Quantum Mechanics

Solutions to the Problems in Homework Assignment 13

Fall, 2019

1. **[C-T Exercise 9-1]** Consider a spin 1/2 particle. Call its spin $\hat{\vec{S}}$, its orbital angular momentum $\hat{\vec{L}}$, and its state vector $|\psi\rangle$. The two functions $\psi_+(\vec{r})$ and $\psi_-(\vec{r})$ are defined by $\psi_\pm(\vec{r}) = \langle\vec{r},\pm|\psi\rangle$. Assume that

$$\psi_+(\vec{r}) = R(r)\left[Y_{00}(\theta,\phi) + \frac{1}{\sqrt{3}}Y_{10}(\theta,\phi)\right],$$
$$\psi_-(\vec{r}) = \frac{R(r)}{\sqrt{3}}\left[Y_{11}(\theta,\phi) - Y_{10}(\theta,\phi)\right],$$

where r, θ, ϕ are the coordinates of the particle and $R(r)$ is a given function of r.

(a) What condition must $R(r)$ satisfy for $|\psi\rangle$ to be normalized?

(b) $\hat{S}_z$ is measured with the particle in the state $|\psi\rangle$. What results can be found, and with what probabilities? Same question for $\hat{L}_z$, then for $\hat{S}_x$.

(c) A measurement of $\hat{\vec{L}}^2$, with the particle in the state $|\psi\rangle$, yielded zero. What state describes the particle just after this measurement? Same question if the measurement of $\hat{\vec{L}}^2$ had given $2\hbar^2$.

(a) From the normalization condition $\langle\psi|\psi\rangle = 1$, we have

$$\begin{aligned}
1 = \langle\psi|\psi\rangle &= \sum_{\varepsilon=\pm}\int d^3r\ \langle\psi|\vec{r},\varepsilon\rangle\,\langle\vec{r},\varepsilon|\psi\rangle = \sum_{\varepsilon=\pm}\int d^3r\ \big|\langle\vec{r},\varepsilon|\psi\rangle\big|^2\\
&= \int d^3r\ \Big[\big|\langle\vec{r},+|\psi\rangle\big|^2 + \big|\langle\vec{r},-|\psi\rangle\big|^2\Big]\\
&= \int_0^\infty dr\ r^2\big|R(r)\big|^2\left(1+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)\\
&= 2\int_0^\infty dr\ r^2\big|R(r)\big|^2.
\end{aligned}$$

Thus, the condition $R(r)$ must satisfy for $|\psi\rangle$ to be normalized is given by

$$\int_0^\infty dr\ r^2\big|R(r)\big|^2 = \frac{1}{2}.$$

(b) The eigenvalues of $\hat{S}_z$ are $\pm\hbar/2$. Thus, if $\hat{S}_z$ is measured, the results that can be found are $+\hbar/2$ and $-\hbar/2$. The probabilities of finding these results are respectively given by

$$\begin{aligned}
\mathscr{P}_{\hat{S}_z}(+\hbar/2) &= \int d^3r\ \big|\langle\vec{r},+|\psi\rangle\big|^2 = \int_0^\infty dr\ r^2\big|R(r)\big|^2\left(1+\frac{1}{3}\right) = \frac{4}{3}\int_0^\infty dr\ r^2\big|R(r)\big|^2 = \frac{2}{3},\\
\mathscr{P}_{\hat{S}_z}(-\hbar/2) &= \int d^3r\ \big|\langle\vec{r},-|\psi\rangle\big|^2 = \frac{1}{3}\int_0^\infty dr\ r^2\big|R(r)\big|^2(1+1) = \frac{2}{3}\int_0^\infty dr\ r^2\big|R(r)\big|^2 = \frac{1}{3}.
\end{aligned}$$

Note that $Y_{1,-1}(\theta,\phi)$ is not contained in $\psi_+(\vec{r})$ or $\psi_-(\vec{r})$. Thus, the results that can be found if $\hat{L}_z$ is measured are only $+\hbar$ and 0. The probabilities of finding these results are respectively given by

$$\mathscr{P}_{\hat{L}_z}(+\hbar) = \sum_{\varepsilon=\pm}\int dr\ r^2\left|\int d\Omega\ Y_{11}^*(\theta,\phi)\psi_\varepsilon(\vec{r})\right|^2 = \frac{1}{3}\int dr\ r^2\big|R(r)\big|^2 = \frac{1}{6},$$

$$\mathscr{P}_{\hat{L}_z}(0) = \sum_{\ell=0,1}\sum_{\varepsilon=\pm}\int dr\ r^2\left|\int d\Omega\ Y_{\ell0}^*(\theta,\phi)\psi_\varepsilon(\vec{r})\right|^2 = \int dr\ r^2\big|R(r)\big|^2\left(1+\frac{1}{3}+\frac{1}{3}\right) = \frac{5}{6}.$$

The eigenvalues of $\hat{S}_x$ are $\pm\hbar/2$. The corresponding eigenvectors of $\hat{S}_x$ in the $\{|\pm\rangle\}$ basis are respectively given by

$$
\begin{aligned}
|+_x\rangle &= \frac{1}{\sqrt{2}}\big[\,|+\rangle + |-\rangle\,\big], \\
|-_x\rangle &= \frac{1}{\sqrt{2}}\big[\,|+\rangle - |-\rangle\,\big].
\end{aligned}
$$

To distinguish the eigenvectors of $\hat{S}_x$ from those of $\hat{S}_z$, we have added the subscript "x" to "$\pm$".
If $\hat{S}_x$ is measured, the results that can be found are $+\hbar/2$ and $-\hbar/2$. The probabilities of finding these results are respectively given by

$$
\begin{aligned}
\mathscr{P}_{\hat{S}_x}(+\hbar/2) &= \int d^3r\,\big|\langle\vec{r},+_x|\psi\rangle\big|^2 = \frac{1}{2}\int d^3r\,\big|\langle\vec{r},+|\psi\rangle + \langle\vec{r},-|\psi\rangle\big|^2 \\
&= \frac{1}{2}\int_0^\infty dr\; r^2\big|R(r)\big|^2\left|Y_{00}(\theta,\phi) + \frac{1}{\sqrt{3}}Y_{11}(\theta,\phi)\right|^2 \\
&= \frac{1}{2}\int_0^\infty dr\; r^2\big|R(r)\big|^2\left(1+\frac{1}{3}\right) = \frac{2}{3}\int_0^\infty dr\; r^2\big|R(r)\big|^2 = \frac{1}{3}, \\
\mathscr{P}_{\hat{S}_x}(-\hbar/2) &= \int d^3r\,\big|\langle\vec{r},-_x|\psi\rangle\big|^2 = \frac{1}{2}\int d^3r\,\big|\langle\vec{r},+|\psi\rangle - \langle\vec{r},-|\psi\rangle\big|^2 \\
&= \frac{1}{2}\int_0^\infty dr\; r^2\big|R(r)\big|^2\left|Y_{00}(\theta,\phi) + \frac{2}{\sqrt{3}}Y_{11}(\theta,\phi) - \frac{1}{\sqrt{3}}Y_{11}(\theta,\phi)\right|^2 \\
&= \frac{1}{2}\int_0^\infty dr\; r^2\big|R(r)\big|^2\left(1+\frac{4}{3}+\frac{1}{3}\right) = \frac{4}{3}\int_0^\infty dr\; r^2\big|R(r)\big|^2 = \frac{2}{3}.
\end{aligned}
$$

(c) The projector operator onto the $\ell = 0$ eigensubspace $\hat{\vec{L}}^2$ is given by

$$
P_0 = |00\rangle\langle 00|\,.
$$

If a measurement of $\hat{\vec{L}}^2$ yielded the result of zero, then the normalized state vector immediately after the measurement is given by

$$
|\psi'\rangle = \frac{\hat{P}_0\,|\psi\rangle}{\sqrt{\langle\psi|\hat{P}_0|\psi\rangle}} = \frac{|00\rangle\langle 00|\psi\rangle}{\sqrt{\langle\psi|\hat{P}_0|\psi\rangle}}.
$$

Evaluating $\langle\psi|\hat{P}_0|\psi\rangle$, we have

$$
\begin{aligned}
\langle\psi|\hat{P}_0|\psi\rangle &= \sum_{\varepsilon}\int dr\; r^2\int d\Omega\int d\Omega'\;\; \langle\psi|r,\Omega,\varepsilon\rangle\langle\Omega|00\rangle\langle 00|\Omega'\rangle\langle r,\Omega',\varepsilon|\psi\rangle \\
&= \int dr\; r^2\int d\Omega\int d\Omega'\;\; \langle\psi|r,\Omega,+\rangle\langle\Omega|00\rangle\langle 00|\Omega'\rangle\langle r,\Omega',+|\psi\rangle \\
&\quad + \int dr\; r^2\int d\Omega\int d\Omega'\;\; \langle\psi|r,\Omega,-\rangle\langle\Omega|00\rangle\langle 00|\Omega'\rangle\langle r,\Omega',-|\psi\rangle \\
&= \int dr\; r^2\int d\Omega\int d\Omega'\;\; \langle\psi|r,\Omega,+\rangle\, Y_{00}(\theta,\phi)Y_{00}^*(\theta',\phi')\,\langle r,\Omega',+|\psi\rangle \\
&\quad + \int dr\; r^2\int d\Omega\int d\Omega'\;\; \langle\psi|r,\Omega,-\rangle\, Y_{00}(\theta,\phi)Y_{00}^*(\theta',\phi')\,\langle r,\Omega',-|\psi\rangle \\
&= \int dr\; r^2\big|R(r)\big|^2 + 0 = \frac{1}{2}.
\end{aligned}
$$

We then have

$$
\begin{aligned}
\psi'_+(\vec{r}) &= \langle \vec{r}, + | \psi' \rangle = \sqrt{2} \int d\Omega' \ \langle \Omega | 00 \rangle \langle 00 | \Omega' \rangle \langle r, \Omega' | \psi \rangle \\
&= \sqrt{2}\, Y_{00}(\theta, \phi) \int d\Omega' \ Y^*_{00}(\theta', \phi') \psi_+(r, \theta', \phi') \\
&= \sqrt{2}\, Y_{00}(\theta, \phi) \int d\Omega' \ Y^*_{00}(\theta', \phi') R(r) \left[Y_{00}(\theta', \phi') + \frac{1}{\sqrt{3}} Y_{10}(\theta', \phi'), \right] \\
&= \sqrt{2}\, R(r) Y_{00}(\theta, \phi),
\end{aligned}
$$

$$
\begin{aligned}
\psi'_-(\vec{r}) &= \langle \vec{r}, - | \psi' \rangle = \sqrt{2} \int d\Omega' \ \langle \Omega | 00 \rangle \langle 00 | \Omega' \rangle \langle r, \Omega' | \psi \rangle \\
&= \sqrt{2}\, Y_{00}(\theta, \phi) \int d\Omega' \ Y^*_{00}(\theta', \phi') \psi_-(r, \theta', \phi') \\
&= \sqrt{2}\, Y_{00}(\theta, \phi) \int d\Omega' \ Y^*_{00}(\theta', \phi') \frac{R(r)}{\sqrt{3}} \left[Y_{11}(\theta', \phi') - Y_{10}(\theta', \phi') \right] \\
&= 0.
\end{aligned}
$$

The projector operator onto the $\ell = 1$ eigensubspace $\hat{\vec{L}}^2$ is given by

$$
P_1 = \sum_{m=0,\pm 1} |1m\rangle\langle 1m| \, .
$$

If a measurement of $\hat{\vec{L}}^2$ yielded the result of $2\hbar^2$, then the normalized state vector immediately after the measurement is given by

$$
|\psi'\rangle = \frac{\hat{P}_1 \, |\psi\rangle}{\sqrt{\langle \psi | \hat{P}_1 | \psi \rangle}} = \frac{1}{\sqrt{\langle \psi | \hat{P}_1 | \psi \rangle}} \sum_{m=0,\pm 1} |1m\rangle\langle 1m|\psi\rangle \, .
$$

Evaluating $\langle \psi | \hat{P}_1 | \psi \rangle$, we have

$$
\begin{aligned}
\langle \psi | \hat{P}_1 | \psi \rangle &= \sum_{m=0,\pm 1} \sum_{\varepsilon} \int dr \ r^2 \int d\Omega \int d\Omega' \ \langle \psi | r, \Omega, \varepsilon \rangle \langle \Omega | 1m \rangle \langle 1m | \Omega' \rangle \langle r, \Omega', \varepsilon | \psi \rangle \\
&= \sum_{m=0,\pm 1} \int dr \ r^2 \int d\Omega \int d\Omega' \ \langle \psi | r, \Omega, + \rangle \langle \Omega | 1m \rangle \langle 1m | \Omega' \rangle \langle r, \Omega', + | \psi \rangle \\
&\quad + \sum_{m=0,\pm 1} \int dr \ r^2 \int d\Omega \int d\Omega' \ \langle \psi | r, \Omega, - \rangle \langle \Omega | 1m \rangle \langle 1m | \Omega' \rangle \langle r, \Omega', - | \psi \rangle \\
&= \int dr \ r^2 \big|R(r)\big|^2 \left(\frac{1}{3} \right) + \frac{1}{3} \int dr \ r^2 \big|R(r)\big|^2 \big(1 + 1\big) \\
&= \int dr \ r^2 \big|R(r)\big|^2 = \frac{1}{2}.
\end{aligned}
$$

We then have

$$
\begin{aligned}
\psi'_+(\vec{r}) &= \langle \vec{r}, + | \psi' \rangle = \sqrt{2} \sum_{m=0,\pm 1} \int d\Omega' \ \langle \Omega | 1m \rangle \langle 1m | \Omega' \rangle \langle r, \Omega' | \psi \rangle \\
&= \sqrt{2} \sum_{m=0,\pm 1} Y_{1m}(\theta, \phi) \int d\Omega' \ Y^*_{1m}(\theta', \phi') \psi_+(r, \theta', \phi') \\
&= \sqrt{2} \sum_{m=0,\pm 1} Y_{1m}(\theta, \phi) \int d\Omega' \ Y^*_{1m}(\theta', \phi') R(r) \left[Y_{00}(\theta', \phi') + \frac{1}{\sqrt{3}} Y_{10}(\theta', \phi)' \right] \\
&= \sqrt{\frac{2}{3}}\, R(r) Y_{10}(\theta, \phi),
\end{aligned}
$$

$$
\begin{aligned}
\psi'_-(\vec{r}) = \langle \vec{r}, -|\psi'\rangle &= \sqrt{2} \sum_{m=0,\pm 1} \int d\Omega' \ \langle \Omega|1m\rangle\langle 1m|\Omega'\rangle\langle r, \Omega'|\psi\rangle \\
&= \sqrt{2} \sum_{m=0,\pm 1} Y_{1m}(\theta,\phi) \int d\Omega' \ Y^*_{1m}(\theta',\phi')\psi_-(r,\theta',\phi') \\
&= \sqrt{2} \sum_{m=0,\pm 1} Y_{1m}(\theta,\phi) \int d\Omega' \ Y^*_{1m}(\theta',\phi') \frac{R(r)}{\sqrt{3}} \Big[Y_{11}(\theta',\phi') - Y_{10}(\theta',\phi') \Big] \\
&= \sqrt{\frac{2}{3}}\, R(r) \big[Y_{11}(\theta,\phi) - Y_{10}(\theta,\phi) \big].
\end{aligned}
$$

2. **[C-T Exercise 9-2]** Consider a spin 1/2 particle. $\hat{\vec{p}}$ and $\hat{\vec{S}}$ designate the observables associated with its momentum and its spin. We choose as the basis of the state space the orthonormal basis $|p_x p_y p_z, \pm\rangle$ of eigenvectors common to $\hat{p}_x$, $\hat{p}_y$, $\hat{p}_z$, and $\hat{S}_z$ (whose eigenvalues are, respectively, p_x, p_y, p_z, and $\pm\hbar/2$). We intend to solve the eigenvalue equation of the operator $\hat{A}$ which is defined by $\hat{A} = \hat{\vec{S}} \cdot \hat{\vec{p}}$.

   (a) Is $\hat{A}$ Hermitian?

   (b) Show that there exists a basis of eigenvectors of $\hat{A}$ which are also eigenvectors of $\hat{p}_x$, $\hat{p}_y$, and $\hat{p}_z$. In the subspace spanned by the kets $|p_x p_y p_z, \pm\rangle$, where p_x, p_y, and p_z are fixed, what is the matrix representing $\hat{A}$?

   (c) What are the eigenvalues of $\hat{A}$, and what is their degree of degeneracy? Find a system of eigenvectors common to $\hat{A}$ and $\hat{p}_x$, $\hat{p}_y$, $\hat{p}_z$.

(a) From the fact that $\hat{\vec{S}}$ and $\hat{\vec{p}}$ are Hermitian operators and they commute, we have

$$\hat{A}^\dagger = \big(\hat{\vec{S}} \cdot \hat{\vec{p}}\big)^\dagger = \hat{\vec{p}}^\dagger \cdot \hat{\vec{S}}^\dagger = \hat{\vec{p}} \cdot \hat{\vec{S}} = \hat{\vec{S}} \cdot \hat{\vec{p}}.$$

Thus, $\hat{A}$ is Hermitian.

(b) The commutator $[\hat{\vec{p}}, \hat{A}]$ is given by

$$
\begin{aligned}
[\hat{\vec{p}}, \hat{A}] &= \sum_{\alpha\beta} [\hat{p}_\alpha, \hat{S}_\beta \hat{p}_\beta] \vec{e}_\alpha \\
&= \sum_{\alpha\beta} \Big\{ \hat{S}_\beta [\hat{p}_\alpha, \hat{p}_\beta] + [\hat{p}_\alpha, \hat{S}_\beta] \hat{p}_\beta \Big\} \vec{e}_\alpha \\
&= \sum_{\alpha\beta} \Big\{ \hat{S}_\beta \cdot 0 + 0 \cdot \hat{p}_\beta \Big\} \vec{e}_\alpha = 0
\end{aligned}
$$

which indicates that $\hat{\vec{p}}$ and $\hat{A}$ commute and thus they have common eigenvectors. That is, there exists a basis of eigenvectors of $\hat{A}$ which are also eigenvectors of $\hat{p}_x$, $\hat{p}_y$, and $\hat{p}_z$.

In the subspace spanned by the kets $|p_x p_y p_z, \pm\rangle$, the matrix representing $\hat{A}$ is given by

$$
\begin{aligned}
A &= \frac{\hbar}{2} \big(\sigma_x p_x + \sigma_y p_y + \sigma_z p_z \big) \\
&= \frac{\hbar}{2} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_x + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} p_y + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p_z \right] \\
&= \frac{\hbar}{2} \begin{pmatrix} p_z & p_x - i p_y \\ p_x + i p_y & -p_z \end{pmatrix}.
\end{aligned}
$$

(c) Let $a = \lambda\hbar/2$ be the eigenvalue of $\hat{A}$. Let $|\varphi\rangle = \alpha\,|p_x p_y p_z, +\rangle + \beta\,|p_x p_y p_z, -\rangle$ be the common eigenvector of $\hat{A}$ and $\hat{p}_x$, $\hat{p}_y$, $\hat{p}_z$. The eigenvalue equation of $\hat{A}$ reads

$$\begin{pmatrix} p_z & p_x - i p_y \\ p_x + i p_y & -p_z \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

That is,

$$(p_z - \lambda)\alpha + (p_x - ip_y)\beta = 0,$$
$$(p_x + ip_y)\alpha - (p_z + \lambda)\beta = 0.$$

The secular equation is given by

$$\det\begin{vmatrix} p_z - \lambda & p_x - ip_y \\ p_x + ip_y & -(p_z + \lambda) \end{vmatrix} = 0.$$

That is,

$$\lambda^2 - \left(p_x^2 + p_y^2 + p_z^2\right) = 0$$

from which it follows that $\lambda_{1,2} = \pm\sqrt{p_x^2 + p_y^2 + p_z^2} = \pm p$ with $p = \sqrt{p_x^2 + p_y^2 + p_z^2}$. The eigenvalues of $\hat{A}$ are then given by

$$a_1 = \frac{\hbar}{2}\lambda_1 = \frac{1}{2}p\hbar,$$
$$a_2 = \frac{\hbar}{2}\lambda_2 = -\frac{1}{2}p\hbar.$$

To obtain the common eigenvector of $\hat{A}$ and $\hat{p}_x$, $\hat{p}_y$, $\hat{p}_z$ with the eigenvalues $a_1 = p\hbar/2$, p_x, p_y, p_z, we insert $\lambda_1 = \sqrt{p_x^2 + p_y^2 + p_z^2} = p$ into the equations for α and β. We have

$$(p_z - p)\alpha + (p_x - ip_y)\beta = 0,$$
$$(p_x + ip_y)\alpha - (p_z + p)\beta = 0.$$

We thus have

$$\beta = \frac{p_x + ip_y}{p + p_z}\alpha.$$

From the normalization condition $\langle\varphi|\varphi\rangle = 1$, we have

$$|\alpha|^2\left(1 + \left|\frac{p_x + ip_y}{p + p_z}\right|^2\right) = 1$$

which leads to

$$|\alpha| = \sqrt{\frac{p + p_z}{2p}}.$$

We choose

$$\alpha = \sqrt{\frac{p + p_z}{2p}},$$

we then have

$$\beta = \frac{p_x + ip_y}{p + p_z}\sqrt{\frac{p + p_z}{2p}} = \frac{p_x + ip_y}{\sqrt{2p(p + p_z)}}.$$

Thus, the common eigenvector of $\hat{A}$ and $\hat{p}_x$, $\hat{p}_y$, $\hat{p}_z$ with the eigenvalues a_1, p_x, p_y, p_z is given by

$$|\varphi_1\rangle = \frac{1}{\sqrt{2p(p + p_z)}}\big[\,(p + p_z)\,|p_xp_yp_z, +\rangle + (p_x + ip_y)\,|p_xp_yp_z, -\rangle\,\big].$$

To obtain the common eigenvector of $\hat{A}$ and $\hat{p}_x$, $\hat{p}_y$, $\hat{p}_z$ with the eigenvalues $a_2 = -p\hbar/2$, p_x, p_y, p_z, we insert $\lambda_2 = -\sqrt{p_x^2 + p_y^2 + p_z^2} = -p$ into the equations for α and β. We have

$$(p + p_z)\alpha + (p_x - ip_y)\beta = 0,$$
$$(p_x + ip_y)\alpha + (p - p_z)\beta = 0.$$

We thus have

$$\beta = -\frac{p_x + ip_y}{p - p_z}\alpha.$$

From the normalization condition $\langle\varphi|\varphi\rangle = 1$, we have

$$|\alpha|^2\left(1 + \left|\frac{p_x + ip_y}{p - p_z}\right|^2\right) = 1$$

which leads to

$$|\alpha| = \sqrt{\frac{p - p_z}{2p}}.$$

We choose

$$\alpha = \sqrt{\frac{p - p_z}{2p}},$$

we then have

$$\beta = -\frac{p_x + ip_y}{p - p_z}\sqrt{\frac{p - p_z}{2p}} = -\frac{p_x + ip_y}{\sqrt{2p(p - p_z)}}.$$

Thus, the common eigenvector of $\hat{A}$ and $\hat{p}_x$, $\hat{p}_y$, $\hat{p}_z$ with the eigenvalues $a_2 = -p\hbar/2$, p_x, p_y, p_z is given by

$$|\varphi_2\rangle = \frac{1}{\sqrt{2p(p - p_z)}}\big[\,(p - p_z)\,|p_x p_y p_z, +\rangle - (p_x + ip_y)\,|p_x p_y p_z, -\rangle\,\big].$$

3. **[C-T Exercise 9-3]** The Hamiltonian of an electron of mass m, charge q, spin $\hbar\vec{\sigma}/2$ with σ_x, σ_y, and σ_z the Pauli matrices, placed in an electromagnetic field described by the vector potential $\vec{A}(\vec{r}, t)$ and the scalar potential $U(\vec{r}, t)$, is written $\hat{H} = \dfrac{1}{2m}\big[\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}}, t)\big]^2 + qU(\hat{\vec{r}}, t) - \dfrac{q\hbar}{2m}\vec{\sigma}\cdot\vec{B}(\hat{\vec{r}}, t)$. The last term represents the interaction between the spin magnetic moment $(q\hbar/2m)\vec{\sigma}$ and the magnetic field $\vec{B}(\hat{\vec{r}}, t) = \vec{\nabla}\times\vec{A}(\hat{\vec{r}}, t)$. Show, using the properties of the Pauli matrices, that this Hamiltonian can also be written in the form ("the Pauli Hamiltonian") $\hat{H} = \dfrac{1}{2m}\Big\{\vec{\sigma}\cdot\big[\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}}, t)\big]\Big\}^2 + qU(\hat{\vec{r}}, t)$.

Making use of

$$\big(\vec{\sigma}\cdot\hat{\vec{A}}\big)\big(\vec{\sigma}\cdot\hat{\vec{B}}\big) = \hat{\vec{A}}\cdot\hat{\vec{B}} + i\vec{\sigma}\cdot\big(\hat{\vec{A}}\times\hat{\vec{B}}\big),$$

we have

$$\begin{aligned}
\Big\{\vec{\sigma}\cdot\big[\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}}, t)\big]\Big\}^2 &= \big[\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}}, t)\big]^2 + i\vec{\sigma}\cdot\Big\{\big[\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}}, t)\big]\times\big[\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}}, t)\big]\Big\}\\
&= \big[\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}}, t)\big]^2 - iq\vec{\sigma}\cdot\big[\hat{\vec{p}}\times\vec{A}(\hat{\vec{r}}, t) + \vec{A}(\hat{\vec{r}}, t)\times\hat{\vec{p}}\big]\\
&= \big[\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}}, t)\big]^2 - iq\sum_{\alpha\beta\gamma}\epsilon_{\alpha\beta\gamma}\sigma_\gamma\big[\hat{p}_\alpha A_\beta(\hat{\vec{r}}, t) - A_\beta(\hat{\vec{r}}, t)\hat{p}_\alpha\big]\\
&= \big[\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}}, t)\big]^2 - iq\sum_{\alpha\beta\gamma}\epsilon_{\alpha\beta\gamma}\sigma_\gamma\big[\hat{p}_\alpha, A_\beta(\hat{\vec{r}}, t)\big].
\end{aligned}$$

We now evaluate the commutator $\big[\,\hat{p}_\alpha, A_\beta(\hat{\vec{r}},t)\,\big]$. Let $\big[\,\hat{p}_\alpha, A_\beta(\hat{\vec{r}},t)\,\big]$ act on any arbitrary wave function $\psi(\vec{r},t)$, we have

$$\begin{aligned}
\big[\,\hat{p}_\alpha, A_\beta(\hat{\vec{r}},t)\,\big]\psi(\vec{r},t) &= \hat{p}_\alpha A_\beta(\vec{r},t)\psi(\vec{r},t) - A_\beta(\vec{r},t)\hat{p}_\alpha\psi(\vec{r},t) \\
&= -i\hbar\frac{\partial}{\partial x_\alpha}\big[\,A_\beta(\vec{r},t)\psi(\vec{r},t)\,\big] + i\hbar A_\beta(\vec{r},t)\frac{\partial\psi(\vec{r},t)}{\partial x_\alpha} \\
&= -i\hbar\frac{\partial A_\beta(\vec{r},t)}{\partial x_\alpha}\psi(\vec{r},t) - i\hbar A_\beta(\vec{r},t)\frac{\partial\psi(\vec{r},t)}{\partial x_\alpha} + i\hbar A_\beta(\vec{r},t)\frac{\partial\psi(\vec{r},t)}{\partial x_\alpha} \\
&= -i\hbar\frac{\partial A_\beta(\vec{r},t)}{\partial x_\alpha}\psi(\vec{r},t)
\end{aligned}$$

from which it follows that

$$\big[\,\hat{p}_\alpha, A_\beta(\hat{\vec{r}},t)\,\big] = -i\hbar\frac{\partial A_\beta(\hat{\vec{r}},t)}{\partial\hat{x}_\alpha}.$$

Making use of the above commutation relation, we have

$$\begin{aligned}
\Big\{\vec{\sigma}\cdot\big[\,\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}},t)\,\big]\Big\}^2 &= \big[\,\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}},t)\,\big]^2 - q\hbar\sum_{\alpha\beta\gamma}\epsilon_{\alpha\beta\gamma}\sigma_\gamma\frac{\partial A_\beta(\hat{\vec{r}},t)}{\partial\hat{x}_\alpha} \\
&= \big[\,\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}},t)\,\big]^2 - q\hbar\vec{\sigma}\cdot\big[\,\vec{\nabla}\times\vec{A}(\hat{\vec{r}},t)\,\big] \\
&= \big[\,\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}},t)\,\big]^2 - q\hbar\vec{\sigma}\cdot\vec{B}(\hat{\vec{r}},t).
\end{aligned}$$

Inserting the above result into

$$\hat{H} = \frac{1}{2m}\Big\{\vec{\sigma}\cdot\big[\,\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}},t)\,\big]\Big\}^2 + qU(\hat{\vec{r}},t),$$

we have

$$\hat{H} = \frac{1}{2m}\big[\,\hat{\vec{p}} - q\vec{A}(\hat{\vec{r}},t)\,\big]^2 + qU(\hat{\vec{r}},t) - \frac{q\hbar}{2m}\vec{\sigma}\cdot\vec{B}(\hat{\vec{r}},t).$$

4. **[C-T Exercise 10-3]** Consider a system composed of two spin 1/2 particles whose orbital variables are ignored. The Hamiltonian of the system is $\hat{H} = \omega_1\hat{S}_{1z} + \omega_2\hat{S}_{2z}$, where $\hat{S}_{1z}$ and $\hat{S}_{2z}$ are the projections of the spins $\hat{S}_1$ and $\hat{S}_2$ of the two particles onto Oz, and ω_1 and ω_2 are real constants.

   (a) The initial state of the system, at time $t = 0$, is $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\big[\,|+-\rangle + |-+\rangle\,\big]$. At time t, $\hat{\vec{S}}^2 = (\hat{\vec{S}}_1 + \hat{\vec{S}}_2)^2$ is measured. What results can be found, and with what probabilities?

   (b) If the initial state of the system is arbitrary, what Bohr frequencies can appear in the evolution of $\langle\hat{\vec{S}}^2\rangle$? Same question for $\hat{S}_x = \hat{S}_{1x} + \hat{S}_{2x}$.

(a) The eigenvalues and the corresponding eigenvectors of the Hamiltonian $\hat{H}$ of the system are respectively given by

$$\begin{aligned}
E_1 &= \frac{\hbar}{2}\big(\omega_1 + \omega_2\big), & |\varphi_1\rangle &= |++\rangle, \\
E_2 &= \frac{\hbar}{2}\big(\omega_1 - \omega_2\big), & |\varphi_2\rangle &= |+-\rangle, \\
E_3 &= \frac{\hbar}{2}\big(-\omega_1 + \omega_2\big), & |\varphi_3\rangle &= |-+\rangle, \\
E_4 &= -\frac{\hbar}{2}\big(\omega_1 + \omega_2\big), & |\varphi_4\rangle &= |--\rangle\,.
\end{aligned}$$

At time t, the state vector of the system is given by

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[\, e^{-iE_2t/\hbar}\,|+-\rangle + e^{-iE_3t/\hbar}\,|-+\rangle\,\right] = \frac{1}{\sqrt{2}}\left[\, e^{-i(\omega_1-\omega_2)t/2}\,|+-\rangle + e^{i(\omega_1-\omega_2)t/2}\,|-+\rangle\,\right].$$

The common eigenvectors of $\hat{\vec{S}}^2$ and $\hat{S}_z$ are

Eigenvalue of $\hat{\vec{S}}^2$	Eigenvalue of $\hat{S}_z$	Common eigenvector
$2\hbar^2$	$\hbar$	$\lvert 11\rangle = \lvert ++\rangle$
$2\hbar^2$	0	$\lvert 10\rangle = \frac{1}{\sqrt{2}}\left[\,\lvert +-\rangle + \lvert -+\rangle\,\right]$
$2\hbar^2$	$-\hbar$	$\lvert 1,-1\rangle = \lvert --\rangle$
0	0	$\lvert 00\rangle = \frac{1}{\sqrt{2}}\left[\,\lvert +-\rangle - \lvert -+\rangle\,\right]$

If $\hat{\vec{S}}^2 = (\hat{\vec{S}}_1 + \hat{\vec{S}}_2)^2$ is measured at time t, then the results that can be found are $2\hbar^2$ and 0. The probabilities of finding these results are respectively given by

$$\begin{aligned}
\mathscr{P}_{\hat{\vec{S}}^2}(2\hbar^2) &= \left|\langle 11|\psi(t)\rangle\right|^2 + \left|\langle 10|\psi(t)\rangle\right|^2 + \left|\langle 1,-1|\psi(t)\rangle\right|^2 \\
&= 0 + \frac{1}{4}\left|e^{-i(\omega_1-\omega_2)t/2} + e^{i(\omega_1-\omega_2)t/2}\right|^2 + 0 \\
&= \cos^2\left[(\omega_1-\omega_2)t/2\right], \\
\mathscr{P}_{\hat{\vec{S}}^2}(0) &= \left|\langle 00|\psi(t)\rangle\right|^2 = \frac{1}{4}\left|e^{-i(\omega_1-\omega_2)t/2} - e^{i(\omega_1-\omega_2)t/2}\right|^2 \\
&= \sin^2\left[(\omega_1-\omega_2)t/2\right]
\end{aligned}$$

(b) For an arbitrary initial state, we have

$$|\psi(0)\rangle = \alpha\,|++\rangle + \beta\,|+-\rangle + \gamma\,|-+\rangle + \delta\,|--\rangle$$

with $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$. At time t, the state vector of the system is given by

$$|\psi(t)\rangle = \alpha e^{-i(\omega_1+\omega_2)t/2}\,|++\rangle + \beta e^{-i(\omega_1-\omega_2)t/2}\,|+-\rangle + \gamma e^{i(\omega_1-\omega_2)t/2}\,|-+\rangle + \delta e^{i(\omega_1+\omega_2)t/2}\,|--\rangle$$

$\hat{\vec{S}}^2$ is given by

$$\hat{\vec{S}}^2 = \hat{\vec{S}}_1^2 + \hat{\vec{S}}_2^2 + \hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+} + 2\hat{S}_{1z}\hat{S}_{2z}.$$

Acting $\hat{\vec{S}}^2$ on $|\psi(t)\rangle$, we have

$$\begin{aligned}
\hat{\vec{S}}^2\,|\psi(t)\rangle &= \left[\,\hat{\vec{S}}_1^2 + \hat{\vec{S}}_2^2 + \hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+} + 2\hat{S}_{1z}\hat{S}_{2z}\,\right] \\
&\quad\times\left[\,\alpha e^{-i(\omega_1+\omega_2)t/2}\,|++\rangle + \beta e^{-i(\omega_1-\omega_2)t/2}\,|+-\rangle + \gamma e^{i(\omega_1-\omega_2)t/2}\,|-+\rangle + \delta e^{i(\omega_1+\omega_2)t/2}\,|--\rangle\,\right] \\
&= \frac{3}{2}\hbar^2\left[\,\alpha e^{-i(\omega_1+\omega_2)t/2}\,|++\rangle + \beta e^{-i(\omega_1-\omega_2)t/2}\,|+-\rangle + \gamma e^{i(\omega_1-\omega_2)t/2}\,|-+\rangle + \delta e^{i(\omega_1+\omega_2)t/2}\,|--\rangle\,\right] \\
&\quad + \hbar^2\left[\,\gamma e^{i(\omega_1-\omega_2)t/2}\,|+-\rangle + \beta e^{-i(\omega_1-\omega_2)t/2}\,|-+\rangle\,\right] \\
&\quad + \frac{1}{2}\hbar^2\left[\,\alpha e^{-i(\omega_1+\omega_2)t/2}\,|++\rangle - \beta e^{-i(\omega_1-\omega_2)t/2}\,|+-\rangle - \gamma e^{i(\omega_1-\omega_2)t/2}\,|-+\rangle + \delta e^{i(\omega_1+\omega_2)t/2}\,|--\rangle\,\right] \\
&= 2\hbar^2\left[\,\alpha e^{-i(\omega_1+\omega_2)t/2}\,|++\rangle + \delta e^{i(\omega_1+\omega_2)t/2}\,|--\rangle\,\right] \\
&\quad + \hbar^2\left[\,\beta e^{-i(\omega_1-\omega_2)t/2}\,|+-\rangle + \gamma e^{i(\omega_1-\omega_2)t/2}\,|-+\rangle\,\right] \\
&\quad + \hbar^2\left[\,\gamma e^{i(\omega_1-\omega_2)t/2}\,|+-\rangle + \beta e^{-i(\omega_1-\omega_2)t/2}\,|-+\rangle\,\right] \\
&= 2\hbar^2\left[\,\alpha e^{-i(\omega_1+\omega_2)t/2}\,|++\rangle + \delta e^{i(\omega_1+\omega_2)t/2}\,|--\rangle\,\right] \\
&\quad + \hbar^2\left[\,\beta e^{-i(\omega_1-\omega_2)t/2} + \gamma e^{i(\omega_1-\omega_2)t/2}\,\right]\left[\,|+-\rangle + |-+\rangle\,\right].
\end{aligned}$$

For $\langle \hat{\vec{S}}^2\rangle(t)/\hbar^2$, we have

$$\begin{aligned}
\frac{1}{\hbar^2}\langle \hat{\vec{S}}^2\rangle(t) &= \langle\psi(t)|\hat{\vec{S}}^2|\psi(t)\rangle \\
&= \big[\,\alpha^* e^{i(\omega_1+\omega_2)t/2}\langle ++| + \beta^* e^{i(\omega_1-\omega_2)t/2}\langle +-| + \gamma^* e^{-i(\omega_1-\omega_2)t/2}\langle -+| + \delta^* e^{-i(\omega_1+\omega_2)t/2}\langle --|\,\big] \\
&\quad\times \Big\{ 2\big[\,\alpha e^{-i(\omega_1+\omega_2)t/2}|++\rangle + \delta e^{i(\omega_1+\omega_2)t/2}|--\rangle\,\big] \\
&\qquad + \big[\,\beta e^{-i(\omega_1-\omega_2)t/2} + \gamma e^{i(\omega_1-\omega_2)t/2}\,\big]\big[\,|+-\rangle + |-+\rangle\,\big]\Big\} \\
&= 2\big(|\alpha|^2+|\delta|^2\big) + |\beta|^2 + |\gamma|^2 + 2\,\mathrm{Re}\big[\,\beta^*\gamma e^{i(\omega_1-\omega_2)t}\,\big].
\end{aligned}$$

An alternative approach to find $\langle \hat{\vec{S}}^2\rangle(t)$ is to first express $|\psi(t)\rangle$ in terms of the common eigenvectors of $\hat{\vec{S}}^2$ and $\hat{S}_z$. We have

$$\begin{aligned}
|\psi(t)\rangle &= \alpha e^{-i(\omega_1+\omega_2)t/2}|11\rangle + \frac{1}{\sqrt{2}}\big[\,\beta e^{-i(\omega_1-\omega_2)t/2} + \gamma e^{i(\omega_1-\omega_2)t/2}\,\big]|10\rangle + \delta e^{i(\omega_1+\omega_2)t/2}|1,-1\rangle \\
&\quad + \frac{1}{\sqrt{2}}\big[\,\beta e^{-i(\omega_1-\omega_2)t/2} - \gamma e^{i(\omega_1-\omega_2)t/2}\,\big]|00\rangle\,.
\end{aligned}$$

From the above expression, we see that $\langle \hat{\vec{S}}^2\rangle(t)/2\hbar^2$ is given by

$$\begin{aligned}
\frac{1}{2\hbar^2}\langle \hat{\vec{S}}^2\rangle(t) &= \big|\alpha e^{-i(\omega_1+\omega_2)t/2}\big|^2 + \left|\frac{1}{\sqrt{2}}\big[\,\beta e^{-i(\omega_1-\omega_2)t/2} + \gamma e^{i(\omega_1-\omega_2)t/2}\,\big]\right|^2 + \big|\delta e^{i(\omega_1+\omega_2)t/2}\big|^2 \\
&= |\alpha|^2 + |\delta|^2 + \frac{1}{2}\big(|\beta|^2+|\gamma|^2\big) + \mathrm{Re}\big[\,\beta^*\gamma e^{i(\omega_1-\omega_2)t}\,\big].
\end{aligned}$$

From the above result, we see that the Bohr frequency that appears in the evolution of $\langle \hat{\vec{S}}^2\rangle$ is $(\omega_1-\omega_2)/2\pi$. Note that the the Bohr frequency that appears in the evolution of $\langle \hat{\vec{S}}^2\rangle$ in an arbitrary initial state is the same as in the initial state given in (a).
Acting $\hat{S}_x = \hat{S}_{1x} + \hat{S}_{2x} = (\hat{S}_{1+} + \hat{S}_{2+} + \hat{S}_{1-} + \hat{S}_{2-})/2$ on $|\psi(t)\rangle$, we have

$$\begin{aligned}
\hat{S}_x|\psi(t)\rangle &= \frac{1}{2}(\hat{S}_{1+} + \hat{S}_{2+} + \hat{S}_{1-} + \hat{S}_{2-}) \\
&\quad\times \big[\,\alpha e^{-i(\omega_1+\omega_2)t/2}|++\rangle + \beta e^{-i(\omega_1-\omega_2)t/2}|+-\rangle + \gamma e^{i(\omega_1-\omega_2)t/2}|-+\rangle + \delta e^{i(\omega_1+\omega_2)t/2}|--\rangle\,\big] \\
&= \frac{1}{2}\hbar\big[\,\gamma e^{i(\omega_1-\omega_2)t/2}|++\rangle + \delta e^{i(\omega_1+\omega_2)t/2}|+-\rangle + \beta e^{-i(\omega_1-\omega_2)t/2}|++\rangle + \delta e^{i(\omega_1+\omega_2)t/2}|-+\rangle \\
&\quad + \alpha e^{-i(\omega_1+\omega_2)t/2}|-+\rangle + \beta e^{-i(\omega_1-\omega_2)t/2}|--\rangle + \alpha e^{-i(\omega_1+\omega_2)t/2}|+-\rangle + \gamma e^{i(\omega_1-\omega_2)t/2}|--\rangle\,\big] \\
&= \frac{1}{2}\hbar\Big\{\big[\,\beta e^{-i(\omega_1-\omega_2)t/2} + \gamma e^{i(\omega_1-\omega_2)t/2}\,\big]\big[\,|++\rangle + |--\rangle\,\big] \\
&\quad + \big[\,\alpha e^{-i(\omega_1+\omega_2)t/2} + \delta e^{i(\omega_1+\omega_2)t/2}\,\big]\big[\,|+-\rangle + |-+\rangle\,\big]\Big\}.
\end{aligned}$$

For $2\langle \hat{S}_x\rangle(t)/\hbar$, we have

$$\begin{aligned}
\frac{2}{\hbar}\langle \hat{S}_x\rangle(t) &= \langle\psi(t)|\hat{S}_x|\psi(t)\rangle \\
&= \big[\,\alpha^* e^{i(\omega_1+\omega_2)t/2}\langle ++| + \beta^* e^{i(\omega_1-\omega_2)t/2}\langle +-| + \gamma^* e^{-i(\omega_1-\omega_2)t/2}\langle -+| + \delta^* e^{-i(\omega_1+\omega_2)t/2}\langle --|\,\big] \\
&\quad\times \Big\{\big[\,\beta e^{-i(\omega_1-\omega_2)t/2} + \gamma e^{i(\omega_1-\omega_2)t/2}\,\big]\big[\,|++\rangle + |--\rangle\,\big] \\
&\quad + \big[\,\alpha e^{-i(\omega_1+\omega_2)t/2} + \delta e^{i(\omega_1+\omega_2)t/2}\,\big]\big[\,|+-\rangle + |-+\rangle\,\big]\Big\} \\
&= \alpha^*\beta e^{i\omega_2 t} + \alpha^*\gamma e^{i\omega_1 t} + \delta^*\beta e^{-i\omega_1 t} + \delta^*\gamma e^{-i\omega_2 t} + \beta^*\alpha e^{-i\omega_2 t} + \beta^*\delta e^{i\omega_1 t} + \gamma^*\alpha e^{-i\omega_1 t} + \gamma^*\delta e^{i\omega_2 t} \\
&= 2\,\mathrm{Re}\Big[\,\big(\alpha^*\gamma + \beta^*\delta\big)e^{i\omega_1 t} + \big(\alpha^*\beta + \gamma^*\delta\big)e^{i\omega_2 t}\,\Big].
\end{aligned}$$

An alternative approach to find $\langle \hat{S}_x \rangle(t)$ is to first express $|\psi(t)\rangle$ in terms of the common eigenvectors of $\hat{\vec{S}}^2$ and $\hat{S}_x$. The details in this approach will not be given here.

From the above result, we see that the Bohr frequencies $\omega_1/2\pi$ and $\omega_2/2\pi$ appear in the evolution of $\langle \hat{S}_x \rangle$.

5. **[C-T Exercise 10-5]** Let $\hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2 + \hat{\vec{S}}_3$ be the total angular momentum of three spin 1/2 particles (whose orbital variables will be ignored). Let $|\varepsilon_1\varepsilon_2\varepsilon_3\rangle$ be the eigenstates common to $\hat{S}_{1z}$, $\hat{S}_{2z}$, and $\hat{S}_{3z}$, of respective eigenvalues $\varepsilon_1\hbar/2$, $\varepsilon_2\hbar/2$, and $\varepsilon_3\hbar/2$. Give a basis of eigenvectors common to $\hat{\vec{S}}^2$ and $\hat{S}_z$, in terms of the kets $|\varepsilon_1\varepsilon_2\varepsilon_3\rangle$. Do these two operators form a CSCO? (Begin by adding two of the spins, then add the partial angular momentum so obtained to the third one.)

We first consider the addition of $\hat{\vec{S}}_1$ and $\hat{\vec{S}}_2$. Let $\hat{\vec{S}}_{12} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$ and $\hat{S}_{12z} = \hat{S}_{1z} + \hat{S}_{2z}$. Let $S_{12}(S_{12}+1)\hbar^2$ be the eigenvalue of $\hat{\vec{S}}_{12}^2$. Let $M_{12}\hbar$ be the eigenvalue of $\hat{S}_{12z}$. The allowed values of S_{12} are $S_{12} = 1, 0$. For $S_{12} = 1$, $M_{12} = 1, 0, -1$; for $S_{12} = 0$, $M_{12} = 0$. From the lecture notes, we have the following common eigenvectors of $\hat{\vec{S}}_1^2$, $\hat{\vec{S}}_2^2$, $\hat{\vec{S}}_{12}^2$, $\hat{S}_{12z}$ for the addition of $\hat{\vec{S}}_1$ and $\hat{\vec{S}}_2$.

$$\begin{aligned}
|11\rangle_{12} &= |++\rangle_{12}, \\
|10\rangle_{12} &= \frac{1}{\sqrt{2}}\big[\,|+-\rangle_{12} + |-+\rangle_{12}\,\big], \\
|1,-1\rangle_{12} &= |--\rangle_{12}, \\
|00\rangle_{12} &= \frac{1}{\sqrt{2}}\big[\,|+-\rangle_{12} - |-+\rangle_{12}\,\big].
\end{aligned}$$

We now add $\hat{\vec{S}}_3$ to $\hat{\vec{S}}_{12}$. Let $\hat{\vec{S}} = \hat{\vec{S}}_{12} + \hat{\vec{S}}_3$. For $S_{12} = 1$, we have $S = 3/2, 1/2$; for $S_{12} = 0$, we have $S = 1/2$. We find the common eigenvectors of $\hat{\vec{S}}_1^2$, $\hat{\vec{S}}_2^2$, $\hat{\vec{S}}_{12}^2$, $\hat{\vec{S}}^2$, and $\hat{S}_z$ in the subspaces $\mathscr{E}(S_{12} = 1, S = 3/2)$, $\mathscr{E}(S_{12} = 1, S = 1/2)$, and $\mathscr{E}(S_{12} = 0, S = 1/2)$, respectively.

Subspace $\mathscr{E}(S_{12} = 1, S = 3/2)$. For $S = 3/2$, $M = 3/2, 1/2, -1/2, -3/2$. The common eigenvector $|S_{12} = 1, S = 3/2, M = 3/2\rangle$ of $\hat{\vec{S}}_1^2$, $\hat{\vec{S}}_2^2$, $\hat{\vec{S}}_{12}^2$, $\hat{\vec{S}}^2$, and $\hat{S}_z$ with the quantum numbers for $\hat{\vec{S}}_1^2$ and $\hat{\vec{S}}_2^2$ suppressed is given by

$$|1, 3/2, 3/2\rangle = |+++\rangle\,.$$

From

$$\hat{S}_-\,|1, 3/2, 3/2\rangle = \hbar\sqrt{(3/2)(3/2+1) - (3/2)(3/2-1)}\,|1, 3/2, 1/2\rangle = \hbar\sqrt{3}\,|1, 3/2, 1/2\rangle,$$

we have

$$|1, 3/2, 1/2\rangle = \frac{1}{\hbar\sqrt{3}}\hat{S}_-\,|1, 3/2, 3/2\rangle\,.$$

Making use of $\hat{S}_- = \hat{S}_{1-} + \hat{S}_{2-} + \hat{S}_{3-}$ and $|1, 3/2, 3/2\rangle = |+++\rangle$, we have

$$|1, 3/2, 1/2\rangle = \frac{1}{\hbar\sqrt{3}}\big(\hat{S}_{1-} + \hat{S}_{2-} + \hat{S}_{3-}\big)\,|+++\rangle\,.$$

Utilizing $\hat{S}_{1-}\,|+++\rangle = \hbar\,|-++\rangle$, $\hat{S}_{2-}\,|+++\rangle = \hbar\,|+-+\rangle$, and $\hat{S}_{3-}\,|+++\rangle = \hbar\,|++-\rangle$, we have

$$|1, 3/2, 1/2\rangle = \frac{1}{\sqrt{3}}\big[\,|-++\rangle + |+-+\rangle + |++-\rangle\,\big].$$

From

$$\hat{S}_-\,|1, 3/2, 1/2\rangle = \hbar\sqrt{(3/2)(3/2+1) - (1/2)(1/2-1)}\,|1, 3/2, -1/2\rangle = 2\hbar\,|1, 3/2, -1/2\rangle,$$

we have

$$
\begin{aligned}
|1,3/2,-1/2\rangle &= \frac{1}{2\hbar}\hat{S}_-\,|1,3/2,1/2\rangle \\
&= \frac{1}{2\sqrt{3}\hbar}\big(\hat{S}_{1-}+\hat{S}_{2-}+\hat{S}_{3-}\big)\big[\,|-++\rangle+|+-+\rangle+|++-\rangle\,\big] \\
&= \frac{1}{2\sqrt{3}}\big[\,|--+\rangle+|-+-\rangle+|--+\rangle+|+--\rangle+|-+-\rangle+|+--\rangle\,\big] \\
&= \frac{1}{\sqrt{3}}\big[\,|--+\rangle+|-+-\rangle+|+--\rangle\,\big].
\end{aligned}
$$

From

$$
\hat{S}_-\,|1,3/2,-1/2\rangle = \hbar\sqrt{(3/2)(3/2+1)-(-1/2)(-1/2-1)}\,|1,3/2,-3/2\rangle = \hbar\sqrt{3}\,|1,3/2,-3/2\rangle,
$$

we have

$$
\begin{aligned}
|1,3/2,-3/2\rangle &= \frac{1}{\hbar\sqrt{3}}\hat{S}_-\,|1,3/2,-1/2\rangle \\
&= \frac{1}{3\hbar}\big(\hat{S}_{1-}+\hat{S}_{2-}+\hat{S}_{3-}\big)\big[\,|--+\rangle+|-+-\rangle+|+--\rangle\,\big] \\
&= \frac{1}{3}\big[\,|---\rangle+|---\rangle\ +|---\rangle\,\big] = |---\rangle\,.
\end{aligned}
$$

Subspace $\mathscr{E}(S_{12}=1, S=1/2)$. For $S=1/2$, $M=1/2,-1/2$. For $M=1/2$, taking into account the fact that $M=M_{12}+M_3$, we have two different combinations $M_{12}=1, M_3=-1/2$ and $M_{12}=0, M_3=1/2$. We thus have

$$
|1,1/2,1/2\rangle = \alpha\,|++-\rangle + \beta\big[\,|+-+\rangle+|-++\rangle\,\big],
$$

where α and β are to be determined. From the orthogonality of $|1,1/2,1/2\rangle$ and $|1,3/2,1/2\rangle$, we have

$$
\begin{aligned}
0 &= \langle 1,3/2,1/2|1,1/2,1/2\rangle \\
&= \frac{1}{\sqrt{3}}\big[\,\langle -++|+\langle +-+|+\langle ++-|\,\big]\big\{\alpha\,|++-\rangle+\beta\big[\,|+-+\rangle+|-++\rangle\,\big]\big\} \\
&= \frac{1}{\sqrt{3}}\big(\alpha+2\beta\big).
\end{aligned}
$$

We thus have

$$
\beta = -\frac{1}{2}\alpha.
$$

From the normalization condition of $|1,1/2,1/2\rangle$, $\langle 1,1/2,1/2|1,1/2,1/2\rangle = 1$, we have

$$
\begin{aligned}
1 &= \langle 1,1/2,1/2|1,1/2,1/2\rangle \\
&= \Big\{\alpha^*\,\langle ++-|+\beta^*\big[\,\langle +-+|+\langle -++|\,\big]\Big\}\Big\{\alpha\,|++-\rangle+\beta\big[\,|+-+\rangle+|-++\rangle\,\big]\Big\} \\
&= |\alpha|^2+2|\beta|^2.
\end{aligned}
$$

Inserting $\beta=-\alpha/2$ into $|\alpha|^2+2|\beta|^2=1$, we obtain $|\alpha|=\sqrt{2/3}$. We choose $\alpha=\sqrt{2/3}$. We then have $\beta=-1/\sqrt{6}$. We thus have

$$
|1,1/2,1/2\rangle = \frac{1}{\sqrt{6}}\big[\,2\,|++-\rangle-|+-+\rangle-|-++\rangle\,\big].
$$

From

$$
\hat{S}_-\,|1,1/2,1/2\rangle = \hbar\,|1,1/2,-1/2\rangle,
$$

we have

$$
\begin{aligned}
|1,1/2,-1/2\rangle &= \frac{1}{\hbar}\hat{S}_- \, |1,1/2,1/2\rangle \\
&= \frac{1}{\hbar\sqrt{6}}\big(\hat{S}_{1-}+\hat{S}_{2-}+\hat{S}_{3-}\big)\big[\, 2\,|++-\rangle - |+-+\rangle - |-++\rangle \,\big] \\
&= \frac{1}{\sqrt{6}}\big[\, 2\,|-+-\rangle - |--+\rangle + 2\,|+--\rangle - |--+\rangle - |+--\rangle - |-+-\rangle \,\big] \\
&= \frac{1}{\sqrt{6}}\big[\, |+--\rangle + |-+-\rangle - 2\,|--+\rangle \,\big].
\end{aligned}
$$

Subspace $\mathscr{E}(S_{12}=0, S=1/2)$. For $S=1/2$, $M=1/2,-1/2$. Since $S_{12}=0$, M_{12} can only take on the value 0. For $M=M_{12}+M_3=1/2$, M_3 can only take on the value $1/2$. We thus have

$$
|0,1/2,1/2\rangle = \frac{1}{\sqrt{2}}\big[\, |+-+\rangle - |-++\rangle \,\big].
$$

From

$$
\hat{S}_- \, |0,1/2,1/2\rangle = \hbar\, |0,1/2,-1/2\rangle,
$$

we have

$$
\begin{aligned}
|0,1/2,-1/2\rangle &= \frac{1}{\hbar}\hat{S}_- \, |0,1/2,1/2\rangle \\
&= \frac{1}{\hbar\sqrt{2}}\big(\hat{S}_{1-}+\hat{S}_{2-}+\hat{S}_{3-}\big)\big[\, |+-+\rangle - |-++\rangle \,\big] \\
&= \frac{1}{\sqrt{2}}\big[\, |+--\rangle - |-+-\rangle \,\big].
\end{aligned}
$$

From the above results, we see that the common eigenvectors can not be completely determined by specifying the eigenvalues of $\hat{\vec{S}}^2$ and $\hat{S}_z$. Thus, $\hat{\vec{S}}^2$ and $\hat{S}_z$ do not form a CSCO.

In summary, we have obtained the following results for the addition of three spin-1/2 angular momenta.

S_{12}	S	M	$\vert S_{12},S,M\rangle$
1	3/2	3/2	$\vert 1,3/2,3/2\rangle = \vert +++\rangle$
		1/2	$\vert 1,3/2,1/2\rangle = \frac{1}{\sqrt{3}}\big[\, \vert -++\rangle + \vert +-+\rangle + \vert ++-\rangle \,\big]$
		−1/2	$\vert 1,3/2,-1/2\rangle = \frac{1}{\sqrt{3}}\big[\, \vert --+\rangle + \vert -+-\rangle + \vert +--\rangle \,\big]$
		−3/2	$\vert 1,3/2,-3/2\rangle = \vert ---\rangle$
1	1/2	1/2	$\vert 1,1/2,1/2\rangle = \frac{1}{\sqrt{6}}\big[\, 2\,\vert ++-\rangle - \vert +-+\rangle - \vert -++\rangle \,\big]$
		−1/2	$\vert 1,1/2,-1/2\rangle = \frac{1}{\sqrt{6}}\big[\, \vert +--\rangle + \vert -+-\rangle - 2\,\vert --+\rangle \,\big]$
0	1/2	1/2	$\vert 0,1/2,1/2\rangle = \frac{1}{\sqrt{2}}\big[\, \vert +-+\rangle - \vert -++\rangle \,\big]$
		−1/2	$\vert 0,1/2,-1/2\rangle = \frac{1}{\sqrt{2}}\big[\, \vert +--\rangle - \vert -+-\rangle \,\big]$