理论力学作业_7

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1、

解：设有心力$F=-\frac{mk}{r^2}$。保守系统，自由度$s=2$，取极坐标系中的θ,r为广义坐标。在极坐标系中，系统的动能为

$$T=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)$$

势能为

$$V=-\frac{mk}{r}$$

系统的拉格朗日函数为

$$L=T-V=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+\frac{mk}{r}$$

代入哈密顿作用量求极值

$$\begin{aligned}
\delta S=&\delta\int_{t_1}^{t_2}Ldt=\int_{t_1}^{t_2}\delta[\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+\frac{mk}{r}]dt\\
=&m\int_{t_1}^{t_2}[\dot{r}\delta\dot{r}+r\dot{\theta}^2\delta r+r^2\dot{\theta}\delta\dot{\theta}-\frac{k}{r^2}\delta r]dt\\
=&m\int_{t_1}^{t_2}[\frac{d}{dt}(\dot{r}\delta r+r^2\dot{\theta}\delta\theta)+(-\ddot{r}+r\dot{\theta}^2-\frac{k}{r^2})\delta r-(2r\dot{\theta}+r^2\ddot{\theta})\delta\theta]dt\\
=&m\int_{t_1}^{t_2}[(-\ddot{r}+r\dot{\theta}^2-\frac{k}{r^2})\delta r-(2\dot{r}\dot{\theta}+r\ddot{\theta})r\delta\theta]dt\\
=&0
\end{aligned}$$

由于广义坐标θ,r独立，得到质点的运动微分方程

$$\ddot{r}-r\dot{\theta}^2+\frac{k}{r^2}=0$$
$$2\dot{r}\dot{\theta}+r\ddot{\theta}=0$$

2、

证明：因为

$$\begin{aligned}\frac{df}{dt}=&[f,H]+\frac{\partial f}{\partial t}=\frac{\partial f}{\partial q_1}\frac{\partial H}{\partial p_1}-\frac{\partial f}{\partial p_1}\frac{\partial H}{\partial q_1}+\frac{\partial f}{\partial q_2}\frac{\partial H}{\partial p_2}-\frac{\partial f}{\partial p_2}\frac{\partial H}{\partial q_2}+\frac{\partial f}{\partial t}\\=&0-2p_1q_2+2q_2p_1-0+0\\=&0\end{aligned}$$

故f为初积分。

因为

$$\begin{aligned}\frac{dg}{dt}=&[g,H]+\frac{\partial g}{\partial t}=\frac{\partial g}{\partial q_1}\frac{\partial H}{\partial p_1}-\frac{\partial g}{\partial p_1}\frac{\partial H}{\partial q_1}+\frac{\partial g}{\partial q_2}\frac{\partial H}{\partial p_2}-\frac{\partial g}{\partial p_2}\frac{\partial H}{\partial q_2}+\frac{\partial g}{\partial t}\\=&2q_1p_2-0+0-2p_2q_1+0\\=&0\end{aligned}$$

故g为初积分。

3、

解：系统自由度$s=1$，取θ为广义坐标。系统的动能为

$$T=\frac{1}{2}m[(a\dot{\theta})^2+(\omega a\sin\theta)^2]=\frac{1}{2}ma^2(\dot{\theta}^2+\omega^2\sin^2\theta)$$

势能为

$$V=mga\cos\theta$$

系统的拉格朗日方程为

$$L=T-V=\frac{1}{2}ma^2(\dot{\theta}^2+\omega^2\sin^2\theta)-mga\cos\theta$$

代入哈密顿作用量求极值

$$\begin{aligned}\delta S=&\delta\int_{t_1}^{t_2}Ldt=\int_{t_1}^{t_2}\delta[\frac{1}{2}ma^2(\dot{\theta}^2+\omega^2\sin^2\theta)-mga\cos\theta]dt\\=&m\int_{t_1}^{t_2}[a^2(\dot{\theta}\delta\dot{\theta}+\omega^2\sin\theta\cos\theta\delta\theta)+ga\sin\theta\delta\theta]dt\\=&m\int_{t_1}^{t_2}[a^2(\dot{\theta}\frac{d}{dt}\delta\theta+\omega^2\sin\theta\cos\theta\delta\theta)+ga\sin\theta\delta\theta]dt\\=&m\int_{t_1}^{t_2}[\frac{d}{dt}(a^2\dot{\theta}\delta\theta)+a^2(-\ddot{\theta}+\omega^2\sin\theta\cos\theta)\delta\theta+ga\sin\theta\delta\theta]dt\\=&m\int_{t_1}^{t_2}(-a^2\ddot{\theta}+a^2\omega^2\sin\theta\cos\theta+ga\sin\theta)\delta\theta dt\\=&0\end{aligned}$$

由于θ是独立的广义坐标，得到小环的运动微分方程

$$a\ddot{\theta}-a\omega^2\sin\theta\cos\theta-g\sin\theta=0$$

4、

解：设两质点质量分别为m_1,m_2，两质点之间引力$F=-\frac{m_1m_2k}{r^2}$。保守系统，系统自由度$s=2$，取两质点之间的距离r和两质点连线转过的角度θ 为广义坐标。两质点到系统质心的距离分别为$\frac{m_2}{m_1+m_2}r,\frac{m_1}{m_1+m_2}r$。系统的动能为

$$\begin{aligned}T=&\frac{1}{2}m_1[(\frac{m_2}{m_1+m_2}\dot{r})^2+(\frac{m_2}{m_1+m_2}r\dot{\theta})^2]\\&+\frac{1}{2}m_2[(\frac{m_1}{m_1+m_2}\dot{r})^2+(\frac{m_1}{m_1+m_2}r\dot{\theta})^2]\\=&\frac{1}{2}\frac{m_1m_2}{m_1+m_2}(\dot{r}^2+r^2\dot{\theta}^2)\end{aligned}$$

势能为

$$V=-\frac{m_1m_2k}{r}$$

系统的拉格朗日函数为

$$L=T-V=\frac{1}{2}\frac{m_1m_2}{m_1+m_2}(\dot{r}^2+r^2\dot{\theta}^2)+\frac{m_1m_2k}{r}$$

代入哈密顿作用量求极值

$$\begin{aligned}\delta S=&\delta\int_{t_1}^{t_2}Ldt=\int_{t_1}^{t_2}\delta[\frac{m_1m_2}{2(m_1+m_2)}(\dot{r}^2+r^2\dot{\theta}^2)+\frac{m_1m_2k}{r}]dt\\=&\int_{t_1}^{t_2}[\frac{m_1m_2}{m_1+m_2}(\dot{r}\delta\dot{r}+r\dot{\theta}^2\delta r+r^2\dot{\theta}\delta\dot{\theta})-\frac{m_1m_2k}{r^2}\delta r]dt\\=&\int_{t_1}^{t_2}[\frac{m_1m_2}{m_1+m_2}(\dot{r}\frac{d}{dt}\delta r+r\dot{\theta}^2\delta r+r^2\dot{\theta}\frac{d}{dt}\delta\theta)-\frac{m_1m_2k}{r^2}\delta r]dt\\=&\int_{t_1}^{t_2}[\frac{d}{dt}(\frac{m_1m_2}{m_1+m_2}(\dot{r}\delta r+r^2\dot{\theta}\delta\theta))+\frac{m_1m_2}{m_1+m_2}(-\ddot{r}\delta r+r\dot{\theta}^2\delta r-2r\dot{r}\dot{\theta}\delta\theta-r^2\ddot{\theta}\delta\theta)-\frac{m_1m_2k}{r^2}\delta r]dt\\=&\int_{t_1}^{t_2}[(-\frac{m_1m_2}{m_1+m_2}\ddot{r}+\frac{m_1m_2}{m_1+m_2}r\dot{\theta}^2-\frac{m_1m_2k}{r^2})\delta r+\frac{m_1m_2}{m_1+m_2}(-2r\dot{r}\dot{\theta}-r^2\ddot{\theta})\delta\theta]dt\\=&0\end{aligned}$$

由于广义坐标r,θ独立，故得到运动微分方程

$$\begin{aligned}&(\ddot{r}-r\dot{\theta}^2)+\frac{(m_1+m_2)k}{r^2}=0\\&2\dot{r}\dot{\theta}+r\ddot{\theta}=0\end{aligned}$$

5、

解：系统的拉格朗日函数为

$$L = T - V = \frac{1}{2}m\dot{q}^2 - mgq$$

旧的广义动量为

$$p = \frac{\partial H}{\partial \dot{q}} = m\dot{q}$$

从而旧的广义速度可表示为

$$\dot{q} = \frac{p}{m}$$

哈密顿函数为

$$\begin{aligned} H = & p\dot{q} - L = \frac{1}{2}m\dot{q}^2 + mgq \\ = & \frac{p^2}{2m} + mgq \end{aligned}$$

由母函数得到

$$\begin{aligned} & p = \frac{\partial U}{\partial q} = mgQ \\ & P = -\frac{\partial U}{\partial Q} = -\frac{1}{2}mg^2Q^2 - mgq \Longrightarrow q = -\frac{P}{mg} - \frac{1}{2}gQ^2 \\ & \frac{\partial U}{\partial t} = 0 \Longrightarrow H^* = H \end{aligned}$$

以上三式代入原广义坐标下的哈密顿函数得到新的哈密顿函数

$$\begin{aligned} H^* = H = & \frac{p^2}{2m} + mgq \\ = & -P \end{aligned}$$

新哈密顿函数的正则方程为

$$\begin{aligned} \dot{Q} = \frac{\partial H}{\partial P} = -1 \Longrightarrow Q = -t + C_1 \\ \dot{P} = -\frac{\partial H}{\partial Q} \Longrightarrow P = C_2 \end{aligned}$$

考虑初始条件当$t = 0$时，$q = 0, \dot{q} = v_0$

$$C_1 = \frac{v_0}{g}, C_2 = -\frac{1}{2}mv_0^2$$

从而竖直上抛物体的运动规律：

$$q = -\frac{P}{mg} - \frac{1}{2}gQ^2 = v_0 t - \frac{1}{2}gt^2$$