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## 1 Probability Theory

### 1.1 Elemantary Probability

**Definition 1.1.1** (Cardano's Principle). A be a random outcome of an experiment that may proceed in various ways. Assume each of these ways is **equally likely**, then, probability  $P[A]$  of outcome A is

$$P[A] = \frac{\text{number of ways leading to outcome } A}{\text{number of ways the experiment can be proceeded}} \quad (1)$$

#### 1.1.1 Basic principles of Counting

- Permutation:

$$A_n^k = \frac{n!}{(n-k)!} \quad (2)$$

- Combination:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} \quad (3)$$

- Permutation of k **Indistinguishable** Objects:

$$\frac{n!}{n_1!n_2!n_3!\dots n_k!} = \binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \dots \binom{n-(n_1+n_2+\dots+n_{k-1})}{n_k} \quad (4)$$

*Remark:*

In permutation of  $k$  indistinguishable objects, the elements has noorder within a group but are different from each other; the groups either has order or are different from each other.

*Example:*

Consider 10 balls, 5 red, 3 green and 2 blue. How many ways can they be arranged on a line?

$$\frac{10!}{5! \cdot 3! \cdot 2!} = 2520 \quad (5)$$

### 1.1.2 Sample Points, Sample Space and $\sigma$ -Field

**Definition 1.1.2** (Sample Points). Mathematical objects are called sample points.

**Definition 1.1.3** (Sample Space). The sample space  $S$  is large enough to accommodate all the sample points.

**Definition 1.1.4** (Event). An outcome in the sense of Cardano's principle is interpreted as a subset  $A$  of a sample space  $S$  and called an event.

**Definition 1.1.5** (Mutually exclusive). Two events  $A_1, A_2$  are called mutually exclusive if  $A_1 \cap A_2 = \emptyset$

**Definition 1.1.6** ( $\sigma$ -Field). Suppose that a non-empty set  $S$  is given. A  $\sigma$ -field  $\mathcal{F}$  on  $S$  is a family of subsets of  $S$  such that:

- $\emptyset \in \mathcal{F}$
- If  $A \in \mathcal{F}$ , then  $S \setminus A \in \mathcal{F}$
- If  $A_1, A_2, \dots \in \mathcal{F}$  is a finite sequence of subsets, then the union  $\cup_k A_k \in \mathcal{F}$

### 1.1.3 Probability Measures and Spaces

**Definition 1.1.7** (Probability Measure). Let  $S$  be the sample space and  $\mathcal{F}$  be a  $\sigma$ -field. Then a function

$$P : \mathcal{F} \rightarrow [0, 1], A \rightarrow P[A], \quad (6)$$

is called *probability measure* / *probability function* on  $S$  if

- $P[S] = 1$
- For any set of events  $A_k \subset \mathcal{F}$  such that  $A_j \cap A_k = \emptyset$  for  $j \neq k$ ,

$$P[\cup_k A_k] = \sum_k P[A_k] \quad (7)$$

**Theorem 1.1.1** (Basic Properties).  $P[A_1 \cup A_2] = P[A_1] + P[A_2] - P[A_1 \cap A_2]$

## 1.2 Conditional Probability

**Definition 1.2.1** (Conditional Probability).  $B$  occurs given that  $A$  has occurred

$$P[B | A] := \frac{P[A \cap B]}{P[A]} \quad (8)$$

### 1.2.1 Independence of Events

**Definition 1.2.2** (independent). Two events are *independent* if

$$P[A \cap B] = P[A]P[B] \quad (9)$$

equivalent to

$$P[A | B] = P[A], P[B] \neq 0 \quad (10)$$

**Definition 1.2.3** (Total Probability).

$$P[B] = P[B | A_1] \cdot P[A_1] + \cdots + P[B | A_n]P[A_n] = \sum_{k=1}^n P[B | A_k] \cdot P[A_k] \quad (11)$$

is called total probability formula for  $P[B]$

### 1.2.2 Bayes' Theorem

**Theorem 1.2.1** (Bayes's Theorem). Let  $A_1, \dots, A_n \subset S$  be a set of pairwise mutually exclusive events whose union is  $S$  and who each have non zero probability of occurring. Let  $B \subset S$  be any events such that  $P[B] \neq 0$ . Then for any  $A_k, k = 1, \dots, n$

$$P[A_k | B] = \frac{P[B \cap A_k]}{P[B]} = \frac{P[B | A_k] \cdot P[A_k]}{\sum_{j=1}^n P[B | A_j] \cdot P[A_j]} \quad (12)$$

## 1.3 Discrete Random Variables

**Definition 1.3.1** (Random Variable). Such function  $X$

$$X : S \rightarrow \mathbb{R} \quad (13)$$

is a random variable.  $X$  has numerical values that are derived from the outcome of a random experiment.

Two types:

- *Discrete Random Variables*: countable range in  $\mathbb{R}$

**Definition 1.3.2** (Discrete Random Variable). Let  $S$  be sample space,  $\Omega$  a countable subset of  $\mathbb{R}$ . A *discrete random variable* is a map

$$X : S \rightarrow \Omega \quad (14)$$

$$f_X : \Omega \rightarrow \mathbb{R} \quad (15)$$

A random variable is often given by the pair  $(X, f_X)$

- *Continuous Random Variables*: having a range equal to  $\mathbb{R}$

*Example:*

Flip a coin three times, sample space can be given by :

$$S = (t, t, t), (t, t, h), (t, h, t), \dots, (h, h, h) \quad (16)$$

Then we can define  $X$  as follows:

$$X(t, t, t) = 0, X(t, t, h) = 1, \dots, X(h, h, h) = 3 \quad (17)$$

$X$  denotes the number of heads

$$P[X = 1] = P[(t, t, h), (t, h, t), (h, t, t)] \quad (18)$$

We can write

$$P[X = x] = P[A] \quad (19)$$

where  $x \in R$  and  $A \subset S$  is the event containing all sample points  $p$  such that  $X(p) = x$ .

### 1.3.1 PDF and CDF

Random variable comes with a probability density function or probability distribution  $f_X$  that allows the calculation of probability directly.

Follows:

- $f_X(x) > 0$  for all  $x$
- $\sum_{x \in \Omega} f_X(x) = 1$

**Definition 1.3.3** (Cumulative distribution function). Cumulative distribution function of a random variable is defined as

$$F_X : \mathbb{R} \rightarrow \mathbb{R}, F_X(x) := P[X \leq x] \quad (20)$$

For discrete random variable,  $F_X(x) = \sum_{y \leq x} f_X(y)$

### 1.3.2 Bernoulli Random Variable

**Definition 1.3.4** (Bernoulli Trial). Consider an experiment can only results in two possible outcomes, such as success or failure. and probability of success  $0 < p < 1$

**Definition 1.3.5** (Bernoulli Random Variable). Let  $S$  be a sample space and  $X : S \rightarrow 0, 1 \in \mathbb{R}$ , Let  $0 < p < 1$ , then define the density function

$$f_X : 0, 1 \rightarrow R \quad f_X(x) = \begin{cases} 1 - p & \text{for } x = 0 \\ p & \text{for } x = 1 \end{cases} \quad (21)$$

Then  $X$  is said to be a *Bernoulli random variable* or follow a *Bernoulli distribution* with parameter  $p$ . We indicate this by writing

$$X \sim \text{Bernoulli}(p) \quad (22)$$

### 1.3.3 Independent and Identical Trials

- *independent* means the outcome of a trial will not influence the following trial
- *identical* means each trial has same probability of success

### 1.3.4 Counting Successes in a Saquence of Trials

$$P[x \text{ successes in } n \text{ trials}] = \binom{n}{x} p^x (1-p)^{n-x} \quad (23)$$

### 1.3.5 Binomal Random Variable

**Definition 1.3.6** (Binomal Random Variable).  $S$ : sample space,  $n \in \mathbb{N} \setminus 0$  and

$$X : S \rightarrow \Omega = \{0, \dots, n\} \in \mathbb{R} \quad (24)$$

Let  $0 < p < 1$  and define the density function

$$f_X : \Omega \rightarrow \mathbb{R}, \quad f_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad (25)$$