4 Restricted-range interpolation

Let $\tau > 0$. Let $E \subset \mathbb{R}$ be a finite set. Let $\varphi : E \to [-\tau, \tau]$. We are interested in finding a function $f : \mathbb{R} \to [-\tau, \tau]$ such that $f = \varphi$ on E, $-\tau \le f \le \tau$ on $\mathbb{R}$, and

$$\|f\|_{C^2(\mathbb{R})} \le C \cdot \inf \left\{ \|\tilde{f}\|_{C^2(\mathbb{R})} : \tilde{f} = \varphi \text{ on } E, \text{ and } -\tau \le \tilde{f} \le \tau. \right\} \tag{4.1}$$

4.1 Extension of a single linear polynomial

Fix a point $x_0 \in \mathbb{R}$ and a polynomial

$$p(x) = k(x - x_0) + b. \tag{4.2}$$

Here $b \in [-\tau, \tau]$, and if $b = \pm\tau$, then $k = 0$. We want to find a function $f \in C^2(\mathbb{R})$ such that f has p as its first-order Taylor polynomial at x_0 (denoted $\jmath_{x_0} f = p$), $-\tau \le f \le \tau$, and

$$\|f\|_{C^2(\mathbb{R})} \le C \cdot \inf \left\{ \|\tilde{f}\|_{C^2(\mathbb{R})} : -\tau \le \tilde{f} \le \tau \text{ and } \jmath_{x_0} \tilde{f} = p. \right\} \tag{4.3}$$

To do this, we have to "complete the square" for both intercepts of p and the lines $y = \pm\tau$. Here are the steps.

If $k = 0$, we just define $f = p$.

If $k \neq 0$. We define two quantities:

$$\mu := \frac{k^2}{\min\{\tau - b, \tau + b\}}. \tag{4.4}$$

$$\delta := \frac{\min\{\tau - b, \tau + b\}}{k}. \tag{4.5}$$

We consider two cases:

(A) Both $\mu < \frac{1}{2}\tau$ and $|b| < \frac{1}{2}\tau$.

(B) Either $\mu \ge \frac{1}{2}\tau$ or $|b| \ge \frac{1}{2}\tau$.

Treatment of Case (A). This is the case where x_0 is sufficiently far away from the intercepts, so we just damp out p. Note that we can write δ as

$$\delta = \mu^{-1/2} \cdot \min \left\{ \sqrt{\tau - b}, \sqrt{\tau + b} \right\} \ge 1. \tag{4.6}$$

thanks to the conditions on μ and b.

Let $\theta(x)$ be the cutoff function defined by

$$\theta(x) = \begin{cases} e \cdot \exp\left[-\frac{1}{1-(x-x_0)^2} \right] & |x - x_0| < 1 \\ 0 & |x - x_0| \ge 1. \end{cases} \tag{4.7}$$

We define

$$f(x) = \theta(x) \cdot p(x). \tag{4.8}$$

Convince yourself that this works and graph it out to see what it looks like.

Treatment of Case (B). This is the case where x_0 is (arbitrarily) close to the intercept so we have to complete the square for at least one side.

We define two auxiliary quadratic polynomials.

$$r_-(x) := p(x) + \frac{\mu}{4}x^2 \quad , \quad r_+(x) := p(x) - \frac{\mu}{4}x^2. \tag{4.9}$$

Graph them to see what they look like. We will patch p, r_-, and r_+ together in suitable regions.

- $I_{\text{mid-left}} := [x_0 - \delta, x_0]$.
- $I_{\text{mid-right}} := [x_0, x_0 + \delta]$.
- $I_{\text{out-left}} := [x_0 - 2\sqrt{2}\delta, x_0 - \delta]$.
- $I_{\text{out-right}} := [x_0 + \delta, x_0 + 2\sqrt{2}\delta]$.
- $I_{-\infty} := (-\infty, x_0 - 2\sqrt{2}\delta]$.
- $I_{+\infty} := [x_0 + 2\sqrt{2}\delta, +\infty)$.

Define the following patching functions.

- $\theta_{\text{mid}}(x) := \begin{cases} e \cdot \exp\left[-\frac{1}{1-((x-x_0)/\delta)^2}\right] & |x - x_0| < \delta \\ 0 & |x - x_0| \geq \delta \end{cases}$. Note that $\theta_{\text{mid}} = 1$ near x_0 and vanishes outside of $(x_0 - \delta, x_0 + \delta)$, with derivatives $\left|\frac{d^m}{dx^m}\theta_{\text{mid}}\right| \leq C\delta^{-m}$.
- $\theta_{\text{left}}(x) := \begin{cases} 1 - \theta_{\text{mid}}(x) & x \in [x_0 - \delta, x_0] \\ e \cdot \exp\left[-\frac{1}{1-\left(\frac{x-(x_0-\delta)}{(2\sqrt{2}-1)\delta}\right)^2}\right] & x \in (x_0 - 2\sqrt{2}\delta, x_0 - \delta) \\ 0 & x \in (-\infty, x_0 - 2\sqrt{2}\delta] \end{cases}$. Note that θ_{left} near $x_0 - \delta$, $\theta_{\text{left}} + \theta_{\text{mid}} = 1$ on $[x_0 - \delta, x_0]$, and $\left|\frac{d^m}{dx^m}\theta_{\text{left}}\right| \leq C\delta^{-m}$.
- $\theta_{\text{right}}(x) := \begin{cases} 1 - \theta_{\text{mid}}(x) & x \in [x_0, x_0 + \delta] \\ e \cdot \exp\left[-\frac{1}{1-\left(\frac{x-(x_0+\delta)}{(2\sqrt{2}-1)\delta}\right)^2}\right] & x \in (x_0 + \delta, x_0 + 2\sqrt{2}\delta) \\ 0 & x \in [x_0 + 2\sqrt{2}\delta, +\infty) \end{cases}$. Note that θ_{right} near $x_0 + \delta$, $\theta_{\text{right}} + \theta_{\text{mid}} = 1$ on $[x_0, x_0 + \delta]$, and $\left|\frac{d^m}{dx^m}\theta_{\text{right}}\right| \leq C\delta^{-m}$.
- $\theta_{-\infty}(x) = \begin{cases} 1 - \theta_{\text{left}}(x) & x \in [x_0 - 2\sqrt{2}\delta, x_0 - \delta] \\ e \cdot \exp\left[-\frac{1}{1-(x-(x_0-2\sqrt{2}\delta))^2}\right] & x \in (-\infty, x_0 - 2\sqrt{2}\delta) \end{cases}$.
- $\theta_{+\infty}(x) = \begin{cases} 1 - \theta_{\text{right}}(x) & x \in [x_0 + \delta, x_0 + 2\sqrt{2}\delta] \\ e \cdot \exp\left[-\frac{1}{1-(x-(x_0+2\sqrt{2}\delta))^2}\right] & x \in (x_0 + 2\sqrt{2}\delta, +\infty) \end{cases}$.

We define

$$f(x) = \theta_{\text{mid}}(x)p(x) + \theta_{\text{left}}(x)r_-(x) + \theta_{\text{right}}(x)r_+(x) - \tau\theta_{-\infty}(x) + \tau\theta_{+\infty}(x). \tag{4.10}$$

Convince yourself that this f works.

4.2 Setting up the quadratic programming problem for three points

Again we are in the situation with $E = \{x_1 < x_2 < x_3\} \subset [0,1]$, $\varphi : E \to [-\tau, \tau]$.

We use the same variables

$$(p_1, p_2, p_3) = (b_1, k_1, b_2, k_2, b_3, k_3) \in \mathbb{R}^6. \tag{4.11}$$

We define L_{cluster} as before, but L_φ is replaced by

$$L_{\varphi,\tau}(p_1, p_2, p_3) = \begin{pmatrix} 0 \\ \frac{k_1}{2\cdot\min\left\{\sqrt{\tau+b_1}, \sqrt{\tau-b_1}\right\}} \\ 0 \\ \frac{k_2}{2\cdot\min\left\{\sqrt{\tau+b_2}, \sqrt{\tau-b_2}\right\}} \\ 0 \\ \frac{k_3}{2\cdot\min\left\{\sqrt{\tau+b_3}, \sqrt{\tau-b_3}\right\}} \end{pmatrix}. \tag{4.12}$$

Again, if any of the $b_j = 0$, we set $k_j = 0$.

4.3 Construct the interpolant

Finally, let (p_1, p_2, p_3) be the minimizer of the quadratic programming problem as in the nonnegative case, with L_φ replaced by $L_{\varphi,\tau}$. Let f_1, f_2, f_3 be the extension above corresponding to p_1, p_2, p_3, respectively. We patch together f_1, f_2, f_3.