



Quantum Mechanics

Homework Assignment 12

Fall, 2019

1. [C-T Exercise 6-2] Consider an arbitrary physical system whose four-dimensional state space is spanned by a basis of four eigenvectors $|jm_z\rangle$ common to \hat{J}^2 and \hat{J}_z ($j = 0$ or 1 ; $-j \leq m_z \leq +j$), of eigenvalues $j(j+1)\hbar^2$ and $m_z\hbar$, such that

$$\begin{aligned}\hat{J}_{\pm}|jm_z\rangle &= \hbar\sqrt{j(j+1)-m_z(m_z\pm 1)}|j, m_z\pm 1\rangle, \\ \hat{J}_+|jj\rangle &= \hat{J}_-|j, -j\rangle = 0.\end{aligned}$$

- (a) Express, in terms of the kets $|jm_z\rangle$, the eigenstates common to \hat{J}^2 and \hat{J}_x , to be denoted by $|jm_x\rangle$.
 (b) Consider a system in the normalized state

$$|\psi\rangle = \alpha|j=1, m_z=1\rangle + \beta|j=1, m_z=0\rangle + \gamma|j=1, m_z=-1\rangle + \delta|j=0, m_z=0\rangle.$$

- i. What is the probability of finding $2\hbar^2$ and \hbar if \hat{J}^2 and \hat{J}_x are measured simultaneously?
 - ii. Calculate the mean value of \hat{J}_z when the system is in the state $|\psi\rangle$, and the probabilities of the various possible results of a measurement bearing only on this observable.
 - iii. Same questions for the observable \hat{J}^2 and \hat{J}_x .
 - iv. \hat{J}_z^2 is now measured; what are the possible results, their probabilities, and their mean value?
2. [C-T Exercise 6-5] A system whose state space is $\mathcal{E}_{\vec{r}}$ has for its wave function $\psi(x, y, z) = N(x+y+z)e^{-r^2/\alpha^2}$, where α , which is real, is given and N is a normalization constant.

- (a) The observables \hat{L}_z and \hat{L}^2 are measured; what are the probabilities of finding 0 and $2\hbar^2$?
 (b) Is it possible to predict directly the probabilities of all possible results of measurements of \hat{L}^2 and \hat{L}_z in the system of wave function $\psi(x, y, z)$?

3. [C-T Exercise 6-8] Consider a particle in three-dimensional space, whose state vector is $|\psi\rangle$, and whose wave function is $\psi(\vec{r}) = \langle\vec{r}|\psi\rangle$. Let \hat{A} be an observable which commutes with $\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$, the orbital angular momentum of the particle. Assuming that \hat{A} , $\hat{\vec{L}}^2$, and \hat{L}_z form a CSCO in $\mathcal{E}_{\vec{r}}$, call $|n\ell m\rangle$ their common eigenkets, whose eigenvalues are, respectively, a_n (the index n is assumed to be discrete), $\ell(\ell+1)\hbar^2$, and $m\hbar$.

Let $\hat{U}(\phi)$ be the unitary operator defined by $\hat{U}(\phi) = e^{-i\phi\hat{L}_z/\hbar}$, where ϕ is a real dimensionless parameter. For an arbitrary operator \hat{K} , we call $\tilde{\hat{K}}$ the transform of \hat{K} by the unitary operator $\hat{U}(\phi)$, $\tilde{\hat{K}} = \hat{U}(\phi)\hat{K}\hat{U}^\dagger(\phi)$.

- (a) We set $\hat{L}_+ = \hat{L}_x + i\hat{L}_y$, $\hat{L}_- = \hat{L}_x - i\hat{L}_y$. Calculate $\tilde{\hat{L}}_+|n\ell m\rangle$ and show that \hat{L}_+ and $\tilde{\hat{L}}_+$ are proportional; calculate the proportionality constant. Same question for \hat{L}_- and $\tilde{\hat{L}}_-$.
 - (b) Express $\tilde{\hat{L}}_x$, $\tilde{\hat{L}}_y$, and $\tilde{\hat{L}}_z$ in terms of \hat{L}_x , \hat{L}_y , and \hat{L}_z . What geometrical transformation can be associated with the transformation of $\hat{\vec{L}}$ into $\tilde{\hat{\vec{L}}}$?
 - (c) Calculate the commutators $[\hat{x} \pm i\hat{y}, \hat{L}_z]$ and $[\hat{z}, \hat{L}_z]$. Show that the kets $(\hat{x} \pm i\hat{y})|n\ell m\rangle$ and $\hat{z}|n\ell m\rangle$ are eigenvectors of \hat{L}_z and calculate their eigenvalues. What relation must exist between m and m' for the matrix element $\langle n'\ell'm'|\hat{x} \pm i\hat{y}|n\ell m\rangle$ to be non-zero? Same question for $\langle n'\ell'm'|\hat{z}|n\ell m\rangle$.
 - (d) By comparing the matrix elements of $\tilde{\hat{x}} \pm i\tilde{\hat{y}}$ and $\tilde{\hat{z}}$ with those of $\hat{x} \pm i\hat{y}$ and \hat{z} , calculate $\tilde{\hat{x}}$, $\tilde{\hat{y}}$, $\tilde{\hat{z}}$ in terms of \hat{x} , \hat{y} , \hat{z} . Give a geometrical interpretation.
4. [C-T Exercise 7-1] Let ρ , ϕ , and z be the cylindrical coordinates of a spinless particle ($x = \rho \cos \phi$, $y = \rho \sin \phi$; $\rho \geq 0$, $0 \leq \phi < 2\pi$). Assume that the potential energy of this particle depends only on ρ , and not on ϕ and z . Recall that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2}.$$

- (a) Write, in cylindrical coordinates, the differential operator associated with the Hamiltonian. Show that \hat{H} commutes with \hat{L}_z and \hat{p}_z . Show from this that the wave functions associated with the stationary states of the particle can be chosen in the form $\varphi_{nmk}(\rho, \phi, z) = f_{nm}(\rho)e^{im\phi}e^{ikz}$, where the values that can be taken on by the indices m and k are to be specified.
- (b) Write, in cylindrical coordinates, the eigenvalue equation of the Hamiltonian \hat{H} of the particle. Derive from it the differential equation which yields $f_{nm}(\rho)$.
- (c) Let $\hat{\Sigma}_y$ be the operator whose action, in the $\{|\vec{r}\rangle\}$ representation, is to change y to $-y$ (reflection with respect to the xOz plane). Does $\hat{\Sigma}_y$ commute with \hat{H} ? Show that $\hat{\Sigma}_y$ anticommutes with \hat{L}_z , and show from this that $\hat{\Sigma}_y|\varphi_{nmk}\rangle$ is an eigenvector of \hat{L}_z . What is the corresponding eigenvalue? What can be concluded concerning the degeneracy of the energy levels of the particle? Could this result be predicted directly from the differential equation established in (b)?

5. [C-T Exercise 7-2] Consider a particle of mass μ , whose Hamiltonian is

$$\hat{H}_0 = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}\mu\omega_0^2\hat{r}^2,$$

where ω_0 is a given positive constant.

- (a) Find the energy levels of the particle and their degrees of degeneracy. Is it possible to construct a basis of eigenstates common to \hat{H}_0 , \hat{L}^2 , \hat{L}_z ?
- (b) Now, assume that the particle, which has a charge q , is placed in a uniform magnetic field \vec{B} parallel to Oz . We set $\omega_L = -qB/2\mu$. If we choose the gauge $\vec{A} = -\vec{r} \times \vec{B}/2$, the Hamiltonian \hat{H} of the particle is then given by $\hat{H} = \hat{H}_0 + \hat{H}_1(\omega_L)$, where \hat{H}_1 is the sum of an operator which is linearly dependent on ω_L (the paramagnetic term) and an operator which is quadratically dependent on ω_L (the diamagnetic term). Show that the new stationary states of the system and their degrees of degeneracy can be determined exactly.
- (c) Show that, if ω_L is much smaller than ω_0 , then the effect of the diamagnetic term is negligible compared to that of the paramagnetic term.
- (d) We now consider the first excited state of the oscillator, that is, the states whose energies approach $5\hbar\omega_0/2$ when $\omega_L \rightarrow 0$. To first order in ω_L/ω_0 , what are the energy levels in the presence of the field \vec{B} and their degrees of degeneracy (the Zeeman effect for a three-dimensional harmonic oscillator)? Same questions for the second excited state.
- (e) Now consider the ground state. How does its energy vary as a function of ω_L (the diamagnetic effect on the ground state)? Calculate the magnetic susceptibility χ of this state. Is the ground state, in the presence of the field \vec{B} , an eigenvector of \hat{L}^2 ? of \hat{L}_z ? of \hat{L}_x ? Give the form of its wave function and the corresponding probability current. Show that the effect of the field \vec{B} is to compress the wave function about Oz (in a ratio $[1 + (\omega_L/\omega_0)^2]^{1/4}$) and to induce a current.