One-dimensional Range-Restricted C^2 Interpolation Algorithms

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Abstract

In this paper, we give an explicit construction of bounded (nonlinear) interpolation operators preserving prescribed range (e.g. nonnegativity, $[0,1]$) as well as a description for the implementation of the interpolation algorithm. The work is inspired by previous results of Jiang-Luli and Fefferman-Jiang-Luli.

1 Introduction

Let $I \subset \mathbb{R}$ be a closed interval. We use $C^2(I)$ to denote the vector space of twice-continuously differentiable functions on I whose derivatives up to the second order are bounded and continuous on I. We equip $C^2(I)$ with the norm

$$\|f\|_{C^2(I)} := \sup_{x\in I} \max_{m=0,1,2} \left| \frac{d^m}{dx^m} f(x) \right|, \tag{1.1}$$

where the derivative is interpreted to be one-sided at the endpoints (if any) of I.

Let X be a finite set. We use $\#X$ to denote the cardinality of X. We use I_X to denote the closed interval $[a-\frac{1}{2}, b+\frac{1}{2}]$, where $[a,b]$ is the closed convex hull of X.

Let $R \subset \mathbb{R}$ be a closed interval and let Ω be a set. We say a map $\mathcal{E} : \Omega \to C^2(\mathbb{R})$ preserves the range R if $\mathcal{E}[\omega](x) \in I$ for all $\omega \in \Omega$ and $x \in \mathbb{R}$. In this paper, we study the case $R = [0,\infty)$ and $R = [\lambda, \Lambda]$ for some $-\infty < \lambda < \Lambda < \infty$. We will refer to operators preserving such R <u>nonnegativity-preserving</u> and <u>finite-range-preserving</u>, respectively.

Our first problem concerns nonnegativity-preserving interpolation.

Problem 1. *Let $E \subset \mathbb{R}$ be a finite set. Let $\varphi : E \to [0,\infty)$. Compute $f \in C^2(I_E)$, $f \geq 0$ on I_E, such that $f = \varphi$ on E and*

$$\|f\|_{C^2(\mathbb{R})} \leq C \cdot \inf\left\{ \|\tilde{f}\|_{C^2(I_E)} : \tilde{f} = \varphi \text{ on } E,\ \tilde{f} \geq 0 \text{ on } I_E \right\} =: \|\varphi\|_{C^2_+(E)}. \tag{1.2}$$

Our second problem concerns finite-range-preserving interpolation. By replacing $R = [\lambda, \Lambda]$ with $R - (\frac{\Lambda+\lambda}{2})$, we may assume that R is symmetric about 0, i.e., $R = [-\tau, \tau]$ with $\tau = \frac{\Lambda-\lambda}{2}$.

Problem 2. *Let $E \subset \mathbb{R}$ be a finite set. Let $\tau > 0$. Let $\gamma : E \to [-\tau, \tau]$. Compute $g \in C^2(I_E)$, $-\tau \leq g \leq \tau$ on I_E, such that $g = \gamma$ on E and*

$$\|g\|_{C^2(\mathbb{R})} \leq C \cdot \inf\left\{ \|\tilde{g}\|_{C^2(I_E)} : \tilde{g} = \gamma \text{ on } E,\ -\tau \leq \tilde{g} \leq \tau \text{ on } I_E \right\} =: \|\gamma\|_{C^2_\tau(E)}. \tag{1.3}$$

Note that Problems 1 and 2 admit equivalent "infinite-domain" formulations if we replace I_E by $\mathbb{R}$ and use a standard damping argument.

In [5, 8–10], the authors solved Problems 1 and 2 by means of constructing bounded range-preserving extension operators and showing the existence of efficient algorithms ($\mathcal{O}(N \log N)$ one-time work and $\mathcal{O}(\log N)$ query work with $N = \#E$) for computing a C-optimal interpolant.

In this paper, we provide the explicit implementation of the algorithms (with some small simplification) in one-dimension. In particular, our solution to Problem 1 can be taken to be a septic spline and our solution to Problem 2 a twelfth-degree spline.

We will see that the only possible obstruction to an extension with small norm is the extension for consecutive three points. This is captured in terms of an improved Finiteness Principle proved in [5, 9], stated in Theorem 2.1 below. Thus, to construct a bounded global range-preserving extension operator, it suffices to construct a sequence of bounded range-preserving extension operators and patch them together via a suitable partition of unity.

The main body of this paper is dedicated to constructing a bounded range-preserving extension operator for three points. The essence of this task consists of constructing a range-preserving (nonlinear) Whitney extension operator (Section 3) and solving a convex quadratic minimization problem with affine constraint (Section 4).

This paper is part of a literature on extension and interpolation, going back to the seminal works of H. Whitney [12–14]. We refer the interested readers to [1–7, 9, 11] and references therein for the history and related problems.

2 Preliminaries

We use C, C', etc., to denote universal constants. In particular, they do not depend on E or f.

2.1 Sorting

We sort E in ascending order $E = \{x_1 < \cdots < x_N\}$. Heap sort algorithm uses at most $\mathcal{O}(N \log N)$ operations and $\mathcal{O}(N)$ storage.

For convenience, we set $x_0 := x_0 - \frac{1}{2}$ and $x_{N+1} := x_N + \frac{1}{2}$. We set

(2.1) $$E_\nu := \{x_{\nu-1}, x_\nu, x_{\nu+1}\} \text{ for } \nu = 2, \cdots, N-1,\ E_1 := E_2, \text{ and } E_N := E_{N-1}.$$

2.2 Model bump function

For the rest of the paper, we fix a function $\theta_{\mathrm{std}} : \mathbb{R} \to [0, \infty)$ we the following properties.

(2.2) $\theta_{\mathrm{std}} \in C^2(\mathbb{R})$, $\theta_{\mathrm{std}}(0) = 1$, and $\theta_{\mathrm{std}}(1) = 0$.

(2.3) θ_{std} is nonnegative and decreasing on $[0, 1]$, and strictly positive on $(0, 1)$,

(2.4) $\theta'_{\mathrm{std}}(0) = \theta''_{\mathrm{std}}(0) = \theta'_{\mathrm{std}}(1) = \theta''_{\mathrm{std}}(1) = 0$, and

(2.5) $\left|\frac{d^m}{dx^m}\theta_{\mathrm{std}}(x)\right| \le C$ for all $x \in \mathbb{R}$.

For instance, we can take θ_{std} to be a quintic spline

$$\theta_{\text{std}}(x) = \begin{cases} 1 & \text{for } x \in (-\infty, 0] \\ -6x^5 + 15x^4 - 10x^3 + 1 & \text{for } x \in (0,1) \\ 0 & \text{for } x \in [1, \infty) \end{cases}$$

or the standard (one-sided) bump function

$$\theta_{\text{std}} = \begin{cases} 1 & \text{for } x \in (-\infty, 0] \\ e \cdot \exp\left(-\frac{1}{1-x^2}\right) & \text{for } x \in (0,1) \\ 0 & \text{for } x \in [1, \infty) \end{cases}.$$

For $x_0 \in \mathbb{R}$ and $\delta > 0$, we define

$$\theta^-_{x_0,\delta}(x) := 1 - \theta_{\text{std}}(\delta^{-1}(x - x_0)) \quad \text{and} \quad \theta^+_{x_0,\delta}(x) := \theta_{\text{std}}(\delta^{-1}(x - x_0)). \tag{2.6}$$

It follows from (2.5) that

$$\left|\frac{d^m}{dx^m}\theta^-_{x_0,\delta}(x)\right| \leq C\delta^{-m} \quad \text{and} \quad \left|\frac{d^m}{dx^m}\theta^+_{x_0,\delta}(x)\right| \leq C\delta^{-m} \text{ for } x \in \mathbb{R},\ m = 0, 1, 2. \tag{2.7}$$

Let $x_1 < x_2 < x_3$, we define

$$\theta_{x_1,x_2,x_3}(x) = \begin{cases} \theta^-_{x_2,|x_2-x_1|}(x) & \text{for } x \in [x_1, x_2] \\ \theta^+_{x_2,|x_3-x_2|}(x) & \text{for } x \in [x_2, x_3] \end{cases}. \tag{2.8}$$

It follows from (2.7) that

$$\left|\frac{d^m}{dx^m}\theta_{x_1,x_2,x_3}(x)\right| \leq \begin{cases} C\,|x_1 - x_2|^{2-m} & x \in (x_1, x_2) \\ C\,|x_3 - x_2|^{2-m} & x \in (x_2, x_3) \end{cases}. \tag{2.9}$$

2.3 Partition of unity

Recall (2.8). For $\nu = 1, \cdots, N$, we set

$$\theta_\nu(x) = \theta_{x_{\nu-1},x_\nu,x_{\nu+1}}(x). \tag{2.10}$$

The θ_ν satisfy the following.

(2.11) $\sum_{\nu=1}^N \theta_\nu(x) \equiv 1$ on $[x_1, x_N]$.

(2.12) For each $\nu = 1, \cdots, N$, $\theta_\nu \equiv 1$ near x_ν, and $\operatorname{supp}(\theta_\nu) \subset J_\nu$. Note that each $x \in \mathbb{R}$ is supported by at most two θ_ν's.

(2.13) For each $\nu = 1, \cdots, N$, $\left|\frac{d^m}{dx^m}\theta_\nu(x)\right| \leq \begin{cases} C\,|x_\nu - x_{\nu-1}|^{-m} & \text{for } x \in (x_{\nu-1}, x_\nu) \\ C\,|x_{\nu+1} - x_\nu|^{-m} & \text{for } x \in (x_\nu, x_{\nu+1}) \end{cases}$, $m = 0, 1, 2$, for some universal constant C.

2.4 Finiteness Principles and Range-preserving extension operators

Let E and E_ν be as in the beginning of the section.

Theorem 2.1. *Let $E \subset \mathbb{R}$ be a finite set with $\#E \geq 3$. Let $E_\nu := \{x_{\nu-1}, x_\nu, x_{\nu+1}\}$ for $\nu = 2, \cdots, N-1$. The following are true.*

(A) Given $\varphi : E \to [0, \infty)$, we have

$$\max_{\nu=2,\cdots,N-1} \|\varphi\|_{C^2_+(E_\nu)} \leq \|\varphi\|_{C^2_+(E)} \leq C \cdot \max_{\nu=2,\cdots,N-1} \|\varphi\|_{C^2_+(E_\nu)}. \tag{2.14}$$

(B) Given $\tau > 0$ and $\gamma : E \to [-\tau, \tau]$, we have

$$\max_{\nu=2,\cdots,N-1} \|\gamma\|_{C^2_\tau(E_\nu)} \leq \|\gamma\|_{C^2_\tau(E)} \leq C \cdot \max_{\nu=2,\cdots,N-1} \|\gamma\|_{C^2_\tau(E_\nu)}. \tag{2.15}$$

Theorem 2.1 is an immediate consequence of the following theorem.

Theorem 2.2. *Let $E \subset \mathbb{R}$ be a finite set with $\#E \geq 3$. Let $E_\nu = \{x_{\nu-1}, x_\nu, x_{\nu+1}\}$ for $\nu = 2, \cdots, N-1$. Set $x_0 = x_1 - \frac{1}{2}$ and $x_{N+1} := x_N + \frac{1}{2}$.*

Let $R \subset \mathbb{R}$ be a closed interval. Let $\varphi : E \to R$.

- *Let $f_2 \in C^2([x_0, x_3])$ satisfy $f_2 = \varphi$ on E_ν and $f_2(x) \in R$ for all $x \in [x_0, x_3]$.*
- *Let $f_{N-1} \in C^2([x_{N-2}, x_{N+1}])$ satisfy $f_{N-1} = \varphi$ on E_ν and $f_{N-1}(x) \in R$ for all $x \in [x_{N-2}, x_{N+1}]$.*
- *Let $f_1 :\equiv f_2$ and $f_N :\equiv f_{N-1}$.*
- *For $\nu = 3, \cdots, N-2$, let $f_\nu \in C^2([x_{\nu-1}, x_\nu])$ satisfy $f_\nu = \varphi$ on E_ν and $f(x) \in R$ for all $x \in [x_{\nu-1}, x_{\nu+1}]$.*

Let $\{\theta_\nu\}$ be as in (2.10). The function

$$f := \sum_{\nu=1}^{N} \theta_\nu \cdot f_\nu$$

satisfies

(A) $f \in C^2(I_E)$,

(B) $f = \varphi$ on E, and

(C) $\|f\|_{C^2(I_E)} \leq C \cdot \max_{\nu=1,\cdots,N} \|f_\nu\|_{C^2(I_{x_\nu})}$.

In light of Theorem 2.2, if we define $\mathcal{E} = \sum_\nu \theta_\nu \cdot \mathcal{E}_\nu$, where each $\mathcal{E}_\nu$ is some bounded range-preserving extension operator for E_ν, $\nu = 1, \cdots, N$, then $\mathcal{E}$ is a bounded range-preserving extension operator for E.

The proof of Theorem 2.2 has been essentially given in [5, 9]. However, we provide the proof here for completeness.

Proof of Theorem 2.2. Theorem 2.2(A) and (B) are clear thanks to the fact that $\{\theta_\nu\}$ is a partition of unity on $[x_1, x_N]$, and $\operatorname{supp}(\theta_\nu) \subset [x_{\nu-1}, x_{\nu+1}]$.

We turn to Theorem 2.2(C). Note that each point $x \in I_E$ supports at most two partition functions $\theta_{\nu(x)}$ and $\theta_{\nu(x)+1}$, with $\theta_{\nu(x)}+\theta_{\nu(x)+1} \equiv 1$ near x. By assumption, $f_{\nu(x)}(x_{\nu(x)}) = \varphi(x_{\nu(x)}) = f_{\nu(x)+1}(x_{\nu(x)})$ and $f_{\nu(x)}(x_{\nu(x)+1}) = \varphi(x_{\nu(x)+1}) = f_{\nu(x)+1}(x_{\nu(x)+1})$. By Rolle's theorem and Taylor's theorem, for $m = 0, 1, 2$,

$$\begin{aligned}\left|\frac{d^m}{dx^m}(f_{\nu(x)} - f_{\nu(x)+1})(x)\right| \leq \\ C \cdot \max\left\{\|f_{\nu(x)}\|_{C^2(I_{x_{\nu(x)}})}, \|f_{\nu(x)+1}\|_{C^2(I_{x_{\nu(x)+1}})}\right\}\left|x_{\nu(x)+1} - x_{\nu(x)}\right|^{2-m}.\end{aligned} \tag{2.16}$$

On the other hand,

$$\frac{d^m}{dx^m} f(x) = \sum_{j\leq m} \binom{m}{j} \frac{d^j}{dx^j}\theta_{\nu(x)} \left[\frac{d^{m-j}}{dx^{m-j}}(f_{\nu(x)} - f_{\nu(x)+1})\right]. \tag{2.17}$$

Theorem 2.2(C) follows from (2.13), (2.16), and (2.17).

□

2.5 Optimality of the parabola for three points

Lemma 2.1. *Given $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$ with $x_1 < x_2 < x_3$ and $|x_1 - x_3| \leq C$. Let π be the unique (possibly degenerate) parabola interpolating $\{(x_i, y_i) : i = 1, 2, 3\}$. Let $I = [x_1, x_3]$. Then*

$$\|\pi\|_{C^2(I)} \leq c \cdot \inf\left\{\|\tilde{f}\|_{C^2(I)} : \tilde{f}(x_i) = y_i \text{ for } i = 1, 2, 3.\right\}. \tag{2.18}$$

Proof. Suppose π is a linear function. By the mean value theorem, any C^2 interpolant $\tilde{f}$ must have $\tilde{f}'(\xi) = \pi'$ for some $\xi \in (x_1, x_3)$. Therefore,

$$\|\tilde{f}\|_{C^2(I)} \geq \max\left\{\left|\tilde{f}'(\xi)\right|, |f(x_1)|, |f(x_3)|\right\} = \|\pi\|_{C^2(I)}.$$

Suppose π is a nondegenerate parabola. Let $\tilde{f}$ be any C^2 interpolant of $\{(x_i, y_i) : i = 1, 2, 3\}$.

Let $D^{(1)}_{12}$, $D^{(1)}_{23}$, and $D^{(2)}$ denote the first and second-order divided differences of $\{(x_i, y_i) : i = 1, 2, 3\}$, and set

$$M = \max\left\{|y_1|, |y_2|, |y_3|, \left|D^{(1)}_{12}\right|, \left|D^{(1)}_{23}\right|, \left|D^{(2)}\right|\right\}.$$

By the mean value theorem for divided difference, there exist $\xi_{ij}, \tilde{\xi}_{ij} \in (x_i, x_j)$, $1 \leq i < j \leq 3$, such that

$$\begin{aligned} \pi'(\xi_{12}) &= D^{(1)}_{12} = \tilde{f}'(\tilde{\xi}_{12}) \\ \pi'(\xi_{23}) &= D^{(1)}_{23} = \tilde{f}'(\tilde{\xi}_{23}) \\ \frac{\pi''}{2} = \frac{\pi''(\xi_{13})}{2} &= D^{(2)} = \frac{\tilde{f}''(\xi_{13})}{2}. \end{aligned} \tag{2.19}$$

We see that

$$\|\tilde{f}\|_{C^2(I)} \geq M.$$

On the other hand, for every $x \in [x_1, x_3]$, we have

$$\left|\pi'(x)\right| \leq \left|\pi'(\xi_{12})\right| + \int_{\xi_{12}}^{x} |\pi''| ds \leq CM$$

and

$$|\pi(x)| \leq |\pi(x_1)| + \int_{x_1}^{x} \left|\pi'(s)\right| ds \leq C'M.$$

This proves Lemma 2.1. $\square$

2.6 Whitney fields

We use $\mathcal{P}$ to denote affine polynomials on $\mathbb{R}$. Each element $p \in \mathcal{P}$ has the form $p(x) = k(x - x_0) + b$ for some $x_0, k, b \in \mathbb{R}$. For a function f twice differentiable near $x_0 \in \mathbb{R}$, we use $\jmath_{x_0} f \in \mathcal{P}$ to denote the degree-one Taylor polynomial at x given by

$$\jmath_{x_0} f(x) = f(x_0) + f'(x_0)(x - x_0).$$

Let $S \subset \mathbb{R}$ be a finite set. A Whitney field on S is an array polynomials $\vec{p} = (p_y)_{y \in S}$, $p_y \in \mathcal{P}$, parametrized by points in S. We denote the vector space of all Whitney fields on S by $W(S)$, equipped with the norm

$$\|(p_y)_{y \in S}\|_{W(S)} = \sum_{y \in S} \sum_{m=0,1} \left| \frac{d^m}{dx^m} p_y(y) \right| + \sum_{y,z \in S,\, y \neq z} \sum_{m=0,1} \frac{\left| \frac{d^m}{dx^m}(p_y - p_z)(y) \right|}{|y - z|^{2-m}} \tag{2.20}$$

Note that every function f differentiable near S generates a Whitney field

$$\jmath_S f := (\jmath_x f)_{x \in S} \in W(S). \tag{2.21}$$

Lemma 2.2. *Let $S \subset \mathbb{R}$ be a finite set with $\#S \leq C$. The following are true.*

(A) $\|\jmath_S f\|_{W(S)} \leq C \|f\|_{C^2(I_S)}$ for all $f \in C^2(I_S)$.

(B) There exists a bounded linear extension operator $\mathcal{E}_S^{\pm} : W(S) \to C^2(I_S)$ such that $\|\mathcal{E}_S^{\pm}[\vec{p}]\|_{C^2(I_S)} \leq C \|\vec{p}\|_{W(S)}$ for all $\vec{p} = (p_y)_{y \in S} \in W(S)$.

Proof. Lemma 2.2(A) is simply Taylor's theorem. Lemma 2.2(B) is the classical Whitney extension theorem, which we prove here for completeness.

We label $S = \{x_1 < \cdots < x_N\}$ with $N = \#S$. Given $\vec{p} = (p_x)_{x \in S} \in W(S)$, we write p_ν in place of p_{x_ν}. Let θ_ν, $\nu = 1, \cdots, N$ be as in Section 2.3. Define

$$\mathcal{E}_{\pm}^{S}[\vec{p}](x) := \sum_{\nu=1}^{N} \theta_\nu(x) \cdot p_\nu(x). \tag{2.22}$$

Note that $\mathcal{E}_{\pm}^{S}[\vec{p}]$ is defined on the entirety of $\mathbb{R}$. Lemma 2.2(B) follows from Taylor's theorem, (2.11)–(2.13), and (2.20). $\square$

3 Whitney Extension

3.1 Nonnegative Whitney extension operator

For $x_0 \in \mathbb{R}$, set

$$\Gamma_+(x_0) = \{p \in \mathcal{P} : p(x_0) \geq 0, \text{ and } p(x_0) = 0 \Rightarrow p \equiv 0.\}. \tag{3.1}$$

Note that $\Gamma_+(x_0)$ consists all the jets at x_0 of functions nonnegative and continuously differentiable near x_0.

Definition 3.1. Let $S \subset \mathbb{R}$ be a finite set. We use $W_+(S)$ to denote the collection of Whitney fields $\vec{p} \in W(S)$ such that $p_y \in \Gamma_+(y)$ for all $y \in S$. Define

$$\|\vec{p}\|_{W_+(S)} := \|\vec{p}\|_{W(S)} + \sum_{y \in S} \frac{(p_y')^2}{p_y(y)} \quad \text{for } p_y \in W_+(S). \tag{3.2}$$

The next few lemmas justify the quadratic terms in (3.2).

Lemma 3.1. *Let $x_0 \in \mathbb{R}$. For each $p \in \Gamma_+(x_0)$, we have*

$$\inf\left\{\|\tilde{f}\|_{C^2(I_{x_0})} : \tilde{f} \geq 0 \text{ on } I_{x_0} \text{ and } \jmath_{x_0}\tilde{f} \equiv p\right\} \geq c \cdot \max\left\{\frac{(p')^2}{2p(x_0)}, \|p\|_{C^2(I)}\right\}. \tag{3.3}$$

Here, we use the convention that $\frac{0}{0} = 0$.

Proof. Write $I = I_{x_0}$. Fix $p \in \Gamma_+(x_0)$. Let $\tilde{f} \in C^2(I)$ with $f \geq 0$ on I and $\jmath_{x_0}\tilde{f} \equiv p$.

Suppose $p' = 0$, then p is a constant. We have

$$\|\tilde{f}\|_{C^2(I)} \geq |f(x_0)| = \|p\|_{C^2(I)}.$$

For the rest of the proof, we assume that $p' \neq 0$.

Suppose $\frac{p(x_0)}{|p'|} \geq \frac{1}{4}$, then

$$\frac{(p')^2}{2p(x_0)} \leq 8|p'| \leq 8\|p\|_{C^2(I)}.$$

Therefore,

$$\|\tilde{f}\|_{C^2(I)} \geq \max\left\{p(x_0), |p'|\right\} \geq c\|p\|_{C^2(I)} \geq c' \cdot \max\left\{\frac{(p')^2}{2p(x_0)}, \|p\|_{C^2(I)}\right\}.$$

Suppose $\frac{p(x_0)}{p'} < \frac{1}{4}$. Taylor's theorem implies

$$\tilde{\pi}(x) := p(x) + \frac{M}{2}(x - x_0)^2 \geq 0 \text{ for all } x \in I, \tag{3.4}$$

where $M = \sup_{x \in I} |f''(x)|$. Note that $M > 0$. We claim that

$$M \geq \frac{(p')^2}{2p(x_0)}. \tag{3.5}$$

If $\tilde{\pi}$ has no roots or two repeated roots (outside the interior of I), we see from computing the discriminant of $\tilde{\pi}$ that $M \geq \frac{(p')^2}{2p(x_0)}$. We claim that $\tilde{\pi}$ cannot have two distinct roots. Suppose otherwise. Since the roots of $\tilde{\pi}$ are given by $x_0 + \frac{-p' \pm \sqrt{(p')^2 - 2Mp(x_0)}}{M}$, we must have

$$
\begin{cases} -p' - \sqrt{(p')^2 - 2Mp(x_0)} \geq \frac{M}{2} \\ p' < 0 \\ -p' > 4p(x_0) \\ M > 0 \end{cases} \quad \text{or} \quad \begin{cases} -p' + \sqrt{(p')^2 - 2Mp(x_0)} \leq -\frac{M}{2} \\ p' > 0 \\ p' > 4p(x_0) \\ M > 0 \end{cases} . \tag{3.6}
$$

However, neither system in (3.6) admits a solution. This is a contradiction, and (3.5) holds. Therefore,

$$
\|\tilde{f}\|_{C^2(I)} \geq \max\left\{M, p(x_0), |p'|\right\} \geq c \cdot \max\left\{\frac{(p')^2}{2p(x_0)}, \|p\|_{C^2(I)}\right\}.
$$

This proves Lemma 3.1.

$\square$

Lemma 3.2. *Let $x_0 \in \mathbb{R}$. There exists an operator $T_+^{x_0} : \Gamma_+(x_0) \to C^2(I_{x_0})$ with the following properties.*

(A) $T_+^{x_0}[p] \geq 0$ on I_{x_0} for all $p \in \Gamma_+(x_0)$.

(B) $\|T_+^{x_0}[p]\|_{C^2(I_{x_0})} \leq C \cdot \inf\left\{\|\tilde{f}\|_{C^2(I_{x_0})} : \tilde{f} \geq 0 \text{ on } I_{x_0} \text{ and } \jmath_{x_0}\tilde{f} \equiv p\right\}$ for all $p \in \Gamma(x_0)$.

Proof. Recall the map $\theta_{x_0-\frac{1}{2},x_0,x_0+\frac{1}{2}}$ from (2.8). We define $T_+^{x_0} : \Gamma(x_0) \to C_+^2([x_0 - \frac{1}{2}, x_0 + \frac{1}{2}])$ by

$$
T_+^{x_0}[p](x) = \theta_{x_0-\frac{1}{2},x_0,x_0+\frac{1}{2}}(x) \cdot \left[p(x) + \frac{1}{2}\frac{(p')^2}{2p(x_0)}(x - x_0)^2\right]. \tag{3.7}
$$

Part (A) follows from computing the discriminant of the parabola

$$
\pi(x) := p(x) + \frac{(p')^2}{4p(x_0)}(x - x_0)^2 \tag{3.8}
$$

and the estimate (2.9).

For part (B), we use Lemma 3.1, the formula (3.7), and the estimate (2.9) and get

$$
\|T_+^{x_0}[p]\|_{C^2(I_{x_0})} \leq C \cdot \max\left\{\frac{(p')^2}{4p(x_0)}, \|p\|_{C^2(I_{x_0})}\right\} \leq C' \inf\left\{\|\tilde{f}\|_{C^2(I_{x_0})} : \tilde{f} \geq 0 \text{ on } I_{x_0} \text{ and } \jmath_{x_0}\tilde{f} \equiv p\right\}.
$$

$\square$

Lemma 3.3. *Let $x_1 < x_2 < x_3$ and $S = \{x_1, x_2, x_3\}$.*

(A) $\|f\|_{C^2(I_S)} \leq C\|\jmath_S f\|_{W_+(S)}$ for all $f \in C^2(I_S)$ with $f \geq 0$ on I.

(B) There exists an operator $T_+^S : W_+(S) \to C^2(I_S)$ such that $\jmath_S T_+^S[\vec{p}] \equiv \vec{p}$ and $\|T_+^S[\vec{p}]\|_{C^2(I)} \leq C\|\vec{p}\|_{W_+(S)}$.

Proof. Lemma 3.3(A) follows from Taylor's theorem and Lemma 3.1.

We turn to Lemma 3.3(B). For convenience, set $x_0 := x_1 - \frac{1}{2}$ and $x_4 := x_3 + \frac{1}{2}$. For $\nu = 1, 2, 3$, let θ_ν be as in (2.10) in (2.3), and let $T_+^{x_\nu}$ be as in Lemma 3.2. We define

$$T_+^S[\vec{p}](x) := \sum_{\nu=1}^{3} \theta_\nu(x) \cdot T_+^{x_\nu}[p_{x_\nu}](x). \tag{3.9}$$

Note that $T_+^S[\vec{p}]$ is defined on all of I_S. Lemma 3.3(B) follows from Lemma 3.2, (2.11)–(2.13), (2.20), and (3.2).

□

3.2 Range-preserving Whitney extension operator

For $x_0 \in \mathbb{R}$ and $\tau > 0$, we define

$$\Gamma_\tau(x_0) := \left\{ p \in \mathcal{P} : \text{ if } p(x_0) = \pm\tau, \text{ then } p' = 0 \right\}. \tag{3.10}$$

Note that $\Gamma_\tau(x_0)$ consists of jets of all functions continuously differentiable with range $[-\tau, \tau]$ near x_0.

Definition 3.2. Let $\tau > 0$ and let $S \subset \mathbb{R}$ be a finite set. We use $W_\tau(S)$ to denote the collection of Whitney fields $\vec{p} \in W(S)$ such that $p_y \in \Gamma_\tau(y)$ for all $y\ inS$. Define

$$\|\vec{p}\|_{W_\tau(S)} := \|\vec{p}\|_{W(S)} + \sum_{y \in S} \frac{(p_y')^2}{\min\{\tau - p_y(y), \tau + p_y(y)\}} \text{ for all } p_y \in W_\tau(S). \tag{3.11}$$

A quick adaptation of the proof of Lemma 3.1 yields the following.

Lemma 3.4. *For each $p \in \Gamma_\tau(x_0)$, we have*

$$\begin{aligned} &\inf\left\{ \|\tilde{f}\|_{C^2(I)} : -\tau \le \tilde{f} \le \tau \text{ on } I \text{ and } \jmath_{x_0}\tilde{f} \equiv p \right\} \ge \\ &\qquad c \cdot \max\left\{ \frac{(p')^2}{\min\{\tau - p(x_0), \tau + p(x_0)\}}, \|p\|_{C^2(I)} \right\}. \end{aligned} \tag{3.12}$$

Lemma 3.5. *Let $x_0 \in \mathbb{R}$. There exists an operator $T_\tau^{x_0} : \Gamma_\tau(x_0) \to C^2(I_{x_0})$ that satisfies the following.*

(A) $-\tau \le T_\tau^{x_0}[p] \le \tau$ on I_{x_0} for all $p \in \Gamma_\tau(x_0)$.

(B) $\|T_\tau^{x_0}[p]\|_{C^2(I_{x_0})} \le C \cdot \inf\left\{ \|\tilde{f}\|_{C^2(I_{x_0})} : -\tau \le \tilde{f} \le \tau \text{ on } I_{x_0} \text{ and } \jmath_{x_0}\tilde{f} \equiv p \right\}$ for all $p \in \Gamma(x_0)$.

Proof. Fix $p \in \Gamma_\tau(x_0)$. Write $p(x) = k(x - x_0) + b$ for some $k \in \mathbb{R}$ and $b \in [-\tau, \tau]$.

If $k = 0$, we just define $\mathcal{E}_{x_0,\tau}[p] \equiv p$. It is clear that both Lemma 3.5(A) and (B) hold.

Suppose $k \ne 0$. For each $p \in \Gamma_\tau(x_0)$, we define

$$\mu := \frac{k^2}{\min\{\tau - b, \tau + b\}} \quad \text{and} \quad \delta := \frac{\min\{\tau - b, \tau + b\}}{k}. \tag{3.13}$$

We consider two cases:

(A) Both $\mu < \frac{1}{2}\tau$ and $|b| < \frac{1}{2}\tau$.

(B) Either $\mu \ge \frac{1}{2}\tau$ or $|b| \ge \frac{1}{2}\tau$.

Extension in Case (A). Note that in this case

$$\operatorname{dist}(x_0, \{p = \pm\tau\}) = \delta = \mu^{-1/2} \cdot \min\left\{\sqrt{\tau - b}, \sqrt{\tau + b}\right\} \geq 1. \tag{3.14}$$

Let $\theta = \theta_{x_0 - \frac{1}{2}, x_0, x_0 + \frac{1}{2}}$ be as in (2.8). We define

$$T_\tau^{x_0}[p](x) := \theta(x) \cdot p(x). \tag{3.15}$$

Lemma 3.5(A) follows from (3.14) and the fact that $\operatorname{supp}(\theta) \subset [x_0 - \frac{1}{2}, x_0 + \frac{1}{2}]$. Lemma 3.5(B) follows from (2.9) and Lemma 3.4.

Extension in Case (B). We define two auxiliary quadratic polynomials.

$$r_-(x) := p(x) + \operatorname{sgn}(k)\frac{\mu}{4}x^2 \quad , \quad r_+(x) := p(x) - \operatorname{sgn}(k)\frac{\mu}{4}x^2. \tag{3.16}$$

Recall θ_{x_1,x_2,x_3} in (2.8). We define the following patching functions.

$$\begin{aligned} \theta_{\text{mid}} &:= \theta_{x_0-\delta, x_0, x_0+\delta}, \\ \theta_{\text{left}} &:= \theta_{x_0 - 2(1+\sqrt{2})\delta, x_0 - \delta, x_0} \\ \theta_{\text{right}} &:= \theta_{x_0, x_0+\delta, x_0 - 2(1+\sqrt{2})\delta}, \\ \theta_{-\infty} &:= \theta_{x_0 - 2(1+\sqrt{2})\delta - \frac{1}{2}, x_0 - 2(1+\sqrt{2})\delta, x_0 - \delta}\text{ , and} \\ \theta_{+\infty}(x) &:= \theta_{x_0+\delta, x_0 + 2(1+\sqrt{2})\delta, x_0 + 2(1+\sqrt{2})\delta + \frac{1}{2}}. \end{aligned} \tag{3.17}$$

Note that

$$\theta_{\text{mid}} + \theta_{\text{left}} + \theta_{\text{right}} + \theta_{-\infty} + \theta_{+\infty} \equiv 1 \text{ on } [x_0 - 2(1+\sqrt{2})\delta, x_0 + 2(1+\sqrt{2})], \tag{3.18}$$

and each point $x \in [x_0 - 2(1+\sqrt{2})\delta, x_0 + 2(1+\sqrt{2})]$ supports at most two patching functions in (3.17).

Finally, we define

$$\begin{aligned} T_\tau^{x_0}[p](x) = \theta_{\text{mid}}(x)p(x) + \theta_{\text{left}}(x)r_-(x) + \theta_{\text{right}}(x)r_+(x) \\ - r_-(x_0 - 2\operatorname{sgn}(k)\delta)\theta_{-\infty}(x) + r_+(x_0 + 2\operatorname{sgn}(k)\delta)\theta_{+\infty}(x). \end{aligned} \tag{3.19}$$

Lemma 3.5(A) follows from the (3.2), the condition on the support of the patching functions, and the observation that that*

$$\begin{aligned} &-\tau \leq p \leq \tau \text{ on } [x_0 - \delta, x_0 + \delta], \\ &-\tau \leq r_- \leq \tau \text{ on } [x_0, x_0 - \operatorname{sgn}(k)2(1+\sqrt{2})\delta], \text{ and} \\ &-\tau \leq r_+ \leq \tau \text{ on } [x_0, x_0 + \operatorname{sgn}(k)2(1+\sqrt{2})\delta]. \end{aligned}$$

We turn to Lemma 3.5(B).

*Here and below, we abuse notation and write $[b, a]$ for $[a, b]$ when $a < b$.

From (2.9) and (3.17), we see that for $* \in \{\text{mid}, \text{left}, \text{right}, -\infty, +\infty\}$,

$$\left|\frac{d^m}{dx^m}\theta_*(x)\right| \leq \begin{cases} C\delta^{-m} & \text{for } x \in [x_0 - 2(1+\sqrt{2})\delta, x_0 + 2(1+\sqrt{2})] \\ C & \text{otherwise} \end{cases}. \tag{3.20}$$

On the other hand, we see from (3.16) that

$$\left|\frac{d^m}{dx^m}(r_\pm - p)(x)\right| \leq C\delta^{2-m} \cdot \max\{\mu, \|p\|_I\} \text{ for } x \in [x_0 - \delta, x_0 + \delta]. \tag{3.21}$$

Moreover, for $x \in x \in [x_0 - 2(1+\sqrt{2})\delta - \frac{1}{2}, x_0 - \delta]$,

$$\left|\frac{d^m}{dx^m}(r_\pm - r_\pm(x_0 \pm 2\text{sgn}(k)\delta))(x)\right| \leq C\delta^{2-m} \cdot \max\{\mu, \|p\|_I\}. \tag{3.22}$$

Lemma 3.5(B) follows from Lemma 3.4, (3.18), and (3.20)–(3.22).

□

Lemma 3.6. *Let $x_1 < x_2 < x_3$ and $S = \{x_1, x_2, x_3\}$. Let $\tau > 0$. The following hold.*

(A) $\|g\|_{C^2(I_S)} \leq C\|\jmath_S g\|_{W_\tau(S)}$ for all $g \in C^2(I_S)$ with $-\tau \leq g \leq \tau$ on I_S.

(B) There exists an operator $T_\tau^S : W_\tau(S) \to C^2(I_S)$ such that $\jmath_S T_\tau^S[\vec{p}] \equiv \vec{p}$ and $\|T_\tau^S[\vec{p}]\|_{C^2(I_S)} \leq C\|\vec{p}\|_{W_\tau(S)}$.

Proof. Lemma 3.6(A) follows from Taylor's theorem and Lemma 3.4.

We turn to Lemma 3.6(B). For convenience, set $x_0 := x_1 - \frac{1}{2}$ and $x_4 := x_3 + \frac{1}{2}$. For $\nu = 1, 2, 3$, let θ_ν be as in (2.10) in (2.3), and let $T_{x_\nu}^\tau$ be as in Lemma 3.5. We define

$$T_\tau^S[\vec{p}](x) := \sum_{\nu=1}^{3} \theta_\nu(x) \cdot T_\tau^{x_\nu}[p_{x_\nu}](x). \tag{3.23}$$

Note that $T_\tau^S[\vec{p}]$ is defined on all of I_S. Lemma 3.6(B) follows from Lemma 3.5, (2.11)–(2.13), (2.20), and (3.11).

□

4 Extension for three points

For the rest of the section, we fix $S = \{x_1 < x_2 < x_3\}$ with $|x_3 - x_1| \leq C$ for some universal constant C. We set $\delta_{ij} = |x_i - x_j|$.

Whitney fields on S are denoted by $\vec{p} = (p_1, p_2, p_3)$, with $p_j = p_{x_j} \in \mathcal{P}$.

4.1 Clustering

Given an ordered triple of linear polynomials $\vec{p} = (p_1, p_2, p_3) \in W(S)$, $p_j(x) = k_j(x - x_j) + b_j$, we identify

$$\vec{p} \sim (b_1, k_1, b_2, k_2, b_3, k_3) \in \mathbb{R}^6. \tag{4.1}$$

We define $L_S \in GL(6)$ by specifying

$$L_S(\vec{p}) = \begin{pmatrix} \delta_{21}^{-2}(b_1 - b_2) + \delta_{21}^{-1} k_2 \\ \delta_{32}^{-2}(b_2 - b_3) + \delta_{32}^{-1} k_3 \\ b_3 \\ \delta_{21}^{-1}(k_1 - k_2) \\ \delta_{32}^{-1}(k_2 - k_3) \\ k_3 \end{pmatrix}. \tag{4.2}$$

With respect to the basis in (4.1), we see that

$$L_S = \begin{pmatrix} \delta_{21}^{-2} & 0 & -\delta_{21}^{-2} & \delta_{21}^{-1} & 0 & 0 \\ 0 & 0 & \delta_{32}^{-2} & 0 & -\delta_{32}^{-2} & \delta_{32}^{-1} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \delta_{21}^{-1} & 0 & -\delta_{21}^{-1} & 0 & 0 \\ 0 & 0 & 0 & \delta_{32}^{-1} & 0 & -\delta_{32}^{-1} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Lemma 4.1. *There exists a universal constant C such that*

$$\|L_S(\vec{p})\|_{\ell_1(\mathbb{R}^6)} \le \|\vec{p}\|_{W(S)} \le C\|L_S(\vec{p})\|_{\ell_1(\mathbb{R}^6)} \tag{4.3}$$

for all $\vec{p} \in W(S)$.

Proof. The first inequality in (4.3) is trivial. We only need to prove the second.

We first express

$$\|\vec{p}\|_{W(S)} = \sum_{j=1,2,3} (|b_j| + |k_j|) + \sum_{i,j=1,2,3, i\neq j} \left(|b_i - b_j + k_j\delta_{ij}|\, \delta_{ij}^{-2} + |k_i - k_j|\, \delta_{ij}^{-1} \right). \tag{4.4}$$

Set $(u, v, w, x, y, z) := L_S(p_1, p_2, p_3)$. Basic algebra gives the following:

$$\begin{aligned} k_2 &= \delta_{32} y + z; \\ k_1 &= \delta_{21} x + \delta_{32} y + z; \\ b_2 &= \delta_{32}^2 v - \delta_{32} z + w; \\ b_1 &= \delta_{21}^2 u - \delta_{21}\delta_{32} y - \delta_{21} z + \delta_{32}^2 v - \delta_{32} z + w = \delta_{21}^2 u + \delta_{32}^2 v + w - \delta_{21}\delta_{32} y - z. \end{aligned}$$

Using the fact that $\delta_{31} \le C$, we see that

$$\sum_{j=1,2,3} (|b_j| + |k_j|) \le C\|L_S(\vec{p})\|_{\ell_1(\mathbb{R}^6)}. \tag{4.5}$$

The analysis for the difference quotients occurring in (4.4) is similar. We provide some typical ones and omit the others:

$$\begin{aligned}
\frac{b_2-[b_1+k_1(x_2-x_1)]}{(x_2-x_1)^2} &= \delta_{21}^{2}(b_2-b_1)-\delta_{21}^{-1}k_1 = -(u+x)\\
\frac{b_3-[b_1+k_1(x_3-x_1)]}{(x_3-x_1)^2} &= \delta_{31}^{-2}(b_3-b_1)-\delta_{31}^{-1}k_1;\\
&= \delta_{31}^{-2}[(b_3-b_2)+(b_2-b_1)]-\delta_{31}^{-1}(\delta_{21}x+\delta_{32}y+z)\\
&= \delta_{31}^{-2}(-\delta_{32}^{2}v+\delta_{32}z-\delta_{21}^{2}u+\delta_{21}\delta_{32}y+\delta_{21}z)-\delta_{31}^{-1}(\delta_{21}x+\delta_{32}y+z)\\
&= \lambda(u,v,w,x,y);\\
\frac{k_3-k_1}{x_3-x_1} &= -\delta_{31}^{-1}\left(\delta_{21}x+\delta_{32}y\right).
\end{aligned}$$

In the above, λ some a linear function with universally bounded coefficients. Thus, we have

$$\sum_{i,j=1,2,3,i\neq j}\left(|b_i-b_j+k_j\delta_{ij}|\,\delta_{ij}^{-2}+|k_i-k_j|\,\delta_{ij}^{-1}\right)\leq C\|L_S(\vec{p})\|_{\ell_1(\mathbb{R}^6)}. \tag{4.6}$$

The second inequality of (4.3) follows form (4.5) and (4.6). Lemma 4.1 is proved.

□

4.2 The quadratic term for nonnegative extension

Let $\imath_X$ denote the indicator function of a set $X\subset\mathbb{R}$. For a given $\varphi:S\to[0,\infty)$, we define $L_\varphi^+:\mathbb{R}^6\to\mathbb{R}^6$ by

$$L_\varphi^+(\vec{p})=\left(0,\frac{k_1(1-\imath_{\{0\}}(\varphi(x_1)))}{\sqrt{2\varphi(x_1)}},0,\frac{k_2(1-\imath_{\{0\}}(\varphi(x_2))}{\sqrt{2\varphi(x_2)}},0,\frac{k_3(1-\imath_{\{0\}}(\varphi(x_3))}{\sqrt{2\varphi(x_3)}}\right)^t, \tag{4.7}$$

with the understanding that $\frac{0}{0}=0$.

Lemma 4.2. *There exists a universal constant C such that*

$$\|L_\varphi^+(\vec{p})\|^2_{\ell_2(\mathbb{R}^6)}+\|L_S(\vec{p})\|_{\ell_1(\mathbb{R}^6)}\leq\|\vec{p}\|_{W_+(S)}\leq C\|L_\varphi^+(\vec{p})\|^2_{\ell_2(\mathbb{R}^6)}+\|L_S(\vec{p})\|_{\ell_1(\mathbb{R}^6)}.$$

In view of Lemma 4.2, in order to minimize $\|\vec{p}\|_{W_+(S)}$ subject to the constraint $p_j(x_j)=\varphi(x_j)$, $j=1,2,3$ up to a universal constant, it suffices solve the following affine-constrained convex quadratic minimization problem:

$$\begin{aligned}
&\text{Minimize } \|L_\varphi^+L_S^{-1}\beta\|^2_{\ell_2}+\|\beta\|_{\ell_1}.\\
&\text{Subject to } [L_S^{-1}\beta]_{2j-1}=\varphi(x_j) \text{ for } j=1,2,3.
\end{aligned} \tag{4.8}$$

4.3 The quadratic term for range-preserving extension

Let $\imath_X$ denote the indicator function of a set $X \subset \mathbb{R}$. For given $\tau > 0$ and $\varphi : S \to [-\tau, \tau]$, we define $L_\varphi^\tau : \mathbb{R}^6 \to \mathbb{R}^6$ by

$$L_\varphi^\tau(\vec{p}) = \begin{pmatrix} 0 \\ \frac{k_1(1-\imath_{\{\pm\tau\}}(\varphi(x_1)))}{\min\left\{\sqrt{\tau+\varphi(x_1)}, \sqrt{\tau-\varphi(x_1)}\right\}} \\ 0 \\ \frac{k_2(1-\imath_{\{\pm\tau\}}(\varphi(x_2)))}{\min\left\{\sqrt{\tau+\varphi(x_2)}, \sqrt{\tau-\varphi(x_2)}\right\}} \\ 0 \\ \frac{k_3(1-\imath_{\{\pm\tau\}}(\varphi(x_3)))}{\min\left\{\sqrt{\tau+\varphi(x_3)}, \sqrt{\tau-\varphi(x_3)}\right\}} \end{pmatrix}, \tag{4.9}$$

with the understanding that $\frac{0}{0} = 0$.

Lemma 4.3. *There exists a universal constant C such that*

$$\|L_\varphi^\tau(\vec{p})\|_{\ell_2(\mathbb{R}^6)}^2 + \|L_S(\vec{p})\|_{\ell_1(\mathbb{R}^6)} \le \|\vec{p}\|_{W_\tau(S)} \le C(\|L_\varphi^\tau(\vec{p})\|_{\ell_2(\mathbb{R}^6)}^2 + \|L_S(\vec{p})\|_{\ell_1(\mathbb{R}^6)}).$$

In view of Lemma 4.3, in order to minimize $\|\vec{p}\|_{W_\tau(S)}$ subject to the constraint $p_j(x_j) = \varphi(x_j)$, $j = 1, 2, 3$ up to a universal constant, it suffices solve the following affine-constrained convex quadratic minimization problem:

$$\begin{aligned} &\text{Minimize } \|L_\varphi^\tau L_S^{-1}\beta\|_{\ell_2}^2 + \|\beta\|_{\ell_1}. \\ &\text{Subject to } [L_S^{-1}\beta]_{2j-1} = \varphi(x_j) \text{ for } j = 1, 2, 3. \end{aligned} \tag{4.10}$$

4.4 Solving the quadratic programming problem

Observe that (4.8) and (4.10) are of the following form:

$$\text{Minimize } \beta^t A \beta + \|\beta\|_{\ell_1(\mathbb{R}^6)} \quad \text{subject to } B\beta = b. \tag{4.11}$$

In (4.11)

- $A = (L_S^{-1})^t (L_\varphi^\star)^t L_\varphi^\star L_S^{-1}$, with $\star = +$ (see (4.7)) or $\star = \tau$ (see (4.9)), is a positive semi-definite matrix;
- $B = VL_S^{-1}$, where $V = \mathrm{diag}(1, 0, 1, 0, 1, 0)$, and $b = (\varphi(x_1), 0, \varphi(x_2), 0, \varphi(x_3), 0)^t$.

To remove the absolute value, we augment the variable β into $(\beta^+, \beta^-) \in \mathbb{R}^{6+6}$, where we would like

$$\beta_i^+ = \max\{0, \beta_i\} \quad \text{and} \quad \beta_i^- = \max\{0, -\beta_i\} \tag{4.12}$$

so that $\beta = \beta^+ - \beta^-$. Note that in this case we have

$$\|\beta\|_{\ell_1(\mathbb{R}^6)} = \underbrace{(1, \cdots, 1)^t}_{\text{twelve copies}} \begin{pmatrix} \beta^+ \\ \beta^- \end{pmatrix}. \tag{4.13}$$

The minimization problem (4.11) is equivalent to the following.

$$
\begin{aligned}
&\text{Minimize } \begin{pmatrix} \beta^+ \\ \beta^- \end{pmatrix}^t \begin{pmatrix} A & -A \\ -A & A \end{pmatrix} \begin{pmatrix} \beta^+ \\ \beta^- \end{pmatrix} + \mathbf{1}^t \begin{pmatrix} \beta^+ \\ \beta^- \end{pmatrix}, \\
&\text{Subject to} \\
&\quad \begin{pmatrix} B & -B \end{pmatrix} \begin{pmatrix} \beta^+ \\ \beta^- \end{pmatrix} = b, \\
&\quad \begin{pmatrix} \beta^+ \\ \beta^- \end{pmatrix} \geq 0, \text{ and} \\
&\quad \beta_i^+ \beta_i^- = 0 \text{ for } i = 1, \cdots, 6.
\end{aligned} \tag{4.14}
$$

The last condition of (4.14) is equivalent to the following statement.

(4.15) For some $J \subset \{1, \cdots, 6\}$, we have $e_j^t \beta^+ = 0$ for $j \in J$, and $e_j^t \beta^- = 0$ for $j \in \{1, \cdots, 6\} \setminus J$.

In the implementation of the actual algorithm, we exhaust all $2^6 = 64$ possibilities of choices of J.

Now, we solve the minimization problem (4.14) by solving for its associated Karush-Kuhn-Tucker (KKT) conditions, which consist of a system of linear inequalities.

For convenience, set

$$
\hat{\beta} = \begin{pmatrix} \beta^+ \\ \beta^- \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} A & -A \\ -A & A \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} B & -B \end{pmatrix} \in \mathbb{R}^{6 \times 12}. \tag{4.16}
$$

Let $\{\hat{e}_j : j = 1, \cdots, 12\}$ be the standard basis for $\mathbb{R}^{6+6}$.

Fix $J_* \subset \{1, \cdots, 6\}$. KKT conditions associated with this particular J_* are specified by:

$$
\begin{aligned}
2\hat{A}\hat{\beta} - \sum_{i=1}^{12} \mu_i \hat{e}_i + \sum_{k=1}^{6} \lambda_k \hat{b}_k^t + \sum_{j \in J_*} v_j \hat{e}_j + \sum_{j \in \{1, \cdots, 6\} \setminus J_*} v_j \hat{e}_{j+6} &= \mathbf{0} \in \mathbb{R}^{12}. \\
\hat{\beta} &\geq \mathbf{0} \\
\hat{B}\hat{\beta} - b &= \mathbf{0}_6 \\
\hat{e}_j^t \hat{\beta} &= 0 \text{ for } j \in J_* \\
\hat{e}_{j+6}^t \hat{\beta} &= 0 \text{ for } j \in \{1, \cdots, 6\} \setminus J_* \\
\mu_k &\geq 0 \text{ for } k = 1, \cdots, 12 \\
\sum_{k=1}^{12} \mu_k (\hat{e}_k^t \hat{\beta}) &= 0.
\end{aligned} \tag{4.17}
$$

In matrix form, (4.17) reads

$$\begin{aligned}
\begin{pmatrix}2\hat{A} & -I_{12} & \hat{B}^t & D_*\end{pmatrix}\cdot(\hat{\beta};\vec{\mu};\vec{\lambda};\vec{v}) &= \mathbf{0}\in\mathbb{R}^{12}\\
\begin{pmatrix}I_{12} & 0 & 0 & 0\end{pmatrix}\cdot(\hat{\beta};\vec{\mu};\vec{\lambda};\vec{v}) &\geq \mathbf{0}\in\mathbb{R}^{12}.\\
\begin{pmatrix}\hat{B} & 0 & 0 & 0\end{pmatrix}\cdot(\hat{\beta};\vec{\mu};\vec{\lambda};\vec{v}) &= b\in\mathbb{R}^{6}\\
\begin{pmatrix}D_* & 0 & 0 & 0\end{pmatrix}\cdot(\hat{\beta};\vec{\mu};\vec{\lambda};\vec{v}) &= \mathbf{0}\in\mathbb{R}^{12}\\
\begin{pmatrix}0 & I_{12} & 0 & 0\end{pmatrix}\cdot(\hat{\beta};\vec{\mu};\vec{\lambda};\vec{v}) &\geq \mathbf{0}\in\mathbb{R}^{12}\\
\begin{pmatrix}0 & 0 & D_* & 0\end{pmatrix}\cdot(\hat{\beta};\vec{\mu};\vec{\lambda};\vec{v}) &= \mathbf{0}\in\mathbb{R}^{12}.
\end{aligned}$$

In the system above, $D_*\in\mathbb{R}^{12\times 12}$ is a diagonal matrix corresponding to the fixed index set $J_*\subset\{1,\cdots,6\}$, given by

$$D_* = \mathrm{diag}(j_1,\cdots,j_6,j_7,\cdots,j_{12}) \text{ where } j_\ell = \begin{cases}1 & \text{if } 1\leq\ell\leq 6 \text{ and } \ell\in J_*\\ 1 & \text{if } 7\leq\ell\leq 12 \text{ and } \ell-6\notin J_*\\ 0 & \text{otherwise}\end{cases}. \tag{4.18}$$

4.5 Three point extension operator

Lemma 4.4. *For a given $\varphi: S\to[0,\infty)$, let β be a minimizer of (4.8), and let $\vec{p}:=L_S^{-1}\beta$. Let T_+^S be as in Lemma 3.3. Define*

$$\mathcal{E}_+^S[\varphi] := T_+^S[\vec{p}].$$

Then $\mathcal{E}_+^S:\{\varphi: S\to[0,\infty)\}\to C^2(I_S)$ satisfies

(A) $\mathcal{E}_+^S[\varphi]=\varphi$ on S,

(B) $\mathcal{E}_+^S[\varphi]\geq 0$ on I_S, and

(C) $\|\mathcal{E}_+^S[\varphi]\|_{C^2(\mathbb{R})}\leq C\|\varphi\|_{C_+^2(S)}$ for all $\varphi: S\to[0,\infty)$.

Proof. This is a direct consequence of Lemma 3.3 and Lemma 4.2. □

Lemma 4.5. *For a given $\gamma: S\to[-\tau,\tau]$, let β be a minimizer of (4.10), and let $\vec{p}:=L_S^{-1}\beta$. Let T_τ^S be as in Lemma 3.6. Define*

$$\mathcal{E}_\tau^S[\varphi] := T_\tau^S[\vec{p}].$$

Then $\mathcal{E}_\tau^S:\{\gamma: S\to[-\tau,\tau]\}\to C^2(I_S)$ satisfies

(A) $\mathcal{E}_+^S[\gamma]=\gamma$ on S,

(B) $-\tau\leq\mathcal{E}_+^S[\gamma]\leq\tau$ on I_S, and

(C) $\|\mathcal{E}_+^S[\gamma]\|_{C^2(I_S)}\leq C\|\gamma\|_{C_\tau^2(S)}$ for all $\gamma: S\to[-\tau,\tau]$.

Proof. This is a direct consequence of Lemma 3.6 and Lemma 4.3. □

5 The global extension operators

Let $E\subset\mathbb{R}$ be a finite set. Recall E_ν from (2.1).

5.1 Nonnegative extension operator

For $\nu = 1, 2, N-1, N$, we define

$$\mathcal{E}_+^\nu := \mathcal{E}_+^{E_\nu}.$$

with $\mathcal{E}_+^{E_\nu}$ as in Lemma 4.4

For $\nu \neq 1, 2, N-1, N$, we define $\mathcal{E}_+^\nu$ as follows. Let $\varphi : E \to [0, \infty)$ and let φ_ν be the restriction of φ to E_ν.

$$\mathcal{E}_+^\nu[\varphi] := \begin{cases} \pi[\varphi_\nu] & \text{if } \pi[\varphi_\nu] \geq 0 \text{ on } [x_{\nu-1}, x_{\nu+1}] \\ \mathcal{E}_+^{E_\nu}[\varphi_\nu] & \text{otherwise} \end{cases}.$$

Here, $\pi[\varphi_\nu]$ is the Lagrange interpolation polynomial of φ_ν and $\mathcal{E}_+^{E_\nu}$ is in Lemma 4.4.

Let θ_ν be as in Section (2.3). We define $\mathcal{E}_+ : \{\varphi : E \to [0, \infty)\} \to C^2(I_E)$ by

$$\mathcal{E}_+[\varphi](x) = \sum_{\nu=1}^{N} \theta_\nu(x) \cdot \mathcal{E}_+^\nu[\varphi](x). \tag{5.1}$$

Note that $\mathcal{E}_+$ preserves nonnegativity since each $\mathcal{E}_+^\nu$ preserves nonnegativity on $[x_{\nu-1}, x_{\nu+1}]$ by construction. Thanks to Lemma 2.1 and Lemma 4.4, each $\mathcal{E}_+^\nu$ is bounded. In view of Theorem 2.2 and the discussion thereafter, we see that $\mathcal{E}_+$ is also bounded.

5.2 Finite Range-preserving operator

For $\nu = 1, 2, N-1, N$, we define

$$\mathcal{E}_\tau^\nu := \mathcal{E}_\tau^{E_\nu}.$$

with $\mathcal{E}_\tau^{E_\nu}$ as in Lemma 4.5

For $\nu \neq 1, 2, N-1, N$, we define $\mathcal{E}_\tau^\nu$ as follows. Let $\gamma : E \to [0, \infty)$ and let γ_ν be the restriction of γ to E_ν.

$$\mathcal{E}_+^\nu[\gamma] := \begin{cases} \pi[\gamma_\nu] & \text{if } -\tau \leq \pi[\gamma_\nu] \leq \tau \text{ on } [x_{\nu-1}, x_{\nu+1}] \\ \mathcal{E}_\tau^{E_\nu}[\varphi_\nu] & \text{otherwise} \end{cases}.$$

Here, $\pi[\gamma_\nu]$ is the Lagrange interpolation polynomial of γ_ν and $\mathcal{E}_\tau^{E_\nu}$ is in Lemma 4.4.

Let θ_ν be as in Section (2.3). We define $\mathcal{E}_\tau : \{\gamma : E \to [-\tau, \tau]\} \to C^2(I_E)$ by

$$\mathcal{E}_+[\gamma](x) = \sum_{\nu=1}^{N} \theta_\nu(x) \cdot \mathcal{E}_\tau^\nu[\gamma](x). \tag{5.2}$$

Note that $\mathcal{E}_\tau$ preserves the range $[-\tau, \tau]$ since each $\mathcal{E}_\tau^\nu$ preserves the range $[-\tau, \tau]$ on $[x_{\nu-1}, x_{\nu+1}]$ by construction. Thanks to Lemma 2.1 and Lemma 4.5, each $\mathcal{E}_\tau^\nu$ is bounded. In view of Theorem 2.2 and the discussion thereafter, we see that $\mathcal{E}_\tau$ is also bounded.

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