

1 Basic Set Theory

Paul R. Halmos, Naive Set Theory.

Definition 1.0.1 (Set). A set is an unordered collection of distinct objects, called *elements* or *members* of the set. A set is said to contain its elements. We write

- $a \in A$ if a is an element of the set A .

1.1 Set Operation

$$\begin{aligned} U &= \{x \mid x \in X, x \in \mathbb{C}\} \\ &= \bigcup_{x \in C} C \end{aligned}$$

1.1.1 Set difference

The set difference of A and B , denoted by $A - B$, or $A \setminus B$

1.1.2 Symmetric Difference

$$A \Delta B = (A - B) \cup (B - A) \quad (1)$$

1.1.3 Power Set

The power set of A is the set of all subsets of A , denoted by $\mathcal{P}(A)$ or 2^A

1.1.4 Set Algebras

De Morgan's Laws

- $C - (A \cup B) = (C - A) \cap (C - B)$
- $C - (A \cap B) = (C - A) \cup (C - B)$

1.1.5 Cartesian Product

Definition 1.1.1 (Cartesian). The Cartesian product of sets A and B is the set of ordered pairs, such that

$$A \times B = \{(a, b) \mid a \in A, b \in B\} \quad (2)$$

Definition 1.1.2 (Cartesian). By Kuratowski
An ordered pair (a, b) is given by

$$(a, b) := \{\{a\}, \{a, b\}\} \quad (3)$$

Theorem 1 (Cartesian). If $a \in C$ and $b \in C$, then $(a, b) \in \mathcal{P}(\mathcal{P}(C))$.
If $a \in A, b \in B$, then take $C = A \cup B$

Theorem 2 (Cartesian). $(x, y) = (a, b)$ iff $x = a$ and $y = b$

Proof

- \Leftarrow : Trivial
- \Rightarrow : By definition, we need to show that:

$$\underbrace{\{\{x\}, \{x, y\}\}}_U = \underbrace{\{\{a\}, \{a, b\}\}}_V \Rightarrow x = a \text{ and } y = b \quad (4)$$

- If $x \neq y$, then $|U| = |V| = 2$, by matching size, we have $\{x\} = \{a\}$ $\{y\} = \{b\}$
- If $x = y$, similarly $x = y = a = b$

Note: a definitio of ordered pairs is rational as long as it can indicate order.

1.1.6 Associative Set Operations

Let A_1, A_1, \dots, A_n be sets, then

- $A_1 \cap A_2 \cap \dots \cap A_n = \bigcap_{i=1}^n A_i$
- for \cup, \times, Δ are the same.

1.2 Simple Graphs

k-element subsets

Let X be a finit set. For a positive integer k , let $\binom{X}{k}$ denote the set of all k-element subsets. Note that $|\binom{X}{k}| = \binom{|X|}{k}$

Definition 1.2.1 (Graph). A finit simple graph G is a pair (V, E) where V is a non-empty finit set and E is a set of 2-element subsets of V , i.e., $E \in \binom{V}{2}$ Elements of V called *vertices*, also denoted as $V(G)$. Elements of E called *edges*, also denoed as $E(G)$.

2 Logic

2.1 CNF and DNF

- Conjunctive Normal Form (CNF):
a junction of one or more clauses, where a clause is a disjunction of literals
like **roduct of sums** or **AND of ORs**
- Disjunction Normal Forn (DNF):
like **sum of products** or **OR of ANDs**

Table 1: Conditional truth table

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

2.2 Conditional Statements

- p : hypothesis
- q : thesis/conclusion

Equivalent forms

- if p , then q
- q is a sufficient condition for p
- q is necessary for p

remark

- either p is false
- or q is true
- i.e. $\neg p \vee q$
- same as $\neg q \rightarrow \neg p$

2.3 Tautology and Contradiction

- Tautology: All cases evaluates to 1.
- Contradiction: All cases evaluates to 0.

2.3.1 Tautological Equivalence

- Absorption

$$p \wedge (p \vee q) \Leftrightarrow p$$

$$p \vee (p \wedge q) \Leftrightarrow p$$