Quantum Mechanics

Solutions to the Problems in Homework Assignment 06

Fall, 2019

1. In a given representation, the matrix representing the Hamiltonian of a particle is given by

$$H = \hbar\omega_0 \begin{pmatrix} -1+\varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & -1-\varepsilon & \sqrt{2}\,\varepsilon & 0 & 0 & 0 \\ 0 & \sqrt{2}\,\varepsilon & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & \sqrt{2}\,\varepsilon & 0 \\ 0 & 0 & 0 & \sqrt{2}\,\varepsilon & -1-\varepsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & -1+\varepsilon \end{pmatrix}$$

with $0 < \varepsilon < 1$. Find the energy eigenvalues and eigenfunctions of the particle in the representation.

First of all, note that the given representation matrix H of the Hamiltonian $\hat{H}$ is a block matrix. Let the basis of the representation be $\{|u_1\rangle, |u_2\rangle, |u_3\rangle, |u_4\rangle, |u_5\rangle, |u_6\rangle\}$. The four blocks in H are respectively in the subspaces $\{|u_1\rangle\}$, $\{|u_2\rangle, |u_3\rangle\}$, $\{|u_4\rangle, |u_5\rangle\}$, and $\{|u_6\rangle\}$. The representation matrices in these subspaces are respectively given by

$$\begin{aligned} h_1 &= -\hbar\omega_0(1-\varepsilon), && 1\times 1 \text{ matrix}, \\ h_2 &= \hbar\omega_0 \begin{pmatrix} -1-\varepsilon & \sqrt{2}\,\varepsilon \\ \sqrt{2}\,\varepsilon & -1 \end{pmatrix}, && 2\times 2 \text{ matrix}, \\ h_3 &= \hbar\omega_0 \begin{pmatrix} -1 & \sqrt{2}\,\varepsilon \\ \sqrt{2}\,\varepsilon & -1-\varepsilon \end{pmatrix}, && 2\times 2 \text{ matrix}, \\ h_4 &= -\hbar\omega_0(1-\varepsilon), && 1\times 1 \text{ matrix}. \end{aligned}$$

We now diagonalize the above four matrices separately.

Diagonalization of h_1. Since h_1 is a 1×1 matrix, its eigenvalue is equal to the sole matrix element, $-(1-\varepsilon)\hbar\omega_0$, and the corresponding eigenvector is the basis vector in this subspace, $|u_1\rangle$.

Diagonalization of h_2. Let the eigenvalue of h_2 be $\hbar\omega_0\lambda$ and the corresponding eigenvector be $a\,|u_2\rangle + b\,|u_3\rangle$. The eigenequation of h_2 reads

$$\begin{pmatrix} -1-\varepsilon & \sqrt{2}\,\varepsilon \\ \sqrt{2}\,\varepsilon & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}.$$

Evaluating the matrix product on the left hand side of the above equation yields

$$\begin{aligned} &-\big(1+\varepsilon+\lambda\big)a + \sqrt{2}\,\varepsilon b = 0, \\ &\sqrt{2}\,\varepsilon a - (1+\lambda)b = 0. \end{aligned}$$

The above equations are homogeneous linear algebraic equations for a and b. The necessary and sufficient condition for the above set of homogeneous linear algebraic equations to possess nontrivial solutions is that the determinant of the coefficients vanish. We then have

$$\begin{vmatrix} -\big(1+\varepsilon+\lambda\big) & \sqrt{2}\,\varepsilon \\ \sqrt{2}\,\varepsilon & -(1+\lambda) \end{vmatrix} = 0.$$

Evaluating the determinant, we have

$$(1+\lambda)^2 + \varepsilon(1+\lambda) - 2\varepsilon^2 = 0.$$

Solving λ from the above equation, we obtain

$$\lambda_1 = -1-2\varepsilon,\ \lambda_2 = -1+\varepsilon.$$

Thus, the eigenvalues of h_2 are $-(1+2\varepsilon)\hbar\omega_0$ and $-(1-\varepsilon)\hbar\omega_0$. They are also the eigenvalues of H. We now find the eigenvectors corresponding to these eigenvalues. Inserting $\lambda_1 = -1-2\varepsilon$ into the above equations for a and b, we obtain

$$b = -\frac{1}{\sqrt{2}}a.$$

From the normalization condition, we have

$$\left(1+\frac{1}{2}\right)|a|^2 = 1.$$

We thus have $|a| = \sqrt{2/3}$. We choose $a = \sqrt{2/3}$. We then have $b = -1/\sqrt{3}$. Thus, the eigenvector of h_2 corresponding to the eigenvalue $-(1+2\varepsilon)\hbar\omega_0$ is given by

$$\frac{1}{\sqrt{3}}\big[\sqrt{2}\,|u_2\rangle - |u_3\rangle\big].$$

Inserting $\lambda_2 = -1+\varepsilon$ into the above equations for a and b, we obtain

$$b = \sqrt{2}a.$$

From the normalization condition, we have

$$\big(1+2\big)|a|^2 = 1.$$

We thus have $|a| = 1/\sqrt{3}$. We choose $a = 1/\sqrt{3}$. We then have $b = \sqrt{2/3}$. Thus, the eigenvector of h_2 corresponding to the eigenvalue $-(1-\varepsilon)\hbar\omega_0$ is given by

$$\frac{1}{\sqrt{3}}\big[\,|u_2\rangle + \sqrt{2}\,|u_3\rangle\big].$$

Diagonalization of h_3. Let the eigenvalue of h_3 be $\hbar\omega_0\lambda$ and the corresponding eigenvector be $a\,|u_4\rangle + b\,|u_5\rangle$. The eigenequation of h_3 reads

$$\begin{pmatrix} -1 & \sqrt{2}\,\varepsilon \\ \sqrt{2}\,\varepsilon & -1-\varepsilon \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix}.$$

Evaluating the matrix product on the left hand side of the above equation yields

$$\begin{aligned} -\big(1+\lambda\big)a + \sqrt{2}\,\varepsilon b &= 0, \\ \sqrt{2}\,\varepsilon a - (1+\varepsilon+\lambda)b &= 0. \end{aligned}$$

The above equations are homogeneous linear algebraic equations for a and b. The necessary and sufficient condition for the above set of homogeneous linear algebraic equations to possess nontrivial solutions is that the determinant of the coefficients vanish. We then have

$$\begin{vmatrix} -\big(1+\lambda\big) & \sqrt{2}\,\varepsilon \\ \sqrt{2}\,\varepsilon & -(1+\varepsilon+\lambda) \end{vmatrix} = 0.$$

Evaluating the determinant, we have

$$(1+\lambda)^2 + \varepsilon(1+\lambda) - 2\varepsilon^2 = 0.$$

Note that the above equation is identical with that for h_2. We thus have the same eigenvalues of h_3 as those of h_2. The eigenvalues of h_3 are $-(1+2\varepsilon)\hbar\omega_0$ and $-(1-\varepsilon)\hbar\omega_0$. They are also the eigenvalues of H. We now find the eigenvectors corresponding to these eigenvalues. Inserting $\lambda_1 = -1-2\varepsilon$ into the above equations for a and b, we obtain

$$b = -\sqrt{2}a.$$

From the normalization condition, we have

$$\big(1+2\big)|a|^2 = 1.$$

We thus have $|a| = 1/\sqrt{3}$. We choose $a = 1/\sqrt{3}$. We then have $b = -\sqrt{2/3}$. Thus, the eigenvector of h_3 corresponding to the eigenvalue $-(1+2\varepsilon)\hbar\omega_0$ is given by

$$\frac{1}{\sqrt{3}}\big[\,|u_4\rangle - \sqrt{2}\,|u_5\rangle\,\big].$$

Inserting $\lambda_2 = -1+\varepsilon$ into the above equations for a and b, we obtain

$$b = \frac{1}{\sqrt{2}}a.$$

From the normalization condition, we have

$$\left(1+\frac{1}{2}\right)|a|^2 = 1.$$

We thus have $|a| = \sqrt{2/3}$. We choose $a = \sqrt{2/3}$. We then have $b = 1/\sqrt{3}$. Thus, the eigenvector of h_3 corresponding to the eigenvalue $-(1-\varepsilon)\hbar\omega_0$ is given by

$$\frac{1}{\sqrt{3}}\big[\,\sqrt{2}\,|u_4\rangle + |u_5\rangle\,\big].$$

Diagonalization of h_4. Since h_4 is a 1×1 matrix, its eigenvalue is equal to the sole matrix element, $-(1-\varepsilon)\hbar\omega_0$, and the corresponding eigenvector is the basis vector in this subspace, $|u_6\rangle$.

Summary: Eigenvalues and eigenvectors of H. From the above results, we see that H has only two distinct eigenvalues, $-(1+2\varepsilon)\hbar\omega_0$ and $-(1-\varepsilon)\hbar\omega_0$. Since there exist two independent eigenvectors corresponding to the eigenvalue $-(1+2\varepsilon)\hbar\omega_0$, the eigenvalue $-(1+2\varepsilon)\hbar\omega_0$ is doubly degenerate. Since there exist four independent eigenvectors corresponding to the eigenvalue $-(1-\varepsilon)\hbar\omega_0$, the eigenvalue $-(1-\varepsilon)\hbar\omega_0$ is four-fold degenerate.

We list the above results in the following table.

Eigenvalue	**Degree of degeneracy**	**Eigenvectors**
$-(1+2\varepsilon)\hbar\omega_0$	2	$\frac{1}{\sqrt{3}}\big[\,\sqrt{2}\,\lvert u_2\rangle - \lvert u_3\rangle\,\big]$ $\frac{1}{\sqrt{3}}\big[\,\lvert u_4\rangle - \sqrt{2}\,\lvert u_5\rangle\,\big]$
$-(1-\varepsilon)\hbar\omega_0$	4	$\lvert u_1\rangle$ $\frac{1}{\sqrt{3}}\big[\,\lvert u_2\rangle + \sqrt{2}\,\lvert u_3\rangle\,\big]$ $\frac{1}{\sqrt{3}}\big[\,\sqrt{2}\,\lvert u_4\rangle + \lvert u_5\rangle\,\big]$ $\lvert u_6\rangle$.

2. **[C-T exercise 2-4]** Let $\hat{K}$ be the operator defined by $\hat{K} = |\varphi\rangle\langle\psi|$, where $|\varphi\rangle$ and $|\psi\rangle$ are two vectors of the state space.
   (a) Under what condition is $\hat{K}$ Hermitian?
   (b) Calculate $\hat{K}^2$. Under what condition is $\hat{K}$ a projector?
   (c) Show that $\hat{K}$ can always be written in the form $\hat{K} = \lambda\hat{P}_1\hat{P}_2$ where λ is a constant to be calculated and $\hat{P}_1$ and $\hat{P}_2$ are projectors.

(a) The Hermitian conjugate of $\hat{K} = |\varphi\rangle\langle\psi|$ is given by

$$\hat{K}^\dagger = |\psi\rangle\langle\varphi| \,.$$

For $\hat{K}$ to be a Hermitian operator, we must have

$$|\psi\rangle\langle\varphi| = |\varphi\rangle\langle\psi| \,.$$

If $|\psi\rangle = C\,|\varphi\rangle$ with C a real number, the above equation holds. Thus, for $\hat{K}$ to be a Hermitian operator, $|\psi\rangle$ and $|\varphi\rangle$ must be proportional with the proportionality constant being a real number.

(b) $\hat{K}^2$ is given by

$$\hat{K}^2 = |\varphi\rangle\langle\psi|\varphi\rangle\langle\psi| = \langle\psi|\varphi\rangle|\varphi\rangle\langle\psi| \,.$$

For $\hat{K}$ to be a projector, we must have $\hat{K}^2 = \hat{K}$. That is,

$$\langle\psi|\varphi\rangle|\varphi\rangle\langle\psi| = |\varphi\rangle\langle\psi| \,.$$

From the above equation, it follows that

$$\langle\psi|\varphi\rangle = 1.$$

Thus, if $\langle\psi|\varphi\rangle = 1$, then $\hat{K} = |\varphi\rangle\langle\psi|$ is a projector.

(c) Assume that $|\varphi\rangle$ and $|\psi\rangle$ are nonzero and not orthogonal to each other. Multiplying and then dividing the right hand side of $\hat{K} = |\varphi\rangle\langle\psi|$ with $\langle\varphi|\psi\rangle$, we have

$$\begin{aligned}\hat{K} &= \frac{|\varphi\rangle\langle\psi|\langle\varphi|\psi\rangle}{\langle\varphi|\psi\rangle} = \frac{|\varphi\rangle\langle\varphi|\psi\rangle\langle\psi|}{\langle\varphi|\psi\rangle} = \frac{1}{\langle\varphi|\psi\rangle}\left[|\varphi\rangle\langle\varphi|\right]\left[|\psi\rangle\langle\psi|\right] \\ &= \frac{\langle\varphi|\varphi\rangle\langle\psi|\psi\rangle}{\langle\varphi|\psi\rangle}\left[\frac{|\varphi\rangle\langle\varphi|}{\langle\varphi|\varphi\rangle}\right]\left[\frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}\right] = \lambda\hat{P}_1\hat{P}_2,\end{aligned}$$

where

$$\begin{aligned}\lambda &= \frac{\langle\varphi|\varphi\rangle\langle\psi|\psi\rangle}{\langle\varphi|\psi\rangle}, \\ \hat{P}_1 &= \frac{|\varphi\rangle\langle\varphi|}{\langle\varphi|\varphi\rangle}, \\ \hat{P}_2 &= \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}.\end{aligned}$$

3. **[C-T exercise 2-5]** Let $\hat{P}_1$ be the orthogonal projector onto the subspace $\mathscr{E}_1$, $\hat{P}_2$ the orthogonal projector on to the subspace $\mathscr{E}_2$. Show that, for the product $\hat{P}_1\hat{P}_2$ to be an orthogonal projector as well, it is necessary and sufficient that $\hat{P}_1$ and $\hat{P}_2$ commute. In this case, what is the subspace onto which $\hat{P}_1\hat{P}_2$ projects?

Proof that $[\hat{P}_1, \hat{P}_2] = 0$ is necessary for $\hat{P}_1\hat{P}_2$ to be an orthogonal projector.

For $\hat{P}_1\hat{P}_2$ to be an orthogonal projector, its square must be equal to itself. Thus, it is necessary that

$$\hat{P}_1\hat{P}_2\hat{P}_1\hat{P}_2 = \hat{P}_1\hat{P}_2.$$

Making use of $\hat{P}_1^2 = \hat{P}_1$ and $\hat{P}_2^2 = \hat{P}_2$, we have

$$\hat{P}_1\hat{P}_2\hat{P}_1\hat{P}_2 = \hat{P}_1\hat{P}_1\hat{P}_2\hat{P}_2.$$

Making a rearrangement of the above equation yields

$$0 = \hat{P}_1\big(\hat{P}_1\hat{P}_2 - \hat{P}_2\hat{P}_1\big)\hat{P}_2 = \hat{P}_1\big[\hat{P}_1, \hat{P}_2\big]\hat{P}_2$$

Thus, $\big[\hat{P}_1, \hat{P}_2\big] = 0$ is necessary for $\hat{P}_1\hat{P}_2$ to be an orthogonal projector.

Proof that $[\hat{P}_1, \hat{P}_2] = 0$ is sufficient for $\hat{P}_1\hat{P}_2$ to be an orthogonal projector.

From $\big[\hat{P}_1, \hat{P}_2\big] = 0$, we have

$$\hat{P}_1\hat{P}_2 = \hat{P}_2\hat{P}_1.$$

Making use of $\hat{P}_1\hat{P}_2 = \hat{P}_2\hat{P}_1$, the square of $\hat{P}_1\hat{P}_2$ is given by

$$\big(\hat{P}_1\hat{P}_2\big)^2 = \hat{P}_1\hat{P}_2\hat{P}_1\hat{P}_2 = \hat{P}_1\hat{P}_2\hat{P}_2\hat{P}_1 = \hat{P}_1\hat{P}_2\hat{P}_1 = \hat{P}_1\hat{P}_1\hat{P}_2 = \hat{P}_1\hat{P}_2$$

from which we see that $\hat{P}_1\hat{P}_2$ is an orthogonal projector. Thus, $\big[\hat{P}_1, \hat{P}_2\big] = 0$ is sufficient for $\hat{P}_1\hat{P}_2$ to be an orthogonal projector.

The subspace onto which $\hat{P}_1\hat{P}_2$ projects is the intersection of $\mathscr{E}_1$ and $\mathscr{E}_2$, $\mathscr{E}_1 \cap \mathscr{E}_2$

4. **[C-T exercise 2-11]** Consider a physical system whose three-dimensional state space is spanned by the orthonormal basis formed by the three kets $|u_1\rangle$, $|u_2\rangle$, and $|u_3\rangle$. In the basis of these three vectors, taken in this order, the two operators $\hat{H}$ and $\hat{B}$ are defined by

$$H = \hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where ω_0 and b are real constants.

(a) Are H and B Hermitian?

(b) Show that H and B commute. Give a basis of eigenvectors common to H and B.

(a) The Hermitian conjugates of H and B are respectively given by

$$H^\dagger = \hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = H, \; B^\dagger = b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = B.$$

Thus, both H and B are Hermitian matrices.

(b) The commutator of H and B is given by

$$\begin{aligned} \big[H, B\big] = HB - BH &= b\hbar\omega_0 \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right] \\ &= b\hbar\omega_0 \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}\right] = b\hbar\omega_0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0. \end{aligned}$$

Thus, H and B commute and they can have common eigenvectors. We now find their common eigenvectors.

Eigenvalues and eigenvectors of H. Since H is a diagonal matrix, its eigenvalues are just the elements on its main diagonal and the corresponding eigenvectors are the basis vectors corresponding to rows or columns of the main diagonal elements. The eigenvalues and the corresponding eigenvectors of H are listed below.

Since the eigenvalue $-\hbar\omega_0$ of H is doubly degenerate, any linear combination of $|u_2\rangle$ and $|u_3\rangle$ is also an eigenvector of H corresponding to the eigenvalue $-\hbar\omega_0$.

TABLE I: Eigenvalues and eigenvectors of H.

Eigenvalue	Eigenvector
$\hbar\omega_0$	$\lvert u_1\rangle$
$-\hbar\omega_0$	$\lvert u_2\rangle$
	$\lvert u_3\rangle$

Eigenvalues and eigenvectors of B. We see that B is a block matrix. We can rewrite B as

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix},$$

where B_1 is 1×1 matrix and B_2 is a 2×2 matrix, They are respectively given by

$$B_1 = b(1),\ B_2 = b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalue of B_1 is b and the eigenvector corresponding to this eigenvalue is $|u_1\rangle$. Since $|u_1\rangle$ is also an eigenvector of H corresponding to the eigenvalue $\hbar\omega_0$, $|u_1\rangle$ is the common eigenvector of H and B corresponding respectively to the eigenvalues $\hbar\omega_0$ and b.
We now diagonalize B_2. Let λb denote the eigenvalue of B_2 and $c\,|u_2\rangle + d\,|u_3\rangle$ the corresponding eigenvector. The eigenequation of B_2 reads

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \lambda\begin{pmatrix} c \\ d \end{pmatrix}$$

from which we have the following homogeneous linear algebraic equations for c and d

$$\begin{aligned} -\lambda c + d &= 0, \\ c - \lambda d &= 0. \end{aligned}$$

The necessary and sufficient condition for the existence of nonzero solutions for c and d is given by

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

from which we have $\lambda = \pm 1$. Thus, the eigenvalues of B_2 are $\pm b$. Inserting $\lambda = 1$ into the above equations for c and d, we have $d = c$. From the normalization condition, we have

$$|c|^2(1+1) = 1$$

from which we have $|c| = 1/\sqrt{2}$. We choose $c = 1/\sqrt{2}$. We then have $d = c = 1/\sqrt{2}$. Thus, the eigenvector of B_2 corresponding to the eigenvalue b is given by

$$\frac{1}{\sqrt{2}}\big[|u_2\rangle + |u_3\rangle\big].$$

Inserting $\lambda = -1$ into the above equations for c and d, we have $d = -c$. From the normalization condition, we have

$$|c|^2(1+1) = 1$$

from which we have $|c| = 1/\sqrt{2}$. We choose $c = 1/\sqrt{2}$. We then have $d = -c = -1/\sqrt{2}$. Thus, the eigenvector of B_2 corresponding to the eigenvalue $-b$ is given by

$$\frac{1}{\sqrt{2}}\big[|u_2\rangle - |u_3\rangle\big].$$

Since any linear combination of $|u_2\rangle$ and $|u_3\rangle$ is an eigenvector of H corresponding to the eigenvalue $-\hbar\omega_0$,

$$\frac{1}{\sqrt{2}}\left[|u_2\rangle + |u_3\rangle\right] \text{ and } \frac{1}{\sqrt{2}}\left[|u_2\rangle - |u_3\rangle\right]$$

are also eigenvectors of H corresponding to the eigenvalue $-\hbar\omega_0$.

Summary: The common eigenvectors of H and B. In summary, we have obtained the following eigenvectors of H and B.

TABLE II: Common eigenvectors of H and B.

Common eigenvector	Eigenvalue of H	Eigenvalue of B
$\lvert u_1\rangle$	$\hbar\omega_0$	b
$\frac{1}{\sqrt{2}}\left[\lvert u_2\rangle + \lvert u_3\rangle\right]$	$-\hbar\omega_0$	b
$\frac{1}{\sqrt{2}}\left[\lvert u_2\rangle - \lvert u_3\rangle\right]$	$-\hbar\omega_0$	$-b$

5. **[C-T exercise 2-12]** In the same state space as that of the preceding exercise, consider two operators $\hat{L}_z$ and $\hat{S}$ defined by

$$\begin{aligned} &\hat{L}_z\,|u_1\rangle = |u_1\rangle, \;\; \hat{L}_z\,|u_2\rangle = 0, \quad \hat{L}_z\,|u_3\rangle = -\,|u_3\rangle; \\ &\hat{S}\,|u_1\rangle = |u_3\rangle, \quad \hat{S}\,|u_2\rangle = |u_2\rangle, \;\; \hat{S}\,|u_3\rangle = |u_1\rangle\,. \end{aligned}$$

(a) Write the matrices which represent, in the $\{|u_1\rangle,\,|u_2\rangle,\,|u_3\rangle\}$ basis, the operators $\hat{L}_z$, $\hat{L}_z^2$, $\hat{S}$, and $\hat{S}^2$. Are these operators observables?

(b) Give the form of the most general matrix which represents an operator which commutes with $\hat{L}_z$. Same question for $\hat{L}_z^2$, then $\hat{S}^2$.

(c) Do $\hat{L}_z^2$ and $\hat{S}$ form a CSCO? Give a basis of common eigenvectors.

(a) From $\hat{L}_z\,|u_1\rangle = |u_1\rangle, \hat{L}_z\,|u_2\rangle = 0, \hat{L}_z\,|u_3\rangle = -\,|u_3\rangle$, we obtain the following matrix elements of $\hat{L}_z$

$$\begin{aligned} &\langle u_1|\hat{L}_z|u_1\rangle = 1, \;\; \langle u_1|\hat{L}_z|u_2\rangle = 0, \;\; \langle u_1|\hat{L}_z|u_3\rangle = 0, \\ &\langle u_2|\hat{L}_z|u_1\rangle = 0, \;\; \langle u_2|\hat{L}_z|u_2\rangle = 0, \;\; \langle u_2|\hat{L}_z|u_3\rangle = 0, \\ &\langle u_3|\hat{L}_z|u_1\rangle = 0, \;\; \langle u_3|\hat{L}_z|u_2\rangle = 0, \;\; \langle u_3|\hat{L}_z|u_3\rangle = -1. \end{aligned}$$

Thus, the representation matrix of $\hat{L}_z$ in the $\{|u_1\rangle,\,|u_2\rangle,\,|u_3\rangle\}$ basis is a diagonal matrix given by

$$L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

From the representation matrix of $\hat{L}_z$, we can obtain the representation matrix of $\hat{L}_z^2$ by squaring the representation matrix of $\hat{L}_z$. We have

$$L_z^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From $\hat{S}\,|u_1\rangle = |u_3\rangle, \hat{S}\,|u_2\rangle = |u_2\rangle, \hat{S}\,|u_3\rangle = |u_1\rangle$, we obtain the following matrix elements of $\hat{S}$

$$\begin{aligned}
&\langle u_1|\hat{S}|u_1\rangle = 0,\ \ \langle u_1|\hat{S}|u_2\rangle = 0,\ \ \langle u_1|\hat{S}|u_3\rangle = 1,\\
&\langle u_2|\hat{S}|u_1\rangle = 0,\ \ \langle u_2|\hat{S}|u_2\rangle = 1,\ \ \langle u_2|\hat{S}|u_3\rangle = 0,\\
&\langle u_3|\hat{S}|u_1\rangle = 1,\ \ \langle u_3|\hat{S}|u_2\rangle = 0,\ \ \langle u_3|\hat{S}|u_3\rangle = 0.
\end{aligned}$$

Thus, the representation matrix of $\hat{S}$ in the $\{|u_1\rangle,\ |u_2\rangle,\ |u_3\rangle\}$ basis is given by

$$S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

From the representation matrix of $\hat{S}$, we can obtain the representation matrix of $\hat{S}^2$ by squaring the representation matrix of $\hat{S}$. We have

$$S^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We see that the representation matrix of $\hat{S}^2$ is a unit matrix.
Because the representation matrices of $\hat{L}_z$, $\hat{L}_z^2$, $\hat{S}$, and $\hat{S}^2$ are all Hermitian matrices, these operators are observables.

(b) Let the general matrix which represents an operator that commutes with $\hat{L}_z$ be

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

with all the matrix elements being complex numbers. From $[A, L_z] = 0$, we have

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Evaluating the matrix products on both sides of the above equation, we have

$$\begin{pmatrix} a_{11} & 0 & -a_{13} \\ a_{21} & 0 & -a_{23} \\ a_{31} & 0 & -a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ -a_{31} & -a_{32} & -a_{33} \end{pmatrix}$$

from which it follows that

$$a_{12} = 0, a_{13} = 0, a_{21} = 0, a_{23} = 0, a_{31} = 0, a_{32} = 0.$$

Thus, the general matrix which represents an operator that commutes with $\hat{L}_z$ is given by

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}.$$

We see that the general matrix which represents an operator that commutes with $\hat{L}_z$ is a diagonal matrix.
Let the general matrix which represents an operator that commutes with $\hat{L}_z^2$ be

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

with all the matrix elements being complex numbers. From $[B, L_z^2] = 0$, we have

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

Evaluating the matrix products on both sides of the above equation, we have

$$\begin{pmatrix} b_{11} & 0 & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & 0 & b_{33} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

from which it follows that

$$b_{12} = 0, b_{21} = 0, b_{23} = 0, b_{32} = 0.$$

Thus, the general matrix which represents an operator that commutes with $\hat{L}_z^2$ is given by

$$B = \begin{pmatrix} b_{11} & 0 & b_{13} \\ 0 & b_{22} & 0 \\ b_{31} & 0 & b_{33} \end{pmatrix}.$$

Note that a unit matrix commutes with all the matrices. Since $\hat{S}^2$ is a unit matrix, any matrix commutes with S^2. Thus, the general matrix which represents an operator that commutes with $\hat{S}^2$ is of the most general form

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

with all the matrix elements being complex numbers.

(c) Let us first see if L_z^2 and S commute. Their commutator is evaluated as follows

$$\left[L_z^2, S\right] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

Thus, L_z^2 and S commute and they can have common eigenvectors. Since L_z^2 is a diagonal matrix, its eigenvalues are the matrix elements on the main diagonal. We see that the eigenvalues of L_z^2 are 0 and 1. The eigenvalue 0 of L_z^2 is nondegenerate with the corresponding eigenvector given by $|u_2\rangle$. The eigenvalue 1 of L_z^2 is doubly degenerate. The two-dimensional eigensubspace of the eigenvalue 1 of L_z^2 is spanned by $|u_1\rangle$ and $|u_3\rangle$. Note that any linear combination of $|u_1\rangle$ and $|u_3\rangle$ is an eigenvector of L_z^2 corresponding to the eigenvalue 1.

For the convenience of diagonalizing S, we rearrange the basis vectors into the order $|u_2\rangle, |u_3\rangle, |u_1\rangle$. From the previous results,

$$\begin{aligned} &\langle u_1|\hat{S}|u_1\rangle = 0, \ \langle u_1|\hat{S}|u_2\rangle = 0, \ \langle u_1|\hat{S}|u_3\rangle = 1, \\ &\langle u_2|\hat{S}|u_1\rangle = 0, \ \langle u_2|\hat{S}|u_2\rangle = 1, \ \langle u_2|\hat{S}|u_3\rangle = 0, \\ &\langle u_3|\hat{S}|u_1\rangle = 1, \ \langle u_3|\hat{S}|u_2\rangle = 0, \ \langle u_3|\hat{S}|u_3\rangle = 0, \end{aligned}$$

we obtain the following representation matrix of S in the $\{|u_2\rangle, |u_3\rangle, |u_1\rangle\}$ basis

$$S' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We see that S' is a block matrix of the form

$$S' = \begin{pmatrix} S'_1 & 0 \\ 0 & S'_2 \end{pmatrix},$$

where

$$S'_1 = (1),\ S'_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

From $S'_1 = (1)$, we see that 1 is an eigenvalue of S' with the corresponding eigenvector given by $|u_2\rangle$. Note that 1 is also an eigenvalue of S with the corresponding eigenvector given by $|u_2\rangle$.

Note that S'_2 is of the form of B_2 in the previous problem. The eigenvalues and the corresponding vectors of S'_2 can be obtained from those of B_2 through setting $b = 1$ and with the proper basis vectors used. Thus, the eigenvalues of S'_2 are ± 1 with the corresponding eigenvectors respectively given

$$\frac{1}{\sqrt{2}}\big[|u_3\rangle + |u_1\rangle\big],\ \frac{1}{\sqrt{2}}\big[|u_3\rangle - |u_1\rangle\big].$$

Note that the eigenvalues and eigenvectors of S'_2 are also those of S. Since the subspace spanned by $|u_3\rangle$ and $|u_1\rangle$ is the eigensubspace of the eigenvalue 1 of L_z^2, the above eigenvectors of S'_2 are also the eigenvectors of L_z^2 corresponding to the eigenvalue 1.

In summary, we have obtained the following common eigenvectors of L_z^2 and S.

TABLE III: Common eigenvectors of L_z^2 and S.

Common eigenvector	Eigenvalue of L_z^2	Eigenvalue of S
$\lvert u_2\rangle$	0	1
$\frac{1}{\sqrt{2}}\big[\lvert u_3\rangle + \lvert u_1\rangle\big]$	1	1
$\frac{1}{\sqrt{2}}\big[\lvert u_3\rangle - \lvert u_1\rangle\big]$	1	-1

From the above table, we see that, specifying a pair of eigenvalues of L_z^2 and S, their common eigenvector can be uniquely determined (within a multiplying numerical factor). Therefore, $\hat{L}_z^2$ and $\hat{S}$ form a CSCO.