Quantum Mechanics
Homework Assignment 10
Fall, 2019

1. **[C-T Exercise 4-1]** Consider a spin 1/2 particle of magnetic moment $\hat{\vec{M}} = \gamma\hat{\vec{S}}$. The spin state space is spanned by the basis of the $|+\rangle$ and $|-\rangle$ vectors, eigenvectors of $\hat{S}_z$ with eigenvalues $+\hbar/2$ and $-\hbar/2$. At time $t = 0$, the state of the system is $|\psi(t = 0)\rangle = |+\rangle$.

   (a) If the observable $\hat{S}_x$ is measured at time $t = 0$, what results can be found, and with what probabilities?

   (b) Instead of performing the preceding measurement, we let the system evolve under the influence of a magnetic field parallel to Oy, of modulus B_0. Calculate, in the $\{|+\rangle, |-\rangle\}$ basis, the state of the system at time t.

   (c) At this time t, we measure the observables $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$. What values can we find, and with what probabilities? What relation must exist between B_0 and t for the result of one of the measurements to be certain? Give a physical interpretation of this condition.

2. **[C-T Exercise 4-3]** Consider a spin 1/2 particle placed in a magnetic field $\vec{B}_0$ with components $B_x = B_0/\sqrt{2}$, $B_y = 0$, and $B_z = B_0/\sqrt{2}$. The notation is the same as that of Problem 1.

   (a) Calculate the matrix representing, in the $\{|+\rangle, |-\rangle\}$ basis, the operator $\hat{H}$, the Hamiltonian of the system.

   (b) Calculate the eigenvalues and the eigenvectors of $\hat{H}$.

   (c) The system at time $t = 0$ is in the state $|-\rangle$. What values can be found if the energy is measured, and with what probabilities?

   (d) Calculate the state vector $|\psi(t)\rangle$ at time t. At this instant, $\hat{S}_x$ is measured; what is the mean value of the results that can be obtained? Give a geometrical interpretation.

3. **[C-T Exercise 4-6]** Consider the system composed of two spin 1/2's, $\hat{\vec{S}}_1$ and $\hat{\vec{S}}_2$, and the basis of four vectors $|\pm,\pm\rangle$. The system at time $t = 0$ is in the state

$$|\psi(0)\rangle = \frac{1}{2}\,|++\rangle + \frac{1}{2}\,|+-\rangle + \frac{1}{\sqrt{2}}\,|--\rangle\,.$$

   (a) At time $t = 0$, $\hat{S}_{1z}$ is measured; what is the probability of finding $-\hbar/2$? What is the state vector after this measurement? If we then measure $\hat{S}_{1x}$, what results can be found, and with what probabilities?

   (b) When the system is in the state $|\psi(0)\rangle$ written above, $\hat{S}_{1z}$ and $\hat{S}_{2z}$ are measured simultaneously. What is the probability of finding opposite results? Identical results?

   (c) Instead of performing the preceding measurements, we let the system evolve under the influence of the Hamiltonian $\hat{H} = \omega_1\hat{S}_{1z} + \omega_2\hat{S}_{2z}$. What is the state vector $|\psi(t)\rangle$ at time t? Calculate at time t the mean values $\langle\hat{\vec{S}}_1\rangle$ and $\langle\hat{\vec{S}}_2\rangle$. Give a physical interpretation.

   (d) Show that the lengths of the vectors $\langle\hat{\vec{S}}_1\rangle$ and $\langle\hat{\vec{S}}_2\rangle$ are less than $\hbar/2$. What must be the form of $|\psi(0)\rangle$ for each of these lengths to be equal to $+\hbar/2$?

4. **[C-T Exercise 5-7]** Consider a one-dimensional harmonic oscillator of Hamiltonian $\hat{H}$ and stationary states $|\varphi_n\rangle$, $\hat{H}\,|\varphi_n\rangle = (n + 1/2)\hbar\omega\,|\varphi_n\rangle$. The operator $\hat{U}(k)$ is defined by $\hat{U}(k) = e^{ik\hat{x}}$, where k is real.

   (a) Is $\hat{U}(k)$ unitary? Show that, for all n, its matrix elements satisfy the relation

$$\sum_{n'}\left|\langle\varphi_n|\hat{U}(k)|\varphi_{n'}\rangle\right|^2 = 1.$$

   (b) Express $\hat{U}(k)$ in terms of the operators $\hat{a}$ and $\hat{a}^\dagger$. Use Glauber's formula to put $\hat{U}(k)$ in the form of a product of exponential operators.

(c) Establish the relations

$$e^{\lambda \hat{a}} |\varphi_0\rangle = |\varphi_0\rangle,$$
$$\langle\varphi_n|e^{\lambda \hat{a}^\dagger}|\varphi_0\rangle = \frac{\lambda^n}{\sqrt{n!}},$$

where λ is an arbitrary complex parameter.

(d) Find the expression, in terms of $E_k = \hbar^2 k^2/2m$ and $E_\omega = \hbar\omega$, for the matrix element $\langle\varphi_0|\hat{U}(k)|\varphi_n\rangle$. What happens when k approaches zero? Could this result have been predicted directly?

5. **[C-T Exercise 5-8]** The evolution operator $\hat{U}(t,0)$ of a one-dimensional harmonic oscillator is written $\hat{U}(t,0) = e^{-i\hat{H}t/\hbar}$ with $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)$.

(a) Consider the operators $\hat{\hat{a}}(t) = \hat{U}^\dagger(t,0)\hat{a}\hat{U}(t,0)$ and $\hat{\hat{a}}^\dagger(t) = \hat{U}^\dagger(t,0)\hat{a}^\dagger\hat{U}(t,0)$. By calculating their action on the eigenkets $|\varphi_n\rangle$ of $\hat{H}$, find the expression for $\hat{\hat{a}}(t)$ and $\hat{\hat{a}}^\dagger(t)$ in terms of $\hat{a}$ and $\hat{a}^\dagger$.

(b) Calculate the operators $\hat{\hat{x}}(t)$ and $\hat{\hat{p}}_x(t)$ obtained from $\hat{x}$ and $\hat{p}_x$ by the unitary transformation $\hat{\hat{x}}(t) = \hat{U}^\dagger(t,0)\hat{x}\hat{U}(t,0)$ and $\hat{\hat{p}}_x(t) = \hat{U}^\dagger(t,0)\hat{p}_x\hat{U}(t,0)$. How can the relations so obtained be interpreted?

(c) Show that $\hat{U}^\dagger(\pi/2\omega,0)\,|x\rangle$ is an eigenvector of $\hat{p}_x$ and specify its eigenvalue. Similarly, establish that $\hat{U}^\dagger(\pi/2\omega,0)\,|p_x\rangle$ is an eigenvector of $\hat{x}$.

(d) At $t=0$, the wave function of the oscillator is $\psi(x,0)$. How can one obtain from $\psi(x,0)$ the wave function of the oscillator at all subsequent times $t_q = q\pi/2\omega$ (where q is a positive integer)?

(e) Choose for $\psi(x,0)$ the wave function $\varphi_n(x)$ associated with a stationary state. From the preceding question derive the relation which must exist between $\varphi_n(x)$ and its Fourier transform $\overline{\varphi}_n(p_x)$.