Quantum Mechanics
Homework Assignment 02
Fall, 2019

1. Consider a particle in a complex potential $V(\vec{r}) = U(\vec{r}) + iW(\vec{r})$, where $U(\vec{r})$ and $W(\vec{r})$ are real functions.
   (a) Derive the continuity equation for the time-dependent Schrödinger equation for a particle of mass m in the above complex potential.
   (b) What is the integral form of the continuity equation?
   (c) What is the condition on $W(\vec{r})$ for it to describe a source? What is the condition on $W(\vec{r})$ for it to describe a sink?

2. Show that
$$\hat{\vec{p}}^2 = \frac{1}{r^2}\hat{\vec{L}}^2 - \hbar^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right).$$

3. (a) Find the Taylor expansion of $\hat{f}(\lambda) = e^{\lambda \hat{A}} \hat{B} e^{-\lambda \hat{A}}$ with respect to λ about $\lambda = 0$. Here the operators $\hat{A}$ and $\hat{B}$ may not commute.
   (b) Setting $\lambda = 1$ in the above Taylor expansion of $\hat{f}(\lambda) = e^{\lambda \hat{A}} \hat{B} e^{-\lambda \hat{A}}$, derive an expansion for $e^{\hat{A}} \hat{B} e^{-\hat{A}}$.
   (c) Using the expansion of $e^{\hat{A}} \hat{B} e^{-\hat{A}}$, evaluate $e^{-i\hat{L}_y \theta/\hbar} \hat{L}_z e^{i\hat{L}_y \theta/\hbar}$.

4. The operators $\hat{A}$ and $\hat{B}$ do not commute, $[\hat{A}, \hat{B}] = \hat{C} \neq 0$, but they both commute with their commutator $\hat{C}$, $[\hat{A}, \hat{C}] = [\hat{B}, \hat{C}] = 0$. Show that
$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\hat{C}/2} = e^{\hat{B}} e^{\hat{A}} e^{\hat{C}/2}.$$

5. Consider a particle of mass m subject to a potential $V(x) = \lambda |x|^n$ with λ a constant, $n \neq -2$, and $-\infty < x < \infty$. The energy of the particle is given by $E = \dfrac{p^2}{2m} + \lambda |x|^n$.
   (a) Making use of $|p| \sim \Delta p$, $\Delta x \Delta p \sim \hbar$, and $|x| \sim \Delta x/2$, express E in terms of Δx.
   (b) To obtain the ground-state energy, minimize E with respect to Δx. Find the value of Δx in the ground state.
   (c) What is the expression of the ground-state energy?