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1 Probability Theory

1.1 Elemantary Probability

Definition 1.1.1 (Cardano's Principle). A be a random outcome of an experiment that may proceed in various ways. Assume each of these ways is **equally likely**, then, probability $P[A]$ of outcome A is

$$P[A] = \frac{\text{number of ways leading to outcome } A}{\text{number of ways the experiment can be proceeded}} \quad (1)$$

1.1.1 Basic principles of Counting

- Permutation:

$$A_n^k = \frac{n!}{(n-k)!} \quad (2)$$

- Combination:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} \quad (3)$$

- Permutation of k **Indistinguishable** Objects:

$$\frac{n!}{n_1!n_2!n_3!\dots n_k!} = \binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \cdot \dots \cdot \binom{n-(n_1+n_2+\dots+n_{k-1})}{n_k} \quad (4)$$

Remark:

In permutation of k indistinguishable objects, the elements has noorder

within a group but are different from each other; the groups either has order or are different from each other.

Example:

Consider 10 balls, 5 red, 3 green and 2 blue. How many ways can they be arranged on a line?

$$\frac{10!}{5! \cdot 3! \cdot 2!} = 2520 \quad (5)$$

1.1.2 Sample Points, Sample Space and σ -Field

Definition 1.1.2 (Sample Points). Mathematical objects are called sample points.

Definition 1.1.3 (Sample Space). The sample space S is large enough to accommodate all the sample points.

Definition 1.1.4 (Event). An outcome in the sense of Cardano's principle is interpreted as a subset A of a sample space S and called an event.

Definition 1.1.5 (Mutually exclusive). Two events A_1, A_2 are called mutually exclusive if $A_1 \cap A_2 = \emptyset$

Definition 1.1.6 (σ -Field). Suppose that a non-empty set S is given. A σ -field \mathcal{F} on S is a family of subsets of S such that:

- $\emptyset \in \mathcal{F}$
- If $A \in \mathcal{F}$, then $S \setminus A \in \mathcal{F}$
- If $A_1, A_2, \dots \in \mathcal{F}$ is a finite sequence of subsets, then the union $\cup_k A_k \in \mathcal{F}$

1.1.3 Probability Measures and Spaces

Definition 1.1.7 (Probability Measure). Let S be the sample space and \mathcal{F} be a σ -field. Then a function

$$P : \mathcal{F} \rightarrow [0, 1], A \mapsto P[A], \quad (6)$$

is called *probability measure* / *probability function* on S if

- $P[S] = 1$
- For any set of events $A_k \subset \mathcal{F}$ such that $A_j \cap A_k = \emptyset$ for $j \neq k$,

$$P[\cup_k A_k] = \sum_k P[A_k] \quad (7)$$

Theorem 1.1.1 (Basic Properties). $P[A_1 \cup A_2] = P[A_1] + P[A_2] - P[A_1 \cap A_2]$

1.2 Conditional Probability

Definition 1.2.1 (Conditional Probability). B occurs given that A has occurred

$$P[B | A] := \frac{P[A \cap B]}{P[A]} \quad (8)$$

1.2.1 Independence of Events

Definition 1.2.2 (independent). Two events are *independent* if

$$P[A \cap B] = P[A]P[B] \quad (9)$$

equivalent to

$$P[A | B] = P[A], P[B] \neq 0 \quad (10)$$

Definition 1.2.3 (Total Probability).

$$P[B] = P[B | A_1] \cdot P[A_1] + \cdots + P[B | A_n] \cdot P[A_n] = \sum_{k=1}^n P[B | A_k] \cdot P[A_k] \quad (11)$$

is called total probability formula for $P[B]$

1.2.2 Bayes' Theorem

Theorem 1.2.1 (Bayes's Theorem). Let $A_1, \dots, A_n \subset S$ be a set of pairwise mutually exclusive events whose union is S and who each have non zero probability of occurring. Let $B \subset S$ be any events such that $P[B] \neq 0$. Then for any $A_k, k = 1, \dots, n$

$$P[A_k | B] = \frac{P[B \cap A_k]}{P[B]} = \frac{P[B | A_k] \cdot P[A_k]}{\sum_{j=1}^n P[B | A_j] \cdot P[A_j]} \quad (12)$$

1.3 Discrete Random Variables

Definition 1.3.1 (Random Variable). Such function X

$$X : S \rightarrow \mathbb{R} \quad (13)$$

is a random variable. X has numerical values that are derived from the outcome of a random experiment.

Two types:

- *Discrete Random Variables*: countable range in \mathbb{R}

Definition 1.3.2 (Discrete Random Variable). Let S be sample space, Ω a countable subset of \mathbb{R} . A *discrete random variable* is a map

$$X : S \rightarrow \Omega \quad (14)$$

$$f_X : \Omega \rightarrow \mathbb{R} \quad (15)$$

A random variable is often given by the pair (X, f_X)

- *Continuous Random Variables*: having a range equal to \mathbb{R}

Example:

Flip a coin three times, sample space can be given by :

$$S = (t, t, t), (t, t, h), (t, h, t), \dots, (h, h, h) \quad (16)$$

Then we can define X as follows:

$$X(t, t, t) = 0, X(t, t, h) = 1, \dots, X(h, h, h) = 3 \quad (17)$$

X denotes the number of heads

$$P[X = 1] = P[(t, t, h), (t, h, t), (h, t, t)] \quad (18)$$

We can write

$$P[X = x] = P[A] \quad (19)$$

where $x \in R$ and $A \subset S$ is the event containing all sample points p such that $X(p) = x$.

1.3.1 PDF and CDF

Random variable comes with a probability density function or probability distribution f_X that allows the calculation of probability directly.

Follows:

- $f_X(x) > 0$ for all x
- $\sum_{x \in \Omega} f_X(x) = 1$

Definition 1.3.3 (Cumulative distribution function). Cumulative distribution function of a random variable is defined as

$$F_X : \mathbb{R} \rightarrow \mathbb{R}, F_X(x) := P[X \leq x] \quad (20)$$

For discrete random variable, $F_X(x) = \sum_{y \leq x} f_X(y)$

1.3.2 Bernoulli Random Variable

Definition 1.3.4 (Bernoulli Trial). Consider an experiment can only results in two possible outcomes, such as success or failure.

and probability of success $0 < p < 1$

Definition 1.3.5 (Bernoulli Random Variable). Let S be a sample space and $X : S \rightarrow 0, 1 \in \mathbb{R}$, Let $0 < p < 1$, then define the density function

$$f_X : 0, 1 \rightarrow \mathbb{R} \quad f_X(x) = \begin{cases} 1 - p & \text{for } x = 0 \\ p & \text{for } x = 1 \end{cases} \quad (21)$$

Then X is said to be a *Bernoulli random variable* or follow a *Bernoulli distribution* with parameter p . We indicate this by writing

$$X \sim \text{Bernoulli}(p) \quad (22)$$

1.3.3 Independent and Identical Trials

- *independent* means the outcome of a trial will not influence the following trial
- *identical* means each trial has same probability of success

1.3.4 Counting Successes in a Saquence of Trials

$$P[x \text{ successes in } n \text{ trials}] = \binom{n}{x} p^x (1-p)^{n-x} \quad (23)$$

1.3.5 Binomal Random Variable

Definition 1.3.6 (Binomal Random Variable). S : sample space, $n \in \mathbb{N} \setminus 0$ and

$$X : S \rightarrow \Omega = \{0, \dots, n\} \in \mathbb{R} \quad (24)$$

Let $0 < p < 1$ and define the density function

$$f_X : \Omega \rightarrow \mathbb{R}, \quad f_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad (25)$$

Then X is said to be a binomial random variable with parameters n and p . Indicate this by writing

$$X \sim B(n, p) \quad (26)$$

Also:

$$B(1, p) = \text{Bernoulli}(p) \quad (27)$$

1.3.6 Cumulative Distribution Function

Definition 1.3.7 (Cumulative Distribution Function).

$$F_X : \mathbb{R} \rightarrow \mathbb{R}, \quad F_X(x) := P[X \leq x] \quad (28)$$

For a discrete random variable

$$F_X(x) = \sum_{y \leq x} f_X(y) \quad (29)$$

In case of binomial distribution,

$$F_X(x) = \sum_{y=0}^{\lfloor x \rfloor} \binom{n}{y} p^y (1-p)^{n-y} \quad (30)$$

$\lfloor x \rfloor$ denote the largest integer not greater than x .

MMA command:

`CDF[BinomialDistribution[n,p],x]`

1.3.7 The Geometric Distribution

Another example: suppose we perform a sequence of Bernoulli trials which continues until a success is obtained. We then define the *geometric random variable* X to denote the number of trials needed to obtain the first success.

Example:

Result (t, t, t, h) , geometric random variable X attains value $X = 4$

Definition 1.3.8 (Geometric Random Variable).

$$X : S \rightarrow \Omega = \mathbb{N} \setminus \{0\} \quad (31)$$

Let $0 < p < 1$ and define density function $f_X(x) : \mathbb{N} \setminus \{0\} \rightarrow \mathbb{R}$

$$f_X(x) = (1 - p)^{x-1} p \quad (32)$$

then, X is a geometric random variable with parameter p

$$X \sim \text{Geom}(p) \quad (33)$$

The cumulative distribution function for a geometrically distributed random variable (X, f_X) with parameter p is given by

$$F(x) = P[X \leq x] = 1 - (1 - p)^{[x]} \quad (34)$$

1.4 Expectation, Variance and Moments