Quantum Mechanics

Solutions to the Problems in Homework Assignment 11

Fall, 2019

1. **[C-T Exercise 5-1]** Consider a harmonic oscillator of mass m and angular frequency ω. At time $t = 0$, the state of this oscillator is given by $|\psi(0)\rangle = \sum_n c_n |\varphi_n\rangle$, where the states $|\varphi_n\rangle$ are stationary states with energies $(n + 1/2)\hbar\omega$.

   (a) What is the probability $\mathscr{P}$ that a measurement of the oscillator's energy performed at an arbitrary time $t > 0$, will yield a result greater than $2\hbar\omega$? When $\mathscr{P} = 0$, what are the non-zero coefficients c_n?

   (b) From now on, assume that only c_0 and c_1 are different from zero. Write normalization condition for $|\psi(0)\rangle$ and the mean value $\langle \hat{H} \rangle$ of the energy in terms of c_0 and c_1. With the additional requirement $\langle \hat{H} \rangle = \hbar\omega$, calculate $|c_0|^2$ and $|c_1|^2$.

   (c) As the normalized state vector $|\psi(0)\rangle$ is defined only to within a global phase factor, we fix this factor by choosing c_0 real and positive. We set $c_1 = |c_1|e^{i\theta_1}$. We assume that $\langle \hat{H} \rangle = \hbar\omega$ and that $\langle \hat{x} \rangle = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}$. Calculate θ_1.

   (d) With $|\psi(0)\rangle$ so determined, write $|\psi(t)\rangle$ for $t > 0$ and calculate the value of θ_1 at t. Deduce the mean value $\langle \hat{x} \rangle(t)$ of the position at t.

(a) The state vector of the harmonic oscillator at time $t > 0$ is given by

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar} |\varphi_n\rangle = \sum_n c_n e^{-i(n+1/2)\omega t} |\varphi_n\rangle .$$

The probability of obtaining the result $E_n = (n + 1/2)\hbar\omega$ when the oscillator's energy is measured is given by

$$\mathscr{P}_{\hat{H}}((n + 1/2)\hbar\omega) = \left|\langle \varphi_n|\psi(t)\rangle\right|^2 = \left|c_n e^{-iE_n t/\hbar}\right|^2 = |c_n|^2.$$

In consideration that $E_1 = 3\hbar\omega/2 < 2\hbar\omega < E_2 = 5\hbar\omega/2$, the probability $\mathscr{P}$ of obtaining a result greater than $2\hbar\omega$ is given by

$$\mathscr{P} = \sum_{n=2}^{\infty} \mathscr{P}_{\hat{H}}((n + 1/2)\hbar\omega) = \sum_{n=2}^{\infty} |c_n|^2.$$

If $\mathscr{P} = 0$, we then have $c_2 = c_3 = \cdots = 0$. Thus, the non-zero coefficients are c_0 and c_1.

(b) With $c_2 = c_3 = \cdots = 0$, we have

$$|\psi(0)\rangle = c_0 |\varphi_0\rangle + c_1 |\varphi_1\rangle .$$

The normalization condition for $|\psi(0)\rangle$ is given by

$$1 = \langle \psi(0)|\psi(0)\rangle = \left[c_0^* \langle \varphi_0| + c_1^* \langle \varphi_1|\right]\left[c_0 |\varphi_0\rangle + c_1 |\varphi_1\rangle\right] = |c_0|^2 + |c_1|^2.$$

The mean value $\langle \hat{H} \rangle$ of the energy is given by

$$\begin{aligned}
\langle \hat{H} \rangle = \langle \psi(0)|\hat{H}|\psi(0)\rangle &= \left[c_0^* \langle \varphi_0| + c_1^* \langle \varphi_1|\right]\left[c_0 \hat{H} |\varphi_0\rangle + c_1 \hat{H} |\varphi_1\rangle\right] \\
&= \left[c_0^* \langle \varphi_0| + c_1^* \langle \varphi_1|\right]\left[c_0 \frac{1}{2}\hbar\omega |\varphi_0\rangle + c_1 \frac{3}{2}\hbar\omega |\varphi_1\rangle\right] \\
&= \frac{1}{2}\hbar\omega\left(|c_0|^2 + 3|c_1|^2\right).
\end{aligned}$$

The mean value $\langle \hat{H} \rangle$ of the energy can be also found quickly in the following manner.

$$\langle \hat{H} \rangle = |c_0|^2 E_0 + |c_1|^2 E_1 = |c_0|^2 \frac{1}{2}\hbar\omega + |c_1|^2 \frac{3}{2}\hbar\omega = \frac{1}{2}\hbar\omega\big(|c_0|^2 + 3|c_1|^2\big).$$

From $\langle \hat{H} \rangle = \hbar\omega$, we have

$$|c_0|^2 + 3|c_1|^2 = 2.$$

Solving for $|c_0|^2$ and $|c_1|^2$ from $|c_0|^2 + |c_1|^2 = 1$ and $|c_0|^2 + 3|c_1|^2 = 2$, we obtain

$$|c_0|^2 = |c_1|^2 = \frac{1}{2}.$$

(c) Choosing c_0 real and positive and setting $c_1 = |c_1| e^{i\theta_1}$, we have

$$c_0 = \frac{1}{\sqrt{2}},\ c_1 = \frac{1}{\sqrt{2}} e^{i\theta_1}.$$

Evaluating $\langle \hat{x} \rangle$, we have

$$\begin{aligned}
\langle \hat{x} \rangle &= \sqrt{\frac{\hbar}{2m\omega}}\, \langle \psi(0) | \hat{a} + \hat{a}^\dagger | \psi(0) \rangle = \sqrt{\frac{\hbar}{2m\omega}} \big[c_0^* \langle \varphi_0 | + c_1^* \langle \varphi_1 | \big] \big[c_0 (\hat{a} + \hat{a}^\dagger) | \varphi_0 \rangle + c_1 (\hat{a} + \hat{a}^\dagger) | \varphi_1 \rangle \big] \\
&= \sqrt{\frac{\hbar}{2m\omega}} \big[c_0^* \langle \varphi_0 | + c_1^* \langle \varphi_1 | \big] \big[c_0 | \varphi_1 \rangle + c_1 \big(| \varphi_0 \rangle + \sqrt{2}\, | \varphi_2 \rangle \big) \big] = \sqrt{\frac{\hbar}{2m\omega}} \big(c_0^* c_1 + c_1^* c_0 \big) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\, \frac{1}{2} \big(e^{i\theta_1} + e^{-i\theta_1} \big) = \sqrt{\frac{\hbar}{2m\omega}}\, \cos\theta_1.
\end{aligned}$$

From

$$\langle \hat{x} \rangle = \frac{1}{2} \sqrt{\frac{\hbar}{m\omega}},$$

we have

$$\cos\theta_1 = \frac{1}{\sqrt{2}}.$$

Thus,

$$\theta_1 = \frac{\pi}{4}.$$

(d) With

$$c_0 = \frac{1}{\sqrt{2}},\ c_1 = \frac{1}{\sqrt{2}} e^{i\pi/4},\ c_2 = c_3 = \cdots = 0,$$

we have

$$| \psi(0) \rangle = \frac{1}{\sqrt{2}} \big[\, | \varphi_0 \rangle + e^{i\pi/4} | \varphi_1 \rangle \,\big].$$

$| \psi(t) \rangle$ is then given by

$$| \psi(t) \rangle = \frac{1}{\sqrt{2}}\, e^{-i\omega t/2} \big[\, | \varphi_0 \rangle + e^{i(\pi/4 - \omega t)} | \varphi_1 \rangle \,\big].$$

The value of θ_1 at t is thus given by

$$\theta_1(t) = \frac{\pi}{4} - \omega t.$$

Since the overall phase factor $e^{-i\omega t/2}$ in $| \psi(t) \rangle$ will be cancelled when the mean value $\langle \hat{x} \rangle(t)$ of the position at t is evaluated, the mean value $\langle \hat{x} \rangle(t)$ is of the same form as $\langle \hat{x} \rangle$ evaluated at $t = 0$ with θ_1 replaced with $\theta_1(t)$. We thus have

$$\langle \hat{x} \rangle(t) = \sqrt{\frac{\hbar}{2m\omega}}\, \cos\theta_1(t) = \sqrt{\frac{\hbar}{2m\omega}}\, \cos\!\left(\frac{\pi}{4} - \omega t \right).$$

2. **[C-T Exercise 5-3]** Two particles of the same mass m, with positions $\hat{x}_1$ and $\hat{x}_2$ and momenta $\hat{p}_1$ and $\hat{p}_2$, are subject to the same potential $\hat{V}(\hat{x}) = \dfrac{1}{2}m\omega^2\hat{x}^2$. The two particles do not interact.

(a) Write the operator $\hat{H}$, the Hamiltonian of the two-particle system. Show that $\hat{H}$ can be written $\hat{H} = \hat{H}_1 + \hat{H}_2$, where $\hat{H}_1$ and $\hat{H}_2$ act respectively only in the state space of particle (1) and in that of particle (2). Calculate the energies of the two-particle system, their degrees of degeneracy, and the corresponding wave functions.

(b) Does $\hat{H}$ form a CSCO? Same question for the set $\{\hat{H}_1, \hat{H}_2\}$. We denote by $|\Phi_{n_1n_2}\rangle$ the eigenvectors common to $\hat{H}_1$ and $\hat{H}_2$. Write the orthonormality and closure relations for the states $|\Phi_{n_1n_2}\rangle$.

(c) Consider a system which, at $t = 0$, is in the state

$$|\psi(0)\rangle = \frac{1}{2}\big[\,|\Phi_{00}\rangle + |\Phi_{10}\rangle + |\Phi_{01}\rangle + |\Phi_{11}\rangle\,\big].$$

If at this time one measures the total energy of the system or the energy of particle (1) or the position of particle (1) or the velocity of particle (1), what results can be found, and with what probabilities?

(a) The Hamiltonian of the two-particle system is given by

$$\hat{H} = \frac{1}{2m}\Big(\hat{p}_1^2 + \hat{p}_2^2\Big) + \frac{1}{2}m\omega^2\Big(\hat{x}_1^2 + \hat{x}_2^2\Big).$$

Grouping the terms respectively for particle (1) and particle (2), we have

$$\hat{H} = \left[\frac{1}{2m}\hat{p}_1^2 + \frac{1}{2}m\omega^2\hat{x}_1^2\right] + \left[\frac{1}{2m}\hat{p}_2^2 + \frac{1}{2}m\omega^2\hat{x}_2^2\right] = \hat{H}_1 + \hat{H}_2,$$

where

$$\begin{aligned}\hat{H}_1 &= \frac{1}{2m}\hat{p}_1^2 + \frac{1}{2}m\omega^2\hat{x}_1^2,\\ \hat{H}_2 &= \frac{1}{2m}\hat{p}_2^2 + \frac{1}{2}m\omega^2\hat{x}_2^2.\end{aligned}$$

Obviously, $\hat{H}_1$ acts only in the state space of particle (1) and $\hat{H}_2$ acts only in the state space of particle (2). Since $[\hat{H}_1, \hat{H}_2] = 0$, $\hat{H}$, $\hat{H}_1$, and $\hat{H}_2$ have common eigenvectors. From the fact that $\hat{H}_1$ and $\hat{H}_2$ are of the form of the Hamiltonian of a one-dimensional harmonic oscillator, the eigenvalues and eigenvectors of $\hat{H}_1$ and $\hat{H}_2$ are respectively given by those of the Hamiltonian of a one-dimensional harmonic oscillator. For $\hat{H}_1$, we have

$$\hat{H}_1\,|\varphi_{n_1}^{(1)}\rangle = E_{n_1}^{(1)}\,|\varphi_{n_1}^{(1)}\rangle,\ E_{n_1}^{(1)} = (n_1 + 1/2)\hbar\omega,\ |\varphi_{n_1}^{(1)}\rangle = \frac{1}{\sqrt{n_1!}}\big(\hat{a}_1^\dagger\big)^{n_1}\,|\varphi_0^{(1)}\rangle,\ n_1 = 0, 1, 2, \cdots.$$

For $\hat{H}_2$, we have

$$\hat{H}_2\,|\varphi_{n_2}^{(2)}\rangle = E_{n_2}^{(2)}\,|\varphi_{n_2}^{(2)}\rangle,\ E_{n_2}^{(2)} = (n_2 + 1/2)\hbar\omega,\ |\varphi_{n_2}^{(2)}\rangle = \frac{1}{\sqrt{n_2!}}\big(\hat{a}_2^\dagger\big)^{n_2}\,|\varphi_0^{(2)}\rangle,\ n_2 = 0, 1, 2, \cdots.$$

The eigenvalues and eigenvectors of $\hat{H}$ are then given by

$$E_{n_1n_2} = E_{n_1} + E_{n_2} = (n_1 + n_2 + 1)\hbar\omega,\ |\Phi_{n_1n_2}\rangle = |\varphi_{n_1}^{(1)}\rangle\,|\varphi_{n_2}^{(2)}\rangle,\ n_1, n_2 = 0, 1, 2, \cdots.$$

Note that $|\Phi_{n_1n_2}\rangle$ are the common eigenvectors of $\hat{H}$, $\hat{H}_1$, and $\hat{H}_2$,

$$\begin{aligned}\hat{H}\,|\Phi_{n_1n_2}\rangle &= E_{n_1n_2}\,|\Phi_{n_1n_2}\rangle,\\ \hat{H}_1\,|\Phi_{n_1n_2}\rangle &= E_{n_1}\,|\Phi_{n_1n_2}\rangle,\\ \hat{H}_2\,|\Phi_{n_1n_2}\rangle &= E_{n_2}\,|\Phi_{n_1n_2}\rangle.\end{aligned}$$

From the expression of $E_{n_1n_2}$, $E_{n_1n_2} = (n_1 + n_2 + 1)\hbar\omega$, we see that, for a given value of $(n_1 + n_2)$, both n_1 and n_2 can take on values $0, 1, \cdots, n_1 + n_2$ with $(n_1 + n_2)$ fixed. Therefore, the degree of degeneracy of $E_{n_1n_2}$ is $(n_1 + n_2 + 1)$.

(b) Because of the presence of the two-fold degeneracy of $E_{n_1 n_2}$ for $n_1 \neq n_2$, $\hat{H}$ alone does not form a CSCO. If both n_1 and n_2 are specified, then a unique common eigenvector of $\hat{H}$, $\hat{H}_1$, and $\hat{H}_2$ can be determined. Thus, the set $\{\hat{H}_1, \hat{H}_2\}$ forms a CSCO.
The orthonormality relation for $|\Phi_{n_1 n_2}\rangle$ is given by

$$\langle \Phi_{n_1 n_2} | \Phi_{n_1' n_2'} \rangle = \delta_{n_1 n_1'} \delta_{n_2 n_2'}.$$

The closure relation for $|\Phi_{n_1 n_2}\rangle$ is given by

$$\sum_{n_1 n_2} |\Phi_{n_1 n_2}\rangle\langle \Phi_{n_1 n_2}| = 1.$$

(c) **Measurement of the total energy.** From the given expression of $|\psi(0)\rangle$, we see that measuring the total energy can yield the results $\hbar\omega$, $2\hbar\omega$, and $3\hbar\omega$. The probabilities of obtaining these results are respectively given by

$$\mathscr{P}_{\hat{H}}(\hbar\omega) = \left(\frac{1}{2}\right)^2 = \frac{1}{4},$$
$$\mathscr{P}_{\hat{H}}(2\hbar\omega) = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2},$$
$$\mathscr{P}_{\hat{H}}(3\hbar\omega) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Measurement of the energy of particle (1). Measuring the energy of particle (1) can yield the results $\hbar\omega/2$ and $3\hbar\omega/2$. The probabilities of obtaining these two results are respectively given by

$$\mathscr{P}_{\hat{H}_1}(\hbar\omega/2) = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2},$$
$$\mathscr{P}_{\hat{H}_1}(3\hbar\omega/2) = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}.$$

Measurement of the position of particle (1). Measuring the position of particle (1) can yield any real value between $-\infty$ and $+\infty$, $-\infty < x_1 < +\infty$. The probability of obtaining the position of particle (1) in the range $(x_1, x_1 + dx_1)$ is given by

$$\begin{aligned}
d\mathscr{P}_{\hat{x}_1}(x_1) &= dx_1 \int_{-\infty}^{+\infty} dx_2 \, \left|\langle x_1 | \langle x_2 | \psi(0)\rangle\right|^2 \\
&= dx_1 \frac{1}{4} \int_{-\infty}^{+\infty} dx_2 \, \left|\varphi_0(x_1)\varphi_0(x_2) + \varphi_1(x_1)\varphi_0(x_2) + \varphi_0(x_1)\varphi_1(x_2) + \varphi_1(x_1)\varphi_1(x_2)\right|^2 \\
&= dx_1 \frac{1}{4} \left|\varphi_0(x_1) + \varphi_1(x_1)\right|^2 \int_{-\infty}^{+\infty} dx_2 \, \left|\varphi_0(x_2) + \varphi_1(x_2)\right|^2 \\
&= dx_1 \frac{1}{4} \left|\varphi_0(x_1) + \varphi_1(x_1)\right|^2 \\
&\quad \times \int_{-\infty}^{+\infty} dx_2 \, \Big[\left|\varphi_0(x_2)\right|^2 + \varphi_0^*(x_2)\varphi_1(x_2) + \varphi_1^*(x_2)\varphi_0(x_2) + \left|\varphi_1(x_2)\right|^2 \Big] \\
&= dx_1 \frac{1}{4} \left|\varphi_0(x_1) + \varphi_1(x_1)\right|^2 \Big[1 + 0 + 0 + 1 \Big] \\
&= dx_1 \frac{1}{2} \left|\varphi_0(x_1) + \varphi_1(x_1)\right|^2.
\end{aligned}$$

Measurement of the velocity v_1 of particle (1). Measuring the velocity of particle (1) is equivalent to measuring the momentum p_1 of particle (1), with v_1 related to p_1 through $v_1 = p_1/\mu$, where μ is the mass of the particle. Measuring the velocity v_1 of particle (1) can yield any real value between $-\infty$ and

$+\infty$, $-\infty < v_1 < +\infty$. The probability of obtaining the velocity of particle (1) in the range $(v_1, v_1 + dv_1)$ is given by

$$
\begin{aligned}
d\mathscr{P}_{\hat{v}_1}(v_1) &= dv_1 \mu \int_{-\infty}^{+\infty} dp_2 \, \left| \langle p_1 = \mu v_1 | \langle p_2 | \psi(0) \rangle \right|^2 \\
&= dv_1 \frac{\mu}{4} \int_{-\infty}^{+\infty} dp_2 \, \left| \overline{\varphi}_0(\mu v_1)\overline{\varphi}_0(p_2) + \overline{\varphi}_1(\mu v_1)\overline{\varphi}_0(p_2) + \overline{\varphi}_0(\mu v_1)\overline{\varphi}_1(p_2) + \overline{\varphi}_1(\mu v_1)\overline{\varphi}_1(p_2) \right|^2 \\
&= dv_1 \frac{\mu}{4} \left| \overline{\varphi}_0(\mu v_1) + \overline{\varphi}_1(\mu v_1) \right|^2 \int_{-\infty}^{+\infty} dp_2 \, \left| \overline{\varphi}_0(p_2) + \overline{\varphi}_1(p_2) \right|^2 \\
&= dv_1 \frac{\mu}{4} \left| \overline{\varphi}_0(\mu v_1) + \overline{\varphi}_1(\mu v_1) \right|^2 \\
&\quad \times \int_{-\infty}^{+\infty} dp_2 \left[\left| \overline{\varphi}_0(p_2) \right|^2 + \overline{\varphi}_0^*(p_2)\overline{\varphi}_1(p_2) + \overline{\varphi}_1^*(p_2)\overline{\varphi}_0(p_2) + \left| \overline{\varphi}_1(p_2) \right|^2 \right] \\
&= dv_1 \frac{\mu}{4} \left| \overline{\varphi}_0(\mu v_1) + \overline{\varphi}_1(\mu v_1) \right|^2 \left[1 + 0 + 0 + 1 \right] \\
&= dv_1 \frac{\mu}{2} \left| \overline{\varphi}_0(\mu v_1) + \overline{\varphi}_1(\mu v_1) \right|^2 .
\end{aligned}
$$

3. **[C-T Exercise 5-5] Continue from the previous problem.** We denote by $|\Phi_{n_1 n_2}\rangle$ the eigenstates common to $\hat{H}_1$ and $\hat{H}_2$, of eigenvalues $(n_1 + 1/2)\hbar\omega$ and $(n_2 + 1/2)\hbar\omega$. The "two particle exchange" operator $\hat{P}_e$ is defined by $\hat{P}_e |\Phi_{n_1 n_2}\rangle = |\Phi_{n_2 n_1}\rangle$.

   (a) Prove that $\hat{P}_e^{-1} = \hat{P}_e$, and that $\hat{P}_e$ is unitary. What are the eigenvalues of $\hat{P}_e$? Let $\hat{B}' = \hat{P}_e \hat{B} \hat{P}_e^\dagger$ be the observable resulting from the transformation by $\hat{P}_e$ of an arbitrary observable $\hat{B}$. Show that the condition $\hat{B}' = \hat{B}$ ($\hat{B}$ invariant under exchange of the two particles) is equivalent to $[\hat{B}, \hat{P}_e] = 0$.

   (b) Show that $\hat{P}_e \hat{H}_1 \hat{P}_e^\dagger = \hat{H}_2$ and $\hat{P}_e \hat{H}_2 \hat{P}_e^\dagger = \hat{H}_1$. Does $\hat{H}$ commute with $\hat{P}_e$? Calculate the action of $\hat{P}_e$ on the observables $\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2$.

   (c) Construct a basis of eigenvectors common to $\hat{H}$ and $\hat{P}_e$. Do these two operators form a CSCO? What happens to the spectrum of $\hat{H}$ and the degeneracy of its eigenvalues if one retains only the eigenvectors $|\Phi\rangle$ of $\hat{H}$ for which $\hat{P}_e |\Phi\rangle = -|\Phi\rangle$?

(a) Applying $\hat{P}_e$ to $\hat{P}_e |\Phi_{n_1 n_2}\rangle = |\Phi_{n_2 n_1}\rangle$, we have

$$\hat{P}_e^2 |\Phi_{n_1 n_2}\rangle = \hat{P}_e |\Phi_{n_2 n_1}\rangle = |\Phi_{n_1 n_2}\rangle$$

from which we have

$$\hat{P}_e^2 = 1.$$

Comparing $\hat{P}_e^2 = 1$ with $\hat{P}_e^{-1}\hat{P}_e = \hat{P}_e \hat{P}_e^{-1} = 1$, we see that

$$\hat{P}_e^{-1} = \hat{P}_e.$$

Assume that $|\Phi_{n_1 n_2}\rangle$ is normalized. Taking the Hermitian conjugation of $\hat{P}_e |\Phi_{n_1 n_2}\rangle = |\Phi_{n_2 n_1}\rangle$, we have

$$\langle \Phi_{n_1 n_2} | \hat{P}_e^\dagger = \langle \Phi_{n_2 n_1} | .$$

Multiplying both sides of the above equation from the right with $\hat{P}_e$ and then taking the scalar product of the resultant with $|\Phi_{n_1 n_2}\rangle$, we have

$$\langle \Phi_{n_1 n_2} | \hat{P}_e^\dagger \hat{P}_e | \Phi_{n_1 n_2} \rangle = \langle \Phi_{n_2 n_1} | \hat{P}_e | \Phi_{n_1 n_2} \rangle = \langle \Phi_{n_2 n_1} | \Phi_{n_2 n_1} \rangle = 1.$$

Since $|\Phi_{n_1 n_2}\rangle$ is an arbitrary common eigenvector of $\hat{H}_1$ and $\hat{H}_2$, we have

$$\hat{P}_e^\dagger \hat{P}_e = 1.$$

Multiplying both sides of the above equation from the left with $\hat{P}_e$ yields

$$\hat{P}_e \hat{P}_e^\dagger \hat{P}_e = \hat{P}_e.$$

Applying the above equation on an arbitrary ket $|\Phi_{n_1 n_2}\rangle$ yields

$$\hat{P}_e \hat{P}_e^\dagger \hat{P}_e |\Phi_{n_1 n_2}\rangle = \hat{P}_e |\Phi_{n_1 n_2}\rangle .$$

Making use of $\hat{P}_e |\Phi_{n_1 n_2}\rangle = |\Phi_{n_2 n_1}\rangle$, we have

$$\hat{P}_e \hat{P}_e^\dagger |\Phi_{n_2 n_1}\rangle = |\Phi_{n_2 n_1}\rangle .$$

Since $|\Phi_{n_2 n_1}\rangle$ is an arbitrary common eigenvector of $\hat{H}_1$ and $\hat{H}_2$, we have

$$\hat{P}_e \hat{P}_e^\dagger = 1.$$

We have thus obtained

$$\hat{P}_e^\dagger \hat{P}_e = \hat{P}_e \hat{P}_e^\dagger = 1.$$

Comparing $\hat{P}_e^\dagger \hat{P}_e = \hat{P}_e \hat{P}_e^\dagger = 1$ with $\hat{P}_e^{-1} \hat{P}_e = \hat{P}_e \hat{P}_e^{-1} = 1$, we see that

$$\hat{P}_e^\dagger = \hat{P}_e^{-1}.$$

Therefore, $\hat{P}_e$ is unitary.
Let λ be the eigenvalue of $\hat{P}_e$. Let $|\chi\rangle$ be the normalized eigenvector of $\hat{P}_e$ corresponding to the eigenvalue λ. The eigenvalue equation of $\hat{P}_e$ reads

$$\hat{P}_e |\chi\rangle = \lambda |\chi\rangle .$$

Acting $\hat{P}_e$ on both sides of the above equation and making use of the eigenvalue equation of $\hat{P}_e$, we have

$$\hat{P}_e^2 |\chi\rangle = \lambda \hat{P}_e |\chi\rangle = \lambda^2 |\chi\rangle .$$

Making use of $\hat{P}_e^2 = 1$ on the left hand side of the above equation and then switching the two sides of the resultant equation, we have

$$\lambda^2 |\chi\rangle = |\chi\rangle .$$

Taking the scalar product of the above equation with $|\chi\rangle$, we have

$$\lambda^2 \langle\chi|\chi\rangle = \langle\chi|\chi\rangle .$$

Since $\langle\chi|\chi\rangle \neq 0$ for an eigenvector $|\chi\rangle$ of $\hat{P}_e$, we have

$$\lambda^2 = 1.$$

Therefore, The eigenvalues of $\hat{P}_e$ are $\lambda_1 = +1$ and $\lambda_2 = -1$. The two corresponding eigenvectors are respectively denoted by $|\chi_1\rangle$ and $|\chi_2\rangle$.
From $\hat{B}' = \hat{B}$, we have

$$\hat{P}_e \hat{B} \hat{P}_e^\dagger = \hat{B}.$$

Multiplying both sides of the above equation from the right with $\hat{P}_e$, we have

$$\hat{P}_e \hat{B} \hat{P}_e^\dagger \hat{P}_e = \hat{B} \hat{P}_e$$

Making use of $\hat{P}_e^\dagger \hat{P}_e = 1$ on the left hand side of the above equation, we have

$$\hat{P}_e \hat{B} = \hat{B} \hat{P}_e$$

which can be written as

$$[\hat{B}, \hat{P}_e] = 0.$$

Therefore, the condition $\hat{B}' = \hat{B}$ is equivalent to $[\hat{B}, \hat{P}_e] = 0$.

(b) The eigenvalue equation of $\hat{H}_1$ reads

$$\hat{H}_1 |\Phi_{n_1n_2}\rangle = E_{n_1} |\Phi_{n_1n_2}\rangle .$$

Applying $\hat{P}_e$ to both sides of the above equation yields

$$\hat{P}_e\hat{H}_1 |\Phi_{n_1n_2}\rangle = \hat{P}_e\big(E_{n_1} |\Phi_{n_1n_2}\rangle\big) = E_{n_2} |\Phi_{n_2n_1}\rangle .$$

Inserting $\hat{P}_e^\dagger\hat{P}_e = 1$ between $\hat{H}_1$ and $|\Phi_{n_1n_2}\rangle$ on the left hand side of the above equation yields

$$\hat{P}_e\hat{H}_1\hat{P}_e^\dagger\hat{P}_e |\Phi_{n_1n_2}\rangle = E_{n_2} |\Phi_{n_2n_1}\rangle .$$

Making use of $\hat{P}_e |\Phi_{n_1n_2}\rangle = |\Phi_{n_2n_1}\rangle$ on the left hand side of the above equation, we have

$$\hat{P}_e\hat{H}_1\hat{P}_e^\dagger |\Phi_{n_2n_1}\rangle = E_{n_2} |\Phi_{n_2n_1}\rangle .$$

Comparing the above equation with the eigenvalue equation of $\hat{H}_2$, $\hat{H}_2 |\Phi_{n_2n_1}\rangle = E_{n_2} |\Phi_{n_2n_1}\rangle$, we obtain

$$\hat{P}_e\hat{H}_1\hat{P}_e^\dagger = \hat{H}_2$$

From $\hat{P}_e^{-1} = \hat{P}_e$ and $\hat{P}_e^\dagger = \hat{P}_e^{-1}$, we have

$$\hat{P}_e^\dagger = \hat{P}_e.$$

That is, $\hat{P}_e$ is a Hermitian operator.
Multiplying both sides of the equation $\hat{P}_e\hat{H}_1\hat{P}_e^\dagger = \hat{H}_2$ with $\hat{P}_e$ from the right and with $\hat{P}_e^\dagger$ from the left and then switching the two sides of the resultant equation, we have

$$\hat{P}_e^\dagger\hat{H}_2\hat{P}_e = \hat{P}_e^\dagger\hat{P}_e\hat{H}_1\hat{P}_e^\dagger\hat{P}_e = \hat{H}_1.$$

Making use of $\hat{P}_e^\dagger = \hat{P}_e$, we can rewrite the above equation as

$$\hat{P}_e\hat{H}_2\hat{P}_e^\dagger = \hat{H}_1.$$

Inserting the explicit expressions of $\hat{H}_1$ and $\hat{H}_2$ into the equation $\hat{P}_e\hat{H}_1\hat{P}_e^\dagger = \hat{H}_2$, we have

$$\frac{1}{2m}\hat{P}_e\hat{p}_1^2\hat{P}_e^\dagger + \frac{1}{2}m\omega^2\hat{P}_e\hat{x}_1^2\hat{P}_e^\dagger = \frac{1}{2m}\hat{p}_2^2 + \frac{1}{2}m\omega^2\hat{x}_2^2.$$

Since the $\hat{x}$ and $\hat{p}$ are independent observabls, we have

$$\hat{P}_e\hat{x}_1^2\hat{P}_e^\dagger = \hat{x}_2^2,$$
$$\hat{P}_e\hat{p}_1^2\hat{P}_e^\dagger = \hat{p}_2^2.$$

Rewriting $\hat{P}_e\hat{x}_1^2\hat{P}_e^\dagger$ and $\hat{P}_e\hat{p}_1^2\hat{P}_e^\dagger$ respectively as

$$\hat{P}_e\hat{x}_1^2\hat{P}_e^\dagger = \hat{P}_e\hat{x}_1\hat{P}_e^\dagger\hat{P}_e\hat{x}_1\hat{P}_e^\dagger = \big(\hat{P}_e\hat{x}_1\hat{P}_e^\dagger\big)^2,$$
$$\hat{P}_e\hat{p}_1^2\hat{P}_e^\dagger = \hat{P}_e\hat{p}_1\hat{P}_e^\dagger\hat{P}_e\hat{p}_1\hat{P}_e^\dagger = \big(\hat{P}_e\hat{p}_1\hat{P}_e^\dagger\big)^2,$$

we see that

$$\hat{P}_e\hat{x}_1\hat{P}_e^\dagger = \hat{x}_2,$$
$$\hat{P}_e\hat{p}_1\hat{P}_e^\dagger = \hat{p}_2,$$

where, in consistency with the definition of $\hat{P}_e$, we have chosen the plus sign for the square roots.
Using the similar line of reasoning used for the derivation of $\hat{P}_e\hat{H}_2\hat{P}_e^\dagger = \hat{H}_1$ from $\hat{P}_e\hat{H}_1\hat{P}_e^\dagger = \hat{H}_2$, from $\hat{P}_e\hat{x}_1\hat{P}_e^\dagger = \hat{x}_2$ and $\hat{P}_e\hat{p}_1\hat{P}_e^\dagger = \hat{p}_2$, we can derive

$$\hat{P}_e\hat{x}_2\hat{P}_e^\dagger = \hat{x}_1,$$
$$\hat{P}_e\hat{p}_2\hat{P}_e^\dagger = \hat{p}_1.$$

(c) From $\hat{P}_e\hat{H}_1\hat{P}_e^\dagger = \hat{H}_2$ and $\hat{P}_e\hat{H}_2\hat{P}_e^\dagger = \hat{H}_1$, we have

$$\hat{P}_e\hat{H}\hat{P}_e^\dagger = \hat{P}_e\hat{H}_1\hat{P}_e^\dagger + \hat{P}_e\hat{H}_2\hat{P}_e^\dagger = \hat{H}_2 + \hat{H}_1 = \hat{H}.$$

That is,

$$\hat{P}_e\hat{H}\hat{P}_e^\dagger = \hat{H}.$$

Multiplying both sides of the above equation with $\hat{P}_e$ from the right and making use of $\hat{P}_e^\dagger\hat{P}_e = 1$, we have

$$\hat{P}_e\hat{H} = \hat{H}\hat{P}_e.$$

Thus, $\hat{H}$ and $\hat{P}_e$ commute so that they have common vectors. Let $|\Phi^{(\varepsilon)}_{n_1n_2}\rangle$ be the common eigenvectors of $\hat{H}$ and $\hat{P}_e$ with $\varepsilon = \pm 1$ the eigenvalues of $\hat{P}_e$. We then have

$$\begin{aligned}
&\hat{H}\,|\Phi^{(\varepsilon)}_{n_1n_2}\rangle = E_{n_1n_2}\,|\Phi^{(\varepsilon)}_{n_1n_2}\rangle,\\
&\hat{P}_e\,|\Phi^{(\varepsilon)}_{n_1n_2}\rangle = \varepsilon\,|\Phi^{(\varepsilon)}_{n_1n_2}\rangle,
\end{aligned}$$

where $E_{n_1n_2} = \big(n_1 + n_2 + 1\big)\hbar\omega$ and

$$\begin{aligned}
&|\Phi^{(+1)}_{n_2n_1}\rangle = |\Phi^{(+1)}_{n_1n_2}\rangle,\\
&|\Phi^{(-1)}_{n_2n_1}\rangle = -\,|\Phi^{(-1)}_{n_1n_2}\rangle,\ n_1 \neq n_2.
\end{aligned}$$

Note that there exist no common eigenvectors for $n_1 = n_2$ and $\varepsilon = -1$.
For $n_1 + n_2 = 2p + 1$ with $p = 0, 1, 2, \cdots$, one half of the $2(p + 1)$ degenerate eigenvectors of $\hat{H}$ are the eigenvectors of $\hat{P}_e$ with the eigenvalue $+1$ and the other half are the eigenvectors of $\hat{P}_e$ with the eigenvalue -1. Note that, for $n_1 + n_2 = 2p + 1$, $n_1 = n_2$ is impossible. We can construct $|\Phi^{(+1)}_{n_1n_2}\rangle$ and $|\Phi^{(-1)}_{n_1n_2}\rangle$ in the following manner

$$\begin{aligned}
|\Phi^{(+1)}_{n_1n_2}\rangle &= \frac{1}{\sqrt{2}}\big[\,|\Phi_{n_1n_2}\rangle + |\Phi_{n_2n_1}\rangle\,\big],\ n_1 + n_2 = 2p + 1,\\
|\Phi^{(-1)}_{n_1n_2}\rangle &= \frac{1}{\sqrt{2}}\big[\,|\Phi_{n_1n_2}\rangle - |\Phi_{n_2n_1}\rangle\,\big],\ n_1 + n_2 = 2p + 1.
\end{aligned}$$

For $n_1 + n_2 = 2q$ with $q = 0, 1, 2, \cdots$, $n_1 = n_2$ is possible. For $n_1 = n_2 = 0$, E_{00} (the ground-state energy) is non-degenerate. For $q > 1$, there are $q + 1$ eigenvectors of $\hat{P}_e$ with the eigenvalue $+1$ and q eigenvectors of $\hat{P}_e$ with the eigenvalue -1. For $n_1 + n_2 = 2q$, we can construct $|\Phi^{(+1)}_{n_1n_2}\rangle$ and $|\Phi^{(-1)}_{n_1n_2}\rangle$ in the following manner

$$\begin{aligned}
|\Phi^{(+1)}_{qq}\rangle &= |\Phi_{qq}\rangle,\\
|\Phi^{(+1)}_{n_1n_2}\rangle &= \frac{1}{\sqrt{2}}\big[\,|\Phi_{n_1n_2}\rangle + |\Phi_{n_2n_1}\rangle\,\big],\ n_1 + n_2 = 2q,\ n_1 \neq n_2,\\
|\Phi^{(-1)}_{n_1n_2}\rangle &= \frac{1}{\sqrt{2}}\big[\,|\Phi_{n_1n_2}\rangle - |\Phi_{n_2n_1}\rangle\,\big],\ n_1 + n_2 = 2q,\ n_1 \neq n_2.
\end{aligned}$$

From the above-constructed common eigenvectors of $\hat{H}$ and $\hat{P}_e$, we see that $\hat{H}$ and $\hat{P}_e$ do not form a CSCO. The eigenvectors of $\hat{H}$ for which the eigenvalue of $\hat{P}_e$ is -1 are $|\Phi^{(-1)}_{n_1n_2}\rangle$. If only the eigenvectors $|\Phi^{(-1)}_{n_1n_2}\rangle$ are retained, then the degree of degeneracy of $E_{n_1n_2}$ is $p+1$ for $n_1+n_2 = 2p+1$ and is q for $n_1+n_2 = 2q$. Since the case $n_1 + n_2 = 0$ is excluded if only the eigenvectors $|\Phi^{(-1)}_{n_1n_2}\rangle$ are retained, the eigenvalue $\hbar\omega$ of $\hat{H}$ will be no longer in the spectrum. All the other eigenvalues of $\hat{H}$ remain to be in the spectrum.

4. **[C-T Exercise 5-6]** A one-dimensional harmonic oscillator is composed of a particle of mass m, charge q, and potential energy $\hat{V}(\hat{x}) = \dfrac{1}{2}m\omega^2\hat{x}^2$. We assume that the particle is placed in an electric field $\mathscr{E}(t)$ parallel to Ox and time-dependent, so that to $\hat{V}(\hat{x})$ must be added the potential energy $\hat{W}(t) = -q\mathscr{E}(t)\hat{x}$.

(a) Write the Hamiltonian $\hat{H}(t)$ of the particle in terms of the operators $\hat{a}$ and $\hat{a}^\dagger$. Calculate the commutators of $\hat{a}$ and $\hat{a}^\dagger$ with $\hat{H}(t)$.

(b) Let $\alpha(t)$ be the number defined by $\alpha(t) = \langle\psi(t)|\hat{a}|\psi(t)\rangle$, where $|\psi(t)\rangle$ is the normalized state vector of the particle under study. Show from the results of the preceding question that $\alpha(t)$ satisfies the differential equation $\frac{d}{dt}\alpha(t) = -i\omega\alpha(t) + i\lambda(t)$, where $\lambda(t)$ is defined by $\lambda(t) = \frac{q}{\sqrt{2m\hbar\omega}}\mathscr{E}(t)$. Integrate this differential equation. At time t, what are the mean values of the position and momentum of the particle?

(c) The ket $|\varphi(t)\rangle$ is defined by $|\varphi(t)\rangle = \left[\hat{a} - \alpha(t)\right]|\psi(t)\rangle$, where $\alpha(t)$ has the value calculated in (b). Using the results of questions (a) and (b), show that the evolution of $|\varphi(t)\rangle$ is given by $i\hbar\frac{d}{dt}|\varphi(t)\rangle = \left[\hat{H}(t) + \hbar\omega\right]|\varphi(t)\rangle$. How does the norm of $|\varphi(t)\rangle$ vary with time?

(d) Assuming that $|\psi(0)\rangle$ is an eigenvector of $\hat{a}$ with the eigenvalue $\alpha(0)$, show that $|\psi(t)\rangle$ is also an eigenvector of $\hat{a}$, and calculate its eigenvalue. Find at time t the mean value of the unperturbed Hamiltonian $\hat{H}_0 = \hat{H}(t) - \hat{W}(t)$ as a function of $\alpha(0)$. Give the root-mean-square deviations Δx, Δp, and ΔH_0; how do they vary with time?

(e) Assume that at $t = 0$, the oscillator is in the ground state $|\varphi(0)\rangle$. The electric field acts between times 0 and T and then falls to zero. When $t > T$, what is the evolution of the mean values $\langle\hat{x}\rangle(t)$ and $\langle\hat{p}\rangle(t)$? Application: Assume that between 0 and T, the field $\mathscr{E}(t)$ is given by $\mathscr{E}(t) = \mathscr{E}_0\cos(\omega' t)$; discuss the phenomena observed (resonance) in terms of $\Delta\omega = \omega' - \omega$. If, at $t > T$, the energy is measured, what results can be found, and with what probabilities?

(a) The Hamiltonian of the system is given by

$$\hat{H}(t) = \frac{1}{2m}\hat{p}^2 + \hat{V}(\hat{x}) + \hat{W}(t) = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 - q\mathscr{E}(t)\hat{x}.$$

Making use of

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right),$$
$$\hat{p} = -i\sqrt{\frac{m\hbar\omega}{2}}\left(\hat{a} - \hat{a}^\dagger\right),$$

we have

$$\hat{H}(t) = \left(\hat{a}^\dagger\hat{a} + 1/2\right)\hbar\omega - \left(\hat{a} + \hat{a}^\dagger\right)\hbar\lambda(t),$$

where

$$\lambda(t) = \frac{q\mathscr{E}(t)}{\sqrt{2m\hbar\omega}}.$$

The commutator $[\hat{a}, \hat{H}(t)]$ is given by

$$[\hat{a}, \hat{H}(t)] = \hbar\omega[\hat{a}, \hat{a}^\dagger\hat{a}] - \hbar\lambda(t)[\hat{a}, \hat{a}^\dagger] = \hbar\omega\hat{a} - \hbar\lambda(t).$$

The commutator $[\hat{a}^\dagger, \hat{H}(t)]$ is given by

$$[\hat{a}^\dagger, \hat{H}(t)] = \hbar\omega[\hat{a}^\dagger, \hat{a}^\dagger\hat{a}] - \hbar\lambda(t)[\hat{a}^\dagger, \hat{a}] = -\hbar\omega\hat{a}^\dagger + \hbar\lambda(t).$$

(b) Differentiating $\alpha(t) = \langle\psi(t)|\hat{a}|\psi(t)\rangle$ with respect to t, we have

$$\frac{d}{dt}\alpha(t) = \left[\frac{d}{dt}\langle\psi(t)|\right]\hat{a}|\psi(t)\rangle + \langle\psi(t)|\hat{a}\left[\frac{d}{dt}|\psi(t)\rangle\right].$$

Making use of

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle,\ -i\hbar\frac{d}{dt}\langle\psi(t)| = \langle\psi(t)|\hat{H}(t),$$

we have

$$\begin{aligned}
\frac{d}{dt}\alpha(t) &= -\frac{1}{i\hbar}\langle\psi(t)|\hat{H}(t)\hat{a}|\psi(t)\rangle + \frac{1}{i\hbar}\langle\psi(t)|\hat{a}\hat{H}(t)|\psi(t)\rangle \\
&= \frac{1}{i\hbar}\langle\psi(t)|[\hat{a},\hat{H}(t)]|\psi(t)\rangle = \frac{1}{i\hbar}\langle\psi(t)|\big[\,\hbar\omega\hat{a} - \hbar\lambda(t)\,\big]|\psi(t)\rangle \\
&= -i\omega\langle\psi(t)|\hat{a}|\psi(t)\rangle + i\lambda(t)\langle\psi(t)|\psi(t)\rangle \\
&= -i\omega\alpha(t) + i\lambda(t).
\end{aligned}$$

For the purpose of cancelling the first term on the right hand side of the equation for $\alpha(t)$, we set $\alpha(t) = e^{-i\omega t}\beta(t)$. We then have

$$-i\omega e^{-i\omega t}\beta(t) + e^{-i\omega t}\frac{d}{dt}\beta(t) = -i\omega e^{-i\omega t}\beta(t) + i\lambda(t)$$

from which the equation for $\beta(t)$ follows

$$\frac{d}{dt}\beta(t) = ie^{i\omega t}\lambda(t).$$

Integrating the above equation from $t = 0$ to t, we have

$$\beta(t) = \beta(0) + i\int_0^t dt'\ e^{i\omega t'}\lambda(t').$$

$\alpha(t)$ is then given by

$$\alpha(t) = e^{-i\omega t}\beta(t) = \beta(0)e^{-i\omega t} + i\int_0^t dt'\ e^{-i\omega(t-t')}\lambda(t').$$

Setting $t = 0$ in the above equation, we have

$$\beta(0) = \alpha(0).$$

We thus have

$$\alpha(t) = \alpha(0)e^{-i\omega t} + i\int_0^t dt'\ e^{-i\omega(t-t')}\lambda(t').$$

From $\alpha(t) = \langle\psi(t)|\hat{a}|\psi(t)\rangle$, we have

$$\alpha^*(t) = \langle\psi(t)|\hat{a}|\psi(t)\rangle^* = \langle\psi(t)|\hat{a}^\dagger|\psi(t)\rangle.$$

From

$$\begin{aligned}
\hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}\,\big(\hat{a} + \hat{a}^\dagger\big), \\
\hat{p} &= -i\sqrt{\frac{m\hbar\omega}{2}}\,\big(\hat{a} - \hat{a}^\dagger\big),
\end{aligned}$$

we have

$$\begin{aligned}
\langle\hat{x}\rangle(t) &= \sqrt{\frac{\hbar}{2m\omega}}\,\big[\,\alpha(t) + \alpha^*(t)\,\big] \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left[\,\alpha(0)e^{-i\omega t} + \alpha^*(0)e^{i\omega t} + i\int_0^t dt'\ e^{-i\omega(t-t')}\lambda(t') - i\int_0^t dt'\ e^{i\omega(t-t')}\lambda(t')\,\right] \\
&= \sqrt{\frac{2\hbar}{m\omega}}\left\{\mathrm{Re}\left[\alpha(0)e^{-i\omega t}\right] + \int_0^t dt'\ \sin[\omega(t-t')]\lambda(t')\right\}, \\
\langle\hat{p}\rangle(t) &= -i\sqrt{\frac{m\hbar\omega}{2}}\,\big[\,\alpha(t) - \alpha^*(t)\,\big] \\
&= -i\sqrt{\frac{m\hbar\omega}{2}}\left[\,\alpha(0)e^{-i\omega t} - \alpha^*(0)e^{i\omega t} + i\int_0^t dt'\ e^{-i\omega(t-t')}\lambda(t') + i\int_0^t dt'\ e^{i\omega(t-t')}\lambda(t')\,\right] \\
&= \sqrt{2m\hbar\omega}\left\{\mathrm{Im}\left[\alpha(0)e^{-i\omega t}\right] + \int_0^t dt'\ \cos[\omega(t-t')]\lambda(t')\right\}.
\end{aligned}$$

(c) Differentiating $|\varphi(t)\rangle$ with respect to t yields

$$\begin{aligned}\frac{d}{dt}|\varphi(t)\rangle &= \big[\,\hat{a}-\alpha(t)\,\big]\frac{d}{dt}|\psi(t)\rangle - \frac{d\alpha(t)}{dt}|\psi(t)\rangle \\ &= \frac{1}{i\hbar}\big[\,\hat{a}-\alpha(t)\,\big]\hat{H}(t)\,|\psi(t)\rangle - \big[-i\omega\alpha(t)+i\lambda(t)\big]\,|\psi(t)\rangle\,.\end{aligned}$$

Multiplying $i\hbar$ on both sides, we have

$$i\hbar\frac{d}{dt}|\varphi(t)\rangle = \big[\,\hat{a}-\alpha(t)\,\big]\hat{H}(t)\,|\psi(t)\rangle - \hbar\big[\,\omega\alpha(t)-\lambda(t)\,\big]\,|\psi(t)\rangle\,.$$

Making use of

$$[\hat{a},\hat{H}(t)] = \hbar\omega\hat{a} - \hbar\lambda(t),$$

we have

$$\begin{aligned}i\hbar\frac{d}{dt}|\varphi(t)\rangle &= \hat{H}(t)\big[\,\hat{a}-\alpha(t)\,\big]\,|\psi(t)\rangle + \big[\hbar\omega\hat{a}-\hbar\lambda(t)\big]\,|\psi(t)\rangle - \hbar\big[\,\omega\alpha(t)-\lambda(t)\,\big]\,|\psi(t)\rangle \\ &= \hat{H}(t)\big[\,\hat{a}-\alpha(t)\,\big]\,|\psi(t)\rangle + \hbar\omega\big[\,\hat{a}-\alpha(t)\,\big]\,|\psi(t)\rangle \\ &= \big[\,\hat{H}(t)+\hbar\omega\,\big]\big[\,\hat{a}-\alpha(t)\,\big]\,|\psi(t)\rangle \\ &= \big[\,\hat{H}(t)+\hbar\omega\,\big]\,|\varphi(t)\rangle\,.\end{aligned}$$

Taking the Hermitian conjugation of the above equation, we have

$$-i\hbar\frac{d}{dt}\langle\varphi(t)| = \langle\varphi(t)|\big[\,\hat{H}(t)+\hbar\omega\,\big].$$

We hence have

$$\begin{aligned}\frac{d}{dt}\langle\varphi(t)|\varphi(t)\rangle &= \left[\frac{d}{dt}\langle\varphi(t)|\right]|\varphi(t)\rangle + \langle\varphi(t)|\frac{d}{dt}|\varphi(t)\rangle \\ &= -\frac{1}{i\hbar}\langle\varphi(t)|\big[\,\hat{H}(t)+\hbar\omega\,\big]|\varphi(t)\rangle + \frac{1}{i\hbar}\langle\varphi(t)|\big[\,\hat{H}(t)+\hbar\omega\,\big]|\varphi(t)\rangle \\ &= 0.\end{aligned}$$

Therefore, the norm of $|\varphi(t)\rangle$ does not vary with time.

(d) Since $|\psi(0)\rangle$ is an eigenvector of $\hat{a}$ with the eigenvalue $\alpha(0)$, we have

$$\big(\hat{a}-\alpha(0)\big)\,|\psi(0)\rangle = 0.$$

Thus,

$$|\varphi(0)\rangle = \big[\,\hat{a}-\alpha(0)\,\big]\,|\psi(0)\rangle = 0.$$

Since the norm of $|\varphi(t)\rangle$ does not vary with time, we have $|\varphi(t)\rangle = \big[\,\hat{a}-\alpha(t)\,\big]\,|\psi(t)\rangle = 0$ from which it follows that

$$\hat{a}\,|\psi(t)\rangle = \alpha(t)\,|\psi(t)\rangle\,.$$

Therefore, $|\psi(t)\rangle$ is also an eigenvector of $\hat{a}$ with the eigenvalue given by

$$\alpha(t) = \alpha(0)e^{-i\omega t} + i\int_0^t dt'\; e^{-i\omega(t-t')}\lambda(t').$$

Taking the Hermitian conjugation of $\hat{a}\,|\psi(t)\rangle = \alpha(t)\,|\psi(t)\rangle$, we have

$$\langle\psi(t)|\,\hat{a}^\dagger = \alpha^*(t)\,\langle\psi(t)|\,.$$

The mean value of $\hat{H}_0$ in the state $|\psi(t)\rangle$ is given by

$$\begin{aligned}\langle \hat{H}_0\rangle &= \langle\psi(t)|\hat{H}_0|\psi(t)\rangle = \langle\psi(t)|(\hat{a}^\dagger\hat{a}+1/2)\hbar\omega|\psi(t)\rangle \\ &= \big(|\alpha(t)|^2+1/2\big)\hbar\omega.\end{aligned}$$

The mean value of $\hat{x}^2$ is given by

$$\begin{aligned}\langle\hat{x}^2\rangle(t) &= \frac{\hbar}{2m\omega}\,\langle(\hat{a}+\hat{a}^\dagger)^2\rangle = \frac{\hbar}{2m\omega}\,\langle\hat{a}^2+(\hat{a}^\dagger)^2+2\hat{a}^\dagger\hat{a}+1\rangle \\ &= \frac{\hbar}{2m\omega}\big\{[\alpha(t)]^2+[\alpha^*(t)]^2+2|\alpha(t)|^2+1\big\}.\end{aligned}$$

$\langle\hat{x}^2\rangle(t)-[\langle\hat{x}\rangle(t)]^2$ is given by

$$\langle\hat{x}^2\rangle(t)-[\langle\hat{x}\rangle(t)]^2 = \frac{\hbar}{2m\omega}\big\{[\alpha(t)]^2+[\alpha^*(t)]^2+2|\alpha(t)|^2+1\big\} - \frac{\hbar}{2m\omega}\big[\,\alpha(t)+\alpha^*(t)\,\big]^2 = \frac{\hbar}{2m\omega}.$$

Δx is then given by

$$\Delta x = \sqrt{\langle\hat{x}^2\rangle(t)-[\langle\hat{x}\rangle(t)]^2} = \sqrt{\frac{\hbar}{2m\omega}}.$$

We see that Δx is independent of t.
For $\langle\hat{p}^2\rangle(t)$, we have

$$\langle\hat{p}^2\rangle(t) = -\frac{m\hbar\omega}{2}\,\langle(\hat{a}-\hat{a}^\dagger)^2\rangle = -\frac{m\hbar\omega}{2}\big\{[\alpha(t)]^2+[\alpha^*(t)]^2-2|\alpha(t)|^2-1\big\}.$$

$\langle\hat{p}^2\rangle(t)-[\langle\hat{p}\rangle(t)]^2$ is given by

$$\langle\hat{p}^2\rangle(t)-[\langle\hat{p}\rangle(t)]^2 = -\frac{m\hbar\omega}{2}\big\{[\alpha(t)]^2+[\alpha^*(t)]^2-2|\alpha(t)|^2-1\big\} + \frac{m\hbar\omega}{2}\big[\,\alpha(t)-\alpha^*(t)\,\big]^2 = \frac{m\hbar\omega}{2}.$$

Δp is then given by

$$\Delta p = \sqrt{\langle\hat{p}^2\rangle(t)-[\langle\hat{p}\rangle(t)]^2} = \sqrt{\frac{m\hbar\omega}{2}}.$$

We see that Δp is also independent of t.
For $\langle\hat{H}_0^2\rangle(t)$, we have

$$\begin{aligned}\langle\hat{H}_0^2\rangle(t) &= \langle(\hat{a}^\dagger\hat{a}+1/2)^2\rangle(\hbar\omega)^2 = \langle(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}+\hat{a}^\dagger\hat{a}+1/4)\rangle \\ &= \langle(\hat{a}^\dagger)^2\hat{a}^2+2\hat{a}^\dagger\hat{a}+1/4)\rangle(\hbar\omega)^2 = \big[\,|\alpha(t)|^4+2|\alpha(t)|^2+1/4\,\big](\hbar\omega)^2\end{aligned}$$

$\langle\hat{H}_0^2\rangle(t)-[\langle\hat{H}_0\rangle(t)]^2$ is given by

$$\langle\hat{H}_0^2\rangle(t)-[\langle\hat{H}_0\rangle(t)]^2 = \big[\,|\alpha(t)|^4+2|\alpha(t)|^2+1/4\,\big](\hbar\omega)^2 - \big(|\alpha(t)|^2+1/2\big)^2(\hbar\omega)^2 = |\alpha(t)|^2(\hbar\omega)^2.$$

ΔH_0 is then given by

$$\Delta H_0 = \sqrt{\langle\hat{H}_0^2\rangle(t)-[\langle\hat{H}_0\rangle(t)]^2} = |\alpha(t)|\hbar\omega.$$

We see that ΔH_0 depends on t.

(e) Since the oscillator is in the ground state $|\varphi(0)\rangle$, we have $\alpha(0)=\langle\varphi(0)|\hat{a}|\varphi(0)\rangle=0$. In consideration that the electric field acts between times 0 and T, $\alpha(t)=\langle\psi(t)|\hat{a}|\psi(t)\rangle$ for $t>T$ is given by

$$\alpha(t) = i\int_0^T dt'\; e^{-i\omega(t-t')}\lambda(t'),\; t>T.$$

From the results for $\langle \hat{x} \rangle(t)$ and $\langle \hat{p} \rangle(t)$ in (b), we have

$$\langle \hat{x} \rangle(t) = \sqrt{\frac{2\hbar}{m\omega}} \int_0^T dt' \; \sin[\omega(t-t')]\lambda(t'),$$
$$\langle \hat{p} \rangle(t) = \sqrt{2m\hbar\omega} \int_0^T dt' \; \cos[\omega(t-t')]\lambda(t').$$

For $\mathscr{E}(t) = \mathscr{E}_0 \cos(\omega' t)$, we have

$$\lambda(t) = \frac{q\mathscr{E}_0}{\sqrt{2m\hbar\omega}} \cos(\omega' t)\theta(t)\theta(T-t).$$

Inserting the above expression of $\lambda(t)$ into the expressions of $\langle \hat{x} \rangle(t)$ and $\langle \hat{p} \rangle(t)$, we can perform the integrals in $\langle \hat{x} \rangle(t)$ and $\langle \hat{p} \rangle(t)$. For $\langle \hat{x} \rangle(t)$, we have

$$\begin{aligned}
\langle \hat{x} \rangle(t) &= \sqrt{\frac{2\hbar}{m\omega}} \frac{q\mathscr{E}_0}{\sqrt{2m\hbar\omega}} \int_0^T dt' \; \sin[\omega(t-t')] \cos(\omega' t') \\
&= \frac{1}{2} \frac{q\mathscr{E}_0}{m\omega} \int_0^T dt' \; \Big\{ \sin\big[(\omega'-\omega)t' + \omega t\big] - \sin\big[(\omega'+\omega)t' - \omega t\big] \Big\} \\
&= \frac{1}{2} \frac{q\mathscr{E}_0}{m\omega} \bigg\{ \frac{\cos(\omega t) - \cos\big[(\omega'-\omega)T + \omega t\big]}{\omega'-\omega} - \frac{\cos(\omega t) - \cos\big[(\omega'+\omega)T - \omega t\big]}{\omega'+\omega} \bigg\}.
\end{aligned}$$

The plot of $\langle \hat{x} \rangle(t)$ as a function of $\Delta\omega T$ for $t > T$ is given in Fig. 1. From Fig. 1, we can see that $\langle \hat{x} \rangle(t)$ takes on peak values at certain values of $\Delta\omega = \omega' - \omega$. This is a resonance phenomenon. Note that the shape of the curve depends on the values of ω, T, and t.

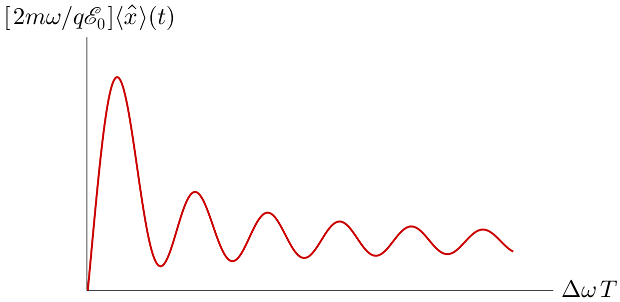

FIG. 1: Plot of $\langle \hat{x} \rangle(t)$ as a function of $\Delta\omega T$ for $t > T$.

For $\langle \hat{p} \rangle(t)$, we have

$$\begin{aligned}
\langle \hat{p} \rangle(t) &= \sqrt{2m\hbar\omega} \frac{q\mathscr{E}_0}{\sqrt{2m\hbar\omega}} \int_0^T dt' \; \cos[\omega(t-t')] \cos(\omega' t') \\
&= \frac{1}{2} q\mathscr{E}_0 \int_0^T dt' \; \Big\{ \cos\big[(\omega'-\omega)t' + \omega t\big] + \cos\big[(\omega'+\omega)t' - \omega t\big] \Big\} \\
&= \frac{1}{2} q\mathscr{E}_0 \bigg\{ \frac{\sin\big[(\omega'-\omega)T + \omega t\big] - \sin(\omega t)}{\omega'-\omega} + \frac{\sin\big[(\omega'+\omega)T - \omega t\big] + \sin(\omega t)}{\omega'+\omega} \bigg\}.
\end{aligned}$$

The plot of $\langle \hat{p} \rangle(t)$ as a function of $\Delta\omega T$ for $t > T$ is given in Fig. 2. From Fig. 2, we can see that $\langle \hat{p} \rangle(t)$ takes on peak values at certain values of $\Delta\omega = \omega' - \omega$. This is a resonance phenomenon. Note that the shape of the curve depends on the values of ω, T, and t.
At $t > T$, the Hamiltonian of the oscillator is $\hat{H}_0$. Thus, if the energy of oscillator is measured at $t > T$, the results obtained will be the eigenvalues of $\hat{H}_0$, $E_n = (n + 1/2)\hbar\omega$ with $n = 0, 1, 2, \cdots$. The probability of obtaining the result $E_n = (n + 1/2)\hbar\omega$ is given by

$$\mathscr{P}_{\hat{H}_0}(E_n) = \big|\langle \varphi_n | \psi(t) \rangle\big|^2, \; t > T.$$

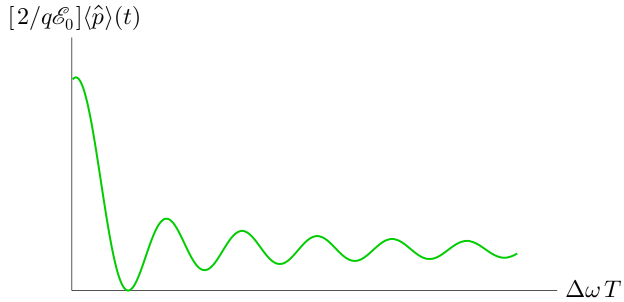

FIG. 2: Plot of $\langle\hat{p}\rangle(t)$ as a function of $\Delta\omega T$ for $t > T$.

5. **[C-T Exercise 6-6]** Consider a system of angular momentum $\ell = 1$. A basis of its state space is formed by the three eigenvectors of $\hat{L}_z$: $|+1\rangle$, $|0\rangle$, $|-1\rangle$, whose eigenvalues are, respectively, $+\hbar$, 0, and $-\hbar$, and which satisfy $\hat{L}_\pm\,|m\rangle = \hbar\sqrt{2}\,|m \pm 1\rangle$, $\hat{L}_+\,|+1\rangle = \hat{L}_-\,|-1\rangle = 0$. This system, which possesses an electric quadrupole moment, is placed in an electric field gradient, so that its Hamiltonian can be written $\hat{H} = \dfrac{\omega_0}{\hbar}\big(\hat{L}_u^2 - \hat{L}_v^2\big)$, where $\hat{L}_u$ and $\hat{L}_v$ are the components of $\hat{\vec{L}}$ along the two directions Ou and Ov of the xOz plane which form angles of 45° with Ox and Oz; ω_0 is a real constant.

   (a) Write the matrix which represents $\hat{H}$ in the $\{|+1\rangle, |0\rangle, |-1\rangle\}$ basis. What are the stationary states of the system, and what are their energies? (These states are to be written $|E_1\rangle$, $|E_2\rangle$, $|E_3\rangle$, in order of decreasing energies.)

   (b) At time $t = 0$, the system is in the state $|\psi(0)\rangle = \dfrac{1}{\sqrt{2}}\big[\,|+1\rangle - |-1\rangle\,\big]$. What is the state vector $|\psi(t)\rangle$ at time t? At t, $\hat{L}_z$ is measured; what are the probabilities of the various possible results?

   (c) Calculate the mean values $\langle\hat{L}_x\rangle(t)$, $\langle\hat{L}_y\rangle(t)$, and $\langle\hat{L}_z\rangle(t)$ at t. What is the motion performed by the vector $\langle\hat{\vec{L}}\rangle$?

   (d) At t, a measurement of $\hat{L}_z^2$ is performed.

      i. Do times exist when only one result is possible?

      ii. Assume that this measurement has yielded the result $\hbar^2$. What is the state of the system immediately after the measurement? Indicate, without calculation, its subsequent evolution.

(a) In terms of $\hat{L}_x$ and $\hat{L}_z$, $\hat{L}_u$ and $\hat{L}_v$ are respectively given by

$$\hat{L}_u = \frac{1}{\sqrt{2}}\big(\hat{L}_z + \hat{L}_x\big),$$
$$\hat{L}_v = \frac{1}{\sqrt{2}}\big(\hat{L}_z - \hat{L}_x\big).$$

Making use of $\hat{L}_x = (\hat{L}_+ + \hat{L}_-)/2$, we have

$$\hat{L}_u = \frac{1}{2\sqrt{2}}\big(2\hat{L}_z + \hat{L}_+ + \hat{L}_-\big),$$
$$\hat{L}_v = \frac{1}{2\sqrt{2}}\big(2\hat{L}_z - \hat{L}_+ - \hat{L}_-\big).$$

The Hamiltonian is then given by

$$\begin{aligned}\hat{H} &= \frac{\omega_0}{8\hbar}\Big[\big(2\hat{L}_z + \hat{L}_+ + \hat{L}_-\big)^2 - \big(2\hat{L}_z - \hat{L}_+ - \hat{L}_-\big)^2\Big] \\ &= \frac{\omega_0}{2\hbar}\Big[2\hat{L}_+\hat{L}_z + 2\hat{L}_-\hat{L}_z + \hbar\big(\hat{L}_+ - \hat{L}_-\big)\Big]\end{aligned}$$

Acting $\hat{H}$ respectively on $|+1\rangle$, $|0\rangle$, and $|-1\rangle$, we have

$$\hat{H}\,|+1\rangle = \frac{\hbar\omega_0}{\sqrt{2}}\,|0\rangle,\ \hat{H}\,|0\rangle = \frac{\hbar\omega_0}{\sqrt{2}}\big(|+1\rangle - |-1\rangle\big),\ \hat{H}\,|-1\rangle = -\frac{\hbar\omega_0}{\sqrt{2}}\,|0\rangle\,.$$

We then have the following matrix elements of $\hat{H}$

$$\begin{aligned}
&\langle +1|\hat{H}|+1\rangle = 0, && \langle +1|\hat{H}|0\rangle = \frac{\hbar\omega_0}{\sqrt{2}}, && \langle +1|\hat{H}|-1\rangle = 0,\\
&\langle 0|\hat{H}|+1\rangle = \frac{\hbar\omega_0}{\sqrt{2}}, && \langle 0|\hat{H}|0\rangle = 0, && \langle 0|\hat{H}|-1\rangle = -\frac{\hbar\omega_0}{\sqrt{2}},\\
&\langle -1|\hat{H}|+1\rangle = 0, && \langle -1|\hat{H}|0\rangle = -\frac{\hbar\omega_0}{\sqrt{2}}, && \langle -1|\hat{H}|-1\rangle = 0.
\end{aligned}$$

The representation matrix of $\hat{H}$ in the $\{|+1\rangle, |0\rangle, |-1\rangle\}$ basis is then given by

$$H = \frac{\hbar\omega_0}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & -1\\ 0 & -1 & 0\end{pmatrix}.$$

Let $|\varphi\rangle = a\,|+1\rangle + b\,|0\rangle + c\,|-1\rangle$ be the eigenvector of $\hat{H}$ corresponding to the eigenvalue E. The eigenvalue equation of $\hat{H}$ in the $\{|+1\rangle, |0\rangle, |-1\rangle\}$ basis reads

$$\frac{\hbar\omega_0}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & -1\\ 0 & -1 & 0\end{pmatrix}\begin{pmatrix} a\\ b\\ c\end{pmatrix} = E\begin{pmatrix} a\\ b\\ c\end{pmatrix}$$

That is,

$$\begin{aligned}
&-Ea + \frac{\hbar\omega_0}{\sqrt{2}}b = 0,\\
&\frac{\hbar\omega_0}{\sqrt{2}}a - Eb - \frac{\hbar\omega_0}{\sqrt{2}}c = 0,\\
&-\frac{\hbar\omega_0}{\sqrt{2}}b - Ec = 0.
\end{aligned}$$

The secular equation is given by

$$\begin{vmatrix} -E & \frac{\hbar\omega_0}{\sqrt{2}} & 0\\ \frac{\hbar\omega_0}{\sqrt{2}} & -E & -\frac{\hbar\omega_0}{\sqrt{2}}\\ 0 & -\frac{\hbar\omega_0}{\sqrt{2}} & -E\end{vmatrix} = 0$$

from which it follows that

$$E^3 - (\hbar\omega_0)^2 E = 0.$$

Thus, the eigenvalues of $\hat{H}$ are

$$E_1 = \hbar\omega_0,\ E_2 = 0,\ E_3 = -\hbar\omega_0.$$

To find the eigenvector of $\hat{H}$ corresponding to the eigenvalue $E_1 = \hbar\omega_0$, we insert $E_1 = \hbar\omega_0$ into the above-obtained equations for a, b, and c. We have

$$\begin{aligned}
&-\hbar\omega_0 a + \frac{\hbar\omega_0}{\sqrt{2}}b = 0,\\
&\frac{\hbar\omega_0}{\sqrt{2}}a - \hbar\omega_0 b - \frac{\hbar\omega_0}{\sqrt{2}}c = 0,\\
&-\frac{\hbar\omega_0}{\sqrt{2}}b - \hbar\omega_0 c = 0.
\end{aligned}$$

We thus have $b = \sqrt{2}\,a$ and $c = -a$. From the normalization condition, we have $|a| = 1/2$. We choose $a = 1/2$. We then have $b = 1/\sqrt{2}$ and $c = -1/2$. The eigenvector $|\varphi_1\rangle$ of $\hat{H}$ corresponding to the eigenvalue $E_1 = \hbar\omega_0$ is given by

$$|\varphi_1\rangle = \frac{1}{2}\big[\,|+1\rangle + \sqrt{2}\,|0\rangle - |-1\rangle\,\big].$$

To find the eigenvector of $\hat{H}$ corresponding to the eigenvalue $E_2 = 0$, we insert $E_2 = 0$ into the above-obtained equations for a, b, and c. We have

$$\begin{aligned}
&\frac{\hbar\omega_0}{\sqrt{2}} b = 0,\\
&\frac{\hbar\omega_0}{\sqrt{2}} a - \frac{\hbar\omega_0}{\sqrt{2}} c = 0,\\
&-\frac{\hbar\omega_0}{\sqrt{2}} b = 0.
\end{aligned}$$

We thus have $b = 0$ and $c = a$. From the normalization condition, we have $|a| = 1/\sqrt{2}$. We choose $a = 1/\sqrt{2}$. We then have $c = 1/\sqrt{2}$. The eigenvector $|\varphi_2\rangle$ of $\hat{H}$ corresponding to the eigenvalue $E_2 = 0$ is given by

$$|\varphi_2\rangle = \frac{1}{\sqrt{2}}\big[\,|+1\rangle + |-1\rangle\,\big].$$

To find the eigenvector of $\hat{H}$ corresponding to the eigenvalue $E_3 = -\hbar\omega_0$, we insert $E_3 = -\hbar\omega_0$ into the above-obtained equations for a, b, and c. We have

$$\begin{aligned}
&\hbar\omega_0 a + \frac{\hbar\omega_0}{\sqrt{2}} b = 0,\\
&\frac{\hbar\omega_0}{\sqrt{2}} a + \hbar\omega_0 b - \frac{\hbar\omega_0}{\sqrt{2}} c = 0,\\
&-\frac{\hbar\omega_0}{\sqrt{2}} b + \hbar\omega_0 c = 0.
\end{aligned}$$

We thus have $b = -\sqrt{2}\,a$ and $c = -a$. From the normalization condition, we have $|a| = 1/2$. We choose $a = 1/2$. We then have $b = -1/\sqrt{2}$ and $c = -1/2$. The eigenvector $|\varphi_3\rangle$ of $\hat{H}$ corresponding to the eigenvalue $E_3 = -\hbar\omega_0$ is given by

$$|\varphi_3\rangle = \frac{1}{2}\big[\,|+1\rangle - \sqrt{2}\,|0\rangle - |-1\rangle\,\big].$$

Expressing $|+1\rangle$, $|0\rangle$, and $|-1\rangle$ in terms of $|\varphi_1\rangle$, $|\varphi_2\rangle$, and $|\varphi_3\rangle$, we have

$$\begin{aligned}
|+1\rangle &= \frac{1}{2}\big[\,|\varphi_1\rangle + \sqrt{2}\,|\varphi_2\rangle + |\varphi_3\rangle\,\big],\\
|0\rangle &= \frac{1}{\sqrt{2}}\big[\,|\varphi_1\rangle - |\varphi_3\rangle\,\big],\\
|-1\rangle &= \frac{1}{2}\big[-|\varphi_1\rangle + \sqrt{2}\,|\varphi_2\rangle - |\varphi_3\rangle\,\big].
\end{aligned}$$

(b) To find the state vector $|\psi(t)\rangle$ at time t, we first reexpress $|\psi(0)\rangle$ in terms of the energy eigenvectors. We have

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\big[\,|\varphi_1\rangle + |\varphi_3\rangle\,\big]$$

The state vector $|\psi(t)\rangle$ at time t is then given by

$$\begin{aligned}
|\psi(t)\rangle &= \frac{1}{\sqrt{2}}\big[\,(e^{-iE_1 t/\hbar}\,|\varphi_1\rangle + e^{-iE_3 t/\hbar}\,|\varphi_3\rangle\,\big]\\
&= \frac{1}{\sqrt{2}}\big[\,e^{-i\omega_0 t}\,|\varphi_1\rangle + e^{i\omega_0 t}\,|\varphi_3\rangle\,\big].
\end{aligned}$$

Expressing $|\psi(t)\rangle$ in terms of $|+1\rangle$, $|0\rangle$, and $|-1\rangle$, we have

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\big[\cos(\omega_0 t)\,|+1\rangle - i\sqrt{2}\sin(\omega_0 t)\,|0\rangle - \cos(\omega_0 t)\,|-1\rangle\big].$$

From the above expression of $|\psi(t)\rangle$, we see that, if $\hat{L}_z$ is measured at t, the possible results are $+\hbar$, 0, and $-\hbar$. The probabilities of obtaining these three results are respectively given by

$$\begin{aligned}
\mathscr{P}_{\hat{L}_z}(+\hbar) &= \big|\langle +1|\psi(t)\rangle\big|^2 = \frac{1}{2}\cos^2(\omega_0 t),\\
\mathscr{P}_{\hat{L}_z}(0) &= \big|\langle 0|\psi(t)\rangle\big|^2 = \sin^2(\omega_0 t),\\
\mathscr{P}_{\hat{L}_z}(-\hbar) &= \big|\langle -1|\psi(t)\rangle\big|^2 = \frac{1}{2}\cos^2(\omega_0 t).
\end{aligned}$$

(c) Making use of $\hat{L}_x = (\hat{L}_+ + \hat{L}_-)/2$, we have

$$\begin{aligned}
\hat{L}_x\,|\psi(t)\rangle &= \frac{1}{2\sqrt{2}}(\hat{L}_+ + \hat{L}_-)\big[\cos(\omega_0 t)\,|+1\rangle - i\sqrt{2}\sin(\omega_0 t)\,|0\rangle - \cos(\omega_0 t)\,|-1\rangle\big]\\
&= \frac{\hbar}{2}\big[-i\sqrt{2}\sin(\omega_0 t)\,|+1\rangle - \cos(\omega_0 t)\,|0\rangle + \cos(\omega_0 t)\,|0\rangle - i\sqrt{2}\sin(\omega_0 t)\,|-1\rangle\big]\\
&= -i\frac{\hbar\sin(\omega_0 t)}{\sqrt{2}}\big[\,|+1\rangle + |-1\rangle\big].
\end{aligned}$$

For $\langle\hat{L}_x\rangle(t)$, we have

$$\begin{aligned}
\langle\hat{L}_x\rangle(t) &= -i\frac{\hbar\sin(\omega_0 t)}{2}\big[\cos(\omega_0 t)\,\langle +1| + i\sqrt{2}\sin(\omega_0 t)\,\langle 0| - \cos(\omega_0 t)\,\langle -1|\big]\big[\,|+1\rangle + |-1\rangle\big]\\
&= -i\frac{\hbar\sin(\omega_0 t)}{2}\big[\cos(\omega_0 t) - \cos(\omega_0 t)\big]\\
&= 0.
\end{aligned}$$

Making use of $\hat{L}_y = (\hat{L}_+ - \hat{L}_-)/2i$, we have

$$\begin{aligned}
\hat{L}_y\,|\psi(t)\rangle &= \frac{1}{2i\sqrt{2}}(\hat{L}_+ - \hat{L}_-)\big[\cos(\omega_0 t)\,|+1\rangle - i\sqrt{2}\sin(\omega_0 t)\,|0\rangle - \cos(\omega_0 t)\,|-1\rangle\big]\\
&= \frac{\hbar}{2i}\big[-i\sqrt{2}\sin(\omega_0 t)\,|+1\rangle - \cos(\omega_0 t)\,|0\rangle - \cos(\omega_0 t)\,|0\rangle + i\sqrt{2}\sin(\omega_0 t)\,|-1\rangle\big]\\
&= -\frac{\hbar}{\sqrt{2}}\sin(\omega_0 t)\,|+1\rangle + i\hbar\cos(\omega_0 t)\,|0\rangle + \frac{\hbar}{\sqrt{2}}\sin(\omega_0 t)\,|-1\rangle\,.
\end{aligned}$$

For $\langle\hat{L}_y\rangle(t)$, we have

$$\begin{aligned}
\langle\hat{L}_y\rangle(t) &= \frac{1}{\sqrt{2}}\big[\cos(\omega_0 t)\,\langle +1| + i\sqrt{2}\sin(\omega_0 t)\,\langle 0| - \cos(\omega_0 t)\,\langle -1|\big]\\
&\qquad\times\left[-\frac{\hbar}{\sqrt{2}}\sin(\omega_0 t)\,|+1\rangle + i\hbar\cos(\omega_0 t)\,|0\rangle + \frac{\hbar}{\sqrt{2}}\sin(\omega_0 t)\,|-1\rangle\right]\\
&= -\frac{\hbar}{2}\sin(\omega_0 t)\cos(\omega_0 t) - \hbar\sin(\omega_0 t)\cos(\omega_0 t) - \frac{\hbar}{2}\sin(\omega_0 t)\cos(\omega_0 t)\\
&= -\hbar\sin(2\omega_0 t).
\end{aligned}$$

Acting $\hat{L}_z$ on $|\psi(t)\rangle$, we have

$$\begin{aligned}
\hat{L}_z\,|\psi(t)\rangle &= \frac{1}{\sqrt{2}}\big[\cos(\omega_0 t)\hat{L}_z\,|+1\rangle - i\sqrt{2}\sin(\omega_0 t)\hat{L}_z\,|0\rangle - \cos(\omega_0 t)\hat{L}_z\,|-1\rangle\big]\\
&= \frac{\hbar}{\sqrt{2}}\big[\cos(\omega_0 t)\,|+1\rangle + \cos(\omega_0 t)\,|-1\rangle\big].
\end{aligned}$$

$\langle \hat{L}_z \rangle(t)$ is then given by

$$\begin{aligned}
\langle \hat{L}_z \rangle(t) &= \frac{\hbar}{2}\big[\cos(\omega_0 t)\,\langle +1| + i\sqrt{2}\sin(\omega_0 t)\,\langle 0| - \cos(\omega_0 t)\,\langle -1|\,\big]\big[\cos(\omega_0 t)\,| +1\rangle + \cos(\omega_0 t)\,| -1\rangle\,\big] \\
&= \frac{\hbar}{2}\big[\cos^2(\omega_0 t) - \cos^2(\omega_0 t)\,\big] \\
&= 0.
\end{aligned}$$

From the above results for $\langle \hat{L}_x \rangle(t)$, $\langle \hat{L}_y \rangle(t)$, and $\langle \hat{L}_z \rangle(t)$, we see that the vector $\langle \hat{\vec{L}} \rangle$ oscillates along the y-axis with an angular frequency of $2\omega_0$.

(d) If $\hat{L}_z^2$ is measured at t, from the expression of $|\psi(t)\rangle$ we see that the possible results are $\hbar^2$ and 0. The probabilities of obtaining these two results are respectively given by

$$\begin{aligned}
\mathscr{P}_{\hat{L}_z^2}(\hbar^2) &= \big|\langle +1|\psi(t)\rangle\big|^2 + \big|\langle -1|\psi(t)\rangle\big|^2 = \frac{1}{2}\cos^2(\omega_0 t) + \frac{1}{2}\cos^2(\omega_0 t) = \cos^2(\omega_0 t), \\
\mathscr{P}_{\hat{L}_z^2}(0) &= \big|\langle 0|\psi(t)\rangle\big|^2 = \sin^2(\omega_0 t).
\end{aligned}$$

i. When $\cos(\omega_0 t) = 1$, only the result $\hbar^2$ is possible; when $\sin(\omega_0 t) = 1$, only the result 0 is possible. Thus, times do exist when only one result is possible.

ii. The state of the system immediately after the measurement $\hat{L}_z^2$ with the result $\hbar^2$ obtained is

$$|\psi'\rangle = \frac{1}{\sqrt{2}}\big[\,| +1\rangle - | -1\rangle\,\big].$$

Since $|\psi'\rangle$ is the same as the initial state $|\psi(0)\rangle$, the evolution of $|\psi'\rangle$ is the same as that of $|\psi(0)\rangle$ except that the initial time is now the time when the measurement was performed.