

第 1 题 (20 分) 得分：_____. 产生湮灭算符 a^\dagger, a 满足对易关系 $[a, a^\dagger] = 1$, 且 $[a, (a^\dagger)^n] = n(a^\dagger)^{n-1}$, 试证:

(i) $[a^\dagger, a^m] = -ma^{m-1}$;

(ii) $a(a^\dagger)^n a^m a^\dagger = (a^\dagger)^{n+1} a^{m+1} + (m+n+1)(a^\dagger)^n a^m + mn(a^\dagger)^{n-1} a^{m-1}$.

证: (i) 利用数学归纳法证明:

– 当 $m = 1$ 时,

$$[a^\dagger, a^1] = -1 = -1 \cdot a^{1-1}. \quad (1)$$

– 假设当 $m = k$ 时,

$$[a^\dagger, a^k] = -ka^{k-1}, \quad (2)$$

则当 $m = k + 1$ 时,

$$[a^\dagger, a^{k+1}] = [a^\dagger, a^k]a + a^k[a^\dagger, a] = -ka^{k-1}a + a^k \cdot (-1) = -(k+1)a^k = -(k+1)a^{(k+1)-1}. \quad (3)$$

综上, $[a^\dagger, a^m] = -ma^{m-1}$.

(ii)

$$\begin{aligned} a(a^\dagger)^n a^m a^\dagger &= [(a^\dagger)^n a + n(a^\dagger)^{n-1}] a^m a^\dagger = (a^\dagger)^n a^{m+1} a^\dagger + n(a^\dagger)^{n-1} a^m a^\dagger \\ &= (a^\dagger)^n [a^\dagger a^{m+1} + (m+1)a^m] + n(a^\dagger)^{n-1} [a^\dagger a^m + ma^{m-1}] \\ &= (a^\dagger)^{n+1} a^{m+1} + (m+n+1)(a^\dagger)^n a^m + mn(a^\dagger)^{n-1} a^{m-1}. \end{aligned} \quad (4)$$

□

第 2 题 (20 分) 得分：_____. 有如下几种单模辐射场, 分别计算它们的光子数分布函数 $p(m)$:

(1) 数态的叠加 $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|10\rangle)$;

(2) $\rho = \sum_{n=0}^{\infty} \frac{e^{-\kappa} \kappa^n}{n!} |n\rangle\langle n|$, $\kappa \in \mathbb{R}^+$;

(3) 湮灭掉一个光子的热光场 $\rho' = \frac{a\rho a^\dagger}{\text{Tr}[a\rho a^\dagger]}$, 其中 ρ 是热光场, 即

$$\rho = \frac{1}{1+\langle n \rangle} \sum_{n=0}^{\infty} \left(\frac{\langle n \rangle}{1+\langle n \rangle} \right)^n |n\rangle\langle n|, \langle n \rangle \text{ 是 } \rho \text{ 光场的平均光子数.}$$

解: (1) 该叠加态的光子数分布函数为

$$p(m) = |\langle m|\psi\rangle|^2 = \frac{1}{2} |\delta_{m0} + \delta_{m,10}|^2. \quad (5)$$

(2) 该辐射场的光子数分布函数为

$$p(m) = \langle m|\rho|m\rangle = \frac{e^{-\kappa} \kappa^m}{m!}. \quad (6)$$

(3)

$$\begin{aligned} a\rho a^\dagger &= \frac{1}{1+\langle n \rangle} \sum_{n=0}^{\infty} \left(\frac{\langle n \rangle}{1+\langle n \rangle} \right)^{n+1} (n+1) |n\rangle\langle n|, \\ \text{Tr}[a\rho a^\dagger] &= \frac{1}{1+\langle n \rangle} \sum_{n=0}^{\infty} \left(\frac{\langle n \rangle}{1+\langle n \rangle} \right)^{n+1} (n+1) = \frac{1}{1+\langle n \rangle} \sum_{n=1}^{\infty} x^n n, \end{aligned} \quad (7)$$

其中 $x = \frac{\langle n \rangle}{1 + \langle n \rangle}$, 令 $f = \sum_{n=1}^{\infty} x^n n$, 由于 $xf = \sum_{n=1}^{\infty} x^{n+1} n$, $f - xf = x + \sum_{n=2}^{\infty} x^n = x + \frac{x^2}{1-x} = \frac{x}{1-x}$, 故 $f = \frac{x}{(1-x)^2}$,

$$\text{Tr}[\rho a^\dagger] = \langle n \rangle. \quad (8)$$

该湮灭掉一个光子的热光场的密度矩阵为

$$\rho' = \frac{a\rho a^\dagger}{\text{Tr}[a\rho a^\dagger]} = \sum_{n=0}^{\infty} \frac{\langle n \rangle^n}{(1 + \langle n \rangle)^{n+2}} (n+1) |n\rangle \langle n|. \quad (9)$$

其光子数分布函数为

$$p(m) = \langle m | \rho' | m \rangle = \frac{\langle n \rangle^m}{(1 + \langle n \rangle)^{m+2}} (m+1). \quad (10)$$

□

第 3 题 (20 分) 得分: _____. 试通过计算判断, 上题 (1) 中的辐射场的光子数分布为何种分布 (Poisson, Sub-Poisson, Super-Poisson)?

解: 上题 (1) 中的辐射场二阶相关度为

$$g^{(2)}(0) = \frac{\langle a^\dagger a^\dagger a a \rangle}{\langle a^\dagger a \rangle^2} = \frac{\frac{1}{\sqrt{2}} (\langle 0 | + i \langle 10 |) a^\dagger a^\dagger a a \frac{1}{\sqrt{2}} (|0\rangle - i |10\rangle)}{\left[\frac{1}{\sqrt{2}} (\langle 0 | + i \langle 10 |) a^\dagger a \frac{1}{\sqrt{2}} (|0\rangle - i |10\rangle) \right]^2} = \frac{9}{5} > 1, \quad (11)$$

故该辐射场的光子数分布为超泊松分布.

□

第 4 题 (20 分) 得分: _____. 增加了一个光子的相干态 (Single-photon-add coherent state (SPACS)) $|\alpha, 1\rangle = \frac{a^\dagger}{\sqrt{1+|\alpha|^2}} |\alpha\rangle$. 考虑该辐射场的两个厄米算符 $X_1 = \frac{1}{2}(a + a^\dagger)$, $X_2 = \frac{1}{2i}(a - a^\dagger)$. 它们分别对应于场的复振幅的实部和虚部. 证明:

SPACS 态 $|\alpha, 1\rangle$ 当 $|\alpha| > 1$ 时是压缩态, (本题取 $\alpha \in \mathbb{R}^+$).

证: X_1 的均值:

$$\begin{aligned} \langle X_1 \rangle &= \langle \alpha | \frac{a}{\sqrt{1+|\alpha|^2}} \frac{1}{2} (a + a^\dagger) \frac{a^\dagger}{\sqrt{1+|\alpha|^2}} | \alpha \rangle = \frac{1}{2(1+\alpha^2)} \langle \alpha | (aaa^\dagger + aa^\dagger a^\dagger) | \alpha \rangle \\ &= \frac{1}{2(1+\alpha^2)} \langle \alpha | [a(a^\dagger a + 1) + (a^\dagger a + 1)a^\dagger] | \alpha \rangle = \frac{1}{2(1+\alpha^2)} \langle \alpha | (aa^\dagger a + a + a^\dagger aa^\dagger + a^\dagger) | \alpha \rangle \\ &= \frac{1}{2(1+\alpha^2)} \langle \alpha | [(a^\dagger a + 1)a + a + a^\dagger(a^\dagger a + 1) + a^\dagger] | \alpha \rangle = \frac{1}{2(1+\alpha^2)} \langle \alpha | (a^\dagger aa + 2a + a^\dagger a^\dagger a + 2a^\dagger) | \alpha \rangle \\ &= \frac{\alpha(2+\alpha^2)}{1+\alpha^2}. \end{aligned} \quad (12)$$

X_1^2 的均值:

$$\begin{aligned} \langle X_1^2 \rangle &= \langle \alpha | \frac{a}{\sqrt{1+|\alpha|^2}} \left[\frac{1}{2} (a + a^\dagger) \right]^2 \frac{a^\dagger}{\sqrt{1+|\alpha|^2}} | \alpha \rangle = \frac{1}{4(1+\alpha^2)} \langle \alpha | (aaaa + aaa^\dagger a^\dagger + aa^\dagger aa^\dagger + aa^\dagger a^\dagger a^\dagger) | \alpha \rangle \\ &= \frac{1}{4(1+\alpha^2)} \langle \alpha | [aaaa + a(a^\dagger a + 1)a^\dagger + aa^\dagger aa^\dagger + (a^\dagger a + 1)a^\dagger a^\dagger] | \alpha \rangle \\ &= \frac{1}{4(1+\alpha^2)} \langle \alpha | (aaaa + 2aa^\dagger aa^\dagger + aa^\dagger + a^\dagger aa^\dagger a^\dagger + a^\dagger a^\dagger) | \alpha \rangle \\ &= \frac{1}{4(1+\alpha^2)} \langle \alpha | (aaaa + 2(a^\dagger a + 1)(a^\dagger a + 1) + (a^\dagger a + 1) + a^\dagger(a^\dagger a + 1)a^\dagger + a^\dagger a^\dagger) | \alpha \rangle \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4(1+\alpha^2)} \langle \alpha | (aaaa + 2a^\dagger aa^\dagger a + 5a^\dagger a + 3 + a^\dagger a^\dagger aa^\dagger + 2a^\dagger a^\dagger) | \alpha \rangle \\
&= \frac{1}{4(1+\alpha^2)} \langle \alpha | (aaaa + 2a^\dagger (a^\dagger a + 1)a + 5a^\dagger a + 3 + a^\dagger a^\dagger (a^\dagger a + 1) + 2a^\dagger a^\dagger) | \alpha \rangle \\
&= \frac{1}{4(1+\alpha^2)} \langle \alpha | (aaaa + 2a^\dagger a^\dagger aa + 7a^\dagger a + 3 + a^\dagger a^\dagger a^\dagger a + 3a^\dagger a^\dagger) | \alpha \rangle \\
&= \frac{4\alpha^4 + 10\alpha^2 + 3}{4(1+\alpha^2)}. \tag{13}
\end{aligned}$$

X_1 的涨落:

$$\Delta X_1 = \sqrt{\langle X_1^2 \rangle - \langle X_1 \rangle^2} = \frac{\sqrt{-2\alpha^4 - 3\alpha^2 + 3}}{2(1+\alpha^2)}. \tag{14}$$

若 $|\alpha, 1\rangle$ 为压缩态, 则

$$\Delta X_1 \neq \frac{1}{2}, \tag{15}$$

$$\implies \alpha \neq 1. \tag{16}$$

故当 $|\alpha| > 1$ 时, $|\alpha, 1\rangle$ 为压缩态. \square

第 5 题 (20 分) 得分: _____. 考虑一个理想的光学腔, 腔里有单模辐射场 $|\phi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|10\rangle)$. 处于基态且与单模场共振的两能级原子 $|\psi_i\rangle = |b\rangle$ 进入该光学腔, 与辐射场发生反应, 反应过程中相互作用的哈密顿量为 $\mathcal{V} = \hbar g(\sigma_+ a + a^\dagger \sigma_-)$. 系统的演化方程为 $\Psi(t)A + F = e^{-\frac{i}{\hbar}\mathcal{V}t}|\psi_i\rangle \otimes |\phi\rangle$. 反应一段时间后原子从腔中逸出. 经探测: 出射原子已经从腔中吸收一个光子而被激发, 且处于 $|\psi_f\rangle = |a\rangle$ 激发态.

- (1) 计算该单模场初始时刻 $|\phi_0\rangle$ 的平均光子数 \bar{n} ;
- (2) 试讨论, 在腔中被吸收一个光子的情况下: 此时腔内的辐射场的平均光子数变为多少? 此时辐射场的光子数分布为何种分布 (Poisson, Sub-Poisson, Super-Poisson)?

解: (1) 该单模场初始时刻 $|\phi_0\rangle$ 的平均光子数为

$$\bar{n} = \langle \phi_0 | a^\dagger a | \phi_0 \rangle = 5. \tag{17}$$

- (2) 无微扰哈密顿量为

$$\hat{H}_0 = \hbar \nu a^\dagger a + \frac{1}{2} \hbar \omega \sigma_z. \tag{18}$$

相互作用绘景中, 相互作用哈密顿量为

$$\hat{V} = e^{i\hat{H}_0 t/\hbar} \mathcal{V} e^{-i\hat{H}_0 t/\hbar} = \hbar g(\sigma_+ a e^{i\Delta t} + a^\dagger \sigma_- e^{-i\Delta t}), \tag{19}$$

其中 $\Delta = \omega - \nu$. 将系统的量子态

$$|\Psi(t)\rangle_{A+F} = \sum_{n=0}^{\infty} (C_{a,n}(t)|a, n\rangle + C_{b,n}(t)|b, n\rangle) \tag{20}$$

代入薛定谔方程

$$i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = \hat{V} |\Psi(t)\rangle \tag{21}$$

有

$$\dot{C}_{a,n}(t) = -ig C_{b,n+1} \sqrt{n+1} e^{i\Delta t}, \tag{22}$$

$$\dot{C}_{b,n}(t) = -igC_{a,n}\sqrt{n+1}e^{-i\Delta t}, \quad (23)$$

解得

$$C_{a,n}(t) = \left\{ C_{a,n}(0) \left[\cos\left(\frac{\Omega_n t}{2}\right) - \frac{i\Delta}{\Omega_n} \sin\left(\frac{\Omega_n t}{2}\right) \right] - \frac{2ig\sqrt{n+1}}{\Omega_n} C_{b,n+1}(0) \sin\left(\frac{\Omega_n t}{2}\right) \right\} e^{i\Delta t/2}, \quad (24)$$

$$C_{b,n+1}(t) = \left\{ C_{b,n+1}(0) \left[\cos\left(\frac{\Omega_n t}{2}\right) + \frac{i\Delta}{\Omega_n} \sin\left(\frac{\Omega_n t}{2}\right) \right] - \frac{2ig\sqrt{n+1}}{\Omega_n} C_{a,n}(0) \sin\left(\frac{\Omega_n t}{2}\right) \right\} e^{-i\Delta t/2}, \quad (25)$$

其中 $\Omega_n^2 = \Delta^2 + 4g^2(n+1)$. 考虑到系统的初始状态

$$|\Psi(0)\rangle_{A+F} = |b\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - i|10\rangle), \quad (26)$$

即 $C_{b,0}(0) = \frac{1}{\sqrt{2}}$, $C_{b,10} = -\frac{i}{\sqrt{2}}$, 故

$$C_{a,9}(t) = -\frac{\sqrt{2}g\sqrt{n+1}}{\Omega_n} \sin\left(\frac{\Omega_n t}{2}\right) e^{i\Delta t/2}, \quad (27)$$

$$C_{b,10}(t) = -\frac{i}{\sqrt{2}} \left[\cos\left(\frac{\Omega_n t}{2}\right) + \frac{i\Delta}{\Omega_n} \sin\left(\frac{\Omega_n t}{2}\right) \right] e^{-i\Delta t/2}, \quad (28)$$

即系统的量子态演化方程为

$$\begin{aligned} |\Psi(t)\rangle_{A+F} = & -\frac{\sqrt{2}g\sqrt{n+1}}{\Omega_n} \sin\left(\frac{\Omega_n t}{2}\right) e^{i\Delta t/2} |a, 9\rangle - \frac{i}{\sqrt{2}} \left[\cos\left(\frac{\Omega_n t}{2}\right) + \frac{i\Delta}{\Omega_n} \sin\left(\frac{\Omega_n t}{2}\right) \right] e^{-i\Delta t/2} |b, 10\rangle \\ & + \frac{1}{\sqrt{2}} |b, 0\rangle. \end{aligned} \quad (29)$$

当探测得 $|\psi_f\rangle = |a\rangle$ 时, 系统的量子态塌缩至

$$|\Psi_f\rangle = |a, 9\rangle. \quad (30)$$

此时腔内的辐射场的平均光子数为 9, 辐射场的光子数分布为亚泊松分布.

□