理论力学作业_5

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Q1

解：大圆盘在桌面上运动，属于刚体的平面平行运动，有3个自由度，小圆盘圆心固定在大圆盘上转动，有1个自由度，整个系统有$s=4$个自由度。取广义坐标：大圆盘的横纵坐标x,y，大圆盘圆心与小圆盘圆心连线与x轴的夹角θ，小圆盘转过的角度φ。设桌面为零势能平面，则系统的重力势能为0，系统的拉格朗日函数即为系统的动能

$$\begin{aligned}L=&\frac{1}{2}M(\dot{x}^2+\dot{y}^2)+\frac{1}{2}(\frac{1}{2}MR^2)\dot{\theta}^2+\frac{1}{2}m[(\dot{x}-b\dot{\theta}\sin\theta)^2+(\dot{y}+b\dot{\theta}\cos\theta)^2]\\&+\frac{1}{2}(\frac{1}{2}mr^2)\dot{\varphi}^2\end{aligned}$$

由于无外力做功，系统动能守恒（或者由上式中不显含时间$\frac{\partial L}{\partial t}=0$也可以看出），上式为一守恒量。

$$\begin{aligned}L=&\frac{1}{2}M(\dot{x}^2+\dot{y}^2)+\frac{1}{2}(\frac{1}{2}MR^2)\dot{\theta}^2+\frac{1}{2}m[(\dot{x}-b\dot{\theta}\sin\theta)^2+(\dot{y}+b\dot{\theta}\cos\theta)^2]\\&+\frac{1}{2}(\frac{1}{2}mr^2)\dot{\varphi}^2=C_1\end{aligned}$$

其中常数C_1由初始条件决定。
拉格朗日函数不显含坐标x,y,φ，故有广义动量守恒

$$\begin{aligned}\frac{\partial L}{\partial x}=0\Longrightarrow p_x=\frac{\partial L}{\partial \dot{x}}=M\dot{x}+m(\dot{x}-b\dot{\theta}\sin\theta)=C_2\\\frac{\partial L}{\partial y}=0\Longrightarrow p_y=\frac{\partial L}{\partial \dot{y}}=M\dot{y}+m(\dot{y}+b\dot{\theta}\cos\theta)=C_3\\\frac{\partial L}{\partial \varphi}=0\Longrightarrow p_\varphi=\frac{\partial L}{\partial \dot{\varphi}}=\frac{1}{2}mr^2\dot{\varphi}=C_4\end{aligned}$$

以上三式分别代表整个系统x方向动量守恒，y方向动量守恒，小圆盘绕自身质心的角动量守恒，其中常数C_2,C_3,C_4由初始条件决定。

Q2

解：系统自由度$s = 1$，如题目图中取杆与竖直方向的夹角θ为广义坐标。取O 点处重力势能为零。小球转动的角速度$\omega = \frac{l\dot{\theta}}{r}$，杆绕$O$ 点的转动惯量$I = \frac{1}{12}m(3l)^2 + m(\frac{l}{2})^2 = ml^2$。系统的拉格朗日函数为

$$
\begin{aligned}
L =& \frac{1}{2}m(l\dot{\theta})^2 + \frac{1}{2}(\frac{1}{2}mr^2)(\frac{l\dot{\theta}}{r})^2 + \frac{1}{2}(ml^2)\dot{\theta}^2 \\
& - mgl\cos\theta + \frac{1}{2}mgl\cos\theta - \frac{1}{2}k(l\sin\theta)^2 \\
=& \frac{5}{4}ml^2\dot{\theta}^2 - \frac{1}{2}mgl\cos\theta - \frac{1}{2}kl^2\sin^2\theta
\end{aligned}
$$

拉格朗日方程为

$$
\begin{aligned}
& \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \\
\Longrightarrow & \frac{5}{2}ml^2\ddot{\theta} - \frac{1}{2}mgl\sin\theta + kl^2\sin\theta\cos\theta = 0 \\
& \because \text{小震动}, \therefore \sin\theta \approx \theta, \cos\theta \approx 1 \\
\Longrightarrow & \ddot{\theta} - \frac{mg - 2kl}{5ml}\theta = 0
\end{aligned}
$$

此即系统的运动微分方程。系统的振动周期为

$$
T = 2\pi\sqrt{\frac{5ml}{2kl - mg}}
$$

Q3.

解：系统自由度为$s = 2$，如题目图中取OC连线与竖直方向夹角θ_1 和Cm和竖直方向夹角θ_2为广义坐标。以O点为零重力势能点。均质环绕O 点的转动惯量为$I = MR^2 + MR^2 = 2MR^2$。系统的拉格朗日函数为

$$
\begin{aligned}
L =& \frac{1}{2}(2MR^2){\dot{\theta}_1}^2 + \frac{1}{2}m[(R\dot{\theta}_1\cos\theta_1 + R\dot{\theta}_2\cos\theta_2)^2 + (R\dot{\theta}_1\sin\theta_1 + R\dot{\theta}_2\sin\theta_2)^2] \\
& + MgR\cos\theta_1 + mg(R\cos\theta_1 + R\cos\theta_2) \\
=& MR^2{\dot{\theta}_1}^2 + \frac{1}{2}mR^2[{\dot{\theta}_1}^2 + {\dot{\theta}_2}^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1)] \\
& + MgR\cos\theta_1 + mgR(\cos\theta_1 + \cos\theta_2)
\end{aligned}
$$

代入拉格朗日方程得到

$$
\begin{cases}
2MR^2\ddot{\theta}_1 + mR^2\{\ddot{\theta}_1 + \ddot{\theta}_2\cos(\theta_2-\theta_1) + \dot{\theta}_2[\sin(\theta_2-\theta_1)\dot{\theta}_2 - \sin(\theta_2-\theta_1)\dot{\theta}_1]\} \\
\quad -mR^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1) + MgR\sin\theta_1 + mgR\sin\theta_1 = 0 \\
mR^2\{\ddot{\theta}_2 + \ddot{\theta}_1\cos(\theta_2-\theta_1) + \dot{\theta}_1[\sin(\theta_2-\theta_1)\dot{\theta}_2 - \sin(\theta_2-\theta_1)\dot{\theta}_1]\} \\
\quad +mR^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1) + mgR\sin\theta_2 = 0
\end{cases}
$$

$\because$ 小振动,$\therefore \cos(\theta_2-\theta_1)\approx 1, \sin(\theta_2-\theta_1)\approx 0, \sin\theta_1\approx\theta_1, \sin\theta_2\approx\theta_2$

$$
\begin{cases}
(2M+m)R\ddot{\theta}_1 + mR\ddot{\theta}_2 + (M+m)g\theta_1 = 0 \\
R(\ddot{\theta}_2+\ddot{\theta}_1) + g\theta_2 = 0
\end{cases}
$$

令上方程组特解为

$$
\begin{aligned}
\theta_1 &= A_1\sin(\omega t+\alpha) \\
\theta_2 &= A_2\sin(\omega t+\alpha)
\end{aligned}
$$

代入方程组中得到

$$
\begin{aligned}
&[-(2M+m)R\omega^2 + (M+m)g]A_1 - mR\omega^2 A_2 = 0 \\
&-R\omega^2 A_1 + (-R\omega^2+g)A_2 = 0
\end{aligned}
$$

解得

$$
\begin{aligned}
\omega_1^2 &= \frac{(M+m)g}{MR} \\
\omega_2^2 &= \frac{g}{2R}
\end{aligned}
$$

回代入方程组得到

$$
\begin{aligned}
\frac{A_2^{(1)}}{A_1^{(1)}} &= -\frac{M+m}{m} \\
\frac{A_2^{(2)}}{A_1^{(2)}} &= 1
\end{aligned}
$$

得到小振动通解

$$
\begin{aligned}
\theta_1 &= A_1^{(1)}\sin(\sqrt{\frac{(M+m)g}{MR}}t+\alpha_1) + A_1^{(2)}\sin(\sqrt{\frac{g}{2R}}t+\alpha_2) \\
\theta_2 &= -\frac{M+m}{m}A_1^{(1)}\sin(\sqrt{\frac{(M+m)g}{MR}}t+\alpha_1) + A_1^{(2)}\sin(\sqrt{\frac{g}{2R}}t+\alpha_2)
\end{aligned}
$$

Q4、

解：系统自由度$s=2$，但含有一个非完整约束，取C点的横纵坐标x,y和杆AB与y轴的夹角θ为广义坐标。设A,B两点坐标分别为$(x_1,y_1),(x_2,y_2)$，A,B两

点坐标可表示为

$$
\begin{aligned}
x_1 &= x - \frac{l}{2}\cos\theta \\
y_1 &= y - \frac{l}{2}\sin\theta \\
x_2 &= x + \frac{l}{2}\cos\theta \\
y_2 &= y + \frac{l}{2}\sin\theta
\end{aligned}
$$

系统的拉格朗日函数为

$$
\begin{aligned}
L =& \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2) - mg(y_1 + y_2) \\
=& \frac{1}{2}m[(\dot{x} + \frac{l}{2}\dot{\theta}\sin\theta)^2 + (\dot{y} - \frac{l}{2}\dot{\theta}\cos\theta)^2] \\
& + \frac{1}{2}m[(\dot{x} - \frac{l}{2}\dot{\theta}\sin\theta)^2 + (\dot{y} + \frac{l}{2}\dot{\theta}\cos\theta)^2] - mg(y_1 + y_2) \\
=& m[\dot{x}^2 + \dot{y}^2 + \frac{l^2}{4}\dot{\theta}^2 - 2gy]
\end{aligned}
$$

杆的中点C的速度只能沿着AB杆的方向，不完整约束为

$$
\begin{aligned}
& \dot{y} = \cos\theta\dot{x} \\
\Longrightarrow & \sin\theta\delta x = 0, -\cos\theta\delta y = 0
\end{aligned}
$$

设拉格朗日不定乘子为λ，非完整体系的拉格朗日方程：

$$
\begin{aligned}
& \begin{cases}
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0
\end{cases} \\
\Longrightarrow & \begin{cases}
2m\ddot{x} - \lambda\sin\theta = 0 \\
2m\ddot{y} + 2mg + \lambda\cos\theta = 0 \\
\frac{ml^2}{2}\ddot{\theta} = 0
\end{cases}
\end{aligned}
$$

上面最后一式解得

$$
\theta = \alpha t + \beta
$$

其中α为初始时刻杆转动的角速度，β为初始时刻杆与y 轴所成夹角。上面前两式得

$$
\ddot{x} = -(\ddot{y} + g)\tan\theta \tag{1}
$$

x,y的一、二阶导数可以表示为

$$\dot{x}=\frac{dx}{d\theta}\frac{d\theta}{dt}=\alpha\frac{dx}{d\theta}$$
$$\ddot{x}=\frac{d\dot{x}}{d\theta}\frac{d\theta}{dt}=\alpha^2\frac{d^2x}{d\theta^2}$$
$$\dot{y}=\alpha\frac{dy}{d\theta}$$
$$\ddot{y}=\alpha^2\frac{d^2y}{d\theta^2}$$

联立非完整约束条件和式(1)解得

$$\sin\theta\frac{dx}{d\theta}-\cos\theta\frac{dy}{d\theta}=0$$
$$\alpha^2\frac{d^2x}{d\theta^2}=-(\alpha^2\frac{d^2y}{d\theta^2}+g)\tan\theta$$

以上两式联立得到

$$\frac{d^y}{d\theta^2}-\cot\theta\frac{dy}{d\theta}+\frac{g}{\alpha^2}\sin^2\theta=0$$

解得

$$y=-\frac{\gamma}{\alpha}\cos\theta-\frac{g}{2\alpha^2}\cos^2\theta+\delta$$
$$x=\frac{\gamma}{\alpha}\sin\theta+\frac{g}{2\alpha^2}(\sin\theta\cos\theta+\theta)+\varepsilon$$

其中$\gamma,\delta,\varepsilon$皆为积分常数。质点$A,B$的运动方程为

$$x_1=\frac{\gamma}{\alpha}\sin\theta+\frac{g}{2\alpha^2}(\sin\theta\cos\theta+\theta)-\frac{l}{2}\cos\theta+\varepsilon$$
$$y_1=-\frac{\gamma}{\alpha}\cos\theta-\frac{g}{2\alpha^2}\cos^2\theta-\frac{l}{2}\sin\theta+\delta$$
$$x_2=\frac{\gamma}{\alpha}\sin\theta+\frac{g}{2\alpha^2}(\sin\theta\cos\theta+\theta)+\frac{l}{2}\cos\theta+\varepsilon$$
$$y_1=-\frac{\gamma}{\alpha}\cos\theta-\frac{g}{2\alpha^2}\cos^2\theta+\frac{l}{2}\sin\theta+\delta$$

其中$\theta=\alpha t+\beta$

Q5、

解：系统自由度$s=1$，取小环的横坐标x为广义坐标。小环沿着抛物线形金属丝滑动，设约束方程为

$$y=\frac{x^2}{4a}$$

则

$$\dot{y}=\frac{x}{2a}\dot{x}$$

系统动能为

$$
\begin{aligned}
T =&\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\omega^2x^2)\\
=&\frac{1}{2}m[(1+\frac{x^2}{4a^2})\dot{x}^2+\omega^2x^2]\\
=&T_2+T_0
\end{aligned}
$$

其中$T_2=\frac{1}{2}m(1+\frac{x^2}{4a^2})\dot{x}^2, T_0=\frac{1}{2}m\omega^2x^2$。设$y=0$处重力势能为零，系统势能为

$$V=mgy=\frac{mgx^2}{4a}$$

系统的拉格朗日函数为

$$L=T-V=\frac{1}{2}m[(1+\frac{x^2}{4a^2})\dot{x}^2+\omega^2x^2-\frac{gx^2}{2a}]$$

势能V中不显含时间t，系统的广义能量积分为

$$H=T_2-T_0+V=\frac{1}{2}m[(1+\frac{x^2}{4a^2})\dot{x}^2-\omega^2x^2+\frac{gx^2}{2a}]$$

广义动量为

$$p_x=\frac{\partial L}{\partial \dot{x}}=m(1+\frac{x^2}{4a^2})\dot{x}$$

代入哈密顿函数中替换$\dot{x}$得

$$H=\frac{p_x^2}{2m(1+\frac{x^2}{4a^2})}-\frac{1}{2}m[\omega^2x^2-\frac{gx^2}{2a}]$$

正则方程：

$$
\begin{aligned}
&\dot{p}_x=-\frac{\partial H}{\partial x}\\
\Longrightarrow& m(1+\frac{x^2}{4a^2})\ddot{x}+m\frac{x}{2a^2}\dot{x}^2=m\frac{x}{4a}\dot{x}^2+m(\omega^2-\frac{g}{2a})x\\
\Longrightarrow&(1+\frac{x^2}{4a^2})\ddot{x}+\frac{x}{4a^2}\dot{x}^2-(\omega^2-\frac{g}{2a})x=0
\end{aligned}
$$

此即小环在x方向的运动微分方程。