Q5

解：(1) $\because r^2 = a^2\cos 2\theta \quad \therefore u = \frac{1}{r} = \frac{1}{a\sqrt{\cos 2\theta}}$

对上式两边求导，得

$$\frac{du}{d\theta} = \frac{1}{a}\sin 2\theta(\cos 2\theta)^{-\frac{3}{2}}$$

$$\frac{d^2u}{d\theta^2} = \frac{1}{a}\left[2(\cos 2\theta)^{-\frac{1}{2}} + 3\sin^2 2\theta(\cos 2\theta)^{-\frac{5}{2}}\right]$$

则 ~~F~~ 由 Binet 公式 $-\frac{F}{m} = h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)$ 知

$$F = \frac{-mh^2}{a^2\cos 2\theta}\frac{1}{a}\left[2(\cos 2\theta)^{-\frac{1}{2}} + 3\sin^2 2\theta(\cos 2\theta)^{-\frac{5}{2}} + (\cos 2\theta)^{-\frac{1}{2}}\right]$$

$$= -\frac{3mh^2}{a^3}(\cos 2\theta)^{-\frac{3}{2}}(1+\sin^2 2\theta\cos^{-2}2\theta)$$

$$= -\frac{3mh^2}{a^3}(\cos 2\theta)^{-\frac{7}{2}} = -\frac{3mh^2}{a^3}\left(\frac{r^2}{a^2}\right)^{-\frac{7}{2}} = -\frac{3mh^2}{a^3}\left(\frac{a^7}{r^7}\right) = -\frac{3mh^2a^4}{r^7}$$

(2) 行星做椭圆轨道运动，设运动周期为 τ.

$$\bar{T} = \frac{1}{\tau}\int_0^\tau \frac{1}{2}mv^2dt = \frac{1}{\tau}\int_0^\tau \frac{1}{2}m\vec{v}\cdot\vec{v}dt = \frac{1}{\tau}\int_0^\tau \frac{1}{2}m\vec{v}\cdot d\vec{r}$$

$$= \frac{1}{\tau}\int_0^\tau \frac{1}{2}md(\vec{v}\cdot\vec{r}) - \frac{1}{\tau}\int_0^\tau \frac{1}{2}m\vec{r}\cdot d\vec{v} = -\frac{1}{2}\frac{m}{\tau}\int_0^\tau \vec{r}\cdot d\vec{v}$$

由于 $m\vec{a} = m\frac{d\vec{v}}{dt} = -\frac{GMm}{r^2}\frac{\vec{r}}{r}$，所以

$$\bar{T} = -\frac{1}{2}\frac{1}{\tau}\int_0^\tau \vec{r}\cdot m d\vec{v} = -\frac{1}{2}\frac{1}{\tau}\int_0^\tau \vec{r}\cdot\left(-\frac{GMm}{r^2}\frac{\vec{r}}{r}\right)dt$$

$$= -\frac{1}{2}\frac{1}{\tau}\int_0^\tau\left(-\frac{GMm}{r^2}\frac{\vec{r}\cdot\vec{r}}{r}\right)dt$$

$$= -\frac{1}{2}\frac{1}{\tau}\int_0^\tau\left(-\frac{GMm}{r}\right)dt = -\frac{1}{2}\bar{V}$$