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Problem 1. [C-T Exercise 2-1] $|\varphi_n\rangle$ are the eigenstates of a Hermitian operator $\hat{H}$ ($\hat{H}$ is, for example, the Hamiltonian of an arbitrary physical system). Assume that the states $|\varphi_n\rangle$ form a discrete orthonormal basis. The operator $\hat{U}(m,n)$ is defined by $\hat{U}(m,n) = |\varphi_m\rangle\langle\varphi_n|$.

(a) Calculate the adjoint $\hat{U}^\dagger(m,n)$ of $\hat{U}(m,n)$.

(b) Calculate the commutator $[\hat{H},\hat{U}(m,n)]$.

(c) Prove the relation $\hat{U}(m,n)\hat{U}^\dagger(p,q) = \delta_{nq}\hat{U}(m,p)$.

(d) Calculate $\mathrm{Tr}\{\hat{U}(m,n)\}$, the trace of the operator $\hat{U}(m,n)$.

(e) Let $\hat{A}$ be an operator, with matrix elements $A_{mn} = \langle\varphi_m|\hat{A}|\varphi_n\rangle$. Prove the relation $\hat{A} = \sum_{m,n} A_{mn}\hat{U}(m,n)$.

(f) Show that $A_{pq} = \mathrm{Tr}\{\hat{A}U^\dagger(p,q)\}$.

Solution:

(a)

$$\begin{aligned}&\langle\psi|U^\dagger(m,n)|\varphi\rangle = \langle\psi|(|\varphi_m\rangle\langle\varphi_n|)^\dagger|\varphi\rangle = [\langle\varphi|(|\varphi_m\rangle\langle\varphi_n|)|\psi\rangle]^* = [\langle\varphi|\varphi_m\rangle\langle\varphi_n|\psi\rangle]^*\\ =&\langle\varphi|\varphi_m\rangle^*\langle\varphi_n|\psi\rangle^* = \langle\varphi_m|\varphi\rangle\langle\psi|\varphi_n\rangle = \langle\psi|\varphi_n\rangle\langle\varphi_m|\varphi\rangle = \langle\psi|(|\varphi_n\rangle\langle\varphi_m|)|\varphi\rangle\end{aligned} \tag{1}$$

Therefore,

$$\hat{U}^\dagger(m,n) = |\varphi_n\rangle\langle\varphi_m| \tag{2}$$

(b)

$$\begin{aligned}[\hat{H},\hat{U}(m,n)]|\varphi\rangle =&\hat{H}\hat{U}(m,n)|\varphi\rangle - \hat{U}(m,n)\hat{H}|\varphi\rangle\\ =&\hat{H}|\varphi_m\rangle\langle\varphi_n|\varphi\rangle - |\varphi_m\rangle\langle\varphi_n|\hat{H}|\varphi\rangle\\ =&\hat{H}|\varphi_m\rangle\langle\varphi_n|\varphi\rangle - |\varphi_m\rangle\langle\varphi_n|\hat{H}^\dagger|\varphi\rangle\\ =&H_m|\varphi_m\rangle\langle\varphi_n|\varphi\rangle - |\varphi_m\rangle\langle\varphi_n|H_n|\varphi\rangle\\ =&(H_m - H_n)|\varphi_m\rangle\langle\varphi_n|\varphi\rangle\end{aligned} \tag{3}$$

Therefore,

$$[\hat{H},\hat{U}(m,n)] = (H_m - H_n)|\varphi_m\rangle\langle\varphi_n| \tag{4}$$

(c)

$$
\begin{aligned}
\hat{U}(m,n)\hat{U}^\dagger(p,q) =&|\varphi_m\rangle\langle\varphi_n|(|\varphi_p\rangle\langle\varphi_q|)^\dagger\\
=&|\varphi_m\rangle\langle\varphi_n|\varphi_q\rangle\langle\varphi_p|\\
=&|\varphi_m\rangle\delta_{nq}\langle\varphi_p|\\
=&\delta_{nq}|\varphi_m\rangle\langle\varphi_p|\\
=&\delta_{nq}\hat{U}(m,p)
\end{aligned}
\tag{5}
$$

(d)

$$
\begin{aligned}
\mathrm{Tr}\{\hat{U}(m,n)\} =&\sum_i[\hat{U}(m,n)]_{ii}\\
=&\sum_i\langle\varphi_i|\varphi_m\rangle\langle\varphi_n|\varphi_i\rangle\\
=&\sum_i\delta_{im}\delta_{ni}\\
=&\delta_{mn}
\end{aligned}
\tag{6}
$$

(e)

$$
\begin{aligned}
\left[\sum_{m,n}A_{mn}\hat{U}(m,n)\right]_{ij} =&\langle\varphi_i|\left[\sum_{m,n}A_{mn}|\varphi_m\rangle\langle\varphi_n|\right]|\varphi_j\rangle\\
=&\sum_{m,n}A_{mn}\langle\varphi_i|\varphi_m\rangle\langle\varphi_n|\varphi_j\rangle\\
=&\sum_{m,n}A_{mn}\delta_{im}\delta_{nj}\\
=&A_{ij}=\langle\varphi_m|\hat{A}|\varphi_n\rangle
\end{aligned}
\tag{7}
$$

Therefore,

$$
\hat{A}=\sum_{m,n}A_{mn}\hat{U}(m,n)
\tag{8}
$$

(f)

$$\begin{aligned}
\mathrm{Tr}\{\hat{A}U^\dagger(p,q)\} =& \sum_i [\hat{A}U^\dagger(p,q)]_{ii} \\
=& \sum_i \langle\varphi_i| \left[\left(\sum_{m,n} A_{mn}\hat{U}(m,n)\right) U^\dagger(p,q)\right] |\varphi_i\rangle \\
=& \sum_i \langle\varphi_i| \left[\left(\sum_{m,n} A_{mn}|\varphi_m\rangle\langle\varphi_n|\right) U^\dagger(p,q)\right] |\varphi_i\rangle \\
=& \sum_{mn} \left[A_{mn}\sum_i \langle\varphi_i|\varphi_m\rangle\langle\varphi_n|\varphi_q\rangle\langle\varphi_p|\varphi_i\rangle\right] \\
=& \sum_{mn} \left[A_{mn}\sum_i \delta_{im}\delta_{nq}\delta_{pi}\right] \\
=& \sum_{mn} A_{mn}\delta_{mp}\delta_{nq} \\
=& A_{pq}
\end{aligned} \tag{9}$$

□

Problem 2. [C-T Exercise 2-2] In a three-dimensional vector space, consider the operator whose matrix, in an orthonormal basis $\{|1\rangle,|2\rangle,|3\rangle\}$, is wirtten as $\hat{L}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$.

(a) Is $\hat{L}_y$ Hermitian? Calculate its eigenvalues and eigenvectors (giving their normalized expansion in terms of the $\{|1\rangle,|2\rangle,|3\rangle\}$ basis).

(b) Calculate the matrices which represent the projectors onto these eigenvectors. Then verify that they satisfy the orthogonality and closure relations.

Solution:

(a) The Hermitian conjugate of $\hat{L}_y$

$$\hat{L}_y^\dagger = (\hat{L}_y^T)^* = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ -i & 0 & i \\ 0 & -i & 0 \end{pmatrix} = \hat{L}_y \tag{10}$$

Therefore, $\hat{L}_y$ is Hermitian.

The secular equation of $\hat{L}_y$

$$\det|A-\lambda I| = \begin{vmatrix} -\lambda & -i & 0 \\ i & -\lambda & -i \\ 0 & i & -\lambda \end{vmatrix} = -\lambda^3 + 2\lambda = 0 \tag{11}$$

The eigenvalues of $\hat{L}_y$ are

$$\lambda_1 = \sqrt{2}, \quad \lambda_2 = 0, \quad \lambda_3 = -\sqrt{2} \tag{12}$$

The normalized eigenvectors of $\hat{L}_y$ are

$$\begin{aligned}
(A - \lambda_1 I)|\varphi_1\rangle &= \begin{pmatrix} -\sqrt{2} & -i & 0 \\ i & -\sqrt{2} & -i \\ 0 & i & -\sqrt{2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\Longrightarrow |\varphi_1\rangle &= \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} i \\ -\frac{1}{2} \end{pmatrix} && (13) \\
(A - \lambda_2 I)|\varphi_2\rangle &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\Longrightarrow |\varphi_2\rangle &= \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix} && (14) \\
(A - \lambda_3 I)|\varphi_3\rangle &= \begin{pmatrix} \sqrt{2} & -i & 0 \\ i & \sqrt{2} & -i \\ 0 & i & \sqrt{2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\Longrightarrow |\varphi_3\rangle &= \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{2}}{2} i \\ -\frac{1}{2} \end{pmatrix} && (15)
\end{aligned}$$

Rewritein them in terms of the $\{|1\rangle, |2\rangle, |3\rangle\}$ basis

$$|\varphi_1\rangle = \frac{1}{2}|1\rangle + \frac{\sqrt{2}}{2} i|2\rangle - \frac{1}{2}|3\rangle \tag{16}$$

$$|\varphi_2\rangle = \frac{\sqrt{2}}{2}|1\rangle + \frac{\sqrt{2}}{2}|3\rangle \tag{17}$$

$$|\varphi_3\rangle = \frac{1}{2}|1\rangle - \frac{\sqrt{2}}{2} i|2\rangle - \frac{1}{2}|3\rangle \tag{18}$$

(b) The matrices which represent the projectors onto these eigenvectors

$$P_1 = |\varphi_1\rangle\langle\varphi_1| = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2}i \\ -\frac{1}{2} \end{pmatrix} (\frac{1}{2} \quad \frac{\sqrt{2}}{2}i \quad -\frac{1}{2}) = \begin{pmatrix} \frac{1}{4} & \frac{\sqrt{2}}{4}i & -\frac{1}{4} \\ \frac{\sqrt{2}}{4}i & -\frac{1}{2} & -\frac{\sqrt{2}}{4}i \\ -\frac{1}{4} & -\frac{\sqrt{2}}{4}i & \frac{1}{4} \end{pmatrix} \tag{19}$$

$$P_2 = |\varphi_2\rangle\langle\varphi_2| = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix} (\frac{\sqrt{2}}{2} \quad 0 \quad \frac{\sqrt{2}}{2}) = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \tag{20}$$

$$P_3 = |\varphi_3\rangle\langle\varphi_3| = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{2}}{2}i \\ -\frac{1}{2} \end{pmatrix} (\frac{1}{2} \quad -\frac{\sqrt{2}}{2}i \quad -\frac{1}{2}) = \begin{pmatrix} \frac{1}{4} & -\frac{\sqrt{2}}{4}i & -\frac{1}{4} \\ -\frac{\sqrt{2}}{4}i & -\frac{1}{2} & \frac{\sqrt{2}}{4}i \\ -\frac{1}{4} & \frac{\sqrt{2}}{4}i & \frac{1}{4} \end{pmatrix} \tag{21}$$

$$P_1P_2 = \begin{pmatrix} \frac{1}{4} & \frac{\sqrt{2}}{4}i & -\frac{1}{4} \\ \frac{\sqrt{2}}{4}i & -\frac{1}{2} & -\frac{\sqrt{2}}{4}i \\ -\frac{1}{4} & -\frac{\sqrt{2}}{4}i & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{22}$$

$$P_2P_3 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{4} & -\frac{\sqrt{2}}{4}i & -\frac{1}{4} \\ -\frac{\sqrt{2}}{4}i & -\frac{1}{2} & \frac{\sqrt{2}}{4}i \\ -\frac{1}{4} & \frac{\sqrt{2}}{4}i & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{23}$$

$$P_3P_1 = \begin{pmatrix} \frac{1}{4} & -\frac{\sqrt{2}}{4}i & -\frac{1}{4} \\ -\frac{\sqrt{2}}{4}i & -\frac{1}{2} & \frac{\sqrt{2}}{4}i \\ -\frac{1}{4} & \frac{\sqrt{2}}{4}i & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{4} & \frac{\sqrt{2}}{4}i & -\frac{1}{4} \\ \frac{\sqrt{2}}{4}i & -\frac{1}{2} & -\frac{\sqrt{2}}{4}i \\ -\frac{1}{4} & -\frac{\sqrt{2}}{4}i & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{24}$$

Therefore, the matrices which represent the projectors onto these eigenvectors satisfy the orthogonality relation.

$$\begin{aligned} P_1 + P_2 + P_3 &= \begin{pmatrix} \frac{1}{4} & \frac{\sqrt{2}}{4}i & -\frac{1}{4} \\ \frac{\sqrt{2}}{4}i & -\frac{1}{2} & -\frac{\sqrt{2}}{4}i \\ -\frac{1}{4} & -\frac{\sqrt{2}}{4}i & \frac{1}{4} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{4} & -\frac{\sqrt{2}}{4}i & -\frac{1}{4} \\ -\frac{\sqrt{2}}{4}i & -\frac{1}{2} & \frac{\sqrt{2}}{4}i \\ -\frac{1}{4} & \frac{\sqrt{2}}{4}i & \frac{1}{4} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{25}$$

□

Problem 3. [C-T Exercise 2-3] The state space of a certain physical system is three-dimensional. Let $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ be an orthonormal basis of this space. The kets $|\psi_0\rangle$

and $|\psi_1\rangle$ are confined by

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{i}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle$$
$$|\psi_1\rangle = \frac{1}{\sqrt{3}}|u_1\rangle + \frac{i}{\sqrt{3}}|u_3\rangle$$

(a) Are these kets normalized?

(b) Calculate the matrices ρ_0 and ρ_1 representing, in the $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ basis, the projection operators onto the state $|\psi_0\rangle$ and onto the state $|\psi_1\rangle$. Verify that these matrices are Hermitian.

Solution:

(a)

$$\begin{aligned}\langle\psi_0|\psi_0\rangle =&(\frac{1}{\sqrt{2}}\langle u_1| - \frac{i}{2}\langle u_2| + \frac{1}{2}\langle u_3|)(\frac{1}{\sqrt{2}}|u_1\rangle + \frac{i}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle)\\ =&\frac{1}{2}\langle u_1|u_1\rangle + \frac{1}{4}\langle u_2|u_2\rangle + \frac{1}{4}\langle u_3|u_3\rangle = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 \end{aligned} \tag{26}$$

$$\begin{aligned}\langle\psi_1|\psi_1\rangle =&(\frac{1}{\sqrt{3}}\langle u_1| - \frac{i}{\sqrt{3}}\langle u_3|)(\frac{1}{\sqrt{3}}|u_1\rangle + \frac{i}{\sqrt{3}}|u_3\rangle)\\ =&\frac{1}{3}\langle u_1|u_1\rangle + \frac{1}{3}\langle u_3|u_3\rangle = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}\end{aligned} \tag{27}$$

Therefore, ket $|\psi_0\rangle$ is normalized while ket $|\psi_1\rangle$ is not normalized.

(b) In the $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ basis, (where ket $|\psi_1\rangle$ got normalized)

$$|\psi_0\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{2} \\ \frac{1}{2} \end{pmatrix} \tag{28}$$

$$|\psi_1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{i}{\sqrt{2}} \end{pmatrix} \tag{29}$$

The projector matrices onto the state $|\psi_0\rangle$ and onto the $|\psi_1\rangle$ are

$$\rho_0 = |\psi_0\rangle\langle\psi_0| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{2} \\ \frac{1}{2} \end{pmatrix} (\frac{1}{\sqrt{2}} \quad -\frac{i}{2} \quad \frac{1}{2}) = \begin{pmatrix} \frac{1}{2} & -\frac{i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{i}{2\sqrt{2}} & \frac{1}{4} & \frac{i}{4} \\ \frac{1}{2\sqrt{2}} & -\frac{i}{4} & \frac{1}{4} \end{pmatrix} \tag{30}$$

$$\rho_1 = |\psi_1\rangle\langle\psi_1| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{i}{\sqrt{2}} \end{pmatrix} (\frac{1}{\sqrt{2}} \quad 0 \quad -\frac{i}{\sqrt{2}}) = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{i}{2} \\ 0 & 0 & 0 \\ \frac{i}{2} & 0 & \frac{1}{2} \end{pmatrix} \tag{31}$$

$$\rho_0^\dagger = (\rho_0^T)^* = \begin{pmatrix} \frac{1}{2} & -\frac{i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{i}{2\sqrt{2}} & \frac{1}{4} & \frac{i}{4} \\ \frac{1}{2\sqrt{2}} & -\frac{i}{4} & \frac{1}{4} \end{pmatrix} = \rho_0 \tag{32}$$

$$\rho_1^\dagger = (\rho_1^T)^* = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{i}{2} \\ 0 & 0 & 0 \\ \frac{i}{2} & 0 & \frac{1}{2} \end{pmatrix} = \rho_1 \tag{33}$$

Therefore, these matrices are Hermitian.

□

Problem 4. [C-T Exercise 2-9] Let $\hat{H}$ be the Hamiltonian operator of a physical system. Denote by $|\varphi_n\rangle$ the eigenvectors of $\hat{H}$, with eigenvalue E_n, $\hat{H}|\varphi_n\rangle = E_n|\varphi_n\rangle$.

(a) For an arbitrary operator $\hat{A}$, prove the relation $\langle\varphi_n|[\hat{A},\hat{H}]|\varphi_n\rangle = 0$.

(b) Consider a one-dimensional problem, where the physical system is a particle of mass m and of potential energy $\hat{V}(\hat{x})$. In this case, $\hat{H}$ is written as $\hat{H} = \frac{1}{2m}\hat{p}^2 + \hat{V}(\hat{x})$.

   i. In terms of $\hat{p}, \hat{x}$, and $\hat{V}(\hat{x})$, find the commutators: $[\hat{H},\hat{p}], [\hat{H},\hat{x}]$ and $[\hat{H},\hat{x}\hat{p}]$.

   ii. Show that the matrix element $\langle\varphi_n|\hat{p}|\varphi_n\rangle$ is zero.

   iii. Establish a relation between $E_k = \langle\varphi_n|\frac{\hat{p}^2}{2m}|\varphi_n\rangle$ and $\langle\varphi_n|\hat{x}\frac{d\hat{V}(\hat{x})}{d\hat{x}}|\varphi_n\rangle$. Apply the derived relation to $\hat{V}(\hat{x}) = V_0\hat{x}^\lambda$ with $\lambda = 2, 4, 6, \cdots$ and $V_0 > 0$?

Solution:

(a)

$$\begin{aligned}\langle\varphi_n|[\hat{A},\hat{H}]|\varphi_n\rangle =&\langle\varphi_n|[\hat{A}\hat{H} - \hat{H}\hat{A}]|\varphi_n\rangle \\ =&\langle\varphi_n|\hat{A}\hat{H}|\varphi_n\rangle - \langle\varphi_n|\hat{H}\hat{A}|\varphi_n\rangle \\ =&\langle\varphi_n|\hat{A}E_n|\varphi_n\rangle - \langle\varphi_n|E_n\hat{A}|\varphi_n\rangle \\ =&E_n\langle\varphi_n|\hat{A}|\varphi_n\rangle - E_n\langle\varphi_n|\hat{A}|\varphi_n\rangle \\ =&0 \end{aligned} \tag{34}$$

(b) i.

$$\begin{aligned}[\hat{H},\hat{p}] =&\hat{H}\hat{p} - \hat{p}\hat{H} \\ =&\left[\frac{1}{2m}\hat{p}^2 + \hat{V}(\hat{x})\right]\hat{p} - \hat{p}\left[\frac{1}{2m}\hat{p}^2 + \hat{V}(\hat{x})\right] \\ =&\frac{1}{2m}\hat{p}^3 + \hat{V}(\hat{x})\hat{p} - \frac{1}{2m}\hat{p}^3 - \hat{p}\hat{V}(\hat{x}) \\ =&\hat{V}(\hat{x})\hat{p} - \hat{p}\hat{V}(\hat{x}) = [\hat{V}(\hat{x}),\hat{p}] \end{aligned} \tag{35}$$

$$\begin{aligned}
[\hat{H},\hat{x}]=&\hat{H}\hat{x}-\hat{x}\hat{H}\\
=&\left[\frac{1}{2m}\hat{p}^2+\hat{V}(\hat{x})\right]\hat{x}-\hat{x}\left[\frac{1}{2m}\hat{p}^2+\hat{V}(\hat{x})\right]\\
=&\frac{1}{2m}\hat{p}^2\hat{x}+\hat{V}(\hat{x})\hat{x}-\frac{1}{2m}\hat{x}\hat{p}^2-\hat{x}\hat{V}(\hat{x})\\
=&\frac{1}{2m}(\hat{p}^2\hat{x}-\hat{x}\hat{p}^2)\\
=&\frac{1}{2m}(\hat{p}^2\hat{x}-\hat{p}\hat{x}\hat{p}+\hat{p}\hat{x}\hat{p}-\hat{x}\hat{p}^2)\\
=&\frac{1}{2m}[\hat{p}(\hat{p}\hat{x}-\hat{x}\hat{p})+(\hat{p}\hat{x}-\hat{x}\hat{p})\hat{p}]\\
=&\frac{1}{2m}[\hat{p}[\hat{p},\hat{x}]+[\hat{p},\hat{x}]\hat{p}]\\
=&\frac{1}{2m}[\hat{p}(-i\hbar)-i\hbar\hat{p}]\\
=&-\frac{i\hbar}{m}\hat{p}
\end{aligned}\tag{36}$$

$$\begin{aligned}
[\hat{H},\hat{x}\hat{p}]=&\hat{H}\hat{x}\hat{p}-\hat{x}\hat{p}\hat{H}\\
=&\left[\frac{1}{2m}\hat{p}^2+\hat{V}(\hat{x})\right]\hat{x}\hat{p}-\hat{x}\hat{p}\left[\frac{1}{2m}\hat{p}^2+\hat{V}(\hat{x})\right]\\
=&\frac{1}{2m}\hat{p}^2\hat{x}\hat{p}+\hat{V}(\hat{x})\hat{x}\hat{p}-\frac{1}{2m}\hat{x}\hat{p}^3-\hat{x}\hat{p}\hat{V}(\hat{x})\\
=&\frac{1}{2m}(\hat{p}^2\hat{x}-\hat{x}\hat{p}^2)\hat{p}+\hat{x}\hat{V}(\hat{x})\hat{p}-\hat{x}\hat{p}\hat{V}(\hat{x})\\
=&-\frac{i\hbar}{m}\hat{p}^2+\hat{x}[\hat{V}(\hat{x}),\hat{p}]
\end{aligned}\tag{37}$$

ii. According to the conclusion derived in (b) i., $\hat{p}=\frac{im}{\hbar}[\hat{H},\hat{x}]$, so

$$\langle\varphi_n|\hat{p}|\varphi_n\rangle=\frac{im}{\hbar}\langle\varphi_n|[\hat{H},\hat{x}]|\varphi_n\rangle=-\frac{im}{\hbar}\langle\varphi_n|[\hat{x},\hat{H}]|\varphi_n\rangle\tag{38}$$

According to the conclusion derived in (a), $\langle\varphi_n|[\hat{A},\hat{H}]|\varphi_n\rangle=0$, therefore

$$\langle\varphi_n|\hat{p}|\varphi_n\rangle=-\frac{im}{\hbar}\langle\varphi_n|[\hat{x},\hat{H}]|\varphi_n\rangle=0\tag{39}$$

iii. According to the conclusion derived in (b) i., $[\hat{H},\hat{x}\hat{p}]=-\frac{i\hbar}{m}\hat{p}^2+[\hat{V}(\hat{x}),\hat{x}\hat{p}]$, so

$$\begin{aligned}
\langle\varphi_n|\frac{\hat{p}^2}{2m}|\varphi_n\rangle=&\langle\varphi_n|\left[-\frac{1}{2i\hbar}([\hat{H},\hat{x}\hat{p}]-\hat{x}[\hat{V}(\hat{x}),\hat{p}])\right]|\varphi_n\rangle\\
=&-\frac{1}{2i\hbar}\left[\langle\varphi_n|[\hat{H},\hat{x}\hat{p}]|\varphi_n\rangle-\langle\varphi_n|\hat{x}[\hat{V}(\hat{x}),\hat{p}]|\varphi_n\rangle\right]\\
=&\frac{1}{2i\hbar}\langle\varphi_n|\hat{x}[\hat{V}(\hat{x}),\hat{p}]|\varphi_n\rangle
\end{aligned}\tag{40}$$

where

$$\begin{aligned}[\hat{V}(\hat{x}),\hat{p}]\psi(x)=&i\hbar[V(x)\frac{d}{dx}\psi(x)-\frac{d}{dx}V(x)\psi(x)]\\=&i\hbar[V(x)\frac{d}{dx}\psi(x)-\left(\frac{d}{dx}V(x)\right)\psi(x)-V(x)\frac{d}{dx}\psi(x)]\\=&-i\hbar\left(\frac{d}{dx}V(x)\right)\psi(x)\end{aligned}\tag{41}$$

$$\Longrightarrow[\hat{V}(\hat{x}),\hat{p}]=-i\hbar\left(\frac{d}{dx}V(x)\right)\tag{42}$$

so

$$\begin{aligned}\langle\varphi_n|\frac{\hat{p}^2}{2m}|\varphi_n\rangle=&\frac{1}{2i\hbar}\langle\varphi_n|\hat{x}[\hat{V}(\hat{x}),\hat{p}]|\varphi_n\rangle\\=&-\frac{1}{2}\langle\varphi_n|\hat{x}\frac{d\hat{V}(\hat{x})}{d\hat{x}}|\varphi_n\rangle\end{aligned}\tag{43}$$

Apply the derived relation to $\hat{V}(\hat{x})=V_0\hat{x}^\lambda$ with $\lambda=2,4,6,\cdots$ and $V_0>0$

$$\begin{aligned}E_k=&-\frac{1}{2}\langle\varphi_n|\hat{x}\frac{\hat{V}(\hat{x})}{d\hat{x}}|\varphi_n\rangle\\=&-\frac{1}{2}\langle\varphi_n|\lambda V_0\hat{x}^\lambda|\varphi_n\rangle\\=&-\frac{\lambda}{2}V_0x^\lambda\\=&-\frac{\lambda}{2}V(x)\end{aligned}\tag{44}$$

□

Problem 5. [C-T Exercise 2-10] Using the relation $\langle x|p\rangle=(2\pi\hbar)^{-1/2}e^{ipx/\hbar}$, find the expression $\langle x|\hat{x}\hat{p}|\psi\rangle$ and $\langle x|\hat{p}\hat{x}|\psi\rangle$ in terms of $\psi(x)$. Can these results be found directly by using the fact that in the $\{|x\rangle\}$ representing, $\hat{p}$ acts like $-i\hbar\frac{d}{dx}$?

Solution:

$$\begin{aligned}
\langle x|\hat{x}\hat{p}|\psi\rangle =&\langle x|x\hat{p}|\psi\rangle\\
=&x\langle x|1\hat{p}1|\psi\rangle\\
=&x\langle x|\int dp|p\rangle\langle p|\hat{p}\int dx|x\rangle\langle x|\psi\rangle\\
=&x\langle x|\int dp|p\rangle\langle p|p\int dx|x\rangle\langle x|\psi\rangle\\
=&x\int dp\langle x|p\rangle p\int dx\langle p|x\rangle\langle x|\psi\rangle\\
=&x\int dp(2\pi\hbar)^{-1/2}e^{-ipx/\hbar}p\int dx(2\pi\hbar)^{-1/2}e^{ipx/\hbar}\psi(x)\\
=&x\int dp(2\pi\hbar)^{-1/2}e^{-ipx/\hbar}p\bar{\psi}_p(x)\\
=&-i\hbar x\frac{d}{dx}\psi(x) \qquad (45)\\
\langle x|\hat{p}\hat{x}|\psi\rangle =&\langle x|1\hat{p}\hat{x}|\psi\rangle\\
=&\langle x|\int dp|p\rangle\langle p|\hat{p}\hat{x}|\psi\rangle\\
=&\int dp\langle x|p\rangle\langle p|\hat{p}\hat{x}|\psi\rangle\\
=&\int dp(2\pi\hbar)^{-1/2}e^{ipx/\hbar}\langle p|p\hat{x}|\psi\rangle\\
=&\int dp(2\pi\hbar)^{-1/2}e^{ipx/\hbar}\langle p|p1\hat{x}|\psi\rangle\\
=&\int dp(2\pi\hbar)^{-1/2}e^{ipx/\hbar}\langle p|p\int dx|x\rangle\langle x|\hat{x}|\psi\rangle\\
=&\int dp(2\pi\hbar)^{-1/2}e^{ipx/\hbar}p\int dx\langle p|x\rangle\langle x|x|\psi\rangle\\
=&\int dp(2\pi\hbar)^{-1/2}e^{ipx/\hbar}p\int dx(2\pi\hbar)^{-1/2}e^{-ipx/\hbar}x\psi(x)\\
=&\int dp(2\pi\hbar)^{-1/2}e^{ipx/\hbar}pi\frac{d}{dp}\bar{\psi}_p(x)\\
=&\int dp(2\pi\hbar)^{-1/2}e^{ipx/\hbar}i[\frac{d}{dp}(p\bar{\psi}_p(x))-\bar{\psi}_p(x)]\\
=&-i\hbar[x\frac{d}{dx}\psi(x)+\psi(x)] \qquad (46)
\end{aligned}$$

These results can be found directly by using the fact that in $|x\rangle$ representing, $\hat{p}$ act like

$-i\hbar\frac{d}{dx}$:

$$\begin{aligned}\langle x|\hat{x}\hat{p}|\psi\rangle=&\int dx'\delta(x'-x)x'(-i\hbar\frac{d}{dx'})\psi(x')\\=&-i\hbar\int dx'\delta(x'-x)x'\frac{d}{dx'}\psi(x')\\=&-i\hbar x\frac{d}{dx}\psi(x)\end{aligned}\tag{47}$$

$$\begin{aligned}\langle x|\hat{p}\hat{x}|\psi\rangle=&\int dx'\delta(x'-x)(-i\hbar\frac{d}{dx'})x'\psi(x')\\=&-i\hbar\int dx'\delta(x'-x)[\psi(x')+x'\frac{d}{dx'}\psi(x')]\\=&-i\hbar[\psi(x)+x\frac{d}{dx}\psi(x)]\end{aligned}\tag{48}$$

$\square$