1 Basic Set Theory

Paul R. Halmos, Naive Set Theory.

Definition 1.0.1 (Set)**.** A set is an unordered colleciton of distinct objects, called *elements* or *menbers* of the set. A set is said to contain its elements. We write

- $a \in A$ if a is an element of the set A.

1.1 Set Operation

$$\begin{aligned} U &= \{x \mid x \in X, x \in \mathbb{C}\} \\ &= U_{x \in C} C \end{aligned}$$

1.1.1 Set difference

The set difference of A and B ,denoted by $A - B$, or $A \backslash B$

1.1.2 Symmetric Difference

$$A \Delta B = (A - B) \cup (B - A) \tag{1}$$

1.1.3 Power Set

The power set of A is the set of all subsets of A, denoted by $\mathcal{P}(A)$ or 2^A

1.1.4 Set Algebras

De Morgan's Laws

- $C - (A \cup B) = (C - A) \cap (C - B)$
- $C - (A \cap B) = (C - A) \cup (C - B)$

1.1.5 Cartesian Product

Definition 1.1.1 (Cartesian)**.** The Cartesian product of sets A and B is the set of ordered pairs, such that

$$A \times B = \{(a, b) \mid a \in A, b \in B\} \tag{2}$$

Definition 1.1.2 (Cartesian)**.** By Kuratowski
An ordered pair (a, b) is given by

$$(a, b) := \{\{a\}, \{a, b\}\} \tag{3}$$

Theorem 1 (Cartesian)**.** If $a \in C$ and $b \in C$, then $(a, b) = \mathcal{P}(\mathcal{P}(C))$.
If $a \in A, b \in B$, then take $C = A \cup B$

Theorem 2 (Cartesian). *$(x, y) = (a, b)$ iff $x = a$ and $y = b$*

Proof

- $\Leftarrow$: Trival
- $\Rightarrow$: By definition, we need to show that:

$$\underbrace{\{\{x\},\{x,y\}\}}_{U} = \underbrace{\{\{a\},\{a,b\}\}}_{V} \Rightarrow x = a \textit{ and } y = b \tag{4}$$

  - If $x \neq y$, then $|U| = |V| = 2$, by matching size, we have $\{x\} = \{a\}$ $\{y\} = \{b\}$
  - If $x = y$, similarly $x = y = a = b$

Note: a definitio of ordered pairs is rational as long as it can indicate order.

1.1.6 Associative Set Operations

Let $A_1, A_1, \cdots, A_n$ be sets, then

- $A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap\limits_{i=1}^{n} A_i$
- for $\cup, \times, \Delta$ are the same.

1.2 Simple Graphs

k-element subsets
Let X be a finit set. For a positive integer k, let $\binom{X}{k}$ denote the set of all k-element subsets. Note that $|\binom{X}{k}| = \binom{|X|}{k}$

Definition 1.2.1 (Graph). A finit simple *graph G is a pair (V, E) where V is a non-empty finit set and E is a set of 2-element subsets of V, i.e., $E \in \binom{V}{2}$ Elements of V called vertics, also denoted as $V(G)$. Elements of E called edges,also denoed as $E(G)$.*

2 Logic

2.1 CNF and DNF

- Conjunctive Normal Form (CNF):
  a junction of one or more clauses, where a clause is a disjunction of literals like **roduct of sums** or **AND of ORs**
- Disjunction Normal Forn (DNF):
  like **sum of products** or **OR of ANDs**

Table 1: Conditional truth table

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \rightarrow q}$
0	0	1
0	1	1
1	0	0
1	1	1

2.2 Conditional Statements

- $\boldsymbol{p}$: hyposhesis
- $\boldsymbol{q}$: thesis/conclusion

Equivalent forms

- if p, then q
- q is a sufficient condition for q
- q is necessary for p

remark

- either $\boldsymbol{p}$ is false
- or $\boldsymbol{q}$ is true
- i.e. $\neg \boldsymbol{p} \vee \boldsymbol{q}$
- same as $\neg \boldsymbol{q} \rightarrow \neg \boldsymbol{p}$

2.3 Tautology and Contradiction

- Tautology: All cases evaluates to 1.
- Contradiction: All cases evaluates to 0.

2.3.1 Tautological Equivalence

- Absorption

$$\boldsymbol{p} \wedge (\boldsymbol{p} \vee \boldsymbol{q}) \Leftrightarrow \boldsymbol{p}$$
$$\boldsymbol{p} \vee (\boldsymbol{p} \wedge \boldsymbol{q}) \Leftrightarrow \boldsymbol{p}$$

- Cases

$$(\boldsymbol{p} \rightarrow \boldsymbol{q}) \wedge (\boldsymbol{p} \rightarrow \boldsymbol{r}) \Leftrightarrow \boldsymbol{p} \rightarrow (\boldsymbol{q} \wedge \boldsymbol{r})$$
$$(\boldsymbol{p} \rightarrow \boldsymbol{q}) \vee (\boldsymbol{p} \rightarrow \boldsymbol{r}) \Leftrightarrow \boldsymbol{p} \rightarrow (\boldsymbol{q} \vee \boldsymbol{r})$$
$$(\boldsymbol{p} \rightarrow \boldsymbol{r}) \wedge (\boldsymbol{p} \rightarrow \boldsymbol{r}) \Leftrightarrow (\boldsymbol{p} \vee \boldsymbol{q}) \rightarrow \boldsymbol{r}$$
$$(\boldsymbol{p} \rightarrow \boldsymbol{r}) \vee (\boldsymbol{p} \rightarrow \boldsymbol{r}) \Leftrightarrow (\boldsymbol{p} \wedge \boldsymbol{q}) \rightarrow \boldsymbol{r}$$

- Added premise

$$\begin{aligned}(\boldsymbol{p} \wedge \boldsymbol{q}) \rightarrow \boldsymbol{r} &\Leftrightarrow \boldsymbol{p} \rightarrow (\boldsymbol{q} \rightarrow \boldsymbol{r}) \\ &\Leftrightarrow \boldsymbol{q} \rightarrow (\boldsymbol{p} \rightarrow \boldsymbol{r})\end{aligned}$$

2.3.2 CNF and DNF

Theorem 3 (Disjunctive Normal Form)**.** For any proposition $\boldsymbol{\varphi}$, there is a proposition $\boldsymbol{\varphi_{dnf}}$ over same Boolean variables and in DNF such that $\boldsymbol{\varphi} \Leftrightarrow \boldsymbol{\varphi_{dnf}}$

Example:

$$\begin{aligned}\boldsymbol{\varphi} &= \boldsymbol{p} \rightarrow \boldsymbol{q} & \boldsymbol{\varphi_{dnf}} &= (\neg \boldsymbol{p}) \vee (\boldsymbol{q}) \\ \boldsymbol{\varphi} &= \boldsymbol{p} \leftrightarrow \boldsymbol{q} & \boldsymbol{\varphi_{dnf}} &= (\boldsymbol{p} \wedge \boldsymbol{q}) \vee (\neg \boldsymbol{q} \wedge \neg \boldsymbol{p}) \\ \boldsymbol{\varphi} &= \boldsymbol{p} \oplus \boldsymbol{q} & \boldsymbol{\varphi_{dnf}} &= (\neg \boldsymbol{p} \wedge \boldsymbol{q}) \vee (\boldsymbol{p} \wedge \neg \boldsymbol{q})\end{aligned}$$

Just like sum of product.

Theorem 4 (Conjunctive Normal Form)**.** For any proposition $\boldsymbol{\varphi}$, there is a proposition $\boldsymbol{\varphi_{cnf}}$ over same Boolean variables and in DNF such that $\boldsymbol{\varphi} \Leftrightarrow \boldsymbol{\varphi_{cnf}}$

Example:

$$\begin{aligned}\boldsymbol{\varphi} &= \boldsymbol{p} \rightarrow \boldsymbol{q} & \boldsymbol{\varphi_{dnf}} &= (\neg \boldsymbol{p} \vee \boldsymbol{q}) \\ \boldsymbol{\varphi} &= \boldsymbol{p} \leftrightarrow \boldsymbol{q} & \boldsymbol{\varphi_{dnf}} &= (\neg \boldsymbol{p} \vee \boldsymbol{q}) \wedge (\neg \boldsymbol{q} \vee \boldsymbol{p}) \\ \boldsymbol{\varphi} &= \boldsymbol{p} \oplus \boldsymbol{q} & \boldsymbol{\varphi_{dnf}} &= (\boldsymbol{p} \vee \boldsymbol{q}) \wedge (\neg \boldsymbol{p} \vee \neg \boldsymbol{q})\end{aligned}$$

Just like product of sum.