Introduction to

Algorithm Design and Analysis

[03] Recursion

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In the Last Class …

- **Asymptotic growth rate**
  - O, Ω, Θ
  - o, ω
- **Brute force algorithms**
  - By iteration
  - By recursion

Recursion

- **Recursion in algorithm design**
  - The divide and conquer strategy
  - Proving the correctness of recursive procedures
- **Solving recurrence equations**
  - Some elementary techniques
  - Master theorem

Recursion in Algorithm Design

- **Computing n! With Fac(n)**
  - if n=1 then return 1 else return Fac(n-1)*n

M(1)=0 and M(n)=M(n-1)+1 for n>0
(critical operation: multiplication)

- **Hanoi Tower**
  - If n=1 then move d(1) to peg3 else Hanoi(n-1, peg1, peg2); move d(n) to peg3; Hanoi(n-1, peg2, peg3)

M(1)=1 and M(n)=2M(n-1)+1 for n>1
(critical operation: move)

Recursion in Algorithm Design

- **Counting the Number of Bits**
  - Input: a positive decimal integer n
  - Output: the number of binary digits in n's binary representation

int BitCounting(int n)

```
1. if(n==1) return 1;
2. else
3.    return BitCounting(n/2) + 1;
```

$$T(n) = \begin{cases} 0 & n = 1 \\ T(\lfloor n/2 \rfloor) + 1 & n > 1 \end{cases}$$

Divide and Conquer

- **Divide**
  - Divide the “big” problem to smaller ones
- **Conquer**
  - Solve the “small” problems by recursion
- **Combine**
  - Combine results of small problems, and solve the original problem

Divide and Conquer

The general pattern

```
solve(I)
    n=size(I);
    if (n≤smallSize)
        solution=directlySolve(I);
    else
        divide I into $I_1$,... $I_k$;
        for each i∈{1,...,k}
            $S_i$=solve($I_i$);
        solution=combine($S_1$ ,... ,$S_k$);
return solution
```

T(n)=B(n) for n≤smallSize

$$T(n) = D(n) + \sum_{i=1}^{k} T(size(I_i)) + C(n)$$

for n>smallSize

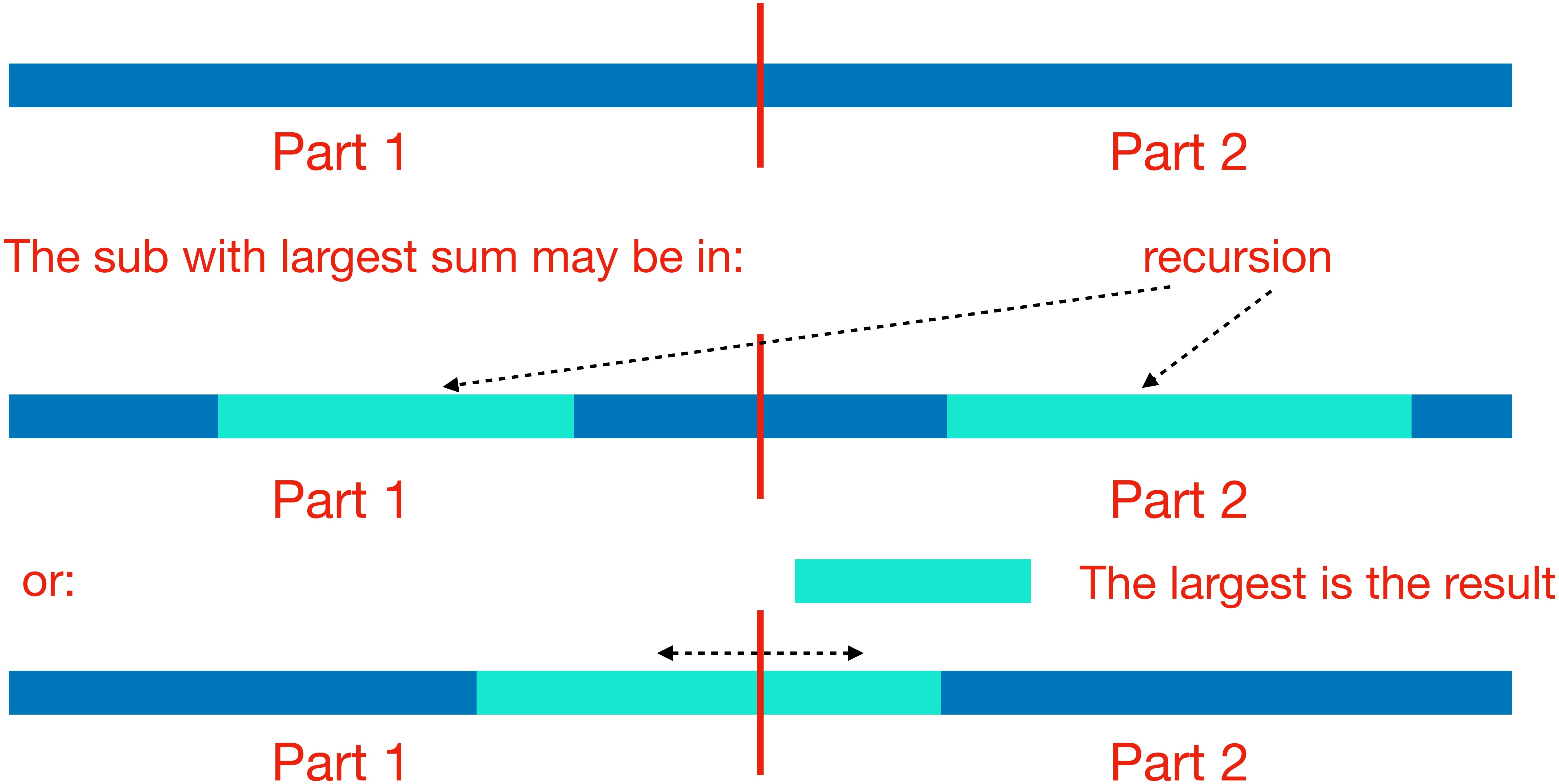
Divide and Conquer

- **The BF recursion**
  - Problem size: often decreases linearly
    - “n, n-1, n-2, …”
- **The D&C recursion**
  - Problem size: often decrease exponentially
    - “n, n/2, n/4, n/8, …”

Examples

Max sum subsequence

$$T(n) = 2T(\frac{n}{2}) + n$$

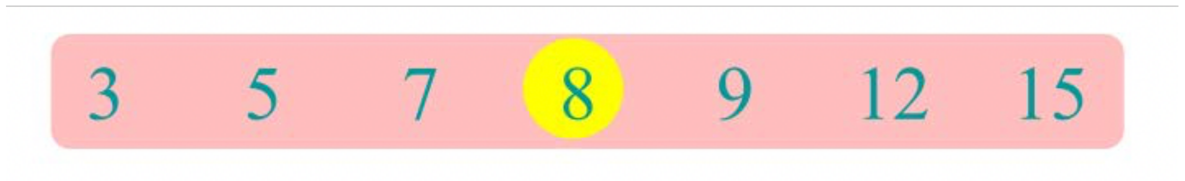

Part 1
Part 2
The sub with largest sum may be in:
recursion
Part 1
Part 2
or:
The largest is the result
Part 1
Part 2

Examples

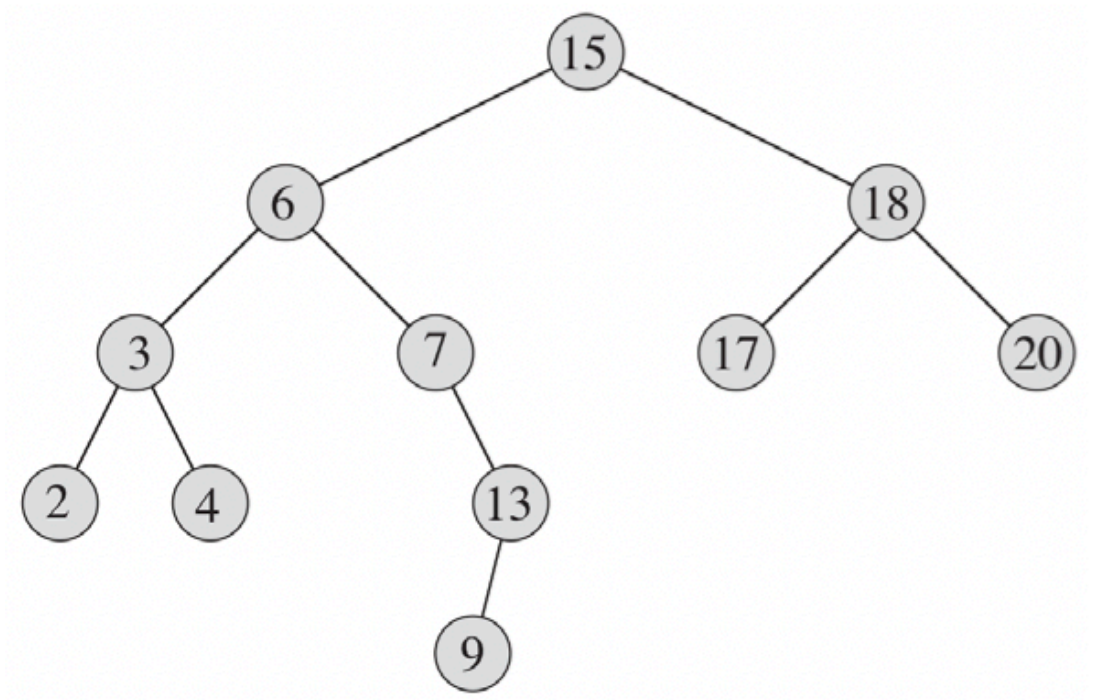
- Maxima
- Frequent element
- Multiplication
  - Integer
  - Matrix
- Nearest point pair

Examples

- Arrays



- Trees

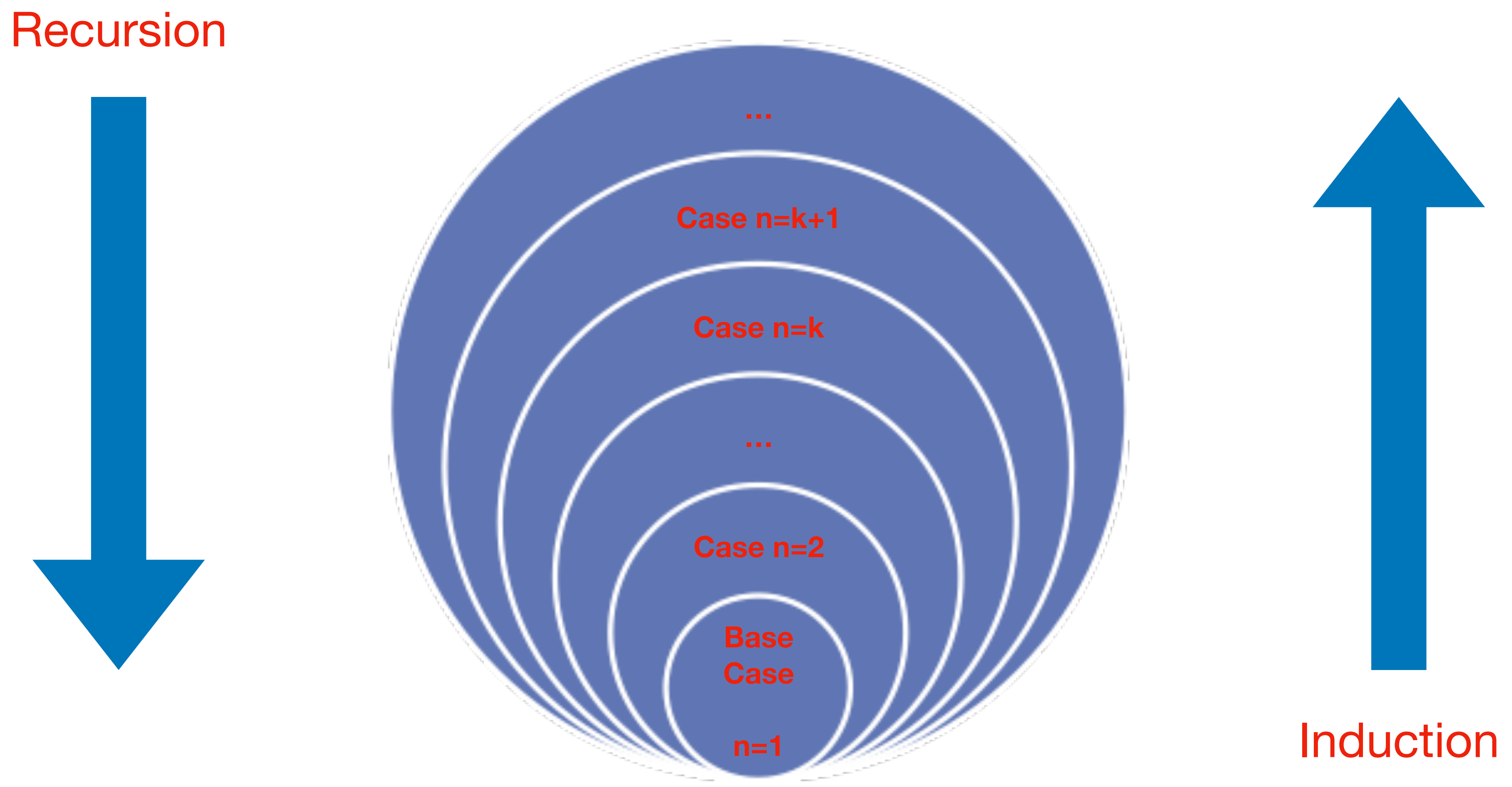


Workhorse

- “Hard division, easy combination”
- “Easy division, hard combination”

Usually,
the “real work”
is in one part.

Correctness of Recursion



Analysis of Recursion

- **Solving recurrence equations**
- **E.g., Bit counting**
    - Critical operation: add
    - The recurrence relation

$$T(n) = \begin{cases} 0 & n = 1 \\ T(\lfloor n/2 \rfloor) + 1 & n > 1 \end{cases}$$

Analysis of Recursion

- **Backward substitutions**

By the recursion equation: $T(n) = T(\lfloor \frac{n}{2} \rfloor) + 1$

For simplicity, let n=2^k (k is a nonnegative integer), that is, k = log n

$$T(n) = T(\frac{n}{2}) + 1 = T(\frac{n}{4}) + 1 + 1 = T(\frac{n}{8}) + 1 + 1 + 1 = \ldots\ldots$$

$$T(n) = T(\frac{n}{2^k}) + \log n = \log n \qquad (T(1) = 0)$$

Smooth Functions

- **f(n)**
  - Nonnegative eventually non-decreasing function defined on the set of natural numbers
- **f(n) is called smooth**
  - If $f(2n) \in \Theta(f(n))$
- **Examples of smooth functions**
  - $\log n$, n, $n\log n$, and n^{α} ($\alpha \geq 0$)
  - E.g., $2n\log 2n = 2n(\log n + \log 2) \in \Theta(n\log n)$

Even Smoother

- **Let f(n) be a smooth function, then, for any fixed integer b≥2, f(bn)∈Θ(f(n))**
    - That is, there exist positive constants c_b and d_b and a nonnegative integer n_0 such that

$$d_b f(n) \le f(bn) \le c_b f(n) \qquad \text{for } n \ge n_0$$

It is easy to prove that the result holds for $b=2^k$,
For the second inequality:

$f(2^k n) \le c_2^k f(n)$ for k=1,2,3… and $n \ge n_0$

For an arbitrary integer b≥2, $2^{k-1} \le b \le 2^k$

Then, $f(bn) \le f(2^k n) \le c_2^k f(n)$, we can use c_2^k as c_b .

Smoothness Rule

- **Let T(n) be an eventually non-decreasing function and f(n) be a smooth function.**
  - If T(n)∈Θ(f(n)) for values of n that are powers of b (b≥2), then T(n)∈Θ(f(n)).

Just proving the big - Oh part:

By the hypothesis: $T(b^k) \leq cf(b^k)$ for $b^k \geq n_0$

By the prior result: $f(bn) \leq c_b f(n)$ for $n \geq n_0$

Let $n_0 \leq b^k \leq n \leq b^{k+1}$

$$T(n) \leq T(b^{k+1}) \leq cf(b^{k+1}) = cf(bb^k) \leq cc_b f(b^k) \leq cc_b f(n)$$

Non-decreasing hypothesis Prior result Non-decreasing

Computing the Fibonacci Number

T(0)=0

T(1)=1 $\Rightarrow$ 0,1,1,2,3,5,8,13,21,34,…

T(0)=T(n-1)+T(n-2)

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k}$$

is called linear homogenous relation of degree k.

For the special case of Fibonacci: $a_n = a_{n-1} + a_{n-2}$

$$r_1 = r_2 = 1$$

Computing the Fibonacci Number

$f_0=0$

$f_1=1$

$f_n=f_{n-1}+f_{n-2}$

$\Rightarrow$ 0,1,1,2,3,5,8,13,21,34,…

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$r_1 = r_2 = 1$

Characteristic Equation

- For a linear homogeneous recurrence relation of degree k

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k}$$

the polynomial of degree k

$$x^k = r_1 x^{k-1} + r_2 x^{k-2} + \cdots + r_k$$

is called its characteristic equation.

- The characteristic equation of linear homogeneous recurrence relation of degree 2 is:

$$x^2 - r_1 x - r_2 = 0$$

Solution of Recurrence Relation

- If the characteristic equation $x^2 - r_1 x - r_2 = 0$ of the recurrence relation $a_n = r_1 a_{n-1} + r_2 a_{n-2}$ has two distinct roots s_1 and s_2, then

$$a_n = u s_1^n + v s_2^n$$

where u and v depend on the initial conditions, is the explicit formula for the sequence.

- If the equation has a single root s, then, both s_1 and s_2 in the formula above are replaced by s.

Proof of the Solution

Remember equation: $x^2 - r_1 x - r_2 = 0$

We need to prove that: $us_1^n + vs_2^n = r_1 a_{n-1} + r_2 a_{n-2}$

$$\begin{aligned}
&us_1^n + vs_2^n = us_1^{n-2}s_1^2 + vs_2^{n-2}s_2^2 \\
&= us_1^{n-2}(r_1 s_1 + r_2) + vs_2^{n-2}(r_1 s_2 + r_2) \\
&= r_1 us_1^{n-1} + r_2 us_1^{n-2} + r_1 vs_2^{n-1} + r_2 vs_2^{n-2} \\
&= r_1(us_1^{n-1} + vs_2^{n-1}) + r_2(us_1^{n-2} + vs_2^{n-2}) \\
&= r_1 a_{n-1} + r_2 a_{n-2}
\end{aligned}$$

Back to Fibonacci Sequence

$f_0=0$

$f_1=1$ ➡ 0,1,1,2,3,5,8,13,21,34,…

$f_n=f_{n-1}+f_{n-2}$ Explicit formula for Fibonacci Sequence

The characteristic equation is $x^2 - x - 1 = 0$,which has roots:

$$s_1 = \frac{1+\sqrt{5}}{2} \text{ and } s_1 = \frac{1-\sqrt{5}}{2}$$

Note: (by initial conditions)

$$f_1 = us_1 + vs_2 = 1 \text{ and } f_2 = us_1^2 + vs_2^2 = 1$$

which means: $f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

Guess and Prove

- Example: $T(n) = 2T(\lfloor n/2 \rfloor) + n$
- Guess
  - $T(n) \in O(n)$?
    - $T(n) \le cn$, to be proved for c large enough
  - $T(n) \in O(n^2)$?
    - $T(n) \le cn^2$, to be proved for c large enough
  - **Or maybe**, $T(n) \in O(n\log n)$?
    - $T(n) \le cn\log n$, to be proved for c large enou
- Prove
  - by substitution

Try to prove $T(n) \le cn$:

$$T(n) = 2T(\lfloor n/2 \rfloor) + n \le 2c(\lfloor n/2 \rfloor) + n$$
$$\le 2c(n/2) + n = (c+1)\text{n},$$ **Fail!**

However:

$$T(n) = 2T(\lfloor n/2 \rfloor) + n \ge 2c\lfloor n/2 \rfloor + n$$
$$\ge 2c[(n-1)/2] + n = cn + (n-c) \ge cn$$

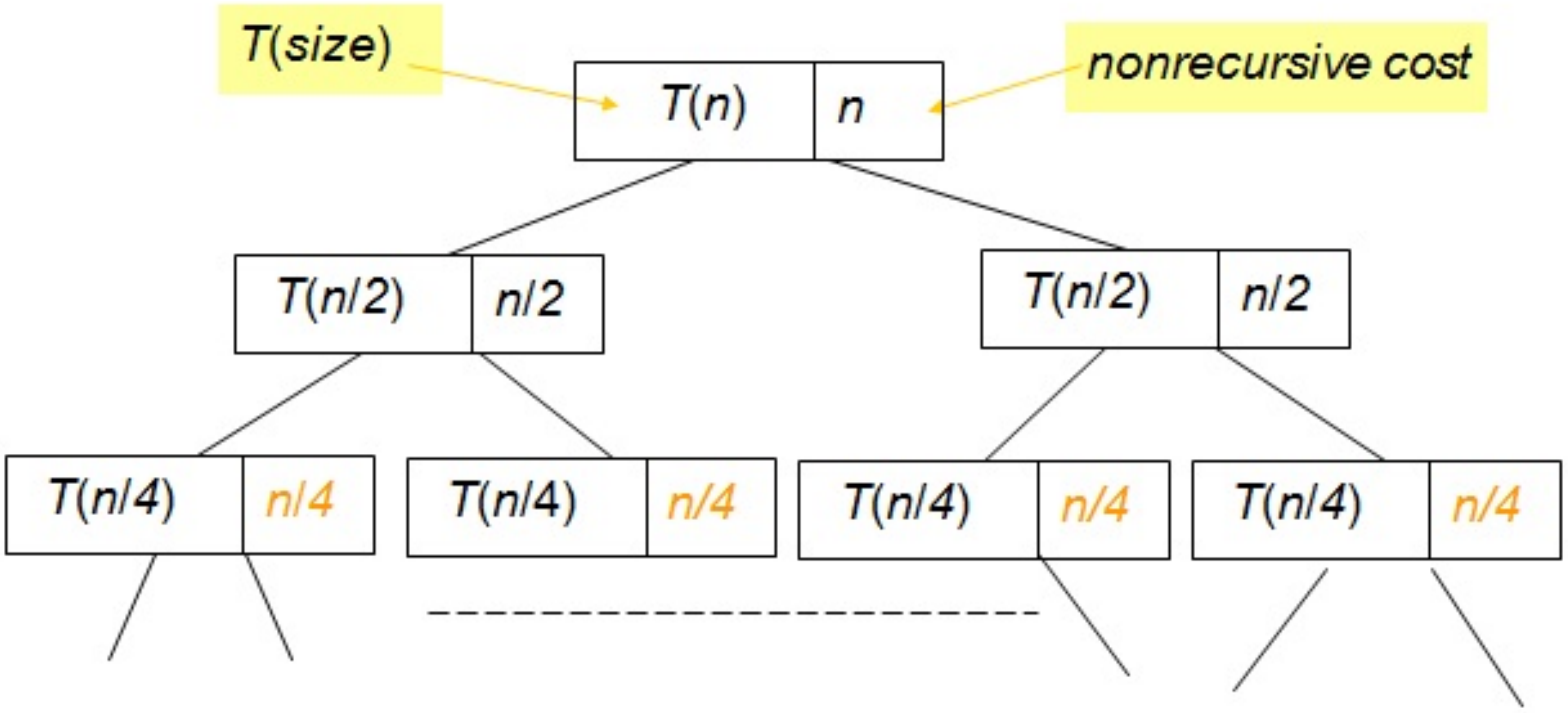
$$\begin{aligned} T(n) &= 2T(\lfloor n/2 \rfloor) + n \\ &\le 2(c\lfloor n/2 \rfloor \log(\lfloor n/2 \rfloor)) + n \\ &\le cn\log(n/2) + n \\ &= cn\log n - cn\log 2 + n \\ &= cn\log n - cn + n \\ &\le cn\log n \quad \text{for } c \ge 1 \end{aligned}$$

Divide and Conquer Recursions

- Divide and conquer
  - Divide the “big” problem to small ones
  - Solve the “small” problems by recursion
  - Combine results of small problems, and solve the original problem
- Divide and conquer recursion

$$T(n) = \underbrace{b}_{\text{divide}}\,\underbrace{T}_{\text{conquer}}(n/\underbrace{c}_{\text{divide}}) + \underbrace{f(n)}_{\text{combine}}$$

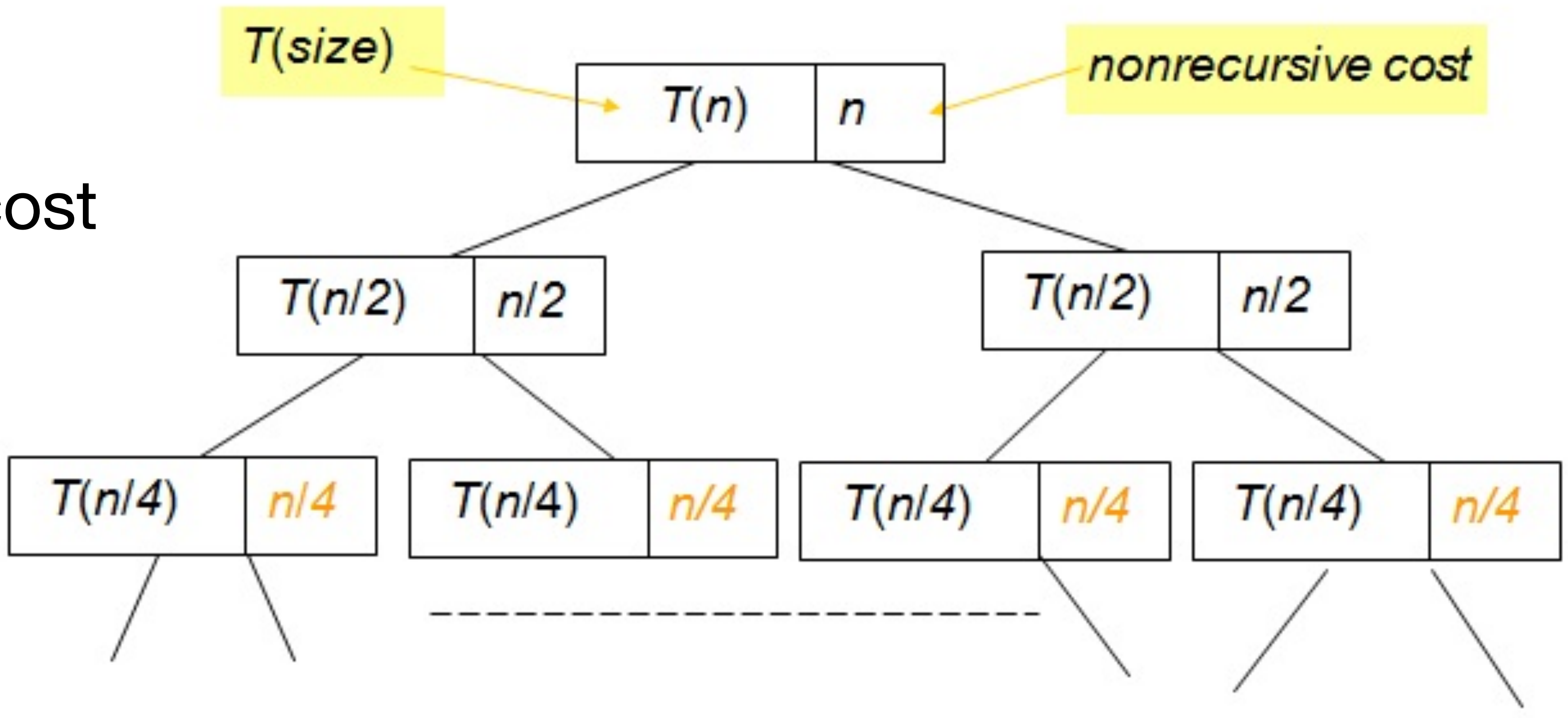
Recursion Tree



The recursion tree for T(n)=T(n/2)+T(n/2)+n

Recursion Tree

- **Node**
  - Non-leaf
    - Non-recursive cost
    - Recursive cost
  - Leaf
    - Base case
- Edge
  - Recursion



The recursion tree for T(n)=T(n/2)+T(n/2)+n

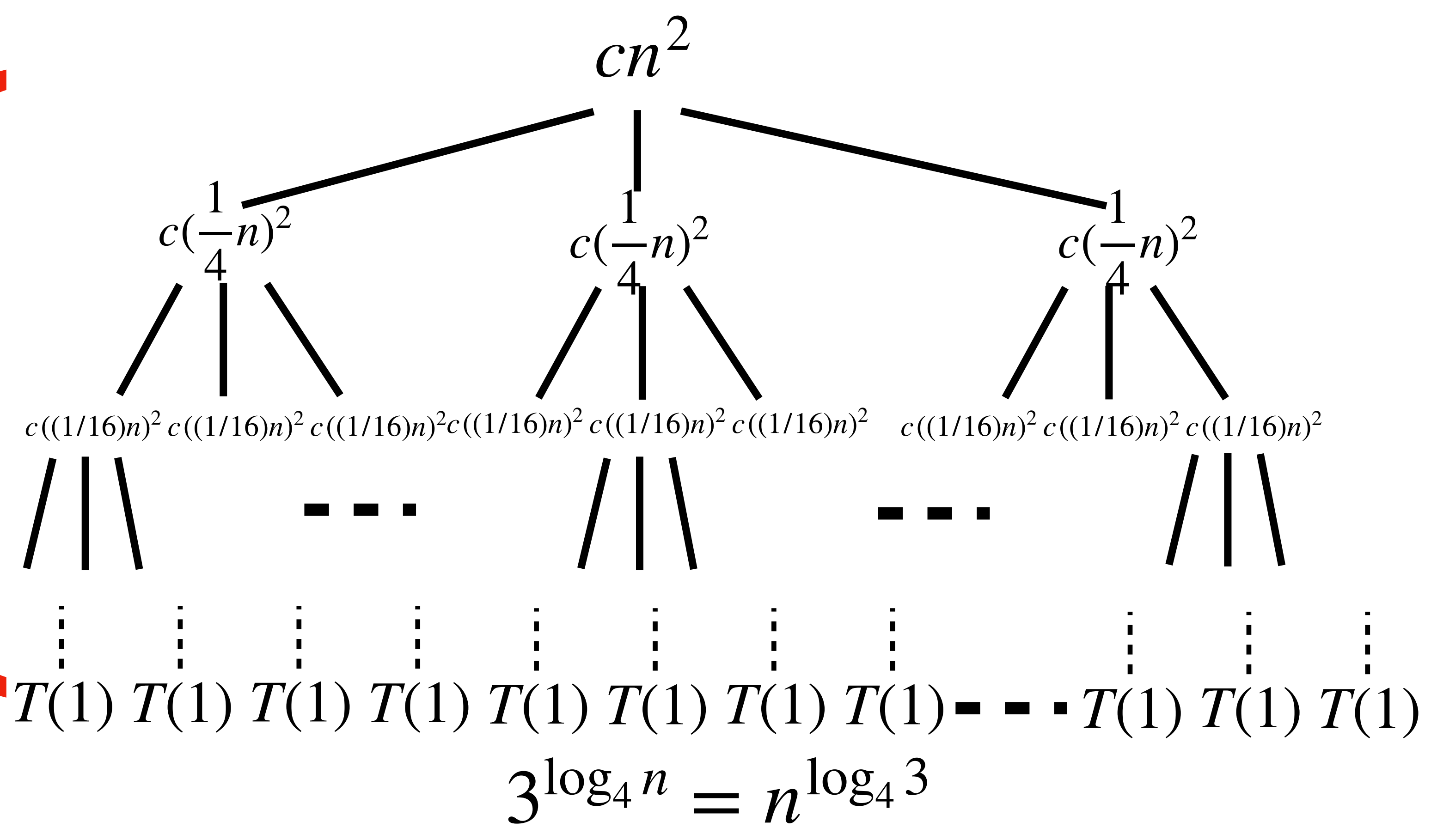
Recursion Tree

$$T(n) = \overbrace{\underbrace{3}_{\text{\# of sub-problems}}T(\lfloor n/\underbrace{4}_{\text{size of sub-problems}} \rfloor)}^{\text{Recursive cost}} + \overbrace{\Theta(n^2)}^{\text{Non-recursive cost}}$$

Total cost

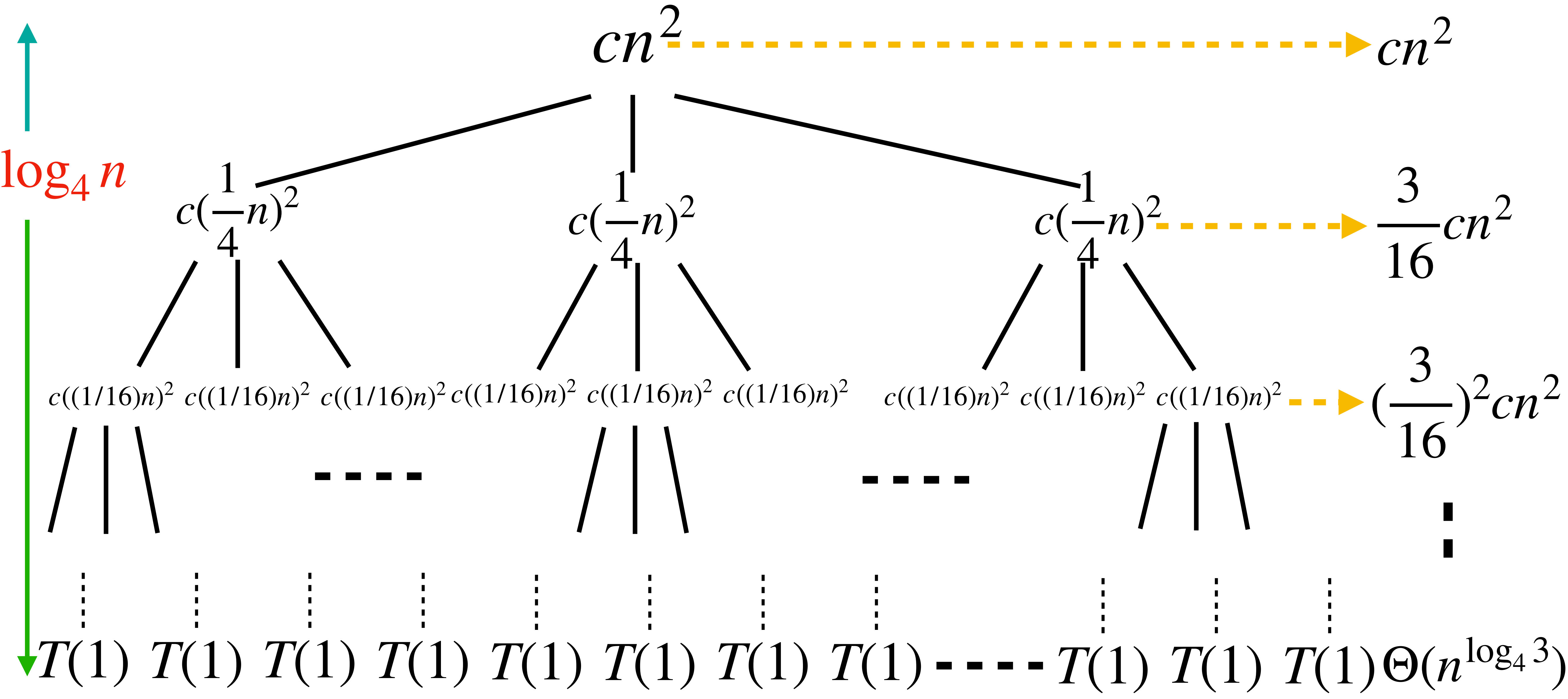
$$\sum$$

Sum of row sums



Recursion Tree for

$$T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2)$$



Note: $3^{\log_4 n} = n^{\log_4 3}$

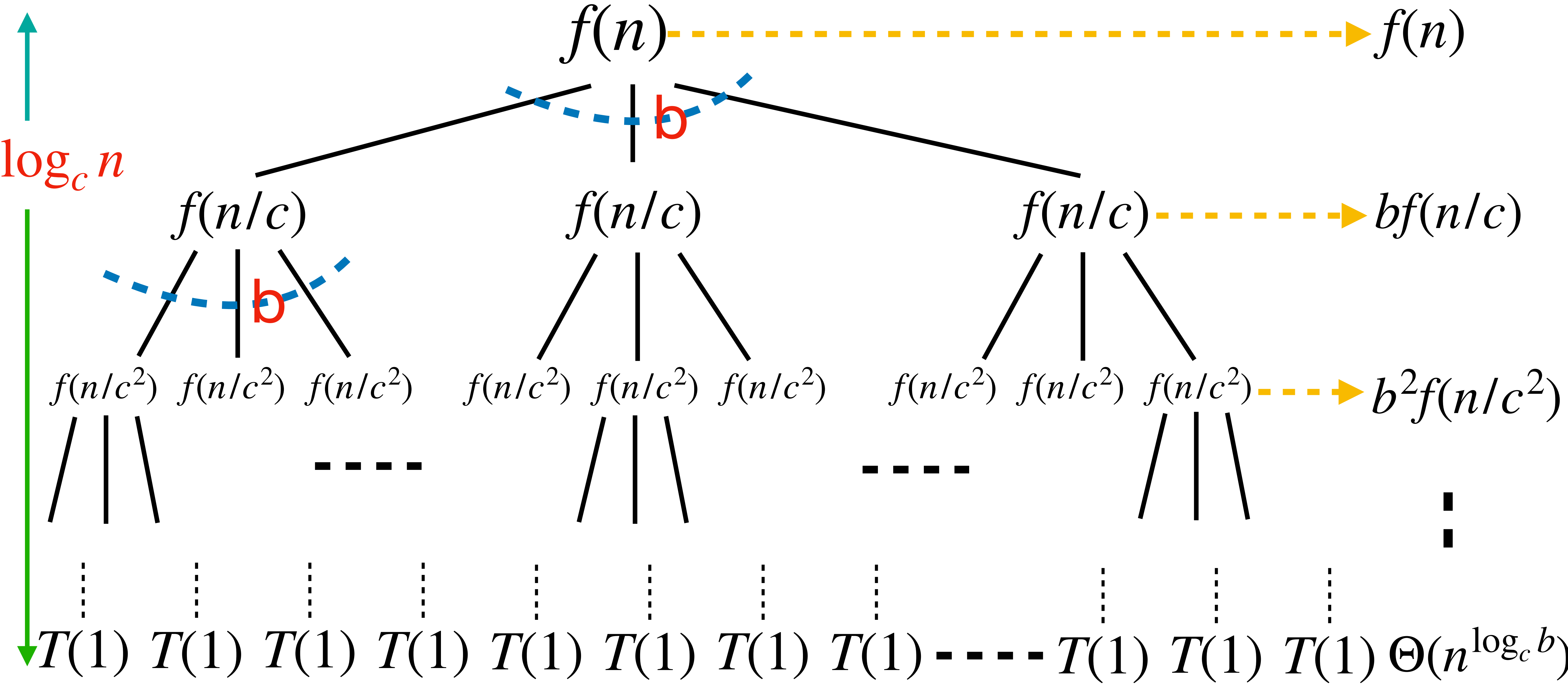
Total?

Solving the Divide-and-Conquer Recurrence

- **The recursion equation for divide-and-conquer, the general case: $T(n)=bT(n/c)+f(n)$**
- **Observations:**
    - Let base-cases occur at depth D(leaf), then $n/c^D=1$, that is $D=\log(n)/\log(c)$
    - Let the number of leaves of the tree be L, then $L=b^D$, that is $L=b^{(\log(n)/\log(c))}$
    - By a little algebra: $L=n^E$, where $E=\log(b)/\log(c)$, called critical exponent.

Recursion Tree for

$$T(n) = bT(n/c) + f(n)$$



Note: $b^{\log_c n} = n^{\log_c b}$

Total?

Divide-and-Conquer - the Solution

- **The solution of divide-and-conquer equation is the non-recursive costs of all nodes in the tree, which is the sum of the row-sums**
    - The recursion tree has depth $D = \log(n)/\log(c)$, so there are about that many row-sums.
- **The 0th row-sum**
    - is f(n), the non recursive cost of the root.
- **The Dth row-sum**
    - is n^E, assuming base cases 1, or $\Theta(n^E)$ in any event.

Solution by Row-sums

- **[Little Master Theorem] Row-sums decide the solution of the equation for divide-and-conquer:**
  - Increasing geometric series: $T(n) \in \Theta(n^E)$
  - Constant: $T(n) \in \Theta(f(n) \log n)$
  - Decreasing geometric series: $T(n) \in \Theta(f(n))$

  This can be generalized to get a result not using explicitly row-sums.

Master Theorem

- **Loosening the restrictions on f(n)**
  - Case 1: $f(n) \in O(n^{E-\varepsilon}), (\varepsilon > 0),$ then:

$$T(n) \in \Theta(n^E)$$

  - Case 2: $f(n) \in \Theta(n^E),$ as all node depth contribute about equally:

$$T(n) \in \Theta(f(n)\log(n))$$

  - Case 3: $f(n) \in \Omega(n^{E+\varepsilon}), (\varepsilon > 0),$ and of $bf(n/c) \leq \theta f(n)$ for some constant θ<1 and all sufficiently large n, then:

$$T(n) \in \Theta(f(n))$$

The positive ε is critical, resulting gaps between cases as well.

Using Master Theorem

- Example 1: $T(n) = 9T(\frac{n}{3}) + n$

  $b = 9, c = 3, E = 2, f(n) = n = O(n^{E-1})$

  Case 1 applies: $T(n) = \Theta(n^2)$

- Example 2: $T(n) = T(\frac{2}{3}n) + 1$

  $b = 1, c = \frac{3}{2}, E = 0, f(n) = 1 = \Theta(n^E)$

  Case 2 applies: $T(n) = \Theta(\log n)$

- Example 3: $T(n) = 3T(\frac{n}{4}) + n \log n$

  $b = 3, c = 4, E = log_4 3, f(n) = n \log n = \Omega(n^{E+\epsilon})$

  Case 3 applies: $T(n) = \Theta(n \log n)$

Using Master Theorem

$$T(n) = 2T(n/2) + n \log n$$

Does Case 3 apply? Why?

$$T(n) = \sqrt{n}T(\sqrt{n}) + n$$

- **The gap between the 3 cases**
  - Often, non of the 3 cases apply
  - Your task: design more non-solvable recursions

Thank you!

Q & A