A course of Financial Modelling and Pricing Analysis for undergraduates

Lecture 2: Review - A binomial model

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Review - A binomial model

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The type of financial models

When we study finance theory we must decide what type of model we will use for the financial markets. Several possibilities exist:

- Discrete-time models
  1. single-period models
  2. multi-period models
- Continuous-time models

Discrete time models vs. continuous time models

- In this course we will focus on discrete-time models.
- The advantage of discrete time models is that less sophisticated mathematics are required to study them.
- On the other hand, while continuous-time models require more sophisticated mathematics, they often lead to explicit solutions to problems.
- These problems would generally have to be solved numerically in discrete-time models.
- All of the important concepts of finance theory can be studied in discrete-time models.

Our plan

- We begin with a single period model which provides a powerful tool to understand arbitrage pricing theory and probability theory.
- Then, we can stitch single periods together to form the Multi-Period Binomial Option Pricing Model.
- The Multi-Period Binomial Option Pricing Model is extremely flexible, hence valuable; it can value American options (which can be exercised early), and most, if not all, exotic options.

The function of financial markets

Financial markets exist to enable the efficient allocation of resources across

- time
- and *states of nature*.

Example 1: allocation across time

Q: Consider a young worker with a very high salary. What should she do with her earnings: spend and consume them all immediately?

A: No, she would probably want to invest them in the financial markets with various objectives related to retirement, home ownership, children's education, capital growth etc.

Example 1: allocation across time, ctd

Q: Assuming that the young worker above has access to financial markets, where exactly does her money go?

A: (Via the stock and bond markets) To corporations and agents who use it to create products / ideas / wealth.

- Moreover, her money will often go to those corporations or agents who are best qualified to use it!
- Why? Contrast this with the the mechanism used by a dictator or central planner?

Example 2: allocation across states of nature

How does an airline hedge against higher fuel costs?

Review - A binomial model

Why do we study one-period binomial model?

Elements of a financial market

1. Uncertainty, generally represented by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ or a stochastic base $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, i.e. a probability space with an information flow (filtration) $\mathbb{F} \equiv \{\mathcal{F}_t, t \geq 0\}$,
2. Financial products (assets) traded.

The one-period binomial model

- One period: The initial time is $t = 0$ and the terminal time is $t = 1$.
- There are three assets:
  1. A risk-free asset (say government **B**ond) with interest rate $r > 0$. The price process is denoted by B: $B_1 = B_0(1 + r)$;
  2. A **S**tock with price process S, where S_0 is a known constant and S_1 is a random variable;
  3. A derivative asset called OPTION with price process V, of which the value (or price) is $V_1 = (S_1 - K)^+$ at time $t = 1$, where K is a known constant and called the strike price of the option.
- The uncertainty is described by the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{H, T\}$, $\mathcal{F} = 2^\Omega$, $0 < \mathbb{P}\{H\} = p < 1$.

The price process of the srock

$t = 0$ $\qquad\qquad$ $t = 1$

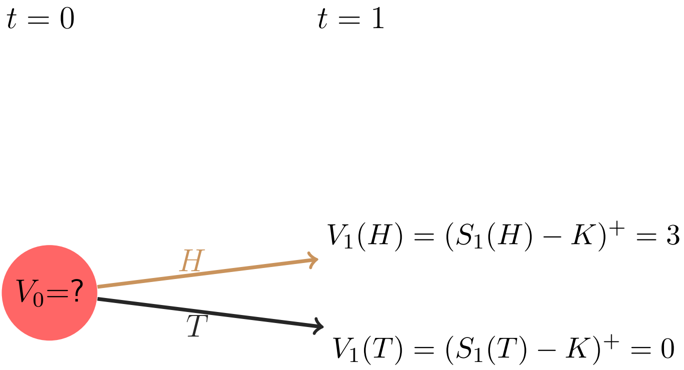
$$\begin{tikzcd}[column sep=huge]
 & S_1(H) = uS_0 = 8 \\
S_0 = 4 \arrow[ur, "H"] \arrow[dr, "T"'] & \\
 & S_1(T) = dS_0 = 2
\end{tikzcd}$$

The price process of the risk-free asset

$t = 0$ $\qquad\qquad$ $t = 1$

$$B_0 = 1 \xrightarrow[T]{H} B_1 = (1+r)B_0 = 1.25$$

The price process of the option if $K = 5$



How much is the fair price V_0 of the option?

The examples of options in practice

- To hedge against higher fuel costs, an airline can buy a call option.
- A firm has an option to invest in a new project. This option is called REAL OPTION.
- If you are buying a book on sale but unfortunately, all the books are sold up, the boss may promise you to buy the book at the sale price once the new books are available. If so, the boss has actually given you a free “option”.
- ...

Review - A binomial model

A portfolio

- To derive the fair price, suppose at time zero you sell the option for V_0 dollars. You now have an obligation to pay off V_1 at time $t = 1$.
- you don't yet know which value ω will take. You hedge your short position in the option by buying Δ_0 shares of stock.
- You can use the proceeds V_0 of the sale of the option for this purpose, and then borrow if necessary at interest rate r to complete the purchase. If V_0 is more than necessary to buy the Δ_0 shares of stock, you invest the residual money at interest rate r. In either case, you will have $V_0 - \Delta_0 S_0$ dollars invested in the money market, where this quantity might be negative.

The fair price and hedging strategy

You need to have the value of your portfolio equal to your obligation, i.e.

$$(V_0 - \Delta_0 S_0)(1+r) + \Delta_0 S_1 = V_1.$$

In other words, you need to choose V_0 and Δ_0, which satisfy the following system of linear equations

$$\begin{cases} (V_0 - \Delta_0 S_0)(1+r) + \Delta_0 S_1(H) = V_1(H); \\ (V_0 - \Delta_0 S_0)(1+r) + \Delta_0 S_1(T) = V_1(T). \end{cases} \tag{1}$$

The fair price and hedging strategy, ctd

The solution of (1) is easily given by

$$\begin{cases} V_0 = \frac{1}{1+r}\left[\tilde{p}V_1(H) + (1-\tilde{p})V_1(T)\right]; \\ \Delta_0 = \frac{V_1(H)-V_1(T)}{S_1(H)-S_1(T)}, \quad \text{(Called delta-hedging foumula)}, \end{cases} \tag{2}$$

where

$$\tilde{p} = \frac{1+r-d}{u-d}.$$

Arbitrage and no-arbitrage price

- What is an arbitrage?
  1. An arbitrage is a transaction that involves no negative cash flow at any probabilistic or temporal state and a positive cash flow in at least one state;
  2. In simple terms, it is the possibility of a risk-free profit at zero cost.
  3. For instance, an arbitrage is present when there is the opportunity to instantaneously buy low and sell high.
- Why $d < 1 + r < u$?
- Why is V_0 given by (2) fair?

Risk-neutral probabilities

If we take $\tilde{p}$ and $1 - \tilde{p}$ as the probabilities of head and tail, respectively, then the fair price equals to the expectation of the discounted revenue generated by the option, i.e.

$$V_0 = \frac{1}{1+r}\mathbb{E}^{\tilde{\mathbb{P}}}(V_1).$$

For this reason, the probability $\tilde{\mathbb{P}}$ is called risk-neutral probability or risk-neutral probability measure.

Thank You !

Q & A