Quantum Mechanics

Solutions to the Problems in Homework Assignment 10

Fall, 2019

1. **[C-T Exercise 4-1]** Consider a spin 1/2 particle of magnetic moment $\hat{\vec{M}} = \gamma\hat{\vec{S}}$. The spin state space is spanned by the basis of the $|+\rangle$ and $|-\rangle$ vectors, eigenvectors of $\hat{S}_z$ with eigenvalues $+\hbar/2$ and $-\hbar/2$. At time $t = 0$, the state of the system is $|\psi(t = 0)\rangle = |+\rangle$.

   (a) If the observable $\hat{S}_x$ is measured at time $t = 0$, what results can be found, and with what probabilities?

   (b) Instead of performing the preceding measurement, we let the system evolve under the influence of a magnetic field parallel to Oy, of modulus B_0. Calculate, in the $\{|+\rangle, |-\rangle\}$ basis, the state of the system at time t.

   (c) At this time t, we measure the observables $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$. What values can we find, and with what probabilities? What relation must exist between B_0 and t for the result of one of the measurements to be certain? Give a physical interpretation of this condition.

(a) The eigenvalues and the corresponding eigenvectors of $\hat{S}_x$ are respectively given by

$$\mu_\pm = \pm\frac{\hbar}{2},\ |\xi_\pm\rangle = \frac{1}{\sqrt{2}}\big[\,|+\rangle \pm |-\rangle\,\big].$$

In terms of $|\xi_\pm\rangle$, $|\psi(t = 0)\rangle = |+\rangle$ is given by

$$|\psi(t = 0)\rangle = |+\rangle = \frac{1}{\sqrt{2}}\big[\,|\xi_+\rangle + |\xi_-\rangle\,\big].$$

Thus, if the observable $\hat{S}_x$ is measured at time $t = 0$, the results that can be found are $\pm\hbar/2$. The probability of finding each result is 1/2,

$$\mathscr{P}_{\hat{S}_x}(+\hbar/2) = \big|\langle\xi_+|\psi(t = 0)\rangle\big|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2},$$
$$\mathscr{P}_{\hat{S}_x}(-\hbar/2) = \big|\langle\xi_-|\psi(t = 0)\rangle\big|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}.$$

(b) The Hamiltonian of the particle is given by

$$\hat{H} = -\hat{\vec{M}}\cdot\vec{B} = -\gamma B_0\hat{S}_y.$$

In consideration that $[\hat{H}, \hat{S}_y] = 0$, $\hat{H}$ and $\hat{S}_y$ possess common eigenstates. From the eigenvalues and eigenvectors of $\hat{S}_y$, we have the following eigenvalues and eigenvectors of $\hat{H}$

$$E_1 = -\frac{1}{2}\gamma\hbar B_0,\ |\varphi_1\rangle = \frac{1}{\sqrt{2}}\big[\,|+\rangle + i\,|-\rangle\,\big];$$
$$E_2 = \frac{1}{2}\gamma\hbar B_0,\ |\varphi_2\rangle = \frac{1}{\sqrt{2}}\big[\,|+\rangle - i\,|-\rangle\,\big].$$

In terms of $|\varphi_1\rangle$ and $|\varphi_2\rangle$, $|\psi(t = 0)\rangle = |+\rangle$ is given by

$$|\psi(t = 0)\rangle = |+\rangle = \frac{1}{\sqrt{2}}\big[\,|\varphi_1\rangle + |\varphi_2\rangle\,\big].$$

At time t, the state of the system is given by

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\big[\,e^{-iE_1t/\hbar}\,|\varphi_1\rangle + e^{-iE_2t/\hbar}\,|\varphi_2\rangle\,\big] = \frac{1}{\sqrt{2}}\big[\,e^{i\gamma B_0t/2}\,|\varphi_1\rangle + e^{-i\gamma B_0t/2}\,|\varphi_2\rangle\,\big]$$
$$= \cos(\gamma B_0t/2)\,|+\rangle - \sin(\gamma B_0t/2)\,|-\rangle\,.$$

(c) **Measuring $\hat{S}_x$.** For the convenience of discussing the measurement of $\hat{S}_x$, we rewrite $|\psi(t)\rangle$ in terms of the eigenvectors of $\hat{S}_x$. We have

$$\begin{aligned}|\psi(t)\rangle &= \cos(\gamma B_0 t/2)\,|+\rangle - \sin(\gamma B_0 t/2)\,|-\rangle \\ &= \cos(\gamma B_0 t/2)\frac{1}{\sqrt{2}}\big[\,|\xi_+\rangle + |\xi_-\rangle\,\big] - \sin(\gamma B_0 t/2)\frac{1}{\sqrt{2}}\big[\,|\xi_+\rangle - |\xi_-\rangle\,\big] \\ &= \frac{1}{\sqrt{2}}\big[\cos(\gamma B_0 t/2) - \sin(\gamma B_0 t/2)\big]\,|\xi_+\rangle + \frac{1}{\sqrt{2}}\big[\cos(\gamma B_0 t/2) + \sin(\gamma B_0 t/2)\big]\,|\xi_-\rangle\,.\end{aligned}$$

Thus, if $\hat{S}_x$ is measured at time t, the values we can find are $\pm\hbar/2$. The probabilities of obtaining these values are respectively given by

$$\begin{aligned}\mathscr{P}_{\hat{S}_x}(+\hbar/2) &= \big|\langle\xi_+|\psi(t)\rangle\big|^2 = \frac{1}{2}\big[\cos(\gamma B_0 t/2) - \sin(\gamma B_0 t/2)\big]^2 = \frac{1}{2}\big[1 - \sin(\gamma B_0 t)\big], \\ \mathscr{P}_{\hat{S}_x}(-\hbar/2) &= \big|\langle\xi_-|\psi(t)\rangle\big|^2 = \frac{1}{2}\big[\cos(\gamma B_0 t/2) + \sin(\gamma B_0 t/2)\big]^2 = \frac{1}{2}\big[1 + \sin(\gamma B_0 t)\big].\end{aligned}$$

If $\sin(\gamma B_0 t) = -1$, that is, if $B_0 t = (4n+3)\pi/2\gamma$ with n an integer, we have $\mathscr{P}_{\hat{S}_x}(+\hbar/2) = 1$ and $\mathscr{P}_{\hat{S}_x}(-\hbar/2) = 0$.
If $\sin(\gamma B_0 t) = 1$, that is, if $B_0 t = (4n+1)\pi/2\gamma$ with n an integer, we have $\mathscr{P}_{\hat{S}_x}(+\hbar/2) = 0$ and $\mathscr{P}_{\hat{S}_x}(-\hbar/2) = 1$.
This is because the system oscillates between $|\xi_+\rangle$ and $|\xi_-\rangle$ as time develops. At certain time instants, the system is in one of these two states with probability one.
Measuring $\hat{S}_y$. The eigenvalues and the corresponding eigenvectors of $\hat{S}_y$ are respectively given by

$$\nu_\pm = \pm\frac{\hbar}{2},\ |\eta_\pm\rangle = \frac{1}{\sqrt{2}}\big[\,|+\rangle \pm i\,|-\rangle\,\big].$$

Note that $|\eta_+\rangle = |\varphi_1\rangle$ and $|\eta_-\rangle = |\varphi_2\rangle$.
For the convenience of discussing the measurement of $\hat{S}_y$, we rewrite $|\psi(t)\rangle$ in terms of the eigenvectors of $\hat{S}_y$. We have

$$\begin{aligned}|\psi(t)\rangle &= \cos(\gamma B_0 t/2)\,|+\rangle - \sin(\gamma B_0 t/2)\,|-1\rangle \\ &= \frac{1}{\sqrt{2}}e^{i\gamma B_0 t/2}\,|\eta_+\rangle + \frac{1}{\sqrt{2}}e^{-i\gamma B_0 t/2}\,|\eta_-\rangle\,.\end{aligned}$$

From the above result, we see that, if $\hat{S}_y$ is measured at time t, the values we can find are $\pm\hbar/2$. The probabilities of obtaining these values are all 1/2,

$$\begin{aligned}\mathscr{P}_{\hat{S}_y}(+\hbar/2) &= \big|\langle\eta_+|\psi(t)\rangle\big|^2 = \left|\frac{1}{\sqrt{2}}e^{i\gamma B_0 t/2}\right|^2 = \frac{1}{2}, \\ \mathscr{P}_{\hat{S}_y}(-\hbar/2) &= \big|\langle\eta_-|\psi(t)\rangle\big|^2 = \left|\frac{1}{\sqrt{2}}e^{-i\gamma B_0 t/2}\right|^2 = \frac{1}{2}.\end{aligned}$$

Note that, if $\hat{S}_y$ is measured, we can not have a certain result.
Measuring $\hat{S}_z$. From

$$|\psi(t)\rangle = \cos(\gamma B_0 t/2)\,|+\rangle - \sin(\gamma B_0 t/2)\,|-1\rangle,$$

we see that, if $\hat{S}_z$ is measured at time t, the values we can find are $\pm\hbar/2$. The probabilities of obtaining these values are respectively given by

$$\begin{aligned}\mathscr{P}_{\hat{S}_z}(+\hbar/2) &= \big|\langle+|\psi(t)\rangle\big|^2 = \big|\cos(\gamma B_0 t/2)\big|^2 = \cos^2(\gamma B_0 t/2), \\ \mathscr{P}_{\hat{S}_z}(-\hbar/2) &= \big|\langle-|\psi(t)\rangle\big|^2 = \big|-\sin(\gamma B_0 t/2)\big|^2 = \sin^2(\gamma B_0 t/2).\end{aligned}$$

If $\cos(\gamma B_0 t/2) = \pm 1$, that is, if $B_0 t = 2n\pi/\gamma$ with n an integer, we have $\mathscr{P}_{\hat{S}_z}(+\hbar/2) = 1$ and $\mathscr{P}_{\hat{S}_z}(-\hbar/2) = 0$.

If $\sin(\gamma B_0 t/2) = \pm 1$, that is, if $B_0 t = (2n+1)\pi/\gamma$ with n an integer, we have $\mathscr{P}_{\hat{S}_z}(+\hbar/2) = 0$ and $\mathscr{P}_{\hat{S}_z}(-\hbar/2) = 1$.

This is because the system oscillates between $|+\rangle$ and $|-\rangle$ as time develops. At certain time instants, the system is in one of these two states with probability one.

2. **[C-T Exercise 4-3]** Consider a spin 1/2 particle placed in a magnetic field $\vec{B}_0$ with components $B_x = B_0/\sqrt{2}$, $B_y = 0$, and $B_z = B_0/\sqrt{2}$. The notation is the same as that of Problem 1.

   (a) Calculate the matrix representing, in the $\{|+\rangle, |-\rangle\}$ basis, the operator $\hat{H}$, the Hamiltonian of the system.

   (b) Calculate the eigenvalues and the eigenvectors of $\hat{H}$.

   (c) The system at time $t = 0$ is in the state $|-\rangle$. What values can be found if the energy is measured, and with what probabilities?

   (d) Calculate the state vector $|\psi(t)\rangle$ at time t. At this instant, $\hat{S}_x$ is measured; what is the mean value of the results that can be obtained? Give a geometrical interpretation.

(a) The Hamiltonian of the system is given by

$$\hat{H} = -\hat{\vec{M}} \cdot \vec{B} = -\frac{1}{\sqrt{2}}\gamma B_0 \big(\hat{S}_x + \hat{S}_z\big).$$

In the $\{|+\rangle, |-\rangle\}$ basis, the representation matrix of $\hat{H}$ is given by

$$H = -\frac{1}{2\sqrt{2}}\gamma\hbar B_0 \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = -\frac{1}{2\sqrt{2}}\gamma\hbar B_0 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

where we have made use of the following representation matrices of $\hat{S}_x$ and $\hat{S}_z$ in the $\{|+\rangle, |-\rangle\}$ basis

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(b) Let $E = -(\gamma\hbar B_0/2\sqrt{2})\lambda$ be the eigenvalue of $\hat{H}$ and $|\varphi\rangle = a\,|+\rangle + b\,|-\rangle$ be the corresponding eigenvector of $\hat{H}$. In the $\{|+\rangle, |-\rangle\}$ basis, the eigenvalue equation of $\hat{H}$ reads

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}.$$

That is,

$$(1-\lambda)a + b = 0,$$
$$a - (1+\lambda)b = 0$$

from which the secular equation follows

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -(1+\lambda) \end{vmatrix} = 0.$$

Evaluating the determinant on the left hand side yields

$$\lambda^2 - 2 = 0$$

from which we obtain

$$\lambda_{1,2} = \pm\sqrt{2}.$$

Thus, the energy eigenvalues are

$$E_1 = -\frac{1}{2\sqrt{2}}\gamma\hbar B_0\lambda_1 = -\frac{1}{2}\gamma\hbar B_0,$$
$$E_2 = -\frac{1}{2\sqrt{2}}\gamma\hbar B_0\lambda_2 = \frac{1}{2}\gamma\hbar B_0.$$

To find the eigenvector of $\hat{H}$ corresponding to the eigenvalue E_1, we insert $\lambda_1 = \sqrt{2}$ into the equations for a and b. We have

$$(1-\sqrt{2})a + b = 0,$$
$$a - (1+\sqrt{2})b = 0$$

from which we have $b = (\sqrt{2}-1)a$. Then the eigenvector of $\hat{H}$ corresponding to the eigenvalue E_1 is given by

$$|\varphi_1\rangle = a\big[\,|+\rangle + (\sqrt{2}-1)\,|-\rangle\,\big].$$

From the normalization condition $\langle\varphi_1|\varphi_1\rangle = 1$, we have

$$|a|^2\big[\,1 + (\sqrt{2}-1)^2\,\big] = 1$$

from which we have $|a|^2 = (\sqrt{2}+1)/2\sqrt{2}$. Choosing $a = (\sqrt{2}+1)^{1/2}/2^{3/4}$, we have the following normalized eigenvector of $\hat{H}$ corresponding to the eigenvalue E_1

$$|\varphi_1\rangle = \frac{(\sqrt{2}+1)^{1/2}}{2^{3/4}}\big[\,|+\rangle + (\sqrt{2}-1)\,|-\rangle\,\big] = \frac{1}{2^{3/4}}\big[\,(\sqrt{2}+1)^{1/2}\,|+\rangle + (\sqrt{2}-1)^{1/2}\,|-\rangle\,\big].$$

To find the eigenvector of $\hat{H}$ corresponding to the eigenvalue E_2, we insert $\lambda_2 = -\sqrt{2}$ into the equations for a and b. We have

$$(1+\sqrt{2})a + b = 0,$$
$$a - (1-\sqrt{2})b = 0$$

from which we have $b = -(\sqrt{2}+1)a$. Then the eigenvector of $\hat{H}$ corresponding to the eigenvalue E_2 is given by

$$|\varphi_2\rangle = a\big[\,|+\rangle - (\sqrt{2}+1)\,|-\rangle\,\big].$$

From the normalization condition $\langle\varphi_2|\varphi_2\rangle = 1$, we have

$$|a|^2\big[\,1 + (\sqrt{2}+1)^2\,\big] = 1$$

from which we have $|a|^2 = (\sqrt{2}-1)/2\sqrt{2}$. Choosing $a = (\sqrt{2}-1)^{1/2}/2^{3/4}$, we have the following normalized eigenvector of $\hat{H}$ corresponding to the eigenvalue E_2

$$|\varphi_2\rangle = \frac{(\sqrt{2}-1)^{1/2}}{2^{3/4}}\big[\,|+\rangle - (\sqrt{2}+1)\,|-\rangle\,\big] = \frac{1}{2^{3/4}}\big[\,(\sqrt{2}-1)^{1/2}\,|+\rangle - (\sqrt{2}+1)^{1/2}\,|-\rangle\,\big].$$

To summarize, we have obtained the following eigenvalues and normalized eigenvectors of $\hat{H}$

$$E_1 = -\tfrac{1}{2}\gamma\hbar B_0,\ \ |\varphi_1\rangle = \frac{1}{2^{3/4}}\big[\,(\sqrt{2}+1)^{1/2}\,|+\rangle + (\sqrt{2}-1)^{1/2}\,|-\rangle\,\big];$$
$$E_2 = \tfrac{1}{2}\gamma\hbar B_0,\quad |\varphi_2\rangle = \frac{1}{2^{3/4}}\big[\,(\sqrt{2}-1)^{1/2}\,|+\rangle - (\sqrt{2}+1)^{1/2}\,|-\rangle\,\big].$$

(c) For the convenience of discussing the measurement of energy, we express the initial state $|\psi(0)\rangle = |-\rangle$ in terms of the eigenvectors of $\hat{H}$. We have

$$|\psi(0)\rangle = |-\rangle = \frac{1}{2^{3/4}}\big[\,(\sqrt{2}-1)^{1/2}\,|\varphi_1\rangle - (\sqrt{2}+1)^{1/2}\,|\varphi_2\rangle\,\big].$$

From the above expression, we see that, if the energy is measured, the values we can find are

$$E_1 = -\frac{1}{2}\gamma\hbar B_0,\ E_2 = \frac{1}{2}\gamma\hbar B_0.$$

The probabilities of finding these values are respectively given by

$$\begin{aligned}
\mathscr{P}_{\hat{H}}(E_1) &= \big|\langle\varphi_1|-\rangle\big|^2 = \left|\frac{(\sqrt{2}-1)^{1/2}}{2^{3/4}}\right|^2 = \frac{\sqrt{2}-1}{2\sqrt{2}},\\
\mathscr{P}_{\hat{H}}(E_2) &= \big|\langle\varphi_2|-\rangle\big|^2 = \left|-\frac{(\sqrt{2}+1)^{1/2}}{2^{3/4}}\right|^2 = \frac{\sqrt{2}+1}{2\sqrt{2}}.
\end{aligned}$$

(d) The state vector $|\psi(t)\rangle$ at time t is given by

$$\begin{aligned}
|\psi(t)\rangle &= \frac{1}{2^{3/4}}\big[\,(\sqrt{2}-1)^{1/2}e^{-iE_1t/\hbar}\,|\varphi_1\rangle - (\sqrt{2}+1)^{1/2}e^{-iE_2t/\hbar}\,|\varphi_2\rangle\,\big]\\
&= \frac{1}{2^{3/4}}\big[\,(\sqrt{2}-1)^{1/2}e^{i\gamma B_0t/2}\,|\varphi_1\rangle - (\sqrt{2}+1)^{1/2}e^{-i\gamma B_0t/2}\,|\varphi_2\rangle\,\big].
\end{aligned}$$

For the convenience of calculating the mean value of $\hat{S}_x$, we express $|\psi(t)\rangle$ in terms of $|\xi_+\rangle$ and $|\xi_-\rangle$. We have

$$\begin{aligned}
|\psi(t)\rangle &= \frac{1}{2\sqrt{2}}\Big\{(\sqrt{2}-1)^{1/2}e^{i\gamma B_0t/2}\big[\,(\sqrt{2}+1)^{1/2}\,|+\rangle + (\sqrt{2}-1)^{1/2}\,|-\rangle\,\big]\\
&\quad - (\sqrt{2}+1)^{1/2}e^{-i\gamma B_0t/2}\big[\,(\sqrt{2}-1)^{1/2}\,|+\rangle - (\sqrt{2}+1)^{1/2}\,|-\rangle\,\big]\Big\}\\
&= \frac{1}{\sqrt{2}}\Big\{i\sin(\gamma B_0t/2)\,|+\rangle + \big[\,\sqrt{2}\cos(\gamma B_0t/2) - i\sin(\gamma B_0t/2)\,\big]\,|-\rangle\Big\}\\
&= \frac{1}{\sqrt{2}}\Big\{\cos(\gamma B_0t/2)\,|\xi_+\rangle - \big[\,\cos(\gamma B_0t/2) - i\sqrt{2}\sin(\gamma B_0t/2)\,\big]\,|\xi_-\rangle\Big\}.
\end{aligned}$$

From the above expression, we see that, if $\hat{S}_x$ is measured at time t, the values we can find are $\pm\hbar/2$. The probability of finding these values are respectively given by

$$\begin{aligned}
\mathscr{P}_{\hat{S}_x}(+\hbar/2) &= \left|\frac{1}{\sqrt{2}}\cos(\gamma B_0t/2)\right|^2 = \frac{1}{2}\cos^2(\gamma B_0t/2),\\
\mathscr{P}_{\hat{S}_x}(-\hbar/2) &= \left|-\frac{1}{\sqrt{2}}\big[\,\cos(\gamma B_0t/2) - i\sqrt{2}\sin(\gamma B_0t/2)\,\big]\right|^2 = \frac{1}{2}\big[\,1+\sin^2(\gamma B_0t/2)\,\big].
\end{aligned}$$

The mean value of $\hat{S}_x$ is given by

$$\begin{aligned}
\langle\hat{S}_x\rangle &= \frac{\hbar}{2}\mathscr{P}_{\hat{S}_x}(+\hbar/2) - \frac{\hbar}{2}\mathscr{P}_{\hat{S}_x}(-\hbar/2)\\
&= \frac{\hbar}{4}\cos^2(\gamma B_0t/2) - \frac{\hbar}{4}\big[\,1+\sin^2(\gamma B_0t/2)\,\big]\\
&= -\frac{\hbar}{4} + \frac{\hbar}{4}\cos(\gamma B_0t)\\
&= -\frac{\hbar}{2}\sin^2(\gamma B_0t/2).
\end{aligned}$$

From the above result, we see that $\langle\hat{S}_x\rangle$ undergoes harmonic oscillations about $-\hbar/4$ with an angular frequency of $|\gamma|B_0$ and an amplitude of $\hbar/4$.

In order to have an overall picture how the mean of $\hat{\vec{S}}$ vary with time, we also evaluate the mean values of $\hat{S}_y$ and $\hat{S}_z$. We do this by utilizing the representation of $\hat{S}_y$ and $\hat{S}_z$ in the $\{|+\rangle, |-\rangle\}$ basis. For $\langle\hat{S}_y\rangle$, we have

$$\begin{aligned}
\langle \hat{S}_y \rangle &= \langle \psi(t)|\hat{S}_y|\psi(t)\rangle \\
&= \frac{\hbar}{4}\begin{pmatrix} -i\sin(\gamma B_0 t/2) & \sqrt{2}\cos(\gamma B_0 t/2) + i\sin(\gamma B_0 t/2)\end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} i\sin(\gamma B_0 t/2) \\ \sqrt{2}\cos(\gamma B_0 t/2) - i\sin(\gamma B_0 t/2)\end{pmatrix} \\
&= \frac{\hbar}{4}\begin{pmatrix} i\sqrt{2}\cos(\gamma B_0 t/2) - \sin(\gamma B_0 t/2) & -\sin(\gamma B_0 t/2)\end{pmatrix}\begin{pmatrix} i\sin(\gamma B_0 t/2) \\ \sqrt{2}\cos(\gamma B_0 t/2) - i\sin(\gamma B_0 t/2)\end{pmatrix} \\
&= -\frac{\hbar}{\sqrt{2}}\sin(\gamma B_0 t/2)\cos(\gamma B_0 t/2) = -\frac{\hbar}{2\sqrt{2}}\sin(\gamma B_0 t).
\end{aligned}$$

For $\langle \hat{S}_z \rangle$, we have

$$\begin{aligned}
\langle \hat{S}_z \rangle &= \langle \psi(t)|\hat{S}_z|\psi(t)\rangle \\
&= \frac{\hbar}{4}\begin{pmatrix} -i\sin(\gamma B_0 t/2) & \sqrt{2}\cos(\gamma B_0 t/2) + i\sin(\gamma B_0 t/2)\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} i\sin(\gamma B_0 t/2) \\ \sqrt{2}\cos(\gamma B_0 t/2) - i\sin(\gamma B_0 t/2)\end{pmatrix} \\
&= \frac{\hbar}{4}\begin{pmatrix} -i\sin(\gamma B_0 t/2) & -\sqrt{2}\cos(\gamma B_0 t/2) - i\sin(\gamma B_0 t/2)\end{pmatrix}\begin{pmatrix} i\sin(\gamma B_0 t/2) \\ \sqrt{2}\cos(\gamma B_0 t/2) - i\sin(\gamma B_0 t/2)\end{pmatrix} \\
&= -\frac{\hbar}{2}\cos^2(\gamma B_0 t/2) = -\frac{\hbar}{4} - \frac{\hbar}{4}\cos(\gamma B_0 t).
\end{aligned}$$

The variation of $\langle \hat{\vec{S}} \rangle$ with the time t is shown in Fig. 1.

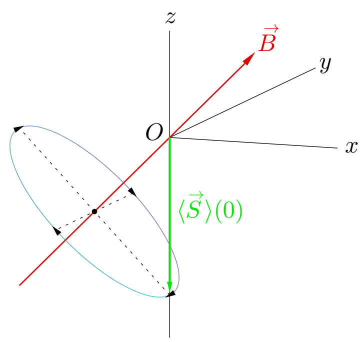

FIG. 1: Variation of $\langle \hat{\vec{S}} \rangle$ with the time t.

From Fig. 1, we see that $\langle \hat{\vec{S}} \rangle$ undergoes a procession with the time t about the magnetic field $\vec{B}$ with the tip of the vector $\langle \hat{\vec{S}} \rangle$ moves along a circle in a plane perpendicular to the magnetic field $\vec{B}$. This motion of $\langle \hat{\vec{S}} \rangle$ yields the time dependence of the above-calculated $\langle \hat{S}_x \rangle$.

3. **[C-T Exercise 4-6]** Consider the system composed of two spin 1/2's, $\hat{\vec{S}}_1$ and $\hat{\vec{S}}_2$, and the basis of four vectors $|\pm,\pm\rangle$. The system at time $t = 0$ is in the state

$$|\psi(0)\rangle = \frac{1}{2}\,|++\rangle + \frac{1}{2}\,|+-\rangle + \frac{1}{\sqrt{2}}\,|--\rangle\,.$$

(a) At time $t = 0$, $\hat{S}_{1z}$ is measured; what is the probability of finding $-\hbar/2$? What is the state vector after this measurement? If we then measure $\hat{S}_{1x}$, what results can be found, and with what probabilities?

(b) When the system is in the state $|\psi(0)\rangle$ written above, $\hat{S}_{1z}$ and $\hat{S}_{2z}$ are measured simultaneously. What is the probability of finding opposite results? Identical results?

(c) Instead of performing the preceding measurements, we let the system evolve under the influence of the Hamiltonian $\hat{H} = \omega_1\hat{S}_{1z} + \omega_2\hat{S}_{2z}$. What is the state vector $|\psi(t)\rangle$ at time t? Calculate at time t the mean values $\langle \hat{\vec{S}}_1 \rangle$ and $\langle \hat{\vec{S}}_2 \rangle$. Give a physical interpretation.

(d) Show that the lengths of the vectors $\langle \hat{\vec{S}}_1 \rangle$ and $\langle \hat{\vec{S}}_2 \rangle$ are less than $\hbar/2$. What must be the form of $|\psi(0)\rangle$ for each of these lengths to be equal to $+\hbar/2$?

(a) If $\hat{S}_{1z}$ is measured at time $t = 0$, the probability of finding $-\hbar/2$ is given by

$$\begin{aligned}
\mathscr{P}_{\hat{S}_{1z}}(-\hbar/2) &= \sum_{\sigma=\pm} \left| \langle 1:- | \langle 2:\sigma | \psi(0) \rangle \right|^2 \\
&= \sum_{\sigma=\pm} \left| \langle 1:- | \langle 2:\sigma | \left[\frac{1}{2} | ++ \rangle + \frac{1}{2} | +- \rangle + \frac{1}{\sqrt{2}} | -- \rangle \right] \right|^2 \\
&= \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}.
\end{aligned}$$

The eigenvalues and eigenstates of $\hat{S}_{1x}$ are given by

$$\mu_{1:\pm} = \pm \frac{\hbar}{2}, \; |1:\xi_\pm\rangle = \frac{1}{\sqrt{2}} \left[\, |1:+\rangle \pm |1:-\rangle \, \right].$$

If $\hat{S}_{1x}$ is measured, the results we can find are $\pm\hbar/2$. The probabilities of finding these results are respectively given by

$$\begin{aligned}
\mathscr{P}_{\hat{S}_{1x}}(+\hbar/2) &= \sum_{\sigma=\pm} \left| \langle 1:\xi_+ | \langle 2:\sigma | \psi(0) \rangle \right|^2 \\
&= \frac{1}{2} \sum_{\sigma=\pm} \left| \left[\langle 1:+| + \langle 1:-| \right] \langle 2:\sigma | \left[\frac{1}{2} | ++ \rangle + \frac{1}{2} | +- \rangle + \frac{1}{\sqrt{2}} | -- \rangle \right] \right|^2 \\
&= \frac{1}{2} \left[\left| \frac{1}{2} \right|^2 + \left| \frac{1}{2} \right|^2 + \left| \frac{1}{\sqrt{2}} \right|^2 \right] = \frac{1}{2}, \\
\mathscr{P}_{\hat{S}_{1x}}(-\hbar/2) &= \sum_{\sigma=\pm} \left| \langle 1:\xi_- | \langle 2:\sigma | \psi(0) \rangle \right|^2 \\
&= \frac{1}{2} \sum_{\sigma=\pm} \left| \left[\langle 1:+| - \langle 1:-| \right] \langle 2:\sigma | \left[\frac{1}{2} | ++ \rangle + \frac{1}{2} | +- \rangle + \frac{1}{\sqrt{2}} | -- \rangle \right] \right|^2 \\
&= \frac{1}{2} \left[\left| \frac{1}{2} \right|^2 + \left| \frac{1}{2} \right|^2 + \left| -\frac{1}{\sqrt{2}} \right|^2 \right] = \frac{1}{2}.
\end{aligned}$$

(b) If $\hat{S}_{1z}$ and $\hat{S}_{2z}$ are measured simultaneously, the probability of finding opposite results is given by

$$\begin{aligned}
\mathscr{P}_{\hat{S}_{1z},\hat{S}_{2z}}(s_{1z} = -s_{2z}) &= \sum_{\sigma=\pm} \left| \langle \sigma, -\sigma | \psi(0) \rangle \right|^2 \\
&= \sum_{\sigma=\pm} \left| \langle \sigma, -\sigma | \left[\frac{1}{2} | ++ \rangle + \frac{1}{2} | +- \rangle + \frac{1}{\sqrt{2}} | -- \rangle \right] \right|^2 \\
&= \left| \frac{1}{2} \right|^2 = \frac{1}{4}.
\end{aligned}$$

If $\hat{S}_{1z}$ and $\hat{S}_{2z}$ are measured simultaneously, the probability of finding identical results is given by

$$\begin{aligned}
\mathscr{P}_{\hat{S}_{1z},\hat{S}_{2z}}(s_{1z} = s_{2z}) &= \sum_{\sigma=\pm} \left| \langle \sigma\sigma | \psi(0) \rangle \right|^2 \\
&= \sum_{\sigma=\pm} \left| \langle \sigma\sigma | \left[\frac{1}{2} | ++ \rangle + \frac{1}{2} | +- \rangle + \frac{1}{\sqrt{2}} | -- \rangle \right] \right|^2 \\
&= \left| \frac{1}{2} \right|^2 + \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{3}{4}.
\end{aligned}$$

(c) The time evolution operator is given by

$$\hat{U}(t,0) = e^{-i\hat{H}t/\hbar} = e^{-i(\omega_1 \hat{S}_{1z} + \omega_2 \hat{S}_{2z})t/\hbar}.$$

Acting $\hat{U}(t,0)$ on $|\psi(0)\rangle$, we obtain the following state vector $|\psi(t)\rangle$ at time t

$$\begin{aligned}
|\psi(t)\rangle = \hat{U}(t,0)\,|\psi(0)\rangle &= e^{-i(\omega_1 \hat{S}_{1z} + \omega_2 \hat{S}_{2z})t/\hbar} \left[\frac{1}{2}\,|++\rangle + \frac{1}{2}\,|+-\rangle + \frac{1}{\sqrt{2}}\,|--\rangle \right] \\
&= \frac{1}{2} e^{-i(\omega_1+\omega_2)t/2}\,|++\rangle + \frac{1}{2} e^{-i(\omega_1-\omega_2)t/2}\,|+-\rangle + \frac{1}{\sqrt{2}} e^{i(\omega_1+\omega_2)t/2}\,|--\rangle,
\end{aligned}$$

where we have made use of

$$e^{-i(\omega_1 \hat{S}_{1z} + \omega_2 \hat{S}_{2z})t/\hbar}\,|\sigma_1\sigma_2\rangle = e^{-i(\sigma_1\omega_1 + \sigma_2\omega_2)/2}\,|\sigma_1\sigma_2\rangle\,.$$

For the convenience of evaluating the mean values, we first find the matrix elements of $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$ between the states $|\pm\rangle$ for a single spin.
The mean value $\langle \hat{S}_{1,x}\rangle$ is given by

$$\begin{aligned}
\langle \hat{S}_{1,x}\rangle &= \left[\frac{1}{2} e^{i(\omega_1+\omega_2)t/2}\,\langle ++| + \frac{1}{2} e^{i(\omega_1-\omega_2)t/2}\,\langle +-| + \frac{1}{\sqrt{2}} e^{-i(\omega_1+\omega_2)t/2}\,\langle --| \right] \\
&\quad \times \hat{S}_{1,x} \left[\frac{1}{2} e^{-i(\omega_1+\omega_2)t/2}\,|++\rangle + \frac{1}{2} e^{-i(\omega_1-\omega_2)t/2}\,|+-\rangle + \frac{1}{\sqrt{2}} e^{i(\omega_1+\omega_2)t/2}\,|--\rangle \right] \\
&= \frac{\hbar}{2}\frac{1}{\sqrt{2}} e^{-i(\omega_1+\omega_2)t/2} \frac{1}{2} e^{-i(\omega_1-\omega_2)t/2} + \frac{\hbar}{2}\frac{1}{2} e^{i(\omega_1-\omega_2)t/2} \frac{1}{\sqrt{2}} e^{i(\omega_1+\omega_2)t/2} \\
&= \frac{\hbar}{4\sqrt{2}} \big(e^{-i\omega_1 t} + e^{i\omega_1 t} \big) = \frac{\hbar}{2\sqrt{2}} \cos(\omega_1 t).
\end{aligned}$$

The mean value $\langle \hat{S}_{1,y}\rangle$ is given by

$$\begin{aligned}
\langle \hat{S}_{1,y}\rangle &= \left[\frac{1}{2} e^{i(\omega_1+\omega_2)t/2}\,\langle ++| + \frac{1}{2} e^{i(\omega_1-\omega_2)t/2}\,\langle +-| + \frac{1}{\sqrt{2}} e^{-i(\omega_1+\omega_2)t/2}\,\langle --| \right] \\
&\quad \times \hat{S}_{1,y} \left[\frac{1}{2} e^{-i(\omega_1+\omega_2)t/2}\,|++\rangle + \frac{1}{2} e^{-i(\omega_1-\omega_2)t/2}\,|+-\rangle + \frac{1}{\sqrt{2}} e^{i(\omega_1+\omega_2)t/2}\,|--\rangle \right] \\
&= i\frac{\hbar}{2}\frac{1}{\sqrt{2}} e^{-i(\omega_1+\omega_2)t/2} \frac{1}{2} e^{-i(\omega_1-\omega_2)t/2} - i\frac{\hbar}{2}\frac{1}{2} e^{i(\omega_1-\omega_2)t/2} \frac{1}{\sqrt{2}} e^{i(\omega_1+\omega_2)t/2} \\
&= i\frac{\hbar}{4\sqrt{2}} \big(e^{-i\omega_1 t} - e^{i\omega_1 t} \big) = \frac{\hbar}{2\sqrt{2}} \sin(\omega_1 t).
\end{aligned}$$

The mean value $\langle \hat{S}_{1,z}\rangle$ is given by

$$\begin{aligned}
\langle \hat{S}_{1,z}\rangle &= \left[\frac{1}{2} e^{i(\omega_1+\omega_2)t/2}\,\langle ++| + \frac{1}{2} e^{i(\omega_1-\omega_2)t/2}\,\langle +-| + \frac{1}{\sqrt{2}} e^{-i(\omega_1+\omega_2)t/2}\,\langle --| \right] \\
&\quad \times \hat{S}_{1,z} \left[\frac{1}{2} e^{-i(\omega_1+\omega_2)t/2}\,|++\rangle + \frac{1}{2} e^{-i(\omega_1-\omega_2)t/2}\,|+-\rangle + \frac{1}{\sqrt{2}} e^{i(\omega_1+\omega_2)t/2}\,|--\rangle \right] \\
&= \frac{\hbar}{2}\frac{1}{2} e^{i(\omega_1+\omega_2)t/2} \frac{1}{2} e^{-i(\omega_1+\omega_2)t/2} + \frac{\hbar}{2}\frac{1}{2} e^{i(\omega_1-\omega_2)t/2} \frac{1}{2} e^{-i(\omega_1-\omega_2)t/2} \\
&\quad - \frac{\hbar}{2}\frac{1}{\sqrt{2}} e^{-i(\omega_1+\omega_2)t/2} \frac{1}{\sqrt{2}} e^{i(\omega_1+\omega_2)t/2} \\
&= \frac{\hbar}{8} + \frac{\hbar}{8} - \frac{\hbar}{4} = 0.
\end{aligned}$$

We thus have

$$\langle \hat{\vec{S}}_1\rangle = \langle \hat{S}_{1,x}\rangle\,\vec{e}_x + \langle \hat{S}_{1,y}\rangle\,\vec{e}_y + \langle \hat{S}_{1,z}\rangle\,\vec{e}_z = \frac{\hbar}{2\sqrt{2}} \big[\cos(\omega_1 t)\vec{e}_x + \sin(\omega_1 t)\vec{e}_y \big].$$

The above result indicates that the spin $\langle \hat{\vec{S}}_1 \rangle$ rotates about the z axis in the xOy plane under the influence of the Hamiltonian $\hat{H} = \omega_1 \hat{S}_{1z} + \omega_2 \hat{S}_{2z}$.

We now evaluate the mean values of the components of the spin $\hat{\vec{S}}_2$. The mean value $\langle \hat{S}_{2,x} \rangle$ is given by

$$\begin{aligned}
\langle \hat{S}_{2,x} \rangle &= \left[\frac{1}{2} e^{i(\omega_1+\omega_2)t/2} \langle + + | + \frac{1}{2} e^{i(\omega_1-\omega_2)t/2} \langle + - | + \frac{1}{\sqrt{2}} e^{-i(\omega_1+\omega_2)t/2} \langle - - | \right] \\
&\quad \times \hat{S}_{2,x} \left[\frac{1}{2} e^{-i(\omega_1+\omega_2)t/2} | + + \rangle + \frac{1}{2} e^{-i(\omega_1-\omega_2)t/2} | + - \rangle + \frac{1}{\sqrt{2}} e^{i(\omega_1+\omega_2)t/2} | - - \rangle \right] \\
&= \frac{\hbar}{2} \frac{1}{2} e^{i(\omega_1-\omega_2)t/2} \frac{1}{2} e^{-i(\omega_1+\omega_2)t/2} + \frac{\hbar}{2} \frac{1}{2} e^{i(\omega_1+\omega_2)t/2} \frac{1}{2} e^{-i(\omega_1-\omega_2)t/2} \\
&= \frac{\hbar}{8} \big(e^{-i\omega_2 t} + e^{i\omega_2 t} \big) = \frac{\hbar}{4} \cos(\omega_2 t).
\end{aligned}$$

The mean value $\langle \hat{S}_{2,y} \rangle$ is given by

$$\begin{aligned}
\langle \hat{S}_{2,y} \rangle &= \left[\frac{1}{2} e^{i(\omega_1+\omega_2)t/2} \langle + + | + \frac{1}{2} e^{i(\omega_1-\omega_2)t/2} \langle + - | + \frac{1}{\sqrt{2}} e^{-i(\omega_1+\omega_2)t/2} \langle - - | \right] \\
&\quad \times \hat{S}_{2,y} \left[\frac{1}{2} e^{-i(\omega_1+\omega_2)t/2} | + + \rangle + \frac{1}{2} e^{-i(\omega_1-\omega_2)t/2} | + - \rangle + \frac{1}{\sqrt{2}} e^{i(\omega_1+\omega_2)t/2} | - - \rangle \right] \\
&= i \frac{\hbar}{2} \frac{1}{2} e^{i(\omega_1-\omega_2)t/2} \frac{1}{2} e^{-i(\omega_1+\omega_2)t/2} - i \frac{\hbar}{2} \frac{1}{2} e^{i(\omega_1+\omega_2)t/2} \frac{1}{2} e^{-i(\omega_1-\omega_2)t/2} \\
&= i \frac{\hbar}{8} \big(e^{-i\omega_2 t} - e^{i\omega_2 t} \big) = \frac{\hbar}{4} \sin(\omega_2 t).
\end{aligned}$$

The mean value $\langle \hat{S}_{2,z} \rangle$ is given by

$$\begin{aligned}
\langle \hat{S}_{2,z} \rangle &= \left[\frac{1}{2} e^{i(\omega_1+\omega_2)t/2} \langle + + | + \frac{1}{2} e^{i(\omega_1-\omega_2)t/2} \langle + - | + \frac{1}{\sqrt{2}} e^{-i(\omega_1+\omega_2)t/2} \langle - - | \right] \\
&\quad \times \hat{S}_{2,z} \left[\frac{1}{2} e^{-i(\omega_1+\omega_2)t/2} | + + \rangle + \frac{1}{2} e^{-i(\omega_1-\omega_2)t/2} | + - \rangle + \frac{1}{\sqrt{2}} e^{i(\omega_1+\omega_2)t/2} | - - \rangle \right] \\
&= \frac{\hbar}{2} \frac{1}{2} e^{i(\omega_1+\omega_2)t/2} \frac{1}{2} e^{-i(\omega_1+\omega_2)t/2} - \frac{\hbar}{2} \frac{1}{2} e^{i(\omega_1-\omega_2)t/2} \frac{1}{2} e^{-i(\omega_1-\omega_2)t/2} \\
&\quad - \frac{\hbar}{2} \frac{1}{\sqrt{2}} e^{-i(\omega_1+\omega_2)t/2} \frac{1}{\sqrt{2}} e^{i(\omega_1+\omega_2)t/2} \\
&= \frac{\hbar}{8} - \frac{\hbar}{8} - \frac{\hbar}{4} = -\frac{\hbar}{4}.
\end{aligned}$$

We thus have

$$\langle \hat{\vec{S}}_2 \rangle = \langle \hat{S}_{2,x} \rangle \vec{e}_x + \langle \hat{S}_{2,y} \rangle \vec{e}_y + \langle \hat{S}_{2,z} \rangle \vec{e}_z = \frac{\hbar}{4} \big[\cos(\omega_2 t) \vec{e}_x + \sin(\omega_2 t) \vec{e}_y - \vec{e}_z \big].$$

The above result indicates that the spin $\hat{\vec{S}}_2$ rotates about the z axis under the influence of the Hamiltonian $\hat{H} = \omega_1 \hat{S}_{1z} + \omega_2 \hat{S}_{2z}$.

(d) From the above-obtained results

$$\begin{aligned}
\langle \hat{\vec{S}}_1 \rangle &= \frac{\hbar}{2\sqrt{2}} \big[\cos(\omega_1 t) \vec{e}_x + \sin(\omega_1 t) \vec{e}_y \big], \\
\langle \hat{\vec{S}}_2 \rangle &= \frac{\hbar}{4} \big[\cos(\omega_2 t) \vec{e}_x + \sin(\omega_2 t) \vec{e}_y - \vec{e}_z \big],
\end{aligned}$$

we have

$$\| \langle \hat{\vec{S}}_1 \rangle \| = \| \langle \hat{\vec{S}}_2 \rangle \| = \frac{\hbar}{2\sqrt{2}} < \frac{\hbar}{2}.$$

Thus, the lengths of the vectors $\langle\hat{\vec{S}}_1\rangle$ and $\langle\hat{\vec{S}}_2\rangle$ are less than $\hbar/2$. If $|\psi(0)\rangle$ can be written as a tensor product of the form $|\psi(0)\rangle = |\chi(1)\rangle \otimes |\varphi(2)\rangle$, then each of the lengths of the vectors $\langle\hat{\vec{S}}_1\rangle$ and $\langle\hat{\vec{S}}_2\rangle$ is equal to $\hbar/2$.
We can show that, if a spin 1/2 is in the state of the form

$$|\chi\rangle = \alpha\,|+\rangle + \beta\,|-\rangle$$

with α and β complex constants satisfying $|\alpha|^2 + |\beta|^2 = 1$, then the length of the average of $\hat{\vec{S}}$ is equal to $\hbar/2$. For the convenience of evaluating the mean values of $\hat{S}_x$ and $\hat{S}_y$, we also express $|\chi\rangle$ in terms of the eigenvectors of $\hat{S}_x$, $|\xi_\pm\rangle$, and the eigenvectors of $\hat{S}_y$, $|\eta_\pm\rangle$. We have

$$\begin{aligned}
|\chi\rangle &= \alpha\,|+\rangle + \beta\,|-\rangle \\
&= \frac{\alpha+\beta}{\sqrt{2}}\,|\xi_+\rangle + \frac{\alpha-\beta}{\sqrt{2}}\,|\xi_-\rangle \\
&= \frac{\alpha-i\beta}{\sqrt{2}}\,|\eta_+\rangle + \frac{\alpha+i\beta}{\sqrt{2}}\,|\eta_-\rangle\,.
\end{aligned}$$

The mean value of $\hat{S}_z$ is given by

$$\langle\hat{S}_z\rangle = \big(|\alpha|^2 - |\beta|^2\big)\frac{\hbar}{2}.$$

The mean value of $\hat{S}_x$ is given by

$$\langle\hat{S}_x\rangle = \big(\alpha\beta^* + \alpha^*\beta\big)\frac{\hbar}{2} = \hbar\,\mathrm{Re}\big(\alpha\beta^*\big).$$

The mean value of $\hat{S}_y$ is given by

$$\langle\hat{S}_y\rangle = i\big(\alpha\beta^* - \alpha^*\beta\big)\frac{\hbar}{2} = -\hbar\,\mathrm{Im}\big(\alpha\beta^*\big).$$

The square of the length of $\langle\hat{\vec{S}}\rangle$ is given by

$$\begin{aligned}
\big|\langle\hat{\vec{S}}\rangle\big|^2 &= \langle\hat{S}_x\rangle^2 + \langle\hat{S}_y\rangle^2 + \langle\hat{S}_z\rangle^2 \\
&= \big(\alpha\beta^* + \alpha^*\beta\big)^2\frac{\hbar^2}{4} - \big(\alpha\beta^* - \alpha^*\beta\big)^2\frac{\hbar^2}{4} + \big(|\alpha|^2 - |\beta|^2\big)^2\frac{\hbar^2}{4} \\
&= \Big[\,\big(\alpha^2\beta^{*2} + 2|\alpha|^2|\beta|^2 + \alpha^{*2}\beta^2\big) - \big(\alpha^2\beta^{*2} - 2|\alpha|^2|\beta|^2 + \alpha^{*2}\beta^2\big) + \big(|\alpha|^4 - 2|\alpha|^2|\beta|^2 + |\beta|^4\big)\,\Big]\frac{\hbar^2}{4} \\
&= \big(|\alpha|^4 + 2|\alpha|^2|\beta|^2 + |\beta|^4\big)\frac{\hbar^2}{4} \\
&= \big(|\alpha|^2 + |\beta|^2\big)^2\,\frac{\hbar^2}{4} \\
&= \frac{\hbar^2}{4}.
\end{aligned}$$

Therefore,

$$\big|\langle\hat{\vec{S}}\rangle\big| = \frac{\hbar}{2}.$$

4. **[C-T Exercise 5-7]** Consider a one-dimensional harmonic oscillator of Hamiltonian $\hat{H}$ and stationary states $|\varphi_n\rangle$, $\hat{H}\,|\varphi_n\rangle = (n+1/2)\hbar\omega\,|\varphi_n\rangle$. The operator $\hat{U}(k)$ is defined by $\hat{U}(k) = e^{ik\hat{x}}$, where k is real.

    (a) Is $\hat{U}(k)$ unitary? Show that, for all n, its matrix elements satisfy the relation

    $$\sum_{n'}\big|\langle\varphi_n|\hat{U}(k)|\varphi_{n'}\rangle\big|^2 = 1.$$

(b) Express $\hat{U}(k)$ in terms of the operators $\hat{a}$ and $\hat{a}^\dagger$. Use Glauber's formula to put $\hat{U}(k)$ in the form of a product of exponential operators.

(c) Establish the relations

$$e^{\lambda\hat{a}}\,|\varphi_0\rangle = |\varphi_0\rangle,$$
$$\langle\varphi_n|e^{\lambda\hat{a}^\dagger}|\varphi_0\rangle = \frac{\lambda^n}{\sqrt{n!}},$$

where λ is an arbitrary complex parameter.

(d) Find the expression, in terms of $E_k = \hbar^2k^2/2m$ and $E_\omega = \hbar\omega$, for the matrix element $\langle\varphi_0|\hat{U}(k)|\varphi_n\rangle$. What happens when k approaches zero? Could this result have been predicted directly?

(a) Making use of the fact that $\hat{x}$ is a Hermitian operator, $\hat{x}^\dagger = \hat{x}$, we have

$$\hat{U}^\dagger(k) = \left[\sum_{j=0}^{\infty}\frac{(ik)^j}{j!}\hat{x}^j\right]^\dagger = \sum_{j=0}^{\infty}\frac{(-ik)^j}{j!}\left(\hat{x}^j\right)^\dagger = \sum_{j=0}^{\infty}\frac{(-ik)^j}{j!}\hat{x}^j = e^{-ik\hat{x}}.$$

We thus have

$$\hat{U}(k)\hat{U}^\dagger(k) = e^{ik\hat{x}}e^{-ik\hat{x}} = e^{ik\hat{x}-ik\hat{x}} = e^0 = 1,$$
$$\hat{U}^\dagger(k)\hat{U}(k) = e^{-ik\hat{x}}e^{ik\hat{x}} = e^{-ik\hat{x}+ik\hat{x}} = e^0 = 1.$$

Therefore, $\hat{U}(k)$ is unitary.

Making use of the closure relation of the eigenvectors of the Hamiltonian $\hat{H}$,

$$\sum_{n'}|\varphi_{n'}\rangle\langle\varphi_{n'}| = 1,$$

we have

$$\begin{aligned}\sum_{n'}\left|\langle\varphi_n|\hat{U}(k)|\varphi_{n'}\rangle\right|^2 &= \sum_{n'}\langle\varphi_n|\hat{U}(k)|\varphi_{n'}\rangle\langle\varphi_n|\hat{U}(k)|\varphi_{n'}\rangle^* \\ &= \sum_{n'}\langle\varphi_n|\hat{U}(k)|\varphi_{n'}\rangle\langle\varphi_{n'}|\hat{U}^\dagger(k)|\varphi_n\rangle \\ &= \langle\varphi_n|\hat{U}(k)\hat{U}^\dagger(k)|\varphi_n\rangle\,.\end{aligned}$$

Making use of $\hat{U}(k)\hat{U}^\dagger(k) = 1$ and $\langle\varphi_n|\varphi_n\rangle = 1$, we have

$$\sum_{n'}\left|\langle\varphi_n|\hat{U}(k)|\varphi_{n'}\rangle\right|^2 = \langle\varphi_n|\varphi_n\rangle = 1.$$

(b) Making use of

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}+\hat{a}^\dagger\right),$$

we have

$$\hat{U}(k) = e^{ik\sqrt{\hbar/2m\omega}\left(\hat{a}+\hat{a}^\dagger\right)}.$$

Utilizing

$$e^{\hat{A}+\hat{B}} = e^{\hat{B}}e^{\hat{A}}e^{[\hat{A},\hat{B}]/2}$$

with

$$\hat{A} = ik\sqrt{\frac{\hbar}{2m\omega}}\,\hat{a},\ \hat{B} = ik\sqrt{\frac{\hbar}{2m\omega}}\,\hat{a}^\dagger,$$
$$[\hat{A},\hat{B}] = -k^2\frac{\hbar}{2m\omega},$$

we have

$$\hat{U}(k) = e^{-\hbar k^2/4m\omega}e^{ik\sqrt{\hbar/2m\omega}\hat{a}^\dagger}e^{ik\sqrt{\hbar/2m\omega}\hat{a}}.$$

(c) Making use of $\hat{a}\,|\varphi_0\rangle = 0$, we have

$$e^{\lambda\hat{a}}\,|\varphi_0\rangle = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\hat{a}^n\,|\varphi_0\rangle = \left(1+\lambda\hat{a}+\frac{1}{2!}\lambda^2\hat{a}^2+\cdots\right)|\varphi_0\rangle = |\varphi_0\rangle\,.$$

Making use of

$$|\varphi_n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n\,|\varphi_0\rangle,$$

we have

$$\langle\varphi_n|e^{\lambda\hat{a}^\dagger}|\varphi_0\rangle = \sum_{n'=0}^{\infty}\frac{\lambda^{n'}}{n'!}\langle\varphi_n|(\hat{a}^\dagger)^{n'}|\varphi_0\rangle = \sum_{n'=0}^{\infty}\frac{\lambda^{n'}}{\sqrt{n'!}}\langle\varphi_n|\varphi_{n'}\rangle = \sum_{n'=0}^{\infty}\frac{\lambda^{n'}}{\sqrt{n'!}}\delta_{nn'} = \frac{\lambda^n}{\sqrt{n!}}.$$

(d) Inserting

$$\hat{U}(k) = e^{-\hbar k^2/4m\omega}e^{ik\sqrt{\hbar/2m\omega}\hat{a}^\dagger}e^{ik\sqrt{\hbar/2m\omega}\hat{a}}$$

into $\langle\varphi_0|\hat{U}(k)|\varphi_n\rangle$, we have

$$\langle\varphi_0|\hat{U}(k)|\varphi_n\rangle = e^{-\hbar k^2/4m\omega}\,\langle\varphi_0|e^{ik\sqrt{\hbar/2m\omega}\,\hat{a}^\dagger}e^{ik\sqrt{\hbar/2m\omega}\,\hat{a}}|\varphi_n\rangle\,.$$

Making use of $\langle\varphi_0|\,e^{\lambda^*\hat{a}^\dagger} = \langle\varphi_0|$ obtained through taking the Hermitian conjugate of $e^{\lambda\hat{a}}\,|\varphi_0\rangle = |\varphi_0\rangle$, we have

$$\langle\varphi_0|\hat{U}(k)|\varphi_n\rangle = e^{-\hbar k^2/4m\omega}\,\langle\varphi_0|e^{ik\sqrt{\hbar/2m\omega}\,\hat{a}}|\varphi_n\rangle = e^{-\hbar k^2/4m\omega}\sum_{n'=0}^{\infty}\frac{(ik)^{n'}(\hbar/2m\omega)^{n'/2}}{n'!}\,\langle\varphi_0|\hat{a}^{n'}|\varphi_n\rangle\,.$$

Making use of

$$\langle\varphi_0|\,\frac{\hat{a}^{n'}}{\sqrt{n'!}} = \langle\varphi_{n'}|\ \text{obtained through taking the Hermitian conjugate of}\ |\varphi_{n'}\rangle = \frac{1}{\sqrt{n'!}}(\hat{a}^\dagger)^{n'}\,|\varphi_0\rangle,$$

we have

$$\langle\varphi_0|\hat{U}(k)|\varphi_n\rangle = e^{-\hbar k^2/4m\omega}\sum_{n'=0}^{\infty}\frac{(ik)^{n'}(\hbar/2m\omega)^{n'/2}}{\sqrt{n'!}}\,\langle\varphi_{n'}|\varphi_n\rangle\,.$$

Making use of $\langle\varphi_{n'}|\varphi_n\rangle = \delta_{n'n}$, we have

$$\langle\varphi_0|\hat{U}(k)|\varphi_n\rangle = e^{-\hbar k^2/4m\omega}\sum_{n'=0}^{\infty}\frac{(ik)^{n'}(\hbar/2m\omega)^{n'/2}}{\sqrt{n'!}}\delta_{n'n} = e^{-\hbar k^2/4m\omega}\,\frac{(ik)^n(\hbar/2m\omega)^{n/2}}{\sqrt{n!}}.$$

Making use of $E_k = \hbar^2k^2/2m$ and $E_\omega = \hbar\omega$, we have

$$\langle\varphi_0|\hat{U}(k)|\varphi_n\rangle = e^{-\hbar k^2/4m\omega}\,\frac{(ik)^n(\hbar/2m\omega)^{n/2}}{\sqrt{n!}} = \frac{i^n}{\sqrt{n!}}e^{-E_k/2E_\omega}(E_k/E_\omega)^{n/2}.$$

When k approaches zero, $\langle\varphi_0|\hat{U}(k)|\varphi_n\rangle \to \delta_{n0}$. Yes, this result could have been predicted directly. From $\hat{U}(k) = e^{ik\hat{x}}$, we see that, when k approaches zero, $\hat{U}(k)\to 1$. We then have

$$\lim_{k\to 0}\langle\varphi_0|\hat{U}(k)|\varphi_n\rangle = \langle\varphi_0|\varphi_n\rangle = \delta_{n0}.$$

5. **[C-T Exercise 5-8]** The evolution operator $\hat{U}(t,0)$ of a one-dimensional harmonic oscillator is written $\hat{U}(t,0) = e^{-i\hat{H}t/\hbar}$ with $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)$.

   (a) Consider the operators $\hat{\tilde{a}}(t) = \hat{U}^\dagger(t,0)\hat{a}\hat{U}(t,0)$ and $\hat{\tilde{a}}^\dagger(t) = \hat{U}^\dagger(t,0)\hat{a}^\dagger\hat{U}(t,0)$. By calculating their action on the eigenkets $|\varphi_n\rangle$ of $\hat{H}$, find the expressions for $\hat{\tilde{a}}(t)$ and $\hat{\tilde{a}}^\dagger(t)$ in terms of $\hat{a}$ and $\hat{a}^\dagger$.

   (b) Calculate the operators $\hat{\tilde{x}}(t)$ and $\hat{\tilde{p}}_x(t)$ obtained from $\hat{x}$ and $\hat{p}_x$ by the unitary transformation $\hat{\tilde{x}}(t) = \hat{U}^\dagger(t,0)\hat{x}\hat{U}(t,0)$ and $\hat{\tilde{p}}_x(t) = \hat{U}^\dagger(t,0)\hat{p}_x\hat{U}(t,0)$. How can the relations so obtained be interpreted?

   (c) Show that $\hat{U}^\dagger(\pi/2\omega,0)\,|x\rangle$ is an eigenvector of $\hat{p}_x$ and specify its eigenvalue. Similarly, establish that $\hat{U}^\dagger(\pi/2\omega,0)\,|p_x\rangle$ is an eigenvector of $\hat{x}$.

   (d) At $t=0$, the wave function of the oscillator is $\psi(x,0)$. How can one obtain from $\psi(x,0)$ the wave function of the oscillator at all subsequent times $t_q = q\pi/2\omega$ (where q is a positive integer)?

   (e) Choose for $\psi(x,0)$ the wave function $\varphi_n(x)$ associated with a stationary state. From the preceding question derive the relation which must exist between $\varphi_n(x)$ and its Fourier transform $\overline{\varphi}_n(p_x)$.

(a) Acting $\hat{\tilde{a}}(t) = \hat{U}^\dagger(t,0)\hat{a}\hat{U}(t,0)$ on $|\varphi_n\rangle$, we have

$$\begin{aligned}
\hat{\tilde{a}}(t)\,|\varphi_n\rangle &= \hat{U}^\dagger(t,0)\hat{a}\hat{U}(t,0)\,|\varphi_n\rangle = \hat{U}^\dagger(t,0)\hat{a}e^{-iE_nt/\hbar}\,|\varphi_n\rangle = e^{-iE_nt/\hbar}\hat{U}^\dagger(t,0)\hat{a}\,|\varphi_n\rangle \\
&= e^{-iE_nt/\hbar}\hat{U}^\dagger(t,0)\sqrt{n}\,|\varphi_{n-1}\rangle = \sqrt{n}\,e^{-iE_nt/\hbar}\hat{U}^\dagger(t,0)\,|\varphi_{n-1}\rangle \\
&= \sqrt{n}\,e^{-iE_nt/\hbar}e^{iE_{n-1}t/\hbar}\,|\varphi_{n-1}\rangle = e^{-i\omega t}\sqrt{n}\,|\varphi_{n-1}\rangle \\
&= e^{-i\omega t}\hat{a}\,|\varphi_n\rangle\,.
\end{aligned}$$

We thus have

$$\hat{\tilde{a}}(t) = e^{-i\omega t}\hat{a}.$$

Acting $\hat{\tilde{a}}^\dagger(t) = \hat{U}^\dagger(t,0)\hat{a}^\dagger\hat{U}(t,0)$ on $|\varphi_n\rangle$, we have

$$\begin{aligned}
\hat{\tilde{a}}^\dagger(t)\,|\varphi_n\rangle &= \hat{U}^\dagger(t,0)\hat{a}^\dagger\hat{U}(t,0)\,|\varphi_n\rangle = \hat{U}^\dagger(t,0)\hat{a}^\dagger e^{-iE_nt/\hbar}\,|\varphi_n\rangle = e^{-iE_nt/\hbar}\hat{U}^\dagger(t,0)\hat{a}^\dagger\,|\varphi_n\rangle \\
&= e^{-iE_nt/\hbar}\hat{U}^\dagger(t,0)\sqrt{n+1}\,|\varphi_{n+1}\rangle = \sqrt{n+1}\,e^{-iE_nt/\hbar}\hat{U}^\dagger(t,0)\,|\varphi_{n+1}\rangle \\
&= \sqrt{n+1}\,e^{-iE_nt/\hbar}e^{iE_{n+1}t/\hbar}\,|\varphi_{n+1}\rangle = e^{i\omega t}\sqrt{n+1}\,|\varphi_{n+1}\rangle \\
&= e^{i\omega t}\hat{a}^\dagger\,|\varphi_n\rangle\,.
\end{aligned}$$

We thus have

$$\hat{\tilde{a}}^\dagger(t) = e^{i\omega t}\hat{a}^\dagger.$$

(b) Making use of

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a} + \hat{a}^\dagger),$$

we have

$$\hat{\tilde{x}}(t) = \hat{U}^\dagger(t,0)\hat{x}\hat{U}(t,0) = \sqrt{\frac{\hbar}{2m\omega}}\,\hat{U}^\dagger(t,0)\big(\hat{a} + \hat{a}^\dagger\big)\hat{U}(t,0).$$

Making use of the above-obtained results, $\hat{\tilde{a}}(t) = e^{-i\omega t}\hat{a}$ and $\hat{\tilde{a}}^\dagger(t) = e^{i\omega t}\hat{a}^\dagger$, we have

$$\hat{\tilde{x}}(t) = \sqrt{\frac{\hbar}{2m\omega}}\,\big(e^{-i\omega t}\hat{a} + e^{i\omega t}\hat{a}^\dagger\big).$$

To express $\hat{\tilde{x}}(t)$ in terms of $\hat{x}$ and $\hat{p}_x$, we make use of

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}_x\right),\ \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}_x\right).$$

We have

$$\begin{aligned}\hat{\tilde{x}}(t) &= \frac{1}{2}\left[e^{-i\omega t}\left(\hat{x} + \frac{i}{m\omega}\hat{p}_x\right) + e^{i\omega t}\left(\hat{x} - \frac{i}{m\omega}\hat{p}_x\right)\right] \\ &= \hat{x}\cos(\omega t) + \frac{1}{m\omega}\,\hat{p}_x\sin(\omega t).\end{aligned}$$

Utilizing

$$\hat{p}_x = -i\sqrt{\frac{m\hbar\omega}{2}}\,\big(\hat{a} - \hat{a}^\dagger\big),$$

we have

$$\hat{\tilde{p}}_x(t) = \hat{U}^\dagger(t,0)\hat{p}_x\hat{U}(t,0) = -i\sqrt{\frac{m\hbar\omega}{2}}\,\hat{U}^\dagger(t,0)\big(\hat{a} - \hat{a}^\dagger\big)\hat{U}(t,0).$$

Utilizing again the above-obtained results, $\hat{\tilde{a}}(t) = e^{-i\omega t}\hat{a}$ and $\hat{\tilde{a}}^\dagger(t) = e^{i\omega t}\hat{a}^\dagger$, we have

$$\hat{\tilde{p}}_x(t) = -i\sqrt{\frac{m\hbar\omega}{2}}\,\big(e^{-i\omega t}\hat{a} - e^{i\omega t}\hat{a}^\dagger\big).$$

To express $\hat{\tilde{p}}_x(t)$ in terms of $\hat{x}$ and $\hat{p}_x$, we again make use of

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}_x\right),\ \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}_x\right).$$

We have

$$\begin{aligned}\hat{\tilde{p}}_x(t) &= -i\frac{m\omega}{2}\left[e^{-i\omega t}\left(\hat{x} + \frac{i}{m\omega}\hat{p}_x\right) - e^{i\omega t}\left(\hat{x} - \frac{i}{m\omega}\hat{p}_x\right)\right] \\ &= -m\omega\,\hat{x}\sin(\omega t) + \hat{p}_x\cos(\omega t).\end{aligned}$$

The results,

$$\begin{aligned}\hat{\tilde{x}}(t) &= \hat{x}\cos(\omega t) + \frac{1}{m\omega}\,\hat{p}_x\sin(\omega t), \\ \hat{\tilde{p}}_x(t) &= -m\omega\,\hat{x}\sin(\omega t) + \hat{p}_x\cos(\omega t),\end{aligned}$$

are the quantum mechanical version of the solutions of Hamilton's equations,

$$\begin{aligned}x(t) &= x_0\cos(\omega t) + \frac{1}{m\omega}p_0\sin(\omega t), \\ p(t) &= -m\omega x_0\sin(\omega t) + p_0\cos(\omega t).\end{aligned}$$

The derivation of the above solutions of Hamilton's equations goes as follows. Hamilton's equations are given by

$$\begin{aligned}\frac{dx}{dt} &= \frac{\partial H}{\partial p}, \\ \frac{dp}{dt} &= -\frac{\partial H}{\partial x}\end{aligned}$$

with the classical Hamiltonian given by

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2.$$

We have

$$\begin{aligned}\frac{dx}{dt} &= \frac{p}{m}, \\ \frac{dp}{dt} &= -m\omega^2 x.\end{aligned}$$

Differentiating the above two equations respectively with respect to t yields

$$\frac{d^2x}{dt^2} + \omega^2 x = 0,$$
$$\frac{d^2p}{dt^2} + \omega^2 p = 0.$$

The general solutions to the above two equations are given by

$$x(t) = A\cos(\omega t) + B\sin(\omega t),$$
$$p(t) = C\cos(\omega t) + D\sin(\omega t).$$

Inserting the above two equations into the initial conditions $x(t=0) = x_0$ and $p(t=0) = p_0$, we obtain

$$A = x_0,\ C = p_0.$$

From Hamilton's equations, we also have

$$\left.\frac{dx}{dt}\right|_{t=0} = \left.\frac{p}{m}\right|_{t=0},$$
$$\left.\frac{dp}{dt}\right|_{t=0} = -m\omega^2 x\big|_{t=0}$$

from which we have

$$B = \frac{1}{m\omega}p_0,\ D = -m\omega x_0.$$

Thus, the solutions of Hamilton's equations are given by

$$x(t) = x_0\cos(\omega t) + \frac{1}{m\omega}p_0\sin(\omega t),$$
$$p(t) = -m\omega x_0\sin(\omega t) + p_0\cos(\omega t).$$

(c) Making use of $\hat{U}^\dagger(\pi/2\omega, 0)\hat{U}(\pi/2\omega, 0) = 1$, we have

$$\hat{p}_x\hat{U}^\dagger(\pi/2\omega, 0)\,|x\rangle = \hat{U}^\dagger(\pi/2\omega, 0)\hat{U}(\pi/2\omega, 0)\hat{p}_x\hat{U}^\dagger(\pi/2\omega, 0)\,|x\rangle\,.$$

Making use of $\hat{U}(t,0)\hat{p}_x\hat{U}^\dagger(t,0) = \hat{U}^\dagger(-t,0)\hat{p}_x\hat{U}(-t,0)$ and the above-obtained result,

$$\hat{\tilde{p}}_x(t) = \hat{U}^\dagger(t,0)\hat{p}_x\hat{U}(t,0) = -m\omega\,\hat{x}\sin(\omega t) + \hat{p}_x\cos(\omega t),$$

we have

$$\begin{aligned}\hat{p}_x\big[\,\hat{U}^\dagger(\pi/2\omega, 0)\,|x\rangle\,\big] &= \hat{U}^\dagger(\pi/2\omega, 0)\hat{U}^\dagger(-\pi/2\omega, 0)\hat{p}_x\hat{U}(-\pi/2\omega, 0)\,|x\rangle \\ &= \hat{U}^\dagger(\pi/2\omega, 0)\big[-m\omega\,\hat{x}\sin(-\pi/2) + \hat{p}_x\cos(-\pi/2)\,\big]\,|x\rangle \\ &= m\omega\hat{U}^\dagger(\pi/2\omega, 0)\hat{x}\,|x\rangle \\ &= m\omega x\big[\,\hat{U}^\dagger(\pi/2\omega, 0)\,|x\rangle\,\big].\end{aligned}$$

Thus, $\hat{U}^\dagger(\pi/2\omega, 0)\,|x\rangle$ is an eigenvector of $\hat{p}_x$ corresponding to the eigenvalue $m\omega x$. Making use of $\hat{U}^\dagger(\pi/2\omega, 0)\hat{U}(\pi/2\omega, 0) = 1$, we have

$$\hat{x}\hat{U}^\dagger(\pi/2\omega, 0)\,|p_x\rangle = \hat{U}^\dagger(\pi/2\omega, 0)\hat{U}(\pi/2\omega, 0)\hat{x}\hat{U}^\dagger(\pi/2\omega, 0)\,|p_x\rangle\,.$$

Making use of $\hat{U}(t,0)\hat{x}\hat{U}^\dagger(t,0) = \hat{U}^\dagger(-t,0)\hat{x}\hat{U}(-t,0)$ and the above-obtained result,

$$\hat{\tilde{x}}(t) = \hat{U}^\dagger(t,0)\hat{x}\hat{U}(t,0) = \hat{x}\cos(\omega t) + \frac{1}{m\omega}\,\hat{p}_x\sin(\omega t),$$

we have

$$\begin{aligned}
\hat{x}\big[\,\hat{U}^\dagger(\pi/2\omega,0)\,|p_x\rangle\,\big] &= \hat{U}^\dagger(\pi/2\omega,0)\hat{U}^\dagger(-\pi/2\omega,0)\hat{x}\hat{U}(-\pi/2\omega,0)\,|p_x\rangle \\
&= \hat{U}^\dagger(\pi/2\omega,0)\big[\,\hat{x}\cos(-\pi/2)+\frac{1}{m\omega}\,\hat{p}_x\sin(-\pi/2)\,\big]\,|p_x\rangle \\
&= -\frac{1}{m\omega}\hat{U}^\dagger(\pi/2\omega,0)\hat{p}_x\,|p_x\rangle \\
&= -\frac{p_x}{m\omega}\big[\,\hat{U}^\dagger(\pi/2\omega,0)\,|p_x\rangle\,\big].
\end{aligned}$$

Thus, $\hat{U}^\dagger(\pi/2\omega,0)\,|p_x\rangle$ is an eigenvector of $\hat{x}$ corresponding to the eigenvalue $-p_x/m\omega$.

(d) From $|\psi(t)\rangle=\hat{U}(t,0)\,|\psi(0)\rangle$, we have

$$\psi(x,t)=\langle x|\psi(t)\rangle=\langle x|\hat{U}(t,0)|\psi(0)\rangle=\int dx'\,\langle x|\hat{U}(t,0)|x'\rangle\langle x'|\psi(0)\rangle=\int dx'\,\langle x|\hat{U}(t,0)|x'\rangle\,\psi(x',0).$$

Note that $\langle x|\hat{U}(t,0)|x'\rangle$ is the propagator, $\langle x|\hat{U}(t,0)|x'\rangle=\langle x|e^{-i\hat{H}t/\hbar}|x'\rangle$. We now evaluate this propagator for $t_q=q\pi/2\omega$. Noting that

$$\langle x|\hat{U}(t,0)|x'\rangle=\langle x'|\hat{U}^\dagger(t,0)|x\rangle^*,$$

we can first evaluate $\langle x'|\hat{U}^\dagger(t,0)|x\rangle$ for $q_q=q\pi/2\omega$. From

$$\begin{aligned}
\hat{\tilde{x}}(t) &= \hat{x}\cos(\omega t)+\frac{1}{m\omega}\,\hat{p}_x\sin(\omega t), \\
\hat{\tilde{p}}_x(t) &= -m\omega\,\hat{x}\sin(\omega t)+\hat{p}_x\cos(\omega t),
\end{aligned}$$

we see that $\hat{\tilde{p}}_x(t)$ is essentially $\hat{x}$ at $t=(2j+1)\pi/2\omega$ for $j=0,1,2,\cdots$ and $\hat{p}_x$ at $t=(2j)\pi/2\omega$ for $j=1,2,3,\cdots$. We thus discuss the $q=2j+1$ and $q=2j$ cases separately.

$q=2j+1$ **case with** $j=0,1,2,\cdots$**.** We already considered $\hat{U}^\dagger(t,0)\,|x\rangle$ for $t=\pi/2\omega$. From $\hat{p}_x\big[\,\hat{U}^\dagger(\pi/2\omega,0)\,|x\rangle\,\big]=m\omega x\big[\,\hat{U}^\dagger(\pi/2\omega,0)\,|x\rangle\,\big]$, we concluded that $\hat{U}^\dagger(\pi/2\omega,0)\,|x\rangle$ is the eigenvector of $\hat{p}_x$ corresponding to the eigenvalue $m\omega x$. From $\hat{\tilde{p}}_x(t)=-m\omega\,\hat{x}\sin(\omega t)+\hat{p}_x\cos(\omega t)$, we have

$$\begin{aligned}
\hat{p}_x\big[\hat{U}^\dagger((2j+1)\pi/2\omega,0)\,|x\rangle\big] &= \hat{U}^\dagger((2j+1)\pi/2\omega,0)\hat{U}^\dagger(-(2j+1)\pi/2\omega,0)\hat{p}_x\hat{U}(-(2j+1)\pi/2\omega,0)\,|x\rangle \\
&= \hat{U}^\dagger((2j+1)\pi/2\omega,0)\big[-m\omega\,\hat{x}\sin(-(2j+1)\pi/2)+\hat{p}_x\cos(-(2j+1)\pi/2)\big]|x\rangle \\
&= (-1)^j m\omega\hat{U}^\dagger((2j+1)\pi/2\omega,0)\hat{x}\,|x\rangle \\
&= (-1)^j m\omega x\big[\hat{U}^\dagger((2j+1)\pi/2\omega,0)\,|x\rangle\big].
\end{aligned}$$

Thus, $\hat{U}^\dagger((2j+1)\pi/2\omega,0)\,|x\rangle$ is the eigenvector of $\hat{p}_x$ corresponding to the eigenvalue $(-1)^j m\omega x$. We can then write

$$|p_x=(-1)^j m\omega x\rangle=\hat{U}^\dagger((2j+1)\pi/2\omega,0)\,|x\rangle\,.$$

Note that $|p_x=(-1)^j m\omega x\rangle$ can only be determined up to a phase factor of modulus one. Hence

$$\langle x|\hat{U}((2j+1)\pi/2\omega,0)|x'\rangle=\langle x'|\hat{U}^\dagger((2j+1)\pi/2\omega,0)|x\rangle^*=\langle x'|p_x=(-1)^j m\omega x\rangle^*\,.$$

From

$$\langle x'|p_x\rangle=\frac{1}{\sqrt{2\pi\hbar}}e^{ip_xx'/\hbar},$$

we have

$$\langle x|\hat{U}((2j+1)\pi/2\omega,0)|x'\rangle=\frac{1}{\sqrt{2\pi\hbar}}e^{-i(-1)^j m\omega xx'/\hbar}.$$

$\psi(x,(2j+1)\pi/2\omega)$ is then given by

$$\psi(x,(2j+1)\pi/2\omega)=\int dx'\,\langle x|\hat{U}((2j+1)\pi/2\omega,0)|x'\rangle\,\psi(x',0)=\frac{1}{\sqrt{2\pi\hbar}}\int dx'\;e^{-i(-1)^j m\omega x x'/\hbar}\psi(x',0).$$

$q=2j$ **case with** $j=1,2,3,\cdots$**.** From $\hat{\tilde{x}}(t)=\hat{x}\cos(\omega t)+(1/m\omega)\,\hat{p}_x\sin(\omega t)$, we have

$$\begin{aligned}\hat{x}\big[\hat{U}^\dagger((2j)\pi/2\omega,0)\,|x\rangle\big]&=\hat{U}^\dagger((2j)\pi/2\omega,0)\hat{U}^\dagger(-(2j)\pi/2\omega,0)\hat{x}\hat{U}(-(2j)\pi/2\omega,0)\,|x\rangle\\&=\hat{U}^\dagger((2j)\pi/2\omega,0)\big[\,\hat{x}\cos(-(2j)\pi/2)+(1/m\omega)\,\hat{p}_x\sin(-(2j)\pi/2)\big]|x\rangle\\&=(-1)^j\hat{U}^\dagger((2j)\pi/2\omega,0)\hat{x}\,|x\rangle\\&=(-1)^j x\big[\hat{U}^\dagger((2j)\pi/2\omega,0)\,|x\rangle\big].\end{aligned}$$

Thus, $\hat{U}^\dagger((2j)\pi/2\omega,0)\,|x\rangle$ is the eigenvector of $\hat{x}$ corresponding to the eigenvalue $(-1)^j x$. We can then write

$$|(-1)^j x\rangle=\hat{U}^\dagger((2j)\pi/2\omega,0)\,|x\rangle\,.$$

Note that $|(-1)^j x\rangle$ can only be determined up to a phase factor of modulus one. Hence,

$$\langle x|\hat{U}((2j)\pi/2\omega,0)|x'\rangle=\langle x'|\hat{U}^\dagger((2j)\pi/2\omega,0)|x\rangle^*=\langle x'|(-1)^j x\rangle^*=\langle(-1)^j x|x'\rangle=\delta(x'-(-1)^j x).$$

$\psi(x,(2j)\pi/2\omega)$ is then given by

$$\psi(x,(2j)\pi/2\omega)=\int dx'\,\langle x|\hat{U}((2j)\pi/2\omega,0)|x'\rangle\,\psi(x',0)=\int dx'\;\delta(x'-(-1)^j x)\psi(x',0)=\psi((-1)^j x,0).$$

(e) In the previous question, we obtained

$$\psi(x,(2j+1)\pi/2\omega)=\frac{1}{\sqrt{2\pi\hbar}}\int dx'\;e^{-ip_x x'/\hbar}\psi(x',0)$$

in which $p_x=(-1)^j m\omega x$ is the value of the momentum variable. If $\psi(x,0)=\varphi_n(x)$, from $\hat{U}(t,0)\varphi_n(x)=e^{-iE_n t/\hbar}\varphi_n(x)$, we have $\psi(x,t)=e^{-iE_n t/\hbar}\varphi_n(x)$. Thus, $\psi(x,(2j+1)\pi/2\omega)$ still describes the eigenstate $|\varphi_n\rangle$ given that $\psi(x,0)=\varphi_n(x)$. Since the variable x on the left hand of the above equation is related to the momentum variable p_x through $p_x=(-1)^j m\omega x$, $\psi(x,(2j+1)\pi/2\omega)$ must be the wave function of the eigenstate $|\varphi_n\rangle$ in momentum space, $\overline{\varphi}_n(p_x)=\langle p_x|\varphi_n\rangle$. We then have

$$\overline{\varphi}_n(p_x)=\frac{1}{\sqrt{2\pi\hbar}}\int dx'\;e^{-ip_x x'/\hbar}\varphi_n(x')=\frac{1}{\sqrt{2\pi\hbar}}\int dx\;e^{-ip_x x/\hbar}\varphi_n(x).$$

The relation between $\varphi_n(x)$ and its Fourier transform $\overline{\varphi}_n(p_x)$ is given by the above equation.