EE5027 Adaptive Signal Processing
Homework Assignment #4

Notice

- Due at 9:00pm, January 16, 2021 (Saturday) $= T_d$ for the electronic copy of your solution.
- Please submit your solution to NTU COOL (`https://cool.ntu.edu.tw/courses/3062`)
- **All answers have to be fully justified**.
- All the figures should include labels for the horizontal and vertical axes, a title for a short description, and grid lines. Add legends and different line styles if there are multiple curves in one plot.
- No extensions, unless granted by the instructor one day before T_d.

Problems

1. (Dependence of the parameters in Kalman filters, 20 points)

   Recall the process equation and the measurement equation in a state-space model:

$$\mathbf{x}(n+1) = \mathbf{F}(n+1,n)\mathbf{x}(n) + \mathbf{v}_1(n), \tag{1}$$
$$\mathbf{y}(n) = \mathbf{C}(n)\mathbf{x}(n) + \mathbf{v}_2(n). \tag{2}$$

   In this problem, you will study the dependence of the parameters in Kalman filters. For example, the state vector $\mathbf{x}(1) = \mathbf{F}(1,0)\mathbf{x}(0) + \mathbf{v}_1(0)$ so $\mathbf{x}(1)$ depends both on $\mathbf{x}(0)$ and on $\mathbf{v}_1(0)$. Therefore we mark two checks (✔) on the first row of Table 1. Similarly, the results for $\mathbf{y}(1)$ and $\boldsymbol{\alpha}(1)$ are also provided.

   Complete the lower part of Table 1 (from $\mathbf{x}(2)$ to $\widehat{\mathbf{x}}(3|\mathscr{Y}_3)$) with the associated reasons.

   Remark: Table 1 is useful in the derivation of Kalman filters. For example, in Table 1, since $\mathbf{v}_1(2)$ and $\boldsymbol{\alpha}(1)$ are independent, we have $\mathbb{E}\left[\mathbf{v}_1(2)\boldsymbol{\alpha}^H(1)\right] = \mathbf{0}$.

2. (Innovations process, 20 points) We consider a scalar, WSS, circularly-symmetric, and complex random process $x(n)$ with zero-mean and autocorrelation function

$$r_x(k) = \frac{1}{4}\delta(k+1) + \delta(k) + \frac{1}{4}\delta(k-1). \tag{3}$$

   We express the measurements $x(1), x(2)$ into the innovations $\alpha(1), \alpha(2)$ by

$$\underbrace{\begin{bmatrix}\alpha(1)\\ \alpha(2)\end{bmatrix}}_{\boldsymbol{\alpha}} = \underbrace{\begin{bmatrix}1 & 0\\ L_{2,1} & 1\end{bmatrix}}_{\mathbf{L}}\underbrace{\begin{bmatrix}x(1)\\ x(2)\end{bmatrix}}_{\mathbf{x}}. \tag{4}$$

	$\mathbf{x}(0)$	$\mathbf{v}_1(0)$	$\mathbf{v}_2(0)$	$\mathbf{v}_1(1)$	$\mathbf{v}_2(1)$	$\mathbf{v}_1(2)$	$\mathbf{v}_2(2)$	$\mathbf{v}_1(3)$	$\mathbf{v}_2(3)$	Reason
$\mathbf{x}(1)$	✔	✔								(1) for $n = 0$
$\mathbf{y}(1)$	✔	✔			✔					(2) for $n = 1$ and the row of $\mathbf{x}(1)$
$\boldsymbol{\alpha}(1)$	✔	✔			✔					$\boldsymbol{\alpha}(1) = \mathbf{y}(1)$
$\mathbf{x}(2)$										
$\mathbf{y}(2)$										
$\boldsymbol{\alpha}(2)$										
$\widehat{\mathbf{x}}(2\|\mathscr{Y}_1)$										
$\widehat{\mathbf{x}}(2\|\mathscr{Y}_2)$										
$\mathbf{x}(3)$										
$\mathbf{y}(3)$										
$\boldsymbol{\alpha}(3)$										
$\widehat{\mathbf{x}}(3\|\mathscr{Y}_2)$										
$\widehat{\mathbf{x}}(3\|\mathscr{Y}_3)$										

Table 1: The dependence of the parameters in Kalman filters.

(a) (5 points) Find $L_{2,1}$.
Hints: You may use the orthogonal property of the innovations process.

(b) (5 points) Find $\mathbf{R}_{\mathbf{x}}$, which is the covariance matrix of $\mathbf{x}$.

(c) (5 points) Find $\mathbf{R}_{\boldsymbol{\alpha}}$, which is the covariance matrix of $\boldsymbol{\alpha}$.

(d) (5 points) Decompose $\mathbf{R}_{\mathbf{x}}$ in the following form

$$\mathbf{R}_{\mathbf{x}} = \underbrace{\begin{bmatrix} \ell_{1,1} & 0 \\ \ell_{2,1} & \ell_{2,2} \end{bmatrix}}_{\boldsymbol{\mathcal{L}}} \underbrace{\begin{bmatrix} \ell_{1,1} & 0 \\ \ell_{2,1} & \ell_{2,2} \end{bmatrix}^H}_{\boldsymbol{\mathcal{L}}^H} = \boldsymbol{\mathcal{L}}\boldsymbol{\mathcal{L}}^H. \tag{5}$$

The lower triangular matrix $\boldsymbol{\mathcal{L}}$ has non-negative diagonals ($\ell_{1,1} \geq 0$ and $\ell_{2,2} \geq 0$). The decomposition in (5) is the *Cholesky decomposition* of $\mathbf{R}_{\mathbf{x}}$.
Hints: You may first relate $\mathbf{R}_{\mathbf{x}}$ to $\mathbf{R}_{\boldsymbol{\alpha}}$ and the matrix $\mathbf{L}$ in (4).

3. (Adaptive beamforming, 30 points) We consider a ULA with N elements and the inter-element spacing $d = \lambda/2$. The array measurements are collected into the following matrix:

$$\texttt{matX} = \begin{bmatrix} \widetilde{\mathbf{x}}(1) & \widetilde{\mathbf{x}}(2) & \widetilde{\mathbf{x}}(3) & \ldots & \widetilde{\mathbf{x}}(L) \end{bmatrix}. \tag{6}$$

The numerical values of matX can be found in ASP_HW4_Problem_3.mat We know *a priori* that the array measurements contain a signal term, an interference term, and a noise term. **The source DOA θ_s satisfies $0^\circ \leq \theta_s \leq 10^\circ$, but the exact value is unknown.**

(a) (5 points) List the numerical values of the sample covariance matrix $\widehat{\mathbf{R}}$.

(b) (5 points) List the eigenvalues of $\widehat{\mathbf{R}}$.

(c) (5 points) Plot the the MVDR spectrum based on $\widehat{\mathbf{R}}$. You may first generate a discrete grid $\theta \in [-90 : 0.01 : 90]$ in degrees, and then evaluate the MVDR spectrum over this grid. **Save your plot to ASP_HW4_Problem_3_MVDR.fig.**

(d) (5 points) Estimate the source DOA by using the MVDR spectrum in Problem 3c. The estimated source DOA is denoted by $\widehat{\theta}_s$.
Hints: You may use the MATLAB command findpeaks in Problem 3d.

(e) (5 points) Plot the beampattern for these beamformers

- Uniform weights
- Array steering with source DOA $\widehat{\theta}_s$
- MVDR beamformer with source DOA $\widehat{\theta}_s$

Save your plot to ASP_HW4_Problem_3_Beampattern.fig.
Hints: See page 28 in 10_Adaptive_Beamforming_Updated.pdf for an example of the beampattern.

(f) (5 points) Let $y(t)$ be the output of a beamformer. Plot the real part and the imaginary part of $y(t)$ against time t, for **all the cases in Problem 3e**. **Save your plot to ASP_HW4_Problem_3_Output.fig.**

4. (Bonus problem, 10 points) You may observe that the beamformer outputs in Problem 3f are noisy. Based on your understanding of array signal processing, adaptive beamforming, and DOA estimation, you are asked to develop your own beamformer for improved performance. In short, the beamformer output should be close to the true signal waveform.

(a) (5 points) Describe the detailed steps in your beamformer. Implement your beamfomer in ASP_HW4_Problem_4.m.

(b) (5 points) Let $\widehat{y}(t)$ be the output of your beamformer based on matX in (6). Save the numerical values of the beamformer output as a *column vector*

$$\texttt{y\_hat} = \begin{bmatrix} \widehat{y}(1) & \widehat{y}(2) & \widehat{y}(3) & \dots & \widehat{y}(L) \end{bmatrix}^T \tag{7}$$

to a mat file ASP_HW4_y_hat_t.mat.
Grading of Problem 4b: Let $s(t)$ be the *true* source waveform, which is unavailable to you. We establish a vector $\mathbf{s} = \begin{bmatrix} s(1) & s(2) & s(3) & \dots & s(L) \end{bmatrix}^T$. Your grade for

Problem 4b is

$$\max\left\{\mathrm{Round}\left(5\times\left(1-\frac{\|\texttt{y\_hat}-\texttt{s}\|_2}{\|\texttt{s}\|_2}\right)\right),0\right\}, \tag{8}$$

where $\mathrm{Round}(\cdot)$ rounds the argument to the nearest integer. The notation $\|\cdot\|_2$ is the vector 2-norm. In other words, if your beamformer output is closer to the true $s(t)$, then you get more grades.

5. (Tracking the state vectors using Kalman filters, 30 points)

   In this problem, we will use Kalman filters to track the state vectors, which reveal the temporal evolution of the source information. We will consider the state-space model of the following form

$$\mathbf{x}(n+1) = \mathbf{F}(n+1,n)\mathbf{x}(n) + \mathbf{v}_1(n), \tag{9}$$

$$\mathbf{y}(n) = \mathbf{C}(n)\mathbf{x}(n) + \mathbf{v}_2(n), \tag{10}$$

   where $\mathbf{F}(n+1,n)$ is the transition matrix and $\mathbf{C}(n)$ is the measurement matrix. The noise terms $\mathbf{v}_1(n)$ and $\mathbf{v}_2(n)$ have zero mean. The correlation matrices of $\mathbf{v}_1(n)$ and $\mathbf{v}_2(n)$ are denoted by $\mathbf{Q}_1(n)$ and $\mathbf{Q}_2(n)$, respectively. The state vector $\mathbf{x}(n)$ is given by

$$\mathbf{x}(n) = \begin{bmatrix} x_1(n) \\ x_2(n) \\ \vdots \\ x_M(n) \end{bmatrix}, \tag{11}$$

   where M is the number of states. All other notations and assumptions are consistent with those in `12_Kalman_filters.pdf`.

   You will be given the measurement vectors in the following layout:

$$\widetilde{\mathbf{Y}} = \begin{bmatrix} \widetilde{\mathbf{y}}(1) & \widetilde{\mathbf{y}}(2) & \widetilde{\mathbf{y}}(3) & \dots & \widetilde{\mathbf{y}}(L) \end{bmatrix} = \begin{bmatrix} \widetilde{y}_1(1) & \widetilde{y}_1(2) & \widetilde{y}_1(3) & \dots & \widetilde{y}_1(L) \\ \widetilde{y}_2(1) & \widetilde{y}_2(2) & \widetilde{y}_2(3) & \dots & \widetilde{y}_2(L) \\ \widetilde{y}_3(1) & \widetilde{y}_3(2) & \widetilde{y}_3(3) & \dots & \widetilde{y}_3(L) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{y}_N(1) & \widetilde{y}_N(2) & \widetilde{y}_N(3) & \dots & \widetilde{y}_N(L) \end{bmatrix}. \tag{12}$$

   Read the numerical values of the measurements and the parameters in the state-space model from `ASP_HW4_Problem_5.mat`, where the names of the variables are listed in Table 2. For simplicity, we assume that all of $\mathbf{F}(n+1,n)$, $\mathbf{C}(n)$, $\mathbf{Q}_1(n)$, and $\mathbf{Q}_2(n)$ are **independent of** n. The initial conditions for Kalman filters are

$$\widehat{\mathbf{x}}(1|\mathscr{Y}_0) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{M\times 1}, \qquad \mathbf{K}(1,0) = \mathbf{I}_M = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}_{M\times M}. \tag{13}$$

Variables in the lecture	Variables in ASP_HW4_Problem_5.mat
$\mathbf{F}(n+1,n)$	F_n1_n
$\mathbf{C}(n)$	C_n
$\mathbf{Q}_1(n)$	Q1_n
$\mathbf{Q}_2(n)$	Q2_n
$\widetilde{\mathbf{Y}}$	Y_tilde

Table 2: A summary of the variables in a Kalman filter.

(a) *(Handwriting)* According to ASP_HW4_Problem_5.mat, what are the numerical values of M, N, and L?

(b) Implement the Kalman filter based on the data in ASP_HW4_Problem_5.mat. The estimated state vector is denoted by

$$\widehat{\mathbf{x}}(n|\mathscr{Y}_n) = \begin{bmatrix} \widehat{x}_1(n|\mathscr{Y}_n) \\ \widehat{x}_2(n|\mathscr{Y}_n) \\ \vdots \\ \widehat{x}_M(n|\mathscr{Y}_n) \end{bmatrix}. \tag{14}$$

For all the components of the estimated state vector (i.e., $\widehat{x}_m(n|\mathscr{Y}_n)$ for $m = 1, 2, \ldots, M$), generate the following plots.

i. A plot for the real part of $\widehat{x}_m(n|\mathscr{Y}_n)$. The horizontal axis is the time index n. Use the MATLAB function `subplot` to display your plots in the form of M rows and 1 column. **Save your plot to `ASP_HW4_Problem_5_Real.fig`.**

ii. A plot for the imaginary part of $\widehat{x}_m(n|\mathscr{Y}_n)$. The horizontal axis is the time index n. Use the MATLAB function `subplot` to display your plots in the form of M rows and 1 column. **Save your plot to `ASP_HW4_Problem_5_Imag.fig`.**

iii. A plot for the magnitude of $\widehat{x}_m(n|\mathscr{Y}_n)$. The horizontal axis is the time index n. Use the MATLAB function `subplot` to display your plots in the form of M rows and 1 column. **Save your plot to `ASP_HW4_Problem_5_Mag.fig`.**

iv. A plot for the unwrapped phase of $\widehat{x}_m(n|\mathscr{Y}_n)$. The horizontal axis is the time index n. Use the MATLAB function `subplot` to display your plots in the form of M rows and 1 column. **Save your plot to `ASP_HW4_Problem_5_Phase.fig`.**

Labels for the horizontal and vertical axes have to be marked clearly. You will generate 4 plots in total.

Hint: It would be involved to implement Kalman filters from scratch. The following architecture is recommended (but not mandatory)

i. Implement the three major blocks (one-step predictor, Kalman gain computer, and the Riccati equation solver) as three separate functions. You are free to design the input/output arguments of these functions on your own.

ii. Combine these functions into one function for Kalman filters.

(c) Design an estimator for **the slope of the unwrapped phase** of $\widehat{x}_m(n|\mathscr{Y}_n)$, where $m = 1, 2, \ldots, M$. Explain the rationale of the proposed estimator and list the numerical values of the estimates.

Hint : You may use the MATLAB routine `unwrap` to evaluate the unwrapped phase.

Remark: Suppose a signal $x(n)$ has the form $A(n)e^{j\phi(n)}$ The slope of the phase $\phi(n)$ is interpreted as the angular frequency of the signal. For example, if $x(n) = e^{j(\omega n+\phi_0)}$, then $\phi(n) = \omega n + \phi_0$ and the slope is ω.

MATLAB Submission Checklist

1. `ASP_HW4_Problem_3.m`: This is the main program. We will obtain all the results in Problem 3 after execution.
2. `ASP_HW4_Problem_3_MVDR.fig`: The figure file for the MVDR spectrum in Problem 3c.
3. `ASP_HW4_Problem_3_Beampattern.fig`: The figure file for the beampatterns in Problem 3e.
4. `ASP_HW4_Problem_3_Output.fig`: The figure file for the beamformer outputs in Problem 3f.
5. [Bonus Problem] `ASP_HW4_Problem_4.m`: This is the main program. We will obtain all the results in Problem 4 after execution.
6. [Bonus Problem] `ASP_HW4_y_hat_t.mat`: Numerical values of the beamformer output.
7. `ASP_HW4_Problem_5.m`: This is the main program. We will obtain all the results in Problem 5 after execution. You may implement one-step predictor, Kalman gain computer, and the Riccati equation solver in separate m-files, but this is optional.
8. `ASP_HW4_Problem_5_Real.fig`: The figure file for the real parts of the state vectors in Problem 5b.
9. `ASP_HW4_Problem_5_Imag.fig`: The figure file for the imaginary parts of the state vectors in Problem 5b.
10. `ASP_HW4_Problem_5_Mag.fig`: The figure file for the magnitudes of the state vectors in Problem 5b.
11. `ASP_HW4_Problem_5_Phase.fig`: The figure file for the phases of the state vectors in Problem 5b.

Last updated December 13, 2020.