Quantum Mechanics
Homework Assignment 08
Fall, 2019

1. **[C-T Exercise 3-1]** In a one-dimensional problem, consider a particle whose wave function is

$$\psi(x) = N \frac{e^{ip_0 x/\hbar}}{\sqrt{x^2 + a^2}},$$

where a and p_0 are real constants and N is a normalization coefficient.

   (a) Determine N so that $\psi(x)$ is normalized.

   (b) The position of the particle is measured. What is the probability of finding a result between $-a/\sqrt{3}$ and $+a/\sqrt{3}$?

   (c) Calculate the mean value of the momentum of a particle which has $\psi(x)$ for its wave function.

2. **[C-T Exercise 3-12]** Consider a particle of mass m submitted to the potential

$$V(x) = \begin{cases} 0, & 0 \leq x \leq a, \\ +\infty, & x < 0,\ x > a. \end{cases}$$

$|\varphi_n\rangle$'s are the eigenstates of the Hamiltonian $\hat{H}$ of the system, and their eigenvalues are $E_n = n^2\pi^2\hbar^2/2ma^2$. The state of the particle at the instant $t = 0$ is

$$|\psi(0)\rangle = a_1 |\varphi_1\rangle + a_2 |\varphi_2\rangle + a_3 |\varphi_3\rangle + a_4 |\varphi_4\rangle .$$

   (a) What is the probability, when the energy of the particle in the state $|\psi(0)\rangle$ is measured, of finding a value smaller than $3\pi^2\hbar^2/ma^2$?

   (b) What is the mean value and what is the root-mean-square deviation of the energy of the particle in the state $|\psi(0)\rangle$?

   (c) Calculate the state vector $|\psi(t)\rangle$ at the instant t. Do the results found in the previous two parts at the instant $t = 0$ remain valid at an arbitrary time t?

   (d) When the energy is measured, the result $8\pi^2\hbar^2/ma^2$ is found. After the measurement, what is the state of the system? What is the result if the energy is measured again?

3. **[C-T Exercise 3-13]** In a two-dimensional problem, consider a particle of mass m; its Hamiltonian $\hat{H}$ is written as $\hat{H} = \hat{H}_x + \hat{H}_y$ with

$$\hat{H}_x = \frac{\hat{p}_x^2}{2m} + \hat{V}(\hat{x}),\ \hat{H}_y = \frac{\hat{p}_y^2}{2m} + \hat{V}(\hat{y}).$$

The potential energy $V(x)$ [or $V(y)$] is zero when x (or y) is included in the interval $[0, a]$ and is infinite everywhere else.

   (a) Of the following sets of operators, which form a CSCO?

$$\{\hat{H}\},\ \{\hat{H}_x\},\ \{\hat{H}_x, \hat{H}_y\},\ \{\hat{H}, \hat{H}_x\}.$$

   (b) Consider a particle whose wave function is

$$\psi(x, y) = N \cos\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi y}{a}\right) \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{2\pi y}{a}\right)$$

   if $0 \leq x \leq a$ and $0 \leq y \leq a$, and is zero everywhere else (where N is a constant).

      i. What is the mean value $\langle \hat{H} \rangle$ of the energy of the particle? If the energy $\hat{H}$ is measured, what results can be found, and with what probabilities?

ii. The observable $\hat{H}_x$ is measured; what results can be found, and with what probabilities? If this measurement yields the result $\pi^2\hbar^2/2ma^2$, what will be the results of a subsequent measurement of $\hat{H}_y$, and with what probabilities?

iii. Instead of performing the preceding measurements, one now performs a simultaneous measurement of $\hat{H}_x$ and $\hat{p}_y$. What are the probabilities of finding the following results?

$$E_x = \frac{9\pi^2\hbar^2}{2ma^2} \text{ and } p_0 \le p_y \le p_0 + dp.$$

4. **[C-T Exercise 3-14]** Consider a physical system whose state space, which is three-dimensional, is spanned by the orthonormal basis formed by the three kets $|u_1\rangle$, $|u_2\rangle$, and $|u_3\rangle$. In this basis, the Hamiltonian operator $\hat{H}$ of the system and the two observables $\hat{A}$ and $\hat{B}$ are written as

$$H = \hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad A = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = b \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where ω_0, a, and b are positive real constants. The physical system at time $t = 0$ is in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} |u_1\rangle + \frac{1}{2} |u_2\rangle + \frac{1}{2} |u_3\rangle .$$

(a) At time $t = 0$, the energy of the system is measured. What values can be found, and with what probabilities? Calculate, for the system in the state $|\psi(0)\rangle$, the mean value $\langle\hat{H}\rangle$ and the root-mean-square deviation ΔH.

(b) Instead of measuring $\hat{H}$ at time $t = 0$, one measures $\hat{A}$; what results can be found, and with what probabilities? What is the state vector immediately after the measurement?

(c) Calculate the state vector $|\psi(t)\rangle$ of the system at time t.

(d) Calculate the mean values $\langle\hat{A}\rangle(t)$ and $\langle\hat{B}\rangle(t)$ of $\hat{A}$ and $\hat{B}$ at time t. What comments can be made?

(e) What results are obtained if the observable $\hat{A}$ is measured at time t? Same question for the observable $\hat{B}$. Interpret.

5. **[C-T Exercise 3-8]** Let $\vec{j}(\vec{r})$ be the probability current density associated with a wave function $\psi(\vec{r})$ describing the state of a particle of mass m.

(a) Show that

$$m \int d^3r \, \vec{j}(\vec{r}) = \langle\hat{\vec{p}}\rangle,$$

where $\langle\hat{\vec{p}}\rangle$ is the mean value of the momentum.

(b) Consider the operator $\hat{\vec{L}}$ (orbital angular momentum) defined by $\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$. Are the three components of $\hat{\vec{L}}$ Hermitian operators? Establish the relation

$$m \int d^3r \, \left[\vec{r} \times \vec{j}(\vec{r})\right] = \langle\hat{\vec{L}}\rangle .$$