期末考试

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成绩：__________

第 1 题 (20 分) 得分: ________. 产生湮灭算符 $a^\dagger$, a 满足对易关系 $[a,a^\dagger]=1$, 且 $[a,(a^\dagger)^n]=n(a^\dagger)^{n-1}$, 试证:

(i) $[a^\dagger,a^m]=-ma^{m-1}$;

(ii) $a(a^\dagger)^n a^m a^\dagger=(a^\dagger)^{n+1}a^{m+1}+(m+n+1)(a^\dagger)^n a^m+mn(a^\dagger)^{n-1}a^{m-1}$.

证: (i) 利用数学归纳法证明:

- 当 $m=1$ 时,

$$[a^\dagger,a^1]=-1=-1\cdot a^{1-1}. \tag{1}$$

- 假设当 $m=k$ 时,

$$[a^\dagger,a^k]=-ka^{k-1}, \tag{2}$$

则当 $m=k+1$ 时,

$$[a^\dagger,a^{k+1}]=[a^\dagger,a^k]a+a^k[a^\dagger,a]=-ka^{k-1}a+a^k\cdot(-1)=-(k+1)a^k=-(k+1)a^{(k+1)-1}. \tag{3}$$

综上, $[a^\dagger,a^m]=-ma^{m-1}$.

(ii)

$$\begin{aligned}
a(a^\dagger)^n a^m a^\dagger=&[(a^\dagger)^n a+n(a^\dagger)^{n-1}]a^m a^\dagger=(a^\dagger)^n a^{m+1}a^\dagger+n(a^\dagger)^{n-1}a^m a^\dagger\\
=&(a^\dagger)^n[a^\dagger a^{m+1}+(m+1)a^m]+n(a^\dagger)^{n-1}[a^\dagger a^m+ma^{m-1}]\\
=&(a^\dagger)^{n+1}a^{m+1}+(m+n+1)(a^\dagger)^n a^m+mn(a^\dagger)^{n-1}a^{m-1}.
\end{aligned} \tag{4}$$

□

第 2 题 (20 分) 得分: ________. 有如下几种单模辐射场, 分别计算它们的光子数分布函数 $p(m)$:

(1) 数态的叠加 $|\psi\rangle=\frac{1}{\sqrt{2}}(|0\rangle-i|10\rangle)$;

(2) $\rho=\sum_{n=0}^\infty\frac{e^{-\kappa}\kappa^n}{n!}|n\rangle\langle n|$, $\kappa\in\mathbb{R}^+$;

(3) 湮灭掉一个光子的热光场 $\rho'=\frac{a\rho a^\dagger}{\mathrm{Tr}[a\rho a^\dagger]}$, 其中 ρ 是热光场, 即 $\rho=\frac{1}{1+\langle n\rangle}\sum_{n=0}^\infty\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^n|n\rangle\langle n|$, $\langle n\rangle$ 是 ρ 光场的平均光子数.

解: (1) 该叠加态的光子数分布函数为

$$p(m)=|\langle m|\psi\rangle|^2=\frac{1}{2}\left|\delta_{m0}+\delta_{m,10}\right|^2. \tag{5}$$

(2) 该辐射场的光子数分布函数为

$$p(m)=\langle m|\rho|m\rangle=\frac{e^{-\kappa}\kappa^m}{m!}. \tag{6}$$

(3)

$$a\rho a^\dagger=\frac{1}{1+\langle n\rangle}\sum_{n=0}^\infty\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{n+1}(n+1)|n\rangle\langle n|, \tag{7}$$

$$\mathrm{Tr}[a\rho a^\dagger]=\frac{1}{1+\langle n\rangle}\sum_{n=0}^\infty\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^{n+1}(n+1)=\frac{1}{1+\langle n\rangle}\sum_{n=1}^\infty x^n n,$$

其中 $x=\frac{\langle n\rangle}{1+\langle n\rangle}$, 令 $f=\sum_{n=1}^{\infty}x^nn$, 由于 $xf=\sum_{n=1}^{\infty}x^{n+1}n$, $f-xf=x+\sum_{n=2}^{\infty}x^n=x+\frac{x^2}{1-x}=\frac{x}{1-x}$, 故 $f=\frac{x}{(1-x)^2}$,

$$\mathrm{Tr}[a\rho a^\dagger]=\langle n\rangle. \tag{8}$$

该湮灭掉一个光子的热光场的密度矩阵为

$$\rho'=\frac{a\rho a^\dagger}{\mathrm{Tr}[a\rho a^\dagger]}=\sum_{n=0}^{\infty}\frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+2}}(n+1)|n\rangle\langle n|. \tag{9}$$

其光子数分布函数为

$$p(m)=\langle m|\rho'|m\rangle=\frac{\langle n\rangle^m}{(1+\langle n\rangle)^{m+2}}(m+1). \tag{10}$$

□

第 3 题 (20 分) 得分: __________. 试通过计算判断, 上题 (1) 中的辐射场的光子数分布为何种分布 (Poisson, Sub-Poisson, Super-Poisson)?

解: 上题 (1) 中的辐射场二阶相关度为

$$g^{(2)}(0)=\frac{\langle a^\dagger a^\dagger aa\rangle}{\langle a^\dagger a\rangle^2}=\frac{\frac{1}{\sqrt{2}}(\langle 0|+i\langle 10|)a^\dagger a^\dagger aa\frac{1}{\sqrt{2}}(|0\rangle-i|10\rangle)}{\left[\frac{1}{\sqrt{2}}(\langle 0|+i\langle 10|)a^\dagger a\frac{1}{\sqrt{2}}(|0\rangle-i|10\rangle)\right]^2}=\frac{9}{5}>1, \tag{11}$$

故该辐射场的光子数分布为超泊松分布. □

第 4 题 (20 分) 得分: __________. 增加了一个光子的相干态 (Single-phtono-add coherent state (SPACS)) $|\alpha,1\rangle=\frac{a^\dagger}{\sqrt{1+|\alpha|^2}}|\alpha\rangle$. 考虑该辐射场的两个厄米算符 $X_1=\frac{1}{2}(a+a^\dagger)$, $X_2=\frac{1}{2i}(a-a^\dagger)$. 它们分别对应于场的复振幅的实部和虚部. 证明:
SPACS 态 $|\alpha,1\rangle$ 当 $|\alpha|>1$ 时是压缩态, (本题取 $\alpha\in\mathbb{R}^+$).

证: X_1 的均值:

$$\begin{aligned}\langle X_1\rangle=&\langle\alpha|\frac{a}{\sqrt{1+|\alpha|^2}}\frac{1}{2}(a+a^\dagger)\frac{a^\dagger}{\sqrt{1+|\alpha|^2}}|\alpha\rangle=\frac{1}{2(1+\alpha^2)}\langle\alpha|(aaa^\dagger+aa^\dagger a^\dagger)|\alpha\rangle\\=&\frac{1}{2(1+\alpha^2)}\langle\alpha|[a(a^\dagger a+1)+(a^\dagger a+1)a^\dagger]|\alpha\rangle=\frac{1}{2(1+\alpha^2)}\langle\alpha|(aa^\dagger a+a+a^\dagger aa^\dagger+a^\dagger)|\alpha\rangle\\=&\frac{1}{2(1+\alpha^2)}\langle\alpha|[(a^\dagger a+1)a+a+a^\dagger(a^\dagger a+1)+a^\dagger]|\alpha\rangle=\frac{1}{2(1+\alpha^2)}\langle\alpha|(a^\dagger aa+2a+a^\dagger a^\dagger a+2a^\dagger)|\alpha\rangle\\=&\frac{\alpha(2+\alpha^2)}{1+\alpha^2}.\end{aligned} \tag{12}$$

X_1^2 的均值:

$$\begin{aligned}\langle X_1^2\rangle=&\langle\alpha|\frac{a}{\sqrt{1+|\alpha|^2}}\left[\frac{1}{2}(a+a^\dagger)\right]^2\frac{a^\dagger}{\sqrt{1+|\alpha|^2}}|\alpha\rangle=\frac{1}{4(1+\alpha^2)}\langle\alpha|(aaaa+aaa^\dagger a^\dagger+aa^\dagger aa^\dagger+aa^\dagger a^\dagger a^\dagger)|\alpha\rangle\\=&\frac{1}{4(1+\alpha^2)}\langle\alpha|[aaaa+a(a^\dagger a+1)a^\dagger+aa^\dagger aa^\dagger+(a^\dagger a+1)a^\dagger a^\dagger]|\alpha\rangle\\=&\frac{1}{4(1+\alpha^2)}\langle\alpha|(aaaa+2aa^\dagger aa^\dagger+aa^\dagger+a^\dagger aa^\dagger a^\dagger+a^\dagger a^\dagger)|\alpha\rangle\\=&\frac{1}{4(1+\alpha^2)}\langle\alpha|(aaaa+2(a^\dagger a+1)(a^\dagger a+1)+(a^\dagger a+1)+a^\dagger(a^\dagger a+1)a^\dagger+a^\dagger a^\dagger)|\alpha\rangle\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{4(1+\alpha^2)}\langle\alpha|(aaaa+2a^\dagger aa^\dagger a+5a^\dagger a+3+a^\dagger a^\dagger aa^\dagger+2a^\dagger a^\dagger)|\alpha\rangle\\
=&\frac{1}{4(1+\alpha^2)}\langle\alpha|(aaaa+2a^\dagger(a^\dagger a+1)a+5a^\dagger a+3+a^\dagger a^\dagger(a^\dagger a+1)+2a^\dagger a^\dagger)|\alpha\rangle\\
=&\frac{1}{4(1+\alpha^2)}\langle\alpha|(aaaa+2a^\dagger a^\dagger aa+7a^\dagger a+3+a^\dagger a^\dagger a^\dagger a+3a^\dagger a^\dagger)|\alpha\rangle\\
=&\frac{4\alpha^4+10\alpha^2+3}{4(1+\alpha^2)}.
\end{aligned}\tag{13}
$$

X_1 的涨落:

$$\Delta X_1=\sqrt{\langle X_1^2\rangle-\langle X_1\rangle^2}=\frac{\sqrt{-2\alpha^4-3\alpha^2+3}}{2(1+\alpha^2)}.\tag{14}$$

若 $|\alpha,1\rangle$ 为压缩态, 则

$$\Delta X_1\neq\frac{1}{2},\tag{15}$$

$$\Longrightarrow\alpha\neq 1.\tag{16}$$

故当 $|\alpha|>1$ 时, $|\alpha,1\rangle$ 为压缩态. □

第 5 题 (20 分) 得分: ________. 考虑一个理想的光学腔, 腔里有单模辐射场 $|\phi_0\rangle=\frac{1}{\sqrt{2}}(|0\rangle-i|1\rangle)$. 处于基态且与单模场共振的两能级原子 $|\psi_i\rangle=|b\rangle$ 进入该光学腔, 与辐射场发生反应, 反应过程中相互作用的哈密顿量为 $\mathscr{V}=\hbar g(\sigma_+a+a^\dagger\sigma_-)$. 系统的演化方程为 $\Psi(t)A+F=e^{-\frac{i}{\hbar}\mathscr{V}t}|\psi_i\rangle\otimes|\phi\rangle$. 反应一段时间后原子从腔中逸出. **经探测: 出射原子已经从腔中吸收一个光子而被激发, 且处于 $|\psi_f\rangle=|a\rangle$ 激发态.**

(1) 计算该单模场初始时刻 $|\phi_0\rangle$ 的平均光子数 $\bar{n}$;

(2) 试讨论, 在腔中被吸收一个光子的情况下: 此时腔内的辐射场的平均光子数变为多少? 此时辐射场的光子数分布为何种分布 (Poisson, Sub-Poisson, Super-Poisson)?

解: (1) 该单模场初始时刻 $|\phi_0\rangle$ 的平均光子数为

$$\bar{n}=\langle\phi_0|a^\dagger a|\phi_0\rangle=\frac{1}{2}.\tag{17}$$

(2) 无微扰哈密顿量为

$$\hat{H}_0=\hbar\nu a^\dagger a+\frac{1}{2}\hbar\omega\sigma_z.\tag{18}$$

相互作用绘景中, 相互作用哈密顿量为

$$\hat{V}=e^{i\hat{H}_0t/\hbar}\mathscr{V}e^{-i\hat{H}_0t/\hbar}=\hbar g(\sigma_+ae^{i\Delta t}+a^\dagger\sigma_-e^{-i\Delta t}),\tag{19}$$

其中 $\Delta=\omega-\nu$. 将系统的量子态

$$|\Psi(t)\rangle_{A+F}=\sum_{n=0}^{\infty}(C_{a,n}(t)|a,n\rangle+C_{b,n}(t)|b,n\rangle)\tag{20}$$

代入薛定谔方程

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle=\hat{V}|\psi(t)\rangle\tag{21}$$

有

$$\dot{C}_{a,n}(t)=-igC_{b,n+1}\sqrt{n+1}e^{i\Delta t}, \tag{22}$$
$$\dot{C}_{b,n}(t)=-igC_{a,n}\sqrt{n+1}e^{-i\Delta t}, \tag{23}$$

解得

$$C_{a,n}(t)=\left\{C_{a,n}(0)\left[\cos\left(\frac{\Omega_n t}{2}\right)-\frac{i\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right]-\frac{2ig\sqrt{n+1}}{\Omega_n}C_{b,n+1}(0)\sin\left(\frac{\Omega_n t}{2}\right)\right\}e^{i\Delta t/2}, \tag{24}$$
$$C_{b,n+1}=\left\{C_{b,n+1}(0)\left[\cos\left(\frac{\Omega_n t}{2}\right)+\frac{i\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right]-\frac{2ig\sqrt{n+1}}{\Omega_n}C_{a,n}(0)\sin\left(\frac{\Omega_n t}{2}\right)\right\}e^{-i\Delta t/2}, \tag{25}$$

其中 $\Omega_n^2=\Delta^2+4g^2(n+1)$. 考虑到系统的初始状态

$$|\Psi(0)\rangle_{A+F}=|b\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle-i|1\rangle), \tag{26}$$

即 $C_{b,0}(0)=\frac{1}{\sqrt{2}}$, $C_{b,1}=-\frac{i}{\sqrt{2}}$, 故

$$C_{a,0}(t)=-\frac{\sqrt{2}g\sqrt{n+1}}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)e^{i\Delta t/2}, \tag{27}$$
$$C_{b,1}(t)=-\frac{i}{\sqrt{2}}\left[\cos\left(\frac{\Omega_n t}{2}\right)+\frac{i\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right]e^{-i\Delta t/2}, \tag{28}$$

即系统的量子态演化方程为

$$\begin{aligned}|\Psi(t)\rangle_{A+F}=&-\frac{\sqrt{2}g\sqrt{n+1}}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)e^{i\Delta t/2}|a,0\rangle-\frac{i}{\sqrt{2}}\left[\cos\left(\frac{\Omega_n t}{2}\right)+\frac{i\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right]e^{-i\Delta t/2}|b,1\rangle\\&+\frac{1}{\sqrt{2}}|b,0\rangle.\end{aligned} \tag{29}$$

当探测得 $|\psi_f\rangle=|a\rangle$ 时, 系统的量子态塌缩至

$$|\Psi_f\rangle=|a,0\rangle. \tag{30}$$

此时腔内的辐射场的平均光子数为 0, 辐射场的光子数分布为亚泊松分布.

□