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PHYS1501
Quantum Mechanics
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Assignment 10

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**Problem 1.** [C-T Exercise 4-1] Consider a spin 1/2 particle of magnetic moment $\hat{\vec{M}} = \gamma\hat{\vec{S}}$. The spin state space is spanned by the basis of the $|+\rangle$ and $|-\rangle$ vectors, eigenvectors of $\hat{S}_z$ with eigenvalues $+\hbar/2$ and $\hbar/2$. At time $t=0$, the state of the system is $|\psi(t=0)\rangle = |+\rangle$.

(a) If the observable $\hat{S}_z$ is measured at time $t=0$, what results can be found, and with what probabilities?

(b) Instead of performing the preceding measurement, we let the system evolve under the influence of a magnetic field parallel to $Oy$, of modulus $B_0$. Calculate, in the $\{|+\rangle, |-\rangle\}$ basis, the state of the system at time $t=0$.

(c) At time $t$, we measure the observables $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$. What values can we find, and with what probabilities? What relation must exist between $B_0$ and $t$ for the result of one of the measurements to be certain? Give a physical interpretation of this condition.

*Solution:*

(a) At time $t$, the state of the system is

$$|\psi(t=0)\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|\xi_+\rangle + |\xi_-\rangle) \tag{1}$$

where the $|\xi_+\rangle$ and $|\xi_-\rangle$ are the two eigenvectors of the observable $\hat{S}_x$ with the eigenvalues $\pm\frac{\hbar}{2}$, respectively.
Since

$$P(S_x = \frac{\hbar}{2}) = |\langle\xi_+|\psi(t=0)\rangle|^2 = \frac{1}{2} \tag{2}$$

$$P(S_x = -\frac{\hbar}{2}) = |\langle\xi_-|\psi(t=0)\rangle|^2 = \frac{1}{2} \tag{3}$$

result $\frac{\hbar}{2}$ can be found with probability $\frac{1}{2}$ and result $-\frac{\hbar}{2}$ can be found with probability $\frac{1}{2}$.

(b) At time $t$, the state of the system is

$$|\psi(t=0)\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|\eta_+\rangle + |\eta_-\rangle) \tag{4}$$

where the $|\eta_+\rangle$ and $|\eta_-\rangle$ are the two eigenvectors of the observable $\hat{S}_y$ with the eigenvalues $\pm\frac{\hbar}{2}$, respectively.
The Hamiltonian of the particle in the magetic field is

$$\hat{H} = \hat{\vec{M}} \cdot \vec{B} = -\gamma B_0 \hat{S}_y \tag{5}$$

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so

$$\hat{H}|\eta_+\rangle = -\gamma B_0\hat{S}_y|\eta_+\rangle = -\frac{1}{2}\gamma\hbar B_0|\eta_+\rangle = E_{\eta_+}|\eta_+\rangle \Longrightarrow E_{\eta_+} = -\frac{1}{2}\gamma\hbar B_0 \tag{6}$$

$$\hat{H}|\eta_-\rangle = -\gamma B_0\hat{S}_y|\eta_-\rangle = \frac{1}{2}\gamma\hbar B_0|\eta_-\rangle = E_{\eta_-}|\eta_-\rangle \Longrightarrow E_{\eta_-} = \frac{1}{2}\gamma\hbar B_0 \tag{7}$$

Therefore, at time $t$, the state of the system is

$$\begin{aligned}|\psi(t)\rangle =& \frac{1}{\sqrt{2}}(e^{-iE_{\eta_+}t/\hbar}|\eta\rangle + e^{-iE_{\eta_+}t/\hbar}|\eta\rangle) = \frac{1}{\sqrt{2}}(e^{i\gamma B_0t/2}|\eta\rangle + e^{-i\gamma B_0t/2}|\eta_-\rangle)\\ =&\frac{1}{\sqrt{2}}[(\cos\frac{\gamma B_0t}{2} + i\sin\frac{\gamma B_0t}{2})\frac{1}{\sqrt{2}}(|+\rangle + i|-\rangle) + (\cos\frac{\gamma B_0t}{2} - i\sin\frac{\gamma B_0t}{2})\frac{1}{\sqrt{2}}(|+\rangle - i|-\rangle)]\\ =&\cos\frac{\gamma B_0t}{2}|+\rangle - \sin\frac{\gamma B_0t}{2}|-\rangle\end{aligned} \tag{8}$$

(c) If we measure the observable $\hat{S}_z$, since

$$P(S_z = \frac{\hbar}{2}) = |\langle+|\psi(t)\rangle|^2 = \cos^2\frac{\gamma B_0t}{2} \tag{9}$$

$$P(S_z = -\frac{\hbar}{2}) = |\langle-|\psi(t)\rangle|^2 = \sin^2\frac{\gamma B_0t}{2} \tag{10}$$

result $\frac{\hbar}{2}$ can be found with probability $\cos^2\frac{\gamma B_0t}{2}$ and result $-\frac{\hbar}{2}$ can be found with probability $\sin^2\frac{\gamma B_0t}{2}$.
If $B_0t = \frac{2n\pi}{\gamma}, n = 0, \pm1, \pm2, \cdots$, the result of the measurement is centain to be $\frac{\hbar}{2}$; if $B_0t = \frac{(2n+1)\pi}{\gamma}, n = 0, \pm1, \pm2, \cdots$, the reuslt of the measurement is certain to be $-\frac{\hbar}{2}$.
A physical interpretation: the spin of the particle will rotate around the $y$ axis under the magnetic field parallel to $Oy$.
The state of the system at time $t$ can be written as

$$\begin{aligned}|\psi(t)\rangle =& \cos\frac{\gamma B_0t}{2}|+\rangle - \sin\frac{\gamma B_0t}{2}|-\rangle\\ =& \cos\frac{\gamma B_0t}{2}\frac{1}{\sqrt{2}}(|\xi_+\rangle + |\xi_-\rangle) - \sin\frac{\gamma B_0t}{2}\frac{1}{\sqrt{2}}(|\xi_+\rangle - |\xi_-\rangle)\\ =&\frac{1}{\sqrt{2}}(\cos\frac{\gamma B_0t}{2} - \sin\frac{\gamma B_0t}{2})|\xi_+\rangle + \frac{1}{\sqrt{2}}(\cos\frac{\gamma B_0t}{2} + \sin\frac{\gamma B_0t}{2})|\xi_-\rangle\end{aligned} \tag{11}$$

If we measure the observable $\hat{S}_x$, since

$$P(S_x = \frac{\hbar}{2}) = |\langle\xi_+|\psi(t)\rangle|^2 = \frac{1}{2}(\cos\frac{\gamma B_0t}{2} - \sin\frac{\gamma B_0t}{2})^2 = \frac{1}{2}[1 - \sin(\gamma B_0t)] \tag{12}$$

$$P(S_x = -\frac{\hbar}{2}) = |\langle\xi_-|\psi(t)\rangle|^2 = \frac{1}{2}(\cos\frac{\gamma B_0t}{2} + \sin\frac{\gamma B_0t}{2})^2 = \frac{1}{2}[1 + \sin(\gamma B_0t)] \tag{13}$$

result $\frac{\hbar}{2}$ can be found with probability $\frac{1}{2}[1 - \sin(\gamma B_0t)]$ and $-\frac{\hbar}{2}$ can be found with probability $\frac{1}{2}[1 + \sin(\gamma B_0t)]$.
If $B_0t = \frac{(4n-1)\pi}{2\gamma}$, the result of the measurement is certain to be $\frac{\hbar}{2}$; if $B_0t = \frac{(4n+1)\pi}{2\gamma}$,

the result of the measurement is certain to be $-\frac{\hbar}{2}$.
A physical interpretation: the spin of the particle will rotate around the $y$ axis under the magnetic field parallel to $Oy$.
The state of the system at time $t$ can be written as

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}(e^{i\gamma B_0 t/2}|\eta\rangle + e^{-i\gamma B_0 t/2}|\eta_-\rangle) \tag{14}$$

If we measure the observable $\hat{S}_y$, since

$$P(S_y = \frac{\hbar}{2}) = |\langle\eta_+|\psi(t)\rangle|^2 = \frac{1}{2} \tag{15}$$

$$P(S_y = -\frac{\hbar}{2}) = |\langle\eta_-|\psi(t)\rangle|^2 = \frac{1}{2} \tag{16}$$

result $\frac{\hbar}{2}$ can be found with probability $\frac{1}{2}$ and $-\frac{\hbar}{2}$ can be found with probability $\frac{1}{2}$. The result of the measurement cannot be certain at any time.

□

**Problem 2.** [C-T Exercise 4-3] Consider a spin 1/2 particle placed in a magnetic field $\hat{B}_0$ with components $B_x = B_0/\sqrt{2}$, $B_y = 0$, and $B_z = B_0/\sqrt{2}$. The notation is the same as that of Problem 1.

(a) Calculate the matrix representing, in the $\{|+\rangle, |-\rangle\}$ basis, the operator $\hat{H}$, the Hamiltonian of the system.

(b) Calculate the eigenvalues and the eigenvectors of $\hat{H}$.

(c) The system at time $t = 0$ is in the state $|-\rangle$. What values can be found if the energy is measured, and with what probabilities?

(d) Calculate the state vector $|\psi(t)\rangle$ at time $t$. At this instant, $\hat{S}_x$ is measured; what is the mean value of the results that can be obtained? Give a geometrical interpretation.

*Solution:*

(a) In the $\{|+\rangle, |-\rangle\}$ basis, the Hamiltonian of the system is

$$\begin{aligned}
\hat{H} =& -\hat{\vec{M}} \cdot \vec{B} = -\frac{1}{\sqrt{2}}\gamma B_0(\hat{S}_x + \hat{S}_z) \\
=& -\frac{1}{\sqrt{2}}\gamma B_0 \left[\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \\
=& -\frac{1}{2\sqrt{2}}\gamma\hbar B_0 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
\end{aligned} \tag{17}$$

(b) The characteristic function of the Hamiltonian is

$$|\hat{H}-EI| = -\frac{1}{2\sqrt{2}}\gamma\hbar B_0 \begin{pmatrix} 1-\lambda & 1 \\ 1 & -(1+\lambda) \end{pmatrix} = -\frac{1}{2\sqrt{2}}\gamma\hbar B_0(\lambda^2-2)=0 \tag{18}$$

$$\Longrightarrow \lambda_{1,2} = \pm\sqrt{2} \tag{19}$$

where $\lambda = \frac{E}{\frac{1}{\sqrt{2\sqrt{2}}}\gamma\hbar B_0}$,
Therefore, the eigenvalues of the Hamiltonian is

$$E_{1,2} = -\frac{1}{2\sqrt{2}}\gamma\hbar B_0\lambda_{1,2} = \mp\frac{1}{2}\gamma\hbar B_0 \tag{20}$$

Assume the eigenvectors of the Hamiltonian is $|\varphi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$, plug the eigenvalues obtained above into the eigenfunction

$$\hat{H}|\varphi\rangle = -\frac{1}{2\sqrt{2}}\gamma\hbar B_0 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = E_{1,2}\begin{pmatrix} a \\ b \end{pmatrix} \tag{21}$$

to get the corresponding normalized eigenvectors

$$\begin{aligned}|\varphi_1\rangle =& \frac{\sqrt{\sqrt{2}+1}}{2^{3/4}}\begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix} = \frac{(\sqrt{2}+1)^{1/2}}{2^{3/4}}[|+\rangle + (\sqrt{2}-1)|-\rangle] \\ =& \frac{1}{2^{3/4}}[(\sqrt{2}+1)^{1/2}|+\rangle + (\sqrt{2}-1)^{1/2}|-\rangle] \end{aligned} \tag{22}$$

$$\tag{23}$$

$$\begin{aligned}|\varphi_2\rangle =& \frac{(\sqrt{2}-1)^{1/2}}{2^{3/4}}\begin{pmatrix} 1 \\ -(\sqrt{2}+1) \end{pmatrix} = \frac{(\sqrt{2}-1)^{1/2}}{2^{3/4}}[|+\rangle - (\sqrt{2}+1)|-\rangle] \\ =& \frac{1}{2^{3/4}}[(\sqrt{2}-1)^{1/2}|+\rangle - (\sqrt{2}+1)^{1/2}|-\rangle] \end{aligned} \tag{24}$$

(c) The state of the system at time 0 can be written as

$$|\psi(0)\rangle = |-\rangle = \frac{1}{2^{3/4}}[(\sqrt{2}-1)^{1/2}|\varphi_1\rangle - (\sqrt{2}+1)^{1/2}|\varphi_2\rangle] \tag{25}$$

Since

$$P(E_1) = |\langle\varphi_1|\psi(0)\rangle|^2 = \frac{\sqrt{2}-1}{2\sqrt{2}} \tag{26}$$

$$P(E_2) = |\langle\varphi_2|\psi(0)\rangle|^2 = \frac{\sqrt{2}+1}{2\sqrt{2}} \tag{27}$$

value $E_1$ can be found with probability $\frac{\sqrt{2}-1}{2\sqrt{2}}$ and value $E_2$ can be found with probability $\frac{\sqrt{2}+1}{2\sqrt{2}}$.

(d) The state of the system at time $t$ is

$$\begin{aligned}
|\psi(t)\rangle =& \frac{1}{2^{3/4}}[(\sqrt{2}-1)^{1/2}e^{-iE_1t/\hbar}|\varphi_1\rangle - (\sqrt{2}+1)^{1/2}e^{-iE_2t/\hbar}|\varphi_2\rangle] \\
=&\frac{1}{2^{3/4}}[(\sqrt{2}-1)^{1/2}e^{i\gamma B_0t}|\varphi_1\rangle - (\sqrt{2}+1)^{1/2}e^{-i\gamma B_0t}|\varphi_2\rangle] \\
=&\frac{1}{2\sqrt{2}}[(\sqrt{2}-1)^{1/2}e^{i\gamma B_0t}((\sqrt{2}+1)^{1/2}|+\rangle + (\sqrt{2}-1)^{1/2}|-\rangle) \\
&+ (\sqrt{2}+1)^{1/2}e^{-i\gamma B_0t}((\sqrt{2}-1)^{1/2}|+\rangle - (\sqrt{2}+1)^{1/2}|-\rangle] \\
=&\frac{1}{\sqrt{2}}[i\sin\frac{\gamma B_0t}{2}|+\rangle + (\sqrt{2}\cos\frac{\gamma B_0t}{2} - i\sin\frac{\gamma B_0t}{2})|-\rangle] \\
=&\frac{1}{\sqrt{2}}[i\sin\frac{\gamma B_0t}{2}\frac{1}{\sqrt{2}}(|\xi_+\rangle + |\xi_-\rangle) + (\sqrt{2}\cos\frac{\gamma B_0t}{2} - i\sin\frac{\gamma B_0t}{2})\frac{1}{\sqrt{2}}(|\xi_+\rangle - |\xi_-\rangle)] \\
=&\frac{1}{\sqrt{2}}[\cos\frac{\gamma B_0t}{2}|\xi_+\rangle - (\cos\frac{\gamma B_0t}{2} - i\sqrt{2}\sin\frac{\gamma B_0t}{2})|\xi_-\rangle]
\end{aligned} \tag{28}$$

if the observable $\hat{S}_x$ is measured, since

$$P(\frac{\hbar}{2}) = |\langle\xi_+|\psi(t)\rangle|^2 = \frac{1}{2}\cos^2\frac{\gamma B_0t}{2} \tag{29}$$

$$P(-\frac{\hbar}{2}) = |\langle\xi_-|\psi(t)\rangle|^2 = \frac{1}{2}[1+\sin^2\frac{\gamma B_0t}{2}] \tag{30}$$

result $\frac{\hbar}{2}$ can be found with probability $\frac{1}{2}\cos^2\frac{\gamma B_0t}{2}$ and result $-\frac{\hbar}{2}$ can be found with probability $\frac{1}{2}[1+\sin^2\frac{\gamma B_0t}{2}]$.
Therefore, the mean value of the result that can be obtained is

$$\begin{aligned}
\bar{S}_x =&P(\frac{\hbar}{2})\frac{\hbar}{2} + P(-\frac{\hbar}{2})(-\frac{\hbar}{2}) \\
=&\frac{\hbar}{4}\cos^2\frac{\gamma B_0t}{2} - \frac{\hbar}{4}[1+\sin^2\frac{\gamma B_0T}{2}] \\
=&\frac{\hbar}{4}\cos(\gamma B_0t) - \frac{\hbar}{4} \\
=&-\frac{\hbar}{2}\sin^2\frac{\gamma B_0t}{2}
\end{aligned} \tag{31}$$

Geometrical interpretation: the magnetic field is along the angular bisector of the $x$ axis and the $z$ axis, the spin is originally along the negative $z$ axis and rotate around the direction of the magetic field, producing the periodically-changing spin project along $x$ axis.

□

**Problem 3.** [C-T Exercise 4-6] Consider the system composed of two spin 1/2's, $\hat{\vec{S}}_1$ and $\hat{\vec{S}}_2$, and the basis of four vectors $|\pm\pm\rangle$. The system at time $t=0$ is in the state

$$|\psi(0)\rangle = \frac{1}{2}|++\rangle + \frac{1}{2}|+-\rangle + \frac{1}{\sqrt{2}}|--\rangle.$$

(a) At time $t = 0$, $\hat{S}_{1z}$ is measured; what is the probability of finding $-\hbar/2$? What is the state vector after this measurement? If we then measure $\hat{S}_{1x}$, what results can be found, and with what probabilities?

(b) When the system is in the state $|\psi(0)\rangle$ written above, $\hat{S}_{1z}$ and $\hat{S}_{2z}$ are measured simultaneously. What is the probability of finding opposite results? Identical results?

(c) Instead of performing the preceding measurements, we let the system evolve under the influence of the Hamiltonian $\hat{H} = \omega_1 \hat{S}_{1z} + \omega_2 \hat{S}_{2z}$. What is the state vector $|\psi(t)\rangle$ at time $t$? Calculate at time $t$ the mean values $\langle\hat{\vec{S}}_1\rangle$ and $\langle\hat{\vec{S}}_2\rangle$. Give a physical interpretation.

(d) Show that the lengths of the vectors $\langle\hat{\vec{S}}_1\rangle$ and $\langle\hat{\vec{S}}_2\rangle$ are less than $\hbar/2$. What must be the form of $|\psi(0)\rangle$ for each of these lengths to be equal to $+\hbar/2$?

*Solution:*

(a) If $\hat{S}_{1z}$ is measured at time $t = 0$, since

$$P(S_{1z} = \frac{\hbar}{2}) = |\langle+|\otimes 1(2)|\psi(0)\rangle|^2 = |\frac{1}{\sqrt{2}}1(1)\otimes|+\rangle_2 + \frac{1}{\sqrt{2}}1(1)\otimes|-\rangle_2|^2 = \frac{1}{2} \quad (32)$$

$$P(S_{1z} = -\frac{\hbar}{2}) = |\langle-|\otimes 1(2)|\psi(0)|^2 = \frac{1}{2} \quad (33)$$

result $\frac{\hbar}{2}$ can be found with probability $\frac{1}{2}$ and result $-\frac{\hbar}{2}$ can be found with probability $\frac{1}{2}$.

If the result $S_z = \frac{\hbar}{2}$ is found at first, then immediately after the measurement, the state of the system is

$$\begin{aligned}\psi_{a,+} =&\frac{1}{\sqrt{2}}(|++\rangle + |+-\rangle) = |+\rangle_1 \otimes \frac{1}{\sqrt{2}}(|+\rangle_2 + |-\rangle_2)\\ =&\frac{1}{\sqrt{2}}(|\xi_+\rangle_1 + |\xi_-\rangle_1)\otimes\frac{1}{\sqrt{2}}(|+\rangle_2 + |-\rangle_2)\end{aligned} \quad (34)$$

If we then measure $\hat{S}_x$, since

$$P(S_{1x} = \frac{\hbar}{2}) = |\langle\xi_+|\otimes 1(2)|\psi_{a,+}\rangle|^2 = \frac{1}{2} \quad (35)$$

$$P(S_{1x} = -\frac{\hbar}{2}) = |\langle\xi_-|\otimes 1(2)|\psi_{a,+}\rangle|^2 = \frac{1}{2} \quad (36)$$

result $\frac{\hbar}{2}$ can be found with probability $\frac{1}{2}$ and result $\frac{\hbar}{2}$ can be found with probability $\frac{1}{2}$.

Similarly, if the result $S_z = \frac{\hbar}{2}$ is found at first, the immediately after the measurement, the state of the system is

$$\psi_{a,+} = |--\rangle = \frac{1}{\sqrt{2}}(|\xi_+\rangle - |\xi_-\rangle)\otimes|-\rangle \quad (37)$$

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If we then measure $\hat{S}_x$, since

$$P(S_{1x}=\frac{\hbar}{2})=|\langle\xi_+|\otimes 1(2)|\psi_{a,-}\rangle|^2=\frac{1}{2} \tag{38}$$

$$P(S_{1x}=-\frac{\hbar}{2})=|\langle\xi_-|\otimes 1(2)|\psi_{a,-}\rangle|^2=\frac{1}{2} \tag{39}$$

result $\frac{\hbar}{2}$ can be found with probability $\frac{1}{2}$ and result $\frac{\hbar}{2}$ can be found with probability $\frac{1}{2}$.

(b) The probability of finding opposite results is

$$\begin{aligned} P(S_{1z}S_{2z}<0)=&P(S_{1z}=\frac{\hbar}{2},S_{2z}=-\frac{\hbar}{2})+P(S_{1z}=-\frac{\hbar}{2},S_{2z}\\ &=\frac{\hbar}{2})=|\langle+-|\psi(0)\rangle|^2+|\langle-+|\psi(0)\rangle|^2=\frac{1}{4} \end{aligned} \tag{40}$$

The probability of finding identical results is

$$\begin{aligned} P(S_{1z}S_{2z}>0)=&P(S_{1z}=\frac{\hbar}{2},S_{2z}=\frac{\hbar}{2})+P(S_{1z}=-\frac{\hbar}{2},S_{2z}\\ &=-\frac{\hbar}{2})=|\langle++|\psi(0)\rangle|^2+|\langle--|\psi(0)\rangle|^2=\frac{1}{4}+\frac{1}{2}=\frac{3}{4} \end{aligned} \tag{41}$$

(c) $|\pm\pm\rangle$ are the eigenvalue of the Hamiltonian

$$\hat{H}|++\rangle=(\omega_1\hat{S}_{1z}\otimes 1(2)+\omega_2 1(1)\otimes\hat{S}_{2z})(|+\rangle_1\otimes|-\rangle_2)=\frac{\hbar(\omega_1+\omega_2)}{2}|++\rangle \tag{42}$$

$$\hat{H}|+-\rangle=\frac{\hbar(\omega_1-\omega_2)}{2}|+-\rangle \tag{43}$$

$$\hat{H}|-+\rangle=\frac{\hbar(-\omega_1+\omega_2)}{2}|-+\rangle \tag{44}$$

$$\hat{H}|--\rangle=\frac{\hbar(-\omega_1-\omega_2)}{2}|--\rangle \tag{45}$$

The eigenvector at time $t$ is

$$|\psi(t)\rangle=\frac{1}{2}e^{-i(\omega_1+\omega_2)t/2}|++\rangle+\frac{1}{2}e^{-i(\omega_1-\omega_2)t/2}|+-\rangle+\frac{1}{\sqrt{2}}e^{i(\omega_1+\omega_2)t/2}|--\rangle \tag{46}$$

The mean values are

$$\begin{aligned} \langle\hat{S}_{1,x}\rangle=&\langle\psi(t)|\hat{\vec{S}}_{1x}|\psi(t)\rangle\\ =&[\frac{1}{2}e^{i(\omega_1+\omega_2)t/2}\langle++|+\frac{1}{2}e^{i(\omega_1-\omega_2)t/2}\langle+-|+\frac{1}{\sqrt{2}}e^{i(\omega_1+\omega_2)t/2}\langle--|]\\ &\times\hat{S}_{1,x}[\frac{1}{2}e^{-i(\omega_1+\omega_2)t/2}|++\rangle+\frac{1}{2}e^{-i(\omega_1-\omega_2)t/2}|+-\rangle+\frac{1}{\sqrt{2}}e^{i(\omega_1+\omega_2)t/2}|--\rangle]\\ =&\frac{\hbar}{2}\frac{1}{\sqrt{2}}e^{-i(\omega_1+\omega_2)t/2}\frac{1}{2}e^{-i(\omega_1-\omega_2)t/2}+\frac{\hbar}{2}\frac{1}{2}e^{i(\omega_1-\omega_2)t/2}\frac{1}{\sqrt{2}}e^{i(\omega_1+\omega_2)t/2}\\ =&\frac{\hbar}{4\sqrt{2}}(e^{-i\omega_1 t}+e^{-i\omega_1 t})=\frac{\hbar}{2\sqrt{2}}\cos(\omega_1 t) \end{aligned} \tag{47}$$

$$\begin{aligned}
\langle\hat{S}_{1,y}\rangle =&\langle\psi(t)|\hat{\vec{S}}_{1,y}|\psi(t)\rangle\\
=&[\frac{1}{2}e^{i(\omega_1+\omega_2)t/2}\langle++|+\frac{1}{2}e^{i(\omega_1-\omega_2)t/2}\langle+-|+\frac{1}{\sqrt{2}}e^{i(\omega_1+\omega_2)t/2}\langle--|]\\
&\times\hat{S}_{1,y}[\frac{1}{2}e^{-i(\omega_1+\omega_2)t/2}|++\rangle+\frac{1}{2}e^{-i(\omega_1-\omega_2)t/2}|+-\rangle+\frac{1}{\sqrt{2}}e^{i(\omega_1+\omega_2)t/2}|--\rangle]\\
=&i\frac{\hbar}{2}\frac{1}{\sqrt{2}}e^{-i(\omega_1+\omega_2)t/2}\frac{1}{2}e^{-i(\omega_1-\omega_2)t/2}-i\frac{\hbar}{2}\frac{1}{2}e^{i(\omega_1-\omega_2)t/2}\frac{1}{\sqrt{2}}e^{i(\omega_1+\omega_2)t/2}\\
=&i\frac{\hbar}{4\sqrt{2}}(e^{-i\omega_1t}-e^{-i\omega_1t})=\frac{\hbar}{2\sqrt{2}}\sin(\omega_1t)
\end{aligned}\tag{48}$$

$$\begin{aligned}
\langle\hat{S}_{1,z}\rangle =&\langle\psi(t)|\hat{\vec{S}}_{1,z}|\psi(t)\rangle\\
=&[\frac{1}{2}e^{i(\omega_1+\omega_2)t/2}\langle++|+\frac{1}{2}e^{i(\omega_1-\omega_2)t/2}\langle+-|+\frac{1}{\sqrt{2}}e^{i(\omega_1+\omega_2)t/2}\langle--|]\\
&\times\hat{S}_{1,y}[\frac{1}{2}e^{-i(\omega_1+\omega_2)t/2}|++\rangle+\frac{1}{2}e^{-i(\omega_1-\omega_2)t/2}|+-\rangle+\frac{1}{\sqrt{2}}e^{i(\omega_1+\omega_2)t/2}|--\rangle]\\
=&\frac{\hbar}{2}\frac{1}{2}e^{i(\omega_1+\omega_2)t/2}\frac{1}{2}e^{-i(\omega_1+\omega_2)t/2}+\frac{\hbar}{2}\frac{1}{2}e^{-i(\omega_1-\omega_2)t/2}\frac{1}{2}e^{-i(\omega_1-\omega_2)t/2}\\
&-\frac{\hbar}{2}\frac{1}{\sqrt{2}}e^{-i(\omega_1+\omega_2)t/2}\frac{1}{\sqrt{2}}e^{i(\omega_1+\omega_2)t/2}\\
=&\frac{\hbar}{8}+\frac{\hbar}{8}-\frac{\hbar}{4}=0
\end{aligned}\tag{49}$$

$$\Longrightarrow\langle\hat{\vec{S}}_1\rangle=\langle\hat{S}_{1,x}\rangle\vec{e}_x+\langle\hat{S}_{1,y}\rangle\vec{e}_y+\langle\hat{S}_{1,z}\rangle\vec{e}_z=\frac{\hbar}{2\sqrt{2}}[\cos(\omega_1t)\vec{e}_x+\sin(\omega_1t)\vec{e}_y]\tag{50}$$

$$\begin{aligned}
\langle\hat{S}_{2,x}\rangle =&\langle\psi(t)|\hat{\vec{S}}_{2,x}|\psi(t)\rangle\\
=&[\frac{1}{2}e^{i(\omega_1+\omega_2)t/2}\langle++|+\frac{1}{2}e^{i(\omega_1-\omega_2)t/2}\langle+-|+\frac{1}{\sqrt{2}}e^{i(\omega_1+\omega_2)t/2}\langle--|]\\
&\times\hat{S}_{2,x}[\frac{1}{2}e^{-i(\omega_1+\omega_2)t/2}|++\rangle+\frac{1}{2}e^{-i(\omega_1-\omega_2)t/2}|+-\rangle+\frac{1}{\sqrt{2}}e^{i(\omega_1+\omega_2)t/2}|--\rangle]\\
=&\frac{\hbar}{2}\frac{1}{2}e^{i(\omega_1-\omega_2)t/2}\frac{1}{2}e^{-i(\omega_1+\omega_2)t/2}+\frac{\hbar}{2}\frac{1}{2}e^{i(\omega_1+\omega_2)t/2}\frac{1}{2}e^{-i(\omega_1-\omega_2)t/2}\\
=&\frac{\hbar}{8}(e^{-i\omega_2t}+e^{i\omega_2t})=\frac{\hbar}{4}\cos(\omega_2t)
\end{aligned}\tag{51}$$

$$\begin{aligned}
\langle\hat{S}_{2,y}\rangle =&\langle\psi(t)|\hat{\vec{S}}_{2,y}|\psi(t)\rangle\\
=&[\frac{1}{2}e^{i(\omega_1+\omega_2)t/2}\langle++|+\frac{1}{2}e^{i(\omega_1-\omega_2)t/2}\langle+-|+\frac{1}{\sqrt{2}}e^{i(\omega_1+\omega_2)t/2}\langle--|]\\
&\times\hat{S}_{2,y}[\frac{1}{2}e^{-i(\omega_1+\omega_2)t/2}|++\rangle+\frac{1}{2}e^{-i(\omega_1-\omega_2)t/2}|+-\rangle+\frac{1}{\sqrt{2}}e^{i(\omega_1+\omega_2)t/2}|--\rangle]\\
=&i\frac{\hbar}{2}\frac{1}{2}e^{i(\omega_1-\omega_2)t/2}\frac{1}{2}e^{-i(\omega_1+\omega_2)t/2}-i\frac{\hbar}{2}\frac{1}{2}e^{i(\omega_1+\omega_2)t/2}\frac{1}{2}e^{-i(\omega_1-\omega_2)/2}\\
=&i\frac{\hbar}{8}(e^{-i\omega_2t}-e^{i\omega_2t})=\frac{\hbar}{4}\sin(\omega_2t)
\end{aligned}\tag{52}$$

$$\begin{aligned}
\langle\hat{S}_{2,z}\rangle =&\langle\psi(t)|\hat{\vec{S}}_{2,z}|\psi(t)\rangle \\
=&[\frac{1}{2}e^{i(\omega_1+\omega_2)t/2}\langle++|+\frac{1}{2}e^{i(\omega_1-\omega_2)t/2}\langle+-|+\frac{1}{\sqrt{2}}e^{i(\omega_1+\omega_2)t/2}\langle--|] \\
&\times\hat{S}_{2,z}[\frac{1}{2}e^{-i(\omega_1+\omega_2)t/2}|++\rangle+\frac{1}{2}e^{-i(\omega_1-\omega_2)t/2}|+-\rangle+\frac{1}{\sqrt{2}}e^{i(\omega_1+\omega_2)t/2}|--\rangle] \\
=&\frac{\hbar}{2}\frac{1}{2}e^{i(\omega_1+\omega_2)t/2}\frac{1}{2}e^{-i(\omega_1+\omega_2)t/2}-\frac{\hbar}{2}\frac{1}{2}e^{-i(\omega_1-\omega_2)t/2}\frac{1}{2}e^{-i(\omega_1-\omega_2)t/2} \\
&-\frac{\hbar}{2}\frac{1}{\sqrt{2}}e^{-i(\omega_1+\omega_2)t/2}\frac{1}{\sqrt{2}}e^{i(\omega_1+\omega_2)t/2} \\
=&\frac{\hbar}{8}-\frac{\hbar}{8}-\frac{\hbar}{4}
\end{aligned}\tag{53}$$

$$\Longrightarrow\langle\hat{\vec{S}}_2\rangle=\langle\hat{S}_{2,x}\rangle\vec{e}_x+\langle\hat{S}_{2,y}\rangle\vec{e}_y+\langle\hat{S}_{2,z}\rangle\vec{e}_z=\frac{\hbar}{4}[\cos(\omega_2 t)\vec{e}_x+\sin(\omega_2 t)\vec{e}_y-\vec{e}_z]\tag{54}$$

(d) Since

$$|\langle\hat{\vec{S}}_1\rangle|=|\langle\hat{\vec{S}}_2\rangle|=\frac{\hbar}{2\sqrt{2}}<\frac{\hbar}{2}\tag{55}$$

the lengths of the vectors $\langle\hat{\vec{S}}_1\rangle$ and $\langle\hat{\vec{S}}_2\rangle$ are less than $\hbar/2$.

If the $|\psi(0)\rangle$ is the form of $|\varphi(1)\rangle\otimes|\varphi(2)\rangle$, then each of these lengths are equal to $\hbar/2$, proof: If

$$|\psi(0)\rangle=|\varphi(1)\rangle\otimes|\varphi(2)\rangle\tag{56}$$

where

$$\begin{aligned}
|\varphi(1)\rangle =&a|+\rangle+b|-\rangle \\
=&\frac{a+b}{\sqrt{2}}|\xi_+\rangle+\frac{a-b}{\sqrt{2}}|\xi_-\rangle \\
=&\frac{a-ib}{\sqrt{2}}|\eta_+\rangle+\frac{a+ib}{\sqrt{2}}|\eta_-\rangle
\end{aligned}\tag{57}$$

and $|a|^2+|b|^2=1$.

The mean values are

$$\langle\hat{S}_z\rangle=(|a|^2-|b|^2)\frac{\hbar}{2}\tag{58}$$

$$\langle\hat{S}_x\rangle=(\frac{|a+b|^2}{2}-\frac{|a-b|^2}{2})\frac{\hbar}{2}=(ab^*+a^*b)\frac{\hbar}{2}=\hbar\mathrm{Re}(ab^*)\tag{59}$$

$$\langle\hat{S}_y\rangle=(\frac{|a-ib|^2}{2}-\frac{|a+ib|^2}{2})\frac{\hbar}{2}=i(ab^*-a^*b)\frac{\hbar}{2}=-\hbar\mathrm{Im}(ab^*)\tag{60}$$

$$\begin{aligned}
|\langle\hat{\vec{S}}\rangle|^2 =&\langle\hat{S}_x\rangle^2+\langle\hat{S}_y\rangle^2+\langle\hat{S}_z\rangle^2\\
=&\mathrm{Re}^2(ab^*)\hbar^2+\mathrm{Im}^2(ab^*)^2+(|a|^2-|b|^2)^2\frac{\hbar^2}{4}\\
=&|ab^*|^2\hbar^2+(|a|^2-|b|^2)^2\frac{\hbar^2}{4}\\
=&(|a|^4+2|a|^2|b|^2+|b|^4)\frac{\hbar^2}{4}\\
=&(|a|^2+|b|^2)^2\frac{\hbar^2}{4}\\
=&\frac{\hbar^2}{4}
\end{aligned}\tag{61}$$

$$\Longrightarrow|\langle\hat{\vec{S}}\rangle|=\frac{\hbar}{2}\tag{62}$$

□

**Problem 4.** [C-T Exercise 5-7] Consider a one-dimensional harmonic oscillator of Hamiltonian $\hat{H}$ and stationary states $|\varphi_n\rangle$, $\hat{H}|\varphi_n\rangle=(n+1/2)\hbar\omega|\varphi_n\rangle$. The operator $\hat{U}(k)$ is defined by $\hat{U}(k)=e^{ik\hat{x}}$, where $k$ is real.

(a) Is $\hat{U}(k)$ unitary? Show that, for all $n$, its matrix elements satisfy the relation

$$\sum_{n'}|\langle\varphi_n|\hat{U}(k)|\varphi_{n'}\rangle|^2=1.$$

(b) Express $\hat{U}(k)$ in terms of the operators $\hat{a}$ and $\hat{a}^\dagger$. Use Glauber's formula to put $\hat{U}(k)$ in the form of a product of exponential operators.

(c) Establish the relations

$$e^{\lambda\hat{a}}|\varphi_0\rangle=|\varphi_0\rangle,$$
$$\langle\varphi_n|e^{\lambda\hat{a}^\dagger}|\varphi_0\rangle=\frac{\lambda^n}{\sqrt{n!}},$$

where $\lambda$ is an arbitrary complex parameter.

(d) Find the expression, in terms of $E_k=\hbar^2k^2/2m$ and $E_\omega=\hbar\omega$, for the matrix element $\langle\varphi_0|\hat{U}(k)|\varphi_n\rangle$. What happens when $k$ approaches zero? Could this result have been predicted directly?

*Solution:*

(a) Since

$$\hat{U}(k)\hat{U}^\dagger(k)=e^{ik\hat{x}}e^{-ik\hat{x}}=1\tag{63}$$
$$\hat{U}(k)^\dagger\hat{U}(k)=e^{-ik\hat{x}}e^{ik\hat{x}}=1\tag{64}$$

$\hat{U}(k)$ is unitary.

$$\begin{aligned}\sum_{n'}|\langle\varphi_n|\hat{U}(k)|\varphi_{n'}\rangle|^2 =&\sum_{n'}\langle\varphi_n|\hat{U}(k)|\varphi_{n'}\rangle\langle[\varphi_n|\hat{U}(k)|\varphi_{n'}\rangle]^*\\ =&\sum_{n'}\varphi_n|\hat{U}(k)|\varphi_{n'}\rangle\langle\varphi_{n'}|\hat{U}^\dagger(k)|\varphi_n\rangle\\ =&\varphi_n|\hat{U}(k)\hat{U}^\dagger(k)|\varphi_n\rangle\\ =&\varphi_n|\varphi_n\rangle=1\end{aligned} \tag{65}$$

(b) The $\hat{U}(k)$ can be expressed as

$$\hat{U}(k)=e^{ik\hat{x}}=e^{ik\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger)} \tag{66}$$

According to Glauber's formula

$$e^{\hat{A}+\hat{B}}=e^{\hat{B}}e^{\hat{A}}e^{[\hat{A},\hat{B}]/2} \tag{67}$$

so

$$\begin{aligned}\hat{U}(k)=&e^{ik\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger)}=e^{ik\sqrt{\frac{\hbar}{2m\omega}}\hat{a}+ik\sqrt{\frac{\hbar}{2m\omega}}\hat{a}^\dagger}\\ =&e^{ik\sqrt{\frac{\hbar}{2m\omega}}\hat{a}^\dagger}e^{ik\sqrt{\frac{\hbar}{2m\omega}}\hat{a}}e^{-k^2\frac{\hbar}{2m\omega}[\hat{a},\hat{a}^\dagger]/2}\\ =&e^{-\frac{\hbar k^2}{4m\omega}}e^{ik\sqrt{\frac{\hbar}{2m\omega}}\hat{a}^\dagger}e^{ik\sqrt{\frac{\hbar}{2m\omega}}\hat{a}}\end{aligned} \tag{68}$$

(c) Since

$$\hat{a}|\varphi_0\rangle=0 \tag{69}$$

$$e^{\lambda\hat{a}}|\varphi_0\rangle=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\hat{a}^n|\varphi_0\rangle=|\varphi_0\rangle \tag{70}$$

Since

$$|\varphi_n\rangle=\frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|\varphi_0\rangle \tag{71}$$

$$\langle\varphi_n|e^{\lambda\hat{a}}|\varphi_0\rangle=\sum_{m=0}^{\infty}\langle\varphi_n|\frac{\lambda^m}{\sqrt{m!}}(\hat{a}^\dagger)^m|\varphi_0\rangle=\sum_{m=0}^{\infty}\frac{\lambda^m}{\sqrt{m!}}\langle\varphi_n|\varphi_m\rangle=\sum_{m=0}^{\infty}\frac{\lambda^m}{\sqrt{m!}}\delta_{mn}=\frac{\lambda^n}{\sqrt{n!}} \tag{72}$$

(d)

$$\begin{aligned}\langle\varphi_0|\hat{U}(k)|\varphi_n\rangle=&e^{-\frac{\hbar k^2}{4m\omega}}\langle\varphi_0|e^{ik\sqrt{\frac{\hbar}{2m\omega}}\hat{a}^\dagger}e^{ik\sqrt{\frac{\hbar}{2m\omega}}\hat{a}}|\varphi_n\rangle\\ =&e^{-\frac{\hbar k^2}{4m\omega}}\langle\varphi_0|e^{ik\sqrt{\frac{\hbar}{2m\omega}}\hat{a}}|\varphi_n\rangle\\ =&e^{-\frac{\hbar k^2}{4m\omega}}\frac{\left(ik\sqrt{\frac{\hbar}{2m\omega}}\right)^n}{\sqrt{n!}}\\ =&\frac{i^n}{\sqrt{n!}}e^{-\frac{E_k}{2E_\omega}}\left(\frac{E_k}{E_\omega}\right)^{n/2}\end{aligned} \tag{73}$$

When $k \to 0$,

$$\langle\varphi_0|\hat{U}(k)|\varphi_n\rangle \to \frac{i^n}{\sqrt{n!}}e^0 0^{n/2} = \delta_{n0} \tag{74}$$

This result can be predicted directly: when $n \to$, $\hat{U}(k) \to 1 \Longrightarrow \lim_{k\to 0}\langle\varphi_0|\hat{U}(k)|\varphi_n\rangle = \langle\varphi_0|\varphi_n\rangle = \delta_{n0}$.

□

**Problem 5.** [C-T Exercise 5-8] The evolution operator $\hat{U}(t,0)$ of a one-dimensional harmonic oscillator is written $\hat{U}(t,0) = e^{-i\hat{H}t/\hbar}$ with $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)$.

(a) Consider the operators $\hat{\tilde{a}}(t) = \hat{U}^\dagger(t,0)\hat{a}\hat{U}(t,0)$ and $\hat{\tilde{a}}^\dagger(t) = \hat{U}(t,0)\hat{a}^\dagger\hat{U}(t,0)$. By calculating their action on the eigenkets $|\varphi_n\rangle$ of $\hat{H}$, find the expression for $\hat{\tilde{a}}(t)$ and $\hat{\tilde{a}}^\dagger(t)$ in terms of $\hat{a}$ and $\hat{a}^\dagger$.

(b) Calculate the operators $\hat{\tilde{x}}(t)$ and $\hat{\tilde{p}}_x(t)$ obtained from $\hat{x}$ and $\hat{p}_x$ by the unitary transformation $\hat{\tilde{x}}(t) = \hat{U}^\dagger(t,0)\hat{x}\hat{U}(t,0)$ and $\hat{\tilde{p}}_x(t) = \hat{U}^\dagger(t,0)\hat{p}_x\hat{U}(t,0)$. How can the relations so obtained be interpreted?

(c) Show that $\hat{U}^\dagger(\pi/2\omega,0)|x\rangle$ is an eigenvector of $\hat{p}_x$ and specify its eigenvalue. Similarly, establish that $\hat{U}^\dagger(\pi/2\omega,0)|p_x\rangle$ is an eigenvector of $\hat{x}$.

(d) At $t=0$, the wave function of the oscillator is $\psi(x,0)$. How can one obtain from $\psi(x,0)$ the wave function of the oscillator at all subsequent times $t_q = q\pi/2\omega$ (where $q$ is a positive integer)?

(e) Choose for $\psi(x,0)$ the wave function $\varphi_n(x)$ associated with a stationary state. From the preceding question derive the relation which must exist between $\varphi_n(x)$ and its Fourier transformation $\bar{\varphi}_n(p_x)$.

*Solution:*

(a)

$$\begin{aligned}
\hat{\tilde{a}}(t)|\varphi_n\rangle =&\hat{U}^\dagger(t,0)\hat{a}\hat{U}(t,0)|\varphi_n\rangle\\
=&\hat{U}^\dagger(t,0)\hat{a}e^{-iE_nt/\hbar}|\varphi_n\rangle\\
=&e^{-iE_nt/\hbar}\hat{U}^\dagger(t,0)\hat{a}|\varphi_n\rangle\\
=&e^{-iE_nt/\hbar}\hat{U}^\dagger(t,0)\sqrt{n}|\varphi_{n-1}\rangle\\
=&e^{-iE_nt/\hbar}\sqrt{n}\hat{U}^\dagger(t,0)|\varphi_{n-1}\rangle\\
=&e^{-iE_nt/\hbar}\sqrt{n}e^{iE_{n-1}t/\hbar}|\varphi_{n-1}\rangle\\
=&e^{-i\omega t}\sqrt{n}|\varphi_{n-1}\rangle = e^{-i\omega t}\hat{a}|\varphi_n\rangle
\end{aligned} \tag{75}$$

$$\Longrightarrow \hat{\tilde{a}}(t) = e^{-i\omega t}\hat{a} \tag{76}$$

$$
\begin{aligned}
\hat{\tilde{a}}^\dagger(t)|\varphi_n\rangle =&\hat{U}^\dagger(t,0)\hat{a}^\dagger\hat{U}(t,0)|\varphi_n\rangle\\
=&\hat{U}^\dagger\hat{a}^\dagger e^{-iE_nt/\hbar}|\varphi_n\rangle\\
=&e^{-iE_nt/\hbar}\hat{U}^\dagger\hat{a}^\dagger|\varphi_n\rangle\\
=&e^{-iE_nt/\hbar}\hat{U}^\dagger\sqrt{n+1}|\varphi_{n+1}\rangle\\
=&e^{-iE_nt/\hbar}\sqrt{n+1}\hat{U}^\dagger|\varphi_{n+1}\rangle\\
=&e^{-iE_nt/\hbar}\sqrt{n+1}e^{iE_{n+1}t/\hbar}|\varphi_{n+1}\rangle\\
=&e^{i\omega t}\sqrt{n+1}|\varphi_{n+1}\rangle=e^{i\omega t}\hat{a}^\dagger|\varphi_n\rangle
\end{aligned}
\tag{77}
$$

$$
\Longrightarrow \hat{\tilde{a}}^\dagger(t)=e^{i\omega t}\hat{a}^\dagger \tag{78}
$$

(b)

$$
\begin{aligned}
\hat{\tilde{x}}(t)=&\hat{U}^\dagger(t,0)\hat{x}\hat{U}(t,0)\\
=&\sqrt{\frac{\hbar}{2m\omega}}\hat{U}^\dagger(t,0)(\hat{a}+\hat{a}^\dagger)\hat{U}(t,0)\\
=&\sqrt{\frac{\hbar}{2m\omega}}(e^{-i\omega t}\hat{a}+e^{i\omega t}\hat{a}^\dagger)\\
=&\frac{1}{2}\left[e^{-i\omega t}\left(\hat{x}+\frac{i}{m\omega}\hat{p}_x\right)+e^{i\omega t}(\hat{x}-\frac{i}{m\omega}\hat{p}_x)\right]\\
=&\hat{x}\cos(\omega t)+\frac{1}{m\omega}\hat{p}_x\sin(\omega t)
\end{aligned}
\tag{79}
$$

$$
\begin{aligned}
\hat{\tilde{p}}(t)=&\hat{U}^\dagger(t,0)\hat{p}\hat{U}(t,0)\\
=&-i\sqrt{\frac{m\hbar\omega}{2}}\hat{U}^\dagger(t,0)(\hat{a}-\hat{a}^\dagger)\hat{U}(t,0)\\
=&-i\sqrt{\frac{m\hbar\omega}{2}}(e^{-i\omega t}\hat{a}-e^{i\omega t}\hat{a}^\dagger)\\
=&-i\frac{m\omega}{2}\left[e^{-i\omega t}\left(\hat{x}+\frac{i}{m\omega}\hat{p}_x\right)-e^{i\omega t}(\hat{x}-\frac{i}{m\omega}\hat{p}_x)\right]\\
=&-m\omega\hat{x}\sin(\omega t)+\hat{p}_x\cos(\omega t)
\end{aligned}
\tag{80}
$$

Interpretation: the results corresponds the solution of Hamilton's equations, proof: Hamilton's equations are

$$
\frac{dx}{dt}=\frac{\partial H}{\partial p} \tag{81}
$$

$$
\frac{dp}{dt}=-\frac{\partial H}{\partial x} \tag{82}
$$

Since Hamiltonian is

$$
H=\frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2 \tag{83}
$$

$$\frac{dx}{dt} = \frac{p}{m} \tag{84}$$

$$\frac{dp}{dp} = -m\omega^2 x \tag{85}$$

Differentiate the two equation above about $t$ to get

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{86}$$

$$\frac{d^2p}{dt^2} + \omega^2 p = 0 \tag{87}$$

The general solutions are

$$x(t) = A\cos(\omega t) + B\sin(\omega t)p(t) = C\cos(\omega t) + D\sin(\omega t) \tag{88}$$

Considering the initial conditions $x(t=0) = x_0, p(t=0) = p_0$, $\frac{dx}{dt}\big|_{t=0} = \frac{p_0}{m}$, $\frac{dp}{dt}\big|_{t=0} = -m\omega x_0$ gives

$$A = x_0, \quad C = p_0 \tag{89}$$

$$B = \frac{1}{m\omega}p_0, \quad D = -m\omega x_0 \tag{90}$$

Therefore,

$$x(t) = x_0\cos(\omega t) + \frac{1}{m\omega}p_0\sin(\omega t) \tag{91}$$

$$p(t) = -m\omega x_0\sin(\omega t) + p_0\cos(\omega t) \tag{92}$$

which has the similar form as the results obtained in the quantum version.

(c)

$$\begin{aligned}
\hat{p}_x\hat{U}^\dagger(\pi/2\omega)|x\rangle =& \hat{U}^\dagger(\pi/2\omega)\hat{U}(\pi/2\omega)\hat{p}_x\hat{U}^\dagger(\pi/2\omega)|x\rangle \\
=& \hat{U}^\dagger(\pi/2\omega)[-m\omega\hat{x}\sin(-\frac{\pi}{2}) + \hat{p}_x\cos(-\frac{\pi}{2})]|x\rangle \\
=& m\omega x\hat{U}^\dagger(\pi/2\omega)
\end{aligned}$$

Therefore, $\hat{U}^\dagger(\pi/2\omega, 0)|x\rangle$ is an eigenvector of $\hat{p}_x$ with eigenvalue $m\omega x$.

$$\begin{aligned}
\hat{x}\hat{U}^\dagger(\pi/2\omega)|p_x\rangle =& \hat{U}^\dagger(\pi/2\omega)\hat{U}(\pi/2\omega)\hat{x}\hat{U}^\dagger(\pi/2\omega)|p_x\rangle \\
=& \hat{U}^\dagger(\pi/2\omega)[\hat{x}\cos(-\frac{\pi}{2}) + \frac{1}{m\omega}\hat{p}_x\sin(-\frac{\pi}{2})]|p_x\rangle \\
=& -\frac{p_x}{m\omega}\hat{U}^\dagger(\pi/2\omega)|p_x\rangle
\end{aligned} \tag{93}$$

Therefore, $\hat{U}^\dagger(\pi/2\omega, 0)|p_x\rangle$ is an eigenvector of $\hat{x}$ with eigenvalue $-\frac{p_x}{m\omega}$.

(d)

$$\begin{aligned}\psi(x,q\pi/2\omega) =&\langle x|\psi(q\pi/2\omega)\rangle = \langle x|U(q\pi/2\omega,t)|\psi(0)\rangle\\ =&\int dx'\langle x|\hat{U}(q\pi/2\omega,t)|x'\rangle\langle x'|\psi(0)\rangle\\ =&\int dx'\langle x|\hat{U}(q\pi/2\omega,t)|x'\rangle\psi(x',0)\end{aligned} \tag{94}$$

(e)

□