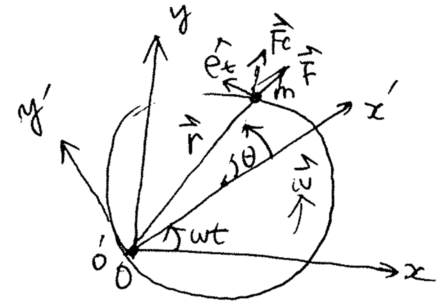
Q4: 解：(1) 以静止的 $Oxyz$ 为 S 系，与圆圈固连的 $O'x'y'z'$ 为 S' 系，如图所示，以 S' 系为参考系。



小环在水平面内受相互作用力为圆圈施与的约束力 $\vec{N} = N_n\hat{e}_n$，受惯性力 $\vec{F}_t = -m\vec{\omega}\times(\vec{\omega}\times\vec{r}')$

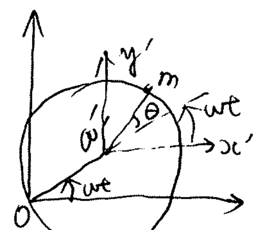
$$= 2ma\omega^2\cos\frac{\theta}{2}\hat{e}_r'$$

$$\vec{F}_c = -2m\vec{\omega}\times\vec{v}' = -2m\vec{\omega}\times(a\dot{\theta}\hat{e}_t) = -2m\omega a\dot{\theta}\hat{e}_n$$

在 S' 系中小环沿切向($\hat{e}_t$)的运动微分方程为

$$ma_t' = ma\ddot{\theta} = -2ma\omega^2\cos\frac{\theta}{2}\sin\frac{\theta}{2}$$

即 $\ddot{\theta} + \omega^2\sin\theta = 0$



(2) 拉格朗日方法

系统自由度为1，以 θ 为广义坐标，以 $O'x'y'$ 为 S' 系

则质点 m 的牵连速度大小为 $a\omega$，与相对速度[大小为 $a(\dot{\theta}+\omega)$]的夹角为 θ，则 $T = \frac{1}{2}mv^2 = \frac{1}{2}m[a^2\omega^2 + a^2(\dot{\theta}+\omega)^2 + 2a^2\omega(\dot{\theta}+\omega)\cos\theta]$

$$= \frac{1}{2}ma^2[\dot{\theta}^2 + 2\omega(1+\cos\theta)\dot{\theta} + 2\omega^2(1+\cos\theta)] = T_2 + T_1 + T_0$$

$$L = T - V = T$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = ma^2\ddot{\theta} - ma^2\omega\dot{\theta}\sin\theta, \quad \frac{\partial L}{\partial\theta} = -ma^2\omega\dot{\theta}\sin\theta - ma^2\omega^2\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = ma^2\ddot{\theta} + ma^2\omega^2\sin\theta = 0, \quad \therefore \ddot{\theta} + \omega^2\sin\theta = 0$$

(3) 正则方程：$L = T - V = \frac{1}{2}ma^2[\dot{\theta}^2 + 2\omega(1+\cos\theta)\dot{\theta} + 2\omega^2(1+\cos\theta)]$

则 $H = T_2 - T_0 + V = \frac{1}{2}ma^2[\dot{\theta}^2 - 2\omega^2(1+\cos\theta)]$

$$p_\theta = \frac{\partial T}{\partial\dot{\theta}} = ma^2[\dot{\theta} + \omega(1+\cos\theta)], \quad \dot{\theta} = \frac{p_\theta}{ma^2} - \omega(1+\cos\theta)$$

$\therefore H = \frac{p_\theta^2}{2ma^2} - p_\theta\omega(1+\cos\theta) - \frac{1}{2}ma^2\omega^2\sin^2\theta$，代入正则方程得

$$\dot{\theta} = \frac{p_\theta}{ma^2} - \omega(1+\cos\theta), \quad \dot{p}_\theta = -p_\theta\omega\sin\theta + ma^2\omega^2\sin\theta\cos\theta$$

$$\therefore \ddot{\theta} = \frac{\dot{p}_\theta}{ma^2} + \omega\dot{\theta}\sin\theta = \frac{-p_\theta\omega\sin\theta + ma^2\omega^2\sin\theta\cos\theta}{ma^2} + \omega\dot{\theta}\sin\theta$$

$= -[\dot{\theta} + \omega(1+\cos\theta)]\omega\sin\theta + \omega^2\sin\theta\cos\theta + \omega\dot{\theta}\sin\theta$，即 $\ddot{\theta} + \omega^2\sin\theta = 0$