Name: 陈稼霖
StudentID: 45875852

PHYS1501
Quantum Mechanics
Semester Fall 2019
Assignment 6

Problem 1. In a given representation, the matrix representing the Hamiltonian of a particle is given by

$$H = \hbar\omega_0 \begin{pmatrix} -1+\varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & -1-\varepsilon & \sqrt{2}\varepsilon & 0 & 0 & 0 \\ 0 & \sqrt{2}\varepsilon & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & \sqrt{2}\varepsilon & 0 \\ 0 & 0 & 0 & \sqrt{2}\varepsilon & -1-\varepsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & -1+\varepsilon \end{pmatrix}$$

with $0 < \varepsilon < 1$. Find the energy eigenvalues and eigenfunctions of the particle in the representation.

Solution: For convenience, let eigenvalue $\lambda = \hbar\omega_0\lambda'$.The characteristic equation

$$\begin{aligned}
|H-\lambda I| =& \hbar\omega_0 \begin{vmatrix} -1+\varepsilon-\lambda' & 0 & 0 & 0 & 0 & 0 \\ 0 & -1-\varepsilon-\lambda' & \sqrt{2}\varepsilon & 0 & 0 & 0 \\ 0 & \sqrt{2}\varepsilon & -1-\lambda' & 0 & 0 & 0 \\ 0 & 0 & 0 & -1-\lambda' & \sqrt{2}\varepsilon & 0 \\ 0 & 0 & 0 & \sqrt{2}\varepsilon & -1-\varepsilon-\lambda' & 0 \\ 0 & 0 & 0 & 0 & 0 & -1+\varepsilon-\lambda' \end{vmatrix} \\
=& \hbar\omega_0[(-1+\varepsilon-\lambda')(-1-\varepsilon-\lambda')(-1-\lambda')(-1-\lambda')(-1-\varepsilon-\lambda')(-1+\varepsilon-\lambda') \\
&-(-1+\varepsilon-\lambda')\sqrt{2}\varepsilon\sqrt{2}\varepsilon(-1-\lambda')(-1-\varepsilon-\lambda')(-1+\varepsilon-\lambda') \\
&-(-1+\varepsilon-\lambda')(-1-\varepsilon-\lambda')(-1-\lambda')\sqrt{2}\varepsilon\sqrt{2}\varepsilon(-1+\varepsilon-\lambda') \\
&+(-1+\varepsilon-\lambda')\sqrt{2}\varepsilon\sqrt{2}\varepsilon\sqrt{2}\varepsilon\sqrt{2}\varepsilon(-1+\varepsilon-\lambda')] \\
=& \hbar\omega_0(-1+\varepsilon-\lambda')^4(-1-2\varepsilon-\lambda')^2 = 0
\end{aligned} \tag{1}$$

gives

$$\lambda_1' = \lambda_2' = \lambda_3' = \lambda_4' = -1+\varepsilon, \quad \lambda_5' = \lambda_6' = -1-2\varepsilon \tag{2}$$

so the eigenvalues are

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \hbar\omega_0(-1+\varepsilon), \quad \lambda_3 = \lambda_4 = \hbar\omega_0(-1-2\varepsilon) \tag{3}$$

When the eigenvalue $\lambda = \hbar\omega_0(-1+\varepsilon)$,

$$(H-\lambda I)\psi = \hbar\omega_0 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\varepsilon & \sqrt{2}\varepsilon & 0 & 0 & 0 \\ 0 & \sqrt{2}\varepsilon & -\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon & \sqrt{2}\varepsilon & 0 \\ 0 & 0 & 0\sqrt{2}\varepsilon & -2\varepsilon & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4}$$

gives four independent normalized eigenvectors

$$\psi_1=\frac{1}{\sqrt{7}}\begin{pmatrix}1\\1\\\sqrt{2}\\\sqrt{2}\\1\\0\end{pmatrix},\quad \psi_2=\frac{1}{\sqrt{7}}\begin{pmatrix}0\\1\\\sqrt{2}\\\sqrt{2}\\1\\1\end{pmatrix},\quad \psi_3=\frac{1}{\sqrt{7}}\begin{pmatrix}1\\1\\\sqrt{2}\\-\sqrt{2}\\-1\\0\end{pmatrix},\quad \psi_4=\frac{1}{\sqrt{7}}\begin{pmatrix}0\\1\\\sqrt{2}\\-\sqrt{2}\\-1\\1\end{pmatrix} \tag{5}$$

When eigenvalue $\lambda=\hbar\omega_0(-1-2\varepsilon)$,

$$(H-\lambda I)\psi=\hbar\omega_0\begin{pmatrix}3\varepsilon & 0 & 0 & 0 & 0 & 0\\ 0 & \varepsilon & \sqrt{2}\varepsilon & 0 & 0 & 0\\ 0 & \sqrt{2}\varepsilon & 2\varepsilon & 0 & 0 & 0\\ 0 & 0 & 0 & 2\varepsilon & \sqrt{2}\varepsilon & 0\\ 0 & 0 & 0\sqrt{2}\varepsilon & 2\varepsilon & 0 & \\ 0 & 0 & 0 & 0 & 0 & 3\varepsilon\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\\c_4\\c_5\\c_6\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix} \tag{6}$$

gives two independent normalized eigenvectors

$$\psi_5=\frac{1}{\sqrt{6}}\begin{pmatrix}0\\\sqrt{2}\\-1\\-1\\\sqrt{2}\\0\end{pmatrix},\quad \psi_6=\frac{1}{\sqrt{6}}\begin{pmatrix}0\\\sqrt{2}\\-1\\1\\-\sqrt{2}\\0\end{pmatrix} \tag{7}$$

□

Problem 2. [C-T exercise 2-4] Let $\hat{K}$ be the operator defined by $\hat{K}=|\varphi\rangle\langle\psi|$, where $|\varphi\rangle$ and $|\psi\rangle$ are two vectors of the state space.

(a) Under what condition is $\hat{K}$ Hermitian?

(b) Calculate $\hat{K}^2$. Under what condition is $\hat{K}$ a projector?

(c) Show that $\hat{K}$ can always be written in the form $\hat{K}=\lambda\hat{P}_1\hat{P}_2$ where λ is a constant to be calculated and $\hat{P}_1$ and $\hat{P}_2$ are projectors.

Solution:

(a) The definition of Hermitian

$$\hat{K}=\hat{K}^\dagger \tag{8}$$

Plugging in definition of $\hat{H}$, we get

$$|\varphi\rangle\langle\psi| = (|\varphi\rangle\langle\psi|)^\dagger \tag{9}$$

Using the relation

$$(|\varphi\rangle\langle\psi|)^\dagger = |\psi\rangle\langle\varphi| \tag{10}$$

we get

$$|\varphi\rangle\langle\psi| = |\psi\rangle\langle\varphi| \tag{11}$$

Therefore, $\hat{H}$ is Hermitian when $|\varphi\rangle\langle\psi| = |\psi\rangle\langle\varphi|$.

(b)

$$\hat{K}^2 = |\varphi\rangle\langle\psi|\varphi\rangle\langle\psi| \tag{12}$$

The definition of projector

$$\hat{K}^2 = |\varphi\rangle\langle\psi|\varphi\rangle\langle\psi| = \hat{K} = |\varphi\rangle\langle\psi| \tag{13}$$

$$\Longrightarrow \langle\psi|\varphi\rangle = 1 \tag{14}$$

Therefore, $\hat{K}$ is a projector when $\langle\psi|\varphi\rangle = 1$.

(c) Rewrite $\hat{K}$ as

$$\hat{K} = \frac{|\varphi\rangle\langle\varphi|\psi\rangle\langle\psi|}{\langle\varphi|\psi\rangle} \tag{15}$$

Therefore, $\hat{K}$ can always be written in the form

$$\hat{K} = \lambda\hat{P}_1\hat{P}_2 \tag{16}$$

where

$$\lambda = \frac{1}{\langle\varphi|\psi\rangle} \tag{17}$$

is a constant to be calculated and

$$\hat{P}_1 = |\varphi\rangle\langle\varphi| \tag{18}$$

$$\hat{P}_2 = |\psi\rangle\langle\psi| \tag{19}$$

are projectors.

□

Problem 3. [C-T exercise 2-5] Let $\hat{P}_1$ be the orthogonal projector onto the subspace $\mathcal{E}_1$, $\hat{P}_2$ the orthogonal projector on to the subspace $\mathcal{E}_2$. Show that, for the product $\hat{P}_1\hat{P}_2$ to be an orthogonal projector as well, it is necessary and sufficient that $\hat{P}_1$ and $\hat{P}_2$ commute. In this case, what is the subspace onto with $\hat{P}_1\hat{P}_2$ projects?

Solution: Since $\hat{P}_1$ and $\hat{P}_2$ are orthogonal projectors,

$$\text{orthogonal: } \hat{P}_1\hat{P}_1^T = I \tag{20}$$

$$\hat{P}_2\hat{P}_2^T = I \tag{21}$$

$$\text{projector: } \hat{P}_1\hat{P}_1 = \hat{P} \tag{22}$$

$$\hat{P}_2\hat{P}_2 = \hat{P}_2 \tag{23}$$

First, let's show that $\hat{P}_1$ and $\hat{P}_2$ commute is necessary for the product $\hat{P}_1\hat{P}_2$ to be orthogonal projector:

Suppose $\hat{P}_1\hat{P}_2$ is orthogonal projector,

$$\text{orthogonal: } (\hat{P}_1\hat{P}_2)(\hat{P}_1\hat{P}_2)^T = I \tag{24}$$

$$\text{projector: } (\hat{P}_1\hat{P}_2)(\hat{P}_1\hat{P}_2) = (\hat{P}_1\hat{P}_2) \tag{25}$$

Premultiplying the equation (25) with $\hat{P}_1^T\hat{P}_2^T$ at both side gives

$$\hat{P}_1\hat{P}_2 = \hat{P}_2^T\hat{P}_1^T(\hat{P}_1\hat{P}_2)(\hat{P}_1\hat{P}_2) = \hat{P}_2^T\hat{P}_1^T(\hat{P}_1\hat{P}_2) = I \tag{26}$$

Premultiplying the equation (25) with $\hat{P}_1^T$ and postmultiplying it with $\hat{P}_2^T$ at both side gives

$$\hat{P}_2\hat{P}_1 = \hat{P}_1^T(\hat{P}_1\hat{P}_2)(\hat{P}_1\hat{P}_2)\hat{P}_2^T = \hat{P}_1^T(\hat{P}_1\hat{P}_2)\hat{P}_2^T = I \tag{27}$$

So

$$[\hat{P}_1, \hat{P}_2] = \hat{P}_1\hat{P}_2 - \hat{P}_2\hat{P}_1 = I - I = 0 \tag{28}$$

$\hat{P}_1$ and $\hat{P}_2$ commute.

Next, let's show that $\hat{P}_1$ and $\hat{P}_2$ commute is sufficient for the product $\hat{P}_1\hat{P}_2$ to be orthogonal projector:

Suppose $\hat{P}_1$ and $\hat{P}_2$ commute,

$$[\hat{P}_1, \hat{P}_2] = \hat{P}_1\hat{P}_2 - \hat{P}_2\hat{P}_1 = 0 \tag{29}$$

$$\Longrightarrow \hat{P}_1\hat{P}_2 = \hat{P}_2\hat{P}_1 \tag{30}$$

Then we have

$$(\hat{P}_1\hat{P}_2)(\hat{P}_1\hat{P}_2) = \hat{P}_1\hat{P}_2\hat{P}_1\hat{P}_2 = \hat{P}_1\hat{P}_1\hat{P}_2\hat{P}_2 = (\hat{P}_1\hat{P}_1)(\hat{P}_2\hat{P}_2) = \hat{P}_1\hat{P}_2 \tag{31}$$

so $\hat{P}_1\hat{P}_2$ is a projector.

And

$$(\hat{P}_1\hat{P}_2)(\hat{P}_1\hat{P}_2)^T = \hat{P}_1\hat{P}_2\hat{P}_2^T\hat{P}_1^T = \hat{P}_1(\hat{P}_2\hat{P}_2^T)\hat{P}_1^T = \hat{P}_1\hat{P}_1^T = I \tag{32}$$

so $\hat{P}_1\hat{P}_2$ is orthogonal.

Therefore, for the product $\hat{P}_1\hat{P}_2$ to be an orthogonal projector as well, it is necessary and sufficient that $\hat{P}_1$ and $\hat{P}_2$ commute.

The subspace onto which $\hat{P}_1\hat{P}_2$ projects is $\mathcal{E}_1 \otimes \mathcal{E}_2$ □

Problem 4. [C-T exercise 2-11] Consider a physical system whose three-dimensional state space is spanned by the orthogonal basis formed by the three kets $|u_1\rangle$, $|u_2\rangle$, and $|u_3\rangle$. In the basis of these three vectors, taken in this order, the two operators $\hat{H}$ and $\hat{B}$ are defined by

$$H = \hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

where ω_0 and b are real constants.

(a) Are H and B Hermitian?

(b) Show that H and B commute. Give a basis of eigenvectors common to H and B.

Solution:

(a)

$$H^\dagger = (H^T)^* = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = H \tag{33}$$

so H is Hermitian.

$$B^\dagger = (B^T)^* = b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = B \tag{34}$$

so B is Hermitian.

(b)

$$\begin{aligned} [H, B] =& HB - BH \\ =& \hbar\omega_0 b \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - \hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ =& \hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} - \hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} = 0 \end{aligned} \tag{35}$$

so H and B commute.

Since H and B commute, they have common eigenspace. Let's first find the eigenvectors of H. Let $\lambda_H=\hbar\omega_0\lambda'_H$, then the characteristic equation of H

$$\begin{aligned}|H-\lambda_H I|=&|H-\hbar\omega\lambda'_H I|=\hbar\omega_0\begin{vmatrix}1-\lambda' & 0 & 0\\ 0 & -1-\lambda' & 0\\ 0 & 0 & -1-\lambda'\end{vmatrix}\\ =&\hbar\omega_0(1-\lambda'_H)(-1-\lambda'_H)^2=0\end{aligned}\tag{36}$$

gives

$$\lambda'_{H1}=1,\quad \lambda'_{H2}=\lambda'_{H3}=-1\tag{37}$$

so the eigenvalues of H

$$\lambda_{H1}=\hbar\omega_0,\quad \lambda_{H2}=\lambda_{H3}=-\hbar\omega_0\tag{38}$$

When the eigenvalue of H $\lambda_H=\hbar\omega_0$,

$$(H-\lambda_H I)\psi=\hbar\omega_0\begin{pmatrix}0 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -2\end{pmatrix}\begin{pmatrix}c_1\\ c_2\\ c_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\tag{39}$$

gives one normalized eigenvector

$$\psi_{H1}=\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}\tag{40}$$

When the eigenvalue of H $\lambda_H=-\hbar\omega_0$,

$$(H-\lambda_H I)\psi=\hbar\omega_0\begin{pmatrix}2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}c_1\\ c_2\\ c_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\tag{41}$$

gives two independent normalized eigenvectors

$$\psi_{H2}=\begin{pmatrix}0\\ 1\\ 0\end{pmatrix},\quad \psi_{H3}=\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\tag{42}$$

Obviously, ψ_1 are also an eigenvector of B

$$B\psi_{H1}=b\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}=b\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}=b\psi_{H1}\tag{43}$$

Let $\lambda_B=b\lambda_B'$, then the characteristic equation of B

$$\begin{aligned}|B-\lambda_B I|=&|B-b\lambda_B' I|=b\begin{vmatrix}1-\lambda_B' & 0 & 0\\ 0 & -\lambda_B' & 1\\ 0 & 1 & -\lambda_B'\end{vmatrix}\\ =&-b(\lambda_B'-1)^2(\lambda_B'+1)=0\end{aligned}\tag{44}$$

gives

$$\lambda_{B1}'=\lambda_{B2}'=1,\quad \lambda_{B3}'=-1\tag{45}$$

so the eigenvalues of B

$$\lambda_{B1}'=\lambda_{B2}'=b,\quad \lambda_{B3}'=-b\tag{46}$$

When eigenvalue of B $\lambda_B=b$,

$$(B-\lambda_B I)\psi_B=b\begin{pmatrix}0 & 0 & 0\\ 0 & -1 & 1\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}d_1\\ d_2\\ d_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\tag{47}$$

Besides, $\psi_{B1}=(1,0,0)$, another normalized eigenvector is

$$\psi_{B2}=\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\tag{48}$$

which can be written as a linear combination of eigenvectors of H: $\psi_{B2}=\frac{1}{\sqrt{2}}\psi_{H2}+\frac{1}{\sqrt{2}}\psi_{H3}$.
When eigenvalue of B $\lambda_B=-b$

$$(B-\lambda_B I)\psi_B=b\begin{pmatrix}2 & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}d_1\\ d_2\\ d_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\tag{49}$$

gives one normalized eigenvector

$$\psi_{B2}=\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}\tag{50}$$

which can be written as a linear combination of eigenvectors of H: $\psi_{B3}=\frac{1}{\sqrt{2}}\psi_{H2}-\frac{1}{\sqrt{2}}\psi_{H3}$

Therefore, $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix},\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}\right\}$ is a basis of eigenvectors common to H and B.

□

Problem 5. [C-T exercise 2-12] In the same state space as that of the preceding exercise, consider two operators $\hat{L}_z$ and $\hat{S}$ defined by

$$\hat{L}_z|u_1\rangle = |u_1\rangle, \quad \hat{L}_z|u_2\rangle = 0, \quad \hat{L}_z = -|u_3\rangle;$$
$$\hat{S}|u_1\rangle = |u_3\rangle, \quad \hat{S}|u_2\rangle = |u_2\rangle, \quad \hat{S}|u_3\rangle = |u_1\rangle.$$

(a) Write the matrices which represent, in the $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ basis, the operator $\hat{L}_z$, $\hat{L}_z^2$, $\hat{S}$, and $\hat{S}^2$. Are these operators observables?

(b) Give the form of the most general matrix which represents an operator which commutes with $\hat{L}_z$. Same question for $\hat{L}_z^2$, then $\hat{S}^2$.

(c) Do $\hat{L}_z^2$ and $\hat{S}$ form a CSCO? Give a basis of common eigenvectors.

Solution:

(a) Diagonalize L_z

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \hat{L}_z \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{51}$$

$$\Longrightarrow \hat{L}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{52}$$

so

$$\hat{L}_z^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{53}$$

Diagonalize S

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \hat{S} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{54}$$

$$\Longrightarrow \hat{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{55}$$

so

$$\hat{S}^2=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\tag{56}$$

$$\hat{L}_z^\dagger=(\hat{L}_z^T)^*=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}=\hat{L}_z\tag{57}$$

so $\hat{L}_z$ is Hermitian.

$$(\hat{L}_z^2)^\dagger=[(\hat{L}_z^2)^T]^*=\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}=\hat{L}_z^2\tag{58}$$

so $\hat{L}_z^2$ is Hermitian.

$$\hat{S}^\dagger=(\hat{S}^T)^*=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}=\hat{S}\tag{59}$$

so $\hat{S}$ is Hermitian.

$$(\hat{S}^2)^\dagger=[(\hat{S}^2)^T]^*=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\hat{S}^2\tag{60}$$

so $\hat{S}^2$ is Hermitian.

$$\begin{aligned}\sum_i|u_i\rangle\langle u_i|=&\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0\end{pmatrix}+\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0&1&0\end{pmatrix}\\&+\begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}0&0&1\end{pmatrix}\\=&\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=I\end{aligned}\tag{61}$$

so their orthonormal system of eigenvectors forms basis in the state space. Therefore, these operators are observables.

(b) Suppose the most general matrix

$$A=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\tag{62}$$

If A commute with $\hat{L}_z$

$$\begin{aligned}[A,\hat{L}_z]&=A\hat{L}_z-\hat{L}_zA\\&=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}-\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\\&=\begin{pmatrix}0&-a_{12}&-2a_{12}\\a_{21}&0&-a_{23}\\2a_{31}&a_{32}&0\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\end{aligned}\tag{63}$$

$$\Longrightarrow a_{12}=a_{13}=a_{21}=a_{23}=a_{31}=a_{32}=0\tag{64}$$

so the most general matrix commuting with $\hat{L}_z$ is

$$\begin{pmatrix}a_{11}&0&0\\0&a_{22}&0\\0&0&a_{33}\end{pmatrix}\tag{65}$$

If A commute with $\hat{L}_z^2$

$$\begin{aligned}[A,\hat{L}_z^2]&=A\hat{L}_z^2-\hat{L}_z^2A\\&=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\\&=\begin{pmatrix}0&a_{12}&0\\a_{21}&0&a_{23}\\0&a_{32}&0\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\end{aligned}\tag{66}$$

$$\Longrightarrow a_{12}=a_{21}=a_{23}=a_{32}=0\tag{67}$$

so the most general matrix commuting with $\hat{L}_z^2$ is

$$\begin{pmatrix}a_{11}&0&a_{13}\\0&a_{22}&0\\a_{31}&0&a_{33}\end{pmatrix}\tag{68}$$

If A commute with $\hat{S}$

$$\begin{aligned}
[A,\hat{S}] &= A\hat{S}-\hat{S}A\\
&=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}-\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\\
&=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}
\end{aligned}\tag{69}$$

$$\Longrightarrow a_{11}=a_{12}=a_{13}=a_{21}=a_{22}=a_{23}=a_{31}=a_{32}=a_{33}=0\tag{70}$$

so the most general matrix commuting with $\hat{S}$ is

$$\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\tag{71}$$

If A commute with $\hat{S}^2$

$$\begin{aligned}
[A,\hat{S}^2] &= A\hat{S}^2-\hat{S}^2A\\
&=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\\
&=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}
\end{aligned}\tag{72}$$

$$\Longrightarrow a_{11}=a_{12}=a_{13}=a_{21}=a_{22}=a_{23}=a_{31}=a_{32}=a_{33}=0\tag{73}$$

so the most general matrix commuting with $\hat{S}^2$ is

$$\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\tag{74}$$

(c)

$$\begin{aligned}
[\hat{L}_z^2,\hat{S}] =&\hat{L}_z^2\hat{S}-\hat{S}\hat{L}_z^2=\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}-\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}\\
=&\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}
\end{aligned}\tag{75}$$

so $\hat{L}_z^2$ and $\hat{S}$ commute.

$|u_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is a common eigenvector of $\hat{L}_z^2$ and $\hat{S}$.

$$\hat{L}_z^2|u_2\rangle = 0|u_2\rangle, \quad \hat{S}|u_2\rangle = 1|u_2\rangle \tag{76}$$

In the subspace spanned by $|u_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $|u_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$,

$$\sum_{i=1,3} |u_i\rangle\langle u_i|\hat{L}_z^2 \sum_{i=1,3} |u_i\rangle\langle u_i| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{77}$$

$$\sum_{i=1,3} |u_i\rangle\langle u_i|\hat{S} \sum_{i=1,3} |u_i\rangle\langle u_i| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{78}$$

The other two common eigenvectors are

$$|\psi_2\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{79}$$

$$|\psi_3\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{80}$$

$$\hat{L}_z^2|\psi_2\rangle = 1|\psi_2\rangle \tag{81}$$

$$\hat{L}_z^2|\psi_3\rangle = 1|\psi_3\rangle \tag{82}$$

$$\hat{S}|\psi_2\rangle = 1|\psi_2\rangle \tag{83}$$

$$\hat{S}|\psi_3\rangle = -1|\psi_3\rangle \tag{84}$$

so specifying the eigenvalues of $\hat{L}_z^2$ and $\hat{S}$ determine a unique set of common eigenvector $\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}\right\}$.

Therefore, $\hat{L}_z^2$ and $\hat{S}$ form a CSCO.

$\square$