作业四

截止时间: 2022 年 4 月 29 日 (周五)

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成绩：__________

第 1 题 得分: __________. $\hat{C}=\frac{1}{2}\Delta\sigma_z+g(\sigma_+a+a^\dagger\sigma_-)$, $\hat{N}=a^\dagger a+\sigma_+\sigma_-$, $\Delta=\omega-\nu$,
证明: $\hat{C}^2=\frac{\Delta^2}{4}+g^2\hat{N}^2$.

证: 等式左边

$$\begin{aligned}\hat{C}^2=&\left[\frac{1}{2}\Delta\sigma_z+g(\sigma_+a+a^\dagger\sigma_-)\right]^2\\=&\frac{\Delta^2}{4}\sigma_z^2+\frac{1}{2}g\Delta[\sigma_z(\sigma_+a+a^\dagger\sigma_-)+(\sigma_+a+a^\dagger\sigma_-)\sigma_z]+g^2(\sigma_+a+a^\dagger\sigma_-)^2\\=&\frac{\Delta^2}{4}+\frac{1}{2}g\Delta[\sigma_+a-a^\dagger\sigma_--\sigma_+a+a^\dagger\sigma_-]+g^2(\sigma_+a\sigma_+a+\sigma_+aa^\dagger\sigma_-+a^\dagger\sigma_-\sigma_+a+a^\dagger\sigma_-a^\dagger\sigma_-)\\=&\frac{\Delta^2}{4}+g^2[\sigma_+\sigma_-(1+a^\dagger a)+(1-\sigma_+\sigma_-)a^\dagger a]\\=&\frac{\Delta^2}{4}+g^2[\sigma_+\sigma_-+a^\dagger a].\end{aligned}\tag{1}$$

等式右边

$$\begin{aligned}\frac{\Delta^2}{4}+g^2\hat{N}^2=&\frac{\Delta^2}{4}+g^2(a^\dagger a+\sigma_+\sigma_-)^2\\=&\frac{\Delta^2}{4}+g^2(a^\dagger aa^\dagger a+a^\dagger a\sigma_+\sigma_-+\sigma_+\sigma_-a^\dagger a+\sigma_+\sigma_-\sigma_+\sigma_-)\\=&\frac{\Delta^2}{4}+g^2[\sigma_+\sigma_-+a^\dagger a].\end{aligned}\tag{2}$$

故等式成立. □

第 2 题 得分: __________. 某一原子-光场相互作用总 Hamiltonian:

$$\hat{H}=\hbar\nu a^\dagger a+\frac{1}{2}\hbar\omega\sigma_z+\hbar g[\sigma_+a(a^\dagger a)^{1/2}+(a^\dagger a)^{1/2}a^\dagger\sigma_-],$$

初态原子处于上能级 $|a\rangle$, 光场处于粒子数态 $|1\rangle$, 求 $W(t)=|c_a(t)|^2-|c_b(t)|^2$.

解: 取哈密顿量

$$\hat{H}=\hat{H}_0+\hat{H}_1,\tag{3}$$

其中

$$\hat{H}_0=\hbar\nu a^\dagger a+\frac{1}{2}\hbar\omega\sigma_z,\tag{4}$$

$$\hat{H}_1=\hbar g[\sigma_+a(a^\dagger a)^{1/2}+(a^\dagger a)^{1/2}a^\dagger\sigma_-].\tag{5}$$

在相互作用绘景下, 微扰哈密顿量为

$$\begin{aligned}\hat{V}=&e^{i\hat{H}_0t/\hbar}\hat{H}_1e^{-i\hat{H}_0t/\hbar}\\=&e^{i\nu a^\dagger at}e^{i\frac{1}{2}\omega\sigma_zt}\hbar g[\sigma_+a(a^\dagger a)^{1/2}+(a^\dagger a)^{1/2}a^\dagger\sigma_-]e^{-i\frac{1}{2}\omega\sigma_zt}e^{-i\nu a^\dagger at}\\=&\hbar g\left[e^{i\nu a^\dagger at}a(a^\dagger a)^{1/2}e^{-i\nu a^\dagger at}e^{i\frac{1}{2}\omega\sigma_zt}\sigma_+e^{-i\frac{1}{2}\omega\sigma_zt}+e^{i\nu a^\dagger at}(a^\dagger a)^{1/2}a^\dagger e^{-i\nu a^\dagger at}e^{i\frac{1}{2}\omega\sigma_zt}\sigma_-e^{-i\frac{1}{2}\omega\sigma_zt}\right]\\=&\hbar g\left[e^{i\nu a^\dagger at}ae^{-i\nu a^\dagger at}(a^\dagger a)^{1/2}e^{i\frac{1}{2}\omega\sigma_zt}\sigma_+e^{-i\frac{1}{2}\omega\sigma_zt}+(a^\dagger a)^{1/2}e^{i\nu a^\dagger at}a^\dagger e^{-i\nu a^\dagger at}e^{i\frac{1}{2}\omega\sigma_zt}\sigma_-e^{-i\frac{1}{2}\omega\sigma_zt}\right]\end{aligned}\tag{6}$$

利用公式 $e^{\alpha A}Be^{-\alpha A}=B+\alpha[A,B]+\frac{\alpha^2}{2!}[A,[A,B]]+\cdots$ 可得

$$e^{i\nu a^\dagger at}ae^{-i\nu a^\dagger at}=ae^{-i\nu t},\tag{7}$$

$$e^{i\nu a^\dagger at}a^\dagger e^{-i\nu a^\dagger at}=a^\dagger e^{i\nu t}, \tag{8}$$
$$e^{i\frac{1}{2}\omega\sigma_z t}\sigma_+ e^{-i\frac{1}{2}\omega\sigma_z t}=\sigma_+ e^{i\omega t}, \tag{9}$$
$$e^{i\frac{1}{2}\omega\sigma_z t}\sigma_- e^{-i\frac{1}{2}\omega\sigma_z t}=\sigma_- e^{-i\omega t}, \tag{10}$$

从而

$$\hat{V}=\hbar g\left[\sigma_+ a(a^\dagger a)^{1/2}e^{i\Delta t}+(a^\dagger a)^{1/2}a^\dagger e^{-i\Delta t}\right], \tag{11}$$

其中失谐频率差 $\Delta=\omega-\nu$. 系统的一般量子态为

$$|\psi(t)\rangle=\sum_{n=0}^{\infty}[C_{a,n}(t)|a,n\rangle+C_{b,n}(t)|b,n\rangle]. \tag{12}$$

相互作用绘景下薛定谔方程为

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle=\hat{V}|\psi(t)\rangle, \tag{13}$$

$$\begin{aligned}\Longrightarrow i\hbar\sum_n[\dot{C}_{a,n}(t)|a,n\rangle+\dot{C}_{b,n}(t)|b,n\rangle]=&\sum_n\hbar g(\sigma_+ a(a^\dagger a)^{1/2}e^{i\Delta t}+(a^\dagger a)^{1/2}a^\dagger\sigma_- e^{-i\Delta t})[C_{a,n}(t)|a,n\rangle+C_{b,n}(t)|b,n\rangle]\\ =&\sum_n\hbar g[C_{a,n}(t)e^{-i\Delta t}\sqrt{n(n+1)}|b,n+1\rangle+C_{b,n}(t)e^{i\Delta t}\sqrt{(n-1)n}|a,n-1\rangle],\end{aligned} \tag{14}$$

$$\dot{C}_{a,n}(t)=-igC_{b,n+1}\sqrt{n(n+1)}e^{i\Delta t}, \tag{15}$$
$$\dot{C}_{b,n+1}(t)=-igC_{a,n}\sqrt{n(n+1)}e^{-i\Delta t}. \tag{16}$$

由 (15) 得

$$C_{b,n+1}(t)=\frac{i\dot{C}_{a,n}(t)}{g\sqrt{n(n+1)}e^{-i\Delta t}}, \tag{17}$$

再代入 (16) 中得

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{i\dot{C}_{a,n}(t)}{g\sqrt{n(n+1)}}e^{-i\Delta t}\right]=-igC_{a,n}\sqrt{n(n+1)}e^{-i\Delta t}, \tag{18}$$

从而解得

$$C_{a,n}(t)=\left\{C_{a,n}(0)\left[\cos\left(\frac{\Omega_n t}{2}\right)-\frac{i\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right]-\frac{2ig\sqrt{n(n+1)}}{\Omega_n}C_{b,n+1}(0)\sin\left(\frac{\Omega_n t}{2}\right)\right\}e^{i\Delta t/2}, \tag{19}$$
$$C_{b,n+1}(t)=\left\{C_{b,n+1}(0)\left[\cos\left(\frac{\Omega_n t}{2}\right)+\frac{i\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right]-\frac{2ig\sqrt{n(n+1)}}{\Omega_n}C_{a,n}(0)\sin\left(\frac{\Omega_n t}{2}\right)\right\}e^{-i\Delta t/2}, \tag{20}$$

其中振荡频率

$$\Omega_n=\sqrt{\Delta^2+4g^2n(n+1)}. \tag{21}$$

当初态原子处于上能级 $|a\rangle$, 光场处于粒子数态 $|1\rangle$, 即系统初态为 $|a,1\rangle$, 即 $C_{a,1}(0)=1$ 时,

$$C_{a,1}(t)=\left[\cos\left(\frac{\Omega_n t}{2}\right)-\frac{i\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right]e^{i\Delta t/2}, \tag{22}$$

$$C_{b,2}(t)=-\frac{2ig\sqrt{n(n+1)}}{\Omega_n}\sin\left(\frac{\Omega_nt}{2}\right)e^{-i\Delta t/2}. \tag{23}$$

故

$$W(t)=\left|C_a(t)\right|^2-\left|C_b(t)\right|^2=\left|C_{a,1}(t)\right|^2-\left|C_{b,2}(t)\right|^2=\cos^2\left(\frac{\Omega_nt}{2}\right)+\frac{\Delta^2-4g^2n(n+1)}{\Omega_n^2}\sin^2\left(\frac{\Omega_nt}{2}\right). \tag{24}$$

□