期末考试

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成绩 : __________

第 1 题　得分: ________. V 型三能级原子与两个经典光场作用. 频率为 ω_1 的经典光场与能级 $|a\rangle$, $|b\rangle$ 耦合, 频率为 ω_2 的经典光场与能级 $|a\rangle$, $|c\rangle$ 耦合. 系统的哈密顿量为 $H = H_0 + H_1$, $H_0 = \hbar\omega_a|a\rangle\langle a| + \hbar\omega_b|b\rangle\langle b| + \hbar\omega_c|c\rangle\langle c|$,

$$H_1 = \frac{\hbar}{2}(\Omega_{R1}e^{-i\phi_1}e^{-i\omega_1 t}|a\rangle\langle b| + \Omega_{R2}e^{-i\phi_2}e^{-i\omega_2 t}|a\rangle\langle c|) + \text{H.c.}$$

$\Omega_{R1}e^{-i\phi_1}$ 和 $\Omega_{R2}e^{-i\phi_2}$ 是复拉比频率. 原子的波函数可以写为 $|\Psi\rangle = c_a(t)e^{-i\omega_a t}|a\rangle + c_b e^{-i\omega_b t}|b\rangle + c_c(t)e^{-i\omega_c t}|c\rangle$. 原子和光场共振, 即: $\omega_a - \omega_b = \omega_1$, $\omega_a - \omega_c = \omega_2$, 通过解薛定谔方程, 可以求得波函数.

(1) 求 $c_a(t)$, $c_b(t)$, $c_c(t)$ 所满足的微分方程;

(2) 假设原子的初态为 $|\Psi(0)\rangle = \cos\frac{\theta}{2}|b\rangle + \sin\frac{\theta}{2}|c\rangle$, 求出 $c_a(t)$, $c_b(t)$, $c_c(t)$;

(3) 当 Ω_{R1}, Ω_{R2}, θ, ϕ_1, ϕ_2 满足什么条件时, 原子在演化过程中始终处于两个能级态 $|b\rangle$, $|c\rangle$ 的叠加态, 而不被激发到激发态上去. 这种现象叫做相干囚禁 (coherent trapping), 从物理上解释这种现象. (见 M. O. Scully, M. S. Zubairy 的书《quantum optics》 223-224 页, 世界图书出版公司出版, 中国, 北京)

解: (1) 在基 $\{|a\rangle, |b\rangle, |c\rangle\}$ 下, 系统哈密顿量的矩阵形式为

$$H = \begin{bmatrix} \hbar\omega_a & \frac{\hbar}{2}\Omega_{R1}e^{-i\phi_1}e^{-i\omega_1 t} & \frac{\hbar}{2}\Omega_{R2}e^{-i\phi_2}e^{-i\omega_2 t} \\ \frac{\hbar}{2}\Omega_{R1}e^{i\phi_1}e^{i\omega_1 t} & \hbar\omega_b & 0 \\ \frac{\hbar}{2}\Omega_{R2}e^{i\phi_2}e^{i\omega_2 t} & 0 & \hbar\omega_c \end{bmatrix}. \tag{1}$$

波函数的矢量形式为

$$|\Psi\rangle = \begin{bmatrix} c_a(t)e^{-i\omega_a t} \\ c_b(t)e^{-i\omega_b t} \\ c_c(t)e^{-i\omega_c t} \end{bmatrix}. \tag{2}$$

薛定谔方程

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = H|\Psi\rangle \tag{3}$$

的矩阵形式可表为

$$i\hbar\frac{\partial}{\partial t}\begin{bmatrix} c_a(t)e^{-i\omega_a t} \\ c_b(t)e^{-i\omega_b t} \\ c_c(t)e^{-i\omega_c t} \end{bmatrix} = \begin{bmatrix} \hbar\omega_a & \frac{\hbar}{2}\Omega_{R1}e^{-i\phi_1}e^{-i\omega_1 t} & \frac{\hbar}{2}\Omega_{R2}e^{-i\phi_2}e^{-i\omega_2 t} \\ \frac{\hbar}{2}\Omega_{R1}e^{i\phi_1}e^{i\omega_1 t} & \hbar\omega_b & 0 \\ \frac{\hbar}{2}\Omega_{R2}e^{i\phi_2}e^{i\omega_2 t} & 0 & \hbar\omega_c \end{bmatrix}\begin{bmatrix} c_a(t)e^{-i\omega_a t} \\ c_b(t)e^{-i\omega_b t} \\ c_c(t)e^{-i\omega_c t} \end{bmatrix}, \tag{4}$$

$$\Longrightarrow i\hbar\begin{bmatrix} \dot{c}_a(t)e^{-i\omega_a t} - i\omega_a c_a(t)e^{-i\omega_a t} \\ \dot{c}_b(t)e^{-i\omega_b t} - i\omega_b c_b(t)e^{-i\omega_b t} \\ \dot{c}_c(t)e^{-i\omega_c t} - i\omega_c c_c(t)e^{-i\omega_c t} \end{bmatrix} = \begin{bmatrix} \hbar\omega_a e^{-i\omega_a t} + \frac{\hbar}{2}\Omega_{R1}e^{-i\phi_1}e^{-i\omega_1 t}c_b(t)e^{-i\omega_b t} + \frac{\hbar}{2}\Omega_{R2}e^{-i\phi_2}e^{-i\omega_2 t}c_c(t)e^{-i\omega_c t} \\ \frac{\hbar}{2}\Omega_{R1}e^{i\phi_1}e^{i\omega_1 t}c_a(t)e^{-i\omega_a t} + \hbar\omega_b c_b(t)e^{-i\omega_b t} \\ \frac{\hbar}{2}\Omega_{R2}e^{i\phi_2}e^{i\omega_2 t}c_a(t)e^{-i\omega_a t} + \hbar\omega_c c_c(t)e^{-i\omega_c t} \end{bmatrix}, \tag{5}$$

$$\Longrightarrow \begin{bmatrix} \dot{c}_a(t) \\ \dot{c}_b(t) \\ \dot{c}_c(t) \end{bmatrix} = \begin{bmatrix} -i\frac{\Omega_{R1}}{2}e^{-i\phi_1}c_b(t) - i\frac{\Omega_{R2}}{2}e^{-i\phi_2}c_c(t) \\ -i\frac{\Omega_{R1}}{2}e^{i\phi_1}c_a(t) \\ -i\frac{\Omega_{R2}}{2}e^{i\phi_2}c_a(t) \end{bmatrix} \tag{6}$$

即 $c_a(t)$, $c_b(t)$, $c_c(t)$ 满足微分方程:

$$\dot{c}_a(t) = -i\frac{\Omega_{R1}}{2}e^{-i\phi_1}c_b(t) - i\frac{\Omega_{R2}}{2}e^{-i\phi_2}c_c(t), \tag{7}$$

$$\dot{c}_b(t) = -i\frac{\Omega_{R1}}{2}e^{i\phi_1}c_a(t), \tag{8}$$

$$\dot{c}_c(t) = -i\frac{\Omega_{R2}}{2}e^{i\phi_2}c_a(t). \tag{9}$$

(2) 解上述微分方程组得

$$c_a(t) = c_a(0)\cos\left(\sqrt{\Omega_{R1}^2+\Omega_{R2}^2}t\right) - i\frac{\Omega_{R1}e^{-i\phi_1}c_b(0)+\Omega_{R2}e^{-i\phi_2}c_c(0)}{\sqrt{\Omega_{R1}^2+\Omega_{R2}^2}}\sin\left(\sqrt{\Omega_{R1}^2+\Omega_{R2}^2}t\right), \tag{10}$$

$$c_b(t) = -i\frac{\Omega_{R1}}{2\sqrt{\Omega_{R1}^2+\Omega_{R2}^2}}\left\{c_a(0)\sin\left(\sqrt{\Omega_{R1}^2+\Omega_{R2}^2}t\right)+i\frac{\Omega_{R1}e^{-i\phi_1}c_b(0)+\Omega_{R2}e^{-i\phi_2}c_c(0)}{\sqrt{\Omega_{R1}^2+\Omega_{R2}^2}}\left[\cos\left(\sqrt{\Omega_{R1}^2+\Omega_{R2}^2}t\right)-1\right]\right\}+c_b(0), \tag{11}$$

$$c_c(t) = -i\frac{\Omega_{R2}}{2\sqrt{\Omega_{R1}^2+\Omega_{R2}^2}}\left\{c_a(0)\sin\left(\sqrt{\Omega_{R1}^2+\Omega_{R2}^2}t\right)+i\frac{\Omega_{R1}e^{-i\phi_1}c_b(0)+\Omega_{R2}e^{-i\phi_2}c_c(0)}{\sqrt{\Omega_{R1}^2+\Omega_{R2}^2}}\left[\cos\left(\sqrt{\Omega_{R1}^2+\Omega_{R2}^2}t\right)-1\right]\right\}+c_c(0). \tag{12}$$

原子的初态为 $|\Psi(0)\rangle = \cos\frac{\theta}{2}|b\rangle + \sin\frac{\theta}{2}|c\rangle$, 即 $c_a(0)=0$, $c_b(0)=\cos\frac{\theta}{2}$, $c_c(0)=\sin\frac{\theta}{2}$, 故

$$c_a(t) = -i\frac{\Omega_{R1}e^{-i\phi_1}\cos\frac{\theta}{2}+\Omega_{R2}e^{-i\phi_2}\sin\frac{\theta}{2}}{\sqrt{\Omega_{R1}^2+\Omega_{R2}^2}}\sin\left(\sqrt{\Omega_{R1}^2+\Omega_{R2}^2}t\right), \tag{13}$$

$$c_b(t) = \frac{\Omega_{R1}^2e^{-i\phi_1}\cos\frac{\theta}{2}+\Omega_{R1}\Omega_{R2}e^{-i\phi_2}\sin\frac{\theta}{2}}{2(\Omega_{R1}^2+\Omega_{R2}^2)}\left[\cos\left(\sqrt{\Omega_{R1}^2+\Omega_{R2}^2}t\right)-1\right]+\cos\frac{\theta}{2}, \tag{14}$$

$$c_c(t) = \frac{\Omega_{R1}\Omega_{R2}e^{-i\phi_1}\cos\frac{\theta}{2}+\Omega_{R2}^2e^{-i\phi_2}\sin\frac{\theta}{2}}{2(\Omega_{R1}^2+\Omega_{R2}^2)}\left[\cos\left(\sqrt{\Omega_{R1}^2+\Omega_{R2}^2}t\right)-1\right]+\sin\frac{\theta}{2}. \tag{15}$$

(3) 当

$$c_a(t)=0, \tag{16}$$

$$\Longrightarrow \Omega_{R1}e^{-i\phi_1}\cos\frac{\theta}{2}+\Omega_{R2}e^{-i\phi_2}\sin\frac{\theta}{2}=0, \tag{17}$$

$$\Longrightarrow \theta = -2\arctan\left[\frac{\Omega_{R1}}{\Omega_{R2}}e^{-i(\phi_1-\phi_2)}\right] \tag{18}$$

时, 发生相干囚禁, 其物理图像为光场激发下, 由 $|b\rangle$ 向 $|a\rangle$ 跃迁和由 $|c\rangle$ 向 $|a\rangle$ 跃迁产生的 $|a\rangle$ 的概率幅变化抵消.

□

第 2 题 得分: ________. 增加了一个光子的相干态 (Single-photon-added coherent state (SPACS)), $|\alpha,1\rangle = \frac{a^\dagger}{\sqrt{1+|\alpha|^2}}|\alpha\rangle$, 考虑该辐射场的两个厄米算符 $X_1=\frac{1}{2}(a+a^\dagger)$, $X_2=\frac{1}{2i}(a-a^\dagger)$, 它们分别对应于场的复振幅的实部和虚部, 满足对易关系 $[X_1,X_2]=\frac{i}{2}$. 当 α 取何值时 (本题 α 取正实数) SPACS 态是压缩态. (提示: 压缩条件 $(\Delta X_1)^2<1/4$ 或 $(\Delta X_2)^2<1/4$).

解: ΔX_1 的涨落计算见 2011 年第 4 题, 当 $|\alpha|>1$ 时, $\Delta X_1<1$, 是压缩态.

X_2 的均值为

$$\begin{aligned}
\langle X_2\rangle =& \langle\alpha|\frac{a}{\sqrt{1+|\alpha|^2}}\frac{1}{2i}(a-a^\dagger)\frac{a^\dagger}{\sqrt{1+|\alpha|^2}}|\alpha\rangle\\
=&\frac{1}{2i(1+|\alpha|^2)}\langle\alpha|a(a-a^\dagger)a^\dagger|\alpha\rangle\\
=&\frac{1}{2i(1+|\alpha|^2)}\langle\alpha|(aaa^\dagger-aa^\dagger a^\dagger)|\alpha\rangle\\
=&\frac{1}{2i(1+|\alpha|^2)}\langle\alpha|[a(a^\dagger a+1)-(a^\dagger a+1)a^\dagger]|\alpha\rangle\\
=&\frac{1}{2i(1+|\alpha|^2)}\langle\alpha|(aa^\dagger a+a-a^\dagger aa^\dagger+a^\dagger)|\alpha\rangle\\
=&\frac{1}{2i(1+|\alpha|^2)}\langle\alpha|[(a^\dagger a+1)a+a-a^\dagger(a^\dagger a+1)+a^\dagger]|\alpha\rangle
\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{2i(1+|\alpha|^2)}\langle\alpha|(a^\dagger aa+2a-a^\dagger a^\dagger a+2a^\dagger)|\alpha\rangle\\
=&\frac{1}{2i(1+|\alpha|^2)}\langle\alpha|(\alpha^3+2\alpha-\alpha^3-2\alpha)|\alpha\rangle\\
=&0.
\end{aligned}\tag{19}
$$

X_2^2 的均值为

$$
\begin{aligned}
\langle X_2^2\rangle=&\langle\alpha|\frac{a}{\sqrt{1+|\alpha|^2}}\left[\frac{1}{2i}(a-a^\dagger)\right]^2\frac{a^\dagger}{\sqrt{1+|\alpha|^2}}|\alpha\rangle\\
=&-\frac{1}{4(1+|\alpha|^2)}\langle\alpha|a(aa-aa^\dagger-a^\dagger a+a^\dagger a^\dagger)a^\dagger|\alpha\rangle\\
=&-\frac{1}{4(1+|\alpha|^2)}\langle\alpha|a(aa-2a^\dagger a-1+a^\dagger a^\dagger)a^\dagger|\alpha\rangle\\
=&-\frac{1}{4(1+|\alpha|^2)}\langle\alpha|(aaaa^\dagger-2aa^\dagger aa^\dagger-aa^\dagger+aa^\dagger a^\dagger a^\dagger)|\alpha\rangle\\
=&-\frac{1}{4(1+|\alpha|^2)}\langle\alpha|[aa(a^\dagger a+1)-2(a^\dagger a+1)(a^\dagger a+1)-(a^\dagger a+1)+(a^\dagger a+1)a^\dagger a^\dagger]|\alpha\rangle\\
=&-\frac{1}{4(1+|\alpha|^2)}\langle\alpha|(aaa^\dagger a+aa-2a^\dagger aa^\dagger a-4a^\dagger a-2-a^\dagger a-1+a^\dagger aa^\dagger a^\dagger+a^\dagger a^\dagger)|\alpha\rangle\\
=&-\frac{1}{4(1+|\alpha|^2)}\langle\alpha|[a(a^\dagger a+1)a+aa-2a^\dagger(a^\dagger a+1)a-4a^\dagger a-2-a^\dagger a-1+a^\dagger(a^\dagger a+1)a^\dagger+a^\dagger a^\dagger]|\alpha\rangle\\
=&-\frac{1}{4(1+|\alpha|^2)}\langle\alpha|(aa^\dagger aa+2aa-2a^\dagger a^\dagger aa-2a^\dagger a-4a^\dagger a-2-a^\dagger a-1+a^\dagger a^\dagger aa^\dagger+2a^\dagger a^\dagger)|\alpha\rangle\\
=&-\frac{1}{4(1+|\alpha|^2)}\langle\alpha|[(a^\dagger a+1)aa+2aa-2a^\dagger a^\dagger aa-2a^\dagger a-4a^\dagger a-2-a^\dagger a-1+a^\dagger a^\dagger(a^\dagger a+1)+2a^\dagger a^\dagger]|\alpha\rangle\\
=&-\frac{1}{4(1+|\alpha|^2)}\langle\alpha|(a^\dagger aaa+3aa-2a^\dagger a^\dagger aa-2a^\dagger a-4a^\dagger a-2-a^\dagger a-1+a^\dagger a^\dagger a^\dagger a+3a^\dagger a^\dagger)|\alpha\rangle\\
=&-\frac{1}{4(1+|\alpha|^2)}\langle\alpha|(\alpha^4+3\alpha^2-2\alpha^4-2\alpha^2-4\alpha^2-2-\alpha^2-1+\alpha^4+3\alpha^2)|\alpha\rangle\\
=&\frac{\alpha^2+3}{4(1+\alpha^2)}>\frac{1}{4}.
\end{aligned}
$$

X_2 的涨落为

$$
\Delta X_2=\sqrt{\langle X_2^2\rangle-\langle X_2\rangle^2}=\sqrt{\langle X_2^2\rangle}>\frac{1}{2},\tag{20}
$$

故无法由 ΔX_2 判断是否存在压缩.

综上, 当 $|\alpha|>1$ 时, SPACS 态为压缩态. □

第 3 题　得分: ________. 考虑一个理想的光学腔, 腔里有单模辐射场 $|\phi(0)\rangle_F=\frac{1}{\sqrt{2}}(|0\rangle-i|10\rangle)$. 处于基态与单模场共振的二能级原子 $|\varphi(0)\rangle_A=|g\rangle$ 进入该光学腔, 与场发生作用, 相互作用的哈密顿量为 $H_I=\hbar g(\sigma_+a^2+\sigma_-(a^\dagger)^2)$ (在相互作用绘景中研究). 系统的演化方程为 $|\Psi\rangle_{AF}=e^{-\frac{i}{\hbar}H_It}|\phi(0)\rangle_R|\varphi(0)\rangle_A$. 作用一段时间后原子从腔中逸出. 经探测: 出射原子处于激发态 $|e\rangle$.

(1) 计算该单模场初始时刻 $|\phi(0)\rangle_F$ 的平均光子数 $\bar{n}$;

(2) 任意时刻系统的态 $|\Psi(t)\rangle_{AF}$;

(3) 原子出射后, 腔内的辐射场的平均光子数变为多少?

解: (1) 该单模场初始时刻的平均光子数为

$$\bar{n} =_F \langle\phi(0)|a^\dagger a|\phi(0)\rangle_F = 5. \tag{21}$$

(2) 设任意时刻系统的态为

$$|\Psi(t)\rangle_{AF} = \sum_{n=0}^{\infty}[C_{e,n}(t)|e,n\rangle + C_{g,n}(t)|g,n\rangle]. \tag{22}$$

将其代入薛定谔方程中得

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle_{AF} = H_I|\Psi(t)\rangle_{AF}, \tag{23}$$

有

$$\dot{C}_{e,n}(t) = -igC_{g,n+2}\sqrt{(n+1)(n+2)}, \tag{24}$$
$$\dot{C}_{g,n+2}(t) = -igC_{e,n}\sqrt{(n+1)(n+2)}, \tag{25}$$

解得

$$C_{e,n}(t) = C_{e,n}(0)\cos[\sqrt{(n+1)(n+2)}gt] - iC_{g,n+2}(0)\sin[\sqrt{(n+1)(n+2)}gt], \tag{26}$$
$$C_{g,n+2}(t) = C_{g,n+2}(0)\cos[\sqrt{(n+1)(n+2)}gt] - iC_{e,n}(0)\sin[\sqrt{(n+1)(n+2)}gt]. \tag{27}$$

系统的初始状态为

$$|\Psi(0)\rangle_{AF} = |\varphi(0)\rangle_A \otimes |\phi(0)\rangle_F = \frac{1}{\sqrt{2}}|g\rangle(|0\rangle - i|10\rangle), \tag{28}$$

即 $C_{g,0} = \frac{1}{\sqrt{2}}$, $C_{g,10} = -\frac{i}{\sqrt{2}}$, 故

$$C_{e,8} = -\frac{1}{\sqrt{2}}\sin(\sqrt{90}gt), \tag{29}$$
$$C_{g,10} = -\frac{i}{\sqrt{2}}\cos(\sqrt{90}gt), \tag{30}$$

即 t 时刻系统的态为

$$|\Psi(t)\rangle_{AF} = -\frac{1}{\sqrt{2}}\sin(\sqrt{90}gt)|e,8\rangle - \frac{i}{\sqrt{2}}\cos(\sqrt{90}gt) + \frac{1}{\sqrt{2}}. \tag{31}$$

(3) 由于探测得出射原子处于激发态 $|e\rangle$, 故系统的态塌缩至 $|8\rangle$, 此时腔内的辐射场的平均光子数为 8.

□

第 4 题 得分: ________. 由一个赝自旋算符 $S_+ = |e\rangle\langle g|$, $S_- = |g\rangle\langle e|$ 和 $S_3 = (|e\rangle\langle e| - |g\rangle\langle g|)/2$ 描述的二能级原子, 可定义两个厄米算符: $S_1 = (S_+ + S_-)/2$, $S_2 = (S_+ - S_-)/(2i)$, 他们的对易关系 $[S_1, S_2] = iS_3$, $S_3 = 1/2\sigma_z$. 相应的海森堡不确定关系为 $(\Delta S_1)^2(\Delta S_2)^2 \geq 1/4\,|\langle S_3\rangle|^2$, 这里 $(\Delta S_i)^2 = \langle S_i^2\rangle - \langle S_i\rangle^2$ 是原子算符 S_i 的量子涨落. 如果量子涨落满足 $(\Delta S_i)^2 < 1/2\,|\langle S_3\rangle|$ ($i = 1$ 或 2), 我们说原子算符的涨落被压缩, 原子出现压缩效应. 当原子处于 $|\Psi\rangle = \cos(\frac{\theta}{2})|e\rangle + \sin(\frac{\theta}{2})|g\rangle$ 时 S_1 的平均值 $\langle S_1\rangle$, 求出分量 S_1 压缩的条件.

解:

$$S_1 = \frac{1}{2}(S_+ + S_-) = \frac{1}{2}(|e\rangle\langle g| + |g\rangle\langle e|). \tag{32}$$

S_1 的平均值为

$$\langle S_1\rangle =\langle\Psi|\frac{1}{2}(|e\rangle\langle g|+|g\rangle\langle e|)|\Psi\rangle=\sin\frac{\theta}{2}\cos\frac{\theta}{2}=\frac{1}{2}\sin\theta. \tag{33}$$

S_1^2 的平均值为

$$\langle S_1^2\rangle =\langle\Psi|S_1^2|\Psi\rangle=\frac{1}{4}\langle\Psi|(|e\rangle\langle e|+|g\rangle\langle g|)|\Psi\rangle=\frac{1}{4}. \tag{34}$$

S_1 的涨落为

$$\Delta S_1=\sqrt{\langle S_1^2\rangle-\langle S_1\rangle^2}=\frac{1}{2}\sqrt{1-\sin^2\theta}. \tag{35}$$

$$\langle S_3\rangle=\langle\Psi|S_3|\Psi\rangle=\frac{1}{2}\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)=\frac{1}{2}\cos\theta. \tag{36}$$

分量 S_1 压缩的条件为

$$(\Delta S_1)^2=\frac{1}{4}(1-\sin^2\theta)<\frac{1}{2}\left|\langle S_3\rangle\right|=\frac{1}{4}\left|\cos\theta\right|, \tag{37}$$

$$\Longrightarrow \theta\neq n\pi,\quad n=1,2,\cdots. \tag{38}$$

□

第 5 题 得分: ________. 设原子初态 $|\varphi(0)\rangle_A=\cos\alpha|e\rangle+\sin\alpha|g\rangle$, 光场初态是粒子数态 $|\varphi(0)\rangle_F=|6\rangle$. 该原子与光场之间的相互作用可用双光子 J-C 模型描述, 在共振条件和相互作用绘景中其哈密顿量表示为 $H_1=\hbar g(\sigma_+a^2+\sigma_-(a^\dagger)^2)$, 求任意时刻 t,

(1) 该复合系统态矢.

(2) 原子处于激发态的概率, 画出概率图形 ($\alpha=\pi/4$, 横坐标表示时间, 纵坐标表示概率). (要求给出程序)

解: (1) 设系统的状态为

$$|\varphi(t)\rangle_{AF}=\sum_{n=0}^{\infty}[C_{e,n}(t)|e,n\rangle+C_{g,n}|g,n\rangle]. \tag{39}$$

相互作用绘景下, 系统的状态演化遵循薛定谔方程:

$$i\hbar\frac{\partial}{\partial t}|\varphi(t)\rangle_{AF}=H_1|\varphi(t)\rangle_{AF}, \tag{40}$$

即

$$\dot{C}_{e,n}(t)=-ig\sqrt{(n+1)(n+2)}C_{g,n+2}(t), \tag{41}$$

$$\dot{C}_{g,n+2}(t)=-ig\sqrt{(n+1)(n+2)}C_{e,n}(t), \tag{42}$$

解得

$$C_{e,n}(t)=C_{e,n}(0)\cos[\sqrt{(n+1)(n+2)}gt]-iC_{g,n+2}(0)\sin[\sqrt{(n+1)(n+2)}gt], \tag{43}$$

$$C_{g,n+2}(t)=C_{g,n+2}(0)\cos[\sqrt{(n+1)(n+2)}gt]-iC_{e,n}(0)\sin[\sqrt{(n+1)(n+2)}gt]. \tag{44}$$

原子初态 $|\varphi(0)\rangle_A=\cos\alpha|e\rangle+\sin\alpha|g\rangle$, 光场初态为粒子数态 $|\varphi(0)\rangle_F=|6\rangle$, 即 $C_{e,6}=\cos\alpha$, $C_{g,6}=\sin\alpha$, 将其代入上面两式得

$$C_{e,6}(t)=\cos\alpha\cos(\sqrt{56}gt), \tag{45}$$

$$C_{g,8}(t) = -i\cos\alpha\sin(\sqrt{56}gt), \tag{46}$$
$$C_{g,6}(t) = \sin\alpha\cos(\sqrt{30}gt), \tag{47}$$
$$C_{e,4}(t) = -i\sin\alpha\cos(\sqrt{30}gt), \tag{48}$$

故任意时刻 t 该复合系统态矢为

$$|\varphi(t)\rangle_{AF} = -i\cos\alpha\sin(\sqrt{56}gt) + \cos\alpha\cos(\sqrt{56}gt)|e,6\rangle + \sin\alpha\cos(\sqrt{30}gt) - i\sin\alpha\cos(\sqrt{30}gt). \tag{49}$$

(2) 任意时刻 t 原子处于激发态的概率为

$$P_e = |C_{e,6}|^2 + |C_{e,4}|^2 = \cos^2\alpha\cos^2(\sqrt{56}gt) + \sin^2\alpha\cos^2(\sqrt{30}gt). \tag{50}$$

概率图形如图 1 所示.

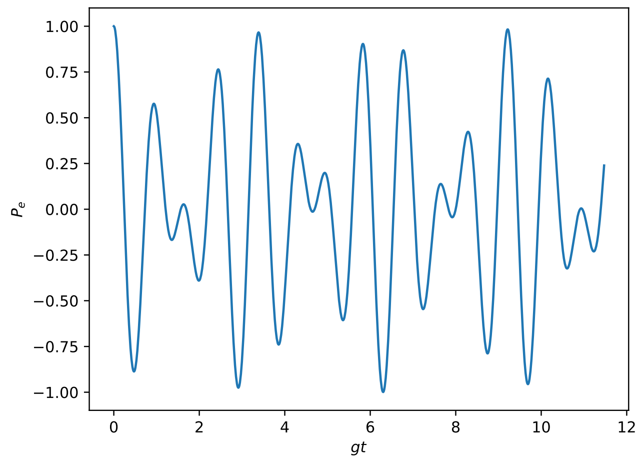

图 1: 任意时刻 t 原子处于激发态的概率.

□

第 6 题 得分: ________. 一个二能级原子 A 与热库 E (环境) 相互作用如下: $U_{AE}|g\rangle_A|0\rangle_E = |e\rangle_A|0\rangle_E$, $U_{AE}|e\rangle_A|0\rangle_E = \sqrt{1-p}|e\rangle_A|0\rangle_E + \sqrt{p}|g\rangle_A|1\rangle_E$, 其中 p 与时间有关, $|0\rangle_E$ 是环境的真空态. 原子的演化可用 Kraus 算符和表示: $\rho_A(t) = \sum_i M_i\rho_A(0)M_i^\dagger$. Kraus 算符的定义是: $M_i =_E \langle i|U_{AE}|0\rangle_E$.

(1) 试求出 M_0 和 M_1.

(2) 设原子初态为 $\rho_A(0) = \begin{bmatrix} a & b \\ d & c \end{bmatrix}$, 求出 $\rho_A(t)$.

解: (1) Kraus 算符:

$$M_0 =_E \langle 0|U_{AE}|0\rangle_E \tag{51}$$
$$=_E \langle 0|U_{AE}(|g\rangle_A\langle g| + |e\rangle_A\langle e|)|0\rangle_E$$
$$=_E \langle 0|(|0\rangle_E|e\rangle_A\langle g| + \sqrt{1-p}|0\rangle_E|e\rangle_A\langle e| + \sqrt{p}|1\rangle_E|g\rangle_A\langle e|)$$
$$= |e\rangle_A\langle g| + \sqrt{1-p}|e\rangle_A\langle e|, \tag{52}$$
$$M_1 =_E \langle 1|U_{AE}|0\rangle_E$$

$$
\begin{aligned}
&=_E \langle 1|U_{AE}(|g\rangle_A\langle g|+|e\rangle_A\langle e|)|0\rangle_E\\
&=_E \langle 1|(|0\rangle_E|e\rangle_A\langle g|+\sqrt{1-p}|0\rangle_E|e\rangle_A\langle e|+\sqrt{p}|1\rangle_E|g\rangle_A\langle e|)\\
&=\sqrt{p}|g\rangle_A\langle e|, \qquad (53)
\end{aligned}
$$

(2) t 时刻原子的状态为

$$
\begin{aligned}
\rho_A(t)=&\sum_{i=0,1}M_i\rho_A(0)M_i^\dagger\\
=&\begin{bmatrix}\sqrt{1-p} & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}a & b\\ d & c\end{bmatrix}\begin{bmatrix}\sqrt{1-p} & 0\\ 1 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\ \sqrt{p} & 0\end{bmatrix}\begin{bmatrix}a & b\\ d & c\end{bmatrix}\begin{bmatrix}0 & \sqrt{p}\\ 0 & 0\end{bmatrix}\\
=&\begin{bmatrix}(1-p)a+\sqrt{1-p}d+\sqrt{1-p}b+c & 0\\ 0 & pa\end{bmatrix}.
\end{aligned}
$$

□

第 7 题 得分: _________. 二能级原子与单模光场发生双光子共振相互作用, 系统的哈密顿量为 $H=\hbar\lambda[\sigma_-(a^\dagger)^2+\sigma_+a^2]$. 假设原子初态 ($t=0$ 时刻的量子态) 为激发态 $|e\rangle$, 光场初态 $|n\rangle$.

(1) 求系统任意时刻的平均光子数;

(2) 画出平均光子数与时间的关系. (要求给出程序)

解: (1) 设系统的状态为

$$
|\varphi(t)\rangle_{AF}=\sum_{n=0}^{\infty}[C_{e,n}|e,n\rangle+C_{g,n}|g,n\rangle]. \qquad (54)
$$

相互作用绘景下, 系统的状态演化遵循薛定谔方程:

$$
i\hbar\frac{\partial}{\partial t}|\varphi(t)\rangle_{AF}=H|\varphi(t)\rangle_{AF}, \qquad (55)
$$

即

$$
\begin{aligned}
\dot{C}_{e,n}(t)=&-i\lambda\sqrt{(n+1)(n+2)}C_{g,n+2}(t), \qquad (56)\\
\dot{C}_{g,n+2}(t)=&-i\lambda\sqrt{(n+1)(n+2)}C_{e,n}(t), \qquad (57)
\end{aligned}
$$

解得

$$
\begin{aligned}
C_{e,n}(t)=&C_{e,n}(0)\cos[\sqrt{(n+1)(n+2)}\lambda t]-iC_{g,n+2}\sin[\sqrt{(n+1)(n+2)}\lambda t], \qquad (58)\\
C_{g,n+2}(t)=&C_{g,n+2}(0)\cos[\sqrt{(n+1)(n+2)}\lambda t]-iC_{e,n}(0)\sin[\sqrt{(n+1)(n+2)}\lambda t]. \qquad (59)
\end{aligned}
$$

原子初态为激发态 $|e\rangle$, 光场初态为 $|n\rangle$, 即 $C_{e,n}=1$, 将其代数上面两式中得

$$
\begin{aligned}
C_{e,n}(t)=&\cos[\sqrt{(n+1)(n+2)}\lambda t], \qquad (60)\\
C_{g,n+2}(t)=&-i\sin[\sqrt{(n+1)(n+2)}\lambda t]. \qquad (61)
\end{aligned}
$$

故系统 t 时刻的平均光子数为

$$
\begin{aligned}
\langle n(t)\rangle=&_{AF}\langle\varphi(t)|a^\dagger a|\varphi(t)\rangle_{AF}=n\cos^2[\sqrt{(n+1)(n+2)}\lambda t]+(n+2)\sin^2[\sqrt{(n+1)(n+2)}\lambda t]\\
=&n+1-\cos[2\sqrt{(n+1)(n+2)}\lambda t]. \qquad (62)
\end{aligned}
$$

(2) 光子数与时间的关系如图 2 所示.

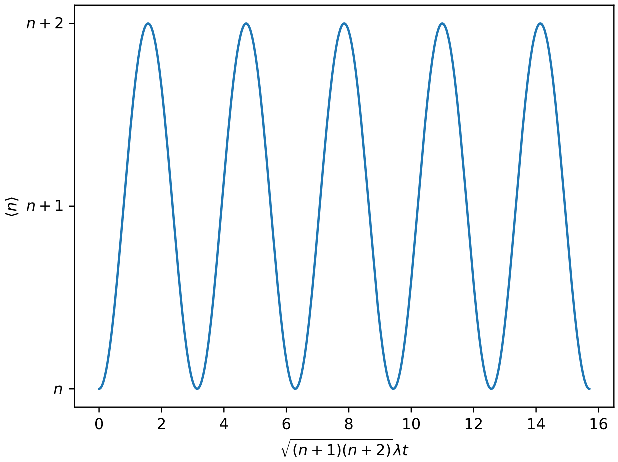

图 2: 光子数与时间的关系.

$\square$

第 8 题 得分: ________. 二能级原子与单模光场发生共振作用, 系统的哈密顿量为 $H=\hbar\lambda(\sigma_-a^\dagger+\sigma_+a)$. 如果原子 $t=0$ 时刻处于激发态 $|e\rangle$, 而光场处于相干态 $|\alpha\rangle$, 计算任意时刻 t 原子处于基态 $|g\rangle$ 的概率 $P_g(t)$, 并作出图形 (横坐标表示时间, 纵坐标为概率. 为方便, $\alpha=1$).

解: 设系统的状态为

$$|\varphi(t)\rangle_{AF}=\sum_{n=0}^{\infty}[C_{e,n}|e,n\rangle+C_{g,n}|g,n\rangle]. \tag{63}$$

相互作用绘景下, 系统的状态演化遵循薛定谔方程:

$$i\hbar\frac{\partial}{\partial t}|\varphi(t)\rangle_{AF}=H|\varphi(t)\rangle_{AF}, \tag{64}$$

即

$$\dot{C}_{e,n}(t)=-i\lambda\sqrt{(n+1)(n+2)}C_{g,n+2}(t), \tag{65}$$
$$\dot{C}_{g,n+2}(t)=-i\lambda\sqrt{(n+1)(n+2)}C_{e,n}(t), \tag{66}$$

解得对于 $n\in\mathbb{Z}^+$,

$$C_{e,n}(t)=C_{e,n}(0)\cos[\sqrt{(n+1)(n+2)}\lambda t]-iC_{g,n+2}\sin[\sqrt{(n+1)(n+2)}\lambda t], \tag{67}$$
$$C_{g,n+2}(t)=C_{g,n+2}(0)\cos[\sqrt{(n+1)(n+2)}\lambda t]-iC_{e,n}(0)\sin[\sqrt{(n+1)(n+2)}\lambda t], \tag{68}$$

此外,

$$C_{g,0}(t)=C_{g,0}(0), \tag{69}$$
$$C_{g,1}(t)=C_{g,1}(0). \tag{70}$$

原子初态为激发态 $|g\rangle$, 光场初态为相干态 $|\alpha\rangle=e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$, 即 $C_{g,n}=e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}$, 将其代数上面四式中得对 $n\in\mathbb{Z}^+$,

$$C_{e,n}(t)=-ie^{-|\alpha|^2/2}\frac{\alpha^{n+2}}{\sqrt{(n+2)!}}\sin[\sqrt{(n+1)(n+2)}\lambda t], \tag{71}$$

$$C_{g,n+2}(t) = e^{-|\alpha|^2/2}\frac{\alpha^{n+2}}{\sqrt{(n+2)!}}\cos[\sqrt{(n+1)(n+2)}\lambda t], \tag{72}$$

此外,

$$C_{g,0}(t) = e^{-|\alpha|^2/2}, \tag{73}$$
$$C_{g,1}(t) = e^{-|\alpha|^2/2}\alpha. \tag{74}$$

若取 $\alpha = 1$, 则任意时刻 t 原子处于基态的概率为

$$P_g(t) = \sum_{n=0}^{\infty}|C_{g,n}(t)|^2 = e^{-1}\left[2 + \sum_{n=2}^{\infty}\frac{\cos^2[\sqrt{n(n-1)}\lambda t]}{n!}\right], \tag{75}$$

如图 3 所示.

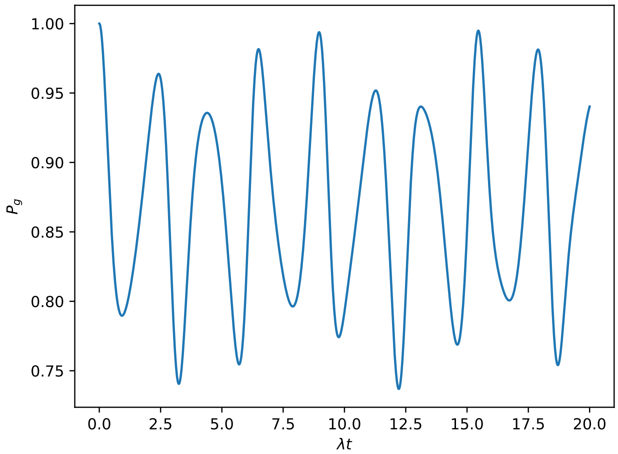

图 3: 任意时刻 t 原子处于基态的概率.

□

第 9 题 得分: _________. 二能级原子与单模光场发生双光子共振相互作用, 系统的哈密顿量为 $H = \hbar\lambda[\sigma_-(a^\dagger)^2 + \sigma_+ a^2]$. 假设原子初态 ($t = 0$ 时刻的量子态) 为激发态 $|e\rangle$, 光场初态为相干态 $|\alpha\rangle$. 求系统任意时刻的量子态.

解: 设系统的状态为

$$|\varphi(t)\rangle_{AF} = \sum_{n=0}^{\infty}[C_{e,n}|e,n\rangle + C_{g,n}|g,n\rangle]. \tag{76}$$

相互作用绘景下, 系统的状态演化遵循薛定谔方程:

$$i\hbar\frac{\partial}{\partial t}|\varphi(t)\rangle_{AF} = H|\varphi(t)\rangle_{AF}, \tag{77}$$

即

$$\dot{C}_{e,n}(t) = -i\lambda\sqrt{(n+1)(n+2)}C_{g,n+2}(t), \tag{78}$$
$$\dot{C}_{g,n+2}(t) = -i\lambda\sqrt{(n+1)(n+2)}C_{e,n}(t), \tag{79}$$

解得对于 $n \in \mathbb{Z}^+$,

$$C_{e,n}(t) = C_{e,n}(0)\cos[\sqrt{(n+1)(n+2)}\lambda t] - iC_{g,n+2}\sin[\sqrt{(n+1)(n+2)}\lambda t], \tag{80}$$

$$C_{g,n+2}(t)=C_{g,n+2}(0)\cos[\sqrt{(n+1)(n+2)}\lambda t]-iC_{e,n}(0)\sin[\sqrt{(n+1)(n+2)}\lambda t], \tag{81}$$

此外,

$$C_{g,0}(t)=C_{g,0}(0), \tag{82}$$
$$C_{g,1}(t)=C_{g,1}(0). \tag{83}$$

原子初态为激发态 $|g\rangle$, 光场初态为相干态 $|\alpha\rangle=e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$, 即 $C_{g,n}=e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}$, 将其代数上面四式中得对 $n\in\mathbb{Z}^+$,

$$C_{e,n}(t)=-ie^{-|\alpha|^2/2}\frac{\alpha^{n+2}}{\sqrt{(n+2)!}}\sin[\sqrt{(n+1)(n+2)}\lambda t], \tag{84}$$
$$C_{g,n+2}(t)=e^{-|\alpha|^2/2}\frac{\alpha^{n+2}}{\sqrt{(n+2)!}}\cos[\sqrt{(n+1)(n+2)}\lambda t], \tag{85}$$

此外,

$$C_{g,0}(t)=e^{-|\alpha|^2/2}, \tag{86}$$
$$C_{g,1}(t)=e^{-|\alpha|^2/2}\alpha. \tag{87}$$

故系统在任意时刻 t 的量子态为

$$\begin{aligned}&|\varphi(t)\rangle_{AF}\\=&e^{-|\alpha|^2/2}\left\{|g,0\rangle+\alpha|g,1\rangle+\sum_{n=0}^{\infty}\frac{\alpha^{n+2}}{\sqrt{(n+2)}}\{-i\sin[\sqrt{(n+1)(n+2)}\lambda t]|e,n\rangle+\cos[\sqrt{(n+1)(n+2)}\lambda t]|g,n+2\rangle\}\right\}.\end{aligned} \tag{88}$$

□

第 10 题 得分: ________. 二能级原子与单模光场发生共振相互作用, 系统的哈密顿量为 $H=\hbar\lambda(\sigma_-a^\dagger+\sigma_+a)$, 如果原子 $t=0$ 时刻处于 $\cos\theta|e\rangle+\sin\theta|g\rangle$, 而光场处于相干态 $|\alpha\rangle$, 定义原子算符 $S_1=1/2(|e\rangle\langle g|+|g\rangle\langle e|)$, 求任意时刻 t, S_1 的平均值.

解: 设系统的状态为

$$|\varphi(t)\rangle_{AF}=\sum_{n=0}^{\infty}[C_{e,n}|e,n\rangle+C_{g,n}|g,n\rangle]. \tag{89}$$

相互作用绘景下, 系统的状态演化遵循薛定谔方程:

$$i\hbar\frac{\partial}{\partial t}|\varphi(t)\rangle_{AF}=H|\varphi(t)\rangle_{AF}, \tag{90}$$

即

$$\dot{C}_{e,n}(t)=-i\lambda\sqrt{(n+1)(n+2)}C_{g,n+2}(t), \tag{91}$$
$$\dot{C}_{g,n+2}(t)=-i\lambda\sqrt{(n+1)(n+2)}C_{e,n}(t), \tag{92}$$

解得对于 $n\in\mathbb{Z}^+$,

$$C_{e,n}(t)=C_{e,n}(0)\cos[\sqrt{(n+1)(n+2)}\lambda t]-iC_{g,n+2}\sin[\sqrt{(n+1)(n+2)}\lambda t], \tag{93}$$
$$C_{g,n+2}(t)=C_{g,n+2}(0)\cos[\sqrt{(n+1)(n+2)}\lambda t]-iC_{e,n}(0)\sin[\sqrt{(n+1)(n+2)}\lambda t], \tag{94}$$

此外,

$$C_{g,0}(t)=C_{g,0}(0),\tag{95}$$
$$C_{g,1}(t)=C_{g,1}(0).\tag{96}$$

原子初态为激发态 $\cos\theta|e\rangle+\sin\theta|g\rangle$, 光场初态为相干态 $|\alpha\rangle=e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$, 即 $C_{g,n}=e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\sin\theta$, $C_{e,n}=e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\cos\theta$, 将其代数上面四式中得对 $n\in\mathbb{Z}^+$,

$$C_{e,n}(t)=e^{-|\alpha|^2/2}\left\{\frac{\alpha^n}{\sqrt{n!}}\cos\theta\cos[\sqrt{(n+1)(n+2)}\lambda t]-i\frac{\alpha^{n+2}}{\sqrt{(n+2)!}}\sin\theta\sin[\sqrt{(n+1)(n+2)}\lambda t]\right\},\tag{97}$$
$$C_{g,n+2}(t)=e^{-|\alpha|^2/2}\left\{\frac{\alpha^{n+2}}{\sqrt{(n+2)!}}\sin\theta\cos[\sqrt{(n+1)(n+2)}\lambda t]-i\frac{\alpha^n}{\sqrt{n!}}\cos\theta\sin[\sqrt{(n+1)(n+2)}\lambda t]\right\},\tag{98}$$

此外,

$$C_{g,0}(t)=e^{-|\alpha|^2/2}\sin\theta,\tag{99}$$
$$C_{g,1}(t)=e^{-|\alpha|^2/2}\alpha\sin\theta.\tag{100}$$

故任意时刻 t, 系统的状态为

$$\begin{aligned}
&|\varphi(t)\rangle_{AF}\\
=&e^{-|\alpha|^2/2}\{\sin\theta|g,0\rangle+\alpha\sin\theta|g,1\rangle\\
&+\sum_{n=0}^{\infty}\left\{\left\{\frac{\alpha^n}{\sqrt{n!}}\cos\theta\cos[\sqrt{(n+1)(n+2)}\lambda t]-i\frac{\alpha^{n+2}}{\sqrt{(n+2)!}}\sin\theta\sin[\sqrt{(n+1)(n+2)}\lambda t]\right\}|e,n\rangle\right.\\
&\left.+\left\{\frac{\alpha^{n+2}}{\sqrt{(n+2)!}}\sin\theta\cos[\sqrt{(n+1)(n+2)}\lambda t]-i\frac{\alpha^n}{\sqrt{n!}}\cos\theta\sin[\sqrt{(n+1)(n+2)}\lambda t]\right\}|g,n+2\rangle\right\}\}\\
=&e^{-|\alpha|^2/2}\left\{|g\rangle\left\{|0\rangle+\alpha|1\rangle+\sum_{n=0}^{\infty}\left\{\frac{\alpha^{n+2}}{\sqrt{(n+2)!}}\sin\theta\cos[\sqrt{(n+1)(n+2)}\lambda t]-i\frac{\alpha^n}{\sqrt{n!}}\cos\theta\sin[\sqrt{(n+1)(n+2)}\lambda t]\right\}|n+2\rangle\right\}\right.\\
&\left.+|e\rangle\sum_{n=0}^{\infty}\left\{\frac{\alpha^n}{\sqrt{n!}}\cos\theta\cos[\sqrt{(n+1)(n+2)}\lambda t]-i\frac{\alpha^{n+2}}{\sqrt{(n+2)!}}\sin\theta\sin[\sqrt{(n+1)(n+2)}\lambda t]\right\}|n\rangle\right\},
\end{aligned}\tag{101}$$

S_1 的平均值为

$$\begin{aligned}
\langle S_1\rangle=&_{AF}\langle\varphi(t)|S_1|\varphi(t)\rangle_{AF}\\
=&e^{-|\alpha|^2}\left\{\cos\theta\cos\sqrt{2}\lambda t+\alpha^2\cos\theta\cos\sqrt{6}\lambda t\right.\\
&+\sum_{n=2}^{\infty}\alpha^{2n}\sin\theta\cos\theta\left\{\frac{1}{n!}\cos[\sqrt{(n-1)n}\lambda t]\cos[\sqrt{(n+1)(n+2)}\lambda t]\right.\\
&\left.\left.+\ +\frac{1}{\sqrt{(n-2)!(n+2)!}}\sin[\sqrt{(n-1)n}\lambda t]\sin[\sqrt{(n+1)(n+2)}\lambda t]\right\}\right\}.
\end{aligned}\tag{102}$$

□

第 11 题 得分: ________. 压缩态的另一种定义: $|\alpha\rangle_g=D(\alpha)S(\xi)|0\rangle$. 我们学过的压缩态为 $|\beta\rangle_g=S(\xi)D(\beta)|0\rangle$. 若 $|\alpha\rangle_g=|\beta\rangle_g$, 利用它们关于 $X_1=1/2(a+a^\dagger)$ 和 $X_2=-i/2(a-a^\dagger)$ 的涨落图, 求出 α 和 β 的关系。

解: 对 $|\beta\rangle_g=S(\xi)D(\beta)|0\rangle$, a 的均值为

$$\langle a\rangle={}_g\langle\beta|a|\beta\rangle_g=\langle 0|D^\dagger(\beta)S^\dagger(\xi)aS(\xi)D(\beta)|0\rangle_g=\langle\beta|S^\dagger(\xi)aS(\xi)|\beta\rangle$$

$$
\begin{aligned}
=&\langle\beta|\exp[-\frac{1}{2}\xi^*a^2+\frac{1}{2}\xi(a^\dagger)^2]a\exp[\frac{1}{2}\xi^*a^2-\frac{1}{2}\xi(a^\dagger)^2]|\beta\rangle\\
=&\langle\beta|(a\cosh r-a^\dagger e^{i\theta}\sinh r)|\beta\rangle\\
=&\beta\cosh r-\beta^*e^{i\theta}\sinh r.
\end{aligned}\tag{103}
$$

对 $|\alpha\rangle_g=S(\xi)D(\alpha)|0\rangle$, a 的均值为

$$
\begin{aligned}
\langle a\rangle=&_g\langle\alpha|a|\alpha\rangle_g=\langle 0|S^\dagger(\xi)D^\dagger(\alpha)aD(\alpha)S(\xi)|0\rangle\\
=&\langle 0|S^\dagger(\xi)(a+\alpha)S(\xi)|0\rangle\\
=&\langle 0|(a\cosh r-a^\dagger e^{i\theta}\sinh r+\alpha)|0\rangle\\
=&\alpha.
\end{aligned}\tag{104}
$$

故

$$
\alpha=\beta\cosh r-\beta^*e^{i\theta}\sinh r. \tag{105}
$$

□

第 12 题 得分: ________. 下图椭圆表示某压缩相干态光场的两正交分量 $X_1=1/2(a+a^\dagger)$ 和 $X_2=-i/2(a-a^\dagger)$ 的涨落范围. 已知椭圆长轴长为 $\Delta X_2=5$, 椭圆中心坐标为 $(0,6)$

(1) 若该压缩相干态 $|\beta\rangle_g=S(\xi)D(\beta)|0\rangle$, 求 β, ξ;

(2) 若压缩相干态 $|\beta\rangle_g=D(\beta)S(\xi)|0\rangle$, 则 β, ξ 又是多少?

解: (1) 若该压缩相干态 $|\beta\rangle_g=S(\xi)D(\beta)|0\rangle$, 则

$$
\langle X_1\rangle=\frac{1}{2}(\langle a\rangle+\langle a^\dagger\rangle)=\frac{1}{2}[(\beta+\beta^*)\cosh r-(\beta^*e^{i\theta}+\beta e^{-i\theta})\sinh r]=0, \tag{106}
$$

$$
\langle X_2\rangle=\frac{-i}{2}(\langle a\rangle-\langle a^\dagger\rangle)=\frac{1}{2}[(\beta-\beta^*)\cosh r-(\beta^*e^{i\theta}-\beta e^{-i\theta})\sinh r]=6, \tag{107}
$$

$$
\Delta X_2=\frac{1}{2}e^r=5, \tag{108}
$$

故

$$
\beta=\frac{3}{5}i, \tag{109}
$$

$$
r=\ln 10, \tag{110}
$$

$$
\theta=0, \tag{111}
$$

$$
\epsilon=re^{i\theta}=\ln 10. \tag{112}
$$

(2) 若压缩相干态 $|\beta\rangle_g=D(\beta)S(\xi)|0\rangle$, 则

$$
\langle X_1\rangle=\frac{1}{2}(\beta+\beta^*)=0, \tag{113}
$$

$$
\langle X_2\rangle=\frac{-i}{2}(\beta-\beta^*)=6, \tag{114}
$$

$$
\Delta X_2=\frac{1}{2}e^r=5, \tag{115}
$$

故

$$
\beta=6i, \tag{116}
$$

$$
\epsilon=re^{i\theta}=\ln 10. \tag{117}
$$

□

第 13 题 得分: ________. 薛定谔猫态 $|\psi\rangle = x[|\alpha\rangle + |-\alpha\rangle]$,

(1) 求归一化系数 x,

(2) 定义光场的两个相位正交的振幅分量 $X_1 = 1/2(a + a^\dagger)$ 和 $X_2 = -i/2(a - a^\dagger)$, 讨论 X_1 的压缩条件.

解: (1) 由归一化条件,

$$\begin{aligned}\langle\psi|\psi\rangle =& |x|^2 (\langle\alpha| + \langle-\alpha|)(|\alpha\rangle + |-\alpha\rangle)\\ =& |x|^2 (\langle\alpha|\alpha\rangle + \langle\alpha| - \alpha\rangle + \langle-\alpha|\alpha\rangle + \langle-\alpha| - \alpha\rangle)\\ =& 2|x|^2 [1 + \exp(-2|\alpha|^2)]\\ =& 1,\end{aligned} \tag{118}$$

$$\Longrightarrow x = [2 + 2\exp(-2|\alpha|^2)]^{-1/2}. \tag{119}$$

(2) X_1 的均值为

$$\begin{aligned}\langle X_1\rangle =& N^*(\langle\alpha| + \langle-\alpha|)\frac{1}{2}(a + a^\dagger)N(|\alpha\rangle + |-\alpha\rangle)\\ =& \frac{|N|^2}{2}[(\langle\alpha| + \langle-\alpha|)(\alpha|\alpha\rangle - \alpha|-\alpha\rangle) + (\alpha^*\langle\alpha| - \alpha^*\langle-\alpha|)(|\alpha\rangle + |-\alpha\rangle)]\\ =& \frac{|N|^2}{2}[\alpha(1 - \exp(-2|\alpha|^2) + \exp(-2|\alpha|^2) - 1) + \alpha^*(1 + \exp(-2|\alpha|^2) - \exp(-2|\alpha|^2) - 1)]\\ =& 0.\end{aligned} \tag{120}$$

X_1^2 的均值为

$$\begin{aligned}\langle X_1^2\rangle =& N^*(\langle\alpha| + \langle-\alpha|)\left[\frac{1}{2}(a + a^\dagger)\right]^2 N(|\alpha\rangle + |-\alpha\rangle)\\ =& \frac{|N|^2}{4}(\langle\alpha| + \langle-\alpha|)(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger)(|\alpha\rangle + |-\alpha\rangle)\\ =& \frac{|N|^2}{4}(\langle\alpha| + \langle-\alpha|)(aa + 2a^\dagger a + 1 + a^\dagger a^\dagger)(|\alpha\rangle + |-\alpha\rangle)\\ =& \frac{|N|^2}{4}[(\langle\alpha| + \langle-\alpha|)(\alpha^2|\alpha\rangle + \alpha^2|-\alpha\rangle) + 2(\alpha^*\langle\alpha| - \alpha^*\langle-\alpha|)(\alpha|\alpha\rangle - \alpha|-\alpha\rangle)\\ &+ (\langle\alpha| + \langle-\alpha|)(|\alpha\rangle + |-\alpha\rangle) + ((\alpha^*)^2\langle\alpha| + (\alpha^*)^2\langle-\alpha|)(|\alpha\rangle + |-\alpha\rangle)]\\ =& \frac{|N|^2}{4}[\alpha^2(1 + \exp(-2|\alpha|^2) + \exp(-2|\alpha|^2) + 1) + 2|\alpha|^2(1 - \exp(-2|\alpha|^2) - \exp(-2|\alpha|^2) + 1)\\ &+ (1 + \exp(-2|\alpha|^2) + \exp(-2|\alpha|^2) + 1) + (\alpha^*)^2(1 + \exp(-2|\alpha|^2) + \exp(-2|\alpha|^2) + 1)]\\ =& \frac{|N|^2}{2}[(\alpha^2 + (\alpha^*)^2 + 1)(1 + \exp(-2|\alpha|^2)) + 2|\alpha|^2(1 - \exp(-2|\alpha|^2)]\\ =& \frac{1}{4}\left[1 + \alpha^2 + (\alpha^*)^2 + 2|\alpha|^2\frac{1 - \exp(-2|\alpha|^2)}{1 + \exp(-2|\alpha|^2)}\right]\\ =& \frac{1}{4}\left[1 - 4\frac{\mathrm{Re}[\alpha]^2 - \mathrm{Im}[\alpha]^2\exp(-2|\alpha|^2)}{1 + \exp(-2|\alpha|^2)}\right].\end{aligned} \tag{121}$$

X_1 的涨落为

$$\Delta X_1 = \sqrt{\langle X_1^2\rangle - \langle X_1\rangle^2} = \sqrt{\langle X_1^2\rangle} = \frac{1}{2}\sqrt{1 - 4\frac{\mathrm{Re}[\alpha]^2 - \mathrm{Im}[\alpha]^2\exp(-2|\alpha|^2)}{1 + \exp(-2|\alpha|^2)}}. \tag{122}$$

X_1 的压缩条件为

$$\Delta X_1 < \frac{1}{2}, \tag{123}$$

$$\mathrm{Re}[\alpha]^2 < \mathrm{Im}[\alpha]^2 \exp(-2\,|\alpha|^2). \tag{124}$$

□