FROM BEREZIN INTEGRAL TO BATALIN–VILKOVISKY FORMALISM: A MATHEMATICAL PHYSICIST'S POINT OF VIEW

ANDREI LOSEV

Institute of Theoretical and Experimental Physics,
B. Cheremushkinskaya 25, 117259, Moscow, Russia
e-mail: losev@itep.ru

Within the path integral approach quantum field theory can be formulated as an integral over the superspace of fields. The Berezin integral is the crucial element in this construction. Replacing an infinite-dimensional supermanifold of fields by an ordinary supermanifold we get a nice toy model of quantum field theory. In this article I motivate the origin of the Berezin integral and trace some of its applications, up to the Batalin–Vilkovisky formalism. The latter was developed as a method for determining the ghost structure in such theories as gravity and supergravity, whose Hamiltonian formalism has constraints not related to a Lie algebra action.

1. Instead of Introduction

1.1. *Mathematical physics as a style of theoretical physics*

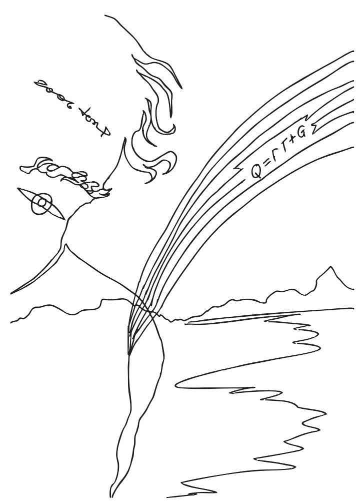
You may like it or hate it, but during the last 40 years the shape of what was previously called theoretical physics has changed considerably. It has diversified in styles.

In the good old days, theoretizing was like sailing between islands of experimental evidence. And, if the trip was not in the vicinity of the shoreline (that was strongly recommended for safety reasons) sailors were continuously looking forward, hoping to see land — the sooner the better. Intellectual gamblers trying to cross the sea were mostly disappointed, lost and forgotten; lucky survivors were proclaimed geniuses, but — do not even think of doing it at home — no one would recommend this path for his friends or relatives.

Nowadays, some theoretical physicists (let us call them sailors) found a way to survive and navigate in the open sea of pure theoretical constructions. Instead of the horizon, they look at the stars,[a] which tell them exactly where they are. Sailors are aware of the fact that the stars will never tell them where the new land is, but they may tell them their position on the globe. In this way sailors — all together — are making a map that will at the end facilitate navigation in the sea and will help to discover a new land.

Theoreticians become *sailors* simply because they just like it. Young people seduced by captains forming crews to go to a Nuevo El Dorado of Unified Quantum Field Theory or Quantum Gravity soon realize that they will spend all their life at sea. Those who do not like sailing desert the voyage, but for true potential sailors the sea becomes their passion. They will probably tell the alluring and frightening truth to their students — and the proper people will join their ranks.

These sailors are known as mathematical physicists; they form one of the styles of modern theoretical physics. The author of this text considers himself to be one of them.



[a]Here by "stars" I mean internal logic organizing the mathematical world, rather than some outstanding members of the community.

Below I will explain my point of view on the *Berezin integral*, a truly remarkable navigating device. The reader may consider this an example of how the mathematical physicist approaches the world, or just share with the author admiration for the unexpected effectiveness of the Berezin integral in mathematical physics. Felix Berezin presented his "integral calculus" in anticommuting variables in his first book *Method of Second Quantization*, published in Russian in 1965 and translated into English in 1966 (see Appendix), which became a cherished handbook for generations of mathematical physicists.

I think that mathematical physics should study all possible quantum field theories (QFTs), by all available tools and methods. At first glance this is an incredibly hard problem, since one can hardly describe all possible QFTs, let alone solve them. However, they are partially ordered by their complexity.

Compare it to geometry which studies all possible spaces. The set of all spaces is intractable; however, the simplest spaces, such as a point, a line or a circle, are easy to imagine and understand. In real dimension 2 we can classify all smooth oriented spaces by genus. In higher dimensions things become increasingly more complicated.

I think that the main task of a mathematical physicist is to study the universal phenomena of QFT as functions on the space of all possible QFTs. In the simplest QFTs most phenomena are tautologies, like $0 = 0$. As complexity increases, interesting phenomena start showing up, at first in their simplest form. At this stage they are tame and their exhaustive quantitative description may be obtained. Increasing complexity makes typical phenomena described by the given QFT yet more complicated, and only an overall qualitative picture survives.

Thus, I assume that understanding the phenomena in QFT is equivalent to finding the simplest (threshold) QFT where nontrivial phenomena "start to fly."

1.2. *Outline of the paper*

One of the approaches to general QFT is the functional integral. It has to be taken over the superspace of fields. It means that some of these fields are fermionic (odd in mathematical terms). These fields include not only spinors corresponding to matter fields, such as electrons, but also ghost fields of gauge symmetries. There are "zero-dimensional" QFTs where the infinite-dimensional space of fields is replaced just by an ordinary finite-dimensional space, and the infinite-dimensional superspace of fields by a finite-dimensional superspace. The functional integral is then replaced by an ordinary integral. However, a part of quantum nature of the "theory" survives in a dependence on $\hbar$ that can be implemented even in this oversimplified model.

In what follows we will discuss the Berezin superspace integral, compare it to other integrals known in mathematics and re-express them in terms of the Berezin integral. We will see how constrained systems can be treated in terms of superspace integrals. The Berezin integral paves the way to understanding the Batalin–Vilkovisky (BV) formalism [1, 2] in QFT. We will show how this formalism arises, as well as some of its simplest applications. We conclude by discussing the gauge systems in the BV language, the geometrical meaning of the Faddeev–Popov ghosts and show how the BV formalism opens the way for generalizing the notion of symmetrical systems.

2. Motivations for and definition of the Berezin integral

2.1. *Geometry, algebra and supergeometry — an outline*

Suppose we have a manifold. Consider the space of smooth, or analytic, or algebraic real (or complex)-valued functions on it. This space forms a commutative ring, where for any point P and functions f_1, f_2 we set

$$(f_1 \cdot f_2)(P) = f_1(P) \cdot f_2(P)\,. \qquad (1)$$

We consider an ideal in this ring formed by functions which vanish on a given subspace. These ideals are partially ordered with respect to inclusion — the smaller the subspace, the bigger the ideal. In particular, the maximal ideals correspond to the smallest subspaces

— points.

Given a ring R we may try to form a space such that functions on this space form the original ring. Let us think that the "points" are maximal ideals I_P. The cosets R/I_P are isomorphic to fields (usually, real or complex numbers or some Galois fields), so the "value" of the element of the ring at a "point" is its image in R/I_P.

A crazy idea that proved to be rather fruitful is to generalize the notion of the manifold by relaxing conditions on the algebra of its functions. In particular, we can consider any commutative ring R or any algebra over any field — the corresponding "manifolds" are schemes from algebraic geometry.

We can replace R by any graded commutative ring — in this case we get the so-called superschemes. One can even completely forget the commutativity condition keeping only associativity — this is the scope of the so-called noncommutative geometry.

Here we will mostly focus on supermanifolds corresponding to $\mathbb{Z}_2$ graded commutative rings (GCR). Such rings can be decomposed as

$$GCR = EVEN \oplus ODD\,. \tag{2}$$

Odd elements mutually anticommute while the rest commute.

The simplest supermanifolds are n-dimensional odd spaces. Their rings are generated by even constants and n odd elements ψ^i. One can show that this space has odd "lines," "planes," etc. but only one point.

2.2. *A warm-up exercise: an integral over circle*

From the standard courses on analysis we know that in order to take an integral over a function F over the interval I we can split the interval in a large number of small subintervals I_i on which the function F is approximately constant, $F \approx F_i$, and then sum up these values multiplied by the lengths of the corresponding intervals,

$$\int_I F\,dx \approx \sum_i F_i\,l_i\,. \tag{3}$$

However, such a definition is totally useless for superspace — odd directions have no points — therefore, no intervals. It means that we should try something else: we must find an appropriate property of the integral and consider it as a definition, provided that this property uniquely determines the value of the integral. Let us try, for instance, the famous relation

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)\,. \tag{4}$$

How does it help? Odd directions have no points, and hence no boundary. Therefore, we should consider functions on the circle, i.e. such that $f(a) = f(b)$. Then

$$\int_{S_1} \frac{df}{dx}\,dx = 0\,. \tag{5}$$

It turns out that this relation determines integral over the circle up to a multiplicative constant. Let x be a coordinate on the circle, $x + 2\pi \sim x$, so we can represent each function on the circle as

$$f(x) = c + \sum_{n>0} \left\{a_n \cos(nx) + b_n \sin(nx)\right\}\,. \tag{6}$$

From (5) we conclude that the integral is independent of a_n and b_n! Indeed,

$$\cos(nx) = \frac{d}{dx}\frac{\sin(nx)}{n}\,, \qquad \sin(nx) = \frac{d}{dx}\frac{-\cos(nx)}{n}\,. \tag{7}$$

Hence, the integral we consider is a linear function in c. We just have to fix normalization. On the circle, the standard normalization is $\int 1 \cdot dx = 2\pi$. This completes the theory of integral over the circle.

Note, that here it is important that we choose the vector field $\partial/\partial X$, rather than $v(X)\partial/\partial X$ with some nonconstant $v(X)$. In the latter case we would get a different answer!

Indeed, in general what we must define is a set of vector fields (first-order differential operators)

$$D_a = v_a^i(X)\frac{\partial}{\partial X^i}$$

such that

$$\int_M \mu\, D_a f = 0\,. \tag{8}$$

Note, that these differential operators form a Lie algebra which can be understood as a Lie algebra of the Lie group of transformations of the space M preserving the measure μ.

2.3. *Definition of the Berezin integral*

The great insight of Felix Berezin was that odd directions may be treated in exactly the same way! The only thing to know is the derivative, and this operation is purely algebraic,

$$f(\psi+\epsilon) = f(\psi) + \varepsilon \, \frac{df}{d\psi}.$$

The expansion in ε stops here since, being odd, there is no ε^2. Higher terms are absent too.

Berezin proposed the following definition (here $\mathcal{D}\psi$ is a measure to be defined):

$$\int_{\text{Berezin}} \mathcal{D}\psi \, \frac{\partial f}{\partial \psi} = 0\,. \tag{9}$$

Consider the space of functions of a single odd variable ψ. This space is just two-dimensional since any function of ψ has the form

$$f(\psi) = c_0 + \psi c_1\,.$$

Since

$$c_0 = \frac{\partial(\psi\, c_0)}{\partial\psi}$$

we conclude that the Berezin integral is independent on c_0. It is proportional to c_1. A nice pick for the normalizing factor is to take the proportionality coefficient 1.

To complete the definition we must define the *multi-dimensional* Berezin integral. This can be done by either generalizing the argument above with the derivative, or demanding the integral over the product of two functions to factorize. Both approaches give the same result. For instance, for two-dimensional odd integral we have

$$\int_{\text{Berezin}} \mathcal{D}\psi_1\, \mathcal{D}\psi_2\, (a + \psi_1\, b_1 + \psi_2\, b_2 + \psi_1\psi_2\, c) = c\,. \tag{10}$$

It corresponds to the Abelian algebra of measure-preserving operators generated by

$$\frac{\partial}{\partial\psi_1} \quad \text{and} \quad \frac{\partial}{\partial\psi_2}\,.$$

The surprising feature of the Berezin integral that follows from its definition is the fact that

$$\int_{\text{Berezin}} \mathcal{D}(c\psi)\, f = \frac{1}{c}\int_{\text{Berezin}} \mathcal{D}\psi\, f\,. \tag{11}$$

This property contrasts a naive expectation that the symbol

$$\int_{\text{Berezin}} \mathcal{D}\psi$$

can be considered as an integral over the differential form $d\psi$ over some virtual space. Indeed, since $d\,(c\psi) = cd\psi$ we would get then c rather than $\frac{1}{c}$ on the right-hand side of Eq. (11).

Thus, the Berezin integral is *not* an integral over a differential top form, as one might naively and mistakenly guess. We will return to this point in Sect. 3.

3. Differential forms as functions on the superspace and their integration

In Sect. 2 we discussed the integral as a linear operation on functions. There is another notion of integrals in classical mathematics, the integral over a differential form over a manifold.

Consider a space M and the space $Fun(M)$ of real-valued functions on M. We may think of M as of the space $Points(M)$, which is the space of points on M. Moreover, we have a *pairing* which takes a point and a function to the value of the function at the given point,

$$Points(M) \times Fun(M) \to \mathbb{R}; \qquad \langle P, f\rangle = f(P)\,. \tag{12}$$

In this pairing the space of functions is a vector space while the space of points is not. We promote it to the vector space of formal linear combination of points. Then the pairing turns out to be bilinear pairing between the vector spaces.

We can consider points as zero-dimensional submanifolds, and look for possible pairings with oriented submanifolds of higher dimension. It turns out that differential forms generalize functions. For example, let us construct an object dual to one-dimensional manifolds and additive with respect to cutting this manifold into pieces. Such object is completely determined by its value on a small interval attached to the point P, i.e. by its value on the tangent vector at the point P.

In fact, this is the definition (up to a smoothness condition) of a differential 1-form. We denote the space of differential 1-forms by $\Omega^1(M)$. By the same token, functions on the surface elements can be shown to be functions on antisymmetrized pairs of tangent vectors at the given point, i.e., 2-forms which span the space denoted $\Omega^2(M)$, and so on.

In this approach, differential forms come equipped with the notion of integral. Let k-chains be the elements of the vector space of formal linear combinations of k-dimensional oriented submanifolds (-1 times a manifold means the manifold with the opposite orientation). There is a pairing

$$\text{(The space of } k\text{-chains)} \times \Omega^k(M) \to \mathbb{R}; \quad \langle C, \omega \rangle = \int_C \omega\,. \tag{13}$$

Note that for k-chains the notion of the boundary ∂ is defined. The boundary of a given k-chain is the $k-1$-chain which is the sum of the boundaries of each summand constituting the chain. Even if we start from a chain which is a single submanifold, its boundary can be a linear combination of submanifolds, i.e., a chain. The simplest example is the oriented interval whose boundary is an end point minus the starting point of the interval.

Now, we can *define* the de Rham exterior differential d as an operation on the differential forms dual to the operator ∂ of taking the boundary,

$$\int_{\partial C} \omega = \int_C d\omega\,. \tag{14}$$

From this point of view d is something that has to be computed from the relation (14).[b]

People advocating this point of view stress the geometrical nature of the integration procedure, while Berezin's approach seems to be algebraic. It seems clear that it is impossible to apply the geometric approach to superspaces since the very notion of superspace is not geometrical.

However, the geometric approach can be easily expressed in terms of the Berezin integral. Indeed, consider the superspace $\Pi(T)M$, which is the tangent bundle to the space M with the reversed parity of the fibers. In simple terms, the tangent bundle TM can be coordinatized by coordinates X^i_α on the base M and coordinates t^i_α along the tangent plane corresponding to the patch U_α on M. On the intersection of patches we have

$$X^i_\alpha = f^i_{\alpha\beta}(X_\beta)\,; \qquad t^i_\alpha = \frac{\partial f^i_{\alpha\beta}}{\partial X^j_\beta}\, t^j_\beta\,. \tag{15}$$

Functions on the space TM correspond just to a set of functions $F_\alpha(X_\alpha, t_\alpha)$ from U_α to $\mathbb{R}$ that are invariant under the transformations (15),

$$F_\alpha(X_\alpha(X_\beta), t_\alpha(X_\beta, t_\beta)) = F_\beta(X_\beta, t_\beta)\,. \tag{16}$$

Expanding in t we see that such functions are symmetric covariant tensors.

The space $\Pi(T)M$ also corresponds to the manifold M covered with the patches and local coordinates X_α and ψ_α. The coordinate transformations are the same,

$$X^i_\alpha = f^i_{\alpha\beta}(X_\beta); \qquad \psi^i_\alpha = \frac{\partial f^i_{\alpha\beta}}{\partial X^j_\beta}\, \psi^j_\beta\,, \tag{17}$$

[b]It is possible, however, to reverse the logic and give an alternative definition of the integral provided the space of differential forms and the exterior differential d are *given*. In this case, the integral over a compact k-dimensional manifold is defined (up to a constant factor) as a linear map from the space of k-forms to reals, vanishing on the image of d.

but now ψ's anticommute. Therefore, the functions on $\Pi(T)M$ correspond to covariant antisymmetric tensors, i.e., differential forms.

From our discussion of the geometrical integral we may anticipate the canonical measure on $\Pi(T)M$. It should correspond to the measure defined on each patch which is invariant under the transformations (17). Such a measure exists indeed and is sometimes called the *canonical Berezin measure*. It has the form

$$\mu_{\mathrm{Ber,\,can}} = \prod_i dX^i_\alpha \prod_i \mathcal{D}\psi^i_\alpha \,. \tag{18}$$

Note, that the first (bosonic) factor in (18) is constructed with the help of the local top differential form, while the second has its distinct meaning, defined above. It is not the superdifferential form.

In order to illustrate how things work consider a one-dimensional case. Then the measure is

$$dX_\alpha \mathcal{D}\psi_\alpha \,,$$

and under the transformations $X_\alpha = cX_\beta, \;\; \psi_\alpha = c\psi_\beta$

$$dX_\alpha = c dX_\beta \,, \quad \text{while} \quad \mathcal{D}\psi_\alpha = \frac{1}{c}\mathcal{D}\psi_\alpha \,. \tag{19}$$

What about the de Rham operator? When we consider differential forms as functions, the de Rham operator becomes just the vector field. Ona can show that this vector field preserves the canonical Berezin measure.

4. The Hodge star operation as an odd Fourier transform

The simplest application of the Berezin integral is the interpretation of the Hodge *-operation. In particular, we have to interpret the property

$$*^2 = (-1)^{\mathrm{deg}} \,,$$

where deg is the degree of the differential form. The Fourier transformation has the same property, and it turns out that the

Hodge star is in fact an odd Fourier transformation. Namely,

$$\tilde{f}(X,\tilde{\psi}) = C(X) \int \mathcal{D}\psi_1 \dots \mathcal{D}\psi_n \exp\left(g_{ij}(X)\tilde{\psi}^i \psi^j \right) f(X,\psi)\,. \quad (20)$$

Note, that if we apply this transformation twice, we will get

$$\begin{aligned} \tilde{\tilde{f}}(\tilde{\tilde{\psi}}) &= C^2 \int \mathcal{D}\psi_1 \dots \mathcal{D}\psi_n \\ &\quad \times \mathcal{D}\tilde{\psi}_1 \dots \mathcal{D}\tilde{\psi}_n \, \exp\left(g_{ij}\tilde{\psi}^i \left(\psi^j + \tilde{\tilde{\psi}}^j \right) \right) f(\psi) \\ &= C^2 \det g_{ij}\, f(-\tilde{\tilde{\psi}}) = C^2 \det g_{ij}\, (-1)^{\deg} f(\tilde{\tilde{\psi}})\,. \end{aligned} \quad (21)$$

That is why it is natural to take

$$C = (\det g_{ij})^{-\frac{1}{2}}\,.$$

5. The Mathai–Quillen representative and supermanifolds as fractions

The space of fields in QFT may be modeled by a manifold. There are two classical ways of constructing new manifolds from the old ones: to take the coset with respect to the action of a group (which will be referred to as the gauge group), and consider zeroes of a function; the procedure is called writing a (system of) *equation*(s) (in mathematics) or imposing a (set of) *constraint*(s) (in physics). The question is how to describe an integral over the new manifold in terms of the integral over the old one. We will return to cosets after we discuss the Batalin–Vilkovisky formalism, while constraints can be considered right here. We will study the simplest possible example, the integral over the zeroes of the function f,

$$\int_{C\in M} \omega = \int_M \delta_C \omega\,, \qquad C = \{p \mid f(p) = 0\}\,. \quad (22)$$

We will present and interpret the integral representation for the δ function in terms of the supermanifolds and the Berezin integral (in the physical literature see Ref. [3]).

The first approximation is to consider a regularized δ function

$$\delta^{\mathrm{naive},m} = \exp\left(-m^2 f^2\right) m \,.$$

Indeed, this function is concentrated along the zeros of f and its integral is independent of m as $m \longrightarrow \infty$. However, it would be an incorrect answer since δ has to be a 1-form rather than a function (in order to match dimensions of the forms that have to be integrated over C and M). Moreover, the integral of $\delta^{\mathrm{naive},m}$ changes as $f \longrightarrow c\, f$ while the set of zeroes does not. These two problems are cured by setting

$$\delta^m = \frac{1}{\sqrt{\pi}} \exp\left(-m^2 f^2\right) m\, df \,. \tag{23}$$

We start from uplifting mdf to the exponent. Namely, we introduce an odd variable λ' such that

$$\delta^m = \frac{1}{\sqrt{\pi}} \int \mathcal{D}\lambda' \exp\left(-m^2 f^2 + \lambda' m df\right) \,. \tag{24}$$

The above expression is correct; however, its geometric meaning is unclear. The terms in the exponent look very different, while purely odd integral is a bit strange and the large-m limit is hard to take.

This can be cured by introducing an auxiliary even variable l', a Lagrange multiplier, so that we arrive at

$$\delta^m = \frac{1}{2\pi} \int \mathcal{D}\lambda'\, dl' \exp\left(il' m f + \lambda' m df - \frac{{l'}^2}{4}\right) \,. \tag{25}$$

Introducing new variables l and λ,

$$2\pi m l = m l' \,, \quad \lambda = m\lambda'$$

we get the final expression

$$\delta^m = \int \mathcal{D}\lambda\, dl \exp\left(2\pi i l f + \lambda df - \frac{(l\pi)^2}{m^2}\right) \,. \tag{26}$$

Now we have the integral over the canonical Berezin measure. Moreover, the expression in the exponent has a naive limit as $m \to \infty$. We will see that this limit has a nice and surprising geometric meaning. If we interpret df as a function on the superspace $\Pi(T)M$, with the coordinates X and ψ, then expression in the exponent has the form

$$d^{\rm super}(f\lambda) = \left(\psi^j \frac{\partial}{\partial X^j} + 2\pi i l \frac{\partial}{\partial \lambda}\right)(f\lambda)\,. \tag{27}$$

The new operator $d^{\rm super}$ may be considered as a de Rham operator on a superspace $M^{\rm super}$ with the even coordinates X^i and odd coordinate λ. All this allows us to rewrite the original integral as follows:

$$\int_{C\in M} \omega = \int_{\Pi(T)M^{\rm super}} \mu_{\rm Ber}\left((\Pi(T)M^{\rm super}\right)\omega \exp\left((-d^{\rm super}(f\lambda)\right)\,. \tag{28}$$

In particular, we can consider supermanifold as something like a *fraction* of the space of variables over the space of equations. The integral (28) is a way to simplify this fraction with the help of the function f. If ω is closed with respect to $d^{\rm super}$, one can show that the result of integration does not depend on the particular choice of f. This can be understood as a manifestation of the fact that different ways to simplify a given "fraction" are equivalent (when applied to "fraction manifolds" with the insertion of a closed form)!

This construction is widely used in topological quantum field theories. In particular, in type A topological sigma models, the space M is an infinite-dimensional space of all maps from the world sheet to the target space, f is replaced by the infinite-dimensional space of holomorphicity equations, the cycle C stands for holomorphic maps — instantons. The ψ and λ fermions are just twisted fermions of the standard supersymmetric sigma model, $d^{\rm super}$ is what is called scalar nilpotent Q symmetry, and the parameter m is often referred to as the $\bar{t}$ coupling constant.

6. The BV master equation as an odd Fourier transform of the de Rham equation

As we will see shortly, the Batalin–Vilkovisky formalism — the most powerful formalism for treating symmetries in (functional) integrals — can be understood in terms of the odd Fourier transformation for the theory of the de Rham cohomology [4] (after Georges de Rham).

Recall that this cohomology, intended for studying integrals over the closed submanifolds up to their deformations, is defined as the quotient space of closed forms ω modulo the image of the exterior differential,

$$d\omega = 0, \qquad \omega \sim \omega + d\nu\,. \tag{29}$$

One of the most important operations on the de Rham cohomologies is the procedure of integration over the compact fiber. Namely, consider the manifold M which is a family of compact fibers F parametrized by the base B. Given a closed form Ω on M one produces a closed form on the base B by integrating it over the compact fiber,

$$\Omega_B = \int_F \Omega\,. \tag{30}$$

One can check that the integral over the exact forms produces exact forms. This follows from the decomposition of the total differential as a sum of differentials over the fiber and the base,

$$d_M = d_F + d_B\,, \tag{31}$$

and

$$\int_F d_F\nu = 0\,, \tag{32}$$

due to the compactness of the fiber. It means that we actually have a map from the cohomology classes of the total space to the cohomology classes of the base.

The BV formalism is an equation on the $\Pi(T^*)M$ which is a cotangent bundle to M with reversed parity of the fibers. It provides equations on functions in X^i and X^*_i, where X^*_i is the odd variable which is the coordinate not on the tangent but rather on the cotangent bundle with reversed parity.

Such functions naturally appear if we perform the odd Fourier transformation over the fiber (as in the theory of the Hodge duality). Now we have no metric to identify the tangent and cotangent directions. Moreover, in performing this Fourier transformation we need a measure on the fiber, which will be denoted as Ω,

$$g(X^*, X) = \int \Omega(X)\, \mathcal{D}\psi^1 \dots \mathcal{D}\psi^n\, f(\psi, X)\,. \tag{33}$$

Under this transformation the de Rham operator considered as a first-order differential operator on the superspace transforms into the following second-order operator:

$$\Delta = \frac{\partial}{\partial X^i} \frac{\partial}{\partial X^*_i}\,. \tag{34}$$

The "closedness" conditions turn into pre-BV equations

$$\Delta g(X^*, X) = 0\,. \tag{35}$$

The operation of integration over the fiber becomes the integration over the Lagrangian submanifold in the fiber corresponding to $X^* = 0$.

The second step in the BV formalism is to consider special functions g of the form

$$g(X^*, X) = \exp\left(\frac{1}{\hbar} S(X^*, X, \hbar)\right), \tag{36}$$

where S is regular at $\hbar = 0$.

Equation (35) leads to the equation for S which is referred to as the *master* equation,

$$\{S, S\}_{\mathrm{BV}} + \hbar \Delta S = 0\,, \tag{37}$$

where the BV bracket is defined as

$$\begin{aligned} \{g_1, g_2\}_{\mathrm{BV}} &= \Delta\left(g_1 g_2\right) - g_1 \Delta g_2 - \left(\Delta g_1\right) g_2 \\ &= \frac{\partial g_1}{\partial X^i} \frac{\partial g_2}{\partial X^*_i} - (1 \leftrightarrow 2)\,. \end{aligned} \tag{38}$$

The BV-bracket was known in mathematics as the Schouten bracket. It generalizes the well-known Lie bracket (the commutator of the vector fields) to the polyvector fields (the polyvector fields are nothing but the contravariant antisymmetric tensors). In particular,

$$\{f_1, f_2\}_{BV} = 0; \quad \{v^i X_i^*, f\}_{\rm BV} = v^i \partial_i f,$$

$$\{v^i X_i^*, u^j X_j^*\}_{\rm BV} = (v^i \partial_i u^j - u^i \partial_i v^j) X_j^* \,, \tag{39}$$

$$\{\pi^{ij} X_i^* X_j^*,\, \pi^{pq} X_p^* X_q^*\}_{\rm BV} = 4\,\pi^{ij}\,\frac{\partial \pi^{pq}}{\partial X^i} X_j^* X_p^* X_q^* \,. \tag{40}$$

7. An exercise in the BV language

Evaluating the master BV equation (37) at $\hbar = 0$ we get the so-called classical BV equation,

$$\{S(X^*, X, 0), S(X^*, X, 0)\}_{BV} = 0\,. \tag{41}$$

In particular,

$$S = \pi^{ij} X_i^* X_j^*$$

solves the classical BV equation if the bivector π^{ij} defines a Poisson bracket. Indeed, if we study the Jacobi equation for the bracket given by the bivector we will obtain the right-hand side of (40).

It is known that for invertible π the Poisson equation is equivalent to the “closedness” of the 2-form,

$$\omega_{ij} = (\pi^{-1})_{ij}\,.$$

This well-known fact always seemed to be a miracle since it connects solutions to the system of quadratic equations with solutions to the linear equation!

However, this miracle has a natural explanation from the BV-integral point of view. Indeed, consider an obvious closed form constructed from the closed 2-form ω

$$d\omega = 0 \Rightarrow d \exp\left(\frac{1}{\hbar}\,\omega\right) = 0\,. \tag{42}$$

Now, let us perform an odd Fourier transformation

$$\int \Omega(X)\mathcal{D}\psi^1 \dots \mathcal{D}\psi^n \exp\left(\frac{1}{\hbar}(\omega_{ij}\psi^i\psi^j + X_i^*\psi^i)\right)$$

$$= \exp\left(\frac{1}{\hbar}S_\omega(\hbar)\right) = \exp\left(\frac{1}{\hbar}(\pi^{ij}X_i^*X_j^* + O(\hbar))\right), \qquad (43)$$

where the $O(\hbar)$ correction is the determinant of the matrix ω_{ij}. As we already explained, $S_\omega(\hbar)$ solves the BV equation. Thus, $S_\omega(0)$ solves the classical BV equation. π is the Poisson bivector field. Miracle explained.

8. The BV integral

The main idea of the BV approach is that the only integrals are the *BV integrals*, i.e. those over the exponent of the BV action over Lagrangian manifolds (a d-dimensional subspace of the $2d$-dimensional BV spaces such that the so-called BV symplectic form

$$\delta X^i \wedge \delta X_i^*$$

vanishes on it). The *only property* of the BV integrals is that the result of the BV integration of

$$\exp\left(\frac{1}{\hbar}S(X^*, X, Y^*, Y, \hbar)\right)$$

gives effective actions which satisfy the master equation (37) on the space of parameters Y, Y^* [1,2].

The simplest example is the action $f(X, Y)$ (which does not depend on the antifields X^* and Y^*). We consider the coordinates Y, Y^* as parameters, and $X^* = 0$ as a Lagrangian submanifold. Clearly, the integral is independent of Y^* and therefore solves the master equations.

The second example comes from rethinking of the Berezin integral which is an odd Fourier transform relating the symplectic and Poisson structures. To this end we introduce the BV space with the coordinates

$$X^i, X_i^*,\ \psi^i,\ \psi_i^*$$

with ψ_i^* being the antifield to the field ψ^i. Hence, ψ_i^* has the opposite parity and is thus even. Then the expression in the exponent in Eq. (43) is just another example of the action solving the BV equations.

Indeed, the term $\psi^i X_i^*$ in the BV action corresponds to the vector field $\psi^i \partial/\partial X_i$ which is just a de Rham exterior differential d. In the BV language, the phrase "the differential 2-form ω is close" simply means that the function $\omega_{ij}\psi^i\psi^j$ is d-invariant.

Now we will interpret the Fourier transformation as a BV integral. Let us consider X, X^* as parameters, ψ, ψ^* as coordinates on the BV space, and $\psi^* = 0$ as the equation of a Lagrangian submanifold. Due to the only propety of the BV integral (mentioned above) the effective action $\pi^{ij}X_i^*X_j^*$ solves the BV equations, i.e., it is the Poisson bivector field.

9. The BV formalism, symmetries and gauge fixing

In gauge theories we start from the space of fields with the action of a symmetry group. We consider actions which are the functional of fields invariant under the action of a symmetry group and perform integration over the space of orbits. In the infinite-dimensional case, the standard way to calculate this integral is to pick up representatives on each orbit (we say that they satisfy the gauge fixing condition) and integrate with a very special measure over the space of representatives. A convenient way to study such measure is to calculate the Berezin integral over special odd fields which are usually referred to as *ghosts*.

As we will explain shortly, the BV formalism is perfectly well suited for this. We will see the geometrical meaning of the ghosts even before the gauge fixing.

9.1. *Symmetric systems in the BV language*

Here we will consider the space of QFT fields as some (super)space M. Certain fields will be coordinates X^i on this space, or functions of these coordinates. The classical system is determined by the action $f(X)$ which is a function on M. The classical symmetries are described by vector fields $v_a^i(X)\frac{\partial}{\partial X^i}$. The subscript a labels these fields. We say that a given action f has a (super)Lie algebra with the structure constants C_{ab}^c as its symmetry algebra if

$$v_a^i(X)\frac{\partial f}{\partial X^i} = 0\,, \tag{44}$$

and

$$\left[v_a^i(X)\frac{\partial}{\partial X^i}, v_b^j(X)\frac{\partial}{\partial X^j}\right]_\pm = C_{ab}^c v_c^i(X)\frac{\partial}{\partial X^i}\,, \tag{45}$$

where $[,]_\pm$ stands for the graded commutator. In the BV language the system of equations (44) and (45) has the following interpretation. Consider the superspace

$$\begin{aligned} M_{\mathrm{BV}} &= \Pi(T^*)M \times \mathcal{G}^* \times \Pi(\mathcal{G}) \\ &= \Pi(T^*)\big(M \times \Pi(\mathcal{G})\big) \end{aligned} \tag{46}$$

with the coordinates

$$X^i,\ X_i^*,\ c_a^*,\ c^a\,.$$

Here the parity of the so-called antifield X_i^* is opposite to that of X^i. The space $\Pi(\mathcal{G})$ is the (super)Lie algebra with inverted parity. In particular, for ordinary (even) symmetries, the coordinates on $\Pi(\mathcal{G})$ are odd fields c^a. They are involved in the parametrization of the general symmetry transformation as $c^a v_a^i \partial/\partial X^i$. We will see that these c^a's (originally introduced in the 1950s in the cohomology theory of the Lie algebras) would turn out to be *c-ghosts* (which form a half of the Faddeev–Popov ghosts).

Note, that c_a^* is a coordinate on $\mathcal{G}^*$ which is even for ordinary symmetries; they are antifields to ghosts c, but they are not what people ordinarily call the Faddeev–Popov antighost (we will reserve the special name *b-ghosts* for the latter; we will encounter b-ghosts below).

The system of equations (44) and (45) is equivalent to the statement that

$$S(X, X^*, c, c^*) = f(X) + c^a v_a^i X_i^* + \frac{1}{2}\, C_{ab}^c c^a c^b c_c^* \tag{47}$$

solves the classical BV equation.

9.2. *Gauge fixing as the BV integral*

Suppose we have a symmetric system and we want to fix a gauge. In the BV language it means that we choose a very special Lagrangian manifold, such that the variables X are restricted by the constraint

$$f_\alpha(X) = 0; \quad \alpha = 1, \dots,\ \dim \mathcal{G}\,. \tag{48}$$

Simultaneously we have to change the $X^* = 0$ condition into

$$X^*_i = \frac{\partial f_\alpha}{\partial X^i}\, b^\alpha\,. \tag{49}$$

Here b^α are the coordinates on the Lagrangian submanifold (they have the same parity as X^*, in particular, they are odd when the coordinates X are even). Let us check that Eq. (49) actually represents a Lagrangian submanifold,

$$\delta X^i \wedge \delta X^*_i = \delta X^i \frac{\partial f_\alpha}{\partial X^i} \delta b^\alpha + \delta X^i \wedge \delta X^j \frac{\partial^2 f_\alpha}{\partial X^i\, \partial X^j} = 0\,, \tag{50}$$

where the first term vanishes due to the gauge fixing condition (48), and the second one vanishes due to the contraction between antisymmetric and symmetric tensors.

In the ghost sector we choose the Lagrangian submanifold

$$c^* = 0\,.$$

Realizing the gauge fixing condition (48) with the help of the Lagrange multipliers β^α we can rewrite the integral in the symmetric system as

$$\begin{aligned} &\int \Pi_\alpha\, Db^\alpha\, d\beta^\alpha\, \Pi_a\, Dc^a\, \Omega(X) \\ &\times \exp\left(f(X) + c^a v^i_a \frac{\partial f_\alpha}{\partial X^i} b^\alpha + 2\pi\, i \beta^\alpha f_\alpha \right)\,, \end{aligned} \tag{51}$$

where $\Pi_a\, Dc^a$ is the canonical measure on the Lie algebra with the inversed parity. In Eq. (51) we can easily recognize the standard

Faddeev–Popov formula, where the coordinates b^α become the Faddeev–Popov b-ghosts.

10. Generalizations of classical symmetries

10.1. *Differential graded Lie algebras and homotopical Lie algebras*

Expression (47) is rather inspiring for generalization of the classical notion of symmetry in the framework of the BV formalism. Let us start from the simplest space M, a point. Then we observe that the only solution to the BV equation is $C^c_{ab} c^a c^b c^*_c$. This solution is linear in antighosts but quadratic in ghosts. We can ask what will happen if we also consider terms linear in ghosts. Naively, this cannot happen since the corresponding term seems to be odd. However, if we extend the Lie algebras to super-Lie algebras this is no longer a problem. As a result, we can try

$$S(c, c^*) = \left\{ q^e_a c^a + \frac{1}{2} C^e_{ab} c^a c^b \right\} c^*_e \,. \tag{52}$$

The first term determines a linear operator q on the Lie algebra. The BV equation means that q is a differential, i.e., $q^2 = 0$, and q differentiates the bracket, i.e., there is a Leibniz rule,

$$q([A, B]) = [q(A), B] + (-1)^{\mathrm{par}\, A} [A, q(B)] \,. \tag{53}$$

Such a structure is known as the *differential graded Lie (DGL) algebra*.

We may also include higher terms in c by adding terms

$$C^e_{a_1 \ldots a_k} c^{a_1} \ldots c^{a_k} c^*_e$$

in Eq. (52). Then the BV equations require $q^2 = 0$, and the Leibniz rule and homotopy rules, with the simplest expression

$$C^b_{[a_1 a_2} C^f_{b a_3]} = q^f_d C^d_{a_1 a_2 a_3} - C^f_{[a_1 a_2 b} q^b_{a_3]} \,. \tag{54}$$

Here the square brackets stand for antisymmetrization (which must include additional signs due to parity of the variables). The left-hand side of Eq. (54) is the Jacobi identity, while the right-hand side can be interpreted as a commutator between the differential q and a new $3 \to 1$ operation $C^d_{a_1 a_2 a_3}$. Thus, Eq. (54) means that the "Jacobi identity" considered as the $3 \to 1$ operation is q-exact. Similar relations exist for higher operations. Such an algebra is called the homotopy Lie algebra or L_∞ algebra.

10.2. *Algebroids*

Another generalization of the notion of symmetry takes place when what was previously called structure constants start to depend on X, namely when we consider a solution to BV equations of the form

$$S(X, X^*, c, c^*) = f(X) + c^a v^i_a X^*_i + (1/2) C^c_{ab}(X) c^a c^b c^*_c \,. \tag{55}$$

Such structure is referred to as the *Lie algebroid* [5]. It may look rather weird but mathematical physicists had already encountered it — believe it or not — in supersymmetric gauge theories in the Wess–Zumino gauge! In these theories the anticommutator of two supersymmetries $\{Q_\alpha, Q_\alpha\}$ is a sum of a shift and a gauge transformation with the gauge parameter $\gamma^m A_m$. In the BV language this means that the action (in the Abelian gauge theory, just for simplicity)

$$\begin{aligned} S = S_{\mathrm{SYM}} &+ c v_{\mathrm{gauge}} + \epsilon^\alpha v_{\mathrm{super},\alpha} + \eta^\mu v_{\mathrm{shift},\mu} \\ &+ \frac{1}{2} \left(\eta^*_\mu + c^* A_\mu \right) \left(\epsilon \gamma^\mu \epsilon \right) \end{aligned} \tag{56}$$

solves the BV equations. Here the vector fields (on the space of fields) v_{super}, v_{shift} and v_{gauge} correspond to supersymmetry, shift symmetry and gauge symmetry, respectively, while ϵ, η and c are the corresponding ghosts, A_μ is the Abelian gauge field.

10.3. *Higher terms in antifields and a new notion of symmetry*

In all examples above the action was linear in antifields. Now, consider the case where it has quadratic terms. This can happen if the BV action is effective, i.e., it is obtained after taking a BV integral.

Consider the simplest example: the space $\mathbb{R}^n$, on which $\mathrm{SO}(n)$ with the generators t_{ab} acts, and the invariant function

$$f = -X_1^2 - \ldots - X_n^2 .$$

The BV action takes the following form:

$$S = -\sum_{i=1}^{n} X_i^2 + \sum_{i,j=1}^{n} c_{ij} X_i^* X_j + \sum_{i,j,k=1}^{n} \frac{1}{2} c_{ik}^* c_{ij} c_{jk} . \tag{57}$$

Now, we integrate S over X_n fixing $X_n^* = 0$ and get

$$S_{\text{eff}} = -\sum_{i=1}^{n-1} X_i^2 + \sum_{i,j=1}^{n-1} c_{ij} X_i^* X_j + \sum_{i,j,k=1}^{n} \frac{1}{2} c_{ik}^* c_{ij} c_{jk} - \frac{1}{4} \sum_{i,j=1}^{n-1} c_{in} c_{jn} X_i^* X_j^* . \tag{58}$$

Without the last term the action (58) would imply something non-existing — a nontrivial linear action of $\mathfrak{o}(n)$ on $\mathbb{R}^{n-1}$. However, the last term saves the game. Surprisingly, it generalizes the notion of systems with symmetries.

Indeed, now the symmetry of a system is not only an invariant function and a representation of the Lie algebra of symmetries by vector fields. The new element is a map from $\Lambda^2(\mathfrak{o}(n))$ to the space of *bivectors*,

$$\pi^{ij}_{kn,ln} = \frac{1}{4} \left(\delta^i_k \delta^j_l - \delta^j_k \delta^i_l \right) , \tag{59}$$

such that the vector fields, bivectors, and the symmetric function are mixed up in a single relation [8] (which we will present in the general case),

$$\{\pi_{ab}, f\} + \{v_a, v_b\} = C^c_{ab} v_c . \tag{60}$$

Once again, one may suspect that relations such as (60) are kind of exotic. However, we do meet them in the description of supersymmetric systems (see e.g. [6, 7]). They appear when we integrate out auxiliary fields, as we did in the description of the regularized delta-form. Indeed, the de Rham operator can be considered as a symmetry that squares to zero (with its ghost called ϵ). Therefore, before integrating out the p-fields we have

$$S = 2\pi i l f(X) + \lambda f'(X)\psi - 2\pi^2 l^2 + \epsilon(X^*\psi + 2\pi i\lambda^* l)\,. \tag{61}$$

Integrating I out we then get

$$S = -\frac{1}{2}\, f^2 + \,\lambda\, f'(X)\,\psi + \epsilon\,(X^*\psi - \lambda^* f) - \frac{1}{2}\,\epsilon^2(\lambda^*)^2\,. \tag{62}$$

The last term is a new bivector. Similar things happen in all supersymmetric theories.

11. Instead of conclusion: Dreams on a BV M-theory

M-theory in modern mathematical physics stands for a hypothetical supersymmetric theory with a complicated space of vacua. Near special points in the space of vacua the theory has distinct reincarnations: either as one of several superstring theories or as eleven-dimensional supergravity with a membrane. We would like to understand M-theory as a phenomenon in a certain unified theory: when the unified theory is defined over a complicated moduli space but it looks more familiar on different boundaries. In the example described above we know no detailed description of the unified theory while we understand, to some extent, its various limits.

Looking at the BV actions we observe a similar pattern. At a generic point the BV action is just some function of even and odd fields and antifields, solving the BV equations. These solutions just have no standard geometrical meaning.

However, as we approach the regions where the BV action degenerates (in particular, when it has no linear term), we may assume the linear structure on the space of fields. Then we can try to understand the limiting solution to the BV equation geometrically. Above we have mentioned several possible patterns: constrains, system with symmetries, and so on.

Moreover, the same solution to the BV equation may have different degeneracies, and therefore different geometrical interpretations. In this case, we say that they are in *duality relation* to each other.

Concluding, I would like to emphasize that all these perspectives became possible due to the development of supergeometry. Felix Berezin gave birth to supergeometry and supercalculus, with far reaching consequences for mathematics and physics, with a long-lasting impact.

Acknowledgements

This work was supported in part by grant RFBR 07-02-01161, by INTAS-03-51-6346, NWO project 047.011.2004.026 and NSh-8065.2006.2.

Appendix: Books by F.A. Berezin

• Метод вторичного квантования, Москва, Наука, 1965

• Ф. А. Березин, М. А. Шубин, Лекции по квантовой механике, Москва, Издательство МГУ, 1972

• Лекции по статистической физике, Москва, Издательство МГУ, 1972

• Лекции по статистической физике, 2-е издание, Ижевск, Институт компьютерных исследований, 2002

• Метод вторичного квантования, Москва, Наука, 2-е дополненное издание, 1986

• Метод вторичного квантования (Шедевры мировой физико-математической литературы), Москва, Издательство ИО НФМИ, 2000

• Ф. А. Березин, М. А. Шубин, Уравнение Шредингера, Москва, Издательство МГУ, 1983

• Введение в алгебру и анализ с антикоммутирующими переменными, Москва, Издательство МГУ, 1983

• Method of Second Quantization, New York, Academic Press, 1966

• Lie supergroups, Moscow, ITEP, 1977

• Introduction to Superanalysis, Dordrecht, D. Reidel Publishing Company, 1987

• Introduction to Superanalysis, Berlin, Springer, 2001

• F. A. Berezin, M.A. Shubin, The Schrödinger Equation, Berlin, Springer, 2002

• F. A. Berezin, Lectures on Statistical Physics, Singapore, World Scientific, 2007

Литература

1. I. A. Batalin and G. A. Vilkovisky, *Gauge Algebra And Quantization,* Phys. Lett. B **102**, 27 (1981); *Quantization Of Gauge Theories With Linearly Dependent Generators,* Phys. Rev. D **28**, 2567 (1983), (E) D **30**, 508 (1984); I. A. Batalin and E. S. Fradkin, *A Generalized Canonical Formalism And Quantization Of Reducible Gauge Theories,* Phys. Lett. B **122**, 157 (1983); B. L. Voronov and I. V. Tyutin, *Formulation Of Gauge Theories Of General Form. I,* Theor. Math. Phys. **50** (1982) 218.
2. A. S. Schwarz, *Geometry of Batalin–Vilkovisky quantization,* Commun. Math. Phys. **155**, 249 (1993) [hep-th/9205088]; *Semiclassical Approximation in Batalin–Vilkovisky Formalism,* Commun. Math. Phys. **158** (1993) 373 [hep-th/9210115].
3. S. Cordes, G. Moore, S. Ramgoolam, *Lectures on Two-Dimensional Yang–Mills Theory, Equivariant Cohomology and Topological Field Theories,* Nucl. Phys. Proc. Suppl. **41**, 184 (1995) [hep-th/9411210].
4. R. Bott, L.W. Tu, *Differential Forms in Algebraic Topology,* (Springer-Verlag, Berlin, 1995).
5. K.Mackenzie, *Lie Groupoids and Lie Algebroids in Differential Geometry*, (Cambridge University Press, 1987).
6. M. B. Green and C. M. Hull, *Quantum Mechanics of a Twisted Superparticle,* Nucl. Phys. B **344**, 115 (1990); *The Quantum Mechanics of an* $n+1$ *Superparticle in an Extended Superspace,* Mod. Phys. Lett. A **5**, 1399 (1990); E. Bergshoeff, R. Kallosh and A. Van Proeyen, *Superparticle Actions And Gauge Fixings,* Class. Quant. Grav. **9**, 321 (1992).
7. V. Aleksandrov, D. Krotov, A. Losev, V. Lysov, *A Pure Spinor Superfield Formalism,* in preparation.
8. D. Krotov, A. Losev, V. Lysov, to appear.