作业二

截止时间: 2022 年 3 月 18 日 (周五)

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成绩：__________

第 1 题 得分: ________. 求单模热光场的平均光子数.

解: 单模热光场的平均光子数为

$$\begin{aligned}
\langle N\rangle =&\mathrm{Tr}(\rho a^\dagger a)\\
=&\sum_m\langle m|\left[1-\exp\left(-\frac{\hbar\omega}{kT}\right)\right]\sum_n\exp\left(-n\frac{\hbar\omega}{kT}\right)|n\rangle\langle n|a^\dagger a|m\rangle\\
=&\left[1-\exp\left(-\frac{\hbar\omega}{kT}\right)\right]\sum_{m,n}\exp\left(-n\frac{\hbar\omega}{kT}\right)\langle m|n\rangle\langle n|a^\dagger a|m\rangle\\
=&\left[1-\exp\left(-\frac{\hbar\omega}{kT}\right)\right]\sum_{n}\exp\left(-n\frac{\hbar\omega}{kT}\right)\langle n|a^\dagger a|n\rangle\\
=&\left[1-\exp\left(-\frac{\hbar\omega}{kT}\right)\right]\sum_{n}\exp\left(-n\frac{\hbar\omega}{kT}\right)n\\
=&\frac{\exp\left(-\frac{\hbar\omega}{kT}\right)}{1-\exp\left(-\frac{\hbar\omega}{kT}\right)}\\
=&\frac{1}{\exp\left(\frac{\hbar\omega}{kT}\right)-1}.
\end{aligned}\tag{1}$$

□

第 2 题 得分: ________. 证明平移算符的性质:

$$D^\dagger=D(-\alpha)=[D(\alpha)]^{-1},$$

$$D^{-1}(\alpha)aD(\alpha)=a+\alpha.$$

证: 位移算符

$$D(\alpha)=e^{\alpha a^\dagger-\alpha^*a}=e^{\alpha a^\dagger}e^{-\alpha^*a}e^{-|\alpha|^2/2}.\tag{2}$$

先证第一个等式: 由于

$$D^\dagger(\alpha)=e^{\alpha^*a-\alpha a^\dagger},\tag{3}$$

$$D(-\alpha)=e^{-\alpha a^\dagger+\alpha^*a},\tag{4}$$

$$[D(\alpha)]^{-1}=e^{-\alpha a^\dagger+\alpha^*a},\tag{5}$$

故

$$D^\dagger(\alpha)=D(-\alpha)=[D(\alpha)]^{-1}.\tag{6}$$

再证第二个等式: 由于 $[a,[a,a^\dagger]]=[a^\dagger,[a,a^\dagger]]=0$, 利用 Baker-Hansdoff 公式, 有

$$D^{-1}(\alpha)=e^{\alpha^*a-\alpha a^\dagger}=e^{\alpha^*a}e^{-\alpha a^\dagger}e^{-[\alpha^*a,\alpha a^\dagger]/2}=e^{\alpha^*a}e^{-\alpha a^\dagger}e^{|\alpha|^2/2},\tag{7}$$

从而

$$D^{-1}(\alpha)aD(\alpha)=e^{\alpha^*a}e^{-\alpha a^\dagger}e^{|\alpha|^2/2}ae^{\alpha a^\dagger}e^{-\alpha^*a}e^{-|\alpha|^2/2}=e^{\alpha^*a}e^{-\alpha a^\dagger}ae^{\alpha a^\dagger}e^{-\alpha^*a},\tag{8}$$

利用 $e^{-xa^\dagger}ae^{xa^\dagger}=a+x$ 得

$$D^{-1}(\alpha)aD(\alpha)=e^{\alpha^*a}(a+x)e^{-\alpha^*a}=a+\alpha.\tag{9}$$

□

第 3 题 得分: ________. 证明:

$$a^\dagger|\alpha\rangle\langle\alpha| = (\alpha^* + \frac{\partial}{\partial\alpha})|\alpha\rangle\langle\alpha|,$$
$$|\alpha\rangle\langle\alpha|a = (\alpha + \frac{\partial}{\partial\alpha^*})|\alpha\rangle\langle\alpha|.$$

证: 相干态

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{10}$$

故

$$|\alpha\rangle\langle\alpha| = e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m|. \tag{11}$$

第一个等式左边:

$$\begin{aligned}a^\dagger|\alpha\rangle\langle\alpha| =& a^\dagger e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m| \\ =& e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}\sqrt{n+1}|n+1\rangle\langle m| \\ =& e^{-|\alpha|^2}\sum_{n=1}^\infty\sum_{m=0}^\infty\frac{\alpha^{n-1}}{\sqrt{(n-1)!}}\frac{(\alpha^*)^m}{\sqrt{m!}}\sqrt{n}|n\rangle\langle m|\end{aligned} \tag{12}$$

第一个等式右边:

$$\begin{aligned}&(\alpha^* + \frac{\partial}{\partial\alpha})|\alpha\rangle\langle\alpha| \\ =&(\alpha^* + \frac{\partial}{\partial\alpha})e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m| \\ =&\alpha^* e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m| - \alpha^* e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m| + e^{-|\alpha|^2}\sum_{n,m}\sqrt{n}\frac{\alpha^{n-1}}{\sqrt{(n-1)!}}\frac{(a^*)^m}{\sqrt{m!}}|n\rangle\langle m| \\ =&e^{-|\alpha|^2}\sum_{n,m}\sqrt{n}\frac{\alpha^{n-1}}{\sqrt{(n-1)!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m| \\ =&e^{-|\alpha|^2}\sum_{n=1}^\infty\sum_{m=0}^\infty\sqrt{n}\frac{\alpha^{n-1}}{\sqrt{(n-1)!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m|,\end{aligned} \tag{13}$$

故

$$a^\dagger|\alpha\rangle\langle\alpha| = (\alpha^* + \frac{\partial}{\partial\alpha})|\alpha\rangle\langle\alpha|. \tag{14}$$

同理, 第二个等式左边:

$$\begin{aligned}|\alpha\rangle\langle\alpha|a =& e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m|a \\ =& e^{-|\alpha|^2}\sum_{n,m}\sqrt{m+1}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m+1| \\ =& e^{-|\alpha|^2}\sum_{n=0}^\infty\sum_{m=1}^\infty\sqrt{m}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^{m-1}}{\sqrt{(m-1)!}}|n\rangle\langle m|\end{aligned} \tag{15}$$

第二个等式右边:

$$(\alpha + \frac{\partial}{\partial\alpha^*})|\alpha\rangle\langle\alpha|$$

$$
\begin{aligned}
&=(\alpha+\frac{\partial}{\partial\alpha^*})e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m|\\
&=\alpha e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m|-\alpha e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m|+e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\sqrt{m}\frac{(\alpha^*)^{m-1}}{\sqrt{(m-1)!}}|n\rangle\langle m|\\
&=e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\sqrt{m}\frac{(\alpha^*)^{m-1}}{\sqrt{(m-1)!}}|n\rangle\langle m|\\
&=e^{-|\alpha|^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\sqrt{m}\frac{(\alpha^*)^{m-1}}{\sqrt{(m-1)!}}|n\rangle\langle m|, \qquad (16)
\end{aligned}
$$

故

$$|\alpha\rangle\langle\alpha|a=(\alpha+\frac{\partial}{\partial\alpha^*})|\alpha\rangle\langle\alpha|. \qquad (17)$$

□