

第 1 题 得分：_____。单模热光场的密度算符为 $\rho = (1 - e^{-\beta}) \exp(-\beta a^\dagger a)$, $\beta = \hbar\omega/kT$ 。求其密度算符的 Q 表示。

解：单模热光场的密度算符可化为

$$\begin{aligned}\rho &= (1 - e^{-\beta}) \exp(-\beta a^\dagger a) \sum_n |n\rangle \langle n| \\ &= (1 - e^{-\beta}) \sum_n \exp(-n\beta) |n\rangle \langle n|.\end{aligned}$$

Q 表示为

$$\begin{aligned}Q(\alpha) &= \frac{1}{\pi} \langle \alpha | \rho | \alpha \rangle = \langle \alpha | (1 - e^{-\beta}) \sum_n \exp(-n\beta) |n\rangle \langle n| \alpha \rangle \\ &= \frac{1 - e^{-\beta}}{\pi} \sum_n \exp(-n\beta) |\langle n | \alpha \rangle|^2 \\ &= \frac{1 - e^{-\beta}}{\pi} \sum_n \exp(-n\beta) \left| e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \right|^2 \\ &= \frac{1 - e^{-\beta}}{\pi} e^{-|\alpha|^2} \sum_n \exp(-n\beta) \frac{|\alpha|^{2n}}{n!} \\ &= \frac{1 - e^{-\beta}}{\pi} e^{-|\alpha|^2 [1 + \exp(-\beta)]}.\end{aligned}\tag{1}$$

□

第 2 题 得分：_____。一光场处于这样的叠加态： $|\psi\rangle = N(|\alpha\rangle + e^{i\theta}|- \alpha\rangle)$ 。

(1) 计算归一化常数 N 。

(2) 若 $\theta = \pi$, 判断其是否为泊松分布, 为什么?

(3) 若 $\theta = 0$, α 为纯虚数, 判断其是否有压缩现象, 为什么? (提示: 计算 $(\Delta X_1)^2$, $X_1 = (\alpha + \alpha^\dagger)/2$, $X_2 = (\alpha - \alpha^\dagger)/2i$.)

解：(1) 由归一化条件,

$$\begin{aligned}\langle \psi | \psi \rangle &= |N|^2 (\langle \alpha | + e^{-i\theta} \langle -\alpha |) (|\alpha\rangle + e^{i\theta} |-\alpha\rangle) \\ &= |N|^2 (\langle \alpha | \alpha \rangle + \langle \alpha | -\alpha \rangle e^{i\theta} + \langle -\alpha | \alpha \rangle e^{-i\theta} + \langle -\alpha | -\alpha \rangle) \\ &= |N|^2 [1 + \exp(-|\alpha|^2) e^{i\theta} + \exp(-|\alpha|^2) e^{-i\theta} + 1] \\ &= 2 |N|^2 [1 + \exp(-|\alpha|^2) \cos \theta] \\ &= 1,\end{aligned}\tag{2}$$

$$\implies N = [2 + 2 \exp(-|\alpha|^2) \cos \theta]^{-1/2}.\tag{3}$$

(2) 若 $\theta = \pi$, 则叠加态为

$$|\psi\rangle = N(|\alpha\rangle - |-\alpha\rangle),\tag{4}$$

二阶相关度

$$g^{(2)}(0) = \frac{\langle a^\dagger a^\dagger a a \rangle}{\langle a^\dagger a \rangle^2} = \frac{\langle \psi | a^\dagger a^\dagger a a | \psi \rangle}{\langle \psi | a^\dagger a | \psi \rangle^2}$$

$$\begin{aligned}
&= \frac{|N|^2 (\langle \alpha | - \langle -\alpha |) a^\dagger a^\dagger a a (|\alpha \rangle - |-\alpha \rangle)}{[|N|^2 (\langle \alpha | - \langle -\alpha |) a^\dagger a (|\alpha \rangle - |-\alpha \rangle)]^2} \\
&= \frac{[(\alpha^*)^2 \langle \alpha | - (\alpha^*)^2 \langle -\alpha |][\alpha^2 |\alpha \rangle - \alpha^2 |-\alpha \rangle]}{|N|^2 \{[\alpha^* \langle \alpha | + \alpha^* \langle -\alpha |][\alpha |\alpha \rangle + \alpha |-\alpha \rangle]\}^2} \\
&= \frac{|\alpha|^4 (1 - \exp(-|\alpha|^2) - \exp(-|\alpha|^2) + 1)}{|N|^2 \{|\alpha|^2 (1 + \exp(-|\alpha|^2) + \exp(-|\alpha|^2) + 1)\}^2} \\
&= \frac{1 - \exp(-|\alpha|^2)}{2 |N|^2 [1 + \exp(-|\alpha|^2)]^2} \\
&= \frac{[1 - \exp(-|\alpha|^2)][1 + \exp(-|\alpha|^2) \cos \theta]}{[1 + \exp(-|\alpha|^2)]^2} < 1,
\end{aligned} \tag{5}$$

故该叠加态为亚泊松分布.

(3) 若 $\theta = 0$, 叠加态为

$$|\psi\rangle = N(|\alpha\rangle + |-\alpha\rangle), \tag{6}$$

其中

$$N = [2 + 2 \exp(-|\alpha|^2)]^{-1/2} \tag{7}$$

X_1 的均值为

$$\begin{aligned}
\langle X_1 \rangle &= N^* (\langle \alpha | + \langle -\alpha |) \frac{1}{2} (a + a^\dagger) N (|\alpha \rangle + |-\alpha \rangle) \\
&= \frac{|N|^2}{2} [(\langle \alpha | + \langle -\alpha |)(\alpha |\alpha \rangle - \alpha |-\alpha \rangle) + (\alpha^* \langle \alpha | - \alpha^* \langle -\alpha |)(|\alpha \rangle + |-\alpha \rangle)] \\
&= \frac{|N|^2}{2} [\alpha (1 - \exp(-|\alpha|^2) + \exp(-|\alpha|^2) - 1) + \alpha^* (1 + \exp(-|\alpha|^2) - \exp(-|\alpha|^2) - 1)] \\
&= 0.
\end{aligned} \tag{8}$$

X_1^2 的均值为

$$\begin{aligned}
\langle X_1^2 \rangle &= N^* (\langle \alpha | + \langle -\alpha |) \left[\frac{1}{2} (a + a^\dagger) \right]^2 N (|\alpha \rangle + |-\alpha \rangle) \\
&= \frac{|N|^2}{4} (\langle \alpha | + \langle -\alpha |) (aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger) (|\alpha \rangle + |-\alpha \rangle) \\
&= \frac{|N|^2}{4} (\langle \alpha | + \langle -\alpha |) (aa + 2a^\dagger a + 1 + a^\dagger a^\dagger) (|\alpha \rangle + |-\alpha \rangle) \\
&= \frac{|N|^2}{4} [(\langle \alpha | + \langle -\alpha |)(\alpha^2 |\alpha \rangle + \alpha^2 |-\alpha \rangle) + 2(\alpha^* \langle \alpha | - \alpha^* \langle -\alpha |)(\alpha |\alpha \rangle - \alpha |-\alpha \rangle) \\
&\quad + (\langle \alpha | + \langle -\alpha |)(|\alpha \rangle + |-\alpha \rangle) + ((\alpha^*)^2 \langle \alpha | + (\alpha^*)^2 \langle -\alpha |)(|\alpha \rangle + |-\alpha \rangle)] \\
&= \frac{|N|^2}{4} [\alpha^2 (1 + \exp(-|\alpha|^2) + \exp(-|\alpha|^2) + 1) + 2|\alpha|^2 (1 - \exp(-|\alpha|^2) - \exp(-|\alpha|^2) + 1) \\
&\quad + (1 + \exp(-|\alpha|^2) + \exp(-|\alpha|^2) + 1) + (\alpha^*)^2 (1 + \exp(-|\alpha|^2) + \exp(-|\alpha|^2) + 1)] \\
&= \frac{|N|^2}{2} [(\alpha^2 + (\alpha^*)^2 + 1)(1 + \exp(-|\alpha|^2)) + |\alpha|^2 (1 - \exp(-|\alpha|^2))] \\
&= \frac{1}{4} \left[1 + \alpha^2 + (\alpha^*)^2 + |\alpha|^2 \frac{1 - \exp(-|\alpha|^2)}{1 + \exp(-|\alpha|^2)} \right],
\end{aligned} \tag{9}$$

由于 α 为纯虚数, 故

$$\langle X_1^2 \rangle = \frac{1}{4} \left[1 - |\alpha|^2 \frac{1 + 3 \exp(-|\alpha|^2)}{1 + \exp(-|\alpha|^2)} \right] \leq \frac{1}{4}. \quad (10)$$

X_1 的涨落为

$$\Delta X_1 = \sqrt{\langle X_1^2 \rangle - \langle X_1 \rangle^2} = \sqrt{\langle X_1^2 \rangle} < \frac{1}{2}, \quad (11)$$

故有压缩.

□

第 3 题 得分: _____. 简述:

- (1) 偶极近似的适用条件;
- (2) 旋转波近似的含义;
- (3) 马尔可夫近似的含义;
- (4) 自发辐射和受激辐射的不同点;
- (5) Hanbry-Brown-Twiss 实验与迈克尔逊干涉实验的不同之处.

解: (1) 偶极近似的适用条件: 讨论的系统尺寸远小于光的波长.

- (2) 旋转波近似的含义: 光场频率相对于系统的本征频率具有大失谐的分量在足够长时间的积分下对系统演化的贡献平均为零.
- (3) 马尔可夫近似: 系统无记忆, 系统的当前状态与其历史无关, 即关联时间为 0, 该近似适用于热库非常大且系统与热库相互作用持续的情形.
- (4) (a) 自发辐射无需外界辐射场的激励就可发生, 受激辐射需要外界辐射场的激励才能发生;
(b) 自发辐射的发生时间、相位、方向和偏振是随机的, 受激辐射的相位、方向和偏振与诱导光子相同.
- (5) HBT 实验是二阶相干, 是光子与光子之间的相干, 体现了光源的光子数分布特性; 迈克尔逊干涉实验是一阶相干, 是光子自身的相干, 体现了光源频谱的单色性.

□

第 4 题 得分: _____. 单个二能级原子 (上下能级分别为 $|a\rangle, |b\rangle$) 同单模光场 (频率 $\nu = \omega_{ab}$) 共振相互作用. 考虑偶极近似和旋转波近似, 假设相互作用系数为实数.

- (1) 写出半经典理论描述的原子-光场系统哈密顿量.
- (2) 写出全量子理论描述的原子-光场哈密顿量.
- (3) 原子初态为 $|a\rangle$, 光场初态为真空态, 利用全量子理论的描述求 t 时刻的原子布居数反转数 $W(t) = |c_a|^2 - |c_b|^2$.
- (4) (附加题) 若原子初态为 $\frac{1}{\sqrt{2}}(|a\rangle - |b\rangle)$, 光场的初态为真空态, 求 $W(t)$.

解: (1) 半经典理论描述的原子-光场系统哈密顿量为

$$H = \hbar\omega_a|a\rangle\langle a| + \hbar\omega_b|b\rangle\langle b| - (\mu_{ab}|a\rangle\langle b| + \mu_{ba}|b\rangle\langle a|)\mathcal{E} \cos \nu t \quad (12)$$

其中电偶极矩算符的对角项

$$\mu_{ab} = \langle a|\mu|b\rangle, \quad \mu_{ba} = \langle b|\mu|a\rangle. \quad (13)$$

(2) 全量子理论描述的原子-光场哈密顿量为

$$H = H_0 + H_1. \quad (14)$$

其中无微扰哈密顿量

$$H_0 = \hbar\nu a^\dagger a + \frac{1}{2}\hbar\omega_{ab}\sigma_z, \quad (15)$$

相互作用哈密顿量

$$H_1 = \hbar g(\sigma_+ a + a^\dagger \sigma_-). \quad (16)$$

(3) 相互作用表象下, 系统的相互作用哈密顿量为

$$\hat{V} = e^{i\hat{H}_0 t/\hbar} H_1 e^{-i\hat{H}_0 t/\hbar} = \hbar g(\sigma_+ a e^{i\Delta t} + a^\dagger \sigma_- e^{-i\Delta t}), \quad (17)$$

其中频率失谐 $\Delta = \omega_{ab} - \nu$. 系统演化遵循相互作用表象下薛定谔方程:

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{V} |\psi(t)\rangle, \quad (18)$$

代入系统的量子态

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} [C_{a,n}(t)|a, n\rangle + C_{b,n}(t)|b, n\rangle], \quad (19)$$

有

$$\dot{C}_{a,n}(t) = -igC_{b,n+1}\sqrt{n+1}e^{i\Delta t}, \quad (20)$$

$$\dot{C}_{b,n+1}(t) = -igC_{a,n}\sqrt{n+1}e^{-i\Delta t}, \quad (21)$$

解得

$$C_{a,n}(t) = \left\{ C_{a,n}(0) \left[\cos\left(\frac{\Omega_n t}{2}\right) - \frac{i\Delta}{\Omega_n} \sin\left(\frac{\Omega_n t}{2}\right) \right] - \frac{2ig\sqrt{n+1}}{\Omega_n} C_{b,n+1}(0) \sin\left(\frac{\Omega_n t}{2}\right) \right\} e^{i\Delta t/2}, \quad (22)$$

$$C_{b,n+1}(t) = \left\{ C_{b,n+1}(0) \left[\cos\left(\frac{\Omega_n t}{2}\right) + \frac{i\Delta}{\Omega_n} \sin\left(\frac{\Omega_n t}{2}\right) \right] - \frac{2ig\sqrt{n+1}}{\Omega_n} C_{a,n}(0) \sin\left(\frac{\Omega_n t}{2}\right) \right\} e^{-i\Delta t/2}, \quad (23)$$

其中

$$\Omega_n^2 = \Delta^2 + 4g^2(n+1). \quad (24)$$

原子初态为 $|a\rangle$, 即 $C_{a,0} = 1$, 将其代入上面两式得

$$C_{a,0}(t) = \left[\cos\left(\frac{\Omega_0 t}{2}\right) - \frac{i\Delta}{\Omega_0} \sin\left(\frac{\Omega_0 t}{2}\right) \right] e^{i\Delta t/2}, \quad (25)$$

$$C_{b,1}(t) = -\frac{2ig}{\Omega_0} \sin\left(\frac{\Omega_0 t}{2}\right) e^{-i\Delta t/2}. \quad (26)$$

t 时刻的原子布居数反转为

$$W(t) = |c_a|^2 - |c_b|^2 = |C_{a,0}(t)|^2 - |C_{b,1}(t)|^2 = \cos^2\left(\frac{\Omega_0 t}{2}\right) + \frac{\Delta^2 - 4g^2}{\Omega_0^2} \sin^2\left(\frac{\Omega_0 t}{2}\right). \quad (27)$$

由于 $\nu = \omega_{ab}$, 故 $\Delta = 0$, $\Omega_0^2 = 4g^2$,

$$W(t) = \cos^2(gt) - \sin^2(gt) = \cos(2gt). \quad (28)$$

(4) 若原子初态为 $\frac{1}{\sqrt{2}}(|a\rangle - |b\rangle)$, 光场的初态为真空态, 则 $C_{a,0}(0) = \frac{1}{\sqrt{2}}, C_{b,0} = \frac{1}{\sqrt{2}},$

$$C_{a,0}(t) = \frac{1}{\sqrt{2}} \left[\cos\left(\frac{\Omega_0 t}{2}\right) - \frac{i\Delta}{\Omega_0} \sin\left(\frac{\Omega_0 t}{2}\right) \right] e^{i\Delta t/2}, \quad (29)$$

$$C_{b,1}(t) = -\frac{\sqrt{2}ig}{\Omega_0} \sin\left(\frac{\Omega_0 t}{2}\right) e^{-i\Delta t/2}, \quad (30)$$

$$C_{b,0}(t) = \frac{1}{\sqrt{2}}. \quad (31)$$

t 时刻的原子布居数反转为

$$\begin{aligned} W(t) &= |c_a(t)|^2 - |c_b(t)|^2 = |C_{a,0}(t)|^2 - |C_{b,1}(t)|^2 - |C_{b,0}(t)|^2 \\ &= \frac{1}{2} \cos^2\left(\frac{\Omega_0 t}{2}\right) + \frac{\Delta^2 - 4g^2}{2\Omega_0^2} \sin^2\left(\frac{\Omega_0 t}{2}\right) - \frac{1}{2}. \end{aligned} \quad (32)$$

由于 $\nu = \omega_{ab}$, 故 $\Delta = 0, \Omega_n^2 = 4g^2(n+1),$

$$W(t) = \frac{1}{2} \cos^2(gt) - \frac{1}{2} \sin^2(gt) - \frac{1}{2} = \frac{\cos(2gt) - 1}{2}. \quad (33)$$

□

第 5 题 得分: _____. 单模光场与热平衡辐射场热库相互作用, 其密度算符的运动方程为:

$$\rho = -\frac{\gamma}{2} \bar{n} (aa^\dagger \rho - 2a^\dagger \rho a + \rho aa^\dagger) - \frac{\gamma}{2} (\bar{n} + 1) (a^\dagger a \rho - 2\rho a^\dagger + \rho a^\dagger a).$$

求 t 时刻光场的粒子数平均值 $\langle a^\dagger(t)a(t) \rangle$. [提示: $\frac{d\langle a^\dagger a \rangle}{dt} = \text{Tr}(\rho \dot{a}^\dagger a)$]

解:

$$\begin{aligned} \frac{d\langle a^\dagger a \rangle}{dt} &= \text{Tr}(\rho \dot{a}^\dagger a) = \text{Tr}(\dot{\rho} a^\dagger a) \\ &= \text{Tr} \left\{ \left[-\frac{\gamma}{2} \bar{n} (aa^\dagger \rho - 2a^\dagger \rho a + \rho aa^\dagger) - \frac{\gamma}{2} (\bar{n} + 1) (a^\dagger a \rho - 2\rho a^\dagger + \rho a^\dagger a) \right] a^\dagger a \right\} \\ &= \text{Tr} \left\{ -\frac{\gamma}{2} \bar{n} (aa^\dagger \rho a^\dagger a - 2a^\dagger \rho a a^\dagger a + \rho a a^\dagger a^\dagger a) - \frac{\gamma}{2} (\bar{n} + 1) (a^\dagger a \rho a^\dagger a - 2\rho a^\dagger a^\dagger a + \rho a^\dagger a a^\dagger a) \right\} \\ &= \text{Tr} \left\{ \rho \left[-\frac{\gamma}{2} \bar{n} (a^\dagger a a a^\dagger - 2a a^\dagger a a^\dagger + a a^\dagger a^\dagger a) - \frac{\gamma}{2} (\bar{n} + 1) (a^\dagger a a^\dagger a - 2a^\dagger a^\dagger a a + a^\dagger a a^\dagger a) \right] \right\} \\ &= \text{Tr} \left\{ \rho \left[-\frac{\gamma}{2} \bar{n} (a^\dagger a (a^\dagger a + 1) - 2(a^\dagger a + 1)(a^\dagger a + 1) + (a^\dagger a + 1)a^\dagger a) - \frac{\gamma}{2} (\bar{n} + 1) (2a^\dagger (a^\dagger a + 1)a - 2a^\dagger a^\dagger a a) \right] \right\} \\ &= \text{Tr} \left\{ \rho [\gamma \bar{n} (a^\dagger a + 1) - \gamma (\bar{n} + 1) a^\dagger a] \right\} \\ &= \text{Tr} \{ \gamma \rho (\bar{n} - a^\dagger a) \} \\ &= \gamma (\bar{n} - \langle a^\dagger a \rangle), \end{aligned} \quad (34)$$

解得 t 时刻光场的粒子数平均值为

$$\langle a^\dagger(t)a(t) \rangle = (\langle a^\dagger(0)a(0) \rangle - \bar{n})e^{-\gamma t} + \bar{n}. \quad (35)$$

□

第 6 题 得分: _____. 单模光场与热平衡辐射场热库相互作用的 Langevin 方程为

$$\dot{\tilde{a}}(t) = -\frac{\gamma}{2} \tilde{a}(t) + F_{\tilde{a}}(t),$$

且已知: $\langle F_{\tilde{a}}^\dagger(t)\tilde{a}(t) \rangle_R + \langle \tilde{a}^\dagger(t)F_{\tilde{a}}(t) \rangle_R = \gamma \bar{n}$. 试求 $\langle \tilde{a}^\dagger(t)\tilde{a}(t) \rangle_R$.

解: Langevin 方程的共轭为

$$\dot{\tilde{a}}^\dagger(t) = -\frac{\gamma^*}{2}\tilde{a}^\dagger(t) + F_a^\dagger(t). \quad (36)$$

$\langle \tilde{a}^\dagger(t)\tilde{a}(t) \rangle$ 遵循演化方程:

$$\begin{aligned} \frac{d\langle \tilde{a}^\dagger(t)\tilde{a}(t) \rangle_R}{dt} &= \left\langle \frac{d\tilde{a}^\dagger(t)}{dt}\tilde{a}(t) \right\rangle_R = \left\langle \frac{d\tilde{a}^\dagger(t)}{dt}\tilde{a}(t) + \tilde{a}^\dagger(t)\frac{d\tilde{a}(t)}{dt} \right\rangle_R \\ &= \left\langle \left[-\frac{\gamma^*}{2}\tilde{a}^\dagger(t) + F_a^\dagger(t) \right] \tilde{a}(t) + \tilde{a}^\dagger(t) \left[-\frac{\gamma}{2}\tilde{a}(t) + F_a(t) \right] \right\rangle_R \\ &= -\frac{\gamma^* + \gamma}{2} \langle \tilde{a}^\dagger(t)\tilde{a}(t) \rangle_R + \langle F_a^\dagger(t)\tilde{a}(t) \rangle_R + \langle \tilde{a}^\dagger(t)F_a(t) \rangle_R \\ &= -\frac{\gamma^* + \gamma}{2} \langle \tilde{a}^\dagger(t)\tilde{a}(t) \rangle_R + \gamma\bar{n}, \end{aligned} \quad (37)$$

解得

$$\langle \tilde{a}^\dagger(t)\tilde{a}(t) \rangle_R = \left[\langle \tilde{a}^\dagger(0)\tilde{a}(0) \rangle - \frac{2\gamma}{\gamma^* + \gamma} \right] e^{-\frac{\gamma^* + \gamma}{2}t} + \frac{2\gamma}{\gamma^* + \gamma}\bar{n}. \quad (38)$$

□

第 7 题 (附加题) 得分: _____. 如何理解原子谱线宽度和寿命的不确定性关系 $\Delta\omega \propto 1/\tau$.

解: 原子上能级具有有限的寿命 τ , 由不确定性原理, 其能量值具有不确定性 ΔE , 满足

$$\Delta E \cdot \tau \sim \frac{\hbar}{2}, \quad (39)$$

故原子谱线, 即原子由上能级跃迁至下能级辐射的光子频率具有不确定性

$$\Delta\omega = \frac{\Delta E}{\hbar} \propto \frac{1}{\tau}. \quad (40)$$

□

第 8 题 (附加题) 得分: _____. 谈谈你对信息的理解.

解: 信息即不确定度的减少, 信息的数量用香农熵

$$H = -\sum_{k=1}^n p_k \log_2 p_k \quad (41)$$

来衡量.

□

第 9 题 (附加题) 得分: _____. 简述量子光学主要是研究什么问题? 在经典物理学中主要用哪一些学科研究这些内容 (例如可举普通物理和四大力学中的学科和工具).

解: 量子光学主要研究光场的量子化和光与物质的相互作用.

经典物理中有光学、原子物理学、电动力学、量子力学来研究这些内容.

□