Quantum Mechanics
Homework Assignment 06
Fall, 2019

1. In a given representation, the matrix representing the Hamiltonian of a particle is given by

$$H = \hbar\omega_0 \begin{pmatrix} -1+\varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & -1-\varepsilon & \sqrt{2}\,\varepsilon & 0 & 0 & 0 \\ 0 & \sqrt{2}\,\varepsilon & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & \sqrt{2}\,\varepsilon & 0 \\ 0 & 0 & 0 & \sqrt{2}\,\varepsilon & -1-\varepsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & -1+\varepsilon \end{pmatrix}$$

with $0 < \varepsilon < 1$. Find the energy eigenvalues and eigenfunctions of the particle in the representation.

2. **[C-T exercise 2-4]** Let $\hat{K}$ be the operator defined by $\hat{K} = |\varphi\rangle\langle\psi|$, where $|\varphi\rangle$ and $|\psi\rangle$ are two vectors of the state space.
   (a) Under what condition is $\hat{K}$ Hermitian?
   (b) Calculate $\hat{K}^2$. Under what condition is $\hat{K}$ a projector?
   (c) Show that $\hat{K}$ can always be written in the form $\hat{K} = \lambda \hat{P}_1 \hat{P}_2$ where λ is a constant to be calculated and $\hat{P}_1$ and $\hat{P}_2$ are projectors.

3. **[C-T exercise 2-5]** Let $\hat{P}_1$ be the orthogonal projector onto the subspace $\mathscr{E}_1$, $\hat{P}_2$ the orthogonal projector on to the subspace $\mathscr{E}_2$. Show that, for the product $\hat{P}_1\hat{P}_2$ to be an orthogonal projector as well, it is necessary and sufficient that $\hat{P}_1$ and $\hat{P}_2$ commute. In this case, what is the subspace onto which $\hat{P}_1\hat{P}_2$ projects?

4. **[C-T exercise 2-11]** Consider a physical system whose three-dimensional state space is spanned by the orthonormal basis formed by the three kets $|u_1\rangle$, $|u_2\rangle$, and $|u_3\rangle$. In the basis of these three vectors, taken in this order, the two operators $\hat{H}$ and $\hat{B}$ are defined by

$$H = \hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where ω_0 and b are real constants.
   (a) Are H and B Hermitian?
   (b) Show that H and B commute. Give a basis of eigenvectors common to H and B.

5. **[C-T exercise 2-12]** In the same state space as that of the preceding exercise, consider two operators $\hat{L}_z$ and $\hat{S}$ defined by

$$\begin{aligned} &\hat{L}_z\,|u_1\rangle = |u_1\rangle, \;\; \hat{L}_z\,|u_2\rangle = 0, \quad \hat{L}_z\,|u_3\rangle = -\,|u_3\rangle; \\ &\hat{S}\,|u_1\rangle = |u_3\rangle, \quad \hat{S}\,|u_2\rangle = |u_2\rangle, \;\; \hat{S}\,|u_3\rangle = |u_1\rangle\,. \end{aligned}$$

   (a) Write the matrices which represent, in the $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ basis, the operators $\hat{L}_z$, $\hat{L}_z^2$, $\hat{S}$, and $\hat{S}^2$. Are these operators observables?
   (b) Give the form of the most general matrix which represents an operator which commutes with $\hat{L}_z$. Same question for $\hat{L}_z^2$, then $\hat{S}^2$.
   (c) Do $\hat{L}_z^2$ and $\hat{S}$ form a CSCO? Give a basis of common eigenvectors.