the previous two parts at the instant $t = 0$ remain valid at an arbitrary time $t$.

(d) Since $8\pi^2\hbar^2/ma^2$ is the energy eigenvalue corresponding to the energy eigenstate $|\varphi_4\rangle$, the state of the system after the measurement of the energy with the result $8\pi^2\hbar^2/ma^2$ found is $|\varphi_4\rangle$. Since the system is in the stationary state $|\varphi_4\rangle$ after the measurement of the energy with the reult $8\pi^2\hbar^2/ma^2$ found, the same result $8\pi^2\hbar^2/ma^2$ will be found if the energy is measured again.

3. **[C-T Exercise 3-13]** In a two-dimensional problem, consider a particle of mass $m$; its Hamiltonian $\hat{H}$ is written as $\hat{H} = \hat{H}_x + \hat{H}_y$ with

$$\hat{H}_x = \frac{\hat{p}_x^2}{2m} + \hat{V}(\hat{x}),\ \hat{H}_y = \frac{\hat{p}_y^2}{2m} + \hat{V}(\hat{y}).$$

The potential energy $V(x)$ [or $V(y)$] is zero when $x$ (or $y$) is included in the interval $[0, a]$ and is infinite everywhere else.

(a) Of the following sets of operators, which form a CSCO?

$$\{\hat{H}\},\ \{\hat{H}_x\},\ \{\hat{H}_x, \hat{H}_y\},\ \{\hat{H}, \hat{H}_x\}.$$

(b) Consider a particle whose wave function is

$$\psi(x, y) = N \cos\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi y}{a}\right) \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{2\pi y}{a}\right)$$

if $0 \le x \le a$ and $0 \le y \le a$, and is zero everywhere else (where $N$ is a constant).

i. What is the mean value $\langle \hat{H} \rangle$ of the energy of the particle? If the energy $\hat{H}$ is measured, what results can be found, and with what probabilities?

ii. The observable $\hat{H}_x$ is measured; what results can be found, and with what probabilities? If this measurement yields the result $\pi^2\hbar^2/2ma^2$, what will be the results of a subsequent measurement of $\hat{H}_y$, and with what probabilities?

iii. Instead of performing the preceding measurements, one now performs a simultaneous measurement of $\hat{H}_x$ and $\hat{p}_y$. What are the probabilities of finding the following results?

$$E_x = \frac{9\pi^2\hbar^2}{2ma^2} \text{ and } p_0 \le p_y \le p_0 + dp.$$

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(a) From the eigenvalues and eigenfunctions of the Hamiltonian of a particle in an infinite-depth potential well, the eigenvalues and eigenfunctions of $\hat{H}_x$ are given by

$$E_{x,n} = \frac{n^2\pi^2\hbar^2}{2ma^2},\ \varphi_{x,n}(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right),\ n = 1, 2, 3, \cdots.$$

Similarly, the eigenvalues and eigenfunctions of $\hat{H}_x$ are given by

$$E_{y,p} = \frac{p^2\pi^2\hbar^2}{2ma^2},\ \varphi_{y,p}(y) = \sqrt{\frac{2}{a}} \sin\left(\frac{p\pi y}{a}\right),\ p = 1, 2, 3, \cdots.$$

Since $[\hat{H}_x, \hat{H}_y] = 0$, $\hat{H}_x$ and $\hat{H}_y$ have common eigenfunctions, with the products $\varphi_{x,n}(x)\varphi_{y,p}(y)$,

$$\phi_{np}(x, y) = \varphi_{x,n}(x)\varphi_{y,p}(y) = \frac{2}{a} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{p\pi y}{a}\right),\ n, p = 1, 2, 3, \cdots.$$

being their common eigenfunctions,

$$\begin{aligned}\hat{H}_x\phi_{np}(x, y) &= E_{x,n}\phi_{np}(x, y),\\ \hat{H}_y\phi_{np}(x, y) &= E_{y,n}\phi_{np}(x, y).\end{aligned}$$

Acting $\hat{H} = \hat{H}_x + \hat{H}_y$ on $\phi_{np}(x, y)$, we have

$$\begin{aligned}\hat{H}\phi_{np}(x, y) &= (\hat{H}_x + \hat{H}_y)\varphi_{x,n}(x)\varphi_{y,p}(y) = \big[\hat{H}_x\varphi_{x,n}(x)\big]\varphi_{y,p}(y) + \varphi_{x,n}(x)\big[\hat{H}_x\varphi_{y,p}(y)\big]\\ &= \big[E_{x,n}\varphi_{x,n}(x)\big]\varphi_{y,p}(y) + \varphi_{x,n}(x)\big[E_{y,p}\varphi_{y,p}(y)\big] = \big[E_{x,n} + E_{y,p}\big]\varphi_{x,n}(x)\varphi_{y,p}(y)\\ &= \big[E_{x,n} + E_{y,p}\big]\phi_{np}(x, y),\ n, p = 1, 2, 3, \cdots.\end{aligned}$$

Thus, the eigenvalues and eigenfunctions of $\hat{H} = \hat{H}_x + \hat{H}_y$ are given by

$$\begin{aligned}
&\mathscr{E}_{np} = E_{x,n} + E_{y,p} = \frac{(n^2+p^2)\pi^2\hbar^2}{2ma^2}, \\
&\phi_{np}(x,y) = \varphi_{x,n}(x)\varphi_{y,p}(y) = \frac{2}{a}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{p\pi y}{a}\right), \\
&n, p = 1, 2, 3, \cdots .
\end{aligned}$$

Note that there exists a two-fold accidental degeneracy $\mathscr{E}_{np} = \mathscr{E}_{pn}$ for $n \neq p$.

**Set** $\{\hat{H}\}$. Because of the presence of the two-fold accidental degeneracy $\mathscr{E}_{np} = \mathscr{E}_{pn}$ for $n \neq p$, the set $\{\hat{H}\}$ is not a CSCO.

**Set** $\{\hat{H}_x\}$. Because $\phi_{np}(x,y)$ can not be completely determined for a given value of $E_{x,n}$, the set $\{\hat{H}_x\}$ is not a CSCO.

**Set** $\{\hat{H}_x, \hat{H}_y\}$. Since $\phi_{np}(x,y)$ is completely completely determined for a given pair of values of $E_{x,n}$ and $E_{y,p}$, the set $\{\hat{H}_x, \hat{H}_y\}$ is a CSCO.

**Set** $\{\hat{H}, \hat{H}_x\}$. Given $\mathscr{E}_{np}$ and $E_{x,n}$, we have $E_{y,p} = \mathscr{E}_{np} - E_{x,n}$. Thus, $\phi_{np}(x,y)$ is completely determined for the given values of $\mathscr{E}_{np}$ and $E_{x,n}$. Therefore, the set $\{\hat{H}, \hat{H}_x\}$ is a CSCO.

(b) For the convenience of performing the calculations in the following, we first normalize the given wave function. From the normalization condition, we have

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\ |\psi(x,y)|^2 \\
&= |N|^2 \int_0^a dx\ \cos^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{2\pi x}{a}\right)\int_0^a dy\ \cos^2\left(\frac{\pi y}{a}\right)\sin^2\left(\frac{2\pi y}{a}\right) \\
&= \frac{1}{16}a^2|N|^2.
\end{aligned}$$

We thus have $|N| = 4/a$. We choose $N = 4/a$. The normalized wave function is then given by

$$\psi(x,y) = \frac{4}{a}\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{2\pi y}{a}\right)$$

We now expand $\psi(x,y)$ in $\{\phi_{np}(x,y)\}$. We have

$$\begin{aligned}
\psi(x,y) &= \frac{4}{a}\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{2\pi y}{a}\right) \\
&= \frac{4}{a}\left[\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\right]\left[\cos\left(\frac{\pi y}{a}\right)\sin\left(\frac{2\pi y}{a}\right)\right] \\
&= \frac{1}{a}\left[\sin\left(\frac{3\pi x}{a}\right) + \sin\left(\frac{\pi x}{a}\right)\right]\left[\sin\left(\frac{3\pi y}{a}\right) + \sin\left(\frac{\pi y}{a}\right)\right] \\
&= \frac{1}{2}\left[\phi_{11}(x,y) + \phi_{13}(x,y) + \phi_{31}(x,y) + \phi_{33}(x,y)\right].
\end{aligned}$$

i. The mean value $\langle\hat{H}\rangle$ of the energy of the particle is given by

$$\begin{aligned}
\langle\hat{H}\rangle &= \frac{1}{4}\left[E_{11} + E_{13} + E_{31} + E_{33}\right] \\
&= \frac{1}{4}\left[(1^2+1^2) + (1^2+3^2) + (3^2+1^2) + (3^2+3^2)\right]\frac{\pi^2\hbar^2}{2ma^2} \\
&= \frac{5\pi^2\hbar^2}{ma^2}
\end{aligned}$$

If the energy $\hat{H}$ is measured, the following three results can be found

$$\mathscr{E}_{11} = \frac{\pi^2\hbar^2}{ma^2}, \mathscr{E}_{13} = \mathscr{E}_{31} = \frac{5\pi^2\hbar^2}{ma^2}, \mathscr{E}_{33} = \frac{9\pi^2\hbar^2}{ma^2}.$$

The probabilities of finding the above three results are respectively given by

$$\mathscr{P}_{\hat{H}}\left(\frac{\pi^2\hbar^2}{ma^2}\right) = \left|\langle\phi_{11}|\psi\rangle\right|^2 = \frac{1}{4},$$
$$\mathscr{P}_{\hat{H}}\left(\frac{5\pi^2\hbar^2}{ma^2}\right) = \left|\langle\phi_{13}|\psi\rangle\right|^2 + \left|\langle\phi_{31}|\psi\rangle\right|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$
$$\mathscr{P}_{\hat{H}}\left(\frac{9\pi^2\hbar^2}{ma^2}\right) = \left|\langle\phi_{33}|\psi\rangle\right|^2 = \frac{1}{4}.$$

ii. From

$$\psi(x,y) = \frac{1}{2}\left[\,\phi_{11}(x,y) + \phi_{13}(x,y) + \phi_{31}(x,y) + \phi_{33}(x,y)\,\right],$$

we see that, if the observable $\hat{H}_x$ is measured, the following two results can be found

$$E_{x,1} = \frac{\pi^2\hbar^2}{2ma^2},\; E_{x,3} = \frac{9\pi^2\hbar^2}{2ma^2}.$$

The probabilities of finding the above two results are respectively given by

$$\mathscr{P}_{\hat{H}_x}\left(\frac{\pi^2\hbar^2}{2ma^2}\right) = \sum_p \left|\langle\phi_{1p}|\psi\rangle\right|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$
$$\mathscr{P}_{\hat{H}_x}\left(\frac{9\pi^2\hbar^2}{2ma^2}\right) = \sum_p \left|\langle\phi_{3p}|\psi\rangle\right|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

If this measurement yields the result $\pi^2\hbar^2/2ma^2$, the state of the particle immediately after the measurement is described by the following wave function by

$$\psi'(x,y) = \frac{1}{\sqrt{2}}\left[\,\phi_{11}(x,y) + \phi_{13}(x,y)\,\right].$$

In a subsequent measurement of $\hat{H}_y$, the following two results will be found

$$E_{y,1} = \frac{\pi^2\hbar^2}{2ma^2},\; E_{y,3} = \frac{9\pi^2\hbar^2}{2ma^2}.$$

The probabilities of finding the above two results are respectively given by

$$\mathscr{P}_{\hat{H}_y}\left(\frac{\pi^2\hbar^2}{2ma^2}\right) = \sum_n \left|\langle\phi_{n1}|\psi\rangle\right|^2 = \frac{1}{2},$$
$$\mathscr{P}_{\hat{H}_y}\left(\frac{9\pi^2\hbar^2}{2ma^2}\right) = \sum_n \left|\langle\phi_{n3}|\psi\rangle\right|^2 = \frac{1}{2}.$$

iii. In a simultaneous measurement of $\hat{H}_x$ and $\hat{p}_y$, the probability of finding the results

$$E_x = E_{x,3} = \frac{9\pi^2\hbar^2}{2ma^2} \text{ and } p_0 \le p_y \le p_0 + dp$$

is given by

$$\mathscr{R}_{\hat{H}_x,\hat{p}_y}\big(E_{x,3}, p_0\big)\,dp = \frac{1}{4}\left|\,{}_y\langle p_0|\varphi_{y,1}\rangle + {}_y\langle p_0|\varphi_{y,3}\rangle\right|^2 dp.$$

Making use of

$$\langle p_y|\varphi_{y,n}\rangle = -\sqrt{\frac{\pi a}{\hbar}}\,\frac{n\left[1-(-1)^n e^{-ip_y a/\hbar}\right]}{(p_y a/\hbar)^2 - n^2\pi^2},\; n = 1, 2, \cdots, \infty,$$

we have

$$\mathscr{R}_{\hat{H}_x,\hat{p}_y}(E_{x,3},p_0)\,dp = \frac{\pi a}{4\hbar}\left|\frac{\left[1+e^{-ip_0a/\hbar}\right]}{(p_0a/\hbar)^2-\pi^2}+\frac{3\left[1+e^{-ip_0a/\hbar}\right]}{(p_0a/\hbar)^2-9\pi^2}\right|^2 dp$$
$$= \frac{16\pi a}{\hbar}\left\{\frac{(p_0a/\hbar)^2-3\pi^2}{[(p_0a/\hbar)^2-\pi^2][(p_0a/\hbar)^2-9\pi^2]}\cos\left(\frac{p_0a}{2\hbar}\right)\right\}^2 dp.$$

4. **[C-T Exercise 3-14]** Consider a physical system whose state space, which is three-dimensional, is spanned by the orthonormal basis formed by the three kets $|u_1\rangle$, $|u_2\rangle$, and $|u_3\rangle$. In this basis, the Hamiltonian operator $\hat{H}$ of the system and the two observables $\hat{A}$ and $\hat{B}$ are written as

$$H = \hbar\omega_0\begin{pmatrix}1&0&0\\0&2&0\\0&0&2\end{pmatrix},\quad A = a\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},\quad B = b\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},$$

where $\omega_0$, $a$, and $b$ are positive real constants. The physical system at time $t=0$ is in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\,|u_1\rangle + \frac{1}{2}\,|u_2\rangle + \frac{1}{2}\,|u_3\rangle\,.$$

(a) At time $t=0$, the energy of the system is measured. What values can be found, and with what probabilities? Calculate, for the system in the state $|\psi(0)\rangle$, the mean value $\langle\hat{H}\rangle$ and the root-mean-square deviation $\Delta H$.

(b) Instead of measuring $\hat{H}$ at time $t=0$, one measures $\hat{A}$; what results can be found, and with what probabilities? What is the state vector immediately after the measurement?

(c) Calculate the state vector $|\psi(t)\rangle$ of the system at time $t$.

(d) Calculate the mean values $\langle\hat{A}\rangle(t)$ and $\langle\hat{B}\rangle(t)$ of $\hat{A}$ and $\hat{B}$ at time $t$. What comments can be made?

(e) What results are obtained if the observable $\hat{A}$ is measured at time $t$? Same question for the observable $\hat{B}$. Interpret.

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(a) Since $H$ is a diagonal matrix in the $\{|u_1\rangle,|u_2\rangle,|u_3\rangle\}$ basis, $|u_1\rangle$, $|u_2\rangle$, and $|u_3\rangle$ are eigenvectors of $H$. From the representation matrix of $H$, we see that the energy eigenvalues are

$$E_1 = \hbar\omega_0,\ E_2 = E_3 = 2\hbar\omega_0.$$

Here, the energy eigenvalue $\hbar\omega_0$ is non-degenerate while the energy eigenvalue $2\hbar\omega_0$ is two-fold degenerate. If the energy is measured at $t=0$, the values can be found are

$$\hbar\omega_0,\, 2\hbar\omega_0.$$

The probabilities of obtaining these values are given by

$$\mathscr{P}_{\hat{H}}(\hbar\omega_0) = \left|\langle u_1|\psi(0)\rangle\right|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2},$$
$$\mathscr{P}_{\hat{H}}(2\hbar\omega_0) = \left|\langle u_2|\psi(0)\rangle\right|^2 + \left|\langle u_3|\psi(0)\rangle\right|^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2},$$

where we have made use of the fact that $|\psi(0)\rangle$ is normalized.

The mean value $\langle\hat{H}\rangle$ is given by

$$\langle\hat{H}\rangle = \mathscr{P}_{\hat{H}}(\hbar\omega_0)\cdot\hbar\omega_0 + \mathscr{P}_{\hat{H}}(2\hbar\omega_0)\cdot 2\hbar\omega_0 = \frac{1}{2}\cdot\hbar\omega_0 + \frac{1}{2}\cdot 2\hbar\omega_0 = \frac{3}{2}\hbar\omega_0.$$

The mean value of $\hat{H}^2$, $\langle\hat{H}^2\rangle$, is given by

$$\langle\hat{H}^2\rangle = \mathscr{P}_{\hat{H}}(\hbar\omega_0)\cdot(\hbar\omega_0)^2 + \mathscr{P}_{\hat{H}}(2\hbar\omega_0)\cdot(2\hbar\omega_0)^2 = \frac{1}{2}\cdot(\hbar\omega_0)^2 + \frac{1}{2}\cdot(2\hbar\omega_0)^2 = \frac{5}{2}(\hbar\omega_0)^2.$$

The root-mean-square deviation $\Delta H$ is given by

$$\Delta H = \sqrt{\langle\hat{H}^2\rangle - \langle\hat{H}\rangle^2} = \sqrt{\frac{5}{2}(\hbar\omega_0)^2 - \left(\frac{3}{2}\hbar\omega_0\right)^2} = \frac{1}{2}\hbar\omega_0.$$

(b) We now find the eigenvalues and eigenvectors of $A$. We first note that $A$ is a block matrix of the form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where $A_1$ is a $1 \times 1$ matrix given by

$$A_1 = a(1),$$

and $A_2$ is a $2 \times 2$ matrix given by

$$A_2 = a \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that $A_1$ is the representation matrix of $A$ in the subspace spanned by $|u_1\rangle$ and that $A_2$ is the representation matrix of $A$ in the subspace spanned by $|u_2\rangle$ and $|u_3\rangle$.
For $A_1$, we see that its sole eigenvalue is $a$ and its sole eigenvector is given by $|u_1\rangle$. Thus, $a$ is an eigenvalue of $A$ with the corresponding eigenvector given by $|u_1\rangle$. We now diagonalize $A_2$. Let $\lambda a$ denote the eigenvalue of $A_2$ and $|\xi\rangle = \beta\,|u_2\rangle + \gamma\,|u_3\rangle$ the corresponding eigenvector of $A_2$. The eigenequation of $A_2$ in the $\{|u_2\rangle, |u_3\rangle\}$ basis reads

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \end{pmatrix} = \lambda \begin{pmatrix} \beta \\ \gamma \end{pmatrix}.$$

That is,

$$\begin{aligned} &-\lambda\beta + \gamma = 0, \\ &\beta - \lambda\gamma = 0 \end{aligned}$$

from which the secular equation follows

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0.$$

Evaluating the determinant on the left hand side, we have

$$\lambda^2 - 1 = 0.$$

The solutions of the above equation are $\lambda_{1,2} = \pm 1$. Thus, the eigenvalues of $A_2$ are $\lambda_{1,2} a = \pm a$. These are also the eigenvalues of $A$. We thus see that the eigenvalue $a$ of $A$ is two-fold degenerate. To find the eigenvector of $A_2$ corresponding to the eigenvalue $a$, we insert $\lambda_1 = 1$ into the equations for $\beta$ and $\gamma$. We have

$$\begin{aligned} &-\beta + \gamma = 0, \\ &\beta - \gamma = 0 \end{aligned}$$

from which we have $\gamma = \beta$. From the normalization condition $\langle \xi | \xi \rangle = 1$, we have $|\beta|^2 + |\gamma|^2 = 2|\beta|^2 = 1$. We thus have $|\beta| = 1/\sqrt{2}$. We choose $\beta = 1/\sqrt{2}$. We then have $\gamma = \beta = 1/\sqrt{2}$. The eigenvector of $A_2$ corresponding to the eigenvalue $a$ is hence given by

$$\frac{1}{\sqrt{2}} \big[\, |u_2\rangle + |u_3\rangle \,\big].$$

To find the eigenvector of $A_2$ corresponding to the eigenvalue $-a$, we insert $\lambda_2 = -1$ into the equations for $\beta$ and $\gamma$. We have $\beta + \gamma = 0$ from which we have $\gamma = -\beta$. From the normalization condition $\langle \xi | \xi \rangle = 1$, we have $|\beta|^2 + |\gamma|^2 = 2|\beta|^2 = 1$. We thus have $|\beta| = 1/\sqrt{2}$. We choose $\beta = 1/\sqrt{2}$. We then have $\gamma = -\beta = -1/\sqrt{2}$. The eigenvector of $A_2$ corresponding to the eigenvalue $-a$ is hence given by

$$\frac{1}{\sqrt{2}} \big[\, |u_2\rangle - |u_3\rangle \,\big].$$

Note that the above-found eigenvalues and eigenvectors of $A_2$ are also the eigenvalues and eigenvectors of $A$. In summary, we have have found the following eigenvalues and eigenvectors of $A$

| Eigenvalue | Degree of degeneracy | Eigenvector |
| --- | --- | --- |
| $a_1 = a$ | 2 | $\lvert\xi_1^{(1)}\rangle = \lvert u_1\rangle$ |
| | | $\lvert\xi_1^{(2)}\rangle = \frac{1}{\sqrt{2}}\big[\,\lvert u_2\rangle + \lvert u_3\rangle\,\big]$ |
| $a_2 = -a$ | 1 | $\lvert\xi_2^{(1)}\rangle = \frac{1}{\sqrt{2}}\big[\,\lvert u_2\rangle - \lvert u_3\rangle\,\big]$ |

For the convenience of discussing the measurements of $\hat{A}$, we reexpress $|\psi(0)\rangle$ in terms of the eigenvectors of $\hat{A}$. From the expression of $|\psi(0)\rangle$ in terms of $|u_1\rangle$, $|u_2\rangle$, and $|u_3\rangle$, we see that $|\psi(0)\rangle$ can be expressed in terms of $|\xi_1^{(1)}\rangle$ and $|\xi_1^{(2)}\rangle$ as follows

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\,|\xi_1^{(1)}\rangle + \frac{1}{\sqrt{2}}\,|\xi_1^{(2)}\rangle\,.$$

From the above expression of $|\psi(0)\rangle$ in terms of $|\xi_1^{(1)}\rangle$ and $|\xi_1^{(2)}\rangle$, both of which correspond to the eigenvalue $a$, we see that, if $\hat{A}$ is measured at $t = 0$, the result $a$ will be found with probability 1. The state vector immediately after the measurement is given by

$$|\psi'\rangle = |\psi(0)\rangle = \frac{1}{\sqrt{2}}\,|\xi_1^{(1)}\rangle + \frac{1}{\sqrt{2}}\,|\xi_1^{(2)}\rangle\,.$$

(c) The state vector $|\psi(t)\rangle$ of the system at time $t$ is given by

$$\begin{aligned}|\psi(t)\rangle &= \frac{1}{\sqrt{2}}e^{-i\omega_0 t}\,|u_1\rangle + \frac{1}{2}e^{-2i\omega_0 t}\,|u_2\rangle + \frac{1}{2}e^{-2i\omega_0 t}\,|u_3\rangle \\ &= \frac{1}{\sqrt{2}}e^{-i\omega_0 t}\,|u_1\rangle + \frac{1}{2}e^{-2i\omega_0 t}\big[\,|u_2\rangle + |u_3\rangle\,\big].\end{aligned}$$

(d) The mean value $\langle\hat{A}\rangle(t)$ is given by

$$\langle\hat{A}\rangle(t) = \langle\psi(t)|\hat{A}|\psi(t)\rangle = \frac{a}{4}\begin{pmatrix}\sqrt{2}e^{i\omega_0 t} & e^{2i\omega_0 t} & e^{2i\omega_0 t}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}\sqrt{2}e^{-i\omega_0 t}\\ e^{-2i\omega_0 t}\\ e^{-2i\omega_0 t}\end{pmatrix}$$

$$= \frac{a}{4}\begin{pmatrix}\sqrt{2}e^{i\omega_0 t} & e^{2i\omega_0 t} & e^{2i\omega_0 t}\end{pmatrix}\begin{pmatrix}\sqrt{2}e^{-i\omega_0 t}\\ e^{-2i\omega_0 t}\\ e^{-2i\omega_0 t}\end{pmatrix} = \frac{a}{4}(2+1+1) = a.$$

The explanation for the time independence of $\langle\hat{A}\rangle(t)$ is that $\hat{A}$ is a constant of motion so that its mean value does not depend on time. That $\hat{A}$ is a constant of motion can be seen from the facts that $\hat{A}$ does not depend explicitly on time and that $\hat{A}$ commutes with $\hat{H}$. Using the representation matrices of $\hat{H}$ and $\hat{A}$ in the $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ basis, we have

$$\begin{aligned}[H, A] &= \hbar\omega_0 a\begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} - \hbar\omega_0 a\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix}\\ &= \hbar\omega_0 a\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 2\\ 0 & 2 & 0\end{pmatrix} - \hbar\omega_0 a\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 2\\ 0 & 2 & 0\end{pmatrix} = \hbar\omega_0 a\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = 0.\end{aligned}$$

Since the eigenvalue $2\hbar\omega_0$ of $\hat{H}$ is degenerate in the subspace spanned by $|u_2\rangle$ and $|u_3\rangle$, the two eigenvectors corresponding to the eigenvalue $2\hbar\omega_0$ of $\hat{H}$ can be chosen as

$$\frac{1}{\sqrt{2}}\big[\,|u_2\rangle \pm |u_3\rangle\,\big].$$

Then, the common eigenvectors of $\hat{H}$ and $\hat{A}$ are given by

$$|\varphi_{11}\rangle = |u_1\rangle,$$
$$|\varphi_{21}\rangle = \frac{1}{\sqrt{2}}\big[\,|u_2\rangle + |u_3\rangle\,\big],$$
$$|\varphi_{22}\rangle = \frac{1}{\sqrt{2}}\big[\,|u_2\rangle - |u_3\rangle\,\big],$$

where the first subscript on $|\varphi_{np}\rangle$ is for the energy eigenvalue of $\hat{H}$ while the second subscript is for the energy eigenvalue of $\hat{A}$. The eigenequations of $\hat{H}$ and $\hat{A}$ read

$$\hat{H}\,|\varphi_{np}\rangle = E_n\,|\varphi_{np}\rangle,$$
$$\hat{A}\,|\varphi_{np}\rangle = a_p\,|\varphi_{np}\rangle,$$
$$np = 11, 21, 22,$$

where

$$E_1 = \hbar\omega_0,\ E_2 = 2\hbar\omega_0,$$
$$a_1 = 1,\ a_2 = -a.$$

Note that $np \neq 12$.
The mean value $\langle \hat{B}\rangle(t)$ is given by

$$\begin{aligned}
\langle \hat{B}\rangle(t) = \langle\psi(t)|\hat{B}|\psi(t)\rangle &= \frac{b}{4}\begin{pmatrix}\sqrt{2}e^{i\omega_0 t} & e^{2i\omega_0 t} & e^{2i\omega_0 t}\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}\sqrt{2}e^{-i\omega_0 t}\\ e^{-2i\omega_0 t}\\ e^{-2i\omega_0 t}\end{pmatrix}\\
&= \frac{b}{4}\begin{pmatrix}e^{2i\omega_0 t} & \sqrt{2}e^{i\omega_0 t} & e^{2i\omega_0 t}\end{pmatrix}\begin{pmatrix}\sqrt{2}e^{-i\omega_0 t}\\ e^{-2i\omega_0 t}\\ e^{-2i\omega_0 t}\end{pmatrix} = \frac{b}{4}\big(\sqrt{2}e^{i\omega_0 t} + \sqrt{2}e^{-i\omega_0 t} + 1\big)\\
&= \frac{b}{4}\big[\,2\sqrt{2}\cos(\omega_0 t) + 1\,\big].
\end{aligned}$$

The time dependence of $\langle \hat{B}\rangle(t)$ is due to the fact that $\hat{B}$ is not a constant of motion. The commutation relation between $\hat{H}$ and $\hat{B}$ can be evaluated through utilizing the representation matrices of $\hat{H}$ and $\hat{B}$ in the $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ basis. We have

$$\begin{aligned}
[H, B] &= \hbar\omega_0 b\begin{pmatrix}1&0&0\\0&2&0\\0&0&2\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix} - \hbar\omega_0 b\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&2&0\\0&0&2\end{pmatrix}\\
&= \hbar\omega_0 b\begin{pmatrix}0&1&0\\2&0&0\\0&0&2\end{pmatrix} - \hbar\omega_0 b\begin{pmatrix}0&2&0\\1&0&0\\0&0&2\end{pmatrix} = \hbar\omega_0 b\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix} \neq 0.
\end{aligned}$$

Since $[H, B] \neq 0$, $\hat{B}$ is not a constant of motion.

(e) In terms of the common eigenvectors of $\hat{H}$ and $\hat{A}$, $|\psi(t)\rangle$ can be written as

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-i\omega_0 t}\,|\varphi_{11}\rangle + \frac{1}{\sqrt{2}}e^{-2i\omega_0 t}\,|\varphi_{21}\rangle\,.$$

If the observable $\hat{A}$ is measured at time $t$, we always obtain the value of $a$ since $|\psi(t)\rangle$ is in the subspace of the eigenvalue $a$ of $\hat{A}$.
To find the results of the measurement of $\hat{B}$ at time $t$, we first find the eigenvalues and eigenvectors of $\hat{B}$. The representation matrix of $\hat{B}$ in the $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ basis is a block matrix of the form

$$B = \begin{pmatrix}B_2 & 0\\ 0 & B_1\end{pmatrix}$$

where $B_2$ is a $2 \times 2$ matrix given by

$$B_2 = b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and $B_1$ is a $1 \times 1$ matrix given by

$$B_1 = b \, (1) \, .$$

From the $1 \times 1$ matrix $B_1$, we see that $b$ is an eigenvalue of $\hat{B}$ with the corresponding eigenvector given by $|u_3\rangle$. The $2 \times 2$ matrix $B_2$ is of the form $A_2$ that was diagonalized in the above. From the results obtained in the above, we see that $\pm b$ are eigenvalues of $B_2$ with the corresponding eigenvectors given by

$$\frac{1}{\sqrt{2}} \big[ \, |u_1\rangle \pm |u_2\rangle \, \big].$$

We thus see that the eigenvalue $b$ of $\hat{B}$ is two-fold degenerate. In summary, the eigenvalues and eigenvectors of $\hat{B}$ are given by

| Eigenvalue | Degree of degeneracy | Eigenvector |
|---|---|---|
| $b_1 = -b$ | 1 | $\vert\zeta_1^{(1)}\rangle = \frac{1}{\sqrt{2}} \big[ \, \vert u_1\rangle - \vert u_2\rangle \, \big]$ |
| $b_2 = b$ | 2 | $\vert\zeta_2^{(1)}\rangle = \frac{1}{\sqrt{2}} \big[ \, \vert u_1\rangle + \vert u_2\rangle \, \big]$ |
| | | $\vert\zeta_2^{(2)}\rangle = \vert u_3\rangle$ |

Expressing $|u_1\rangle$, $|u_2\rangle$, and $|u_3\rangle$ in terms of $|\zeta_1^{(1)}\rangle$, $|\zeta_2^{(1)}\rangle$, and $|\zeta_2^{(2)}\rangle$, we have

$$\begin{aligned} |u_1\rangle &= \frac{1}{\sqrt{2}} \big[ \, |\zeta_2^{(1)}\rangle + |\zeta_1^{(1)}\rangle \, \big], \\ |u_2\rangle &= \frac{1}{\sqrt{2}} \big[ \, |\zeta_2^{(1)}\rangle - |\zeta_1^{(1)}\rangle \, \big], \\ |u_3\rangle &= |\zeta_2^{(2)}\rangle \, . \end{aligned}$$

In terms of $|\zeta_1^{(1)}\rangle$, $|\zeta_2^{(1)}\rangle$, and $|\zeta_2^{(2)}\rangle$, $|\psi(t)\rangle$ is written as

$$\begin{aligned} |\psi(t)\rangle &= \frac{1}{2} e^{-i\omega_0 t} \big[ \, |\zeta_2^{(1)}\rangle + |\zeta_1^{(1)}\rangle \, \big] + \frac{1}{2\sqrt{2}} e^{-2i\omega_0 t} \big[ \, |\zeta_2^{(1)}\rangle - |\zeta_1^{(1)}\rangle \, \big] + \frac{1}{2} e^{-2i\omega_0 t} \, |\zeta_2^{(2)}\rangle \\ &= \frac{e^{-i\omega_0 t}}{2\sqrt{2}} \big( \sqrt{2} - e^{-i\omega_0 t} \big) \, |\zeta_1^{(1)}\rangle + \frac{e^{-i\omega_0 t}}{2\sqrt{2}} \big( \sqrt{2} + e^{-i\omega_0 t} \big) \, |\zeta_2^{(1)}\rangle + \frac{1}{2} e^{-2i\omega_0 t} \, |\zeta_2^{(2)}\rangle \, . \end{aligned}$$

If the observable $\hat{B}$ is measured at time $t$, the results $\pm b$ can be obtained. The probability of obtaining the result $-b$ is given by

$$\begin{aligned} \mathscr{P}_{\hat{B}}(-b) &= \big| \langle \zeta_1^{(1)} | \psi(t) \rangle \big|^2 = \frac{1}{8} \big| \sqrt{2} - e^{-i\omega_0 t} \big|^2 \\ &= \frac{1}{8} \Big\{ \big[ \, \sqrt{2} - \cos(\omega_0 t) \, \big]^2 + \sin^2(\omega_0 t) \Big\} \\ &= \frac{1}{8} \big[ \, 3 - 2\sqrt{2} \cos(\omega_0 t) \, \big] = \frac{1}{8} - \frac{1}{2\sqrt{2}} \cos(\omega_0 t). \end{aligned}$$

The probability of obtaining the result $b$ is given by

$$\begin{aligned} \mathscr{P}_{\hat{B}}(b) &= \big| \langle \zeta_2^{(1)} | \psi(t) \rangle \big|^2 + \big| \langle \zeta_2^{(2)} | \psi(t) \rangle \big|^2 = \frac{1}{8} \big| \sqrt{2} + e^{-i\omega_0 t} \big|^2 + \frac{1}{4} \\ &= \frac{1}{8} \Big\{ \big[ \, \sqrt{2} + \cos(\omega_0 t) \, \big]^2 + \sin^2(\omega_0 t) \Big\} + \frac{1}{4} \\ &= \frac{1}{8} \big[ \, 5 + 2\sqrt{2} \cos(\omega_0 t) \, \big] = \frac{5}{8} + \frac{1}{2\sqrt{2}} \cos(\omega_0 t). \end{aligned}$$

5. **[C-T Exercise 3-8]** Let $\vec{j}(\vec{r})$ be the probability current density associated with a wave function $\psi(\vec{r})$ describing the state of a particle of mass $m$.

(a) Show that

$$m\int d^3r\,\vec{j}(\vec{r}) = \langle\hat{\vec{p}}\rangle,$$

where $\langle\hat{\vec{p}}\rangle$ is the mean value of the momentum.

(b) Consider the operator $\hat{\vec{L}}$ (orbital angular momentum) defined by $\hat{\vec{L}} = \hat{\vec{r}}\times\hat{\vec{p}}$. Are the three components of $\hat{\vec{L}}$ Hermitian operators? Establish the relation

$$m\int d^3r\,\big[\vec{r}\times\vec{j}(\vec{r})\big] = \langle\hat{\vec{L}}\rangle\,.$$

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(a) The probability current density $\vec{j}(\vec{r})$ is given by

$$\vec{j}(\vec{r}) = \frac{\hbar}{2im}\big[\,\psi^*(\vec{r})\vec{\nabla}\psi(\vec{r}) - \psi(\vec{r})\vec{\nabla}\psi^*(\vec{r})\,\big].$$

Multiplying both sides of the above equation with $m$ and then integrating the resultant equation, we have

$$m\int d^3r\,\vec{j}(\vec{r}) = \frac{\hbar}{2i}\int d^3r\,\big[\,\psi^*(\vec{r})\vec{\nabla}\psi(\vec{r}) - \psi(\vec{r})\vec{\nabla}\psi^*(\vec{r})\,\big].$$

Performing an integral by parts to the second term in the square brackets and making use of $\lim_{|\vec{r}|\to\infty}\psi(\vec{r}) = 0$, we have

$$\begin{aligned}
m\int d^3r\,\vec{j}(\vec{r}) &= \frac{\hbar}{2i}\int d^3r\,\big[\,\psi^*(\vec{r})\vec{\nabla}\psi(\vec{r}) + \psi^*(\vec{r})\vec{\nabla}\psi(\vec{r})\,\big] \\
&= -i\hbar\int d^3r\,\psi^*(\vec{r})\vec{\nabla}\psi(\vec{r}) \\
&= \int d^3r\,\psi^*(\vec{r})\big(-i\hbar\vec{\nabla}\big)\psi(\vec{r}) \\
&= \langle\hat{\vec{p}}\rangle\,.
\end{aligned}$$

(b) Writing down the components of $\hat{\vec{L}} = \hat{\vec{r}}\times\hat{\vec{p}}$, we have

$$\begin{aligned}
\hat{L}_x &= \hat{y}\hat{p}_z - \hat{z}\hat{p}_y,\\
\hat{L}_y &= \hat{z}\hat{p}_x - \hat{x}\hat{p}_z,\\
\hat{L}_z &= \hat{x}\hat{p}_y - \hat{y}\hat{p}_x.
\end{aligned}$$

Making use of the facts that $\hat{\vec{r}}$ and $\hat{\vec{p}}$ are Hermitian operators and that different components of $\hat{\vec{r}}$ and $\hat{\vec{p}}$ commute, we have

$$\begin{aligned}
\hat{L}_x^\dagger &= \hat{p}_z^\dagger\hat{y}^\dagger - \hat{p}_y^\dagger\hat{z}^\dagger = \hat{p}_z\hat{y} - \hat{p}_y\hat{z} = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = \hat{L}_x,\\
\hat{L}_y^\dagger &= \hat{p}_x^\dagger\hat{z}^\dagger - \hat{p}_z^\dagger\hat{x}^\dagger = \hat{p}_x\hat{z} - \hat{p}_z\hat{x} = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z = \hat{L}_y,\\
\hat{L}_z^\dagger &= \hat{p}_y^\dagger\hat{x}^\dagger - \hat{p}_x^\dagger\hat{y}^\dagger = \hat{p}_y\hat{x} - \hat{p}_x\hat{y} = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = \hat{L}_z.
\end{aligned}$$

Thus, the three components of $\hat{\vec{L}}^\dagger$ are Hermitian operators. For $\hat{\vec{L}}^\dagger$, we have

$$\hat{\vec{L}}^\dagger = \hat{L}_x^\dagger\vec{e}_x + \hat{L}_y^\dagger\vec{e}_y + \hat{L}_z^\dagger\vec{e}_z = \hat{L}_x\vec{e}_x + \hat{L}_y\vec{e}_y + \hat{L}_z\vec{e}_z = \hat{\vec{L}}.$$

Multiplying both sides of the equation

$$\vec{j}(\vec{r}\,) = \frac{\hbar}{2im}\big[\,\psi^*(\vec{r}\,)\vec{\nabla}\psi(\vec{r}\,) - \psi(\vec{r}\,)\vec{\nabla}\psi^*(\vec{r}\,)\,\big].$$

with $m$ and then taking the vector product of $\vec{r}$ with the resultant equation, we have

$$m\vec{r}\times\vec{j}(\vec{r}\,) = \frac{\hbar}{2i}\vec{r}\times\big[\,\psi^*(\vec{r}\,)\vec{\nabla}\psi(\vec{r}\,) - \psi(\vec{r}\,)\vec{\nabla}\psi^*(\vec{r}\,)\,\big].$$

Integrating the above equation yields

$$m\int d^3r\;\vec{r}\times\vec{j}(\vec{r}\,) = \frac{\hbar}{2i}\int d^3r\;\vec{r}\times\big[\,\psi^*(\vec{r}\,)\vec{\nabla}\psi(\vec{r}\,) - \psi(\vec{r}\,)\vec{\nabla}\psi^*(\vec{r}\,)\,\big].$$

Noting that each term of the vector product $\vec{r}\times\vec{\nabla}$ contains different components of $\vec{r}$ and $\vec{\nabla}$, we can perform an integral by parts to the second term in the square brackets. We then have

$$\begin{aligned}
m\int d^3r\;\vec{r}\times\vec{j}(\vec{r}\,) &= \frac{\hbar}{i}\int d^3r\;\vec{r}\times\psi^*(\vec{r}\,)\vec{\nabla}\psi(\vec{r}\,) = \int d^3r\;\vec{r}\times\psi^*(\vec{r}\,)\hat{\vec{p}}\psi(\vec{r}\,)\\
&= \int d^3r\;\psi^*(\vec{r}\,)\vec{r}\times\hat{\vec{p}}\psi(\vec{r}\,) = \int d^3r\;\psi^*(\vec{r}\,)\hat{\vec{L}}\psi(\vec{r}\,)\\
&= \langle\hat{\vec{L}}\,\rangle\,.
\end{aligned}$$