Contents

1 Signal & Systems (Fundamental)

1.1 Signal Definition

Definition 1.1.1 (Signal)**.** Any "physical" quantity that varies with time or space (or other independetn variables).

Example 1.1 (Ambulance Siren:)**.**

$$s(t) = (1 + t)sin(2\pi[1000t + 10t^2 + 300sin(2\pi t/2)]) \tag{1}$$

- $(1 + t)$: amplitude term represents incresing loudness as ambulance approaches
- $1000t$: represents 1kHz siren oscillatione
- $10t^2$: increasing pitch due to the *Doppler effect* as the ambulance approaches.

- $300sim(2\pi 2t)$: the eeh-ooh-eeh-ohh periodic variation in pitch.

Definition 1.1.2 (Systems)**.** A physical "devicec" that performs an operation on a signal.

Definition 1.1.3 (Signal Processing)**.** Take some input signals and produce some related output signals.

Example 1.2 (audio amplifier)**.**

$$S_{out}(t) = aS_{in}(t) \tag{2}$$

Emphasize on continuous-time of analog signals.

1.2 Classification of Signals

1.2.1 Dimensionality

- By the domain of the function, i.e. how many arguments a function has.
- By the dimension of the range of the function, i.e. the space of values the funciton can take.

1.2.2 Time characteristics

Definition 1.2.1 (Continuous-time Signal)**.** A function defined for all times $t \in (-\infty, \infty)$, or at least some interval (a, b).

Classify signals by time characteristics

- Continuous-11:10 signals or analog signals
- Discrete-time signals

1.2.3 Value Characteristics

Definition 1.2.2 (continuous-valued signal or continuous-amplitude signal)**.** Can take any value in some continuous interval.

Definition 1.2.3 (discrete-valued signal or discrete-amplitude signal)**.** Only takes values from a discrete set of possible values.

1.2.4 Determinstic vs Random Signals

Definition 1.2.4 (Determinstic signals)**.** Can be described by an explicit mathematical representation.

Definition 1.2.5 (Random signals)**.** Evolve over time in an unpredictable manner.

1.3 Transformation of CT Signals

1.3.1 Transformations

- Time transformations
  - Folding/reflecting/time-reversal
$$y(t) = x(-t) \tag{3}$$
  - Time-scaling
$$y(t) = x(at) \tag{4}$$
  - Time-shifting
$$y(t) = x(t - t_0) \tag{5}$$
  - General time transformations
    Involves all three of the above time transformations.
$$y(t) = x(at - b) = x(\frac{t - t_0}{w}) \tag{6}$$
    where $t_0 = b/a$, $w = 1/a$
- Amplitude transformations
  - reverse
$$y(t) = -x(t) \tag{7}$$
  - scaling
$$y(t) = ax(t) \tag{8}$$
  - shifting
$$y(t) = x(t) + b \tag{9}$$
- Differentiator
$$y(t) = \frac{d}{dt}x(t) \tag{10}$$
  Example 1.3.
$$y(t) = -RC\frac{d}{dt}x(t) \tag{11}$$
- Integrator
$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau \tag{12}$$
  Example 1.4.
$$y(t) = -\frac{1}{RC}\int_{-\infty}^{t} x(\tau)d\tau \tag{13}$$
- Operation with two signals
  Sum or product at any point

1.4 Signal Characteristic

1.4.1 Periodic signals

$$x(t+T) = x(t) \forall t \tag{14}$$

if no T exists, called aperiodic.

Definition 1.4.1 (Fundamental Period)**.** Smallest T_0 of T.

Theorem 1.4.1. With period $T > 0$,

$$x(t+nT) = x(t) \tag{15}$$

Sum of two periodic signals
Suppose a value $T > 0$ satisfies $T = n_1 T_1$ and $T = n_2 T_2$
then, x(t) is periodic with period T.

Theorem 1.4.2. A sum of two periodic signals is period iff the ratio od their periods is rational.

1.4.2 Even and Odd Symmetry

Definition 1.4.2 (Even Symmetry)**.** iff $x(-t) = x(t) \forall t$

Definition 1.4.3 (Odd Symmetry)**.** iff $x(-t) = x(t) \forall t$

Even and Odd Components
We can decompose any signal into even and odd components:

$$x(t) = x_e(t) + x_o(t) \tag{16}$$

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)], \quad x_o(t) = \frac{1}{2}[x(t) - x(-t)] \tag{17}$$

1.4.3 Average value and energy

Definition 1.4.4 (Average Value)**.**

$$A = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) dt \tag{18}$$

Definition 1.4.5 (Energy)**.**

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \tag{19}$$

Definition 1.4.6 (Average Power)**.**

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \tag{20}$$

Definition 1.4.7 (Energy Signal)**.** If E is finite, then $x(t)$ is called energy signal and $P = 0$.

Definition 1.4.8 (Power Signal)**.** If E is infinite and P is finite and nonzero, $x(t)$ is called power signal.

1.5 Exponential signals

To be done

1.6 Sigularity functions

1.6.1 Transformed rect functions

$rect(\frac{t-t_0}{T})$ centered ar t_0 with width T.

1.6.2 Unit impulse function

$\delta(t)$, area is 1 and width is 0
relationships: $\delta(t) = \frac{d}{dt}u(t),\ u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$
scaling property: $\delta(at+b) = \frac{1}{|a|}\delta(t+b/a)$

1.6.3 Practical impulse function

$$\delta_\Delta(t) = \begin{cases} \dfrac{1}{\Delta}, & 0 < t < \Delta \\ 0, & otherwise \end{cases} \tag{21}$$

width approaches zero as $\Delta \to 0$; height approaches infinity as $\Delta \to 0$

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t) \tag{22}$$

1.7 Continuous-time system

Definition 1.7.1 (Continuous-time(CT) System)**.** a device that transforms input CT signal into another output CT signal.

$$y(\cdot) = \mathcal{T}(x(\cdot)) \tag{23}$$

Definition 1.7.2 (Input-output Relationship)**.** Precisely defines how the output signal is related to the input signal.

- Series connection

$$x(t) \to \boxed{\mathcal{T}_1} \to \boxed{\mathcal{T}_2} \to y(t) \tag{24}$$

$$y(t) = \mathcal{T}_2[\mathcal{T}_1[x(t)]] \tag{25}$$

- Parallel connection

$$y(t) = \mathcal{T}_1[x(t)] + \mathcal{T}_2[x(t)] \tag{26}$$

1.7.1 Classification of CT Systems

- Amplitude properties
  - A-1 linearity

$$\mathcal{T}[a_1x_1(t) + a_2x_2(t)] = a_1\mathcal{T}[x_1(t)] + a_2\mathcal{T}[x_2(t)] \tag{27}$$

    Property

$$\mathcal{T}[ax(t)] = a\mathcal{T}[x(t)] \tag{28}$$

    superpostion property

$$\mathcal{T}[\sum_{k=1}^{K}\mathcal{T}[x_k(t)]] \tag{29}$$

$$\mathcal{T}[\int x(t;v)dv] = \int \mathcal{T}[x(t;v)]dv \tag{30}$$

  - A-2 stability
    Satisfy BIBO

    Definition 1.7.3 (Bounded-input Bounded-output (BIBO) stable)**.** Every bounded input produces a bounded output

    Triangle Inquality

$$|\sum_{n} a_n| \leq \sum_{n} |a_n| \tag{31}$$

  - invertibility

    Definition 1.7.4 (invertible)**.** each output signal is the response to only one input signal.

    Property

$$\mathcal{T}^{-1}[\mathcal{T}[x(t)]] = x(t) \tag{32}$$

- Time properties
  - T-1 causality

    Definition 1.7.5 (Casual System)**.** The output $y(t)$ at time t only depends on the present and (possibly) past inputs, not on future inputs.

    Noncasual systems arise often when t is other variables than time, such as space.

  - T-2 memory

Definition 1.7.6 (Static system or Memoryless System)**.** The output $y(t)$ at time t only depends on the current input $x(t)$.
Otherwise is a dynamic system.

– T-3 time-invariance Systems whose input-output behavior does not change with time

Definition 1.7.7 (Time Invariant)**.**

$$x(t) \xrightarrow{\mathcal{T}} y(t) \quad \textit{implies that} \quad x(t-t_0) \xrightarrow{\mathcal{T}} y(t-t_0) \tag{33}$$

2 CT LTI Systems

2.1 Introduction

Primary forcus: **CT linear-time-variant (LTI)** systems. Overview:

$$x(t) \rightarrow \boxed{LTI\ with\ imulse\ response\ h(t)} \rightarrow y(t) = x(t) * h(t) \tag{34}$$

where $\delta(t) \xrightarrow{\mathcal{T}} h(t)$
Input-output relationships (given by convolution intergral):

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{35}$$

- Any system whose input and output can be discribed by the above form is LTI system
- An LTI system is completely described by its impulse response $h(t)$
- We can determine the response $y(t)$ due to any input signal with impulse response.

2.2 LTI system properties

2.2.1 Properties of Convolution and Impulse Functions

- commutative property

$$x(t) * h(t) = h(t) * x(t) \tag{36}$$

- associative property

$$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)] \tag{37}$$

- distributive property
-

$$x(t) * \delta(t) = x(t) \tag{38}$$

- delay property
$$x(t) * \delta(t - t_0) = x(t - t_0) \tag{39}$$
- If $y(t) = x(t) * h(t)$, then $x(t - t_0) * h(t - t_1) = y(t - t_0 - t_1)$

there are four remaining properties in terms of $h(t)$.

- T-1 casusality
$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau + \int_{-\infty}^{0^-} h(\tau)x(t-\tau)d\tau \end{aligned}$$

  Definition 2.2.1 (LTI causal)**.** An LTI system is causal iff its impulse response $h(t) = 0$ for all $t < 0$.

  then using $\tau' = t - \tau$
$$y(t) = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{t} x(\tau')h(t-\tau')d\tau' \tag{40}$$

- T-2 memory

  Definition 2.2.2 (LTI memoyless)**.** Iff its impulse response is $h(t) = a\delta(t)$. Otherwise is dynamic.

  In this case, the response is $y(t) = ax(t)$
  There are two classes of dynamic systems:

  –

    Definition 2.2.3 (finite impulse response (FIR))**.** has$h(t)$ tha is nonzero only within some finite interval $t_1 < t < t_2$.

  –

    Definition 2.2.4 (infinite impulse response (IIR))**.** has $h(t)$ that persists indefinitely.

- A-2 stability Suppose $x(t)$ is a bounded input signal $|x(t)| \leq M_x < \infty \ \ \forall t$.
$$\begin{aligned} |y(t)| &= |\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau| \\ &\leq \int_{-\infty}^{\infty} |h(\tau)x(t-\tau)|d\tau \quad (\textit{trangle inequality}) \\ &= \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)|d\tau \\ &\leq M_x \int_{-\infty}^{\infty} |h(\tau)|d\tau \end{aligned}$$

So, sufficient condition:

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty \tag{41}$$

Definition 2.2.5 (LTI BIBO stable)**.** Iff its imppluse response is absolutely integrable, i.e. $\int_{-\infty}^{\infty} |h(t)| dt < \infty$

- A-3 invertibility Fact: if a system is LTI, then if it is also invertible, the inverse system is also LTI.

$$x(t) \to \boxed{LTI\ h(t)} \to y(t) \to \boxed{LTI h_i(t)} \to z(t) = x(t) \tag{42}$$

The cascade of two LTI system is also LTI.

Definition 2.2.6 (LTI invertible)**.** If the system is invertible, then

$$h(t) * h_i(t) = \delta(t) \tag{43}$$

2.3 Step response

A way to find the impluse response $h(t)$ of an LTI system in practice.