3.3 Setting up the quadratic minimization: clustering

Let $E = \{x_1 < x_2 < x_3\}$. We set

$$\delta_{ij} = |x_i - x_j|$$

We can assume $\delta_{ij} \leq 1$.

Given a triple of linear polynomials (p_1, p_2, p_3), $p_j(x) = k_j(x - x_j) + b_j$ we identify

$$(p_1, p_2, p_3) \sim (b_1, k_1, b_2, k_2, b_3, k_3) \in \mathbb{R}^6. \tag{3.8}$$

The Whitney norm on the triple (p_1, p_2, p_3) is

$$\begin{aligned} \|(p_1,p_2,p_3)\| &= \sum_{j=1,2,3} \sum_{m=0,1} \left| \frac{d^m}{dx^m} p_j(x_j) \right| + \sum_{i,j=1,2,3, i\neq j} \sum_{m=0,1} \frac{\left|\frac{d^m}{dx^m}(p_i - p_j)(x_i)\right|}{|x_i - x_j|^{2-m}} \\ &= \sum_{j=1,2,3} (|b_j| + |k_j|) \\ &\quad + \sum_{i,j=1,2,3,i\neq j} \left(|b_i - b_j + k_j\delta_{ij}|\, \delta_{ij}^{-2} + |k_i - k_j|\, \delta_{ij} \right). \end{aligned} \tag{3.9}$$

The key matrix is given by:

$$L_{\text{cluster}}(p_1,p_2,p_3) = \begin{pmatrix} \delta_{21}^{-2}(b_1 - b_2) + \delta_{21}^{-1} k_2 \\ \delta_{32}^{-2}(b_2 - b_3) + \delta_{32}^{-1} k_3 \\ b_3 \\ \delta_{21}^{-1}(k_1 - k_2) \\ \delta_{32}^{-1}(k_2 - k_3) \\ k_3 \end{pmatrix}. \tag{3.10}$$

Problem 3. *Write out L_{cluster} in the basis $(b_1, k_2, b_2, k_2, b_3, k_3)$ (the coefficients should only consist of combinations of δ_{ij}). Compute L_{cluster}^{-1}.*

Let $\varphi : E \to [0, \infty)$ be some given data. We define L_φ by

$$L_\varphi = \left(0, \frac{k_1}{2\sqrt{\varphi(x_1)}}, 0, \frac{k_2}{2\sqrt{\varphi(x_2)}}, 0, \frac{k_3}{2\sqrt{\varphi(x_3)}}\right) \tag{3.11}$$

when $\varphi > 0$. When $\varphi(x_j) = 0$ for some x_j, we set $k_j = 0$.

Let $\beta = L_{\text{cluster}}(p_1, p_2, p_3)$. The quadratic minimization problem is

$$\begin{aligned} &\text{Minimize } \|L_\varphi \beta\|_{\ell_2}^2 + \|\beta\|_{\ell_1} = \beta^t A \beta + \|\beta\|_{\ell_1}. \\ &\text{Subject to } V L_{\text{cluster}}^{-1} \beta = \begin{pmatrix} \varphi(x_1) \\ 0 \\ \varphi(x_2) \\ 0 \\ \varphi(x_3) \\ 0 \end{pmatrix} \end{aligned} \tag{3.12}$$

Here is the dictionary:

- $A = L_\varphi^t L_\varphi$.
- $V = \text{diag}(1, 0, 1, 0, 1, 0)$. It fixes the function value.
- $B = V L_{\text{cluster}}^{-1}$ and $b = \begin{pmatrix} \varphi(x_1) \\ 0 \\ \varphi(x_2) \\ 0 \\ \varphi(x_3) \\ 0 \end{pmatrix}$.

Finally, let β_* be the minimizer of the problem above. We can find the slope (k_1, k_2, k_3):

$$(0, k_1, 0, k_2, 0, k_3) = L_{\text{cluster}}^{-1} \beta_*$$

If you have time, consider the new triple of *quadratic* polynomials $(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3)$ associated with the data φ, given by

$$\tilde{p}_j(x) = \frac{k_j^2}{4\varphi(x_j)}(x - x_j)^2 + k_j(x - x_j) + \varphi(x_j),$$

where k_j are solved as above. Patch them together using the old technique.

Problem 4. *Given data*

- $E = \{0, 0.2, 1\}$,
- $\varphi(0) = 4,\ \varphi(0.2) = 0,\ \varphi(1) = 1.$

Find a nonnegative interpolant on $[-1, 2]$ *using all the techniques.*