# 作业五

截止时间: 2022 年 5 月 20 日 (周五)

姓名：陈　稼　霖
学号：SA21038052
成绩：__________

**第 1 题 得分: ________.**

$$\dot{\rho}=-\frac{\mathscr{C}}{2}\bar{n}_{\mathrm{th}}(aa^\dagger\rho-2a^\dagger\rho a+\rho aa^\dagger)-\frac{\mathscr{C}}{2}(\bar{n}_{\mathrm{th}}+1)(a^\dagger a\rho-2a\rho a^\dagger+\rho a^\dagger a)$$

单模场同热库耦合, 求 $\langle a^\dagger a\rangle(t)$, 即 $\langle\hat{n}(t)\rangle$, 初始为 $n(0)$.

**解:** $\langle a^\dagger a\rangle$ 的运动方程为

$$\begin{aligned}
&\frac{\mathrm{d}\langle a^\dagger a\rangle}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Tr}(\rho a^\dagger a)=\operatorname{Tr}(\dot{\rho}a^\dagger a)\\
&=\operatorname{Tr}\left\{\left[-\frac{\mathscr{C}}{2}\bar{n}_{\mathrm{th}}(aa^\dagger\rho-2a^\dagger\rho a+\rho aa^\dagger)-\frac{\mathscr{C}}{2}(\bar{n}_{\mathrm{th}}+1)(a^\dagger a\rho-2a\rho a^\dagger+\rho a^\dagger a)\right]a^\dagger a\right\}\\
&=\operatorname{Tr}\left\{-\frac{\mathscr{C}}{2}\bar{n}_{\mathrm{th}}(aa^\dagger\rho a^\dagger a-2a^\dagger\rho aa^\dagger a+\rho aa^\dagger a^\dagger a)-\frac{\mathscr{C}}{2}(\bar{n}_{\mathrm{th}}+1)(a^\dagger a\rho a^\dagger a-2a\rho a^\dagger a^\dagger a+\rho a^\dagger aa^\dagger a)\right\}\\
&=\operatorname{Tr}\left\{\rho\left[-\frac{\mathscr{C}}{2}\bar{n}_{\mathrm{th}}(a^\dagger aaa^\dagger-2aa^\dagger aa^\dagger+aa^\dagger a^\dagger a)-\frac{\mathscr{C}}{2}(\bar{n}_{\mathrm{th}}+1)(a^\dagger aa^\dagger a-2a^\dagger a^\dagger aa+a^\dagger aa^\dagger a)\right]\right\}\\
&=\operatorname{Tr}\left\{\rho\left[-\frac{\mathscr{C}}{2}\bar{n}_{\mathrm{th}}(a^\dagger a(a^\dagger a+1)-2(a^\dagger a+1)(a^\dagger a+1)+(a^\dagger a+1)a^\dagger a)-\frac{\mathscr{C}}{2}(\bar{n}_{\mathrm{th}}+1)(2a^\dagger(a^\dagger a+1)a-2a^\dagger a^\dagger aa)\right]\right\}\\
&=\operatorname{Tr}\left\{\rho\left[-\frac{\mathscr{C}}{2}\bar{n}_{\mathrm{th}}(-2a^\dagger a-2)-\frac{\mathscr{C}}{2}(\bar{n}_{\mathrm{th}}+1)(2a^\dagger a)\right]\right\}\\
&=\operatorname{Tr}\{\mathscr{C}\rho(\bar{n}_{\mathrm{th}}-a^\dagger a)\}\\
&=\mathscr{C}(n_{\mathrm{th}}-\langle a^\dagger a\rangle).
\end{aligned}\tag{1}$$

考虑到 $\langle a^\dagger a\rangle$ 的初始值为 $n(0)$, 解上式得

$$\langle a^\dagger a\rangle(t)=[n(0)-\bar{n}_{\mathrm{th}}]e^{-\mathscr{C}t}+\bar{n}_{\mathrm{th}}.\tag{2}$$

□

**第 2 题 得分: ________.** 某系统与环境相互作用 Hamiltonian 为:

$$\hat{V}=\Delta\sigma_z\sum_{\boldsymbol{k}}(a_{\boldsymbol{k}}^\dagger e^{-i\omega t}+a_{\boldsymbol{k}}e^{i\omega t})$$

环境算符 $(a_{\boldsymbol{k}}^\dagger,a_{\boldsymbol{k}})$, 环境为热平衡谐振子热库, 求系统演化的主方程 $\dot{\rho}_s$ (依据课本 (8.1.7) 式).

**解:** 将 $\rho_S$ 的刘维尔方程截断至关于 $\hat{V}$ 的二阶项并对热库做偏迹得到课本 (8.1.7) 式:

$$\dot{\rho}_S=-\frac{i}{\hbar}\operatorname{Tr}_R[\hat{V}(t),\rho_S(t_i)\otimes\rho_R(t_i)]-\frac{1}{\hbar^2}\operatorname{Tr}_R\int_{t_i}^t[\hat{V}(t),[\hat{V}(t'),\rho_S(t')\otimes\rho_R(t')]]\,\mathrm{d}t',\tag{3}$$

其中热平衡谐振子热库的密度算符

$$\rho_R=\prod_{\boldsymbol{k}}\left[1-\exp\left(-\frac{\hbar\nu_k}{k_BT}\right)\right]\exp\left(-\frac{\hbar\nu_k a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}}{k_BT}\right),\tag{4}$$

其中 $k_B$ 为玻尔兹曼常数, $T$ 为温度. 将相互作用 Hamiltonian 的具体表达式代入课本 (8.1.7) 式得

$$\begin{aligned}
\dot{\rho}_S=&-\frac{i\Delta}{\hbar}\sum_{\boldsymbol{k}}\langle a_{\boldsymbol{k}}^\dagger\rangle[\sigma_z,\rho_S(t)]e^{-i\omega t}\\
&-\frac{\Delta^2}{\hbar^2}\int_{t_i}^t\mathrm{d}t'\sum_{\boldsymbol{k},\boldsymbol{k}'}\{[\sigma_z\sigma_z\rho_S(t')-2\sigma_z\rho_S(t')\sigma_z+\rho_S(t')\sigma_z\sigma_z]e^{-i\omega(t+t')}\langle a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}'}^\dagger\rangle\\
&+[\sigma_z\sigma_z\rho_S(t')-\sigma_z\rho_S(t')\sigma_z]e^{-i\omega(t-t')}\langle a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}'}\rangle
\end{aligned}$$

$$
\begin{aligned}
&+[\sigma_z\sigma_z\rho_S(t')-\sigma_z\rho_S(t')\sigma_z]e^{i\omega(t-t')}\langle a_{\boldsymbol{k}}a_{\boldsymbol{k}'}^\dagger\rangle\}+\text{H.C.}\\
=&-\frac{i\Delta}{\hbar}\sum_{\boldsymbol{k}}\langle a_{\boldsymbol{k}}^\dagger\rangle[\sigma_z,\rho_S(t)]e^{-i\omega t}\\
&-\frac{\Delta^2}{\hbar^2}\int_{t_i}^t\mathrm{d}t'\sum_{\boldsymbol{k},\boldsymbol{k}'}\{[\rho_S(t')-2\sigma_z\rho_S(t')\sigma_z+\rho_S(t')]e^{-i\omega(t+t')}\langle a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}'}^\dagger\rangle\\
&+[\rho_S(t')-\sigma_z\rho_S(t')\sigma_z]e^{-i\omega(t-t')}\langle a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}'}\rangle\\
&+[\rho_S(t')-\sigma_z\rho_S(t')\sigma_z]e^{i\omega(t-t')}\langle a_{\boldsymbol{k}}a_{\boldsymbol{k}'}^\dagger\rangle\}+\text{H.C.}
\end{aligned}\tag{5}
$$

其中

$$
\begin{aligned}
\langle a_{\boldsymbol{k}}\rangle=\text{Tr}_R(\rho_Ra_{\boldsymbol{k}})=\langle a_{\boldsymbol{k}}^\dagger\rangle=\text{Tr}_R(\rho_Ra_{\boldsymbol{k}}^\dagger)=&0, &(6)\\
\langle a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}'}\rangle=\text{Tr}_R(\rho_Ra_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}'})=&\bar{n}_{\boldsymbol{k}}\delta_{\boldsymbol{k}\boldsymbol{k}'}, &(7)\\
\langle a_{\boldsymbol{k}}a_{\boldsymbol{k}'}^\dagger\rangle=\text{Tr}_R(\rho_Ra_{\boldsymbol{k}}a_{\boldsymbol{k}'}^\dagger)=&(\bar{n}_{\boldsymbol{k}}+1)\delta_{\boldsymbol{k}\boldsymbol{k}'}, &(8)\\
\langle a_{\boldsymbol{k}}a_{\boldsymbol{k}'}\rangle=\text{Tr}_R(\rho_Ra_{\boldsymbol{k}}a_{\boldsymbol{k}'})=\langle a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}'}^\dagger\rangle=\text{Tr}_R(\rho_Ra_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}'}^\dagger)=&0, &(9)
\end{aligned}
$$

模式 $\boldsymbol{k}$ 的平均光子数

$$
\bar{n}_{\boldsymbol{k}}=\frac{1}{\exp\left(\frac{\hbar\nu_k}{k_BT}\right)-1},\tag{10}
$$

故

$$
\dot{\rho}_S=-\frac{\Delta^2}{\hbar^2}\int_{t_i}^t\mathrm{d}t'\sum_{\boldsymbol{k}}\{[\rho_S(t')-\sigma_z\rho_S(t')\sigma_z]e^{-i\omega(t-t')}\bar{n}_{\boldsymbol{k}}+[\rho_S(t')-\sigma_z\rho_S(t')\sigma_z]e^{i\omega(t-t')}(\bar{n}_{\boldsymbol{k}}+1)\}+\text{H.C.}.\tag{11}
$$

将对 $\boldsymbol{k}$ 的求和换成积分, 即

$$
\sum_{\boldsymbol{k}}\to2\frac{V}{(2\pi)^3}\int_0^{2\pi}\mathrm{d}\phi\int_0^\pi\mathrm{d}\theta\,\sin\theta\int_0^\infty\mathrm{d}k\,k^2=2\frac{V}{(2\pi)^3c^3}\int_0^{2\pi}\mathrm{d}\phi\int_0^\pi\mathrm{d}\theta\,\sin\theta\int_0^\infty\mathrm{d}\nu_k\,\nu_k^2,\tag{12}
$$

可得

$$
\begin{aligned}
\dot{\rho}_S=&-\frac{\Delta^2}{\hbar^2}\frac{V}{\pi^2c^3}\int_{t_i}^t\mathrm{d}t'\int_0^{+\infty}\mathrm{d}\nu_k\,\nu_k^2\{[\rho_S(t')-\sigma_z\rho_S(t')\sigma_z]e^{-i\omega(t-t')}\bar{n}_{\boldsymbol{k}}+[\rho_S(t')-\sigma_z\rho_S(t')\sigma_z]e^{i\omega(t-t')}(\bar{n}_{\boldsymbol{k}}+1)\}\\
&+\text{H.C.}.
\end{aligned}\tag{13}
$$

在马尔科夫近似下, $\rho_S(t')=\rho_S(t)$, 利用

$$
\int_{t_i}^t\mathrm{d}t'\,e^{-i\omega(t-t')}=\pi\delta(\omega),\tag{14}
$$

可得

$$
\dot{\rho}_S=-\frac{\Delta^2}{\hbar^2}\frac{V}{\pi^2c^3}\pi\delta(\omega)\int_0^{+\infty}\mathrm{d}\nu_k\,\nu_k^2\{[\rho_S(t')-\sigma_z\rho_S(t')\sigma_z]\bar{n}_{\boldsymbol{k}}+[\rho_S(t')-\sigma_z\rho_S(t')\sigma_z](\bar{n}_{\boldsymbol{k}}+1)\}+\text{H.C.}.\tag{15}
$$

□