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Problem 1. [C-T Exercise 9-1] Consider a spin 1/2 particle. Call its spin $\hat{\vec{S}}$, its orbital angular momentum $\hat{\vec{L}}$, and its state vector $|\psi\rangle$. The two functions $\psi_+(\vec{r})$ and $\psi_-(\vec{r})$ are defined by $\psi_\pm(\vec{r}) = \langle\vec{r},\pm|\psi\rangle$. Assume that

$$\psi_+(\vec{r}) = R(r)\left[Y_{00}(\theta,\phi) + \frac{1}{\sqrt{3}}Y_{10}(\theta,\phi)\right] \tag{1}$$

$$\psi_-(\vec{r}) = \frac{R(r)}{\sqrt{3}}\left[Y_{11}(\theta,\phi) - Y_{10}(\theta,\phi)\right] \tag{2}$$

where r, θ, ϕ are the coordinates of the particle and $R(r)$ is a given function of r.

(a) What condition must $R(r)$ satisfy for $|\psi\rangle$ to be normalized?

(b) $\hat{S}_z$ is measured with the particle in the state $|\psi\rangle$. What results can be found, and with what probabilities? Same question for $\hat{L}_z$, then for $\hat{S}_x$.

(c) A measurement of $\hat{\vec{L}}^2$, with the particle in the state $|\psi\rangle$, yielded zero. What state describes the particle just after this measurement? Same question if the measurement of $\hat{\vec{L}}^2$ had given $2\hbar^2$.

Solution:

(a) The normalization condition is

$$\begin{aligned}
\langle\psi|\psi\rangle =& \int d^3\vec{r}[|\psi_+(\vec{r})|^2 + |\psi_-(\vec{r})|^2] \\
=& \int_0^{+\infty}|R(r)|^2r^2dr\int_0^\pi \sin\theta d\theta\int_0^{2\pi}d\phi \\
&\left[\left|Y_{00}(\theta,\phi)+\frac{1}{\sqrt{3}}Y_{10}(\theta,\phi)\right|^2 + \left|\frac{1}{\sqrt{3}}Y_{11}(\theta,\phi)-\frac{1}{\sqrt{3}}Y_{10}(\theta,\phi)\right|^2\right] \\
=&2\int_0^{+\infty}|R(r)|^2r^2dr = 1
\end{aligned} \tag{3}$$

Therefore, $R(r)$ must satisfy the condition that

$$\int_0^\infty |R(r)|^2r^2dr = \frac{1}{2} \tag{4}$$

for $|\psi\rangle$ to be normalized.

(b) In the basis $\{|R,l,m,\varepsilon\rangle\}$, where $|R,l,m,\varepsilon\rangle = |R\rangle^{(r)}\otimes|l,m\rangle^{(\Omega)}\otimes|\varepsilon\rangle^{(s)}$, and $\langle R|R\rangle^{(r)} = \frac{1}{2}$, the state of the system can be written as

$$|\psi\rangle = |R,0,0,+\rangle + \frac{1}{\sqrt{3}}|R,1,0,+\rangle + \frac{1}{\sqrt{3}}|R,1,1,-\rangle - \frac{1}{\sqrt{3}}|R,1,0,-\rangle \tag{5}$$

If $\hat{S}_z$ is measured with the particle in the state $|\psi\rangle$, the result $\hat{S}_z = \frac{\hbar}{2}$ can be found with probability

$$\begin{aligned}
\mathscr{P}(\hat{S}_z = \frac{\hbar}{2}) =&|\langle+|\psi\rangle|^2 \\
=&\left||R,0,0\rangle + \frac{1}{\sqrt{3}}|R,1,0\rangle\right|^2 \\
=&\frac{4}{3}\langle R|R\rangle \\
=&\frac{2}{3}
\end{aligned} \tag{6}$$

and the result $\hat{S}_z = -\frac{\hbar}{2}$ can be found with probability

$$\mathscr{P}(\hat{S}_z = -\frac{\hbar}{2}) = 1 - \mathscr{P}(\hat{S}_z = \frac{\hbar}{2}) = \frac{1}{3} \tag{7}$$

If $\hat{L}_z$ is measured, the result $\hat{L}_z = \hbar$ can be found with probability

$$\begin{aligned}\mathscr{P}(\hat{L}_z = 0)&|\langle m = 1|\psi\rangle|^2\\ &= \left|\frac{1}{\sqrt{3}}|R, 1, -\rangle\right|^2\\ &=\frac{1}{3}\langle R|R\rangle\\ &=\frac{1}{6}\end{aligned} \tag{8}$$

and the result $\hat{L}_z = 0$ can be found with probability

$$\mathscr{P}(\hat{L}_z = 0) = 1 - \mathscr{P}(\hat{L}_z = \hbar) = \frac{5}{6} \tag{9}$$

If $\hat{S}_x$ is measured, the result $\hat{S}_x = \frac{\hbar}{2}$ can be found with probability

$$\begin{aligned}\mathscr{P}(\hat{S}_x = \frac{\hbar}{2}) =&|\langle \uparrow_x |\psi\rangle|^2\\ =&\left|\frac{1}{\sqrt{2}}(\langle+| + \langle-|)|\psi\rangle\right|^2\\ =&\frac{1}{2}\left||R, 0, 0\rangle + \frac{1}{\sqrt{3}}|R, 1, 1\rangle\right|^2\\ =&\frac{2}{3}\langle R|R\rangle\\ =&\frac{1}{3}\end{aligned} \tag{10}$$

the result $\hat{S}_x = \frac{\hbar}{2}$ can be found with probability

$$\mathscr{P}(\hat{S}_x = -\frac{\hbar}{2}) = 1 - \mathscr{P}(\hat{S}_x = \frac{\hbar}{2}) = \frac{2}{3} \tag{11}$$

(c) The state describes the particle just after meansurement of $\hat{\vec{L}}^2$ yielding zero is

$$|\psi'\rangle = \frac{|l = 0\rangle\langle l = 0|\psi\rangle}{|\langle l = 0|\psi\rangle|^2} = \sqrt{2}|R, 0, 0, +\rangle \tag{12}$$

The state describes the particle just after measurement of $\hat{\vec{L}}^2$ yeilding $2\hbar^2$ is

$$|\psi'\rangle = \frac{|l = 1\rangle\langle l = 1|\psi\rangle}{|\langle l = 1|\psi\rangle|^2} = \sqrt{\frac{2}{3}}(|R, 1, 0, +\rangle + |R, 1, 1, -\rangle - |R, 1, 0, -\rangle) \tag{13}$$

Reference: https://web.pa.msu.edu/people/mmoore/851HW13_09Solutions.pdf □

Problem 2. [C-T Exercise 9-2] Consider a spin 1/2 particle. $\hat{\vec{p}}$ and $\hat{\vec{S}}$ designate the observables associated with its momentum and its spin. We choose as the basis of the state space the orthonormal basis $|p_xp_yp_z, \pm\rangle$ of eigenvectors common to $\hat{p}_x$, $\hat{p}_y$, $\hat{p}_z$, and $\hat{S}_z$ (whose eigenvalues are, respectively, p_x, p_y, p_z, and $\pm\hbar/2$). We intend to solve the eigenvalue equation of the operator $\hat{A}$ which is defined by $\hat{A} = \hat{\vec{S}} \cdot \hat{\vec{p}}$.

(a) Is $\hat{A}$ Hermitian?

(b) Show that there exists a basis of eigenvectors of $\hat{A}$ which are also eigenvectors of $\hat{p}_x$, $\hat{p}_y$, and $\hat{p}_z$. In the subspace spanned by the kets $|p_xp_yp_z,\pm\rangle$, where p_x, p_y, and p_z are fixed, what is the matrix representing $\hat{A}$?

(c) What are the eigenvalues of $\hat{A}$, and what is their degree of degeneracy? Find a system of eigenvectors common to $\hat{A}$ and $\hat{p}_x$, $\hat{p}_y$, $\hat{p}_z$.

Solution:

(a) The operator $\hat{A}$ can be expressed as

$$\hat{A} = (\hat{S}_x\vec{e}_x + \hat{S}_y\vec{e}_y + \hat{S}_z\vec{e}_z)\cdot(\hat{p}_x\vec{e}_x + \hat{p}_y\vec{e}_y + \hat{p}_z\vec{e}_z) = \hat{S}_x\hat{p}_x + \hat{S}_y\hat{p}_y + \hat{S}_z\hat{p}_z \tag{14}$$

The Hermitian conjugate of the operator $\hat{A}$ is

$$\hat{A}^\dagger = \hat{p}_x^\dagger\hat{S}_x^\dagger + \hat{p}_y^\dagger\hat{S}_y^\dagger + \hat{p}_z^\dagger\hat{S}_z^\dagger = \hat{p}_x\hat{S}_x + \hat{p}_y\hat{S}_y + \hat{p}_z\hat{S}_z \tag{15}$$

Since the momentum operator and the spin operator act on two different Hilbert spaces, they commute

$$[\hat{S}_x,\hat{p}_x] = \hat{S}_x\hat{p}_x - \hat{p}_x\hat{S}_x = 0 \tag{16}$$
$$[\hat{S}_y,\hat{p}_y] = \hat{S}_y\hat{p}_y - \hat{p}_y\hat{S}_y = 0 \tag{17}$$
$$[\hat{S}_z,\hat{p}_z] = \hat{S}_z\hat{p}_z - \hat{p}_z\hat{S}_z = 0 \tag{18}$$

so the Hermitian conjugate of the operator $\hat{A}$ can be written as

$$\hat{A}^\dagger = \hat{S}_x\hat{p}_x + \hat{S}_y\hat{p}_y + \hat{S}_z\hat{p}_z = \hat{A} \tag{19}$$

Therefore, $\hat{A}$ is Hermitian.

(b)

$$\begin{aligned}
\langle p_xp_yp_z,+|\hat{A}|p_xp_yp_z,+\rangle =&\langle p_xp_yp_z,+|(\hat{S}_x\hat{p}_x + \hat{S}_y\hat{p}_y + \hat{S}_z\hat{p}_z)|p_xp_yp_z,+\rangle\\
=&\langle p_xp_yp_z,+|(\frac{\hat{S}_+ + \hat{S}_-}{2}\hat{p}_x + \frac{\hat{S}_+ - \hat{S}_-}{2i}\hat{p}_y + \hat{S}_z\hat{p}_z)|p_xp_yp_z,+\rangle\\
=&\langle p_xp_yp_z,+|(p_x\frac{\hat{S}_+ + \hat{S}_-}{2} + p_y\frac{\hat{S}_+ - \hat{S}_-}{2i} + p_z\hat{S}_z)|p_xp_yp_z,+\rangle\\
=&\frac{\hbar}{2}p_z
\end{aligned} \tag{20}$$

$$\begin{aligned}
\langle p_xp_yp_z,+|\hat{A}|p_xp_yp_z,-\rangle =&\langle p_xp_yp_z,+|(p_x\frac{\hat{S}_+ + \hat{S}_-}{2} + p_y\frac{\hat{S}_+ - \hat{S}_-}{2i} + p_z\hat{S}_z)|p_xp_yp_z,-\rangle\\
=&\frac{\hbar}{2}(p_x - ip_y)
\end{aligned} \tag{21}$$

$$\begin{aligned}
\langle p_xp_yp_z,-|\hat{A}|p_xp_yp_z,+\rangle =&\langle p_xp_yp_z,-|(p_x\frac{\hat{S}_+ + \hat{S}_-}{2} + p_y\frac{\hat{S}_+ - \hat{S}_-}{2i} + p_z\hat{S}_z)|p_xp_yp_z,+\rangle\\
=&\frac{\hbar}{2}(p_x + ip_y)
\end{aligned} \tag{22}$$

$$\begin{aligned}
\langle p_xp_yp_z,-|\hat{A}|p_xp_yp_z,-\rangle =&\langle p_xp_yp_z,-|(p_x\frac{\hat{S}_+ + \hat{S}_-}{2} + p_y\frac{\hat{S}_+ - \hat{S}_-}{2i} + p_z\hat{S}_z)|p_xp_yp_z,-\rangle\\
=&-\frac{\hbar}{2}p_z
\end{aligned} \tag{23}$$

Therefore, in the subspace spanned by the kets $|p_xp_yp_z,\pm\rangle$, the matrix representing $\hat{A}$ is

$$\hat{A}=\frac{\hbar}{2}\begin{pmatrix} p_z & p_x-ip_y \\ p_x+ip_y & -p_z \end{pmatrix} \tag{24}$$

The characteristic equation of $\hat{A}$ is

$$|\hat{A}-\lambda I|=\begin{vmatrix} \frac{\hbar}{2}p_z-\lambda & \frac{\hbar}{2}(p_x-ip_y) \\ \frac{\hbar}{2}(p_x+ip_y) & -\frac{\hbar}{2}p_z-\lambda \end{vmatrix}=\lambda^2-\frac{\hbar^2}{4}(p_x^2+p_y^2+p_z^2)=0 \tag{25}$$

The eigenvalues of $\hat{A}$ is

$$\lambda_{1,2}=\pm\frac{\hbar}{2}\sqrt{p_x^2+p_y^2+p_z^2} \tag{26}$$

Since $\hat{A}$ has two different eigenvalues given p_x, p_y and p_z fixed, it has two linear-independent eigenvector in the basis of $\{|p_xp_yp_z,\pm\rangle\}$, which are also the eigenvectors of $\hat{p}_x$, $\hat{p}_y$ and $\hat{p}_z$. This means that any arbitrary ket $|p_xp_yp_z,\pm\rangle$ can be wirtten as one and only one linear combination of the eigenvectors of $\hat{A}$. Since $\{|p_xp_yp_z,\pm\rangle\}$ is a basis of the state space, any state can be written as one and only one combination of $|p_xp_yp_z,\pm\rangle$. In this way, any state can be written as one and only one combination of the eigenvectors of $\hat{A}$. Therefore, there exists a basis of eigenvectors of $\hat{A}$ which are also eigenvectors of $\hat{p}_x$, $\hat{p}_y$ and $\hat{p}_z$.

(c) As obtained in (b) above, the eigenvalues of $\hat{A}$ are

$$\lambda_1=\frac{\hbar}{2}\sqrt{p_x^2+p_y^2+p_z^2} \tag{27}$$

$$\lambda_2=-\frac{\hbar}{2}\sqrt{p_x^2+p_y^2+p_z^2} \tag{28}$$

Their degeneracy is infinite since infinite sets of $\{p_x,p_y,p_z\}$ can make equal $\sqrt{p_x^2+p_y^2+p_z^2}$. The eigenvector corresponding to λ_1 is

$$\begin{aligned}|\psi_1\rangle=&\begin{pmatrix} \frac{p_x-ip_y}{\sqrt{2(p_x^2+p_z^2-ip_xp_y-p_z\sqrt{p_x^2+p_y^2+p_z^2})}} \\ \frac{-p_z+\sqrt{p_x^2+p_y^2+p_z^2}}{\sqrt{2(p_x^2+p_z^2-ip_xp_y-p_z\sqrt{p_x^2+p_y^2+p_z^2})}} \end{pmatrix}\\ =&\frac{(p_x-ip_y)|p_xp_yp_z,+\rangle+(p_z-\sqrt{p_x^2+p_y^2+p_z^2})|p_xp_yp_z,-\rangle}{\sqrt{2(p_x^2+p_z^2-ip_xp_y-p_z\sqrt{p_x^2+p_y^2+p_z^2})}}\end{aligned} \tag{29}$$

The eigenvector corresponding to λ_2 is

$$\begin{aligned}|\psi_2\rangle=&\begin{pmatrix} \frac{p_x-ip_y}{\sqrt{2(p_x^2+p_z^2-ip_xp_y+p_z\sqrt{p_x^2+p_y^2+p_z^2})}} \\ \frac{-p_z-\sqrt{p_x^2+p_y^2+p_z^2}}{\sqrt{2(p_x^2+p_z^2-ip_xp_y+p_z\sqrt{p_x^2+p_y^2+p_z^2})}} \end{pmatrix}\\ =&\frac{(p_x-ip_y)|p_xp_yp_z,+\rangle-(p_z+\sqrt{p_x^2+p_y^2+p_z^2})|p_xp_yp_z,-\rangle}{\sqrt{2(p_x^2+p_z^2-ip_xp_y+p_z\sqrt{p_x^2+p_y^2+p_z^2})}}\end{aligned} \tag{30}$$

These eigenvectors are common to $\hat{A}$ and $\hat{p}_x$, $\hat{p}_y$, $\hat{p}_z$.

□

Problem 3. [C-T Exercise 9-3] The Hamiltonian of an electron of mass m, charge q, spin $\hbar\vec{\sigma}/2$ with σ_x, σ_y, and σ_z the Pauli matrices, placed in an electromagnetic field described by the vector potential

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$\hat{A}(\vec{r},t)$ and the scalar potential $U(\vec{r},t)$, is written $\hat{H}=\frac{1}{2m}[\hat{\vec{p}}-q\vec{A}(\hat{\vec{r}},t)]^2+qU(\hat{\vec{r}},t)-\frac{q\hbar}{2m}\vec{\sigma}\cdot\vec{B}(\hat{\vec{r}},t)$. The last term represents the interaction between the spin magnetic moment $(q\hbar/2m)\vec{\sigma}$ and the magnetic field $\vec{B}(\hat{\vec{r}},t)=\vec{\nabla}\times\vec{A}(\hat{\vec{r}},t)$. Show, using the properties of the Pauli matrices, that this Hamiltonian can also be written in the form ("the Pauli Hamiltonian") $\hat{H}=\frac{1}{2m}\left\{\vec{\sigma}\cdot[\hat{\vec{p}}-q\vec{A}(\hat{\vec{r}},t)]\right\}^2+qU(\hat{\vec{r}},t)$.

Solution: Let the Hamiltonian operates on a state function

$$\begin{aligned}\hat{H}\psi=&\frac{1}{2m}[\hat{\vec{p}}-q\vec{A}(\hat{\vec{r}},t)]^2\psi+qU(\hat{\vec{r}},t)\psi-\frac{q\hbar}{2m}\vec{\sigma}\cdot\vec{B}(\hat{\vec{r}},t)\\=&\frac{1}{2m}\{[\hat{\vec{p}}-q\vec{A}(\hat{\vec{r}},t)]^2-iq\vec{\sigma}\cdot[-i\hbar\nabla\times\vec{A}(\hat{\vec{r}},t)]\}\psi+qU(\hat{\vec{r}},t)\psi\\=&\frac{1}{2m}\{[\hat{\vec{p}}-q\vec{A}(\hat{\vec{r}},t)]^2\psi-iq\vec{\sigma}\cdot[\hat{\vec{p}}\times\vec{A}(\hat{\vec{r}},t)]\psi\}+qU(\hat{\vec{r}},t)\psi\\=&\frac{1}{2m}\{[\hat{\vec{p}}-q\vec{A}(\hat{\vec{r}},t)]^2\psi-i\vec{\sigma}\cdot[\hat{\vec{p}}\times q\vec{A}(\hat{\vec{r}},t)]\psi\}+qU(\hat{\vec{r}},t)\psi\end{aligned}\tag{31}$$

where

$$[\hat{\vec{p}}\times q\hat{A}(\hat{\vec{r}},t)]\psi=[\hat{\vec{p}}\times q\hat{A}(\hat{\vec{r}},t)]\psi+(\hat{\vec{p}}\psi)\times q\vec{A}(\hat{\vec{r}},r)-(\hat{\vec{p}}\psi)\times q\vec{A}(\hat{\vec{r}},r)$$

Using

$$\vec{A}\times\vec{B}\phi=(\vec{A}\times\vec{B})\phi+(\hat{A}\phi)\times\vec{B}\tag{32}$$

we have

$$\begin{aligned}[\hat{\vec{p}}\times q\hat{A}(\hat{\vec{r}},t)]\psi=&\hat{\vec{p}}\times q\vec{A}(\hat{\vec{r}},t)\psi-(\hat{\vec{p}}\psi)\times q\vec{A}(\hat{\vec{r}},t)\\=&-[\hat{\vec{p}}\times\hat{\vec{p}}-\vec{p}\times q\vec{A}(\hat{\vec{r}},t)-q\vec{A}(\hat{\vec{r}},t)\times\hat{\vec{p}}+q\vec{A}(\hat{\vec{r}},t)\times q\vec{A}(\hat{\vec{r}},t)]\psi\\=&-[\hat{\vec{p}}-q\vec{A}(\hat{\vec{r}},t)]\times[\hat{\vec{p}}-q\vec{A}(\hat{\vec{r}},t)]\psi\end{aligned}\tag{33}$$

Plugging the equation above into the equation (31) gives

$$\hat{H}\psi=\frac{1}{2m}\{[\hat{\vec{p}}-q\vec{A}(\hat{\vec{r}},t)]^2+i\vec{\sigma}\cdot[\hat{\vec{p}}-q\vec{A}(\hat{\vec{r}},t)]\times[\hat{\vec{p}}-q\vec{A}(\hat{\vec{r}},t)]\}\psi+qU(\hat{\vec{r}},t)\psi\tag{34}$$

Using

$$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B})=\vec{A}\times\vec{B}+i\vec{\sigma}\cdot\vec{A}\times\vec{B}\tag{35}$$

we have

$$\hat{H}\psi=\frac{1}{2m}\{\vec{\sigma}\cdot[\hat{\vec{p}}-q\vec{A}(\hat{\vec{r}},t)]\}^2\psi+qU(\hat{\vec{r}},t)\tag{36}$$

Therefore, the Hamiltonian can be written in the form

$$\hat{H}=\frac{1}{2m}\{\vec{\sigma}\cdot[\hat{\vec{p}}-q\vec{A}(\hat{\vec{r}},t)]\}^2+qU(\hat{\vec{r}},t)\tag{37}$$

□

Problem 4. [C-T Exercise 10-3] Consider a system composed of two spin 1/2 particles whose orbital variables are ignored. The Hamiltonian of the system is $\hat{H}=\omega_1\hat{S}_{1z}+\omega_2\hat{S}_{2z}$, where $\hat{S}_{1z}$ and $\hat{S}_{2z}$ are the projections of the spins $\hat{S}_1$ and $\hat{S}_2$ of the two particles onto Oz, and ω_1 and ω_2 are real constants.

(a) The initial state of the system, at time $t=0$, is $|\psi(0)\rangle=\frac{1}{\sqrt{2}}[|+-\rangle+|-+\rangle]$. At time t, $\hat{\vec{S}}^2=(\hat{\vec{S}}_1+\hat{\vec{S}}_2)^2$ is measured. What results can be found, and with what probabilities?

(b) If the initial state of the system is arbitrary, what Bohr frequencies can appear in the evolution of $\langle\hat{\vec{S}}^2\rangle$? Same question for $\hat{S}_x=\hat{S}_{1x}+\hat{S}_{2x}$.

Solution:

(a) The energy of the two eigenstates $|+-\rangle$ and $|-+\rangle$ are

$$E_{+-}=\langle +-\hat{H}|+-\rangle=\langle +-|(\omega_1\hat{S}_{1z}+\omega_2\hat{S}_{2z})|+-\rangle=\frac{\hbar}{2}(\omega_1-\omega_2) \tag{38}$$

$$E_{-+}=\langle -+\hat{H}|-+\rangle=\langle -+|(\omega_1\hat{S}_{1z}+\omega_2\hat{S}_{2z})|-+\rangle=\frac{\hbar}{2}(-\omega_1+\omega_2) \tag{39}$$

At time t, the state of the system is

$$\begin{aligned}|\psi(t)\rangle=&\frac{1}{\sqrt{2}}[e^{iE_{+-}t/\hbar}|+-\rangle+e^{iE_{-+}t/\hbar}|-+\rangle]\\=&\frac{1}{\sqrt{2}}[e^{i(\omega_1-\omega_2)t/2}|+-\rangle+e^{i(-\omega_1+\omega_2)t/2}|-+\rangle]\end{aligned} \tag{40}$$

If $\hat{\vec{S}}^2$ is measured, result $2\hbar^2$ can be found with probability

$$\begin{aligned}\mathscr{P}(\hat{\vec{S}}^2=2\hbar^2)=&|\langle 11|\psi(t)\rangle|^2+|\langle 10|\psi(t)\rangle|^2+|\langle 1,-1|\psi(t)\rangle|^2\\=&\frac{1}{2}[|\langle ++|(e^{i(\omega_1-\omega_2)t/2}|+-\rangle+e^{i(-\omega_1+\omega_2)t/2}|-+\rangle)|^2\\&+|\frac{1}{\sqrt{2}}(\langle +-|+\langle +-|)(e^{i(\omega_1-\omega_2)t/2}|+-\rangle+e^{i(-\omega_1+\omega_2)t/2}|-+\rangle)|^2\\&+|\langle --|(e^{i(\omega_1-\omega_2)t/2}|+-\rangle+e^{i(-\omega_1+\omega_2)t/2}|-+\rangle)|^2]\\=&\frac{1}{2}|\frac{1}{\sqrt{2}}(e^{i(\omega_1-\omega_2)t/2}+e^{i(-\omega_1+\omega_2)t/2})|^2\\=&\cos^2\frac{\omega_1-\omega_2}{2}t\end{aligned} \tag{41}$$

Result 0 can be found with probability

$$\mathscr{P}(\hat{\vec{S}}^2=0)=1-\mathscr{P}(\hat{\vec{S}}^2=2\hbar^2)=\sin^2\frac{\omega_1-\omega_2}{2}t$$

(b) Using Ehrenfest Theorem we have

$$\frac{d\langle\hat{\vec{S}}^2\rangle}{dt}=\frac{1}{i\hbar}\langle[\hat{H},\hat{\vec{S}}^2]\rangle \tag{42}$$

Since

$$\begin{aligned}[\hat{H},\hat{\vec{S}}^2]=&[\omega_1\hat{S}_{1z}+\omega_2\hat{S}_{2z},(\hat{\vec{S}}_1+\hat{\vec{S}}_2)^2]\\=&\omega_1\{[\hat{S}_{1z},\hat{\vec{S}}_1^2]+[\hat{S}_{1z},\hat{\vec{S}}_1\hat{\vec{S}}_2]+[\hat{S}_{1z},\hat{\vec{S}}_2\hat{\vec{S}}_1]+[\hat{S}_{1z},\hat{\vec{S}}_2^2]\}\\&+\omega_2\{[\hat{S}_{2z},\hat{\vec{S}}_1^2]+[\hat{S}_{2z},\hat{\vec{S}}_1\hat{\vec{S}}_2]+[\hat{S}_{2z},\hat{\vec{S}}_2\hat{\vec{S}}_1]+[\hat{S}_{2z},\hat{\vec{S}}_2^2]\}\\=&0\end{aligned} \tag{43}$$

$\hat{H}$ and $\hat{\vec{S}}^2$ commute, we have

$$\frac{d\langle\hat{\vec{S}}^2\rangle}{dt}=0 \tag{44}$$

Therefore, $\langle\hat{\vec{S}}^2\rangle$ remains a constant and does not evolve with time and no Bohr frequency can appear.

As for the same question for $\langle\hat{S}_x\rangle=\langle\hat{S}_{1x}+\hat{S}_{2x}\rangle$: the initial arbitrary state of the system can be written as

$$|\psi(0)\rangle=\alpha|++\rangle+\beta|+-\rangle+\gamma|-+\rangle+\delta|--\rangle \tag{45}$$

The energy of the four eigenstates are

$$E_{++} = \frac{\hbar}{2}(\omega_1 + \omega_2) \tag{46}$$

$$E_{+-} = \frac{\hbar}{2}(\omega_1 - \omega_2) \tag{47}$$

$$E_{-+} = \frac{\hbar}{2}(-\omega_1 + \omega_2) \tag{48}$$

$$E_{--} = \frac{\hbar}{2}(-\omega_1 - \omega_2) \tag{49}$$

The state of the system at time t is

$$\begin{aligned}|\psi(t)\rangle =&\alpha e^{iE_{++}t/\hbar}|++\rangle + \beta e^{iE_{+-}t/\hbar}|+-\rangle + \gamma e^{iE_{-+}t/\hbar}|-+\rangle + \delta e^{iE_{--}t/\hbar}|--\rangle\\ =&\alpha e^{i(\omega_1+\omega_2)t/2}|++\rangle + \beta e^{i(\omega_1-\omega_2)t/2}|+-\rangle + \gamma e^{i(-\omega_1+\omega_2)t/2}|-+\rangle + \delta e^{i(-\omega_1-\omega_2)t/2}|--\rangle\end{aligned} \tag{50}$$

$$\begin{aligned}\langle \hat{S}_x\rangle =&\langle\psi(t)|\hat{S}_x|\psi(t)\rangle\\ =&(\alpha^* e^{-i(\omega_1+\omega_2)t/2}\langle ++| + \beta^* e^{-i(\omega_1-\omega_2)t/2}\langle +-| + \gamma^* e^{-i(-\omega_1+\omega_2)t/2}\langle -+| + \delta^* e^{-i(-\omega_1-\omega_2)t/2}\langle --|)\\ &(\hat{S}_{1x} + \hat{S}_{2x})\\ &(\alpha e^{i(\omega_1+\omega_2)t/2}|++\rangle + \beta e^{i(\omega_1-\omega_2)t/2}|+-\rangle + \gamma e^{i(-\omega_1+\omega_2)t/2}|-+\rangle + \delta e^{i(-\omega_1-\omega_2)t/2}|--\rangle)\\ =&(\alpha^* e^{-i(\omega_1+\omega_2)t/2}\langle ++| + \beta^* e^{-i(\omega_1-\omega_2)t/2}\langle +-| + \gamma^* e^{-i(-\omega_1+\omega_2)t/2}\langle -+| + \delta^* e^{-i(-\omega_1-\omega_2)t/2}\langle --|)\\ &(\frac{\hat{S}_{1+}+\hat{S}_{1-}}{2} + \frac{\hat{S}_{2+}+\hat{S}_{2-}}{2})\\ &(\alpha e^{i(\omega_1+\omega_2)t/2}|++\rangle + \beta e^{i(\omega_1-\omega_2)t/2}|+-\rangle + \gamma e^{i(-\omega_1+\omega_2)t/2}|-+\rangle + \delta e^{i(-\omega_1-\omega_2)t/2}|--\rangle)\\ =&\frac{\hbar}{2}[(\alpha^*\gamma e^{-i\omega_1 t} + \alpha\gamma^* e^{i\omega_1 t}) + (\beta^*\delta e^{-i\omega_1 t} + \beta\delta^* e^{i\omega_1 t}) + (\alpha^*\beta e^{-i\omega_2 t} + \alpha\beta^* e^{i\omega_2 t}) + (\gamma^*\delta e^{-i\omega_2 t} + \gamma\delta^* e^{i\omega_2 t})]\\ =&\frac{\hbar}{2}[(\mathrm{Re}(\alpha^*\gamma) + \mathrm{Re}(\beta^*\delta))\cos\omega_1 t + (\mathrm{Im}(\alpha^*\gamma) + \mathrm{Im}(\beta^*\delta))\sin\omega_1 t\\ &+ (\mathrm{Re}(\alpha^*\beta) + \mathrm{Re}(\gamma^*\delta))\cos\omega_2 t + (\mathrm{Im}(\alpha^*\beta) + \mathrm{Im}(\gamma^*\delta))\sin\omega_2 t]\end{aligned} \tag{51}$$

Both ω_1 and ω_2 can appear in the revolution of $\langle\hat{S}_x\rangle$.

Reference: https://phys.cst.temple.edu/∼meziani/homework3s_5702_2016.pdf □

Problem 5. [C-T Exercise 10-5] Let $\hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2 + \hat{\vec{S}}_3$ be the total angular momentum of three spin 1/2 particles (whose orbital variables will be ignored). Let $|\varepsilon_1\varepsilon_2\varepsilon_3\rangle$ be the eigenstates common to $\hat{S}_{1z}$, $\hat{S}_{2z}$, and $\hat{S}_{3z}$, of respective eigenvalues $\varepsilon_1\hbar/2$, $\varepsilon_2\hbar/2$, and $\varepsilon_3\hbar/2$. Give a basis of eigenvectors common to $\hat{\vec{S}}^2$ and $\hat{S}_z$, in terms of the kets $|\varepsilon_1\varepsilon_2\varepsilon_3\rangle$. Do these two operators form a CSCO? (Begin by adding two of the spins, then add the partial angular momentum so obtained to the third one.)

Solution: Add two of the spin $\hat{\vec{S}}_1$ and $\hat{\vec{S}}_2$ first. Let $\hat{S}_{12} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$, the eigenvecors common to $\hat{\vec{S}}_{12}^2$ and $\hat{\vec{S}}_{12z}$ are $|s_{12}m_{12}\rangle$.

$$\hat{\vec{S}}^2|s_{12}m_{12}\rangle = s_{12}(s_{12}+1)\hbar^2|s_{12}m_{12}\rangle \tag{52}$$

$$\hat{S}_{12z}|s_{12}m_{12}\rangle = m_{12}\hbar|s_{12}m_{12}\rangle \tag{53}$$

In subspace $\mathscr{E}(s_{12}=1)$, $m_{12} = 1, 0, -1$. Let

$$|11\rangle = |++\rangle \tag{54}$$

Then

$$\hat{S}_{12-}|11\rangle = \hbar\sqrt{1(1+1)-1(1-1)}|10\rangle = \hbar\sqrt{2}|10\rangle \tag{55}$$

so

$$|10\rangle = \frac{1}{\hbar\sqrt{2}}\hat{S}_{12-}|11\rangle = \frac{1}{\hbar\sqrt{2}}(\hat{S}_{1-}+\hat{S}_{2-})|++\rangle = \frac{1}{\sqrt{2}}[|+-\rangle+|-+\rangle] \tag{56}$$

And

$$\hat{S}_{12-}|10\rangle = \hbar\sqrt{1(1+1)-0(0-1)}|1,-1\rangle = \hbar\sqrt{2}|1,-1\rangle \tag{57}$$

so

$$|1,-1\rangle = \frac{1}{\hbar\sqrt{2}}\hat{S}_{12-}|10\rangle = \frac{1}{2\hbar}(\hat{S}_{1-}+\hat{S}_{2-})[|+-\rangle+|-+\rangle] = |--\rangle \tag{58}$$

In subspace $\mathscr{E}(s_{12}=0)$, $m_{12}=0$. Since $|00\rangle$ is orthogonal to $|11\rangle = |++\rangle$ and $|1,-1\rangle = |--\rangle$, it is in the form

$$|00\rangle = a_1|+-\rangle + a_2|-+\rangle \tag{59}$$

Since $|00\rangle$ is orthogonal to $|10\rangle = \frac{1}{\sqrt{2}}[|+-\rangle+|-+\rangle]$ and the normalization condition requires $|\alpha|^2+|\beta|^2 = 1$,

$$|00\rangle = \frac{1}{\sqrt{2}}[|+-\rangle-|-+\rangle] \tag{60}$$

Now add the partial angular momentum to obtain the third one: Wirte the eigenvectors common to $\hat{\vec{S}}^2$ and $\hat{S}_z$ as $|sm\rangle$.

$$\hat{\vec{S}}^2|sm\rangle = s(s+1)\hbar^2|sm\rangle\hat{S}_z|sm\rangle = m\hbar|sm\rangle \tag{61}$$

In subspace $\mathscr{E}(s=\frac{3}{2})$, $m=\frac{3}{2},\frac{1}{2},-\frac{1}{2},-\frac{3}{2}$. Let

$$|\frac{3}{2}\frac{3}{2}\rangle = |+++\rangle \tag{62}$$

Then

$$\hat{S}_-|\frac{3}{2}\frac{3}{2}\rangle = \hbar\sqrt{\frac{3}{2}(\frac{3}{2}+1)-\frac{3}{2}(\frac{3}{2}-1)}|\frac{3}{2}\frac{1}{2}\rangle = \hbar\sqrt{3}|\frac{3}{2}\frac{1}{2}\rangle \tag{63}$$

so

$$|\frac{3}{2}\frac{1}{2}\rangle = \frac{1}{\hbar\sqrt{2}}\hat{S}_-|11\rangle = \frac{1}{\hbar\sqrt{3}}(\hat{S}_{1-}+\hat{S}_{2-}+\hat{S}_{3-})|+++\rangle = \frac{1}{\sqrt{3}}[|++-\rangle+|+-+\rangle+|-++\rangle] \tag{64}$$

And

$$\hat{S}_-|\frac{3}{2},\frac{1}{2}\rangle = \hbar\sqrt{\frac{3}{2}(\frac{3}{2}+1)-\frac{1}{2}(\frac{1}{2}-1)}|\frac{3}{2},-\frac{1}{2}\rangle = 2\hbar|\frac{3}{2},-\frac{1}{2}\rangle \tag{65}$$

so

$$\begin{aligned}|\frac{3}{2},-\frac{1}{2}\rangle =&\frac{1}{2\hbar}\hat{S}_-|\frac{3}{2}\frac{1}{2}\rangle = \frac{1}{2\sqrt{3}\hbar}(\hat{S}_{1-}+\hat{S}_{2-}+\hat{S}_{3-})[|++-\rangle+|+-+\rangle+|-++\rangle]\\ =&\frac{1}{\sqrt{3}}[|+--\rangle+|-+-\rangle+|--+\rangle]\end{aligned} \tag{66}$$

And

$$\hat{S}_-|\frac{3}{2},-\frac{1}{2}\rangle = \hbar\sqrt{\frac{3}{2}(\frac{3}{2}+1)-(-\frac{1}{2})(-\frac{1}{2}-1)} = \hbar\sqrt{3}|\frac{3}{2},-\frac{3}{2}\rangle \tag{67}$$

so

$$|\frac{3}{2},-\frac{3}{2}\rangle = \frac{1}{\hbar\sqrt{3}}\hat{S}_-|\frac{3}{2},-\frac{1}{2}\rangle = \frac{1}{3\hbar}(\hat{S}_{1-}+\hat{S}_{2-}+\hat{S}_{3-})[|+--\rangle+|-+-\rangle+|--+\rangle] = |---\rangle \tag{68}$$

In subspace $\mathscr{E}(s=\frac{1}{2})$, $m=\frac{1}{2},-\frac{1}{2}$. $|\frac{1}{2}\frac{1}{2}\rangle$ is orthogonal to $|\frac{3}{2}\frac{3}{2}\rangle$, $|\frac{3}{2}\frac{1}{2}\rangle$, $|\frac{3}{2},-\frac{1}{2}\rangle$, $|\frac{3}{2},-\frac{3}{2}\rangle$ and is normalized, so it can be

$$|\frac{1}{2}\frac{1}{2}\rangle_1 = \frac{1}{\sqrt{2}}[|++-\rangle-|-++\rangle] \tag{69}$$

or

$$|\frac{1}{2}\frac{1}{2}\rangle_2 = -\frac{1}{\sqrt{6}}|++-\rangle + \frac{2}{\sqrt{6}}|+-+\rangle - \frac{1}{\sqrt{6}}|-++\rangle \tag{70}$$

(degree of degeneracy is 2.)

Then

$$\hat{S}_-|\frac{1}{2}\frac{1}{2}\rangle_{1,2} = \sqrt{\frac{1}{2}(\frac{1}{2}+1) - \frac{1}{2}(\frac{1}{2}-\frac{1}{2})} = \hbar|\frac{1}{2},-\frac{1}{2}\rangle_{1,2} \tag{71}$$

so

$$|\frac{1}{2},-\frac{1}{2}\rangle_1 = \frac{1}{\hbar\sqrt{2}}\hat{S}_-|\frac{1}{2}\frac{1}{2}\rangle_1 = \frac{1}{\hbar\sqrt{2}}(\hat{S}_{1-}+\hat{S}_{2-}+\hat{S}_{3-})[|++-\rangle - |-++\rangle] = \frac{1}{\sqrt{2}}[|+--\rangle - |--+\rangle] \tag{72}$$

or

$$\begin{aligned}|\frac{1}{2},-\frac{1}{2}\rangle_2 =&\frac{1}{\hbar}\hat{S}_-|\frac{1}{2}\frac{1}{2}\rangle = \frac{1}{\hbar}(\hat{S}_{1-}+\hat{S}_{2-}+\hat{S}_{3-})[-\frac{1}{\sqrt{6}}|++-\rangle + \frac{2}{\sqrt{6}}|+-+\rangle - \frac{1}{\sqrt{6}}|-++\rangle]\\ =&\frac{1}{\sqrt{6}}|+--\rangle - \frac{2}{\sqrt{6}}|-+-\rangle + \frac{1}{\sqrt{6}}|--+\rangle\end{aligned} \tag{73}$$

These two operators do not form a CSCO. □