A course of Financial Modelling and Pricing Analysis for undergraduates

Lecture 3: A general single-period model

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A general single-period model

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The security market model

- **Uncertainty** is represented by a finite set $\{1, \cdots, S\}$ of states, one of which will be revealed as true. That is, $\Omega = \{1, \cdots, S\}$.
- The N securities are given by an $N \times S$ matrix D, with D_{ij} denoting the number of units of account paid by security i in state j. The **security prices** are given by some q in $\Re^N$.
- **A portfolio** $\theta \in \Re^N$ has **market value** $q \cdot \theta$ and **payoff** $D^T\theta$ in $\Re^S$.

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What are arbitrage and state prices?

Definition (Arbitrage)

An **arbitrage** *is a portfolio θ in $\Re^N$ with $q \cdot \theta \leq 0$ and $D^T\theta > 0$, or $q \cdot \theta < 0$ and $D^T\theta \geq 0$.*

An arbitrage is therefore, in effect, a portfolio offering something for nothing.

Definition (State-Price)

A **state-price** *vector is a vector ψ in $\Re^S_{++}$ with $q = D\psi$.*

We can think of ψ_j as the marginal cost of obtaining an additional unit of account in state j.

The relation between arbitrage and state-price

Theorem

There is no arbitrage $\Longleftrightarrow$ *there is a state-price vector.*

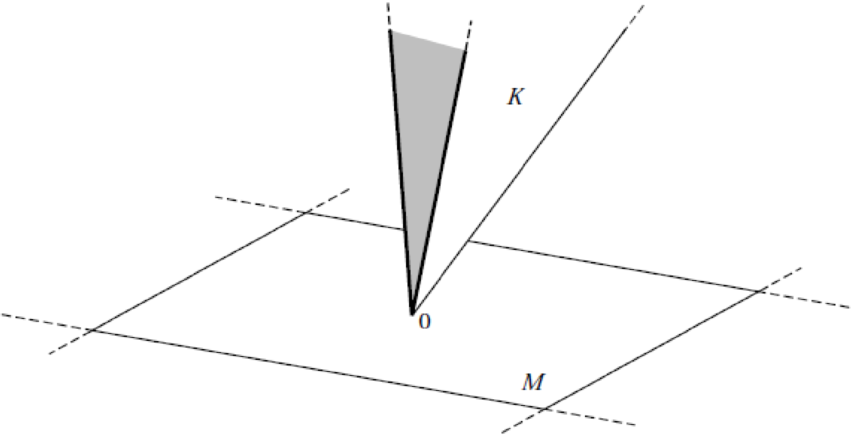

Figure1.1: Separating a Cone from a Linear Subspace

The Proof of the necessity

- Let $\mathcal{L} = \Re \times \Re^S$ and $\mathcal{M} = \{(-q \cdot \theta, D^T\theta) : \theta \in \Re^N\}$. $\mathcal{K} = \Re_+ \times \Re_+^S$. Clearly, No arbitrage $\iff$ $\mathcal{K} \bigcap \mathcal{M} = \{0\}$. The remaining of the proof is finished by an application of the Separating Hyperplane Theorem (SHT).
- Suppose $K \bigcap M = 0$, the SHT implies the existence of a linear functional $F : L \to \Re$, s.t. $F(z) < F(x)$ for all $z \in M$ and all nonzero x in K. This means that $F(z) = 0$ for all $z \in M$ and $F(x) > 0$ for all nonzero x in K, since M is a linear space. The former fact implies that there is some $\alpha > 0$ in $\Re$ and $\psi \gg 0$ in $\Re^S$ s.t. $F(v, c) = \alpha v + \psi \cdot c$, for any $(v, c) \in L$.
- This in turn implies that $-\alpha q \cdot \theta + \psi \cdot (D^T\theta) = 0$ for all $\theta \in \Re^N$. That is, $\alpha q = D\psi$. The vector ψ/α is therefore a state-price vector. $\square$

In-Class Exercise 1

Please produce a proof of the sufficiency of the theorem.

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Risk-neutral probabilities

- We can view any p in $\Re_{+}^{S}$ with $p_1 + \cdots + p_s = 1$ as a vector of probabilities of the corresponding states.
- Given a state-pricing vector ψ for the dividend-price pair (D, q), let $\psi_0 = \psi_1 + \cdots + \psi_s$ and, for any state j, let $\hat{\psi}_j = \psi_j / \psi_0$.
- We now have a vector $(\hat{\psi}_1, \cdots, \hat{\psi}_s)$ of probabilities and can write, for an arbitrary security i,

$$\frac{q_i}{\psi_0} = \hat{\mathbb{E}}(D_i) \equiv \sum_{j=1}^{S} \hat{\psi}_j D_{ij}, i = 1, \cdots, N.$$

- So, the normalized price of the security is its expected payoff under specially chosen risk-neutral probabilities.

The case with a risk-free portfolio

- If there exists a portfolio $\bar{\theta}$ with $D^T\bar{\theta} = (1, \cdots, 1)^T$, then we have
$$\bar{\theta} \cdot q = q^T\bar{\theta} = (D\psi)^T\bar{\theta} = \psi^T D^T\bar{\theta} = \psi_0.$$
- Thus, $\psi_0 = \bar{\theta} \cdot q$ is the discount factor on riskless borrowing.
- Therefore, for any securities i, $q_i = \psi_0\hat{\mathbb{E}}(D_i)$, showing any security's price to be its discounted expected payoff (**using risk-free discount rates!**) in this sense of **artificially constructed probabilities** (i.e. **Risk-Neutral Probability Measure**).

A general single-period model

Review: Idea—why we can price assets by optimality

- The objective of an agent is to maximize his utility from consumption.
- The agent trades because the marginal utility is decreasing with consumption and the trading can lead to a better consumption profile.
- Given a price process S, the agent will choose the best trading strategy θ^* and get the best consumption.
- On the contrary, if an agent consumes as he does, you can think this phenomenon corresponds to an equilibrium, which is, of course, optimal, for this reason, we can recover the price process S from the consumption choice he has made!
- This is why we can price assets by optimality.

The optimization model

- Suppose the dividend-price pair (D, q) is given. An **agent** is defined by a strictly increasing **utility function** $U : \Re^S_+ \to \Re$ and an **endowment** e in $\Re^S_+$.
- This leaves the **budget-feasible set**

$$X(q, e) = \{e + D^T\theta \in \mathcal{R}^S_+ : \theta \in \Re^N, q \cdot \theta \leq 0\},$$

and the optimality problem

$$\sup_{c \in X(q,e)} U(c). \tag{1}$$

Why $q \cdot \theta \leq 0$ must be imposed?

- To answer this question, I provide an example here.
- First, I emphasize that in our model, only resources which have improved the terminal consumption are useful or valuable.
- Suppose, for simplicity's sake, there are two assets (securities) with two states. The dividend-price pair (D, q) is given by

$$\begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$$

and the security prices are $(4, 2)^T$.

- An agent wanting to smooth his consumption with an endowment $e = (1, 100)^T$ will prefer consumption $c = (31, 40)^T$ to consumption e without trading.

Why $q \cdot \theta \le 0$ must be imposed? ctd

- To realize the consumption c, the agent turns to the market. He must sell the second security 6 shares and using the payoff 6×2 to buy 12/4=3 shares of the first security, i.e. $\theta = (3, -6)^T$. Obviously, $q \cdot \theta = 0$ and his aim is reached.
- If, instead, he sell the same but buy 5 rather than 3 shares of the first security or $\theta = (5, -6)^T$, then he can consume $c = (51, 40)^T$ instead of $c = (31, 40)^T$, which is clearly better than the latter for him. However, this portfolio is unfeasible since $q \cdot \theta = 8 > 0$.
- On the other hand, if he sell the same but buy less, i.e. $\theta = (2, -6)^T$, then he can consume only $c = (21, 40)^T$, which is obviously worse than the former. This portfolio is feasible but not optimal since $q \cdot \theta = -4 < 0$. In fact, he completely waste the amount 4 of his wealth since it plays no role to improve his terminal consumption.

No arbitrage and the existence of the solution

- Suppose there is some portfolio θ^0 with payoff $D^T\theta^0 > 0$.
- Because U is strictly increasing, the wealth constraint $q \cdot \theta \leq 0$ is then **binding** at an optimum. **Why?**
- That is, if $c^* = e + D^T\theta^*$ solves (1), then $q \cdot \theta^* = 0$.

Proposition

If there is a solution to (1), then there is no arbitrage. If U is continuous and there is no arbitrage, then there is a solution to (1).

Proof sketch

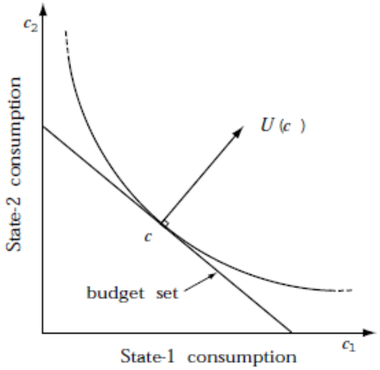

Figure1.2: First-Order Conditions for Optimal Consumption Choice

Proof sketch, ctd

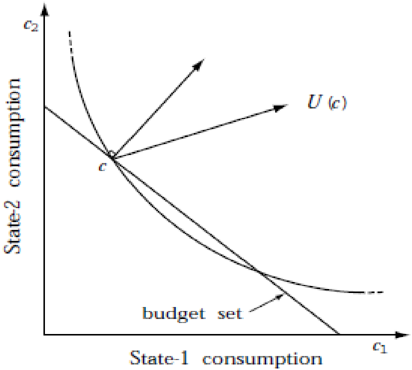

Figure1.3: A Strictly Suboptimal Consumption Choice

Optimality determines state prices

Theorem

Suppose that c^* is a strictly positive solution to (1), that U is continuously differentiable at c^*, and that the vector $\partial U(c^*)$ of partial derivatives of U at c^* is strictly positive. Then there is some scale $\lambda > 0$ such that $\lambda \partial U(c^*)$ is a state-price vector.

Ideas for the proof: The proof is simple since it is just a problem of finding extreme values of a multivariate function with a linear equality constraint. The key is to note that the first-order condition for optimality is that for any θ with $q \cdot \theta = 0$, the marginal utility for buy the portfolio θ is zero. In addition, one conclusion of matrix algebra is needed.

The proof of the Theorem

- We can express more precisely in the following way: The strictly positivity of c^* implies that for any portfolio θ, there is some scale $k > 0$ such that $c^* + \alpha D^T\theta \geq 0$ for all α in $[-k, k]$. Let $g_\theta : [-k, k] \to \Re$ be defined by

$$g_\theta(\alpha) = U(c^* + \alpha D^T\theta).$$

- Suppose $q \cdot \theta = 0$. The optimality of c^* implies that g_θ is maximized at $\alpha = 0$. The first-order condition for this is that $g'_\theta(0) = \partial U(c^*)^T D^T\theta = 0$. We can concluded that, for any θ in $\Re^N$, if $q \cdot \theta = 0$, then $\partial U(c^*)^T D^T\theta = 0$. From this, there is some scale μ such that $D\partial U(c^*) = \mu q$.

The proof of the theorem, continued

- By assumption, there is some portfolio θ^0 with $D^T\theta^0 > 0$. From the existence of a solution to (1), there is no arbitrage, implying that $q \cdot \theta^0 > 0$. We have

  $$\mu q \cdot \theta^0 = \partial U(c^*)^T D^T \theta^0 > 0.$$

  Thus $\mu > 0$. We let $\lambda = 1/\mu$, obtaining

  $$q = \lambda D \partial U(c^*), \tag{2}$$

  implying that $\lambda \partial U(c^*)$ is a state-price vector. $\square$

Optimality determines state prices, a corollary

- Although we have assumed that U is strictly increasing, this does not necessarily mean that $\partial U(c^*) \gg 0$ **(such as function** $f(x) = x^3$**)**.
- If U is concave and strictly increasing, however, it is always true that $\partial U(c^*) \gg 0$.

Corollary

Suppose U is concave and differentiable at some $c^* = e + D^T\theta^* \gg 0$, with $q \cdot \theta^* = 0$. Then c^* is optimal $\iff$ $\lambda\partial U(c^*)$ is a state-price vector for some scale $\lambda > 0$.

One example—the case of expected utility function

- We consider the special case of an **expected utility function** U, defined by a given vector p of probabilities and by some $u : \Re_+ \to \Re$ according to

$$U(c) = \mathbb{E}[u(c)] \equiv \sum_{j=1}^{S} p_j u(c_j).$$

.

- One can check that for $c \geq 0$, if u is differentiable, then we get a more explicit result: $\partial U(c)_j = p_j u'(c_j)$.

Change of Measure

- For this expected utility function, (2) therefore applies if and only if

$$q = \lambda \mathbb{E}[Du'(c^*)],$$

which says that the prices are not the discounted expectation of the payments with regard to the objective probabilities.

- But if we take the "risk-neutral" probability defined by $\hat{\psi}_j = \frac{u'(c_j^*)p_j}{\mathbb{E}[u'(c^*)]}$, then we can also write (2) in the following form

$$q_i = \psi_0 \hat{\mathbb{E}}(D_i) = \psi_0 \sum_{j=1}^{S} D_{ij} \hat{\psi}_j, \quad 1 \leq i \leq N,$$

as expected.

Exercise in class

Suppose there are two assets with three states. The dividend-price pair (D, q) is given by

$$D = \begin{pmatrix} 10 & 0 & 2 \\ 0 & 10 & 1 \end{pmatrix}$$

and $q = (4, 2)^T$. Please fix all the possible state price vectors? In addition, An agent is defined by a utility function $U : x \in \Re^3_+ \to U(x) = \log(x_1) + \log(x_2) + \log(x_3)$ and an endowment $e = (1, 50, 100)^T$. This leaves the budget-feasible set

$$X(q, e) = \{e + D^T\theta \in \mathcal{R}^3_+ : \theta \in \Re^2, q \cdot \theta \leq 0\}.$$

Please solve the optimality problem

$$\sup_{c \in X(q,e)} U(c). \tag{3}$$

Fix the state-price vector $\lambda \partial U(c^*)$ and the scalar λ.

Homework 1

Suppose there are two assets with two states. The dividend-price pair (D, q) is given by

$$D = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$$

and $q = (4, 2)^T$. Please fix all the possible state price vectors? An agent is defined by a utility function
$U : x \in \Re^2_+ \to U(x) = \log(x_1) + \log(x_2)$ and an endowment
$e = (1, 100)^T$. This leaves the budget-feasible set

$$X(q, e) = \{e + D^T\theta \in \mathcal{R}^2_+ : \theta \in \Re^2, q \cdot \theta \leq 0\}.$$

Please solve the optimality problem

$$\sup_{c \in X(q,e)} U(c). \tag{4}$$

Fix the state-price vector $\lambda\partial U(c^*)$ and the scalar λ. Compute the risk-neutral probabilities. For an asset defined by $X = (20, 5)^T$, determine its price.

Homework 2

Suppose there are two assets with three states. The dividend-price pair (D, q) is given by

$$D = \begin{pmatrix} 10 & 0 & 2 \\ 0 & 10 & 1 \end{pmatrix}$$

and $q = (4, 2)^T$. An agent is defined by a utility function $U : x \in \Re^3_+ \to U(x) = x_1^\alpha + x_2^\alpha + x_3^\alpha$ for $0 < \alpha < 1$ and an endowment $e = (1, 50, 100)^T$. This leaves the budget-feasible set

$$X(q, e) = \{e + D^T\theta \in \mathcal{R}^3_+ : \theta \in \Re^2, q \cdot \theta \leq 0\}.$$

Please solve the optimality problem

$$\sup_{c \in X(q,e)} U(c). \tag{5}$$

Fix the state-price vector $\lambda \partial U(c^*)$ and the scalar λ. Compute the risk-neutral probabilities. For an asset defined by $X = (20, 5, 10)^T$, determine its price.

Thank You !

Q & A