Quantum Mechanics

Solutions to the Problems in Homework Assignment 01

Fall, 2019

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1. The scattering of a photon by an energetic electron can result in a transfer of energy from the electron to the photon. When this is the case, the scattering process is known as *the inverse Compton scattering*. Assume that, before the scattering, the wavelength of the photon is λ and the speed of the electron is v that is a relativistic speed. Also assume that the momentum vectors of the electron and photon are all in the same plane before and after the scattering.

   (a) Find an expression for the wavelength shift of the photon in the inverse Compton scattering.

   (b) What is the condition under which the inverse Compton scattering occurs.

(a) Let λ' and v' denote respectively the wavelength of the photon and the speed of the electron after the scattering. The scattering angles of the photon and the electron are respectively denoted by θ and φ as shown in Fig. 1.

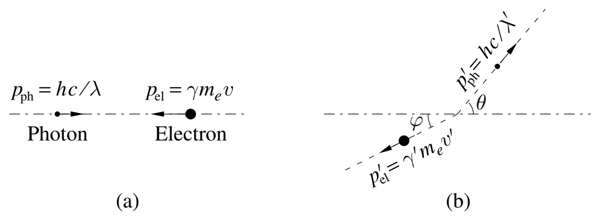

FIG. 1: Inverse Compton scattering. (a) Before the scattering. A photon moves toward an energetic electron. (b) After the scattering.

The collision between the photon and the electron is taken to be elastic. Hence, the momentum and the energy are conserved. We will make use of the conservation of momentum and energy to solve this problem. From the conservation of momentum components in the parallel and perpendicular directions with respect to the direction of relative motion before the collision, we have

$$\gamma m_e v - h/\lambda = \gamma' m_e v' \cos\varphi - (h/\lambda')\cos\theta, \tag{1a}$$

$$0 = (h/\lambda')\sin\theta - \gamma' m_e v' \sin\varphi, \tag{1b}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$ and $\gamma' = 1/\sqrt{1 - v'^2/c^2}$. From the above two equations, we eliminate φ and obtain the following relation between λ' and v'

$$(\gamma m_e v - h/\lambda)^2 = (\gamma'^2 - 1)m_e^2 c^2 - (h/\lambda')^2 - 2(h/\lambda')(\gamma m_e v - h/\lambda)\cos\theta, \tag{2}$$

where we have made use of $(\gamma' v')^2 = (\gamma'^2 - 1)c^2$. From the conservation of energy, we have

$$hc/\lambda + \gamma m_e c^2 = hc/\lambda' + \gamma' m_e c^2. \tag{3}$$

Eliminating γ' from Eqs. (2) and (3), we obtain

$$\Delta\lambda \equiv \lambda' - \lambda = -\frac{h}{\gamma m_e c(1 + v/c)}\left(\frac{\lambda\gamma m_e v}{h} - 1\right)(1 - \cos\theta). \tag{4}$$

Note that we recover the result for the wavelength shift for the electron initially at rest (Compton's formula) if we set $v = 0$ in the above equation.

(b) We can easily deduce from Eq. 4 the condition on the speed of the electron for the inverse Compton scattering to occur. Demanding that $\Delta\lambda < 0$, we have

$$v > \frac{h}{\lambda\gamma m_e}. \tag{5}$$

From the above equation, we see that the speed of the electron must be higher than $h/\lambda\gamma m_e$ for the inverse Compton scattering to occur. If we use the momentum instead of the speed, we can put the condition for the occurrence of the inverse Compton scattering in a more compact form. Realizing that $\gamma m_e v$ is the magnitude of the momentum of the electron before the scattering and that h/λ is the magnitude of the momentum of the photon before the scattering, we can rewrite Eq. 5 as

$$p_{\rm el} > p_{\rm ph}, \tag{6}$$

that is, the initial momentum of the electron must be greater than the initial momentum of the photon for the inverse Compton scattering to occur.

General treatment of the inverse Compton scattering

Here we consider the case in which the electron and the photon do not move in the opposite direction before their collision as shown in the following figure.

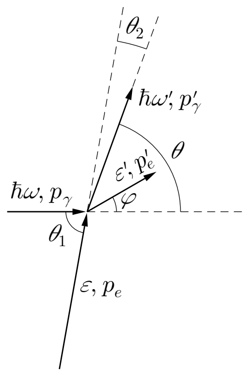

FIG. 2: General treatment of the inverse Compton scattering.

The symbols we will use are given in the following.

$\vec{p}_\gamma$: The momentum of the photon before the collision;

$\vec{p}_e$: The momentum of the electron before the collision;

$\vec{p}'_\gamma$: The momentum of the photon after the collision;

$\vec{p}'_e$: The momentum of the electron after the collision;

$\hbar\omega$: The energy of the photon before the collision;

ε: The energy of the electron before the collision;

$\hbar\omega'$: The energy of the photon after the collision;

ε': The energy of the electron after the collision;

$\theta_1 = \angle(\vec{p}_e, \vec{p}_\gamma)$;

$\theta_2 = \angle(\vec{p}'_\gamma, \vec{p}_e)$;

$\theta = \angle(\vec{p}'_\gamma, \vec{p}_\gamma) = \theta_1 - \theta_2$;

$\phi = \angle(\vec{p}'_e, \vec{p}_\gamma)$.

From the conservation of momentum, we have

$$\begin{aligned} p_\gamma + p_e \cos\theta_1 &= p'_\gamma \cos\theta + p'_e \cos\phi, \\ p_e \sin\theta_1 &= p'_\gamma \sin\theta + p'_e \sin\phi. \end{aligned}$$

Eliminating ϕ from the above two equations yields

$${p'_e}^2 = p_\gamma^2 + p_e^2 + {p'_\gamma}^2 - 2p_\gamma p'_\gamma \cos\theta + 2p_e p_\gamma \cos\theta_1 - 2p_e p'_\gamma \cos\theta_2.$$

We now express the momenta in terms of the energies for the electron using the momentum-energy relation $E^2 = p^2c^2 + m_e^2c^4$. For p_e^2, we have

$$p_e^2 = \frac{1}{c^2}\varepsilon^2 - m_e^2c^2.$$

For ${p_e'}^2$, we have

$${p_e'}^2 = \frac{1}{c^2}{\varepsilon'}^2 - m_e^2c^2 = \frac{1}{c^2}(\hbar\omega - \hbar\omega' + \varepsilon)^2 - m_e^2c^2,$$

where we have made use of the conservation of energy

$$\hbar\omega + \varepsilon = \hbar\omega' + \varepsilon'.$$

For the photon, we have

$$p_\gamma = \frac{\hbar\omega}{c},\ p_\gamma' = \frac{\hbar\omega'}{c}.$$

Comparing the above-obtained two expressions for ${p_e'}^2$ with use of the relations between the momenta and the energies, we have

$$\begin{aligned}
&\frac{1}{c^2}(\hbar\omega)^2 + \frac{1}{c^2}\varepsilon^2 - m_e^2c^2 + \frac{1}{c^2}(\hbar\omega')^2 - \frac{2}{c^2}(\hbar\omega)(\hbar\omega')\cos\theta + 2p_e\frac{\hbar\omega}{c}\cos\theta_1 - 2p_e\frac{\hbar\omega'}{c}\cos\theta_2 \\
&\quad = \frac{1}{c^2}(\hbar\omega - \hbar\omega' + \varepsilon)^2 - m_e^2c^2 \\
&\quad = \frac{1}{c^2}(\hbar\omega)^2 + \frac{1}{c^2}(\hbar\omega')^2 + \frac{1}{c^2}\varepsilon^2 + \frac{2}{c^2}\varepsilon(\hbar\omega) - \frac{2}{c^2}\varepsilon(\hbar\omega') - \frac{2}{c^2}(\hbar\omega)(\hbar\omega') - m_e^2c^2
\end{aligned}$$

from which it follows that

$$-\frac{2}{c^2}(\hbar\omega)(\hbar\omega')\cos\theta + 2p_e\frac{\hbar\omega}{c}\cos\theta_1 - 2p_e\frac{\hbar\omega'}{c}\cos\theta_2 = \frac{2}{c^2}\varepsilon(\hbar\omega) - \frac{2}{c^2}\varepsilon(\hbar\omega') - \frac{2}{c^2}(\hbar\omega)(\hbar\omega').$$

Multiplying both sides of the above equation with $c^2/2$, we have

$$-(\hbar\omega)(\hbar\omega')\cos\theta + p_ec(\hbar\omega)\cos\theta_1 - p_ec(\hbar\omega')\cos\theta_2 = \varepsilon(\hbar\omega) - \varepsilon(\hbar\omega') - (\hbar\omega)(\hbar\omega').$$

Solving for $\hbar\omega'$, we obtain

$$\hbar\omega' = \frac{\varepsilon - p_ec\cos\theta_1}{\varepsilon - p_ec\cos\theta_2 + \hbar\omega(1-\cos\theta)}\hbar\omega.$$

Utilizing

$$\frac{p_e}{\varepsilon} = \frac{\gamma m_e v}{\gamma m_e c^2} = \frac{v}{c^2} = \frac{\beta}{c}$$

with $\beta = v/c$, we have

$$\hbar\omega' = \frac{1 - \beta\cos\theta_1}{1 - \beta\cos\theta_2 + (\hbar\omega/\varepsilon)(1-\cos\theta)}\hbar\omega.$$

Making use of

$$\hbar\omega = \frac{hc}{\lambda},\ \hbar\omega' = \frac{hc}{\lambda'},$$

we have

$$\lambda' = \frac{1 - \beta\cos\theta_2 + (\hbar\omega/\varepsilon)(1-\cos\theta)}{1 - \beta\cos\theta_1}\lambda.$$

If the electron is at rest before the collision, we have $\beta = 0$ and $\varepsilon = m_e c^2$. We then have

$$\lambda' = \lambda + \frac{h}{m_e c}(1 - \cos\theta)$$

which is the result for the normal Compton scattering for which λ' is always not smaller than λ.
The change in wavelength is given by

$$\begin{aligned}
\Delta\lambda = \lambda' - \lambda &= \lambda \frac{\beta(\cos\theta_1 - \cos\theta_2) + (\hbar\omega/\varepsilon)(1-\cos\theta)}{1 - \beta\cos\theta_1} \\
&= \lambda \frac{-2\beta \sin\big[(\theta_1+\theta_2)/2\big] \sin\big[(\theta_1 - \theta_2)/2\big] + (2\hbar\omega/\varepsilon)\sin^2(\theta/2)}{1-\beta\cos\theta_1} \\
&= -2\lambda \frac{\beta\sin(\theta_1 - \theta/2)\sin(\theta/2) - (\hbar\omega/\varepsilon)\sin^2(\theta/2)}{1-\beta\cos\theta_1} \\
&= -2\lambda\sin^2(\theta/2)\frac{\beta[\sin\theta_1\cot(\theta/2) - \cos\theta_1] - (h/\gamma\lambda m_e c)}{1-\beta\cos\theta_1}.
\end{aligned}$$

Here $\gamma = 1/\sqrt{1-\beta^2}$. From the above expression of $\Delta\lambda$, we see that, for the inverse Compton scattering to occur, we must have

$$\sin\theta_1 \cot\left(\frac{\theta}{2}\right) - \cos\theta_1 > \frac{h}{\beta\gamma\lambda m_e c} = \frac{p_\gamma}{p_e}.$$

The above condition constrains the momenta of the photon and the electron before the collision in a complicated manner that depends on the angles θ_1 and θ (or θ_1 and θ_2). In the case that the momenta of the photon and the electron before the collision are perpendicular, we have

$$\cot\left(\frac{\theta}{2}\right) > \frac{p_\gamma}{p_e}. \quad (\vec{p}_\gamma \perp \vec{p}_e)$$

2. The average energy of a mode in Planck's quantized theory of radiation in a cavity is given by

$$\overline{E} = \frac{\displaystyle\sum_{n=0}^{\infty} nh\nu e^{-nh\nu/k_B T}}{\displaystyle\sum_{n=0}^{\infty} e^{-nh\nu/k_B T}}.$$

   (a) Calculate the value of the sum in the denominator by rewriting it as a geometric series of the form $\sum_{n=0}^{\infty} q^n$ for some value of q (to be determined) and then evaluating the sum using the standard formula for an infinite geometric series.

   (b) Prove the identity

$$\sum_{n=0}^{\infty} ne^{-nh\nu/k_B T} = -\frac{k_B T}{h}\frac{\partial}{\partial\nu}\sum_{n=0}^{\infty} e^{-nh\nu/k_B T}.$$

   Using the answer to the previous part, further prove that

$$\sum_{n=0}^{\infty} ne^{-nh\nu/k_B T} = \frac{e^{h\nu/k_B T}}{(e^{h\nu/k_B T} - 1)^2}.$$

   (c) Find an explicit expression for $\overline{E}$.

   (d) Find an expression for the spectral energy density u_ν with $u_\nu d\nu$ equal to the energy per unit volume in the frequency interval $(\nu, \nu + d\nu)$, given that the number of modes per unit volume in the frequency interval $(\nu, \nu + d\nu)$ is $D(\nu)d\nu = 8\pi\nu^2 d\nu/c^3$.

(e) The relationship between the spectral irradiance E_ν and the spectral energy density u_ν is given by $E_\nu = cu_\nu/4$. Derive Planck's law expressed in terms of E_ν.

(a) Let $q = e^{-h\nu/k_BT}$. We see that $q < 1$ for $\nu > 0$. In terms of q, we can rewrite the denominator $\sum_{n=0}^{\infty} e^{-nh\nu/k_BT}$ as $\sum_{n=0}^{\infty} q^n$. Evaluating the sum using the standard formula for an infinite geometric series, we have

$$\sum_{n=0}^{\infty} e^{-nh\nu/k_BT} = \sum_{n=0}^{\infty} q^n = \frac{1}{1-q} = \frac{1}{1-e^{-h\nu/k_BT}}. \tag{7}$$

Differentiating $\sum_{n=0}^{\infty} e^{-nh\nu/k_BT}$ with respect to ν, we have

$$\frac{\partial}{\partial\nu}\sum_{n=0}^{\infty} e^{-nh\nu/k_BT} = \sum_{n=0}^{\infty}\frac{\partial e^{-nh\nu/k_BT}}{\partial\nu} = -\frac{h}{k_BT}\sum_{n=0}^{\infty} ne^{-nh\nu/k_BT}.$$

We thus have

$$\sum_{n=0}^{\infty} ne^{-nh\nu/k_BT} = -\frac{k_BT}{h}\frac{\partial}{\partial\nu}\sum_{n=0}^{\infty} e^{-nh\nu/k_BT}. \tag{8}$$

From Eqs. (8) and (7), we have

$$\begin{aligned}\sum_{n=0}^{\infty} ne^{-nh\nu/k_BT} &= -\frac{k_BT}{h}\frac{\partial}{\partial\nu}\sum_{n=0}^{\infty} e^{-nh\nu/k_BT} \\ &= -\frac{k_BT}{h}\frac{\partial}{\partial\nu}\frac{1}{1-e^{-h\nu/k_BT}} \\ &= \frac{e^{-h\nu/k_BT}}{(1-e^{-h\nu/k_BT})^2} = \frac{e^{h\nu/k_BT}}{(e^{h\nu/k_BT}-1)^2}. \end{aligned} \tag{9}$$

(b) Having evaluated both the denominator and the numerator in the definition of $\overline{E}$, we can now write down an explicit expression for $\overline{E}$. We have

$$\begin{aligned}\overline{E} &= \frac{\sum_{n=0}^{\infty} nh\nu e^{-nh\nu/k_BT}}{\sum_{n=0}^{\infty} e^{-nh\nu/k_BT}} = \frac{h\nu e^{h\nu/k_BT}(1-e^{-h\nu/k_BT})}{(e^{h\nu/k_BT}-1)^2} \\ &= \frac{h\nu}{e^{h\nu/k_BT}-1}. \end{aligned} \tag{10}$$

(c) The energy density $u_\nu d\nu$ in the frequency interval $(\nu, \nu + d\nu)$ in the cavity is given by the product of $\overline{E}$ and the number of modes per unit volume in the frequency interval $(\nu, \nu + d\nu)$. We have

$$u_\nu d\nu = \overline{E}\cdot D(\nu)d\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/k_BT}-1}d\nu.$$

From the above equation, we see that u_ν is given by

$$u_\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/k_BT}-1}. \tag{11}$$

(d) Making use of the fact that the spectral irradiance E_ν is related to the energy density per unit frequency u_ν through $E_\nu = cu_\nu/4$, we have

$$E_\nu = \frac{c}{4}u_\nu = \frac{2\pi h\nu^3}{c^2}\frac{1}{e^{h\nu/k_BT}-1}. \tag{12}$$

The above expression is Planck's law in terms of frequency ν.

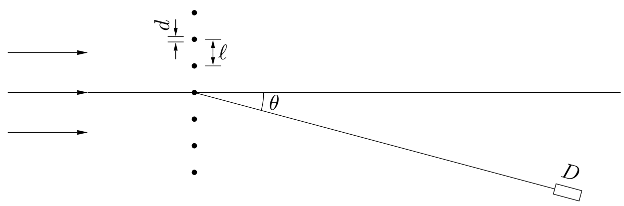

FIG. 3: Diffraction of neutrons on a chain of atoms.

3. A beam of neutrons of constant velocity, mass M_n ($M_n \approx 1.67 \times 10^{-27}$ kg) and energy E, is incident on a linear chain of atomic nuclei, arranged in a regular fashion as shown in the figure (these nuclei could be, for example, those of a long linear molecule). We call ℓ the distance between two consecutive nuclei, and d, their size ($d \ll \ell$). A neutron detector D is placed far away, in a direction which makes an angle of θ with the direction of the incident neutrons.

   (a) Describe qualitatively the phenomena observed at D when the energy E of the incident neutrons is varied.

   (b) The counting rate, as a function of E, presents a resonance about $E = E_1$. Knowing that there are no other resonances for $E < E_1$, show that one can determine ℓ. Calculate ℓ for $\theta = 30^{\rm o}$ and $E_1 = 1.3 \times 10^{-20}$ J.

   (c) At about what value of E must we begin to take the finite size of the nuclei into account?

(a) When the energy of the incident neutrons is varied, the wavelength of the neutrons is varied according to

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2M_n T}}$$

with T the kinetic energy or

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{E^2/c^2 - M_n^2 c^2}}$$

with E the total energy, depending on whether the energy of the neutrons is in the non-relativistic regime or in the relativistic regime.

- When the energy of the neutrons is extremely high so that $\lambda \ll d$, no diffraction of the neutrons on the nuclei (no resonances) will be observed.
- When the energy of the neutrons is so high that $\lambda \sim d$, diffraction of the neutrons on the nuclei (resonances) will be observed.
- When the energy of the neutrons is high enough so that $d \ll \lambda \ll \ell$, no resonances will be observed at the detector D.
- When the energy of the neutrons is decreased to such a value so that $\lambda \sim \ell$, diffraction of the neutrons by the slits between nuclei (resonances) will be observed at the detector D with the observed diffraction pattern similar to that of the grating diffraction of light.
- When the energy of the neutrons is very low so that $\lambda \gg \ell$, the only resonance occurring at $\theta = 0$ will be observed at the detector D.

(b) A resonance at the detector D at an angle of θ implies that a diffraction maximum appears at the angle θ. The intensity at a maximum is proportional to the counting rate. Since no other resonances occur for $E < E_1$, the resonance appearing at θ is of the first order. From the grating equation

$$\ell \sin\theta_m = m\lambda,$$

we have for $m = 1$

$$\ell = \frac{\lambda}{\sin\theta_1} = \frac{h}{\sqrt{2M_n E}\sin\theta_1}.$$

For $\theta_1 = \theta = 30^{\rm o}$ and $E = E_1 = 1.3 \times 10^{-20}$ J, we have

$$\ell = \frac{6.626 \times 10^{-34}}{\sqrt{2 \times 1.67 \times 10^{-27} \times 1.3 \times 10^{-20}}\sin 30^{\rm o}} \approx 2.01 \times 10^{-10}\ \text{m} = 0.201\ \text{nm}.$$

(c) When the wavelength λ of the neutrons is of the order of the size of the nuclei, we must take the finite size of the nuclei into account. We take the size of the nuclei to be of the order of 1 fm so that we have $\lambda \sim 1$ fm $= 10^{-15}$ m. For such a short wavelength, we expect the relativistic effect to be substantial. Making use of the expression for the relativistic momentum

$$p = \sqrt{E^2/c^2 - M_n^2 c^2}$$

with E the total energy, we have

$$\lambda = \frac{h}{\sqrt{E^2/c^2 - M_n^2 c^2}}.$$

Solving for E from the above equation yields

$$E = c\sqrt{h^2/\lambda^2 + M_n^2 c^2}.$$

The kinetic energy T of the neutrons is then given by

$$T = E - M_n c^2 = c\sqrt{h^2/\lambda^2 + M_n^2 c^2} - M_n c^2 \approx 9.87 \times 10^{-11}.$$

The velocity of such neutrons is given by

$$v = \frac{c}{\sqrt{1 + M_n^2 c^2 \lambda^2 / h^2}} \approx 2.39 \times 10^8 \text{ m/s} \approx 0.80c.$$

From the above value of the velocity, we see that the neutrons with such a short wavelength must be indeed treated relativistically.

4. Consider the motion of a particle of mass m in a one-dimensional harmonic potential $U(x) = m\omega^2 x^2/2$. As an approximation, the momentum p in the de Broglie relation $p = h/\lambda$ for the particle in the harmonic potential can be taken as $\sqrt{2m\overline{K}}$, where $\overline{K}$ is the average of the kinetic energy given by $\overline{K} = E/2$ with E the mechanical energy of the particle.

(a) If the mechanical energy of the particle is E, what are the two extremal points that the particle can reach according to Newtonian mechanics?

(b) From the condition that the matter wave of the particle must be fitted between the two extremal points, determine the allowed values of the energy of the particle. The numerical factor $4\sqrt{2}$ can be set approximately equal to 2π in the final result.

(a) The extremal points that the particle can reach are to be determined from the fact that the mechanical energy of the particle is equal to its potential energy at these extremal points. We have

$$\frac{1}{2} m\omega^2 x^2 = E$$

from which we obtain

$$x_1 = -\sqrt{\frac{2E}{m\omega^2}}, \quad x_2 = \sqrt{\frac{2E}{m\omega^2}}. \tag{13}$$

These two extremal points are called *classical turning points*. Note that $\sqrt{2E/m\omega^2}$ is the amplitude of a classical harmonic oscillator with mechanical energy E.

(b) From the fact that the distance between two successive nodes of a standing wave is $\lambda/2$ with λ the wavelength of the standing wave, we see that, in order to fit the matter wave of the particle between the two extremal points, we must have

$$n\frac{\lambda}{2} = x_2 - x_1 = 2\sqrt{\frac{2E}{m\omega^2}}, \quad n = 1, 2, 3, \cdots.$$

From the above result, the de Broglie relation $p = h/\lambda$, and the given condition that $p = \sqrt{2mK} = \sqrt{mE}$, we have

$$\frac{4}{n}\sqrt{\frac{2E}{m\omega^2}} = \lambda = \frac{h}{p} = \frac{h}{\sqrt{mE}}.$$

Solving for E from the above equation, we obtain

$$E_n = \frac{1}{4\sqrt{2}} nh\omega \approx n\hbar\omega, \;\; n = 1, 2, 3, \cdots , \tag{14}$$

where we have set $4\sqrt{2} \approx 2\pi$ ($4\sqrt{2} \approx 0.9 \times 2\pi$).

5. Consider two quantum states described by the following wave functions

$$\psi_1(x,0) = \left(\frac{2}{\pi a^2}\right)^{1/4} e^{-(x-a)^2/a^2},$$
$$\psi_2(x,0) = \left(\frac{2}{\pi a^2}\right)^{1/4} e^{-(x+a)^2/a^2}.$$

Let the state of a free particle in one-dimensional space be given at time $t = 0$ by

$$\psi(x,0) = A\big[\psi_1(x,0) - \psi_2(x,0)\big],$$

where A is a normalization constant.

(a) Without performing any calculation, deduce an exact value for the probability density $|\psi(0,t)|^2$ for finding the particle near $x = 0$ at an arbitrary time t.

(b) Find the value of the normalization constant A.

(c) Without solving exactly for $\psi(x,t)$, find an exact expression for the probability density $|\psi(x,t)|^2$ for finding the particle near any point x at an arbitrary time t.

(a) Note that $\psi_1(x,0)$ and $\psi_2(x,0)$ are Gaussian wave packets with centers respectively at $x = a$ and $x = -a$ and that they take on the same value at $x = 0$. Therefore, $\psi(0,0) = A\big[\psi_1(0,0) - \psi_2(0,0)\big] = 0$. As time develops, the two wave packets expand. Because the expansion of the two wave packets is symmetric about their centers, the values of $\psi_1(0,t)$ and $\psi_2(0,t)$ remain to be the same at any arbitrary time $t > 0$. We thus have $\psi(0,t) = A\big[\psi_1(0,t) - \psi_2(0,t)\big] = 0$ and $|\psi(0,t)|^2 = 0$.

(b) From the normalization condition, we have

$$\begin{aligned}1 &= \int_{-\infty}^{\infty} |\psi(x,0)|^2 dx = |A|^2 \int_{-\infty}^{\infty} |\psi_1(x,0) - \psi_2(x,0)|^2 dx \\ &= |A|^2 \left(\frac{2}{\pi a^2}\right)^{1/2} \int_{-\infty}^{\infty} \left| e^{-(x-a)^2/a^2} - e^{-(x+a)^2/a^2} \right|^2 dx.\end{aligned}$$

Simplifying yields

$$1 = 2|A|^2 \left(\frac{2}{\pi a^2}\right)^{1/2} \left(1 - e^{-2}\right) \int_{-\infty}^{\infty} e^{-2x^2/a^2} dx.$$

Making use of the integral formula

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}},$$

we have

$$1 = 2|A|^2 \big(1 - e^{-2}\big)$$

from which we obtain $|A| = 1/\sqrt{2\big(1-e^{-2}\big)}$. We take

$$A = \frac{1}{\sqrt{2\big(1-e^{-2}\big)}} = \frac{e}{\sqrt{2\big(e^2-1\big)}}. \tag{15}$$

(c) To obtain the time dependence of the wave function of the superimposed state, let us first find the wave functions of the two quantum states in momentum space. For quantum state 1, we have

$$\begin{aligned}\varphi_1(p) &= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\psi_1(x)e^{-ipx/\hbar}dx \\ &= \left(\frac{1}{2\pi^3a^2\hbar^2}\right)^{1/4}\int_{-\infty}^{\infty}e^{-(x-a)^2/a^2-ipx/\hbar}dx \\ &= \left(\frac{1}{2\pi^3a^2\hbar^2}\right)^{1/4}e^{-ipa/\hbar}\int_{-\infty}^{\infty}e^{-x^2/a^2-ipx/\hbar}dx,\end{aligned}$$

where we have changed the integration variables from x to $x-a$ with the latter denoted by x afterwards. Utilizing Euler's formula, $e^{iz} = \cos(z) + i\sin(z)$, we have

$$\varphi_1(p) = \left(\frac{1}{2\pi^3a^2\hbar^2}\right)^{1/4}e^{-ipa/\hbar}\int_{-\infty}^{\infty}e^{-x^2/a^2}\cos(px/\hbar)dx$$

in which the integral of the term containing $\sin(px/\hbar)$ vanishes because the integrand is an odd function of x. Making use of the integral formula

$$\int_{-\infty}^{\infty}e^{-\alpha x^2}\cos(\beta x)dx = \sqrt{\frac{\pi}{\alpha}}\,e^{-\beta^2/4\alpha}, \tag{16}$$

we have

$$\varphi_1(p) = \left(\frac{a^2}{2\pi\hbar^2}\right)^{1/4}e^{-p^2a^2/4\hbar^2-ipa/\hbar}.$$

For quantum state 2, we similarly have

$$\varphi_2(p) = \left(\frac{a^2}{2\pi\hbar^2}\right)^{1/4}e^{-p^2a^2/4\hbar^2+ipa/\hbar}.$$

Note that $\varphi_1(p)$ and $\varphi_2(p)$ describe states with definite momentum. Since the particle is free, the time dependence of $\varphi_1(p)$ or $\varphi_2(p)$ is simply given by the factor $e^{-ip^2t/2m\hbar}$ with m the mass of the particle. We thus have the following time-dependent wave functions in momentum space

$$\begin{aligned}\varphi_1(p,t) &= \left(\frac{a^2}{2\pi\hbar^2}\right)^{1/4}e^{-p^2(a^2/4\hbar^2+it/2m\hbar)-ipa/\hbar}, \\ \varphi_2(p,t) &= \left(\frac{a^2}{2\pi\hbar^2}\right)^{1/4}e^{-p^2(a^2/4\hbar^2+it/2m\hbar)+ipa/\hbar}.\end{aligned}$$

We can now obtain $\psi_1(x,t)$ and $\psi_2(x,t)$ respectively from $\varphi_1(p,t)$ and $\varphi_2(p,t)$. For $\psi_1(x,t)$, we have

$$\begin{aligned}\psi_1(x,t) &= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\varphi_1(p,t)e^{ipx/\hbar}dp \\ &= \left(\frac{a^2}{8\pi^3\hbar^4}\right)^{1/4}\int_{-\infty}^{\infty}e^{-p^2(a^2/4\hbar^2+it/2m\hbar)+ip(x-a)/\hbar}\,dp \\ &= \left(\frac{a^2}{8\pi^3\hbar^4}\right)^{1/4}\int_{-\infty}^{\infty}e^{-p^2(a^2/4\hbar^2+it/2m\hbar)}\cos[\,p(x-a)/\hbar\,]\,dp.\end{aligned}$$

Making use of the integral formula in Eq. 16, we have

$$\psi_1(x,t) = \left(\frac{2a^2}{\pi}\right)^{1/4} \frac{e^{-(x-a)^2/(a^2+2i\hbar t/m)}}{\left(a^2+2i\hbar t/m\right)^{1/2}}.$$

For $\psi_2(x,t)$, we similarly have

$$\psi_2(x,t) = \left(\frac{2a^2}{\pi}\right)^{1/4} \frac{e^{-(x+a)^2/(a^2+2i\hbar t/m)}}{\left(a^2+2i\hbar t/m\right)^{1/2}}$$

Finally, $|\psi(x,t)|^2 = |A|^2|\psi_1(x,t) - \psi_2(x,t)|^2$ is given by

$$\begin{aligned}
|\psi(x,t)|^2 &= |A|^2 \left(\frac{2a^2}{\pi}\right)^{1/2} \left|\frac{1}{\left(a^2+2i\hbar t/m\right)^{1/2}}\right|^2 \\
&\quad \times \left|e^{-(x-a)^2/(a^2+2i\hbar t/m)} - e^{-(x+a)^2/(a^2+2i\hbar t/m)}\right|^2 \\
&= \frac{2e^2}{e^2-1}\left(\frac{2a^2}{\pi}\right)^{1/2} \frac{e^{-2a^2(x^2+a^2)/[a^4+(2\hbar t/m)^2]}}{[a^4+(2\hbar t/m)^2]^{1/2}} \\
&\quad \times \left\{\sinh^2\left[\frac{2xa^3}{a^4+(2\hbar t/m)^2}\right] + \sin^2\left[\frac{4\hbar axt/m}{a^4+(2\hbar t/m)^2}\right]\right\}.
\end{aligned} \tag{17}$$

Note that $|\psi(0,t)|^2 = 0$ as asserted in Part (1). The probability density $|\psi(x,t)|^2$ is plotted in Fig. 4 at four different time instants: $t = 0$, $ma^2/\hbar$, $2ma^2/\hbar$, and $3ma^2/\hbar$.

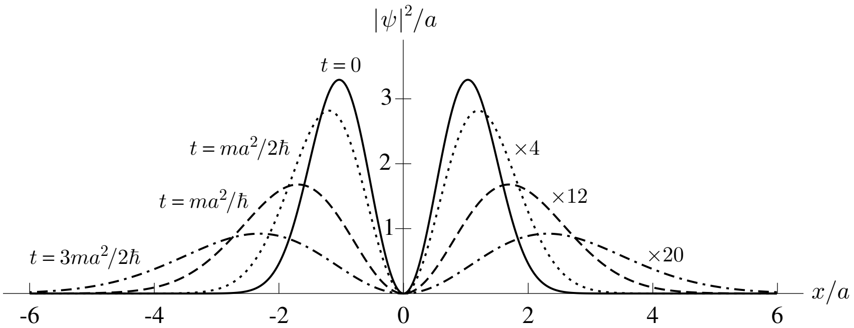

FIG. 4: Temporal dependence of the probability density for a particle in a state that is a superposition of two states with Gaussian-shaped wave functions. The solid, dotted, dashed, and dot-dashed lines are respectively at $t = 0$, $ma^2/\hbar$, $2ma^2/\hbar$, and $3ma^2/\hbar$. Note that, in order to make all the curves legible, the values of the probability density are multiplied respectively by factors of 4, 12, and 20 for the last three time instants.

From Fig. 4, we can see clearly that, as time develops, (1) the two peaks in the probability density shift respectively to left and right with the peak values decreasing and (2) the peaks become broader. We have thus seen the spreading of the wave packet. However, because the wave packet consists of two symmetrically-centered Gaussian-shaped sub-wave-packets, the center of the wave packet does not move and the probability density at the center remains to be zero.