Problem 4-1:

[Ref: HW4_林柏宇 林柏宇_DCS_HW04_20190426_stability, controllability, observability]

The characteristic equation of the closed-loop system is:

$$z(z - 0.2)(z - 0.4) + K = 0 \tag{1.1}$$

where $K > 0$

Using root locus, starting points are z=0, 0.2, 0.4.

To find the crossing point that roots cross the **unit circle**, let:

$$z = a + bi$$
$$where \ a^2 + b^2 = 1 \tag{1.2}$$

Apply Equation (1.2) to Equation (1.1), we can obtain:

$$(a + bi)(a + bi - 0.2)(a + bi - 0.4) = -K \tag{1.3}$$

Multiply with $(a - bi)$, and reorganize Equation (1.3):

$$a^2 - b^2 - 0.6a + 0.08 + (2ab - 0.6b)i = -K(a - bi) \tag{1.4}$$

Reorganize Equation (1.4), separate real and imaginary parts:

$$\begin{cases} a^2 - b^2 - 0.6a + 0.08 = -Ka & (1.5) \\ b(2a - 0.6) = Kb & (1.6) \end{cases}$$

For the case when $b \neq 0$, replace b^2 in Equation (1.5) by $b^2 = 1 - a^2$, we get Equation (1.7)

$$a^2 - (1 - a^2)^2 - 0.6a + 0.08 = -a(2a - 0.6) \tag{1.7}$$

Therefore,

$$a^2 - 0.3a - 0.23 = 0 \tag{1.8}$$

So,

$$a \approx 0.652 \ or - o.352 \tag{1.9}$$

From Equation (1.6), and apply Equation (1.9):

$$K = 2a - 0.6 \approx 0.704 \ or - 1.304 \tag{1.10}$$

For another case that root locus may also cross the unit circle for $b = 0$ and $a = \pm 1$.

Follow Equation (1.1), when $b = 0, a = -1$

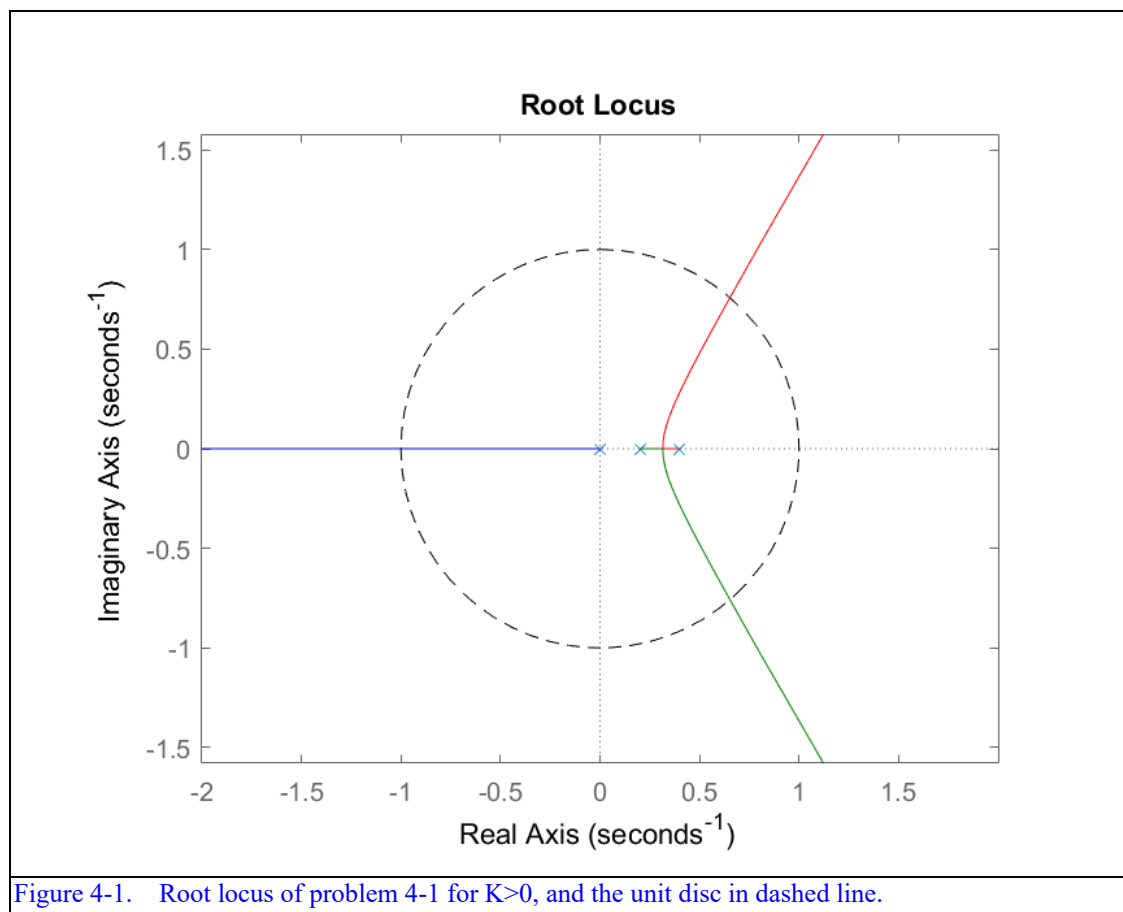| $$-1(-1 - 0.2)(-1 - 0.4) + K = 0$$ $$K = 1.68$$	(1.11)

Follow Equation (1.1), when $b = 0, a = 1$

$$K = -0.48$$	(1.12)

Therefore, consider Equation (1.10) and Equation (1.11) and $K > 0$:

$$\begin{cases} K > 0 \\ K \le 0.704 \\ K \le 1.68 \end{cases}$$	(1.13)

We can conclude the answer:

$$\boldsymbol{0 < K \le 0.704}$$	**Ans. of Pro. 4-1**

Figure 4-1.   Root locus of problem 4-1 for K>0, and the unit disc in dashed line.

Problem 4-2:

[Ref: HW4_楊軒昱 楊軒昱_R07921006_DCS_HW4_Stability, Controllability, Observability]

As we know, the discrete time state-space model can be written like below:

$$x[k + 1] = Fx[k] + Hu[k]$$ $$y[k] = Cx[k] + Du[k]$$	(2.1)

Compare Equation (2.1) to the question, we can get:

$$F = \begin{bmatrix} 0.5 & -0.5 \\ 0 & 0.25 \end{bmatrix}$$ $$H = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$ $$C = \begin{bmatrix} 2 & -4 \end{bmatrix}$$	(2.2)

(a)

To check whether the state-space model is stable, we have to find the eigenvalue of matrix F in Equation (2.2), which means we have to solve

$$\lambda I - F = 0$$	(2.3)

And we can get

$$\lambda = \ 0.5, 0.25$$	(2.4)

Because **the magnitude of both eigenvalues are smaller than 1**, which means **the eigenvalues are inside the unit circle**. **Hence, the system is stable according to Theorem 3.1 on page 11 of the lecture note: 107-2_dcs22_Stability.pdf.**

(b)

   To check whether the state-space model is observable, we can check it by

calculating W_o in Equation (2.5)

$$W_o = \begin{bmatrix} C \\ CF \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 2 & -4 \end{bmatrix} \\ \begin{bmatrix} 2 & -4 \end{bmatrix} \begin{bmatrix} 0.5 & -0.5 \\ 0 & 0.25 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \qquad (2.5)$$

   Because the **rank of W_o is 1**, which is **not full rank**, the **state-space model is**

not observable according to Theorem 3.8 on page 17 of the lecture note: 107-

2_dcs23_ControllabilityObservability.pdf.

(c)

   To check whether the state-space model is reachable, we can check it by

calculating W_c in Equation (2.6)

$$W_c = \begin{bmatrix} H & FH \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} & \begin{bmatrix} 0.5 & -0.5 \\ 0 & 0.25 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 4 & 1 \end{bmatrix} \qquad (2.6)$$

Because the **rank of W_c is 2**, which is **full rank**, the **state-space model is reachable**

according to Theorem 3.7 on page 7 of the lecture note: 107-

2_dcs23_ControllabilityObservability.pdf.

Problem 4-3

[Ref: HW4_林柏宇 林柏宇_DCS_HW04_20190426_stability, controllability, observability]

(a)

The original state function is denoted in Equation (3.1)

$$x(k+1) = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(k)$$	(3.1)

Because we want to get a control sequence, first, we can take k = 0 and k = 1 into
Equation (3.1), and we can get Equation (3.2) and Equation (3.3)

$$x(1) = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(0) = \begin{bmatrix} 3 \\ 3+u(0) \\ 0 \end{bmatrix}$$	(3.2)
$$x(2) = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 3+u(0) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(1) = \begin{bmatrix} 3+u(0) \\ u(1) \\ 0 \end{bmatrix}$$	(3.3)

Follow Equation (3.3), if we set $\begin{cases} u(0) = -3 \\ u(1) = 0 \end{cases}$, we can reach the origin from the initial

state $x(0)$. That is, $x(2) = \begin{bmatrix} 3+u(0) \\ u(1) \\ 0 \end{bmatrix} = \begin{bmatrix} 3-3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. So the answer for the control

sequence is

$$\begin{cases} u(0) = -3 \\ u(1) = 0 \end{cases}$$	Ans. of 4-3 (a)

(b)

$$F^2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$	(3.4)
$$F^3 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$	(3.5)

In general, it would take 3 steps because it is a 3rd order system.

However, we reach the origin by **2 steps** in this case.

(c) Let's check out the reachability,

$$W_c = [H \quad FH \quad F^2H] = \left[\begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}0&1&2\\0&0&3\\0&0&0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}0&0&3\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\right]$$

$$= \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}$$

(3.6)

We can conclude from Equation (3.6) that W_c is **not full rank**, that is, this system is **unreachable, but may be controlled.**

$[1 \quad 1 \quad 1]^T$ is not in the column space of W_c and can therefore not be reached from the origin. It's easily seen from the state space description that x_3 **can't be changed from its initial value**.

Problem 4-4

[Ref: HW4_林柏宇 林柏宇_DCS_HW04_20190426_stability, controllability, observability]

(a)
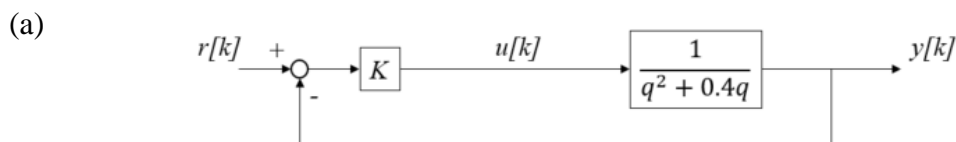
For closed-loop system, the transfer function is in Equation (4.1):

$$T = \frac{\dfrac{K}{q^2 + 0.4q}}{1 + \dfrac{K}{q^2 + 0.4q}} = \frac{K}{q^2 + 0.4q + K} \qquad (4.1)$$

The characteristic equation of the closed-loop system is in Equation (4.2):

$$q^2 + 0.4q + K = 0 \qquad (4.2)$$

Using **Jury's stability test** on Equation (4.2),

$$
\begin{array}{ccc}
1 & 0.4 & K \\
K & 0.4 & 1 \\
\hline
1 - K^2 & 0.4(1-K) & \\
0.4(1-K) & 1 - K^2 & \\
\hline
1 - K^2 - \dfrac{0.16(1-K)}{1+K} & &
\end{array}
\qquad (4.3)
$$

The system is stable if $\begin{cases} 1 - K^2 > 0 \\ 1 - K^2 - \frac{0.16(1-K)}{1+k} > 0 \end{cases}$, that is,

$$\begin{cases} 1 - K^2 > 0 \\ (1-K)(1+K)(K+0.6)(K+1.4) > 0 \end{cases} \qquad (4.4)$$

Therefore, follow the Equation (4.3), we can obtain:

$$\begin{cases} -1 < K < 1 \\ -1.4 < K < -1 \ or -0.6 < K < 1 \end{cases} \qquad (4.5)$$

Finally, we can conclude the answer in Equation (4.6):

$\boldsymbol{-0.6 < K < 1}$	**Ans. of 4-4 a**

(b) Since $e(k) = r(k) - y(k)$, we can obtain:

$$E(z) = \big(1 - T(z)\big)R(z) \qquad (4.7)$$

When $r(k)$ is a step function, and $K = 0.5$, the system is stable, the final-value theorem can be used:

$$\lim_{k \to \infty} e(k) = \lim_{z \to 1} \frac{z-1}{z} E(z) = \lim_{z \to 1} \frac{z-1}{z} \left(1 - \frac{K}{z^2 + 0.4z + K}\right)\left(\frac{z}{z-1}\right)$$ $$= \lim_{z \to 1} \frac{z^2 + 0.4z}{z^2 + 0.4z + K} = \frac{1.4}{1.4 + K} = \mathbf{0.737}$$	**Ans. of** **4-4 b**