

第 1 题 得分: _____. 求单模热光场的平均光子数.

解: 单模热光场的平均光子数为

$$\begin{aligned}
 \langle N \rangle &= \text{Tr}(\rho a^\dagger a) \\
 &= \sum_m \langle m | \left[1 - \exp\left(-\frac{\hbar\omega}{kT}\right) \right] \sum_n \exp\left(-n\frac{\hbar\omega}{kT}\right) |n\rangle \langle n| a^\dagger a |m\rangle \\
 &= \left[1 - \exp\left(-\frac{\hbar\omega}{kT}\right) \right] \sum_{m,n} \exp\left(-n\frac{\hbar\omega}{kT}\right) \langle m|n\rangle \langle n| a^\dagger a |m\rangle \\
 &= \left[1 - \exp\left(-\frac{\hbar\omega}{kT}\right) \right] \sum_n \exp\left(-n\frac{\hbar\omega}{kT}\right) \langle n| a^\dagger a |n\rangle \\
 &= \left[1 - \exp\left(-\frac{\hbar\omega}{kT}\right) \right] \sum_n \exp\left(-n\frac{\hbar\omega}{kT}\right) n \\
 &= \frac{\exp\left(-\frac{\hbar\omega}{kT}\right)}{1 - \exp\left(-\frac{\hbar\omega}{kT}\right)} \\
 &= \frac{1}{\exp\left(\frac{\hbar\omega}{kT}\right) - 1}.
 \end{aligned} \tag{1}$$

□

第 2 题 得分: _____. 证明平移算符的性质:

$$\begin{aligned}
 D^\dagger &= D(-\alpha) = [D(\alpha)]^{-1}, \\
 D^{-1}(\alpha) a D(\alpha) &= a + \alpha.
 \end{aligned}$$

证: 位移算符

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} = e^{\alpha a^\dagger} e^{-\alpha^* a} e^{-|\alpha|^2/2}. \tag{2}$$

先证第一个等式: 由于

$$D^\dagger(\alpha) = e^{\alpha^* a - \alpha a^\dagger}, \tag{3}$$

$$D(-\alpha) = e^{-\alpha a^\dagger + \alpha^* a}, \tag{4}$$

$$[D(\alpha)]^{-1} = e^{-\alpha a^\dagger + \alpha^* a}, \tag{5}$$

故

$$D^\dagger(\alpha) = D(-\alpha) = [D(\alpha)]^{-1}. \tag{6}$$

再证第二个等式: 由于 $[a, [a, a^\dagger]] = [a^\dagger, [a, a^\dagger]] = 0$, 利用 Baker-Hansdoff 公式, 有

$$D^{-1}(\alpha) = e^{\alpha^* a - \alpha a^\dagger} = e^{\alpha^* a} e^{-\alpha a^\dagger} e^{-[\alpha^* a, \alpha a^\dagger]/2} = e^{\alpha^* a} e^{-\alpha a^\dagger} e^{|\alpha|^2/2}, \tag{7}$$

从而

$$D^{-1}(\alpha) a D(\alpha) = e^{\alpha^* a} e^{-\alpha a^\dagger} e^{|\alpha|^2/2} a e^{\alpha a^\dagger} e^{-\alpha^* a} e^{-|\alpha|^2/2} = e^{\alpha^* a} e^{-\alpha a^\dagger} a e^{\alpha a^\dagger} e^{-\alpha^* a}, \tag{8}$$

利用 $e^{-xa^\dagger} a e^{xa^\dagger} = a + x$ 得

$$D^{-1}(\alpha) a D(\alpha) = e^{\alpha^* a} (a + x) e^{-\alpha^* a} = a + \alpha. \tag{9}$$

□

第 3 题 得分: _____. 证明:

$$a^\dagger|\alpha\rangle\langle\alpha|=(\alpha^*+\frac{\partial}{\partial\alpha})|\alpha\rangle\langle\alpha|,$$

$$|\alpha\rangle\langle\alpha|a=(\alpha+\frac{\partial}{\partial\alpha^*})|\alpha\rangle\langle\alpha|.$$

证: 相干态

$$|\alpha\rangle=e^{-|\alpha|^2/2}\sum_n\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \quad (10)$$

故

$$|\alpha\rangle\langle\alpha|=e^{-|\alpha|^2}\sum_{n,m}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}|n\rangle\langle m|. \quad (11)$$

第一个等式左边:

$$\begin{aligned} a^\dagger|\alpha\rangle\langle\alpha| &= a^\dagger e^{-|\alpha|^2} \sum_{n,m} \frac{\alpha^n}{\sqrt{n!}} \frac{(\alpha^*)^m}{\sqrt{m!}} |n\rangle\langle m| \\ &= e^{-|\alpha|^2} \sum_{n,m} \frac{\alpha^n}{\sqrt{n!}} \frac{(\alpha^*)^m}{\sqrt{m!}} \sqrt{n+1} |n+1\rangle\langle m| \\ &= e^{-|\alpha|^2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha^{n-1}}{\sqrt{(n-1)!}} \frac{(\alpha^*)^m}{\sqrt{m!}} \sqrt{n} |n\rangle\langle m| \end{aligned} \quad (12)$$

第一个等式右边:

$$\begin{aligned} &(\alpha^* + \frac{\partial}{\partial\alpha})|\alpha\rangle\langle\alpha| \\ &= (\alpha^* + \frac{\partial}{\partial\alpha}) e^{-|\alpha|^2} \sum_{n,m} \frac{\alpha^n}{\sqrt{n!}} \frac{(\alpha^*)^m}{\sqrt{m!}} |n\rangle\langle m| \\ &= \alpha^* e^{-|\alpha|^2} \sum_{n,m} \frac{\alpha^n}{\sqrt{n!}} \frac{(\alpha^*)^m}{\sqrt{m!}} |n\rangle\langle m| - \alpha^* e^{-|\alpha|^2} \sum_{n,m} \frac{\alpha^n}{\sqrt{n!}} \frac{(\alpha^*)^m}{\sqrt{m!}} |n\rangle\langle m| + e^{-|\alpha|^2} \sum_{n,m} \sqrt{n} \frac{\alpha^{n-1}}{\sqrt{(n-1)!}} \frac{(\alpha^*)^m}{\sqrt{m!}} |n\rangle\langle m| \\ &= e^{-|\alpha|^2} \sum_{n,m} \sqrt{n} \frac{\alpha^{n-1}}{\sqrt{(n-1)!}} \frac{(\alpha^*)^m}{\sqrt{m!}} |n\rangle\langle m| \\ &= e^{-|\alpha|^2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sqrt{n} \frac{\alpha^{n-1}}{\sqrt{(n-1)!}} \frac{(\alpha^*)^m}{\sqrt{m!}} |n\rangle\langle m|, \end{aligned} \quad (13)$$

故

$$a^\dagger|\alpha\rangle\langle\alpha|=(\alpha^*+\frac{\partial}{\partial\alpha})|\alpha\rangle\langle\alpha|. \quad (14)$$

同理, 第二个等式左边:

$$\begin{aligned} |\alpha\rangle\langle\alpha|a &= e^{-|\alpha|^2} \sum_{n,m} \frac{\alpha^n}{\sqrt{n!}} \frac{(\alpha^*)^m}{\sqrt{m!}} |n\rangle\langle m|a \\ &= e^{-|\alpha|^2} \sum_{n,m} \sqrt{m+1} \frac{\alpha^n}{\sqrt{n!}} \frac{(\alpha^*)^m}{\sqrt{m!}} |n\rangle\langle m+1| \\ &= e^{-|\alpha|^2} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sqrt{m} \frac{\alpha^n}{\sqrt{n!}} \frac{(\alpha^*)^{m-1}}{\sqrt{(m-1)!}} |n\rangle\langle m| \end{aligned}$$

第二个等式右边:

$$(\alpha + \frac{\partial}{\partial\alpha^*})|\alpha\rangle\langle\alpha|$$

$$\begin{aligned}
&= (\alpha + \frac{\partial}{\partial \alpha^*}) e^{-|\alpha|^2} \sum_{n,m} \frac{\alpha^n}{\sqrt{n!}} \frac{(\alpha^*)^m}{\sqrt{m!}} |n\rangle \langle m| \\
&= \alpha e^{-|\alpha|^2} \sum_{n,m} \frac{\alpha^n}{\sqrt{n!}} \frac{(\alpha^*)^m}{\sqrt{m!}} |n\rangle \langle m| - \alpha e^{-|\alpha|^2} \sum_{n,m} \frac{\alpha^n}{\sqrt{n!}} \frac{(\alpha^*)^m}{\sqrt{m!}} |n\rangle \langle m| + e^{-|\alpha|^2} \sum_{n,m} \frac{\alpha^n}{\sqrt{n!}} \sqrt{m} \frac{(\alpha^*)^{m-1}}{\sqrt{(m-1)!}} |n\rangle \langle m| \\
&= e^{-|\alpha|^2} \sum_{n,m} \frac{\alpha^n}{\sqrt{n!}} \sqrt{m} \frac{(\alpha^*)^{m-1}}{\sqrt{(m-1)!}} |n\rangle \langle m| \\
&= e^{-|\alpha|^2} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{m} \frac{(\alpha^*)^{m-1}}{\sqrt{(m-1)!}} |n\rangle \langle m|, \tag{15}
\end{aligned}$$

故

$$|\alpha\rangle \langle \alpha| a = (\alpha + \frac{\partial}{\partial \alpha^*}) |\alpha\rangle \langle \alpha|. \tag{16}$$

□