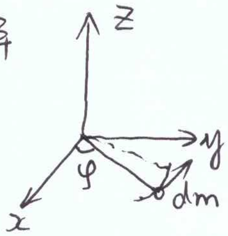
Solutions for Homework 4th.

Q1 解：

(1) 如图所示，取Oxyz坐标系固连于刚体，刚体定轴转动的自由度为1，取广义坐标为φ. 考虑刚体上任一质元 $dm\,(x,y,z)$，其受达朗贝尔惯性力为



$$-\vec{a}\,dm = -[(\ddot{r}-r\dot{\varphi}^2)\hat{e}_r + (r\ddot{\varphi}+2\dot{r}\dot{\varphi})\hat{e}_\theta]dm$$

∵ $\dot{r}=\ddot{r}=0$，而 $r=\sqrt{x^2+y^2}$，∴ 质点实际受 $\dot{\varphi}^2\sqrt{x^2+y^2}\,dm\,\hat{e}_r$ 及 $-\ddot{\varphi}\sqrt{x^2+y^2}\,dm\,\hat{e}_\theta$ 两个惯性力，只有后者做虚功不为零，为

$$\delta W = -\ddot{\varphi}\sqrt{x^2+y^2}\,dm\sqrt{x^2+y^2}\,\delta\varphi = -\ddot{\varphi}(x^2+y^2)\,dm\,\delta\varphi$$

设主动力对转轴的力矩为M，则根据动力学普遍性方程有

$$\left[M-\int\ddot{\varphi}(x^2+y^2)\,dm\right]\delta\varphi=0$$

$\Rightarrow M=\ddot{\varphi}\int(x^2+y^2)\,dm = I_z\ddot{\varphi}$，即为所求。

(2) 刚体定点转动时各质点的虚位移可设想由绕过固定点O的任何轴的一个无限小转动引起，表示为

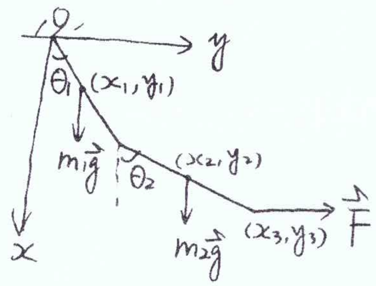
$$\delta\vec{r}_i = \delta\vec{\theta}\times\vec{r}_i$$

则由动力学普遍性方程，得

$$\begin{aligned}
\sum_{i=1}^{n}(\vec{F}_i-m_i\ddot{\vec{r}}_i)\cdot\delta\vec{r}_i &= \sum_{i=1}^{n}(\vec{F}_i-m_i\ddot{\vec{r}}_i)\cdot(\delta\vec{\theta}\times\vec{r}_i)\\
&= \sum_{i=1}^{n}\delta\vec{\theta}\cdot\vec{r}_i\times(\vec{F}_i-m_i\ddot{\vec{r}}_i)\\
&= \delta\vec{\theta}\cdot\left[\sum_{i=1}^{n}(\vec{r}_i\times\vec{F}_i)-\sum_{i=1}^{n}(\vec{r}_i\times m_i\ddot{\vec{r}}_i)\right]\\
&= \delta\vec{\theta}\cdot\left\{\vec{M}-\left[\frac{d}{dt}\sum_{i=1}^{n}(\vec{r}_i\times m_i\dot{\vec{r}}_i)-\sum_{i=1}^{n}\dot{\vec{r}}_i\times m_i\dot{\vec{r}}_i\right]\right\}\\
&= \delta\vec{\theta}\cdot\left(\vec{M}-\frac{d}{dt}\vec{L}\right)=0
\end{aligned}$$

$\therefore \frac{d}{dt}\vec{L}=\vec{M}$ 即为所求。

Q2 解：选取下杆质心的坐标 x_2、y_2 为广义坐标，另选上杆位置 θ_1 及下杆位置 θ_2 为广义坐标，由虚功原理：



$$m_1 g\delta x_1 + m_2 g\delta x_2 + F\delta y_3 = 0$$

用广义坐标：

$$m_1 g\delta\left(\frac{1}{2}l_1\cos\theta_1\right) + m_2 g\delta x_2 + F\delta\left(\frac{1}{2}l_2\sin\theta_2 + y_2\right) = 0$$

$$-\frac{1}{2}l_1 m_1 g\sin\theta_1\delta\theta_1 + m_2 g\delta x_2 + \frac{1}{2}l_2\cos\theta_2 F\delta\theta_2 + F\delta y_2 = 0 \quad (a)$$

对应于 δx_2 和 δy_2 的主动力分别是 $m_2\vec{g}$ 和 $\vec{F}$，由于广义坐标不独立，受约束：

$$\begin{cases} f(x_2,y_2,\theta_1,\theta_2) = l_1\cos\theta_1 + \frac{1}{2}l_2\cos\theta_2 - x_2 = 0 \\ f(x_2,y_2,\theta_1,\theta_2) = l_1\sin\theta_1 + \frac{1}{2}l_2\sin\theta_2 - y_2 = 0 \end{cases}$$

$$\therefore -l_1\sin\theta_1\delta\theta_1 - \frac{1}{2}l_2\sin\theta_2\delta\theta_2 - \delta x_2 = 0 \quad (b)$$

$$l_1\cos\theta_1\delta\theta_1 + \frac{1}{2}l_2\cos\theta_2\delta\theta_2 - \delta y_2 = 0 \quad (c)$$

用 λ_1 遍乘 (b) 的各项，λ_2 遍乘 (c) 的各项，并与 (a) 相加，得

$$\left(-\frac{1}{2}m_1 g\sin\theta_1 + \lambda_1\sin\theta_1 + \lambda_2\cos\theta_1\right)l_1\delta\theta_1 + (m_2 g - \lambda_1)\delta x_2 +$$
$$\left(\frac{1}{2}\cos\theta_2 F - \frac{1}{2}\lambda_1\sin\theta_2 + \frac{1}{2}\lambda_2\cos\theta_2\right)l_2\delta\theta_2 + (F - \lambda_2)\delta y_2 = 0$$

令各系数分别为零，得平衡方程，其中对应 δx_2 和 δy_2 的两个平衡方程为

$$\begin{cases} m_2 g - \lambda_1 = 0 \\ F - \lambda_2 = 0 \end{cases} \qquad \therefore R_P = \lambda^{p}\nabla f^{p}$$

∴ 下杆所受的 x 方向的力为 $-m_2\vec{g}$，y 方向的力为 $-\vec{F}$.

Q3解: 小圆柱体作平面平行运动, 自由度为3个,

又 $x_c^2+y_c^2=(R-r)^2$, $r\varphi=(R-r)\theta$

∵ $s=1$, 令 $q_1=\theta$

∵ $T=\frac{1}{2}mu_c^2+\frac{1}{2}I_c\omega^2$, $u_c=(R-r)\dot{\theta}$

$\dot{\varphi}=\omega=\frac{R-r}{r}\dot{\theta}$, $I_c=\frac{1}{2}mr^2$

$T=\frac{1}{2}m(R-r)^2\dot{\theta}+\frac{1}{4}m(R-r)\dot{\theta}^2=\frac{3}{4}m(R-r)^2\dot{\theta}^2$

设 $\theta=0$ 处的势能为零, 则 $V=mg(R-r)(1-\cos\theta)$

$L=T-V=\frac{3}{4}m(R-r)^2\dot{\theta}^2-mg(R-r)(1-\cos\theta)$

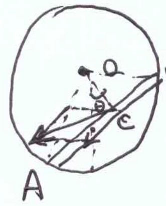
$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)=\frac{d}{dt}\left[\frac{3}{2}m(R-r)^2\dot{\theta}\right]=\frac{3}{2}m(R-r)^2\ddot{\theta}$

$\frac{\partial L}{\partial\theta}=-mg(R-r)\sin\theta$, 得 $\frac{3}{2}m(R-r)^2\ddot{\theta}+mg(R-r)\sin\theta=0$

即 $\ddot{\theta}+\frac{2}{3}\frac{g}{(R-r)}\sin\theta=0$, 又对于小振动, $\sin\theta\approx\theta$, 故

方程为: $\ddot{\theta}+\frac{2g}{3(R-r)}\theta=0$, 得 $\omega=\sqrt{\frac{2g}{3(R-r)}}$

解：系统自由度为2，取圆环中心O的坐标x_0及OC与垂直线的夹角θ为广义坐标。圆环作纯滚动，其转动角速度为 $\omega=\frac{\dot{x}_0}{r}$



圆环动能 $T_2=\frac{1}{2}m_2\dot{x}_0^2+\frac{1}{2}(m_2r^2)\omega^2=m_2\dot{x}_0^2$

杆AB动能 $T_1=\frac{1}{2}m_1v_c^2+\frac{1}{2}I_c\dot{\theta}^2$

其中 $v_c^2=\dot{x}_0^2+\frac{1}{2}r^2\dot{\theta}^2+\sqrt{2}r\dot{x}_0\dot{\theta}\cos\theta$，$I_c=\frac{1}{12}m_1(\sqrt{2}r)^2$

$\therefore T_1=\frac{1}{2}m_1(\dot{x}_0^2+\frac{1}{2}r^2\dot{\theta}^2+\sqrt{2}r\dot{x}_0\dot{\theta}\cos\theta)+\frac{1}{12}m_1r^2\dot{\theta}^2$

则系统 $T=T_1+T_2=(m_2+\frac{m_1}{2})\dot{x}_0^2+\frac{1}{3}m_1r^2\dot{\theta}^2+\frac{\sqrt{2}}{2}m_1r\dot{x}_0\dot{\theta}\cos\theta$

系统势能 $V=-\frac{\sqrt{2}}{2}m_1rg\cos\theta$，选O点的水平面为势能面零

则拉式函数为 $L=T-V$

$$=(m_2+\frac{m_1}{2})\dot{x}_0^2+\frac{1}{3}m_1r^2\dot{\theta}^2+\frac{\sqrt{2}}{2}m_1r\dot{x}_0\dot{\theta}\cos\theta+\frac{\sqrt{2}}{2}m_1rg\cos\theta$$

代入拉格朗日方程得：

$$\begin{cases}(2m_2+m_1)\ddot{x}_0+\frac{\sqrt{2}}{2}m_1r(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta)=0\\ 4r\ddot{\theta}+3\sqrt{2}\ddot{x}_0\cos\theta+3\sqrt{2}g\sin\theta=0\end{cases}$$

若只考虑微振动则可略去$\dot{\theta}^2$，且$\cos\theta\approx1$，$\sin\theta\approx\theta$，则

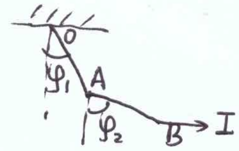
$$\begin{cases}(2m_2+m_1)\ddot{x}_0+\frac{\sqrt{2}}{2}m_1r\ddot{\theta}=0\\ 4r\ddot{\theta}+3\sqrt{2}\ddot{x}_0+3\sqrt{2}g\theta=0\end{cases}$$

消去x_0得到关于θ的方程：$\frac{8m_2+m_1}{2m_2+m_1}r\ddot{\theta}+3\sqrt{2}g\theta=0$

即 $\ddot{\theta}+3\sqrt{2}\frac{g}{r}\frac{2m_2+m_1}{8m_2+m_1}\theta=0$

显然为一简谐振动，则 $\omega^2=3\sqrt{2}\frac{g}{r}\frac{2m_2+m_1}{8m_2+m_1}$

解：系统自由度为2，取 $q_1=\varphi_1$ 和 $q_2=\varphi_2$



则OA和OB杆的质心速度分别为

$$u_1=\frac{1}{2}l\dot{\varphi}_1,\quad u_2=l\dot{\varphi}_1+\frac{l}{2}\dot{\varphi}_2$$

以广义坐标表示的系统的动能为 $T=\frac{1}{2}I_O\dot{\varphi}_1^2+\frac{1}{2}mu_2^2+\frac{1}{2}I_{C_2}\dot{\varphi}_2^2$

$I_O=\frac{1}{3}ml^2$，$I_{C_2}=\frac{1}{12}ml^2$，$\therefore T=\frac{1}{2}ml^2\left(\frac{4}{3}\dot{\varphi}_1^2+\dot{\varphi}_1\dot{\varphi}_2+\frac{1}{3}\dot{\varphi}_2^2\right)$

系统所受广义冲量 $I_k=\sum_{i=1}^{n}\vec{I}_i\cdot\frac{\partial\vec{r}_i}{\partial q_k}=\vec{I}\cdot\frac{\partial\vec{r}_B}{\partial q_k}=I\frac{\partial x_B}{\partial q_k}$

如图所示 $x_B=l\sin\varphi_1+l\sin\varphi_2$

则 $I_{\varphi_1}=I\frac{\partial x_B}{\partial\varphi_1}=Il\cos\varphi_1\xrightarrow{\varphi_1\to 0}Il$

同理，$I_{\varphi_2}=Il$，则代入 $\left[\frac{\partial T}{\partial\dot{q}_k}\right]_{t_2}-\left[\frac{\partial T}{\partial\dot{q}_k}\right]_{t_2}=I_k$，得

$$\dot{\varphi}_1=-\frac{6I}{7ml},\quad\dot{\varphi}_2=\frac{30I}{7ml}$$