Quantum Mechanics

Solutions to the Problems in Homework Assignment 12

Fall, 2019

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1. **[C-T Exercise 6-2]** Consider an arbitrary physical system whose four-dimensional state space is spanned by a basis of four eigenvectors $|jm_z\rangle$ common to $\hat{\vec{J}}^2$ and $\hat{J}_z$ ($j = 0$ or 1; $-j \leq m_z \leq +j$), of eigenvalues $j(j+1)\hbar^2$ and $m_z\hbar$, such that

$$\hat{J}_\pm |jm_z\rangle = \hbar\sqrt{j(j+1) - m_z(m_z \pm 1)}\, |j, m_z \pm 1\rangle,$$
$$\hat{J}_+ |jj\rangle = \hat{J}_- |j, -j\rangle = 0.$$

(a) Express, in terms of the kets $|jm_z\rangle$, the eigenstates common to $\hat{\vec{J}}^2$ and $\hat{J}_x$, to be denoted by $|jm_x\rangle$.

(b) Consider a system in the normalized state

$$|\psi\rangle = \alpha\,|j=1, m_z=1\rangle + \beta\,|j=1, m_z=0\rangle + \gamma\,|j=1, m_z=-1\rangle + \delta\,|j=0, m_z=0\rangle\,.$$

i. What is the probability of finding $2\hbar^2$ and $\hbar$ if $\hat{\vec{J}}^2$ and $\hat{J}_x$ are measured simultaneously?

ii. Calculate the mean value of $\hat{J}_z$ when the system is in the state $|\psi\rangle$, and the probabilities of the various possible results of a measurement bearing only on this observable.

iii. Same questions for the observable $\hat{\vec{J}}^2$ and $\hat{J}_x$.

iv. $\hat{J}_z^2$ is now measured; what are the possible results, their probabilities, and their mean value?

(a) All the three components of an angular momentum have the eigenvalue spectrum. Thus, the eigenvalue of $\hat{J}_x$ for $j = 0$ is 0, $m_x\hbar = 0$; the eigenvalues of $\hat{J}_x$ for $j = 1$ are $m_x\hbar = \hbar, 0, -\hbar$. Let $|jm_x\rangle_x$ be the common eigenvector of $\hat{\vec{J}}^2$ and $\hat{J}_x$ corresponding respectively to the eigenvalues $j(j+1)\hbar^2$ and $m_x\hbar$. To avoid confusion, a subscript x is added to the ket $|jm_x\rangle_x$. The eigenvalue equations for $\hat{\vec{J}}^2$ and $\hat{J}_x$ read respectuvely

$$\hat{\vec{J}}^2\,|jm_x\rangle_x = j(j+1)\hbar^2\,|jm_x\rangle_x,$$
$$\hat{J}_x\,|jm_x\rangle_x = m_x\hbar\,|jm_x\rangle_x\,.$$

Since the subspace $\mathscr{E}(j=0)$ is one-dimensional, the single common eigenvector $|00\rangle$ of $\hat{\vec{J}}^2$ and $\hat{J}_z$ in this subspace is also the common eigenvector of $\hat{\vec{J}}^2$ and $\hat{J}_x$. That is,

$$|00\rangle_x = |00\rangle\,.$$

It is trivial to verify that $|00\rangle_x = |00\rangle$ is indeed the common eigenvector of $\hat{\vec{J}}^2$ and $\hat{J}_x$. we have

$$\hat{\vec{J}}^2\,|00\rangle_x = \hat{\vec{J}}^2\,|00\rangle = 0 = 0(0+1)\hbar^2\,|00\rangle_x,$$
$$\hat{J}_x\,|00\rangle_x = \frac{1}{2}(\hat{J}_+ + \hat{J}_-)\,|00\rangle = 0 = 0\hbar\,|00\rangle_x\,.$$

We now find the common eigenvectors $|1m_x\rangle_x$ of $\hat{\vec{J}}^2$ and $\hat{J}_x$ in subspace $\mathscr{E}(j=1)$. Let

$$|1m_x\rangle_x = a_{m_x}\,|11\rangle + b_{m_x}\,|10\rangle + c_{m_x}\,|1,-1\rangle\,.$$

It is trivial to verify that $|1m_x\rangle_x$ for $m_x = 1, 0, -1$ are eigenvectors of $\hat{\vec{J}}^2$ corresponding to the eigenvalue $2\hbar^2$ since $|11\rangle$, $|10\rangle$, and $|1,-1\rangle$ are all the eigenvectors of $\hat{\vec{J}}^2$ corresponding to the same eigenvalue $2\hbar^2$. We have

$$\begin{aligned}
\hat{\vec{J}}^2\,|1m_x\rangle_x &= \hat{\vec{J}}^2\big[\,a_{m_x}\,|11\rangle + b_{m_x}\,|10\rangle + c_{m_x}\,|1,-1\rangle\,\big] \\
&= 2\hbar^2\big[\,a_{m_x}\,|11\rangle + b_{m_x}\,|10\rangle + c_{m_x}\,|1,-1\rangle\,\big] \\
&= 2\hbar^2\,|1m_x\rangle_x\,.
\end{aligned}$$

We now find the values of a_{m_x}, b_{m_x}, and b_{m_x} respectively for $m_x = 1, 0, -1$. From the fact that $|1m_x\rangle_x$ for $m_x = 1, 0, -1$ are eigenvectors of $\hat{J}_x$, we have

$$\begin{aligned} m_x\hbar\big[\,a_{m_x}|11\rangle + b_{m_x}|10\rangle + c_{m_x}|1,-1\rangle\,\big] &= \hat{J}_x\big[\,a_{m_x}|11\rangle + b_{m_x}|10\rangle + c_{m_x}|1,-1\rangle\,\big] \\ &= \frac{1}{2}\big(\hat{J}_+ + \hat{J}_-\big)\big[\,a_{m_x}|11\rangle + b_{m_x}|10\rangle + c_{m_x}|1,-1\rangle\,\big] \\ &= \frac{\hbar}{\sqrt{2}}\big[\,b_{m_x}|11\rangle + \big(a_{m_x} + c_{m_x}\big)|10\rangle + b_{m_x}|1,-1\rangle\,\big] \end{aligned}$$

from which it follows that

$$\begin{aligned} \sqrt{2}\,m_x a_{m_x} &= b_{m_x}, \\ \sqrt{2}\,m_x b_{m_x} &= a_{m_x} + c_{m_x}, \\ \sqrt{2}\,m_x c_{m_x} &= b_{m_x}. \end{aligned}$$

For $m_x = 1$, we have

$$\begin{aligned} \sqrt{2}\,a_1 &= b_1, \\ \sqrt{2}\,b_1 &= a_1 + c_1, \\ \sqrt{2}\,c_1 &= b_1. \end{aligned}$$

We then have

$$a_1 = c_1 = \frac{b_1}{\sqrt{2}}.$$

From the normalization condition ${}_x\langle 11|11\rangle_x = 1$, we have $|a| = 1/2$. Choosing $a_1 = 1/2$, we have $c_1 = 1/2$ and $b_1 = 1/\sqrt{2}$. The common eigenvector $|11\rangle_x$ of $\hat{\vec{J}}^2$ and $\hat{J}_x$ corresponding respectively to the eigenvalues $2\hbar^2$ and $\hbar$ is then given by

$$|11\rangle_x = \frac{1}{2}\big[\,|11\rangle + \sqrt{2}\,|10\rangle + |1,-1\rangle\,\big].$$

For $m_x = 0$, we have

$$b_0 = 0,\ \ c_0 = -a_0.$$

From the normalization condition ${}_x\langle 10|10\rangle_x = 1$, we have $|a_0| = 1/\sqrt{2}$. Choosing $a_0 = 1/\sqrt{2}$, we have $c_0 = -1/\sqrt{2}$. The common eigenvector $|10\rangle_x$ of $\hat{\vec{J}}^2$ and $\hat{J}_x$ corresponding respectively to the eigenvalues $2\hbar^2$ and 0 is then given by

$$|10\rangle_x = \frac{1}{\sqrt{2}}\big[\,|11\rangle - |1,-1\rangle\,\big].$$

For $m_x = -1$, we have

$$\begin{aligned} -\sqrt{2}a_{-1} &= b_{-1}, \\ -\sqrt{2}b_{-1} &= a_{-1} + c_{-1}, \\ -\sqrt{2}c_{-1} &= b_{-1}. \end{aligned}$$

We then have

$$a_{-1} = c_{-1} = -\frac{b_{-1}}{\sqrt{2}}.$$

From the normalization condition ${}_x\langle 1,-1|1,-1\rangle_x = 1$, we have $|a_{-1}| = 1/2$. Choosing $a_{-1} = 1/2$, we have $c_{-1} = 1/2$ and $b_{-1} = -1/\sqrt{2}$. The common eigenvector $|1,-1\rangle_x$ of $\hat{\vec{J}}^2$ and $\hat{J}_x$ corresponding respectively to the eigenvalues $2\hbar^2$ and $-\hbar$ is then given by

$$|1,-1\rangle_x = \frac{1}{2}\big[\,|11\rangle - \sqrt{2}\,|10\rangle + |1,-1\rangle\,\big].$$

(b) i. For the convenience of discussing the simultaneous measurement of $\hat{\vec{J}}^2$ and $\hat{J}_x$, we express $|\psi\rangle$ in terms of $|jm_x\rangle_x$. We first express $|jm_z\rangle$ in terms of $|jm_x\rangle_x$. From

$$\begin{aligned}
|00\rangle_x &= |00\rangle, \\
|11\rangle_x &= \frac{1}{2}\big[\,|11\rangle + \sqrt{2}\,|10\rangle + |1,-1\rangle\,\big], \\
|10\rangle_x &= \frac{1}{\sqrt{2}}\big[\,|11\rangle - |1,-1\rangle\,\big], \\
|1,-1\rangle_x &= \frac{1}{2}\big[\,|11\rangle - \sqrt{2}\,|10\rangle + |1,-1\rangle\,\big],
\end{aligned}$$

we have

$$\begin{aligned}
|00\rangle &= |00\rangle_x, \\
|11\rangle &= \frac{1}{2}\big[\,|11\rangle_x + \sqrt{2}\,|10\rangle_x + |1,-1\rangle_x\,\big], \\
|10\rangle &= \frac{1}{\sqrt{2}}\big[\,|11\rangle_x - |1,-1\rangle_x\,\big], \\
|1,-1\rangle &= \frac{1}{2}\big[\,|11\rangle_x - \sqrt{2}\,|10\rangle_x + |1,-1\rangle_x\,\big].
\end{aligned}$$

Note the similar forms of the above two sets of expressions. This is due to the equivalence of the components of the angular momentum in an isotropic space. In terms of $|jm_x\rangle_x$, $|\psi\rangle$ is given by

$$\begin{aligned}
|\psi\rangle &= \alpha\,|j=1, m_z=1\rangle + \beta\,|j=1, m_z=0\rangle + \gamma\,|j=1, m_z=-1\rangle + \delta\,|j=0, m_z=0\rangle \\
&= \frac{\alpha}{2}\big[\,|11\rangle_x + \sqrt{2}\,|10\rangle_x + |1,-1\rangle_x\,\big] + \frac{\beta}{\sqrt{2}}\big[\,|11\rangle_x - |1,-1\rangle_x\,\big] + \frac{\gamma}{2}\big[\,|11\rangle_x - \sqrt{2}\,|10\rangle_x + |1,-1\rangle_x\,\big] \\
&\quad + \delta\,|00\rangle_x \\
&= \frac{1}{2}\big(\alpha + \sqrt{2}\,\beta + \gamma\big)\,|11\rangle_x + \frac{1}{\sqrt{2}}\big(\alpha - \gamma\big)\,|10\rangle_x + \frac{1}{2}\big(\alpha - \sqrt{2}\,\beta + \gamma\big)\,|1,-1\rangle_x + \delta\,|00\rangle_x\,.
\end{aligned}$$

If $\hat{\vec{J}}^2$ and $\hat{J}_x$ are measured simultaneously, the probability of finding $2\hbar^2$ and $\hbar$ is given by

$$\mathscr{P}_{\hat{\vec{J}}^2,\hat{J}_x}(2\hbar^2, \hbar) = \big|{}_x\langle 11|\psi\rangle\big|^2 = \frac{1}{4}\big|\alpha + \sqrt{2}\,\beta + \gamma\big|^2.$$

ii. If $\hat{J}_z$ is measured, the possible results are $0, \hbar, -\hbar$. The probabilities of obtaining these results are respectively given by

$$\begin{aligned}
\mathscr{P}_{\hat{J}_z}(0) &= \big|\langle 10|\psi\rangle\big|^2 + \big|\langle 00|\psi\rangle\big|^2 = |\beta|^2 + |\delta|^2, \\
\mathscr{P}_{\hat{J}_z}(\hbar) &= \big|\langle 11|\psi\rangle\big|^2 = |\alpha|^2, \\
\mathscr{P}_{\hat{J}_z}(-\hbar) &= \big|\langle 1,-1|\psi\rangle\big|^2 = |\gamma|^2.
\end{aligned}$$

The mean value of $\hat{J}_z$ is then given by

$$\langle \hat{J}_z\rangle = 0\cdot\mathscr{P}_{\hat{J}_z}(0) + \hbar\mathscr{P}_{\hat{J}_z}(\hbar) + (-\hbar)\mathscr{P}_{\hat{J}_z}(-\hbar) = |\alpha|^2\hbar - |\gamma|^2\hbar = \big(|\alpha|^2 - |\gamma|^2\big)\hbar.$$

iii. If $\hat{\vec{J}}^2$ is measured, the possible results are 0 and $2\hbar^2$. The probabilities of obtaining these results are respectively given by

$$\begin{aligned}
\mathscr{P}_{\hat{\vec{J}}^2}(0) &= \big|\langle 00|\psi\rangle\big|^2 = |\delta|^2, \\
\mathscr{P}_{\hat{\vec{J}}^2}(2\hbar^2) &= \sum_{m_z=0,\pm 1}\big|\langle 1m_z|\psi\rangle\big|^2 = |\alpha|^2 + |\beta|^2 + |\gamma|^2.
\end{aligned}$$

The mean value of $\hat{\vec{J}}^2$ is then given by

$$\langle\hat{\vec{J}}^2\rangle = 0\cdot\mathscr{P}_{\hat{\vec{J}}^2}(0) + 2\hbar^2\mathscr{P}_{\hat{\vec{J}}^2}(2\hbar^2) = 2\hbar^2\big(|\alpha|^2 + |\beta|^2 + |\gamma|^2\big).$$

If $\hat{J}_x$ is measured, the possible results are 0 and $\pm\hbar$. The probabilities of obtaining these results are respectively given by

$$\begin{aligned}
\mathscr{P}_{\hat{J}_x}(0) &= \big|{}_x\langle 10|\psi\rangle\big|^2 + \big|{}_x\langle 00|\psi\rangle\big|^2 = \frac{1}{2}|\alpha-\gamma|^2 + |\delta|^2, \\
\mathscr{P}_{\hat{J}_x}(\hbar) &= \big|{}_x\langle 11|\psi\rangle\big|^2 = \frac{1}{4}\big|\alpha + \sqrt{2}\,\beta + \gamma\big|^2, \\
\mathscr{P}_{\hat{J}_x}(-\hbar) &= \big|{}_x\langle 1,-1|\psi\rangle\big|^2 = \frac{1}{4}\big|\alpha - \sqrt{2}\,\beta + \gamma\big|^2.
\end{aligned}$$

The mean value of $\hat{J}_x$ is then given by

$$\begin{aligned}
\langle \hat{J}_x\rangle &= 0\cdot\mathscr{P}_{\hat{J}_x}(0) + \hbar\mathscr{P}_{\hat{J}_x}(\hbar) + (-\hbar)\mathscr{P}_{\hat{J}_x}(-\hbar) \\
&= \frac{\hbar}{4}\Big[\big|\alpha+\sqrt{2}\,\beta+\gamma\big|^2 - \big|\alpha-\sqrt{2}\,\beta+\gamma\big|^2\Big] \\
&= \hbar\sqrt{2}\,\mathrm{Re}\big(\alpha\beta^* + \beta\gamma^*\big).
\end{aligned}$$

iv. If $\hat{J}_z^2$ is measured, the possible results are 0 and $\hbar^2$. The probabilities of obtaining these results are respectively given by

$$\begin{aligned}
\mathscr{P}_{\hat{J}_z^2}(0) &= \big|\langle 10|\psi\rangle\big|^2 + \big|\langle 00|\psi\rangle\big|^2 = |\beta|^2 + |\delta|^2, \\
\mathscr{P}_{\hat{J}_z^2}(\hbar^2) &= \big|\langle 11|\psi\rangle\big|^2 + \big|\langle 1,-1|\psi\rangle\big|^2 = |\alpha|^2 + |\gamma|^2.
\end{aligned}$$

The mean value of $\hat{J}_z^2$ is then given by

$$\langle \hat{J}_z^2\rangle = 0\cdot\mathscr{P}_{\hat{J}_z^2}(0) + \hbar^2\mathscr{P}_{\hat{J}_z^2}(\hbar^2) = \big(|\alpha|^2+|\gamma|^2\big)\hbar^2.$$

2. **[C-T Exercise 6-5]** A system whose state space is $\mathscr{E}_{\vec{r}}$ has for its wave function $\psi(x,y,z) = N(x+y+z)e^{-r^2/\alpha^2}$, where α, which is real, is given and N is a normalization constant.

   (a) The observables $\hat{L}_z$ and $\hat{\vec{L}}^2$ are measured; what are the probabilities of finding 0 and $2\hbar^2$?

   (b) Is it possible to predict directly the probabilities of all possible results of measurements of $\hat{\vec{L}}^2$ and $\hat{L}_z$ in the system of wave function $\psi(x,y,z)$?

(a) From the explicit expressions of the spherical harmonic functions

$$\begin{aligned}
Y_{11}(\theta,\phi) &= -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}, \\
Y_{10}(\theta,\phi) &= \sqrt{\frac{3}{4\pi}}\cos\theta, \\
Y_{1,-1}(\theta,\phi) &= \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi},
\end{aligned}$$

we have

$$\begin{aligned}
Y_{11}(\theta,\phi) &= -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi} = -\sqrt{\frac{3}{8\pi}}(\sin\theta\cos\phi + i\sin\theta\sin\phi) = -\sqrt{\frac{3}{8\pi}}\frac{x+iy}{r}, \\
Y_{10}(\theta,\phi) &= \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\frac{z}{r}, \\
Y_{1,-1}(\theta,\phi) &= \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi} = \sqrt{\frac{3}{8\pi}}(\sin\theta\cos\phi - i\sin\theta\sin\phi) = \sqrt{\frac{3}{8\pi}}\frac{x-iy}{r}.
\end{aligned}$$

Expressing x, y, and z in terms of the spherical harmonic functions, we have

$$\begin{aligned}
x &= \sqrt{\frac{2\pi}{3}}\, r\big[-Y_{11}(\theta,\phi) + Y_{1,-1}(\theta,\phi)\big], \\
y &= \sqrt{\frac{2\pi}{3}}\, ir\big[\,Y_{11}(\theta,\phi) + Y_{1,-1}(\theta,\phi)\big], \\
z &= \sqrt{\frac{4\pi}{3}}\, rY_{10}(\theta,\phi).
\end{aligned}$$

In the spherical coordinate system, the wave function is given by

$$\psi(r,\theta,\phi) = \sqrt{\frac{2\pi}{3}} N\big[\,(-1+i)Y_{11}(\theta,\phi) + (1+i)Y_{1,-1}(\theta,\phi) + \sqrt{2}\,Y_{10}(\theta,\phi)\,\big] re^{-r^2/\alpha^2}.$$

Before we proceed any further, let us normalize the wave function. From the normalization condition, we have

$$\begin{aligned}
1 &= \int_0^\infty dr\, r^2 \int d\Omega\, \big|\psi(r,\theta,\phi)\big|^2 = \frac{2\pi}{3}|N|^2\big[\,|-1+i|^2 + |1+i|^2 + 2\,\big] \int_0^\infty dr\, r^4 e^{-2r^2/\alpha^2} \\
&= 4\pi|N|^2 \int_0^\infty dr\, r^4 e^{-2r^2/\alpha^2} = 4\pi|N|^2 \cdot \frac{3}{8}\frac{\sqrt{\pi}}{(2/\alpha^2)^{5/2}} = \frac{3}{8}\sqrt{\frac{\pi^3}{2}}\alpha^5|N|^2.
\end{aligned}$$

Choosing N to be a positive number, we have

$$N = \frac{2}{(3\alpha^5)^{1/2}}\left(\frac{2}{\pi}\right)^{3/4}.$$

The normalized wave function is then given by

$$\begin{aligned}
\psi(r,\theta,\phi) &= \frac{4}{3\alpha^{5/2}}\left(\frac{2}{\pi}\right)^{1/4} re^{-r^2/\alpha^2}\big[\,(-1+i)Y_{11}(\theta,\phi) + (1+i)Y_{1,-1}(\theta,\phi) + \sqrt{2}\,Y_{10}(\theta,\phi)\,\big] \\
&= R(r)\mathscr{Y}(\theta,\phi),
\end{aligned}$$

where

$$\begin{aligned}
R(r) &= \frac{4}{(3\alpha^5)^{1/2}}\left(\frac{8}{\pi}\right)^{1/4} re^{-r^2/\alpha^2}, \\
\mathscr{Y}(\theta,\phi) &= \frac{1}{\sqrt{6}}\big[\,(-1+i)Y_{11}(\theta,\phi) + (1+i)Y_{1,-1}(\theta,\phi) + \sqrt{2}\,Y_{10}(\theta,\phi)\,\big].
\end{aligned}$$

Note that the radial wave function $R(r)$ and the angular wave function $\mathscr{Y}(\theta,\phi)$ are respectively normalized. When the measurements of the angular momentum are discussed, we can use only the angular wave function. If $\hat{L}_z$ is measured, the probability of finding 0 is given by

$$\mathscr{P}_{\hat{L}_z}(0) = \big|\langle 10|\mathscr{Y}\rangle\big|^2 = \left|\frac{1}{\sqrt{6}}\sqrt{2}\right|^2 = \frac{1}{3}.$$

If $\hat{\vec{L}}^2$ is measured, the probability of finding $2\hbar^2$ is given by

$$\mathscr{P}_{\hat{\vec{L}}^2}(2\hbar^2) = \sum_{m_z=0,\pm1}\big|\langle 1m_z|\mathscr{Y}\rangle\big|^2 = \left|\frac{1}{\sqrt{6}}(-1+i)\right|^2 + \left|\frac{1}{\sqrt{6}}(1+i)\right|^2\left|\frac{1}{\sqrt{6}}\sqrt{2}\right|^2 = 1.$$

(b) Yes, it is possible to predict directly the probabilities of all possible results of measurements of $\hat{\vec{L}}^2$ and $\hat{L}_z$ in the system of wave function $\psi(x,y,z)$. If $\hat{\vec{L}}^2$ is measured, we can only obtain the result $2\hbar^2$ and hence the probability of finding $2\hbar^2$ is 1. If $\hat{L}_z$ is measured, all the possible results are $\hbar$, 0, and $-\hbar$. The probabilities of obtaining these results are respectively given by

$$\begin{aligned}
\mathscr{P}_{\hat{L}_z}(\hbar) &= \left|\langle 11|\mathscr{Y}\rangle\right|^2 = \left|\frac{1}{\sqrt{6}}(-1+i)\right|^2 = \frac{1}{3}, \\
\mathscr{P}_{\hat{L}_z}(0) &= \left|\langle 10|\mathscr{Y}\rangle\right|^2 = \left|\frac{1}{\sqrt{6}}\sqrt{2}\right|^2 = \frac{1}{3}, \\
\mathscr{P}_{\hat{L}_z}(-\hbar) &= \left|\langle 1,-1|\mathscr{Y}\rangle\right|^2 = \left|\frac{1}{\sqrt{6}}(1+i)\right|^2 = \frac{1}{3}.
\end{aligned}$$

3. **[C-T Exercise 6-8]** Consider a particle in three-dimensional space, whose state vector is $|\psi\rangle$, and whose wave function is $\psi(\vec{r}) = \langle \vec{r}|\psi\rangle$. Let $\hat{A}$ be an observable which commutes with $\hat{\vec{L}} = \hat{\vec{r}}\times\hat{\vec{p}}$, the orbital angular momentum of the particle. Assuming that $\hat{A}$, $\hat{\vec{L}}^2$, and $\hat{L}_z$ form a CSCO in $\mathscr{E}_{\vec{r}}$, call $|n\ell m\rangle$ their common eigenkets, whose eigenvalues are, respectively, a_n (the index n is assumed to be discrete), $\ell(\ell+1)\hbar^2$, and $m\hbar$.

   Let $\hat{U}(\phi)$ be the unitary operator defined by $\hat{U}(\phi) = e^{-i\phi\hat{L}_z/\hbar}$, where ϕ is a real dimensionless parameter. For an arbitrary operator $\hat{K}$, we call $\tilde{\hat{K}}$ the transform of $\hat{K}$ by the unitary operator $\hat{U}(\phi)$, $\tilde{\hat{K}} = \hat{U}(\phi)\hat{K}\hat{U}^\dagger(\phi)$.

   (a) We set $\hat{L}_+ = \hat{L}_x + i\hat{L}_y$, $\hat{L}_- = \hat{L}_x - i\hat{L}_y$. Calculate $\tilde{\hat{L}}_+|n\ell m\rangle$ and show that $\hat{L}_+$ and $\tilde{\hat{L}}_+$ are proportional; calculate the proportionality constant. Same question for $\hat{L}_-$ and $\tilde{\hat{L}}_-$.

   (b) Express $\tilde{\hat{L}}_x$, $\tilde{\hat{L}}_y$, and $\tilde{\hat{L}}_z$ in terms of $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$. What geometrical transformation can be associated with the transformation of $\hat{\vec{L}}$ into $\tilde{\hat{\vec{L}}}$?

   (c) Calculate the commutators $[\hat{x}\pm i\hat{y}, \hat{L}_z]$ and $[\hat{z}, \hat{L}_z]$. Show that the kets $(\hat{x}\pm i\hat{y})|n\ell m\rangle$ and $\hat{z}|n\ell m\rangle$ are eigenvectors of $\hat{L}_z$ and calculate their eigenvalues. What relation must exist between m and m' for the matrix element $\langle n'\ell'm'|\hat{x}\pm i\hat{y}|n\ell m\rangle$ to be non-zero? Same question for $\langle n'\ell'm'|\hat{z}|n\ell m\rangle$.

   (d) By comparing the matrix elements of $\widetilde{\hat{x}\pm i\hat{y}}$ and $\tilde{\hat{z}}$ with those of $\hat{x}\pm i\hat{y}$ and $\hat{z}$, calculate $\tilde{\hat{x}}$, $\tilde{\hat{y}}$, $\tilde{\hat{z}}$ in terms of $\hat{x}$, $\hat{y}$, $\hat{z}$. Give a geometrical interpretation.

(a) $\tilde{\hat{L}}_+|n\ell m\rangle$ is given by

$$\begin{aligned}
\tilde{\hat{L}}_+|n\ell m\rangle &= \hat{U}(\phi)\hat{L}_+\hat{U}^\dagger(\phi)|n\ell m\rangle = e^{-i\phi\hat{L}_z/\hbar}\hat{L}_+e^{i\phi\hat{L}_z/\hbar}|n\ell m\rangle \\
&= e^{-i\phi\hat{L}_z/\hbar}\hat{L}_+e^{im\phi}|n\ell m\rangle = e^{im\phi}e^{-i\phi\hat{L}_z/\hbar}\hbar\sqrt{\ell(\ell+1)-m(m+1)}\,|n\ell,m+1\rangle \\
&= e^{im\phi}\hbar\sqrt{\ell(\ell+1)-m(m+1)}\,e^{-i(m+1)\phi}|n\ell,m+1\rangle \\
&= \hbar\sqrt{\ell(\ell+1)-m(m+1)}\,e^{-i\phi}|n\ell,m+1\rangle.
\end{aligned}$$

Comparing the above result with $\hat{L}_+|n\ell m\rangle = \hbar\sqrt{\ell(\ell+1)-m(m+1)}\,|n\ell,m+1\rangle$, we see that $\tilde{\hat{L}}_+$ and $\hat{L}_+$ are proportional with the proportionality constant given by $e^{-i\phi}$,

$$\tilde{\hat{L}}_+ = e^{-i\phi}\hat{L}_+.$$

For $\tilde{\hat{L}}_-$, we have

$$\begin{aligned}
\tilde{\hat{L}}_-|n\ell m\rangle &= \hat{U}(\phi)\hat{L}_-\hat{U}^\dagger(\phi)|n\ell m\rangle = e^{-i\phi\hat{L}_z/\hbar}\hat{L}_-e^{i\phi\hat{L}_z/\hbar}|n\ell m\rangle \\
&= e^{-i\phi\hat{L}_z/\hbar}\hat{L}_-e^{im\phi}|n\ell m\rangle = e^{im\phi}e^{-i\phi\hat{L}_z/\hbar}\hbar\sqrt{\ell(\ell+1)-m(m-1)}\,|n\ell,m-1\rangle \\
&= e^{im\phi}\hbar\sqrt{\ell(\ell+1)-m(m+1)}\,e^{-i(m-1)\phi}|n\ell,m-1\rangle \\
&= \hbar\sqrt{\ell(\ell+1)-m(m-1)}\,e^{i\phi}|n\ell,m-1\rangle.
\end{aligned}$$

Comparing the above result with $\hat{L}_-|n\ell m\rangle = \hbar\sqrt{\ell(\ell+1)-m(m-1)}\,|n\ell,m-1\rangle$, we see that $\tilde{\hat{L}}_-$ and $\hat{L}_-$ are proportional with the proportionality constant given by $e^{i\phi}$,

$$\tilde{\hat{L}}_- = e^{i\phi}\hat{L}_-.$$

(b) For $\tilde{\hat{L}}_z$, we have

$$\tilde{\hat{L}}_z = \hat{U}(\phi)\hat{L}_z\hat{U}^\dagger(\phi) = e^{-i\phi\hat{L}_z/\hbar}\hat{L}_z e^{i\phi\hat{L}_z/\hbar} = e^{-i\phi\hat{L}_z/\hbar}e^{i\phi\hat{L}_z/\hbar}\hat{L}_z = \hat{L}_z.$$

Thus, $\hat{L}_z$ remains unchanged under the transformation by the unitary operator $\hat{U}(\phi)$.
From

$$\hat{L}_x = \frac{1}{2}\big(\hat{L}_+ + \hat{L}_-\big),\ \hat{L}_y = \frac{1}{2i}\big(\hat{L}_+ - \hat{L}_-\big),$$

we have

$$\begin{aligned}
\tilde{\hat{L}}_x &= \frac{1}{2}\big(\tilde{\hat{L}}_+ + \tilde{\hat{L}}_-\big) = \frac{1}{2}\big(e^{-i\phi}\hat{L}_+ + e^{i\phi}\hat{L}_-\big) \\
&= \frac{1}{2}\big[\, e^{-i\phi}\big(\hat{L}_x + i\hat{L}_y\big) + e^{i\phi}\big(\hat{L}_x - i\hat{L}_y\big)\,\big] = \hat{L}_x\cos\phi + \hat{L}_y\sin\phi, \\
\tilde{\hat{L}}_y &= \frac{1}{2i}\big(\tilde{\hat{L}}_+ - \tilde{\hat{L}}_-\big) = \frac{1}{2i}\big(e^{-i\phi}\hat{L}_+ - e^{i\phi}\hat{L}_-\big) \\
&= \frac{1}{2i}\big[\, e^{-i\phi}\big(\hat{L}_x + i\hat{L}_y\big) - e^{i\phi}\big(\hat{L}_x - i\hat{L}_y\big)\,\big] = -\hat{L}_x\sin\phi + \hat{L}_y\cos\phi.
\end{aligned}$$

From the above-obtained results,

$$\begin{aligned}
\tilde{\hat{L}}_x &= \hat{L}_x\cos\phi + \hat{L}_y\sin\phi, \\
\tilde{\hat{L}}_y &= -\hat{L}_x\sin\phi + \hat{L}_y\cos\phi, \\
\tilde{\hat{L}}_z &= \hat{L}_z,
\end{aligned}$$

we see that the transformation of $\hat{\vec{L}}$ into $\tilde{\hat{\vec{L}}}$ is a rotation about the z-axis through an angle of ϕ.

(c) The commutators $[\hat{x} \pm i\hat{y}, \hat{L}_z]$ are given by

$$[\hat{x} \pm i\hat{y}, \hat{L}_z] = [\hat{x} \pm i\hat{y}, \hat{x}\hat{p}_y - \hat{y}\hat{p}_x] = -\hat{y}[\hat{x}, \hat{p}_x] \pm i\hat{x}[\hat{y}, \hat{p}_y] = -i\hbar\hat{y} \mp \hbar\hat{x} = \mp\hbar(\hat{x} \pm i\hat{y}).$$

The commutator $[\hat{z}, \hat{L}_z]$ is given by

$$[\hat{z}, \hat{L}_z] = [\hat{z}, \hat{x}\hat{p}_y - \hat{y}\hat{p}_x] = 0.$$

For $(\hat{x} \pm i\hat{y})\,|n\ell m\rangle$, we have

$$\hat{L}_z\big[(\hat{x} \pm i\hat{y})\,|n\ell m\rangle\big] = \big[(\hat{x} \pm i\hat{y})\hat{L}_z \pm \hbar(\hat{x} \pm i\hat{y})\big]\,|n\ell m\rangle = (m \pm 1)\hbar\big[(\hat{x} \pm i\hat{y})\,|n\ell m\rangle\big].$$

For $\hat{z}\,|n\ell m\rangle$, we have

$$\hat{L}_z\big[\hat{z}\,|n\ell m\rangle\big] = \hat{z}\hat{L}_z\,|n\ell m\rangle = m\hbar\big[\hat{z}\,|n\ell m\rangle\big].$$

Thus, the kets $(\hat{x} \pm i\hat{y})\,|n\ell m\rangle$ and $\hat{z}\,|n\ell m\rangle$ are eigenvectors of $\hat{L}_z$ with the eigenvalues given respectively by $(m \pm 1)\hbar$ and $m\hbar$.

From the fact that $(\hat{x} \pm i\hat{y})\,|n\ell m\rangle$ are the eigenvectors of $\hat{L}_z$ with the eigenvalues given respectively by $(m \pm 1)\hbar$, we see that, for $\langle n'\ell' m'|\hat{x} \pm i\hat{y}|n\ell m\rangle \neq 0$, we must have

$$m' = m \pm 1.$$

That is,

$$\Delta m = m' - m = \pm 1.$$

The above result can be also proved in the following manner. Making use of $[\hat{x} \pm i\hat{y}, \hat{L}_z] = \mp\hbar(\hat{x} \pm i\hat{y})$, the matrix elements $\langle n'\ell' m'|\hat{x} \pm i\hat{y}|n\ell m\rangle$ are given by

$$\begin{aligned}
\langle n'\ell' m'|\hat{x} \pm i\hat{y}|n\ell m\rangle &= \mp\frac{1}{\hbar}\,\langle n'\ell' m'|[\hat{x} \pm i\hat{y}, \hat{L}_z]|n\ell m\rangle \\
&= \mp\frac{1}{\hbar}\,\langle n'\ell' m'|(\hat{x} \pm i\hat{y})\hat{L}_z - \hat{L}_z(\hat{x} \pm i\hat{y})|n\ell m\rangle \\
&= \mp(m - m')\,\langle n'\ell' m'|\hat{x} \pm i\hat{y}|n\ell m\rangle\,.
\end{aligned}$$

From the above equation, we have

$$\left[1 \pm (m - m')\right] \langle n'\ell'm'|\hat{x} \pm i\hat{y}|n\ell m\rangle = 0.$$

For $\langle n'\ell'm'|\hat{x} \pm i\hat{y}|n\ell m\rangle \neq 0$, we must have

$$1 \pm (m - m') = 0.$$

That is,

$$\Delta m = m' - m = \pm 1.$$

From the fact that $\hat{z}\,|n\ell m\rangle$ is an eigenvector of $\hat{L}_z$ with the eigenvalue $m\hbar$, we see that, for $\langle n'\ell'm'|\hat{z}|n\ell m\rangle \neq 0$, we must have

$$m' = m.$$

That is,

$$\Delta m = m' - m = 0.$$

(d) From

$$\widetilde{\hat{x} \pm i\hat{y}} = e^{-i\phi\hat{L}_z/\hbar}(\hat{x} \pm i\hat{y})e^{i\phi\hat{L}_z/\hbar},$$

we have

$$\begin{aligned}\langle n'\ell'm'|\widetilde{\hat{x} \pm i\hat{y}}|n\ell m\rangle &= \langle n'\ell'm'|e^{-i\phi\hat{L}_z/\hbar}(\hat{x} \pm i\hat{y})e^{i\phi\hat{L}_z/\hbar}|n\ell m\rangle \\ &= e^{-i(m'-m)\phi}\,\langle n'\ell'm'|(\hat{x} \pm i\hat{y})|n\ell m\rangle \\ &= e^{\mp i\phi}\,\langle n'\ell'm'|(\hat{x} \pm i\hat{y})|n\ell m\rangle,\end{aligned}$$

where we have made use of the fact that $\langle n'\ell'm'|\hat{x} \pm i\hat{y}|n\ell m\rangle \neq 0$ only for $\Delta m = m' - m = \pm 1$. We thus have

$$\widetilde{\hat{x} \pm i\hat{y}} = e^{\mp i\phi}(\hat{x} \pm i\hat{y}).$$

For $\tilde{\hat{z}}$, making use of $[\hat{z}, \hat{L}_z] = 0$, we have

$$\tilde{\hat{z}} = e^{-i\phi\hat{L}_z/\hbar}\hat{z}e^{i\phi\hat{L}_z/\hbar} = e^{-i\phi\hat{L}_z/\hbar}e^{i\phi\hat{L}_z/\hbar}\hat{z} = \hat{z}.$$

Geometrically, $\widetilde{\hat{x} \pm i\hat{y}}$ and $\tilde{\hat{z}}$ are respectively the rotations of $\hat{x} \pm i\hat{y}$ and $\hat{z}$ about the z-axis through an angle of ϕ.

4. **[C-T Exercise 7-1]** Let ρ, ϕ, and z be the cylindrical coordinates of a spinless particle ($x = \rho\cos\phi$, $y = \rho\sin\phi$; $\rho \geq 0$, $0 \leq \phi < 2\pi$). Assume that the potential energy of this particle depends only on ρ, and not on ϕ and z. Recall that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}.$$

(a) Write, in cylindrical coordinates, the differential operator associated with the Hamiltonian $\hat{H}$. Show that $\hat{H}$ commutes with $\hat{L}_z$ and $\hat{p}_z$. Show from this that the wave functions associated with the stationary states of the particle can be chosen in the form $\varphi_{nmk}(\rho, \phi, z) = f_{nm}(\rho)e^{im\phi}e^{ikz}$, where the values that can be taken on by the indices m and k are to be specified.

(b) Write, in cylindrical coordinates, the eigenvalue equation of the Hamiltonian $\hat{H}$ of the particle. Derive from it the differential equation which yields $f_{nm}(\rho)$.

(c) Let $\hat{\Sigma}_y$ be the operator whose action, in the $\{|\vec{r}\rangle\}$ representation, is to change y to $-y$ (reflection with respect to the xOz plane). Does $\hat{\Sigma}_y$ commute with $\hat{H}$? Show that $\hat{\Sigma}_y$ anticommutes with $\hat{L}_z$, and show from this that $\hat{\Sigma}_y\,|\varphi_{nmk}\rangle$ is an eigenvector of $\hat{L}_z$. What is the corresponding eigenvalue? What can be concluded concerning the degeneracy of the energy levels of the particle? Could this result be predicted directly from the differential equation established in (b)?

(a) Let $V(\rho)$ be the potential energy of the particle. Let μ be the mass of the particle. The Hamiltonian of the particle is given by

$$\hat{H} = -\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right) + V(\rho).$$

$\hat{L}_z$ is given by

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}.$$

Note that both $\hat{H}$ and $\hat{L}_z$ depends on ϕ only through the partial derivatives with respect to ϕ. Since the partial derivatives with respect to ϕ commute, $\hat{H}$ and $\hat{L}_z$ commute. That is,

$$[\hat{H}, \hat{L}_z] = 0.$$

$\hat{p}_z$ is given by

$$\hat{p}_z = -i\hbar\frac{\partial}{\partial z}.$$

Note that both $\hat{H}$ and $\hat{p}_z$ depends on z only through the partial derivatives with respect to z. Since the partial derivatives with respect to z commute, $\hat{H}$ and $\hat{p}_z$ commute. That is,

$$[\hat{H}, \hat{p}_z] = 0.$$

Since $[\hat{H}, \hat{L}_z] = 0$, $[\hat{H}, \hat{p}_z] = 0$, and $[\hat{L}_z, \hat{p}_z] = 0$, $\hat{H}$, $\hat{L}_z$, and $\hat{p}_z$ have common eigenfunctions. We know that the eigenfunction $\hat{L}_z$ corresponding to the eigenvalue $m\hbar$ is given by

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi},\ m = 0, \pm 1, \pm 2, \cdots.$$

We also know that the eigenfunction of $\hat{p}_z$ corresponding to the eigenvalue $\hbar k$ is given by

$$\overline{\psi}_k(z) = \frac{1}{\sqrt{2\pi}}e^{ikz},\ -\infty < k < \infty.$$

Thus, the common eigenfunctions of $\hat{H}$, $\hat{L}_z$, and $\hat{p}_z$ are of the form

$$\varphi_{nmk}(\rho, \phi, z) = f_{nm}(\rho)e^{im\phi}e^{ikz},\ m = 0, \pm 1, \pm 2, \cdots,\ -\infty < k < \infty.$$

(b) The eigenvalue equation of the Hamiltonian $\hat{H}$ of the particle is given by

$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right) + V(\rho)\right]\varphi_{nmk}(\rho, \phi, z) = E_{nmk}\varphi_{nmk}(\rho, \phi, z).$$

Inserting $\varphi_{nmk}(\rho, \phi, z) = f_{nm}(\rho)e^{im\phi}e^{ikz}$ into the above equation, we obtain the differential equation for $f_{nm}(\rho)$

$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho}\right) + \frac{m^2\hbar^2}{2\mu\rho^2} + V(\rho)\right]f_{nm}(\rho) = \varepsilon_{nm}f_{nm}(\rho),$$

where

$$E_{nmk} = \varepsilon_{nm} + \frac{\hbar^2 k^2}{2\mu}.$$

(c) In the cylindrical coordinates, the reflection with respect to the xOz plane can be described by

$$\phi \to 2\pi - \phi.$$

Let $\phi' = 2\pi - \phi$. Since

$$\frac{\partial^2}{\partial \phi'^2} = \frac{\partial^2}{\partial \phi^2},$$

$\hat{\Sigma}_y$ commutes with $\hat{H}$.
Consider an arbitrary function $\psi(\rho, \phi, z)$. For $\hat{\Sigma}_y \hat{L}_z \psi(\rho, \phi, z)$, we have

$$\hat{\Sigma}_y \hat{L}_z \psi(\rho, \phi, z) = -i\hbar \hat{\Sigma}_y \frac{\partial \psi(\rho, \phi, z)}{\partial \phi} = -i\hbar \frac{\partial \psi(\rho, 2\pi - \phi, z)}{\partial (2\pi - \phi)} = i\hbar \frac{\partial \psi(\rho, 2\pi - \phi, z)}{\partial \phi}.$$

For $\hat{L}_z \hat{\Sigma}_y \psi(\rho, \phi, z)$, we have

$$\hat{L}_z \hat{\Sigma}_y \psi(\rho, \phi, z) = \hat{L}_z \psi(\rho, 2\pi - \phi, z) = -i\hbar \frac{\partial \psi(\rho, 2\pi - \phi, z)}{\partial \phi}.$$

Comparing the above two results, we have

$$\hat{\Sigma}_y \hat{L}_z \psi(\rho, \phi, z) = -\hat{L}_z \hat{\Sigma}_y \psi(\rho, \phi, z).$$

Since $\psi(\rho, \phi, z)$ is an arbitrary function, we have

$$\hat{\Sigma}_y \hat{L}_z = -\hat{L}_z \hat{\Sigma}_y.$$

That is, $\hat{\Sigma}_y$ anticommutes with $\hat{L}_z$,

$$[\hat{\Sigma}_y, \hat{L}_z] = 0.$$

From $\hat{\Sigma}_y \hat{L}_z = -\hat{L}_z \hat{\Sigma}_y$, we have

$$\hat{L}_z \big[\hat{\Sigma}_y \left|\varphi_{nmk}\right\rangle\big] = \hat{L}_z \hat{\Sigma}_y \left|\varphi_{nmk}\right\rangle = -\hat{\Sigma}_y \hat{L}_z \left|\varphi_{nmk}\right\rangle = -m\hbar \hat{\Sigma}_y \left|\varphi_{nmk}\right\rangle = -m\hbar \big[\hat{\Sigma}_y \left|\varphi_{nmk}\right\rangle\big].$$

Thus, $\hat{\Sigma}_y \left|\varphi_{nmk}\right\rangle$ is an eigenvector of $\hat{L}_z$ corresponding to the eigenvalue $-m\hbar$. From this, we can conclude that the energy levels of the particle is degenerate with respect to m and $-m$. This result could be predicted directly from the differential equation established in (b). That the form of the differential equation for $f_{nm}(\rho)$ is invariant under the change $m \to -m$ implies that the energy levels of the particle is degenerate with respect to m and $-m$.

5. **[C-T Exercise 7-2]** Consider a particle of mass μ, whose Hamiltonian is

$$\hat{H}_0 = \frac{1}{2m} \hat{\vec{p}}^2 + \frac{1}{2} \mu \omega_0^2 \hat{\vec{r}}^2,$$

where ω_0 is a given positive constant.

(a) Find the energy levels of the particle and their degrees of degeneracy. Is it possible to construct a basis of eigenstates common to $\hat{H}_0$, $\hat{\vec{L}}^2$, $\hat{L}_z$?

(b) Now, assume that the particle, which has a charge q, is placed in a uniform magnetic field $\vec{B}$ parallel to Oz. We set $\omega_L = -qB/2\mu$. If we choose the gauge $\vec{A} = -\vec{r} \times \vec{B}/2$, the Hamiltonian $\hat{H}$ of the particle is then given by $\hat{H} = \hat{H}_0 + \hat{H}_1(\omega_L)$, where $\hat{H}_1$ is the sum of an operator which is linearly dependent on ω_L (the paramagnetic term) and an operator which is quadratically dependent on ω_L (the diamagnetic term). Show that the new stationary states of the system and their degrees of degeneracy can be determined exactly.

(c) Show that, if ω_L is much smaller than ω_0, then the effect of the diamagnetic term is negligible compared to that of the paramagnetic term.

(d) We now consider the first excited state of the oscillator, that is, the states whose energies approach $5\hbar\omega_0/2$ when $\omega_L \to 0$. To first order in ω_L/ω_0, what are the energy levels in the presence of the field $\vec{B}$ and their degrees of degeneracy (the Zeeman effect for a three-dimensional harmonic oscillator)? Same questions for the second excited state.

(e) Now consider the ground state. How does its energy vary as a function of ω_L (the diamagnetic effect on the ground state)? Calculate the magnetic susceptibility χ of this state. Is the ground state, in the presence of the field $\vec{B}$, an eigenvector of $\hat{\vec{L}}^2$? of $\hat{L}_z$? of $\hat{L}_x$? Give the form of its wave function and the corresponding probability current. Show that the effect of the field $\vec{B}$ is to compress the wave function about Oz (in a ratio $[1+(\omega_L/\omega_0)^2]^{1/4}$) and to induce a current.

(a) We first consider the problem in the Cartesian coordinate system and then reconsider the problem in the spherical coordinate system. In the Cartesian coordinate system, the eigenvalue equation of the Hamiltonian $\hat{H}_0$ reads

$$-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+\frac{1}{2}\mu\omega_0^2(x^2+y^2+z^2)\varphi(x,y,z)=E\varphi(x,y,z).$$

The above equation can be solved by separating variables. Let $\varphi(x,y,z)=X(x)Y(y)Z(z)$. We then have

$$\begin{aligned}
&-\frac{\hbar^2}{2\mu}\frac{d^2X(x)}{dx^2}+\frac{1}{2}\mu\omega_0^2x^2X(x)=E_1X(x),\\
&-\frac{\hbar^2}{2\mu}\frac{d^2Y(y)}{dy^2}+\frac{1}{2}\mu\omega_0^2y^2Y(y)=E_2Y(y),\\
&-\frac{\hbar^2}{2\mu}\frac{d^2Z(z)}{dz^2}+\frac{1}{2}\mu\omega_0^2z^2Z(z)=E_3Z(z)
\end{aligned}$$

with

$$E=E_1+E_2+E_3.$$

The equations for $X(x)$, $Y(y)$, and $Z(z)$ are of the form of the eigenvalue equation of the Hamiltonian of the one-dimensional oscillator. Thus, the energy levels are given by

$$E_{n_1n_2n_3}=\left(n_1+n_2+n_3+3/2\right)\hbar\omega_0,\ n_1,n_2,n_3=0,1,2,\cdots.$$

Let $n=n_1+n_2+n_3$. For a given value of n, n_1 can take on the values 0, 1, 2, $\cdots$, n. For fixed values of n and n_1, the number of different values of the pair (n_2,n_3) is $n-n_1+1$. Thus, the degree of degeneracy g_n of the nth energy level, $E_n=(n+3/2)\hbar\omega_0$ with $n=n_1+n_2+n_3=0,1,2,\cdots$, is given by

$$\begin{aligned}
g_n&=\sum_{n_1=0}^{n}\left(n-n_1+1\right)=(n+1)\sum_{n_1=0}^{n}1-\sum_{n_1=1}^{n}n_1\\
&=(n+1)^2-\frac{1}{2}n(n+1)=\frac{1}{2}(n+1)(n+2).
\end{aligned}$$

The expression of $\hat{H}_0$ in the spherical coordinate system is given by

$$\hat{H}_0=-\frac{\hbar^2}{2\mu r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{\hat{\vec{L}}^2}{2\mu r^2}+\frac{1}{2}\mu\omega_0^2r^2.$$

The eigenvalue equation of $\hat{H}_0$ in the spherical coordinate system reads

$$\left[-\frac{\hbar^2}{2\mu r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{\hat{\vec{L}}^2}{2\mu r^2}+\frac{1}{2}\mu\omega_0^2r^2\right]\varphi(r,\theta,\phi)=E\varphi(r,\theta,\phi).$$

From the expression of $\hat{H}_0$, we see that

$$[\hat{H}_0, \hat{\vec{L}}^2] = 0, \; [\hat{H}_0, \hat{L}_z] = 0$$

Thus, it is possible to construct a basis of eigenstates common to $\hat{H}_0$, $\hat{\vec{L}}^2$, $\hat{L}_z$. Let $\varphi_{n\ell m}(r,\theta,\phi) = R_{n\ell}(r)Y_{\ell m}(\theta,\phi)$ with $Y_{\ell m}(\theta,\phi)$ a normalized spherical harmonic function. The equation for $R_{n\ell}(r)$ is given by

$$-\frac{\hbar^2}{2\mu r}\frac{d^2}{dr^2}\big[rR_{n\ell}(r)\big] + \left[\frac{\ell(\ell+1)\hbar^2}{2\mu r^2} + \frac{1}{2}\mu\omega_0^2 r^2\right]R_{n\ell}(r) = E_{n\ell}R_{n\ell}(r).$$

Let $u_{n\ell}(r) = rR_{n\ell}(r)$. The equation for $u_{n\ell}(r)$ reads

$$-\frac{\hbar^2}{2\mu}\frac{d^2 u_{n\ell}(r)}{dr^2} + \left[\frac{\ell(\ell+1)\hbar^2}{2\mu r^2} + \frac{1}{2}\mu\omega_0^2 r^2\right]u_{n\ell}(r) = E_{n\ell}u_{n\ell}(r).$$

Note that, for $R_{n\ell}(r)$ to be finite at $r=0$, we must have $u_{n\ell}(r=0)=0$; for $R_{n\ell}(r)$ to be zero at $r=\infty$, we must have $u_{n\ell}(r\to\infty)=0$. To solve the equation for $u_{n\ell}(r)$, we first examine its behavior in the $r\to 0$ and $r\to\infty$ limits. In the $r\to 0$ limit, the equation for $u_{n\ell}(r)$ becomes

$$-\frac{d^2 u_{n\ell}(r)}{dr^2} + \frac{\ell(\ell+1)}{r^2}u_{n\ell}(r) = 0, \; r\to 0.$$

The general solution to the above equation is given by

$$u_{n\ell}(r) = A_{n\ell}r^{\ell+1} + B_{n\ell}r^{-\ell}, \; r\to 0.$$

To satisfy $u_{n\ell}(r=0)=0$, we must have

$$u_{n\ell}(r) = A_{n\ell}r^{\ell+1}, \; r\to 0.$$

In the $r\to\infty$ limit, the equation for $u_{n\ell}(r)$ becomes

$$\frac{d^2 u_{n\ell}(r)}{dr^2} = \left(\frac{\mu\omega_0}{\hbar}\right)^2 r^2 u_{n\ell}(r), \; r\to\infty.$$

The approximate solution of the above equation that goes to 0 as $r\to\infty$ is given by

$$u_{n\ell}(r) \sim e^{-\alpha^2 r^2/2}, \; r\to\infty.$$

Here

$$\alpha = \sqrt{\frac{\mu\omega_0}{\hbar}}.$$

Taking into account the behaviors of the solution both at $r=0$ and at $r=\infty$, we put the solution into the form

$$u_{n\ell}(r) = (\alpha r)^{\ell+1}e^{-\alpha^2 r^2/2}f_{n\ell}(\alpha r).$$

The equation for $f_{n\ell}(\xi)$ with $\xi = \alpha r$ is given by

$$\frac{d^2 f_{n\ell}(\xi)}{d\xi^2} + 2\left(\frac{\ell+1}{\xi} - \xi\right)\frac{df_{n\ell}(\xi)}{d\xi} + \left[\beta_{n\ell} - (2\ell+3)\right]f_{n\ell}(\xi) = 0$$

with $\beta_{n\ell} = 2E_{n\ell}/\hbar\omega_0$. The above equation can be converted into the generalized Laguerre equation through a change of variables from ξ to $\eta = \xi^2$. Let $f_{n\ell}(\eta)$ denote $f_{n\ell}(\xi)$. We have

$$\eta\frac{d^2 f_{n\ell}(\eta)}{d\eta^2} + \big(\ell + 3/2 - \eta\big)\frac{df_{n\ell}(\eta)}{d\eta} + \frac{1}{4}\big(\beta_{n\ell} - 2\ell - 3\big)f_{n\ell}(\eta) = 0.$$

The above equation has a solution that goes to 0 as $\eta \to \infty$ only if

$$\frac{1}{4}\big(\beta_{n\ell} - 2\ell - 3\big) = n$$

with $n = 0, 1, 2, \cdots$. The solution is given by

$$f_{n\ell}(\eta) = L_n^{\ell+1/2}(\eta),$$

where $L_n^{\ell+1/2}(\eta)$ is a generalized Laguerre polynomial of degree n. The energy eigenfunction is then given by

$$\begin{aligned}\varphi_{n\ell m}(r,\theta,\phi) &= R_{r\ell}(r)Y_{\ell m}(\theta,\phi) = \frac{u_{n\ell}(r)}{r}Y_{\ell m}(\theta,\phi) \\ &= \alpha(\alpha r)^{\ell} e^{-(\alpha r)^2/2} f_{n\ell}(\alpha r) Y_{\ell m}(\theta,\phi) \\ &= \alpha(\alpha r)^{\ell} e^{-(\alpha r)^2/2} L_n^{\ell+1/2}\big((\alpha r)^2\big) Y_{\ell m}(\theta,\phi).\end{aligned}$$

Let $N_{n\ell}$ be the normalization constant. We then have

$$\varphi_{n\ell m}(r,\theta,\phi) = N_{n\ell}(\alpha r)^{\ell} e^{-(\alpha r)^2/2} L_n^{\ell+1/2}\big((\alpha r)^2\big) Y_{\ell m}(\theta,\phi).$$

Note that $Y_{\ell m}(\theta,\phi)$ is already normalized. The normalization condition for the radial part is given by

$$|N_{n\ell}|^2 \int_0^\infty dr\, r^2 (\alpha r)^{2\ell} e^{-(\alpha r)^2} \big[L_n^{\ell+1/2}\big((\alpha r)^2\big)\big]^2 = 1.$$

Making a change of integral variables from r to $x = (\alpha r)^2$, we have

$$\frac{1}{2\alpha^3}|N_{n\ell}|^2 \int_0^\infty dx\, x^{\ell+1/2} e^{-x} \big[L_n^{\ell+1/2}(x)\big]^2 = 1.$$

From the orthogonality relation

$$\int_0^\infty x^{\alpha} e^{-x} L_n^{\alpha}(x) L_m^{\alpha}(x) dx = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{nm},$$

we have

$$\frac{1}{2\alpha^3}|N_{n\ell}|^2 \frac{\Gamma(n+\ell+3/2)}{n!} = 1$$

from which we have

$$|N_{n\ell}| = \sqrt{\frac{2\alpha^3 n!}{\Gamma(n+\ell+3/2)}} = \left(\frac{\alpha^6}{\pi}\right)^{1/4} \sqrt{\frac{2^{n+\ell+2} n!}{(2n+2\ell+1)!!}}.$$

Choosing

$$N_{n\ell} = \sqrt{\frac{2\alpha^3 n!}{\Gamma(n+\ell+3/2)}} = \sqrt{\frac{2^{n+\ell+2} n! \alpha^3}{(2n+2\ell+1)!!\sqrt{\pi}}},$$

the normalized common eigenfunctions are given by

$$\begin{aligned}&\varphi_{n\ell m}(r,\theta,\phi) = \sqrt{\frac{2^{n+\ell+2} n! \alpha^3}{(2n+2\ell+1)!!\sqrt{\pi}}}\,(\alpha r)^{\ell} e^{-(\alpha r)^2/2} L_n^{\ell+1/2}\big((\alpha r)^2\big) Y_{\ell m}(\theta,\phi), \\ &n, \ell = 0, 1, 2, \cdots,\ m = -\ell, -\ell+1, \cdots, \ell-1, \ell.\end{aligned}$$

We now consider the degeneracy of the energy levels. From $\beta_{n\ell} = 2E_{n\ell}/\hbar\omega_0$, we have.

$$\begin{aligned}E_{n\ell} &= (2n+\ell+3/2)\hbar\omega_0,\ n, \ell = 0, 1, 2, \cdots \\ &= (n'+3/2),\ n' = 2n+\ell,\ n, \ell = 0, 1, 2, \cdots.\end{aligned}$$

From $n' = 2n + \ell$, we see that, if n' is even, then ℓ is even; if n' is odd, then ℓ is odd. There are $2\ell + 1$ eigenvectors corresponding to a given value of ℓ. The degree of degeneracy of the energy level can be found as follows.

For n' being an even integer, then ℓ can take on even integral values, $\ell = 0, 2, 4, \cdots, n'$. Let $\ell = 2\ell'$. Then, $\ell' = 0, 1, 2, \cdots, n'/2$. We have

$$\begin{aligned} g_{n'} &= \sum_{\ell=0,2,4,\cdots,n'} (2\ell+1) = \sum_{\ell'=0}^{n'/2}(4\ell'+1) = 4\cdot\frac{1}{2}\frac{n'}{2}\left(\frac{n'}{2}+1\right) + \left(\frac{n'}{2}+1\right) \\ &= (n'+1)\left(\frac{n'}{2}+1\right) = \frac{1}{2}(n'+1)(n'+2). \end{aligned}$$

For n' being an odd integer, then ℓ can take on odd integral values, $\ell = 1, 3, 5, \cdots, n'$. Let $\ell = 2\ell' + 1$. Then, $\ell' = 0, 1, 2, \cdots, (n'-1)/2$. We have

$$\begin{aligned} g_{n'} &= \sum_{\ell=1,3,5,\cdots,n'} (2\ell+1) = \sum_{\ell'=0}^{(n'-1)/2}(4\ell'+3) = 4\cdot\frac{1}{2}\frac{n'-1}{2}\left(\frac{n'-1}{2}+1\right) + 3\left(\frac{n'-1}{2}+1\right) \\ &= \left(\frac{n'-1}{2}+1\right)\left(n'-1+3\right) = \frac{1}{2}(n'+1)(n'+2). \end{aligned}$$

Thus, no matter whether n' is an even or odd integer, the degree of degeneracy of the energy level for a given value of n' is given by

$$g_{n'} = \frac{1}{2}(n'+1)(n'+2).$$

In terms of n and ℓ, the degree of degeneracy $g_{n\ell}$ of the energy level $E_{n\ell}$ is given by

$$g_{n\ell} = \frac{1}{2}(2n+\ell+1)(2n+\ell+2).$$

(b) In the magnetic field $\vec{B} = B\vec{e}_z$, the vector potential is given by

$$\vec{A} = -\frac{1}{2}B\vec{r}\times\vec{e}_z = \frac{1}{2}B(x\vec{e}_y - y\vec{e}_x).$$

The Hamiltonian of the particle is now given by

$$\begin{aligned} \hat{H} &= \frac{1}{2\mu}\big(\hat{\vec{p}} - q\vec{A}\big)^2 + \frac{1}{2}\mu\omega_0^2\hat{\vec{r}}^2 \\ &= \frac{1}{2\mu}\left[\hat{\vec{p}} - \frac{1}{2}qB(x\vec{e}_y - y\vec{e}_x)\right]^2 + \frac{1}{2}\mu\omega_0^2\hat{\vec{r}}^2 \\ &= \hat{H}_0 + \hat{H}_1(\omega_L), \end{aligned}$$

where $\hat{H}_1(\omega_L)$ is given by

$$\hat{H}_1(\omega_L) = \omega_L\hat{L}_z + \frac{1}{2}\mu\omega_L^2(x^2+y^2).$$

We see that the first term in $\hat{H}_1(\omega_L)$ is linear in ω_L, while the second term is quadratic in ω_L.

Because the Hamiltonian $\hat{H}$ of the particle possesses the cylindrical symmetry, it can be broken into three commuting parts. We have

$$\begin{aligned} \hat{H} &= \hat{h}_1 + \hat{h}_2 + \hat{h}_3, \\ \hat{h}_1 &= -\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}\mu\Omega^2(x^2+y^2), \\ \hat{h}_2 &= -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial z^2} + \frac{1}{2}\mu\omega_0^2 z^2, \\ \hat{h}_3 &= \omega_L\hat{L}_z. \end{aligned}$$

The first part $\hat{h}_1$ describes a two-dimensional harmonic oscillator of angular frequency $\Omega = \sqrt{\omega_0^2 + \omega_L^2}$, the second part $\hat{h}_2$ describe a one-dimensional harmonic oscillator of angular frequency ω_0, and the third part $\hat{h}_3$ describes a particle with a magnetic moment due to its orbital angular momentum in a magnetic field. Since the three parts commute, they and $\hat{H}$ have common eigenvectors.

The eigenvalues, the degrees of degeneracy of the eigenvalues, and the eigenvectors of $\hat{h}_1$, $\hat{h}_2$, and $\hat{h}_3$ are given in the following table.

Hamiltonian	Eigenvalue	Degree of degeneracy	Eigenvector
$\hat{h}_1$	$(n_1 + n_2 + 1)\hbar\Omega$, $n_1, n_2 = 0, 1, 2, \cdots$	$n_1 + n_2 + 1$	$\|n_1 n_2\rangle$ of 2D HO
$\hat{h}_2$	$(n_3 + 1/2)\hbar\omega_0$, $n_3 = 0, 1, 2, \cdots$	1	$\|n_3\rangle$ of 1D HO
$\hat{h}_3$	$m\hbar\omega_L$, $m = 0, \pm 1, \pm 2, \cdots$	1	$\|m\rangle$ of $\hat{L}_z$

(c) If $\omega_L \ll \omega_0$,

$$\Omega = \sqrt{\omega_0^2 + \omega_L^2} \approx \omega_0.$$

Thus, the effect of the diamagnetic term is negligible compared to that of the paramagnetic term.

(d) **First excited state of the oscillator.** With $\omega_L \to 0$, we neglect the diamagnetic term. In the absence of the magnetic field, the three degenerate eigenfunctions of the first excited state are given by

$$\begin{aligned}
\varphi_{100}(x, y, z) &= \left(\frac{\alpha^2}{\pi}\right)^{3/4} \sqrt{2}\,\alpha x\, e^{-\alpha^2 r^2/2}, \\
\varphi_{010}(x, y, z) &= \left(\frac{\alpha^2}{\pi}\right)^{3/4} \sqrt{2}\,\alpha y\, e^{-\alpha^2 r^2/2}, \\
\varphi_{001}(x, y, z) &= \left(\frac{\alpha^2}{\pi}\right)^{3/4} \sqrt{2}\,\alpha z\, e^{-\alpha^2 r^2/2}.
\end{aligned}$$

Making use of

$$\begin{aligned}
x &= \sqrt{\frac{2\pi}{3}}\, r\big[-Y_{11}(\theta, \phi) + Y_{1,-1}(\theta, \phi)\big], \\
y &= \sqrt{\frac{2\pi}{3}}\, ir\big[Y_{11}(\theta, \phi) + Y_{1,-1}(\theta, \phi)\big], \\
z &= \sqrt{\frac{4\pi}{3}}\, rY_{10}(\theta, \phi),
\end{aligned}$$

we can reexpress the above eigenfunctions as

$$\begin{aligned}
\varphi_{100}(r, \theta, \phi) &= 2\left(\frac{\alpha^6}{9\pi}\right)^{1/4} \alpha r\, e^{-\alpha^2 r^2/2}\big[-Y_{11}(\theta, \phi) + Y_{1,-1}(\theta, \phi)\big], \\
\varphi_{010}(r, \theta, \phi) &= 2\left(\frac{\alpha^6}{9\pi}\right)^{1/4} i\alpha r\, e^{-\alpha^2 r^2/2}\big[Y_{11}(\theta, \phi) + Y_{1,-1}(\theta, \phi)\big], \\
\varphi_{001}(r, \theta, \phi) &= 2\sqrt{2}\left(\frac{\alpha^6}{9\pi}\right)^{1/4} \alpha r\, e^{-\alpha^2 r^2/2} Y_{10}(\theta, \phi).
\end{aligned}$$

In the presence of the magnetic field, the new energy levels, the degrees of degeneracy, and the eigenfunctions can be inferred through expressing the spherical harmonic functions in terms of $\varphi_{100}(r, \theta, \phi)$, $\varphi_{010}(r, \theta, \phi)$, and $\varphi_{001}(r, \theta, \phi)$, and are given in the following table. We see that the original three-fold degeneracy is completely lifted.

Eigenvalue	Degree of degeneracy	Eigenfunction
$5\hbar\omega_0/2 + \hbar\omega_L$	1	$-\frac{1}{\sqrt{2}}\big(\varphi_{100} + i\varphi_{010}\big)$
$5\hbar\omega_0/2$	1	φ_{001}
$5\hbar\omega_0/2 - \hbar\omega_L$	1	$\frac{1}{\sqrt{2}}\big(\varphi_{100} - i\varphi_{010}\big)$

Second excited state of the oscillator. With $\omega_L \to 0$, we neglect the diamagnetic term. In the absence of the magnetic field, the six degenerate eigenfunctions of the second excited state are given by

$$\begin{aligned}
\varphi_{110}(x,y,z) &= 2\left(\frac{\alpha^2}{\pi}\right)^{3/4} \alpha^2 xy\, e^{-\alpha^2 r^2/2},\\
\varphi_{101}(x,y,z) &= 2\left(\frac{\alpha^2}{\pi}\right)^{3/4} \alpha^2 zx\, e^{-\alpha^2 r^2/2},\\
\varphi_{011}(x,y,z) &= 2\left(\frac{\alpha^2}{\pi}\right)^{3/4} \alpha^2 yz\, e^{-\alpha^2 r^2/2},\\
\varphi_{200}(x,y,z) &= \frac{1}{\sqrt{2}}\left(\frac{\alpha^2}{\pi}\right)^{3/4} \big(2\alpha^2 x^2 - 1\big) e^{-\alpha^2 r^2/2},\\
\varphi_{020}(x,y,z) &= \frac{1}{\sqrt{2}}\left(\frac{\alpha^2}{\pi}\right)^{3/4} \big(2\alpha^2 y^2 - 1\big) e^{-\alpha^2 r^2/2},\\
\varphi_{002}(x,y,z) &= \frac{1}{\sqrt{2}}\left(\frac{\alpha^2}{\pi}\right)^{3/4} \big(2\alpha^2 z^2 - 1\big) e^{-\alpha^2 r^2/2}.
\end{aligned}$$

Making use of

$$\begin{aligned}
Y_{22}(\theta,\phi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\frac{(x+iy)^2}{r^2},\\
Y_{21}(\theta,\phi) &= -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi} = -\sqrt{\frac{15}{8\pi}}\frac{(x+iy)z}{r^2},\\
Y_{20}(\theta,\phi) &= \frac{1}{2}\sqrt{\frac{5}{4\pi}}\big(3\cos^2\theta - 1\big) = \frac{1}{2}\sqrt{\frac{5}{4\pi}}\frac{2z^2 - x^2 - y^2}{r^2},\\
Y_{2,-1}(\theta,\phi) &= \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{-i\phi} = \sqrt{\frac{15}{8\pi}}\frac{(x-iy)z}{r^2},\\
Y_{2,-2}(\theta,\phi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{-2i\phi} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\frac{(x-iy)^2}{r^2},
\end{aligned}$$

we can reexpress the above eigenfunctions as

$$\begin{aligned}
\varphi_{110}(r,\theta,\phi) &= -2i\sqrt{\frac{2}{15}}\left(\frac{\alpha^6}{\pi}\right)^{1/4} \alpha^2 r^2\, e^{-\alpha^2 r^2/2}\big[\,Y_{22}(\theta,\phi) - Y_{2,-2}(\theta,\phi)\,\big],\\
\varphi_{101}(r,\theta,\phi) &= -2\sqrt{\frac{2}{15}}\left(\frac{\alpha^6}{\pi}\right)^{1/4} \alpha^2 r^2\, e^{-\alpha^2 r^2/2}\big[\,Y_{21}(\theta,\phi) - Y_{2,-1}(\theta,\phi)\,\big],\\
\varphi_{011}(r,\theta,\phi) &= 2i\sqrt{\frac{2}{15}}\left(\frac{\alpha^6}{\pi}\right)^{1/4} \alpha^2 r^2\, e^{-\alpha^2 r^2/2}\big[\,Y_{21}(\theta,\phi) + Y_{2,-1}(\theta,\phi)\,\big],\\
\varphi_{200}(r,\theta,\phi) - \varphi_{020}(r,\theta,\phi) &= \frac{4}{\sqrt{15}}\left(\frac{\alpha^6}{\pi}\right)^{1/4} \alpha^2 r^2\, e^{-\alpha^2 r^2/2}\big[\,Y_{22}(\theta,\phi) + Y_{2,-2}(\theta,\phi)\,\big],\\
2\varphi_{002}(r,\theta,\phi) - \varphi_{200}(r,\theta,\phi) - \varphi_{020}(r,\theta,\phi) &= 4\sqrt{\frac{2}{5}}\left(\frac{\alpha^6}{\pi}\right)^{1/4} \alpha^2 r^2\, e^{-\alpha^2 r^2/2} Y_{20}(\theta,\phi).
\end{aligned}$$

Expresssing $Y_{22}(\theta,\phi)$, $Y_{21}(\theta,\phi)$, $Y_{20}(\theta,\phi)$, $Y_{2,-1}(\theta,\phi)$, and $Y_{2,-2}(\theta,\phi)$ in terms of $\varphi_{110}(r,\theta,\phi)$, $\varphi_{101}(r,\theta,\phi)$, $\varphi_{011}(r,\theta,\phi)$, $\varphi_{200}(r,\theta,\phi)$, $\varphi_{020}(r,\theta,\phi)$, and $\varphi_{002}(r,\theta,\phi)$, we can infer the new energy levels, the degrees of degeneracy, and the eigenfunctions in the presence of the magnetic field. Note that an independent combination independent of the angles can be also obtained. The results are given in the following table. We see that the degeneracy is not completely lifted.

Eigenvalue	Degree of degeneracy	Eigenfunction
$7\hbar\omega_0/2 + 2\hbar\omega_L$	1	$\frac{1}{2}\big(\varphi_{200} + i\sqrt{2}\,\varphi_{110} - \varphi_{020}\big)$
$7\hbar\omega_0/2 + \hbar\omega_L$	1	$\frac{1}{\sqrt{2}}\big(\varphi_{101} + i\varphi_{011}\big)$
$7\hbar\omega_0/2$	2	$\frac{1}{\sqrt{6}}\big(2\varphi_{002} - \varphi_{200} - \varphi_{020}\big)$ $\frac{1}{\sqrt{3}}\big(\varphi_{200} + \varphi_{020} + \varphi_{002}\big)$
$7\hbar\omega_0/2 - \hbar\omega_L$	1	$\frac{1}{\sqrt{2}}\big(\varphi_{101} - i\varphi_{011}\big)$
$7\hbar\omega_0/2 - 2\hbar\omega_L$	1	$\frac{1}{2}\big(\varphi_{200} - i\sqrt{2}\,\varphi_{110} - \varphi_{020}\big)$

(e) The ground-state wave function in the cylindrical coordinate system is given by

$$\varphi_g(\rho,\phi,z) = \left(\frac{\mu}{\pi\hbar}\right)^{3/4}\big(\Omega^2\omega_0\big)^{1/4} e^{-\mu\Omega\rho^2/2\hbar - \mu\omega_0 z^2/2\hbar}.$$

Because of the independence of ϕ of $\varphi_g(\rho,\phi,z)$, the z-component of the angular momentum is equal to zero in the ground state. Thus, the paramagnetic term does not make a contribution in the ground state. The ground-state energy is given by

$$E_g = \hbar\big(\Omega + \omega_0/2\big) = \hbar\Big(\sqrt{\omega_0^2 + \omega_L^2} + \omega_0/2\Big).$$

The above equation gives us the dependence of E_g on ω_L. Inserting $\omega_L = -qB/2\mu$, we have the following dependence of E_g on B

$$E_g = \hbar\Big(\sqrt{\omega_0^2 + q^2B^2/4\mu^2} + \omega_0/2\Big).$$

From E_g the magnetization can be calculated by taking the derivative of E_g with respect to B. We have

$$M_z = -\frac{\partial E_g}{\partial B} = -\frac{\hbar q^2 B}{4\mu^2\sqrt{\omega_0^2 + q^2B^2/4\mu^2}}.$$

The magnetic susceptibility is then given by

$$\chi = \frac{\partial M_z}{\partial B} = -\frac{\hbar q^2}{4\mu^2\sqrt{\omega_0^2 + q^2B^2/4\mu^2}} + \frac{\hbar(q^2B)^2}{16\mu^4(\omega_0^2 + q^2B^2/4\mu^2)^{3/2}} = -\frac{\hbar q^2\omega_0^2}{4\mu^2(\omega_0^2 + q^2B^2/4\mu^2)^{3/2}}.$$

Note that the dimensionless magnetic susceptibility is defined by $\chi = \partial M_z/\partial H$. That $\chi < 0$ indicates that diamagnetism exists in the ground state. In the linear regime, we have

$$\chi = \lim_{B\to 0}\frac{\partial M_z}{\partial B} = -\frac{\hbar q^2\omega_0^2}{4\mu^2\omega_0^3} = -\frac{\hbar q^2}{4\mu^2\omega_0}.$$

Since the ground-state wave function does not depend on ϕ, the ground state is an eigenvector of $\hat{L}_z$ corresponding to the eigenvalue 0. From the expression of the ground-state wave function, we can see that the ground state is not an eigenvector of $\hat{\vec{L}}^2$ or $\hat{L}_x$.

The probability current density in the ground state is given by

$$\begin{aligned}
\vec{J} &= \frac{1}{\mu}\,\mathrm{Re}\big[\,\varphi_g^*\big(-i\hbar\vec{\nabla} - q\vec{A}\,\big)\varphi_g\,\big] = -\frac{q}{\mu}\big|\varphi_g\big|^2\vec{A} \\
&= -\frac{qB}{2\mu}\big|\varphi_g\big|^2\big(x\vec{e}_y - y\vec{e}_x\big) = \omega_L\big|\varphi_g\big|^2\rho\vec{e}_\phi \\
&= \omega_L\left(\frac{\mu}{\pi\hbar}\right)^{3/2}\big(\Omega^2\omega_0\big)^{1/2}e^{-\mu\Omega\rho^2/\hbar-\mu\omega_0 z^2/\hbar}\rho\vec{e}_\phi.
\end{aligned}$$

which indicates that the induced probability current flows around the z-axis with the electric current density given by

$$\vec{J}_q = q\vec{J} = q\omega_L\left(\frac{\mu}{\pi\hbar}\right)^{3/2}\big(\Omega^2\omega_0\big)^{1/2}e^{-\mu\Omega\rho^2/\hbar-\mu\omega_0 z^2/\hbar}\rho\vec{e}_\phi.$$

The electric current within a ring of area $d\rho dz$ and of a distance ρ to the z-axis is given by

$$dI = \vec{J}_q\cdot\vec{e}_\phi d\rho dz.$$

The magnetic moment produced by the current dI is given by

$$dM_z = \pi\rho^2 dI = \pi\rho^2\vec{J}_q\cdot\vec{e}_\phi d\rho dz = q\omega_L\left(\frac{\mu}{\hbar}\right)^{3/2}\left(\frac{\Omega^2\omega_0}{\pi}\right)^{1/2}e^{-\mu\Omega\rho^2/\hbar-\mu\omega_0 z^2/\hbar}\rho^3 d\rho dz.$$

Integrating, we have

$$\begin{aligned}
M_z &= q\omega_L\left(\frac{\mu}{\hbar}\right)^{3/2}\left(\frac{\Omega^2\omega_0}{\pi}\right)^{1/2}\int_0^\infty d\rho\;\rho^3 e^{-\mu\Omega\rho^2/\hbar}\int_{-\infty}^{\infty} dz\;{}^{-\mu\omega_0 z^2/\hbar} \\
&= q\omega_L\left(\frac{\mu}{\hbar}\right)^{3/2}\left(\frac{\Omega^2\omega_0}{\pi}\right)^{1/2}\cdot\frac{1}{2}\left(\frac{\hbar}{\mu\Omega}\right)^2\cdot\left(\frac{\pi\hbar}{\mu\omega_0}\right)^{1/2} \\
&= \frac{\hbar\omega_L q}{2\mu\Omega} = -\frac{\hbar q^2 B}{4\mu^2\sqrt{\omega_0^2 + q^2B^2/4\mu^2}}
\end{aligned}$$

which is identical with the result obtained in the above through taking the derivative of the ground-state energy with respective to B.

The uncertainties Δx, Δy, and Δz can be obtained from the ground-state wave function and they are respectively given by

$$\Delta x = \Delta y = \sqrt{\frac{\hbar}{2\mu\Omega}},\ \Delta z = \sqrt{\frac{\hbar}{2\mu\omega_0}}.$$

From the above results, we see that Δx and Δy become smaller in the magnetic field because $\Omega > \omega_0$. In the absence of the magnetic field, Δx and Δy are given by

$$(\Delta x)_0 = (\Delta y)_0 = \sqrt{\frac{\hbar}{2\mu\omega_0}}.$$

We then have

$$\frac{(\Delta x)_0}{\Delta x} = \frac{(\Delta y)_0}{\Delta y} = \sqrt{\frac{\Omega}{\omega_0}} = (1+\omega_L^2/\omega_0^2)^{1/4} > 1.$$

Thus, the wave function is compressed about the z-axis with the ratio given by $(1+\omega_L^2/\omega_0^2)^{1/4}$.