Quantum Mechanics

Solutions to the Problems in the Midterm Exam

Fall, 2019

1. **(25 points) Complete the following five problems (5 points each).**

(1) Planck's law of blackbody radiation is given by $E_\lambda(T) = \dfrac{2\pi hc^2}{\lambda^5}\dfrac{1}{e^{hc/\lambda k_B T}-1}$. Find the total radiation intensity $E(T) = \displaystyle\int_0^\infty d\lambda\ E_\lambda(T)$. An integral that yields a numerical constant can be left unevaluated in your answer.

The total radiation intensity is given by

$$E(T) = \int_0^\infty d\lambda\ E(\lambda) = 2\pi hc^2 \int_0^\infty \frac{d\lambda}{\lambda^5}\frac{1}{e^{hc/\lambda k_B T}-1}.$$

Making a change of integral variables from λ to $x = hc/\lambda k_B T$, we have

$$E(T) = 2\pi hc^2\left(\frac{k_B T}{hc}\right)^4 \int_0^\infty dx\ \frac{x^3}{e^x-1} = \left[\frac{2\pi k_B^4}{h^3c^2}\int_0^\infty dx\ \frac{x^3}{e^x-1}\right]T^4 = \sigma T^4,$$

where $\sigma = \dfrac{2\pi k_B^4}{h^3c^2}\displaystyle\int_0^\infty dx\ \frac{x^3}{e^x-1}$ is the Stefan-Boltzmann constant. $E(T) = \sigma T^4$ is referred to as the Stefan-Boltzmann law.

(2) For a wave packet of a particle, $g(k) = \left(\dfrac{a}{\sqrt{2\pi}}\right)^{1/2} e^{-a^2(k-k_0)^2/4}$. Using $\psi(x,0) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^\infty dk\ g(k)e^{ikx}$, find $\psi(x,0)$. What is the probability density of finding the particle at $t=0$?

From $\psi(x,0) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^\infty dk\ g(k)e^{ikx}$, we have

$$\begin{aligned}\psi(x,0) &= \frac{\sqrt{a}}{(2\pi)^{3/4}}\int_{-\infty}^\infty dk\ e^{-a^2(k-k_0)^2/4}e^{ikx}\\ &= \frac{\sqrt{a}}{(2\pi)^{3/4}}\int_{-\infty}^\infty dk\ e^{-a^2[(k-k_0)-2ix/a^2]^2/4}e^{-x^2/a^2}e^{ik_0x}\\ &= \left(\frac{2}{\pi a^2}\right)^{1/4} e^{-x^2/a^2}e^{ik_0x}.\end{aligned}$$

From the above result, we see that the Fourier transform of a Gaussian function is also a Gaussian function. $\psi(x,0)$ is normalized as can be seen from the following

$$\int_{-\infty}^\infty dx\ |\psi(x,0)|^2 = \sqrt{\frac{2}{\pi a^2}}\int_{-\infty}^\infty dx\ e^{-2x^2/a^2} = \sqrt{\frac{2}{\pi a^2}}\cdot\sqrt{\frac{\pi}{2/a^2}} = 1.$$

The probability density of finding the particle at $t=0$ is given by $|\psi(x,0)|^2 = \sqrt{\dfrac{2}{\pi a^2}}\,e^{-2x^2/a^2}$.

(3) The component operators of the orbital angular momentum are respectively given by $\hat{L}_x = y\hat{p}_z - z\hat{p}_y$, $\hat{L}_y = z\hat{p}_x - x\hat{p}_z$, and $\hat{L}_z = x\hat{p}_y - y\hat{p}_x$. Show that $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$.

$$\begin{aligned}[\hat{L}_x,\hat{L}_y] &= [y\hat{p}_z - z\hat{p}_y, z\hat{p}_x - x\hat{p}_z] = [y\hat{p}_z, z\hat{p}_x] - [y\hat{p}_z, x\hat{p}_z] - [z\hat{p}_y, z\hat{p}_x] + [z\hat{p}_y, x\hat{p}_z]\\ &= y[\hat{p}_z, z]\hat{p}_x - 0 - 0 + x[z,\hat{p}_z]\hat{p}_y = -i\hbar y\hat{p}_x + i\hbar x\hat{p}_y = i\hbar\big(x\hat{p}_y - y\hat{p}_x\big) = i\hbar\hat{L}_z.\end{aligned}$$

(4) It is known that the volume occupied by a quantum state in phase space is $h^3 = (2\pi\hbar)^3$. Find the density of states per unit volume, $g(E)$, for a free particle of mass m.

From the fact that the volume occupied by a quantum state in phase space is $h^3 = (2\pi\hbar)^3$, the number of states in the volume $V(4\pi p^2 dp) = V(4\pi\hbar^3 k^2 dk)$ in phase space is given by

$$dN = \frac{V(4\pi\hbar^3 k^2 dk)}{h^3} = \frac{V}{2\pi^2} k^2 dk.$$

For a free particle of mass m, we have $E = \hbar^2 k^2/2m$ from which we obtain

$$k^2 dk = \frac{1}{2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{E}\, dE.$$

The density of states per unit volume for the free particle is then given by

$$g(E) = \frac{1}{V}\frac{dN}{dE} = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{E} = 2\pi\left(\frac{m}{2\pi^2\hbar^2}\right)^{3/2}\sqrt{E}.$$

(5) Find the eigenvalues and normalized eigenfunctions of the z-component operator $\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$ of the orbital angular momentum operator.

The eigenequation of $\hat{L}_z$ reads $-i\hbar\frac{\partial\Phi(\phi)}{\partial\phi} = \lambda\Phi(\phi)$. The general solution to the eigenequation of $\hat{L}_z$ is given by $\Phi(\phi) = Ce^{i\lambda\phi/\hbar}$.

The single-valuedness of $\Phi(\phi)$ indicates that $\Phi(\phi) = \Phi(\phi+2\pi)$. From $\Phi(\phi) = \Phi(\phi+2\pi)$, we have $e^{2\pi i\lambda/\hbar} = 1$ from which we obtain the following discrete eigenvalues of $\hat{L}_z$

$$\lambda_m = m\hbar,\ m = 0, \pm 1, \pm 2, \cdots.$$

Making use of the normalization condition, we have

$$1 = |C_m|^2 \int_0^{2\pi} d\phi = 2\pi|C_m|^2.$$

Choosing $C_m = \frac{1}{\sqrt{2\pi}}$, we have the following normalized eigenfunctions

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi},\ m = 0, \pm 1, \pm 2, \cdots.$$

2. **(15 points)** $\hat{A}$ and $\hat{B}$ are two Hermitian operators. The commutation relation between them is given by $[\hat{A}, \hat{B}] = i\hat{C}$ with $\hat{C}$ a Hermitian operator. The averages of $\hat{A}$ and $\hat{B}$ in the state described by the normalized ket $|\psi\rangle$ are respectively given by $\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$ and $\langle\hat{B}\rangle = \langle\psi|\hat{B}|\psi\rangle$. The deviations of $\hat{A}$ and $\hat{B}$ from their respective averages are given by $\Delta\hat{A} = \hat{A} - \langle\hat{A}\rangle$ and $\Delta\hat{B} = \hat{B} - \langle\hat{B}\rangle$. Through considering the function of the real variable λ, $I(\lambda) = \langle(\lambda\Delta\hat{A} - i\Delta\hat{B})\psi|(\lambda\Delta\hat{A} - i\Delta\hat{B})\psi\rangle$, derive the general Heisenberg uncertainty relation $\Delta A \Delta B \geq \left|\langle\hat{C}\rangle\right|/2$ with ΔA and ΔB given respectively by $\Delta A = \left\langle(\Delta\hat{A})^2\right\rangle^{1/2}$ and $\Delta B = \left\langle(\Delta\hat{B})^2\right\rangle^{1/2}$.

We first evaluate $I(\lambda)$. We have

$$\begin{aligned} I(\lambda) &= \big((\lambda\Delta\hat{A} - i\Delta\hat{B})\psi, (\lambda\Delta\hat{A} - i\Delta\hat{B})\psi\big) = \big(\psi, (\lambda\Delta\hat{A} - i\Delta\hat{B})^\dagger(\lambda\Delta\hat{A} - i\Delta\hat{B})\psi\big) \\ &= \big(\psi, (\lambda\Delta\hat{A} + i\Delta\hat{B})(\lambda\Delta\hat{A} - i\Delta\hat{B})\psi\big) = \big(\psi, \{\lambda^2(\Delta\hat{A})^2 - i\lambda[\Delta\hat{A}, \Delta\hat{B}] + (\Delta\hat{B})^2\}\psi\big) \\ &= \big(\psi, [\lambda^2(\Delta\hat{A})^2 + \lambda\hat{C} + (\Delta\hat{B})^2]\psi\big) = \lambda^2\left\langle(\Delta\hat{A})^2\right\rangle + \lambda\langle\hat{C}\rangle + \left\langle(\Delta\hat{B})^2\right\rangle. \end{aligned}$$

Making use of $\Delta A = \langle(\Delta\hat{A})^2\rangle^{1/2}$ and $\Delta B = \langle(\Delta\hat{B})^2\rangle^{1/2}$, we have

$$I(\lambda) = \lambda^2(\Delta A)^2 + \lambda\,\langle\hat{C}\rangle + (\Delta B)^2.$$

Completing the square for λ yields

$$I(\lambda) = (\Delta A)^2\left[\lambda + \frac{\langle\hat{C}\rangle}{2(\Delta A)^2}\right]^2 + (\Delta B)^2 - \frac{\langle\hat{C}\rangle^2}{4(\Delta A)^2}.$$

From the above expression of $I(\lambda)$, we see that the minimum value of $I(\lambda)$ is given by

$$\min I(\lambda) = (\Delta B)^2 - \frac{\langle\hat{C}\rangle^2}{4(\Delta A)^2}.$$

Because $I(\lambda) \geq 0$, $\min I(\lambda) \geq 0$. From $\min I(\lambda) \geq 0$, we have

$$(\Delta B)^2 - \frac{\langle\hat{C}\rangle^2}{4(\Delta A)^2} \geq 0$$

from which we obtain the following general Heisenberg uncertainty relation

$$\Delta A \Delta B \geq \frac{1}{2}\left|\langle\hat{C}\rangle\right|.$$

3. **(15 points)** The potential energy of a particle of mass m in a δ-function potential well is given by $V(x) = -\alpha\delta(x)$ with $\alpha > 0$. Find the energy and normalized wave function of the particle in the bound state of the δ-function potential well.

The stationary Schrödinger equation of the particle in the δ-function potential well reads in the two regions given in Fig. 1

$$\begin{aligned}
\frac{d^2\varphi_{\rm I}(x)}{dx^2} - \kappa^2\varphi_{\rm I}(x) &= 0,\\
\frac{d^2\varphi_{\rm II}(x)}{dx^2} - \kappa^2\varphi_{\rm II}(x) &= 0.
\end{aligned}$$

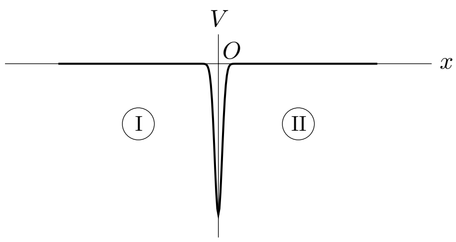

FIG. 1: δ-function potential well.

The boundary conditions at $x = \pm\infty$ are given by

$$\begin{aligned}
\varphi_{\rm I}(x = -\infty) &= 0,\\
\varphi_{\rm II}(x = +\infty) &= 0.
\end{aligned}$$

The matching conditions at $x = 0$ are given by

$$\begin{aligned}
\varphi_{\rm II}(0) &= \varphi_{\rm I}(0),\\
\varphi_{\rm II}'(0) - \varphi_{\rm I}'(0) &= -\frac{2m\alpha}{\hbar^2}\varphi_{\rm I}(0).
\end{aligned}$$

The general solutions of the stationary Schrödinger equation in the two regions are respectively given by

$$\varphi_{\mathrm{I}}(x) = Ae^{\kappa x} + Be^{-\kappa x},$$
$$\varphi_{\mathrm{II}}(x) = Ce^{\kappa x} + De^{-\kappa x}.$$

From the boundary conditions at $x = \pm\infty$, $\varphi_{\mathrm{I}}(x = -\infty) = 0$ and $\varphi_{\mathrm{II}}(x = +\infty) = 0$, we have $B = 0$ and $C = 0$. We then have

$$\varphi_{\mathrm{I}}(x) = Ae^{\kappa x},$$
$$\varphi_{\mathrm{II}}(x) = De^{-\kappa x}.$$

From the matching condition $\varphi_{\mathrm{II}}(0) = \varphi_{\mathrm{I}}(0)$, we have $A = D$. We then have

$$\varphi_{\mathrm{I}}(x) = Ae^{\kappa x},$$
$$\varphi_{\mathrm{II}}(x) = Ae^{-\kappa x}.$$

From the matching condition $\varphi'_{\mathrm{II}}(0) - \varphi'_{\mathrm{I}}(0) = -\dfrac{2m\alpha}{\hbar^2}\varphi_{\mathrm{I}}(0)$, we have

$$\left(\kappa - \frac{m\alpha}{\hbar^2}\right)A = 0.$$

Since $A \neq 0$, we have

$$\kappa = \frac{m\alpha}{\hbar^2}$$

from which we obtain

$$E = -\frac{m\alpha^2}{2\hbar^2}.$$

Putting the wave function into a single expression, we have

$$\varphi(x) = Ae^{-\kappa|x|}.$$

From the normalization condition, we have

$$1 = \int_{-\infty}^{\infty} dx\, |\varphi(x)|^2 = |A|^2\left[\int_{-\infty}^{0} dx\, e^{2\kappa x} + \int_{0}^{\infty} dx\, e^{-2\kappa x}\right] = \frac{1}{\kappa}|A|^2.$$

Choosing $A = \sqrt{\kappa}$, we have the following normalized wave function

$$\varphi(x) = \sqrt{\kappa}\, e^{-\kappa|x|}.$$

4. **(15 points)** The state space of a certain physical system is three-dimensional. Let $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ be an orthonormal basis of this space. The kets $|\psi_0\rangle$ and $|\psi_1\rangle$ are defined by

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{i}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle,\ |\psi_1\rangle = \frac{1}{\sqrt{3}}|u_1\rangle + \frac{i}{\sqrt{3}}|u_3\rangle.$$

   (a) **(5 points)** Are these kets normalized?
   (b) **(10 points)** Calculate the matrices ρ_0 and ρ_1 representing, in the $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ basis, the projection operators onto the state $|\psi_0\rangle$ and onto the state $|\psi_1\rangle$. Verify that these matrices are Hermitian.

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(1) **(5 points)** The square of the norm of $|\psi_0\rangle$ is given by

$$\begin{aligned}\langle\psi_0|\psi_0\rangle &= \left[\frac{1}{\sqrt{2}}\langle u_1| - \frac{i}{2}\langle u_2| + \frac{1}{2}\langle u_3|\right]\left[\frac{1}{\sqrt{2}}|u_1\rangle + \frac{i}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle\right]\\ &= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{i}{2}\right)\left(\frac{i}{2}\right) + \left(\frac{1}{2}\right)^2 = 1.\end{aligned}$$

$\langle\psi_0|\psi_0\rangle = 1$ indicates that the norm of $|\psi_0\rangle$ is equal to unity. Thus, $|\psi_0\rangle$ is normalized. The square of the norm of $|\psi_1\rangle$ is given by

$$\begin{aligned}\langle\psi_1|\psi_1\rangle &= \left[\frac{1}{\sqrt{3}}\langle u_1| - \frac{i}{\sqrt{3}}\langle u_3|\right]\left[\frac{1}{\sqrt{3}}|u_1\rangle + \frac{i}{\sqrt{3}}|u_3\rangle\right] \\ &= \left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{i}{\sqrt{3}}\right)\left(\frac{i}{\sqrt{3}}\right) = \frac{2}{3}.\end{aligned}$$

Thus, $|\psi_1\rangle$ is not normalized. The normalized $|\psi_1\rangle$ is given by

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{i}{\sqrt{2}}|u_3\rangle.$$

The above normalized $|\psi_1\rangle$ will be used in the following calculations.

(2) **(10 points)** The projection operator $\hat{\rho}_0$ is given by

$$\begin{aligned}\hat{\rho}_0 = |\psi_0\rangle\langle\psi_0| &= \left[\frac{1}{\sqrt{2}}|u_1\rangle + \frac{i}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle\right]\left[\frac{1}{\sqrt{2}}\langle u_1| - \frac{i}{2}\langle u_2| + \frac{1}{2}\langle u_3|\right] \\ &= \frac{1}{2}|u_1\rangle\langle u_1| - \frac{i}{2\sqrt{2}}|u_1\rangle\langle u_2| + \frac{1}{2\sqrt{2}}|u_1\rangle\langle u_3| \\ &\quad + \frac{i}{2\sqrt{2}}|u_2\rangle\langle u_1| + \frac{1}{4}|u_2\rangle\langle u_2| + \frac{i}{4}|u_2\rangle\langle u_3| \\ &\quad + \frac{1}{2\sqrt{2}}|u_3\rangle\langle u_1| - \frac{i}{4}|u_3\rangle\langle u_2| + \frac{1}{4}|u_3\rangle\langle u_3|.\end{aligned}$$

From the above expression, we can easily infer the matrix representing $\hat{\rho}_0$. We have

$$\rho_0 = \begin{pmatrix} \frac{1}{2} & -\frac{i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{i}{2\sqrt{2}} & \frac{1}{4} & \frac{i}{4} \\ \frac{1}{2\sqrt{2}} & -\frac{i}{4} & \frac{1}{4} \end{pmatrix}.$$

From the above matrix representing $\hat{\rho}_0$, we see that ρ_0 is a Hermitian matrix. The projection operator $\hat{\rho}_1$ is given by

$$\begin{aligned}\hat{\rho}_1 = |\psi_1\rangle\langle\psi_1| &= \left[\frac{1}{\sqrt{2}}|u_1\rangle + \frac{i}{\sqrt{2}}|u_3\rangle\right]\left[\frac{1}{\sqrt{2}}\langle u_1| - \frac{i}{\sqrt{2}}\langle u_3|\right] \\ &= \frac{1}{2}|u_1\rangle\langle u_1| - \frac{i}{2}|u_1\rangle\langle u_3| \\ &\quad + \frac{i}{2}|u_3\rangle\langle u_1| + \frac{1}{2}|u_3\rangle\langle u_3|.\end{aligned}$$

From the above expression, we can easily infer the matrix representing $\hat{\rho}_1$. We have

$$\rho_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 1 \end{pmatrix}$$

From the above matrix representing $\hat{\rho}_1$, we see that ρ_1 is a Hermitian matrix. Let us check the properties of ρ_0 and ρ_1. For ρ_0^2, we have

$$\rho_0^2 = \begin{pmatrix} \frac{1}{2} & -\frac{i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{i}{2\sqrt{2}} & \frac{1}{4} & \frac{i}{4} \\ \frac{1}{2\sqrt{2}} & -\frac{i}{4} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{i}{2\sqrt{2}} & \frac{1}{4} & \frac{i}{4} \\ \frac{1}{2\sqrt{2}} & -\frac{i}{4} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{i}{2\sqrt{2}} & \frac{1}{4} & \frac{i}{4} \\ \frac{1}{2\sqrt{2}} & -\frac{i}{4} & \frac{1}{4} \end{pmatrix} = \rho_0.$$

For ρ_1^2, we have

$$\rho_1^2 = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{i}{2} \\ 0 & 0 & 0 \\ \frac{i}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & -\frac{i}{2} \\ 0 & 0 & 0 \\ \frac{i}{2} & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{i}{2} \\ 0 & 0 & 0 \\ \frac{i}{2} & 0 & \frac{1}{2} \end{pmatrix} = \rho_1.$$

Thus, ρ_0 and ρ_1 indeed possess the property of projection operators that the square of a projection operator is equal to itself.

5. **(15 points)** In the three-dimensional state space spanned by the orthonormal basis formed by the three kets $|u_1\rangle$, $|u_2\rangle$, and $|u_3\rangle$, consider two operators $\hat{L}_z$ and $\hat{S}$ defined by

$$\begin{aligned} &\hat{L}_z |u_1\rangle = |u_1\rangle, \ \ \hat{L}_z |u_2\rangle = 0, \quad \hat{L}_z |u_3\rangle = -|u_3\rangle; \\ &\hat{S} |u_1\rangle = |u_3\rangle, \quad \hat{S} |u_2\rangle = |u_2\rangle, \ \ \hat{S} |u_3\rangle = |u_1\rangle . \end{aligned}$$

(1) **(5 points)** Write the matrices which represent, in the $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ basis, the operators $\hat{L}_z$, $\hat{L}_z^2$, $\hat{S}$, and $\hat{S}^2$. Are these operators observables?

(2) **(10 points)** Do $\hat{L}_z^2$ and $\hat{S}$ form a CSCO? Give a basis of common eigenvectors.

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(1) **(5 points)** From $\hat{L}_z |u_1\rangle = |u_1\rangle, \hat{L}_z |u_2\rangle = 0, \hat{L}_z |u_3\rangle = -|u_3\rangle$, we obtain the following matrix elements of $\hat{L}_z$

$$\begin{aligned} &\langle u_1|\hat{L}_z|u_1\rangle = 1, \ \langle u_1|\hat{L}_z|u_2\rangle = 0, \ \langle u_1|\hat{L}_z|u_3\rangle = 0, \\ &\langle u_2|\hat{L}_z|u_1\rangle = 0, \ \langle u_2|\hat{L}_z|u_2\rangle = 0, \ \langle u_2|\hat{L}_z|u_3\rangle = 0, \\ &\langle u_3|\hat{L}_z|u_1\rangle = 0, \ \langle u_3|\hat{L}_z|u_2\rangle = 0, \ \langle u_3|\hat{L}_z|u_3\rangle = -1. \end{aligned}$$

Thus, the representation matrix of $\hat{L}_z$ in the $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ basis is a diagonal matrix given by

$$L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

From the representation matrix of $\hat{L}_z$, we can obtain the representation matrix of $\hat{L}_z^2$ by squaring the representation matrix of $\hat{L}_z$. We have

$$L_z^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From $\hat{S} |u_1\rangle = |u_3\rangle, \hat{S} |u_2\rangle = |u_2\rangle, \hat{S} |u_3\rangle = |u_1\rangle$, we obtain the following matrix elements of $\hat{S}$

$$\begin{aligned} &\langle u_1|\hat{S}|u_1\rangle = 0, \ \langle u_1|\hat{S}|u_2\rangle = 0, \ \langle u_1|\hat{S}|u_3\rangle = 1, \\ &\langle u_2|\hat{S}|u_1\rangle = 0, \ \langle u_2|\hat{S}|u_2\rangle = 1, \ \langle u_2|\hat{S}|u_3\rangle = 0, \\ &\langle u_3|\hat{S}|u_1\rangle = 1, \ \langle u_3|\hat{S}|u_2\rangle = 0, \ \langle u_3|\hat{S}|u_3\rangle = 0. \end{aligned}$$

Thus, the representation matrix of $\hat{S}$ in the $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ basis is given by

$$S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

From the representation matrix of $\hat{S}$, we can obtain the representation matrix of $\hat{S}^2$ by squaring the representation matrix of $\hat{S}$. We have

$$S^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We see that the representation matrix of $\hat{S}^2$ is a unit matrix.

Because the representation matrices of $\hat{L}_z$, $\hat{L}_z^2$, $\hat{S}$, and $\hat{S}^2$ are all Hermitian matrices, these operators are observables.

(2) **(10 points)** Let us first see if L_z^2 and S commute. Their commutator is evaluated as follows

$$\left[L_z^2, S\right] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

Thus, L_z^2 and S commute and they can have common eigenvectors. Since L_z^2 is a diagonal matrix, its eigenvalues are the matrix elements on the main diagonal. We see that the eigenvalues of L_z^2 are 0 and 1. The eigenvalue 0 of L_z^2 is nondegenerate with the corresponding eigenvector given by $|u_2\rangle$. The eigenvalue 1 of L_z^2 is doubly degenerate. The two-dimensional eigensubspace of the eigenvalue 1 of L_z^2 is spanned by $|u_1\rangle$ and $|u_3\rangle$. Note that any linear combination of $|u_1\rangle$ and $|u_3\rangle$ is an eigenvector of L_z^2 corresponding to the eigenvalue 1.

For the convenience of diagonalizing S, we rearrange the basis vectors into the order $|u_2\rangle, |u_3\rangle, |u_1\rangle$. From the previous results,

$$\begin{aligned} &\langle u_1|\hat{S}|u_1\rangle = 0,\ \langle u_1|\hat{S}|u_2\rangle = 0,\ \langle u_1|\hat{S}|u_3\rangle = 1, \\ &\langle u_2|\hat{S}|u_1\rangle = 0,\ \langle u_2|\hat{S}|u_2\rangle = 1,\ \langle u_2|\hat{S}|u_3\rangle = 0, \\ &\langle u_3|\hat{S}|u_1\rangle = 1,\ \langle u_3|\hat{S}|u_2\rangle = 0,\ \langle u_3|\hat{S}|u_3\rangle = 0, \end{aligned}$$

we obtain the following representation matrix of S in the $\{|u_2\rangle, |u_3\rangle, |u_1\rangle\}$ basis

$$S' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We see that S' is a block matrix of the form

$$S' = \begin{pmatrix} S_1' & 0 \\ 0 & S_2' \end{pmatrix},$$

where

$$S_1' = (1),\ S_2' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

From $S_1' = (1)$, we see that 1 is an eigenvalue of S' with the corresponding eigenvector given by $|u_2\rangle$. Note that 1 is also an eigenvalue of S with the corresponding eigenvector given by $|u_2\rangle$.

Note that S_2' is of the form of B_2 in the previous problem. The eigenvalues and the corresponding vectors of S_2' can be obtained from those of B_2 through setting $b = 1$ and with the proper basis vectors used. Thus, the eigenvalues of S_2' are ± 1 with the corresponding eigenvectors respectively given

$$\frac{1}{\sqrt{2}}\big[|u_3\rangle + |u_1\rangle\big],\ \frac{1}{\sqrt{2}}\big[|u_3\rangle - |u_1\rangle\big].$$

Note that the eigenvalues and eigenvectors of S_2' are also those of S. Since the subspace spanned by $|u_3\rangle$ and $|u_1\rangle$ is the eigensubspace of the eigenvalue 1 of L_z^2, the above eigenvectors of S_2' are also the eigenvectors of L_z^2 corresponding to the eigenvalue 1.

In summary, we have obtained the following common eigenvectors of L_z^2 and S.

From the above table, we see that, specifying a pair of eigenvalues of L_z^2 and S, their common eigenvector can be uniquely determined (within a multiplying numerical factor). Therefore, $\hat{L}_z^2$ and $\hat{S}$ form a CSCO.

TABLE I: Common eigenvectors of L_z^2 and S.

Common eigenvector	Eigenvalue of L_z^2	Eigenvalue of S
$\lvert u_2\rangle$	0	1
$\frac{1}{\sqrt{2}}\left[\lvert u_3\rangle + \lvert u_1\rangle\right]$	1	1
$\frac{1}{\sqrt{2}}\left[\lvert u_3\rangle - \lvert u_1\rangle\right]$	1	-1

6. **(15 points)**

(1) **(7 points)** Starting from the time-dependent Schrödinger equation $i\hbar\frac{d\,|\psi(t)\rangle}{dt} = \left[\frac{\hat{\vec{p}}^2}{2m} + \hat{V}(\hat{\vec{r}})\right]|\psi(t)\rangle$ in the Dirac notation, derive the time-dependent Schrödinger equation in the $\{|\vec{p}\rangle\}$ representation.

(2) **(8 points)** Starting from the time-dependent Schrödinger equation in the $\{|\vec{p}\rangle\}$ representation, derive the time-dependent Schrödinger equation in the $\{|\vec{r}\rangle\}$ representation.

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(1) **(7 points)** Taking the scalar product of $i\hbar\frac{d\,|\psi(t)\rangle}{dt} = \left[\frac{\hat{\vec{p}}^2}{2m} + \hat{V}(\hat{\vec{r}})\right]|\psi(t)\rangle$ with $|\vec{p}\rangle$, we have

$$i\hbar\,\langle\vec{p}|\,\frac{d}{dt}\,|\psi(t)\rangle = \langle\vec{p}|\left[\frac{\hat{\vec{p}}^2}{2m} + \hat{V}(\hat{\vec{r}})\right]|\psi(t)\rangle\,.$$

From $\langle\vec{p}|\psi(t)\rangle = \overline{\psi}(\vec{p},t)$, we have

$$i\hbar\,\langle\vec{p}|\,\frac{d}{dt}\,|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}\,\langle\vec{p}|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}\overline{\psi}(\vec{p},t).$$

From $\langle\vec{p}|\hat{\vec{p}}|\psi(t)\rangle = \vec{p}\,\langle\vec{p}|\psi(t)\rangle = \vec{p}\,\overline{\psi}(\vec{p},t)$, we have

$$\frac{1}{2m}\,\langle\vec{p}|\hat{\vec{p}}^2|\psi(t)\rangle = \frac{1}{2m}\vec{p}\cdot\langle\vec{p}|\hat{\vec{p}}|\psi(t)\rangle = \frac{\vec{p}^2}{2m}\,\langle\vec{p}|\psi(t)\rangle = \frac{\vec{p}^2}{2m}\overline{\psi}(\vec{p},t).$$

From $\langle\vec{p}|\hat{\vec{r}}|\psi(t)\rangle = i\hbar\vec{\nabla}_{\vec{p}}\,\langle\vec{p}|\psi(t)\rangle = i\hbar\vec{\nabla}_{\vec{p}}\,\overline{\psi}(\vec{p},t)$, we have

$$\langle\vec{p}|\hat{V}(\hat{\vec{r}})|\psi(t)\rangle = V(i\hbar\vec{\nabla}_{\vec{p}})\overline{\psi}(\vec{p},t).$$

Making use of the above-obtained results, we obtain the following time-dependent Schrödinger equation in the $\{|\vec{p}\rangle\}$ representation

$$i\hbar\frac{\partial}{\partial t}\overline{\psi}(\vec{p},t) = \left[\frac{\vec{p}^2}{2m} + V(i\hbar\vec{\nabla}_{\vec{p}})\right]\overline{\psi}(\vec{p},t).$$

(2) **(8 points)** Writing $\overline{\psi}(\vec{p},t)$ as $\langle\vec{p}|\psi(t)\rangle$ and making use of $\vec{p}^2\,\langle\vec{p}|\psi(t)\rangle = \langle\vec{p}|\hat{\vec{p}}^2|\psi(t)\rangle$ and $V(i\hbar\vec{\nabla}_{\vec{p}})\,\langle\vec{p}|\psi(t)\rangle = \langle\vec{p}|\hat{V}(\hat{\vec{r}})|\psi(t)\rangle$, we can rewrite the time-dependent Schrödinger equation in the $\{|\vec{p}\rangle\}$ representation as

$$i\hbar\frac{\partial}{\partial t}\,\langle\vec{p}|\psi(t)\rangle = \langle\vec{p}|\left[\frac{\hat{\vec{p}}^2}{2m} + \hat{V}(\hat{\vec{r}})\right]|\psi(t)\rangle\,.$$

Utilizing the magic one, $\int d^3r'\;|\vec{r}\,'\rangle\,\langle\vec{r}\,'| = 1$, in the $\{|\vec{r}\rangle\}$ representation, we have

$$i\hbar\int d^3r'\;\frac{\partial}{\partial t}\left[\,\langle\vec{p}|\vec{r}\,'\rangle\,\langle\vec{r}\,'|\psi(t)\rangle\,\right] = \int d^3r'\;\langle\vec{p}|\vec{r}\,'\rangle\,\langle\vec{r}\,'|\left[\frac{\hat{\vec{p}}^2}{2m} + \hat{V}(\hat{\vec{r}})\right]|\psi(t)\rangle\,.$$

Utilizing $\langle \vec{r}\,' | \psi(t) \rangle = \psi(\vec{r}\,', t)$, $\langle \vec{r}\,' | \hat{\vec{p}}^2 | \psi(t) \rangle = -\hbar^2 \vec{\nabla}'^2 \langle \vec{r}\,' | \psi(t) \rangle = -\hbar^2 \vec{\nabla}'^2 \psi(\vec{r}\,', t)$, and $\langle \vec{r}\,' | \hat{V}(\hat{\vec{r}}) | \psi(t) \rangle = V(\vec{r}\,') \langle \vec{r}\,' | \psi(t) \rangle = V(\vec{r}\,') \psi(\vec{r}\,', t)$, we have

$$i\hbar \int d^3r' \ \langle \vec{p} | \vec{r}\,' \rangle \frac{\partial}{\partial t} \psi(\vec{r}\,', t) = \int d^3r' \ \langle \vec{p} | \vec{r}\,' \rangle \Big[-\frac{\hbar^2}{2m} \vec{\nabla}'^2 + V(\vec{r}\,') \Big] \psi(\vec{r}\,', t).$$

Multiplying both sides of the above equation with $\langle \vec{r} | \vec{p} \rangle$ and then integrating both sides of the resulting equation over $\vec{p}$, we have

$$i\hbar \int d^3r' \int d^3p \ \langle \vec{r} | \vec{p} \rangle \langle \vec{p} | \vec{r}\,' \rangle \frac{\partial}{\partial t} \psi(\vec{r}\,', t) = \int d^3r' \int d^3p \ \langle \vec{r} | \vec{p} \rangle \langle \vec{p} | \vec{r}\,' \rangle \Big[-\frac{\hbar^2}{2m} \vec{\nabla}'^2 + V(\vec{r}\,') \Big] \psi(\vec{r}\,', t).$$

Utilizing the magic one, $\displaystyle\int d^3p \ |\vec{p}\rangle \langle \vec{p}| = 1$, in the $\{|\vec{p}\rangle\}$ representation, we have

$$i\hbar \int d^3r' \ \langle \vec{r} | \vec{r}\,' \rangle \frac{\partial}{\partial t} \psi(\vec{r}\,', t) = \int d^3r' \ \langle \vec{r} | \vec{r}\,' \rangle \Big[-\frac{\hbar^2}{2m} \vec{\nabla}'^2 + V(\vec{r}\,') \Big] \psi(\vec{r}\,', t).$$

Utilizing $\langle \vec{r} | \vec{r}\,' \rangle = \delta(\vec{r} - \vec{r}\,')$ yields

$$i\hbar \int d^3r' \ \delta(\vec{r} - \vec{r}\,') \frac{\partial}{\partial t} \psi(\vec{r}\,', t) = \int d^3r' \ \delta(\vec{r} - \vec{r}\,') \Big[-\frac{\hbar^2}{2m} \vec{\nabla}'^2 + V(\vec{r}\,') \Big] \psi(\vec{r}\,', t).$$

Performing the integration over $\vec{r}\,'$ on both sides of the above equation, we obtain the following time-dependent Schrödinger equation in the $\{|\vec{r}\rangle\}$ representation

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = \Big[-\frac{\hbar^2}{2m} \vec{\nabla}^2 + V(\vec{r}) \Big] \psi(\vec{r}, t).$$