作业二

截止时间: 2022 年 3 月 18 日 (周五)

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成绩：__________

第 1 题 得分: __________. 求单模热光场的平均光子数.

解: 单模热光场的平均光子数为

$$\begin{aligned}\langle N\rangle =&\mathrm{Tr}(\rho a^\dagger a)\\ =&\sum_m\langle m|\left[1-\exp\left(-\frac{\hbar\omega}{kT}\right)\right]\sum_n\exp\left(-n\frac{\hbar\omega}{kT}\right)|n\rangle\langle n|a^\dagger a|m\rangle\\ =&\left[1-\exp\left(-\frac{\hbar\omega}{kT}\right)\right]\sum_{m,n}\exp\left(-n\frac{\hbar\omega}{kT}\right)\langle m|n\rangle\langle n|a^\dagger a|m\rangle\\ =&\left[1-\exp\left(-\frac{\hbar\omega}{kT}\right)\right]\sum_n\exp\left(-n\frac{\hbar\omega}{kT}\right)\langle n|a^\dagger a|n\rangle\\ =&\left[1-\exp\left(-\frac{\hbar\omega}{kT}\right)\right]\sum_n\exp\left(-n\frac{\hbar\omega}{kT}\right)n\\ =&\frac{\exp\left(-\frac{\hbar\omega}{kT}\right)}{1-\exp\left(-\frac{\hbar\omega}{kT}\right)}.\end{aligned}$$

□

第 2 题 得分: __________. 证明平移算符的性质:

$$D^\dagger=D(-\alpha)=[D(\alpha)]^{-1},$$

$$D^{-1}(\alpha)aD(\alpha)=a+\alpha.$$

证: 位移算符

$$D(\alpha)=e^{\alpha a^\dagger-\alpha^*a}=e^{\alpha a^\dagger}e^{-\alpha^*a}e^{-|\alpha|^2/2}. \tag{1}$$

先证第一个等式: 由于

$$\begin{aligned}D^\dagger(\alpha)=&e^{\alpha^*a-\alpha a^\dagger}, &(2)\\ D(-\alpha)=&e^{-\alpha a^\dagger+\alpha^*a}, &(3)\\ [D(\alpha)]^{-1}=&e^{-\alpha a^\dagger+\alpha^*a}, &(4)\end{aligned}$$

故

$$D^\dagger(\alpha)=D(-\alpha)=[D(\alpha)]^{-1}. \tag{5}$$

再证第二个等式: 由于 $[a,[a,a^\dagger]]=[a^\dagger,[a,a^\dagger]]=0$, 利用 Baker-Hansdoff 公式, 有

$$D^{-1}(\alpha)=e^{\alpha^*a-\alpha a^\dagger}=e^{\alpha^*a}e^{-\alpha a^\dagger}e^{-[\alpha^*a,\alpha a^\dagger]/2}=e^{\alpha^*a}e^{-\alpha a^\dagger}e^{|\alpha|^2/2}, \tag{6}$$

从而

$$D^{-1}(\alpha)aD(\alpha)=e^{\alpha^*a}e^{-\alpha a^\dagger}e^{|\alpha|^2/2}ae^{\alpha a^\dagger}e^{-\alpha^*a}e^{-|\alpha|^2/2}=e^{\alpha^*a}e^{-\alpha a^\dagger}ae^{\alpha a^\dagger}e^{-\alpha^*a}, \tag{7}$$

利用 $e^{-xa^\dagger}ae^{xa^\dagger}=a+x$ 得

$$D^{-1}(\alpha)aD(\alpha)=e^{\alpha^*a}(a+x)e^{-\alpha^*a}=a+\alpha. \tag{8}$$

□

第 3 题 得分: ________. 证明:

$$a^\dagger|\alpha\rangle\langle\alpha|=(\alpha^*+\frac{\partial}{\partial\alpha})|\alpha\rangle\langle\alpha|,$$
$$|\alpha\rangle\langle\alpha|a=(\alpha+\frac{\partial}{\partial\alpha^*})|\alpha\rangle\langle\alpha|.$$

证: 相干态

$$|\alpha\rangle=e^{-|\alpha|^2/2}\sum_n\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{9}$$

故

$$|\alpha\rangle\langle\alpha|=e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m|. \tag{10}$$

第一个等式左边:

$$\begin{aligned}a^\dagger|\alpha\rangle\langle\alpha|=&a^\dagger e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m|\\=&e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}\sqrt{n+1}|n+1\rangle\langle m|\\=&e^{-|\alpha|^2}\sum_{n=1}^\infty\sum_{m=0}^\infty\frac{\alpha^{n-1}}{\sqrt{(n-1)!}}\frac{(\alpha^*)^m}{\sqrt{m!}}\sqrt{n}|n\rangle\langle m|\end{aligned} \tag{11}$$

第一个等式右边:

$$\begin{aligned}&(\alpha^*+\frac{\partial}{\partial\alpha})|\alpha\rangle\langle\alpha|\\=&(\alpha^*+\frac{\partial}{\partial\alpha})e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m| &(12)\\=&\alpha^*e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m|-\alpha^*e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m|+e^{-|\alpha|^2}\sum_{n,m}\sqrt{n}\frac{\alpha^{n-1}}{\sqrt{(n-1)!}}\frac{(a^*)^m}{\sqrt{m!}}|n\rangle\langle m|\\=&e^{-|\alpha|^2}\sum_{n,m}\sqrt{n}\frac{\alpha^{n-1}}{\sqrt{(n-1)!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m|\\=&e^{-|\alpha|^2}\sum_{n=1}^\infty\sum_{m=0}^\infty\sqrt{n}\frac{\alpha^{n-1}}{\sqrt{(n-1)!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m|, &(13)\end{aligned}$$

故

$$a^\dagger|\alpha\rangle\langle\alpha|=(\alpha^*+\frac{\partial}{\partial\alpha})|\alpha\rangle\langle\alpha|. \tag{14}$$

同理, 第二个等式左边:

$$\begin{aligned}|\alpha\rangle\langle\alpha|a=&e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m|a\\=&e^{-|\alpha|^2}\sum_{n,m}\sqrt{m+1}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m+1|\\=&e^{-|\alpha|^2}\sum_{n=0}^\infty\sum_{m=1}^\infty\sqrt{m}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^{m-1}}{\sqrt{(m-1)!}}|n\rangle\langle m|\end{aligned}$$

第二个等式右边:

$$(\alpha+\frac{\partial}{\partial\alpha^*})|\alpha\rangle\langle\alpha|$$

$$
\begin{aligned}
=&(\alpha+\frac{\partial}{\partial\alpha^*})e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m|\\
=&\alpha e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m|-\alpha e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m|+e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\sqrt{m}\frac{(\alpha^*)^{m-1}}{\sqrt{(m-1)!}}|n\rangle\langle m|\\
=&e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\sqrt{m}\frac{(\alpha^*)^{m-1}}{\sqrt{(m-1)!}}|n\rangle\langle m|\\
=&e^{-|\alpha|^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\sqrt{m}\frac{(\alpha^*)^{m-1}}{\sqrt{(m-1)!}}|n\rangle\langle m|, \qquad (15)
\end{aligned}
$$

故

$$
|\alpha\rangle\langle\alpha|a=(\alpha+\frac{\partial}{\partial\alpha^*})|\alpha\rangle\langle\alpha|. \qquad (16)
$$

□