Quantum Mechanics
Homework Assignment 13
Fall, 2019

1. **[C-T Exercise 9-1]** Consider a spin 1/2 particle. Call its spin $\hat{\vec{S}}$, its orbital angular momentum $\hat{\vec{L}}$, and its state vector $|\psi\rangle$. The two functions $\psi_+(\vec{r})$ and $\psi_-(\vec{r})$ are defined by $\psi_\pm(\vec{r}) = \langle \vec{r}, \pm|\psi\rangle$. Assume that

$$\psi_+(\vec{r}) = R(r)\left[Y_{00}(\theta,\phi) + \frac{1}{\sqrt{3}}Y_{10}(\theta,\phi)\right],$$
$$\psi_-(\vec{r}) = \frac{R(r)}{\sqrt{3}}\left[Y_{11}(\theta,\phi) - Y_{10}(\theta,\phi)\right],$$

where r, θ, ϕ are the coordinates of the particle and $R(r)$ is a given function of r.

(a) What condition must $R(r)$ satisfy for $|\psi\rangle$ to be normalized?

(b) $\hat{S}_z$ is measured with the particle in the state $|\psi\rangle$. What results can be found, and with what probabilities? Same question for $\hat{L}_z$, then for $\hat{S}_x$.

(c) A measurement of $\hat{\vec{L}}^2$, with the particle in the state $|\psi\rangle$, yielded zero. What state describes the particle just after this measurement? Same question if the measurement of $\hat{\vec{L}}^2$ had given $2\hbar^2$.

2. **[C-T Exercise 9-2]** Consider a spin 1/2 particle. $\hat{\vec{p}}$ and $\hat{\vec{S}}$ designate the observables associated with its momentum and its spin. We choose as the basis of the state space the orthonormal basis $|p_x p_y p_z, \pm\rangle$ of eigenvectors common to $\hat{p}_x$, $\hat{p}_y$, $\hat{p}_z$, and $\hat{S}_z$ (whose eigenvalues are, respectively, p_x, p_y, p_z, and $\pm\hbar/2$). We intend to solve the eigenvalue equation of the operator $\hat{A}$ which is defined by $\hat{A} = \hat{\vec{S}}\cdot\hat{\vec{p}}$.

(a) Is $\hat{A}$ Hermitian?

(b) Show that there exists a basis of eigenvectors of $\hat{A}$ which are also eigenvectors of $\hat{p}_x$, $\hat{p}_y$, and $\hat{p}_z$. In the subspace spanned by the kets $|p_x p_y p_z, \pm\rangle$, where p_x, p_y, and p_z are fixed, what is the matrix representing $\hat{A}$?

(c) What are the eigenvalues of $\hat{A}$, and what is their degree of degeneracy? Find a system of eigenvectors common to $\hat{A}$ and $\hat{p}_x$, $\hat{p}_y$, $\hat{p}_z$.

3. **[C-T Exercise 9-3]** The Hamiltonian of an electron of mass m, charge q, spin $\hbar\vec{\sigma}/2$ with σ_x, σ_y, and σ_z the Pauli matrices, placed in an electromagnetic field described by the vector potential $\vec{A}(\vec{r},t)$ and the scalar potential $U(\vec{r},t)$, is written $\hat{H} = \frac{1}{2m}\big[\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}},t)\big]^2 + qU(\hat{\vec{r}},t) - \frac{q\hbar}{2m}\vec{\sigma}\cdot\vec{B}(\hat{\vec{r}},t)$. The last term represents the interaction between the spin magnetic moment $(q\hbar/2m)\vec{\sigma}$ and the magnetic field $\vec{B}(\hat{\vec{r}},t) = \vec{\nabla}\times\vec{A}(\hat{\vec{r}},t)$. Show, using the properties of the Pauli matrices, that this Hamiltonian can also be written in the form ("the Pauli Hamiltonian") $\hat{H} = \frac{1}{2m}\Big\{\vec{\sigma}\cdot\big[\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}},t)\big]\Big\}^2 + qU(\hat{\vec{r}},t)$.

4. **[C-T Exercise 10-3]** Consider a system composed of two spin 1/2 particles whose orbital variables are ignored. The Hamiltonian of the system is $\hat{H} = \omega_1\hat{S}_{1z} + \omega_2\hat{S}_{2z}$, where $\hat{S}_{1z}$ and $\hat{S}_{2z}$ are the projections of the spins $\hat{S}_1$ and $\hat{S}_2$ of the two particles onto Oz, and ω_1 and ω_2 are real constants.

(a) The initial state of the system, at time $t=0$, is $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\big[\,|+-\rangle + |-+\rangle\,\big]$. At time t, $\hat{\vec{S}}^2 = (\hat{\vec{S}}_1 + \hat{\vec{S}}_2)^2$ is measured. What results can be found, and with what probabilities?

(b) If the initial state of the system is arbitrary, what Bohr frequencies can appear in the evolution of $\langle\hat{\vec{S}}^2\rangle$? Same question for $\hat{S}_x = \hat{S}_{1x} + \hat{S}_{2x}$.

5. **[C-T Exercise 10-5]** Let $\hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2 + \hat{\vec{S}}_3$ be the total angular momentum of three spin 1/2 particles (whose orbital variables will be ignored). Let $|\varepsilon_1\varepsilon_2\varepsilon_3\rangle$ be the eigenstates common to $\hat{S}_{1z}$, $\hat{S}_{2z}$, and $\hat{S}_{3z}$, of respective eigenvalues $\varepsilon_1\hbar/2$, $\varepsilon_2\hbar/2$, and $\varepsilon_3\hbar/2$. Give a basis of eigenvectors common to $\hat{\vec{S}}^2$ and $\hat{S}_z$, in terms of the kets $|\varepsilon_1\varepsilon_2\varepsilon_3\rangle$. Do these two operators form a CSCO? (Begin by adding two of the spins, then add the partial angular momentum so obtained to the third one.)