Problem 5-1:

[Ref: HW5_劉仲恩 Chung-En Liu_Digital Control_HW5_Design in State-Space Model_20190510]

In this problem, we have to determine a linear state-feedback controller such that the closed-loop poles match those given in the problem.

The system given in the problem is indicated in Equation (1-1) and Equation (1-2).

$x(k+1)=\begin{bmatrix}1.0 & 0.1\\ 0.5 & 0.1\end{bmatrix}x(k)+\begin{bmatrix}1\\0\end{bmatrix}u(k)$	(1-1)
$y(k)=\begin{bmatrix}1 & 1\end{bmatrix}x(k)$	(1-2)

Since the closed-loop poles are situated at **0.1 and 0.25**, the desired characteristic equation can be written out as in Equation (1-3). We then get Equation (1-4) by expanding Equation (1-3).

$(z-0.1)(z-0.25)$	(1-3)
$z^2-0.35z+0.025$	(1-4)

We then set the state-feedback controller K as in Equation (1-5).

$K=\begin{bmatrix}K_1\\K_2\end{bmatrix}^T$	(1-5)

For state-feedback control system, the closed-loop characteristic equation can be expressed as of line two on page 14 in [1: Lian 2019], which is also shown here in Equation (1-6). By modifying Equation (1-6) and adopting the Eigenvalue Assignment as mentioned on page 14 and 24 in [1: Lian 2019], we obtain the closed-loop characteristic equation of the system as in Equation (1-7).

$\det(\lambda I-(F-HK))=\lambda^n+(a_1+k_1)\lambda^{n-1}+\cdots+(a_n+k_n)$	(1-6)
$\det(zI-(F-HK))$	(1-7)

According to page 11 in [1: Lian 2019], we can see that the discrete time model with given h can be expressed as in Equation (1-8). By comparing Equation (1-8) with the given system as indicated in Equation (1-1) and Equation (1-2), we can set matrices F and H as in Equation (1-9) and Equation (1-10) respectively.

$x(k+1)=Fx(k)+Hu(k)$	(1-8)
$F=\begin{bmatrix}1.0 & 0.1\\ 0.5 & 0.1\end{bmatrix}$	(1-9)
$H=\begin{bmatrix}1\\ 0\end{bmatrix}$	(1-10)

By substituting the values of matrices F and H from Equation (1-9) and Equation (1-10) into Equation (1-7), Equation (1-7) can then be rewritten as Equation (1-11).

$\det(zI-(\begin{bmatrix}1.0 & 0.1\\ 0.5 & 0.1\end{bmatrix}-\begin{bmatrix}1\\ 0\end{bmatrix}\begin{bmatrix}K_1 & K_2\end{bmatrix}))$	(1-11)

By expanding Equation (1-11), we get Equation (1-12).

$z^2+z(K_1-1.1)+(0.5K_2-0.1K_1+0.05)$	(1-12)

As Equation (1-12) should be equivalent to the desired characteristic equation as in Equation (1-4), we get the following Equation (1-13) and Equation (1-14).

$K_1 - 1.1 = -0.35$	(1-13)
$0.5K_2 - 0.1K_1 + 0.05 = 0.025$	(1-14)

By solving Equation (1-13) and Equation (1-14), we obtain the values of K_1 and K_2 as in Equation (1-15) and Equation (1-16).

$K_1 = 0.75$	(1-15)
$K_2 = 0.1$	(1-16)

The linear state-feedback controller can thus be expressed in the following.

$K = \begin{bmatrix} 0.75 \\ 0.1 \end{bmatrix}^T$	**Ans. of Pro. 5-1**

Problem 5-2:

Refs:
[HW5_王喻民 王喻民_DCS_HW-5_Design in State-Space Model] → a
[HW5_林柏佑_HW-5_DesignInStateSpaceModel] → b

(a)

The given system is denoted in Equation (2-1) and Equation (2-2)

$x(k+1)=\begin{bmatrix}0.55 & 0.12\\ 0 & 0.67\end{bmatrix}x(k)+\begin{bmatrix}0.01\\ 0.16\end{bmatrix}u(k)$	(2-1)
$y(k)=\begin{bmatrix}1 & 0\end{bmatrix}x(k)$	(2-2)

The discrete time state-space model can be written as Equation (2-3) and Equation (2-4)

$x(k+1)=Fx(k)+Hu(k)$	(2-3)
$y(k)=Cx(k)+Du(k)$	(2-4)

Hence, we can get $F=\begin{bmatrix}0.55 & 0.12\\ 0 & 0.67\end{bmatrix}$, $H=\begin{bmatrix}0.01\\ 0.16\end{bmatrix}$, and $C=\begin{bmatrix}1 & 0\end{bmatrix}$.

A linear state-feedback controller is to be determined such that the desired closed-loop characteristic equation in Equation (2-5):

$z^2-0.63z+0.21=0$	(2-5)

By the state-feedback law, $u(k)=-Kx(k)$. Then, Equation (2-5) can be rewritten as

$\begin{aligned} x(k+1)&=\begin{bmatrix}0.55 & 0.12\\ 0 & 0.67\end{bmatrix}x(k)+\begin{bmatrix}0.01\\ 0.16\end{bmatrix}(-Kx(k))\\ &=\left\{\begin{bmatrix}0.55 & 0.12\\ 0 & 0.67\end{bmatrix}-\begin{bmatrix}0.01\\ 0.16\end{bmatrix}\begin{bmatrix}k_1 & k_2\end{bmatrix}\right\}x(k)\end{aligned}$	(2-6)

The characteristic equation is

$$\begin{aligned} &\det(zI-(F-HK)) \\ &=\det(zI-\begin{bmatrix}0.55 & 0.12\\ 0 & 0.67\end{bmatrix}+\begin{bmatrix}0.01\\ 0.16\end{bmatrix}\begin{bmatrix}k_1 & k_2\end{bmatrix}) \\ &=\det(\begin{bmatrix}z-0.55+0.01k_1 & -0.12+0.01k_2\\ 0.16k_1 & z-0.67+0.16k_2\end{bmatrix}) \\ &=(z-0.55+0.01k_1)(z-0.67+0.16k_2) \\ &\quad-(0.16k_1)(-0.12+0.01k_2) \\ &=z^2+(0.01k_1+0.16k_2-1.22)z \\ &\quad+(0.3685+0.0125k_1-0.088k_2) \\ &=0 \end{aligned}$$	(2-7)

By comparing the parameters of Equation (2-5) and Equation (2-7), we can get

$$\begin{cases}0.01k_1+0.16k_2-1.22=-0.63\\ 0.3685+0.0125k_1-0.088k_2=0.21\end{cases}\Rightarrow\begin{cases}k_1=9.22\\ k_2=3.11\end{cases}$$	**Ans. of Pro 5-2 a**

Therefore, the linear state-feedback controller is $K=\begin{bmatrix}9.22 & 3.11\end{bmatrix}$

(b) The following figures are some code snippets and the simulation result.

The code in Figure 2-1 construct the original discrete-time system, and then use the state-feedback controller gain K, which is found in problem (a), to construct the closed-loop discrete-time system. Then, use Matlab function ***d2c()*** to construct the closed-loop continuous-time system.

Result 1: Simulation DT Controller only

A = [-3, 1; 0, -2]; B = [0; 1]; C = [1, 0]; K = [9.22, 3.11]; x0 = [1; 0]; Ts = 0.2;	(2-8)

System in continuous time is derived:

sys = ss(A, B, C, []);	(2-9)

Transform the continuous system into discrete time in h = 0.2:

sys = c2d(sys,Ts);	(2-10)

Obtain the discrete state-space model:

[F,H,C,D] = ssdata(sys);	(2-11)

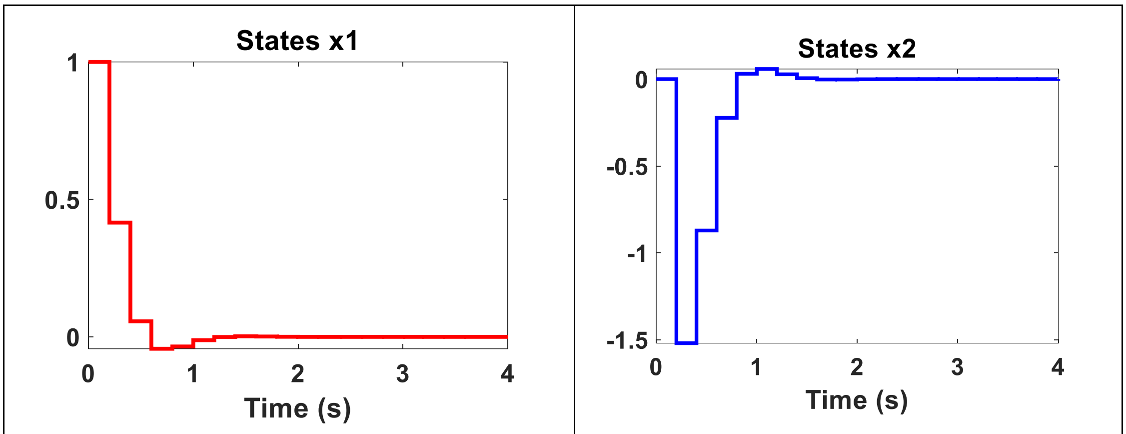
Implement the controller:

F2 = F-H*K;	(2-12)

Obtain the input, output and states with initial condition:

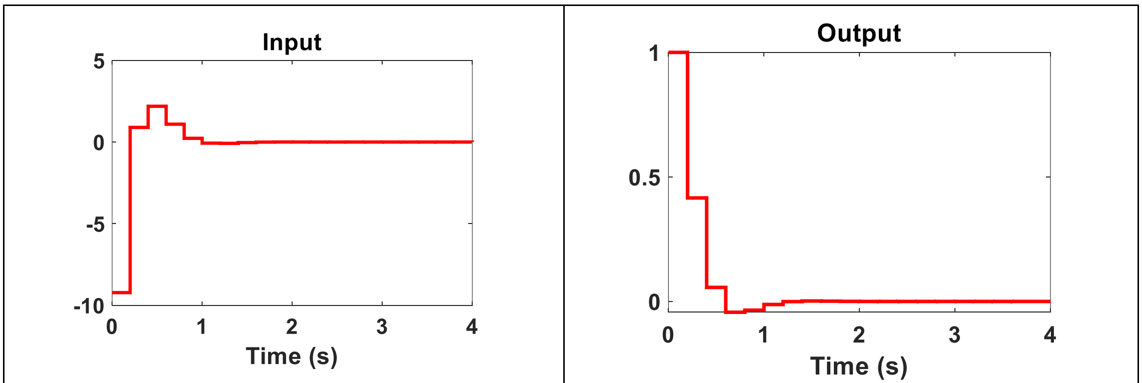
sys2 = dss(F2, [] , C, [] , [] ,Ts);
t = 0 : Ts : 4;
[y, t, x] = initial(sys2, x0, t);
u = -K * x';
x1 = x(:,1);
x2 = x(:,2);
	(2-13)

Finally, plot the figures for states, input, and output:



(a) (b)

Figure 2-1. The states of the system. (a) State x1 figure (b) State x2 figure.



(a) (b)

Figure 2-2. The input and output of the system

Result 2: Simulation of DT Controller -> CT Controller

```
% Original discrete-time system
F = [0.55 0.12; 0 0.67];
H = [0.01; 0.16];
C = [1 0];
D = 0;

sys_ori = ss(F, H, C, D, 0.2);

% Linear state-feedback controller
K = [9.2222 3.1111];

% Closed-loop discrete-time system, h = 0.2
sys_cl = ss(F-H*K, H, C, D, 0.2)
% Closed-loop continuous-time system
sys_cl_ct = d2c(sys_cl)
```

Figure 2-1. System construction

The code in Figure 2-2 generates the input signal to do simulation. Because the simulated system has already considered the state-feedback input, here the input signal is zero.

```
% Generate input signal to simulate the system
t = 0:0.01:4;
u = zeros(size(t));
x0 = [1 0];
```

Figure 2-2. Input signal generation

The code in Figure 2-3 shows how to simulate the system and extract the input signals from states. Here use Matlab function ***lsim()*** to simulate the closed-loop continuous-time system. Store the output in variable y, time vector in t, and states in x.

Then use x and the relation $\boldsymbol{u} = -\boldsymbol{K}\boldsymbol{x}$ to extract input signal every ***0.2 second***. Finally, the simulation results are shown in Figure 2-4.

```
% Simulation
figure(1)
[y, t, x] = lsim(sys_cl_ct, u, t, x0);
subplot(2, 2, 1)
input = zeros(fix(size(t)/20) + 1);
for i = 1:size(t)
    if mod(i-1,20) == 0
        input(fix(i/20)+mod(i,20)) = - K * x(i,:).';
    end
end
stairs(input, 'r', 'LineWidth', 2)
```

Figure 2-3. Simulation and extract the input signals

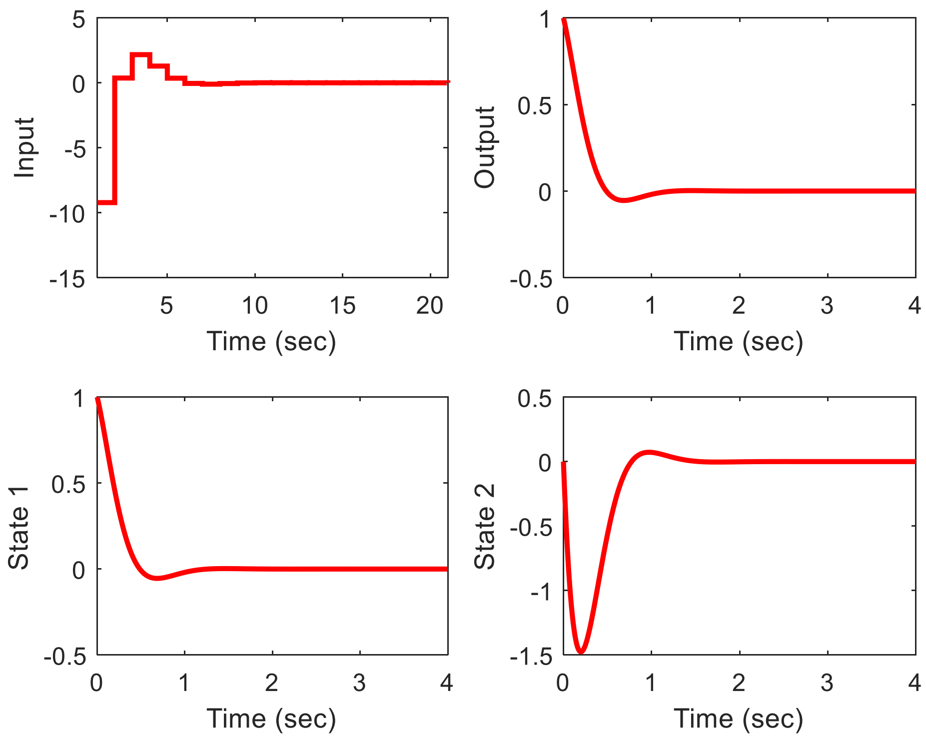


Figure 2-4. Simulation Results [**Ans. of Pro 5-2 b**]

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Result 3: Simulation of DT Controller + CT Plant

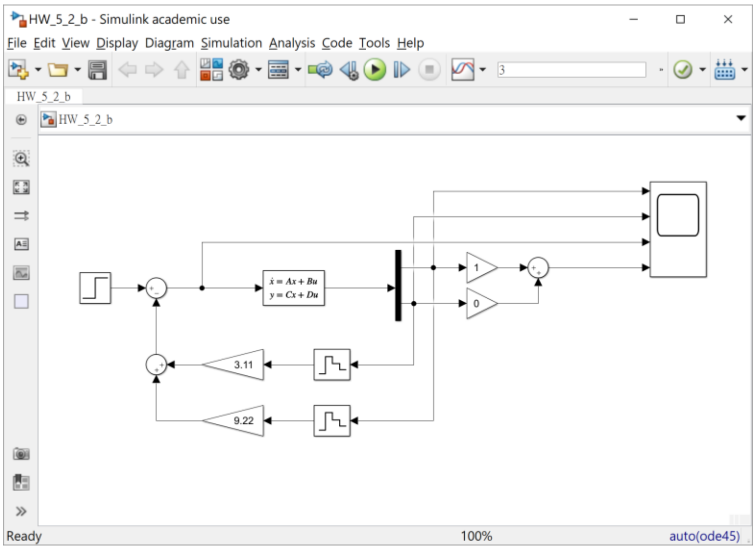

Figure: Simulink Model

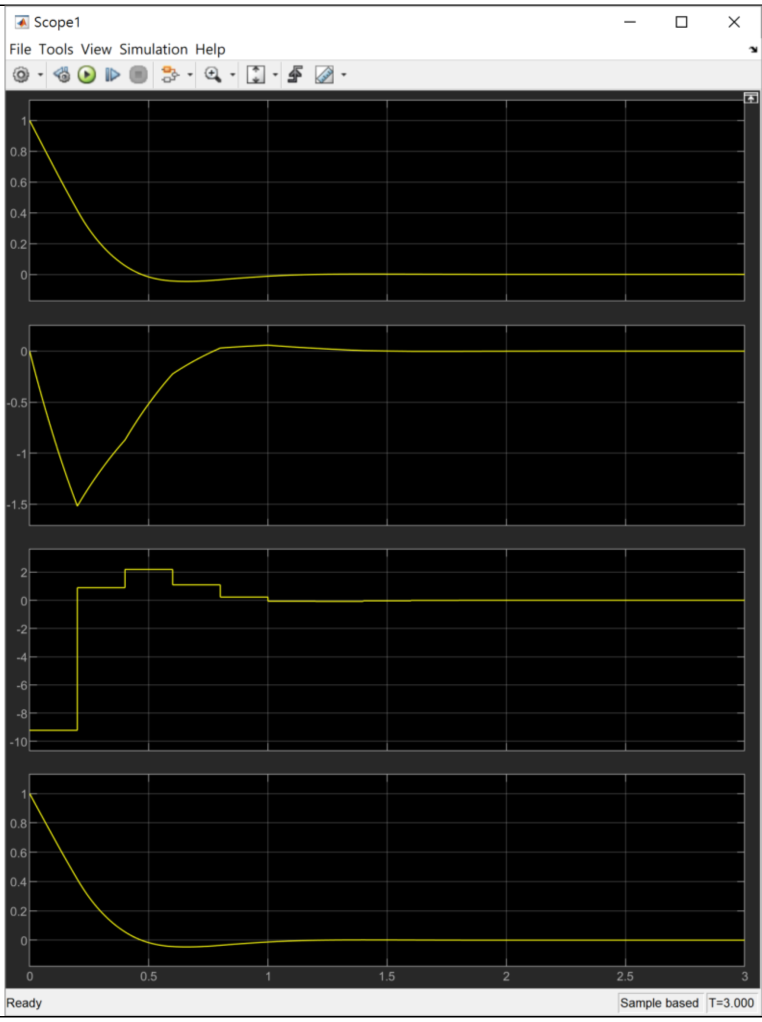

Figure: States, x1 and x2; Input, Output

Problem 5-3

Refs:
[HW5_張峻豪 r07921012 張峻豪_DCS_HW5_107 0510_Design in State-Space Model] → a, c
[HW5_林柏宇 林柏宇_DCS_HW05_20190510_Design in state-space model] → b

(a)

The system is described as:

$$x(k+1) = Fx(k) + Hu(k)$$	(3-1)
$$y(k) = Cx(k)$$	(3-2)

Where

$$F = \begin{bmatrix} 0.25 & 0.5 \\ 1 & 2 \end{bmatrix}, H = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$	(3-3)

(a) To bring the state to origin in two sampling intervals in closed-loop system, the condition of deadbeat control should be satisfied.

The linear state-feedback controller is given:

$$u(k) = -Kx(k) + K_r r(k)$$	(3-4)

Where

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$	(3-5)

The condition of deadbeat control for two time intervals:

$$\det[\lambda I - F + HK] = \lambda^2$$	(3-6)

Substitute Equation (3-3) and Equation (3-5) into Equation (3-6):

$$\det \begin{bmatrix} \lambda - 0.25 + k_1 & -0.5 + k_2 \\ 4k_1 - 1 & \lambda - 2 + 4k_2 \end{bmatrix} = \lambda^2$$	(3-7)

Summarize Equation (3-7):

$\lambda^2 + (k_1 + 4k_2 - 2.25)\lambda = \lambda^2$	(3-8)

Therefore, the following condition should be met to satisfy Equation (3-8):

$k_1 + 4k_2 - 2.25 = 0$	(3-9)

There are infinite solutions to Equation (3-9), one of solutions is:

$K = [k_1 \quad k_2] = [\mathbf{-1.75} \quad \mathbf{1}]$	**Ans. of Pro 5-3 a**

(b)

First, check out the controllability.

The controllability matrix is denoted in Equation (3-10):

$W_c = [H \quad FH] = \left[\begin{bmatrix}1\\4\end{bmatrix} \quad \begin{bmatrix}0.25 & 0.5\\1 & 2\end{bmatrix}\begin{bmatrix}1\\4\end{bmatrix}\right] = \begin{bmatrix}1 & 2.25\\4 & 9\end{bmatrix}$	(3.10)

As we can see, $\det \boldsymbol{W_c} = \mathbf{0}$. Which means the system is **not reachable** and we **cannot reach arbitrary states from the origin**.

However, $\mathrm{x(k)} = \begin{bmatrix}2\\8\end{bmatrix}$ happen to be in the column space of W_c, so this point can be reached.

Let's check out the following calculation process:

$x(1) = Hx(0) + Fu(0)$ $= \begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}1\\4\end{bmatrix}u(0)$	(3.11)

$$\begin{aligned} x(2) &= Hx(1) + Fu(1) \\ &= H[Hx(0) + Fu(0)] + Fu(1) \\ &= H^2x(0) + HFu(0) + Fu(1) \\ &= \begin{bmatrix} 0.5625 & 1.125 \\ 2.25 & 4.5 \end{bmatrix} x(0) + \begin{bmatrix} 2.25 \\ 9 \end{bmatrix} u(0) + \begin{bmatrix} 1 \\ 4 \end{bmatrix} u(1) \end{aligned}$$	(3.12)

Where $x(2) = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$ and $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, we can obtain:

$$\begin{cases} 2 = 2.25u(0) + u(1) \\ 8 = 9u(0) + 4u(1) \end{cases}$$	(3.13)

Therefore, we can find out the answer as:

$$\begin{cases} \boldsymbol{u(0) = 0} \\ \boldsymbol{u(1) = 2} \end{cases}$$	**Ans. of Pro. 5-3 b**

(c)

To estimate the states of system, the system should be observable. In advance, the observer can be design to let the system meet the desired characteristic polynomial.

First of all, the observability matrix is:

$$W_o = \begin{bmatrix} C \\ CF \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.25 & 0.5 \end{bmatrix}$$	(3-14)

From Equation (3-14), the observability matrix is full rank such that the system is observable.

The observer is designed to estimate the state:

$$\hat{x}(k+1) = F\hat{x}(k) + Hu(k) + L[y(k) - C\hat{x}(k)]$$	(3-15)

Where

$$\mathrm{L} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$	(3-16)

The characteristic polynomial can be derived from:

$$\det[\lambda I - F + LC] = 0$$	(3-17)

Substitute Equation (3-3) and Equation (3-16) into Equation (3-17), and the

characteristic polynomial is:

$$\lambda^2 + (-2.25 + l_1)\lambda + (-2l_1 + 0.5l_2) = 0$$	(3-18)

On the other hands, the desired characteristic polynomial:

$$(\lambda - 0.2)^2 = \lambda^2 - 0.4\lambda + 0.04 = 0$$	(3-19)

Compare Equation (3-19) to Equation (3-20), the observer is derived:

$$\mathrm{L} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} \mathbf{1.85} \\ \mathbf{7.48} \end{bmatrix}$$	**Ans. of Pro. 5-3 c**

In conclusion, the states can be estimated by observer design. In advance, the original characteristic polynomial can be brought to desired one.

References

[1: Lian 2019]

Feng-Li Lian. (2019, May). 107-2_dcs31_StateSpaceDesign. [Online]. Available: http://cc.ee.ntu.edu.tw/~fengli/Teaching/DigitalControl/107-2_dcs31_StateSpaceDesign.pdf