PHYS1501
Quantum Mechanics
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Assignment 7

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Problem 1. Starting from the time-dependent Schrödinger equation in the Dirac notation, $i\hbar\frac{d|\psi(t)\rangle}{dt}=[\frac{\hat{\vec{p}}^2}{2m}+\hat{V}(\hat{\vec{r}})]|\psi(t)\rangle$, derive the time-dependent Schrödinger equation in the $\{|\vec{p}\rangle\}$ representation.

$$i\hbar\frac{\partial}{\partial t}\bar{\psi}(\vec{p},t)=\left[\frac{\vec{p}^2}{2m}+\hat{V}(i\hbar\vec{\nabla}_{\vec{p}})\right]\bar{\psi}(\vec{p},t)$$

Solution: The scalar product of

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t}=\left[\frac{\hat{\vec{p}}^2}{2m}+\hat{V}(\hat{\vec{r}})\right]|\psi(t)\rangle \tag{1}$$

by $|\vec{p}\rangle$ is

$$i\hbar\langle\vec{p}|\frac{d}{dt}|\psi(t)\rangle=\langle\vec{p}|\left[\frac{\hat{\vec{p}}^2}{2m}+\hat{V}(\hat{\vec{r}})\right]|\psi(t)\rangle \tag{2}$$

where

$$\langle\vec{p}|\frac{d}{dt}|\psi(t)\rangle=\frac{\partial}{\partial t}\langle\vec{p}|\psi(t)\rangle=\frac{\partial}{\partial t}\bar{\psi}(\vec{p},t) \tag{3}$$

and

$$\langle\vec{p}|\hat{\vec{p}}^2|\psi(t)\rangle=\langle\vec{p}|\vec{p}^2|\psi\rangle=\vec{p}^2\langle\vec{p}|\psi(t)\rangle=\vec{p}^2\bar{\psi}(\vec{p},t) \tag{4}$$

and

$$\langle\vec{p}|\hat{V}(\hat{\vec{r}})|\psi(t)\rangle=\langle|\hat{V}(i\hbar\vec{\nabla}_{\vec{p}})|\psi(t)\rangle=\hat{V}(i\hbar\vec{\nabla}_{\vec{p}})\langle\vec{p}|\psi(t)\rangle=\hat{V}(i\hbar\vec{\nabla}_{\vec{p}})\bar{\psi}(\vec{p},t) \tag{5}$$

Therefore,

$$i\hbar\frac{\partial}{\partial t}\bar{\psi}(\vec{p},t)=\left[\frac{\vec{p}^2}{2m}+\hat{V}(i\hbar\vec{\nabla}_{\vec{p}})\right]\bar{\psi}(\vec{p},t) \tag{6}$$

□

Problem 2. Introducing the Fourier transform of the potential energy $V(\vec{r})$ in the $\{|\vec{r}\rangle\}$ representation, $\bar{V}(\vec{p})=\frac{1}{(2\pi\hbar)^{3/2}}\int d^3re^{-i\vec{p}\cdot\vec{r}/\hbar}V(\vec{r})$, show that the time-dependent Schrödinger equation in the $\{|\vec{p}\rangle\}$ representation can be also written as

$$i\hbar\frac{\partial}{\partial t}\bar{\psi}(\vec{p},t)=\frac{\vec{p}^2}{2m}\bar{\psi}(\vec{p},t)+\frac{1}{(2\pi\hbar)^{3/2}}\int d^3p'\bar{V}(\vec{p}-\vec{p}')\bar{\psi}(\vec{p}',t)$$

Solution:

$$
\begin{aligned}
\langle\vec{p}|\hat{V}(\hat{\vec{r}})|\psi(t)\rangle =& \int d^3p' \langle\vec{p}|\hat{V}(\hat{\vec{r}})\cdot 1|\psi(t)\rangle \\
=& \int d^3p' \langle\vec{p}|\hat{V}(\hat{\vec{r}})|\vec{p}'\rangle\langle\vec{p}'|\psi(t)\rangle \\
=& \int d^3p' \langle\vec{p}|\hat{V}(\hat{\vec{r}})|\vec{p}'\rangle\bar{\psi}(\vec{p}',t) \\
=& \int d^3p' \left[\int d^3r \frac{1}{(2\pi\hbar)^{3/2}}e^{-i\vec{p}\cdot\vec{r}}\hat{V}(\hat{\vec{r}})\frac{1}{(2\pi\hbar)^{3/2}}e^{i\vec{p}'\cdot\vec{r}}\right]\bar{\psi}(\vec{p}',t) \\
=& \int d^3p' \left[\int d^3r \frac{1}{(2\pi\hbar)^{3/2}}e^{-i\vec{p}\cdot\vec{r}}V(\vec{r})\frac{1}{(2\pi\hbar)^{3/2}}e^{i\vec{p}'\cdot\vec{r}}\right]\bar{\psi}(\vec{p}',t) \\
=& \frac{1}{(2\pi\hbar)^{3/2}}\int d^3p' \left[\frac{1}{(2\pi\hbar)^{3/2}}\int d^3r e^{-i(\vec{p}-\vec{p}')\cdot\vec{r}}V(\vec{r})\right]\bar{\psi}(\vec{p}',t) \\
=& \frac{1}{(2\pi\hbar)^{3/2}}\int d^3p' V(\vec{p}-\vec{p}')\bar{\psi}(\vec{p}',t)
\end{aligned}
$$

Plugging the equation above into the time-dependent Schrödinger equation in the $\{|\vec{p}\rangle\}$ representation (the conclusion derived in problem 1) gives

$$
i\hbar\frac{\partial}{\partial t}\bar{\psi}(\vec{p},t) = \frac{\vec{p}^2}{2m}\bar{\psi}(\vec{p},t) + \frac{1}{(2\pi\hbar)^{3/2}}\int d^3p'\bar{V}(\vec{p}-\vec{p}')\bar{\psi}(\vec{p}',t) \tag{7}
$$

□

Problem 3. In the $\{|p_x\rangle\}$ representation, find the energy eigenvalue and eigenfunction of a particle of mass m in the one-dimensional δ-function potential well

$$
V(x) = -\lambda\delta(x), \quad \lambda > 0
$$

Solution: The Fourier transformation of the potential energy $V(\vec{r})$ in the representation is

$$
\bar{V}(\vec{p}) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} dx e^{-ip_x x/\hbar}V(x) = -\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} dx e^{-ip_x x/\hbar}\lambda\delta(x) = -\frac{\lambda}{\sqrt{2\pi\hbar}} \tag{8}
$$

The time-dependent Schrödinger equation in the $\{|p_x\rangle\}$ representation

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\bar{\psi}(p_x,t) =& \frac{p_x^2}{2m}\bar{\psi}(p_x,t) + \frac{1}{\sqrt{2\pi\hbar}}\int dp_x'\bar{V}(p_x-p_x')\psi(p_x',t) \\
=& \frac{p_x^2}{2m}\bar{\psi}(p_x,t) - \frac{1}{\sqrt{2\pi\hbar}}\int dp_x'\frac{\lambda}{\sqrt{2\pi\hbar}}\bar{\psi}(p_x',t) \\
=& \frac{p_x^2}{2m}\bar{\psi}(p_x,t) - \frac{\lambda}{2\pi\hbar}\int dp_x'\bar{\psi}(p_x',t)
\end{aligned} \tag{9}
$$

gives the stationary Schrödinger equation in the $\{|p_x\rangle\}$ representation

$$
E\bar{\psi}(p_x) = \hat{H}\bar{\psi}(p_x,t) = \frac{p_x^2}{2m}\bar{\psi}(p_x) - \frac{\lambda}{2\pi\hbar}\int dp_x'\bar{\psi}(p_x')
$$

Differentiating both sides of the equation above about the p_x gives

$$(E-\frac{p_x^2}{2m})\frac{d}{dp_x}\bar{\psi}(p_x)=\frac{p_x}{m}\psi(\bar{p}_x) \tag{10}$$

$$\Longrightarrow \frac{d\psi(\bar{p}_x)}{\psi(p_x)}=\frac{2p_x}{2mE-p_x^2}dp_x \tag{11}$$

Integral both sides of the equation above gives

$$\ln\psi(p_x)=-\ln(2mE-p_x^2)+C_1 \tag{12}$$

$$\Longrightarrow \psi(p_x)=\frac{C}{p_x^2-2mE} \tag{13}$$

where C is the normalization constant.
The normalization condition is

$$\begin{aligned}\int_{-\infty}^{+\infty}dp_x\bar{\psi}^*(p_x)p\bar{s}i(p_x)=&C^2\int dp_x\frac{1}{(p_x^2-2mE)^2}\\ =&2\pi iC^2\text{Res}\left[\frac{1}{(p_x^2-2mE)^2},i\sqrt{-2mE}\right]\\ =&2\pi iC^2\lim_{p\to i\sqrt{-2mE}}\frac{d}{dp_x}\frac{1}{(p_x+i\sqrt{-2mE})^2}\\ =&-2\pi iC^2\lim_{p\to i\sqrt{-2mE}}\frac{2}{(p_x+i\sqrt{-2mE})^3}\\ =&\frac{\pi C^2}{2(-2mE)^{3/2}}=1\end{aligned} \tag{14}$$

$$\Longrightarrow C=\left(\frac{2}{\pi}\right)^{1/2}(-2mE)^{3/4} \tag{15}$$

Therefore, the eigenfunction of a particle of mass m in the one-dimensional δ-function potential well is

$$\bar{\psi}(p_x)=\left(\frac{2}{\pi}\right)^{1/2}\frac{(-2mE)^{3/4}}{(p_x^2-2mE)} \tag{16}$$

Plugging the eigenfunction into the stationary Schrödinger equation gives the energy eigenvalue

$$\begin{aligned}
&\frac{p_x^2}{2m}\bar{\psi}(p_x)+\frac{\lambda}{2\pi\hbar}\int dp'_x\bar{\psi}(p'_x)\\
=&\frac{p_x^2}{2m}\left(\frac{2}{\pi}\right)^{1/2}\frac{(-2mE)^{3/4}}{(p_x^2-2mE)}+\frac{\lambda}{2\pi\hbar}\int dp'_x\left(\frac{2}{\pi}\right)^{1/2}\frac{(-2mE)^{3/4}}{(p_x'^2-2mE)}\\
=&\frac{p_x^2}{2m}\left(\frac{2}{\pi}\right)^{1/2}\frac{(-2mE)^{3/4}}{(p_x^2-2mE)}+\frac{\lambda(-2mE)^{3/4}}{2^{1/2}\pi^{3/2}\hbar}2\pi i\mathrm{Res}\left[\frac{1}{(p_x'^2-2mE)},i\sqrt{-2mE}\right]\\
=&\frac{p_x^2}{2m}\left(\frac{2}{\pi}\right)^{1/2}\frac{(-2mE)^{3/4}}{(p_x^2-2mE)}+\left(\frac{2}{\pi}\right)^{1/2}\frac{\lambda(-2mE)^{3/4}}{\hbar}i\lim_{p'_x\to i\sqrt{-2mE}}\frac{1}{(p'_x+i\sqrt{-2mE})}\\
=&\frac{p_x^2}{2m}\left(\frac{2}{\pi}\right)^{1/2}\frac{(-2mE)^{3/4}}{(p_x^2-2mE)}+\left(\frac{2}{\pi}\right)^{1/2}\frac{\lambda(-2mE)^{3/4}}{\hbar}i\lim_{p_x\to i\sqrt{-2mE}}\frac{1}{(p'_x+i\sqrt{-2mE})}\\
=&\frac{p_x^2}{2m}\left(\frac{2}{\pi}\right)^{1/2}\frac{(-2mE)^{3/4}}{(p_x^2-2mE)}+\left(\frac{2}{\pi}\right)^{1/2}\frac{\lambda(-2mE)^{1/4}}{2\hbar}\\
=&E\bar{\psi}(p_x)=E\left(\frac{2}{\pi}\right)^{1/2}\frac{(-2mE)^{3/4}}{(p_x^2-2mE)}
\end{aligned}\tag{17}$$

$$\Longrightarrow E=-\frac{m\lambda^2}{2\hbar^2}\tag{18}$$

□

Problem 4. In the $\{|\vec{p}\rangle\}$ representation, the wave function of a particle at a given time is given by $\bar{\psi}(\vec{p})=Ne^{-\alpha|\vec{p}|/\hbar}$ with $\alpha>0$. Find the value of the normalization constant N and the wave function $\psi(\vec{r})$ in the $\{|\vec{r}\rangle\}$ representation.

Solution: The normalization condition is

$$\begin{aligned}\int d^3p\bar{\psi}^*(\vec{p})\bar{\psi}(\vec{p}) =& N^2\int d^3pe^{-2\alpha|\vec{p}|/\hbar}\\=&N^2\int_0^{+\infty}p^2e^{-2\alpha p/\hbar}dp\int_0^{2\pi}d\phi\int_0^\pi\sin\theta d\theta\\=&-\frac{2\pi\hbar}{\alpha}N^2\int_0^{+\infty}p^2d\left(e^{-2\alpha p/\hbar}\right)\\=&-\frac{2\pi\hbar}{\alpha}N^2\left[\left(p^2e^{-2\alpha p/\hbar}\right)\Big|_0^{+\infty}-\int_0^{+\infty}e^{-2\alpha p/\hbar}dp^2\right]\\=&\frac{4\pi\hbar}{\alpha}N^2\int_0^{+\infty}pe^{-2\alpha p/\hbar}dp\\=&-\frac{2\pi\hbar^2}{\alpha^2}N^2\int_0^{+\infty}pd\left(e^{-2\alpha p/\hbar}\right)\\=&-\frac{2\pi\hbar^2}{\alpha^2}N^2\left[\left(pe^{-2\alpha p/\hbar}\right)\Big|_0^{+\infty}-\int_0^{+\infty}e^{-2\alpha p/\hbar}dp\right]\\=&\frac{2\pi\hbar^2}{\alpha^2}N^2\int_0^{+\infty}e^{-2\alpha p/\hbar}dp\\=&\frac{\pi\hbar^3}{\alpha^3}N^2=1\end{aligned}\tag{19}$$

$$\Longrightarrow N=\sqrt{\frac{\alpha^3}{\pi\hbar^3}}\tag{20}$$

The wave function in $\{|\vec{p}\rangle\}$ representation is

$$\bar{\psi}(\vec{p})=\sqrt{\frac{\alpha^3}{\pi\hbar^3}}e^{-\alpha|\vec{p}|/\hbar}\tag{21}$$

The wave function in $\{|\vec{r}\rangle\}$ representation is

$$\begin{aligned}\psi(\vec{r})=&\frac{1}{(2\pi\hbar)^{3/2}}\int d^3pe^{i\vec{p}\cdot\vec{r}/\hbar}\bar{\psi}(\vec{p})\\=&\frac{\alpha^{3/2}}{2^{3/2}\pi^2\hbar^3}\int d^3pe^{i\vec{p}\cdot\vec{r}/\hbar}e^{-\alpha|\vec{p}|/\hbar}\\=&\frac{\alpha^{3/2}}{2^{3/2}\pi^2\hbar^3}\int_0^{+\infty}\int_0^{2\pi}\int_0^\pi e^{ipr\cos\theta/\hbar}e^{-\alpha p/\hbar}p^2\sin\theta d\theta d\phi dp\\=&-\frac{\alpha^{3/2}}{2^{1/2}\pi\hbar^3}\int_0^{+\infty}\int_0^\pi e^{irp\cos\theta/\hbar}e^{-\alpha p/\hbar}p^2d\cos\theta dp\\=&\frac{2^{1/2}\alpha^{3/2}}{\pi\hbar^2r}\int_0^{+\infty}pe^{-\alpha p/\hbar}\sin\left(\frac{pr}{\hbar}\right)dp\end{aligned}$$

where

$$\begin{aligned}\int_0^{+\infty} pe^{-\alpha/\hbar}\sin\left(\frac{pr}{\hbar}\right)dp =&\mathrm{Im}\left[\int_0^{+\infty} pe^{-\alpha p/\hbar}e^{ipr/\hbar}dp\right]\\ =&\mathrm{Im}\left[\int_0^{+\infty} pe^{p(ir-\alpha)/\hbar}dp\right]\\ =&\mathrm{Im}\left[\frac{\hbar}{ir-\alpha}\int_0^{+\infty} pde^{p(ir-\alpha)/\hbar}\right]\\ =&\mathrm{Im}\left[\frac{\hbar}{ir-\alpha}\left(pe^{p(ir-\alpha)/\hbar}\right)\big|_0^{+\infty}-\frac{\hbar}{ir-\alpha}\int_0^{+\infty} e^{p(ir-\alpha)/\hbar}dp\right]\\ =&\mathrm{Im}\left[-\frac{\hbar^2}{(ir-\alpha)^2}e^{p(ir-\alpha)/\hbar}\bigg|_0^{+\infty}\right]\\ =&\mathrm{Im}\left[\frac{\hbar^2}{r^2-\alpha^2-2ir\alpha}\right]\\ =&\mathrm{Im}\left[\frac{\hbar^2(r^2-\alpha^2+2ir\alpha)}{r^4+\alpha^4+2r^2\alpha^2}\right]\\ =&\frac{2\hbar^2r\alpha}{(r^2+\alpha^2)^2}\end{aligned} \tag{22}$$

Therefore, the wavefunction $\psi(\vec{r})$ in the $\{|\vec{r}\rangle\}$ representation is

$$\psi(\vec{r}) = \frac{2^{3/2}\alpha^{5/2}}{\pi(r^2+\alpha^2)} \tag{23}$$

□

Problem 5. For a particle in one-dimensional space, find the expression of the operator $\hat{x}^{-1}=\frac{1}{\hat{x}}$ in the $\{|p_x\rangle\}$ representation and the expression of the operator $\hat{p}_x^{-1}=\frac{1}{\hat{p}_x}$ in the $\{|x\rangle\}$ representation.
Note that $\hat{x}^{-1}$ is the inverse of $\hat{x}$ and that $\hat{p}_x^{-1}$ is the inverse of $\hat{p}_x$.

Solution: Since

$$\hat{x}\hat{x}^{-1}=1 \tag{24}$$

$$\hat{x}\hat{x}^{-1}\bar{\psi}(p_x)=ih\frac{d}{dp_x}[\hat{x}^{-1}\bar{\psi}(p_x)]=\psi(\bar{p}_x) \tag{25}$$

$$\Longrightarrow \hat{x}^{-1}\bar{\psi}(p_x)=\frac{1}{i\hbar}\int_{-\infty}^{p_x}dp_x\bar{\psi}(p_x) \tag{26}$$

Therefore,

$$\hat{x}^{-1}=\frac{1}{i\hbar}\int_{-\infty}^{p_x}dp_x \tag{27}$$

Since

$$\hat{p}_x\hat{p}_x^{-1}=1 \tag{28}$$

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$$\hat{p}_x\hat{p}_x^{-1}\psi(x) = -i\hbar\frac{d}{dx}[\hat{p}_x^{-1}\psi(x)] = \psi(x) \tag{29}$$

$$\Longrightarrow \hat{p}_x^{-1}\psi(p_x) = -\frac{1}{i\hbar}\int_{-\infty}^{x} dx\psi(x) \tag{30}$$

Therefore,

$$\hat{p}_x^{-1} = -\frac{1}{i\hbar}\int_{-\infty}^{x} dx \tag{31}$$

□