HW 2: Discrete-Time Models	Digital Control Systems, Spring 2019, NTU-EE
Name: 參考答案	Date: 3/27, 2019

Problem 2-1:

[Ref: HW2_林柏宇_林柏宇_DCS_HW02_20190322_Discrete-Time Models]

Derive the formulas of the DT State-Space System with Inner Time Delay.

Consider the LTI system with inner time delay:

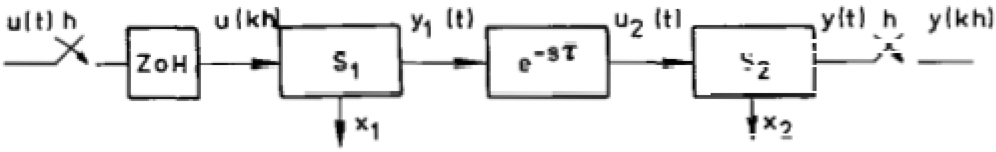

Fig. 1. The time delay system.

[1: Wittenmark 1985]

$$S_1:\ \begin{cases}\dot{x}_1(t) = A_1x_1(t) + B_1u_1(t)\\ y_1(t) = C_1x_1(t) + D_1u_1(t)\end{cases} \tag{2.1}$$

$$S_2:\ \begin{cases}\dot{x}_2(t) = A_2x_2(t) + B_2u_2(t)\\ y_2(t) = C_2x_2(t) + D_2u_2(t)\end{cases} \tag{2.2}$$

$$u_2(t) = y_1(t-\tau) \tag{2.3}$$

Where τ is the delay time

And let

$$\tau = (\mathrm{d}-1)\mathrm{h} + \tau' \tag{2.4}$$

Where d is an integer, h is the sampling period, and τ' is a fraction of the sampling interval, i.e.,

$$0 < \tau' \leq \mathrm{h}$$

Three cases will be considered:

Case 1 – time delay before the system, i.e., $A_1 = B_1 = C_1 = D_2 = 0,\ D_1 = I$

Case 2 – time delay after the system, i.e., $A_2 = B_2 = C_2 = D_1 = 0,\ D_2 = I$

Case 3 – time delay between the subsystems. Assume $D_1 = D_2 = 0$

Case 1 – time delay before the system:

Assume subsystem S_1 has a unity transfer function:

$\rightarrow\ A_1 = B_1 = C_1 = D_2 = 0,\ D_1 = I$

Assume state of S_2 is known at $t = kh$,

The state at $kh + h$ is given by solving S_2:

$x_2(kh+h) = \mathrm{e}^{A_2 h} x_2(kh) + \int_{kh}^{kh+h} \mathrm{e}^{A_2(kh+h-s)} B_2 u_2(\mathrm{s})\ \mathrm{ds}$	

From $u_2(t) = y_1(t-\tau) = \mathrm{u}(\mathrm{t}-\tau)$

The integral can be separated into $u_2(t) = u(kh - dh)$ & $u_2(t) = u(kh-(d-1)h)$

$x_2(kh+h) = \mathrm{e}^{A_2 h} x_2(kh) + \int_0^{t'} \mathrm{e}^{A_2(h-\tau')}\mathrm{e}^{A_2 s'} B_2\ \mathrm{d}s' \cdot u(kh-dh)$ $+ \int_0^{h-t'} \mathrm{e}^{A_2 s'} B_2\ \mathrm{d}s' \cdot u(kh-(d-1)h)$	

Introduce the notations:

$\Phi_i(t) = \mathrm{e}^{A_i t}$ $\Gamma_i(t) = \int_0^t \mathrm{e}^{A_i s} B_i\ ds$	(3.1) (3.2)

Then,

$\boldsymbol{x_2(kh+h) = \Phi_2(h)x_2(kh) + \Phi_2(h-\tau')\Gamma_2(\tau')u(kh-dh) + \Gamma_2(h-\tau')u(kh-(d-1)h)}$	**Ans. of Case 1 (3.3)**

Case 2 – time delay after the system:

Assume subsystem S_2 has a unity transfer function.

$\rightarrow A_2 = B_2 = C_2 = D_1 = 0,\ D_2 = I$

$\boldsymbol{x_1(kh+h) = \Phi_1(h)x_1(kh) + \Gamma_1(h)u(kh)}$	**Ans. of Case 2 (3.4)**

Where

$\boldsymbol{\Phi_1(h) = \mathrm{e}^{A_1 t}}$ $\boldsymbol{\Gamma_1(t) = \int_0^t \mathrm{e}^{A_1 s} B_1\ ds}$	

Case 3 – time delay between the subsystems:

(1) For $\tau = 0$, system =

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}A_1 & 0\\ A_{21} & A_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}B_1\\ 0\end{bmatrix}u$$

$$y = \begin{bmatrix}0 & C_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{3.6}$$

Where

$$A_{21} = B_2C_1$$

Then,

$$x_2(kh+h) = \mathrm{e}^{A_2h}x_2(kh) + \int_{kh}^{kh+h} \mathrm{e}^{A_2(kh+h-s)}\, A_{21}x_1(\mathrm{s})\ \mathrm{ds} \tag{3.7}$$

In the interval $kh < s \le kh + h$

$$x_1(s) = \mathrm{e}^{A_1(s-kh)}x_1(kh) + \int_{kh}^{s} \mathrm{e}^{A_1(s-s')}\, B_1\ \mathrm{d}s' \cdot u(kh) \tag{3.8}$$

Thus,

$$\begin{aligned} x_2(kh+h) &= \Phi_2(h)x_2(kh) \\ &\quad + \int_{kh}^{kh+h} \mathrm{e}^{A_2(kh+h-s)}\, A_{21}\mathrm{e}^{A_1(s-kh)}\ \mathrm{ds}x_1(kh) \\ &\quad + \int_{kh}^{kh+h} \mathrm{e}^{A_2(kh+h-s)}\, A_{21} \int_{kh}^{s} \mathrm{e}^{A_1(s-s')}\, B_1\ \mathrm{d}s'\ \mathrm{ds}u(kh) \\ &= \Phi_2(h)x_2(kh) + \Phi_{21}(h)x_1(kh) + \Gamma_2'(h)u(kh) \end{aligned} \tag{3.9}$$

Where

$$\Phi_{21}(h) = \int_0^h \mathrm{e}^{A_2s'}\, A_{21}\mathrm{e}^{A_1(h-s')}\ \mathrm{d}s' \tag{3.10}$$

$$\Gamma_2'(h) = \int_0^h \mathrm{e}^{A_2s''}\, A_{21}\Gamma_1^{(h-s'')}\ ds'' \tag{3.11}$$

The sampled version of (3.6):

$$\begin{bmatrix}x_1(kh+h)\\ x_2(kh+h)\end{bmatrix} = \begin{bmatrix}\Phi_1(h) & 0\\ \Phi_{21}(h) & \Phi_2(h)\end{bmatrix}\begin{bmatrix}x_1(kh)\\ x_2(kh)\end{bmatrix} + \begin{bmatrix}\Gamma_1(h)\\ \Gamma_2'(h)\end{bmatrix}u(kh)$$

$$y(kh) = \begin{bmatrix}0 & C_2\end{bmatrix}\begin{bmatrix}x_1(kh)\\ x_2(kh)\end{bmatrix} \tag{3.12}$$

If there is a delay in the system, then $x_1(\mathrm{s})$ in (3.7) is replaced by $x_1(\mathrm{s}-\tau)$, i.e., if the delay is a multiple of sampling interval, then (3.9) is changed to:

$$x_2(kh+h) = \Phi_2(h)x_2(kh) + \Phi_{21}(h)x_1(kh-(d-1)h) + \Gamma_2'(h)u(kh-(d-1)h)$$

When delay is not a multiple of sampling period, it can be assumed that

$$0 < \tau \leq \mathrm{h}$$

Assume the states of S_1 are denoted x_1' and are related to the state of S_1 through

$$x_1'(t) = x_1(\mathrm{t}-\tau)$$

Then,

$$\begin{bmatrix} x_1'(kh+h) \\ x_2(kh+h) \end{bmatrix} = \begin{bmatrix} \Phi_1(h) & 0 \\ \Phi_{21}(h) & \Phi_2(h) \end{bmatrix} \begin{bmatrix} x_1'(kh) \\ x_2(kh) \end{bmatrix} + \begin{bmatrix} \Phi_1(h-\tau) & 0 \\ \Phi_{21}(h-\tau) & \Phi_2(h-\tau) \end{bmatrix} \begin{bmatrix} \Gamma_1(\tau) \\ \Gamma_2'(\tau) \end{bmatrix} u(kh-h) + \begin{bmatrix} \Gamma_1(h-\tau) \\ \Gamma_2'(h-\tau) \end{bmatrix} u(kh)$$

Since $x_1'(kh) = x_1(kh-\tau)$

$$\begin{aligned} x_2(kh+h) = \Phi_2(h)x_2(kh) &+ \Phi_{21}(h)[\Phi_1(h-\tau)x_1(kh-h) \\ &+ \Gamma_1(h-\tau)u(kh-h)] + \Gamma_2'(h-\tau)u(kh) \\ &+ [\Phi_{21}(h-\tau)\Gamma_1(\tau) + \Phi_2(h-\tau)\Gamma_2'(\tau)]u(kh-h) \end{aligned}$$

Therefore, we have:

$$\boldsymbol{x_1(kh+h) = \Phi_1(h)x_1(kh) + \Gamma_1(h)u(kh)}$$

$$\boldsymbol{\begin{aligned} x_2(kh+h) = \Phi_2(h)x_2(kh) &+ \Phi_{21}(h)[\Phi_1(h-\tau)x_1(kh-h) \\ &+ \Gamma_1(h-\tau)u(kh-h)] + \Gamma_2'(h-\tau)u(kh) \\ &+ [\Phi_{21}(h-\tau)\Gamma_1(\tau) + \Phi_2(h-\tau)\Gamma_2'(\tau)]u(kh-h) \end{aligned}}$$

Final Answer

Where

$$\Phi_1(h) = \mathrm{e}^{A_1 t}$$

$$\Gamma_1(t) = \int_0^t \mathrm{e}^{A_1 s} B_1 \, ds$$

$$\Phi_2(h) = \mathrm{e}^{A_2 t}$$

$$\Gamma_2(t) = \int_0^t \mathrm{e}^{A_2 s} B_2 \, ds$$

$$\Phi_{21}(h) = \int_0^h \mathrm{e}^{A_2 s'} A_{21} \mathrm{e}^{A_1(h-s')} \, \mathrm{d}s'$$

$\Gamma_2'(h) = \int_0^h \mathrm{e}^{A_2 s''} A_{21} \Gamma_1^{(h-s'')} \, ds''$	

Problem 2-2:

[Ref: HW2_張立揚_數位控制 HW2_R07921008_張立揚]

From the given equation of y and $\frac{d}{dt}x$, we get A, B, C and D in (2-1), while our goal is to get F and H in (2-2).

$\dot{x}(t) = Ax(t) + Bu(t)$ $y(t) = Cx(t) + Du(t)$ where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $C = 1$ $D = 0$	(2-1)

$x[k+1] = Fx[k] + Hu[k]$ $y[k] = Cx[k] + Du[k]$	(2-2)

Let "h" be the sampling period, we can calculate F and H with the equations in Lecture Note: DCS-11 page 11, as shown in (2-3) and (2-4).

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$$F = e^{Ah} = I + Ah + \frac{A^2h^2}{2!} + ...$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} h + \frac{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2 h^2}{2!} + \dots$$

$$= \begin{bmatrix} 1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \dots & h - \frac{h^3}{3!} + \frac{h^5}{5!} + \dots \\ -h + \frac{h^3}{3!} - \frac{h^5}{5!} + \dots & 1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \dots \end{bmatrix}$$

approximated F to the inverse of a Taylor series, we get:

$$F \approx \begin{bmatrix} \cos(h) & \sin(h) \\ -\sin(h) & \cos(h) \end{bmatrix}$$

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<td>Ans. of F
(2-3)</td>
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$$H = \left(\int_0^h e^{A\eta} d\eta \right) B = \left(\sum_{k=0}^{\infty} \frac{A^k h^{k+1}}{(k+1)!} \right) B$$

$$= \left(Ih + A\frac{h^2}{2!} + A^2 \frac{h^3}{3!} + \dots \right) B$$

with similar approximation in (2-3), we get:

$$H \approx \begin{bmatrix} \sin(h) & 1 - \cos(h) \\ \cos(h) - 1 & \sin(h) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - \cos(h) \\ \sin(h) \end{bmatrix}$$

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<td>Ans. of H
(2-4)</td>
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With F and H being calculated, we can present the discrete-time system as (2-2).

Problem 2-4

[Ref: HW2_張峻豪_r07921012 張峻豪_DCS_HW2_107 0322_Discrete-Time Models]

From the state space equation:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ h = 0.3,\ \tau = 0.2$$

Substitute A into the formulation of F:

$$F = e^{Ah} = \sum_{k=0}^{\infty} \frac{A^k h^k}{k!} = \sum_{k=0}^{\infty} \frac{h^k}{k!} \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{h^k}{k!} & 0 \\ h\sum_{k=0}^{\infty} \frac{h^k}{k!} & \sum_{k=0}^{\infty} \frac{h^k}{k!} \end{bmatrix}$$

F is derived:

$$F = \begin{bmatrix} e^h & 0 \\ he^h & e^h \end{bmatrix} \tag{1}$$

Substitute h = 0.3 into (1):

$$\mathbf{F} = \begin{bmatrix} \mathbf{1.3499} & \mathbf{0} \\ \mathbf{0.4050} & \mathbf{1.3499} \end{bmatrix} \approx \begin{bmatrix} \mathbf{1.350} & \mathbf{0} \\ \mathbf{0.405} & \mathbf{1.350} \end{bmatrix}$$ **Ans. of F (2)**

For H_1:

$$H_1 = e^{A(h-\tau)} \int_0^{\tau} e^{A\eta} B d\eta$$

$$H_1 = \begin{bmatrix} e^{h-\tau} & 0 \\ (h-\tau)e^{h-\tau} & e^{h-\tau} \end{bmatrix} \sum_{k=0}^{\infty} \frac{A^k \tau^{k+1}}{(k+1)!} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$H_1 = \begin{bmatrix} e^{h-\tau} & 0 \\ (h-\tau)e^{h-\tau} & e^{h-\tau} \end{bmatrix} \begin{bmatrix} \sum_{k=0}^{\infty} \frac{\tau^k}{k!} - 1 & 0 \\ \tau \sum_{k=0}^{\infty} \frac{\tau^k}{k!} - \sum_{k=0}^{\infty} \frac{\tau^k}{k!} + 1 & \sum_{k=0}^{\infty} \frac{\tau^k}{k!} - 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$H_1 = \begin{bmatrix} e^{h-\tau} & 0 \\ (h-\tau)e^{h-\tau} & e^{h-\tau} \end{bmatrix} \begin{bmatrix} e^{\tau}-1 & 0 \\ \tau e^{\tau} - e^{\tau} + 1 & e^{\tau}-1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{h-\tau}(-1+e^{\tau}) \\ e^{h-\tau}(1-h+\tau) + (h-1)e^{h} \end{bmatrix}$	(3)

Substitute h = 0.3, d = 0.2 into (3):

$\boldsymbol{H_1} = \begin{bmatrix} \mathbf{0.2447} \\ \mathbf{0.0498} \end{bmatrix} \approx \begin{bmatrix} \mathbf{0.245} \\ \mathbf{0.050} \end{bmatrix}$	**Ans. of H_1 (4)**

Similarly, for H_0:

$H_0 = \int_0^{h-\tau} e^{A\eta} B d\eta$	
$H_0 = \sum_{k=0}^{\infty} \frac{A^k (h-\tau)^{k+1}}{(k+1)!} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	
$H_0 = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{(h-\tau)^k}{k!} - 1 & 0 \\ (h-\tau)\sum_{k=0}^{\infty} \frac{(h-\tau)^k}{k!} - \sum_{k=0}^{\infty} \frac{(h-\tau)^k}{k!} + 1 & \sum_{k=0}^{\infty} \frac{(h-\tau)^k}{k!} - 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	
$H_0 = \begin{bmatrix} e^{h-\tau} - 1 & 0 \\ (h-\tau)e^{h-\tau} - e^{h-\tau} + 1 & e^{h-\tau} - 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	(5)

Substitute h = 0.3, d = 0.2 into (5):

$\boldsymbol{H_0} = \begin{bmatrix} \mathbf{0.1052} \\ \mathbf{0.0053} \end{bmatrix} \approx \begin{bmatrix} \mathbf{0.105} \\ \mathbf{0.005} \end{bmatrix}$	**Ans. of H_0 (6)**

For the formulation of the pulse-transfer operator:

$\mathrm{G(q)} = \mathrm{C}(qI - F)^{-1}(H_0 + H_1 q^{-1})$	(7)

Combine the (2), (4), (6), and (7), the pulse-transfer operator G(q) is derived:

$\mathrm{G(q)} = \mathrm{C} \frac{\begin{bmatrix} 0.1052q^2 + 0.1027q - 0.3303 \\ 0.0053q^2 + 0.0853q + 0.0319 \end{bmatrix}}{q(q-1.3499)^2}$	(8)

References

[1: Wittenmark 1985]

Bjorn Wittenmark, "Sampling of a system with a time delay,“ IEEE Transactions on Automatic Control, Vol. 30, No. 5, pp. 507-510, May 1985.