Quantum Mechanics
Homework Assignment 11
Fall, 2019

1. **[C-T Exercise 5-1]** Consider a harmonic oscillator of mass m and angular frequency ω. At time $t=0$, the state of this oscillator is given by $|\psi(0)\rangle = \sum_n c_n |\varphi_n\rangle$, where the states $|\varphi_n\rangle$ are stationary states with energies $(n+1/2)\hbar\omega$.

   (a) What is the probability $\mathscr{P}$ that a measurement of the oscillator's energy performed at an arbitrary time $t>0$, will yield a result greater than $2\hbar\omega$? When $\mathscr{P}=0$, what are the non-zero coefficients c_n?

   (b) From now on, assume that only c_0 and c_1 are different from zero. Write normalization condition for $|\psi(0)\rangle$ and the mean value $\langle\hat{H}\rangle$ of the energy in terms of c_0 and c_1. With the additional requirement $\langle\hat{H}\rangle = \hbar\omega$, calculate $|c_0|^2$ and $|c_1|^2$.

   (c) As the normalized state vector $|\psi(0)\rangle$ is defined only to within a global phase factor, we fix this factor by choosing c_0 real and positive. We set $c_1 = |c_1|e^{i\theta_1}$. We assume that $\langle\hat{H}\rangle = \hbar\omega$ and that $\langle\hat{x}\rangle = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}$. Calculate θ_1.

   (d) With $|\psi(0)\rangle$ so determined, write $|\psi(t)\rangle$ for $t>0$ and calculate the value of θ_1 at t. Deduce the mean value $\langle\hat{x}\rangle(t)$ of the position at t.

2. **[C-T Exercise 5-3]** Two particles of the same mass m, with positions $\hat{x}_1$ and $\hat{x}_2$ and momenta $\hat{p}_1$ and $\hat{p}_2$, are subject to the same potential $\hat{V}(\hat{x}) = \frac{1}{2}m\omega^2\hat{x}^2$. The two particles do not interact.

   (a) Write the operator $\hat{H}$, the Hamiltonian of the two-particle system. Show that $\hat{H}$ can be written $\hat{H} = \hat{H}_1 + \hat{H}_2$, where $\hat{H}_1$ and $\hat{H}_2$ act respectively only in the state space of particle (1) and in that of particle (2). Calculate the energies of the two-particle system, their degrees of degeneracy, and the corresponding wave functions.

   (b) Does $\hat{H}$ form a CSCO? Same question for the set $\{\hat{H}_1, \hat{H}_2\}$. We denote by $|\Phi_{n_1 n_2}\rangle$ the eigenvectors common to $\hat{H}_1$ and $\hat{H}_2$. Write the orthonormality and closure relations for the states $|\Phi_{n_1 n_2}\rangle$.

   (c) Consider a system which, at $t=0$, is in the state

   $$|\psi(0)\rangle = \frac{1}{2}\big[\,|\Phi_{00}\rangle + |\Phi_{10}\rangle + |\Phi_{01}\rangle + |\Phi_{11}\rangle\,\big].$$

   If at this time one measures the total energy of the system or the energy of particle (1) or the position of particle (1) or the velocity of particle (1), what results can be found, and with what probabilities?

3. **[C-T Exercise 5-5] Continue from the previous problem.** We denote by $|\Phi_{n_1 n_2}\rangle$ the eigenstates common to $\hat{H}_1$ and $\hat{H}_2$, of eigenvalues $(n_1+1/2)\hbar\omega$ and $(n_2+1/2)\hbar\omega$. The "two particle exchange" operator $\hat{P}_e$ is defined by $\hat{P}_e|\Phi_{n_1 n_2}\rangle = |\Phi_{n_2 n_1}\rangle$.

   (a) Prove that $\hat{P}_e^{-1} = \hat{P}_e$, and that $\hat{P}_e$ is unitary. What are the eigenvalues of $\hat{P}_e$? Let $\hat{B}' = \hat{P}_e\hat{B}\hat{P}_e^\dagger$ be the observable resulting from the transformation by $\hat{P}_e$ of an arbitrary observable $\hat{B}$. Show that the condition $\hat{B}' = \hat{B}$ ($\hat{B}$ invariant under exchange of the two particles) is equivalent to $[\hat{B}, \hat{P}_e] = 0$.

   (b) Show that $\hat{P}_e\hat{H}_1\hat{P}_e^\dagger = \hat{H}_2$ and $\hat{P}_e\hat{H}_2\hat{P}_e^\dagger = \hat{H}_1$. Does $\hat{H}$ commute with $\hat{P}_e$? Calculate the action of $\hat{P}_e$ on the observables $\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2$.

   (c) Construct a basis of eigenvectors common to $\hat{H}$ and $\hat{P}_e$. Do these two operators form a CSCO? What happens to the spectrum of $\hat{H}$ and the degeneracy of its eigenvalues if one retains only the eigenvectors $|\Phi\rangle$ of $\hat{H}$ for which $\hat{P}_e|\Phi\rangle = -|\Phi\rangle$?

4. **[C-T Exercise 5-6]** A one-dimensional harmonic oscillator is composed of a particle of mass m, charge q, and potential energy $\hat{V}(\hat{x}) = \frac{1}{2}m\omega^2\hat{x}^2$. We assume that the particle is placed in an electric field $\mathscr{E}(t)$ parallel to Ox and time-dependent, so that to $\hat{V}(\hat{x})$ must be added the potential energy $\hat{W}(t) = -q\mathscr{E}(t)\hat{x}$.

(a) Write the Hamiltonian $\hat{H}(t)$ of the particle in terms of the operators $\hat{a}$ and $\hat{a}^\dagger$. Calculate the commutators of $\hat{a}$ and $\hat{a}^\dagger$ with $\hat{H}(t)$.

(b) Let $\alpha(t)$ be the number defined by $\alpha(t) = \langle\psi(t)|\hat{a}|\psi(t)\rangle$, where $|\psi(t)\rangle$ is the normalized state vector of the particle under study. Show from the results of the preceding question that $\alpha(t)$ satisfies the differential equation $\frac{d}{dt}\alpha(t) = -i\omega\alpha(t) + i\lambda(t)$, where $\lambda(t)$ is defined by $\lambda(t) = \frac{q}{\sqrt{2m\hbar\omega}}\mathscr{E}(t)$. Integrate this differential equation. At time t, what are the mean values of the position and momentum of the particle?

(c) The ket $|\varphi(t)\rangle$ is defined by $|\varphi(t)\rangle = \left[\,\hat{a} - \alpha(t)\,\right]|\psi(t)\rangle$, where $\alpha(t)$ has the value calculated in (b). Using the results of questions (a) and (b), show that the evolution of $|\varphi(t)\rangle$ is given by $i\hbar\frac{d}{dt}|\varphi(t)\rangle = \left[\,\hat{H}(t) + \hbar\omega\,\right]|\varphi(t)\rangle$. How does the norm of $|\varphi(t)\rangle$ vary with time?

(d) Assuming that $|\psi(0)\rangle$ is an eigenvector of $\hat{a}$ with the eigenvalue $\alpha(0)$, show that $|\psi(t)\rangle$ is also an eigenvector of $\hat{a}$, and calculate its eigenvalue. Find at time t the mean value of the unperturbed Hamiltonian $\hat{H}_0 = \hat{H}(t) - \hat{W}(t)$ as a function of $\alpha(0)$. Give the root-mean-square deviations Δx, Δp, and ΔH_0; how do they vary with time?

(e) Assume that at $t = 0$, the oscillator is in the ground state $|\varphi(0)\rangle$. The electric field acts between times 0 and T and then falls to zero. When $t > T$, what is the evolution of the mean values $\langle\hat{x}\rangle(t)$ and $\langle\hat{p}\rangle(t)$? Application: Assume that between 0 and T, the field $\mathscr{E}(t)$ is given by $\mathscr{E}(t) = \mathscr{E}_0\cos(\omega' t)$; discuss the phenomena observed (resonance) in terms of $\Delta\omega = \omega' - \omega$. If, at $t > T$, the energy is measured, what results can be found, and with what probabilities?

5. **[C-T Exercise 6-6]** Consider a system of angular momentum $\ell = 1$. A basis of its state space is formed by the three eigenvectors of $\hat{L}_z$: $|+1\rangle$, $|0\rangle$, $|-1\rangle$, whose eigenvalues are, respectively, $+\hbar$, 0, and $-\hbar$, and which satisfy $\hat{L}_\pm|m\rangle = \hbar\sqrt{2}\,|m \pm 1\rangle$, $\hat{L}_+|+1\rangle = \hat{L}_-|-1\rangle = 0$. This system, which possesses an electric quadrupole moment, is placed in an electric field gradient, so that its Hamiltonian can be written $\hat{H} = \frac{\omega_0}{\hbar}\left(\hat{L}_u^2 - \hat{L}_v^2\right)$, where $\hat{L}_u$ and $\hat{L}_v$ are the components of $\hat{\vec{L}}$ along the two directions Ou and Ov of the xOz plane which form angles of 45° with Ox and Oz; ω_0 is a real constant.

(a) Write the matrix which represents $\hat{H}$ in the $\{|+1\rangle, |0\rangle, |-1\rangle\}$ basis. What are the stationary states of the system, and what are their energies? (These states are to be written $|E_1\rangle$, $|E_2\rangle$, $|E_3\rangle$, in order of decreasing energies.)

(b) At time $t = 0$, the system is in the state $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left[\,|+1\rangle - |-1\rangle\,\right]$. What is the state vector $|\psi(t)\rangle$ at time t? At t, $\hat{L}_z$ is measured; what are the probabilities of the various possible results?

(c) Calculate the mean values $\langle\hat{L}_x\rangle(t)$, $\langle\hat{L}_y\rangle(t)$, and $\langle\hat{L}_z\rangle(t)$ at t. What is the motion performed by the vector $\langle\hat{\vec{L}}\rangle$?

(d) At t, a measurement of $\hat{L}_z^2$ is performed.

i. Do times exist when only one result is possible?

ii. Assume that this measurement has yielded the result $\hbar^2$. What is the state of the system immediately after the measurement? Indicate, without calculation, its subsequent evolution.