5 A quadratic programming problem

The question at hand is:

$$\text{Minimize}\beta^t A\beta + \|\beta\|_1 \quad \text{subject to } B\beta = b. \tag{5.1}$$

Here $A \in \mathbb{R}^{6\times 6}$ positive semidefinite, $B \in \mathbb{R}^{6\times 6}$, and $b \in \mathbb{R}^6$ are given. Say

$$A = \operatorname{diag}(1/2, 1/3, 1/4, 1, 2, 3) \quad \text{and} \quad B = \operatorname{diag}(2, 3, 6, 0, 0, 0). \tag{5.2}$$

We are solving for $\beta = (\beta_1, \cdots, \beta_6) \in \mathbb{R}^6$, and

$$\|\beta\|_1 = |\beta_1| + \cdots + |\beta_6|. \tag{5.3}$$

To deal with the absolute value, we augment the variable β into $(\beta^+, \beta^-) \in \mathbb{R}^{6+6}$, where we would like

$$\beta_i^+ = \max\{0, \beta_i\} \quad \text{and} \quad \beta_i^- = \max\{0, -\beta_i\} \tag{5.4}$$

so that $\beta = \beta^+ - \beta^-$. Note that in this case we have

$$\|\beta\|_1 = \underbrace{(1, \cdots, 1)^t}_{\text{twelve copies}} \begin{pmatrix} \beta^+ \\ \beta^- \end{pmatrix} = \mathbf{1}^t. \tag{5.5}$$

Consider the new equivalent problem:

$$\begin{aligned}
&\text{Minimize } \begin{pmatrix} \beta^+ \\ \beta^- \end{pmatrix}^t \begin{pmatrix} A & -A \\ -A & A \end{pmatrix} \begin{pmatrix} \beta^+ \\ \beta^- \end{pmatrix} + \mathbf{1}^t \begin{pmatrix} \beta^+ \\ \beta^- \end{pmatrix}. \\
&\text{Subject to} \\
&\quad \begin{pmatrix} B & -B \end{pmatrix} \begin{pmatrix} \beta^+ \\ \beta^- \end{pmatrix} = b \\
&\quad \begin{pmatrix} \beta^+ \\ \beta^- \end{pmatrix} \geq 0 \\
&\quad \beta_i^+ \beta_i^- = 0 \text{ for } i = 1, \cdots, 6.
\end{aligned} \tag{5.6}$$

The system is almost easy to work with except for the last condition, which we can reformulate into the following:

- For some $J \subset \{1, \cdots, 6\}$, we have

$$e_j^t \beta^+ = 0 \text{ for } j \in J, \text{ and } e_j^t \beta^- = 0 \text{ for } j \in \{1, \cdots, 6\} \setminus J. \tag{5.7}$$

We will exhaust all these possible J in our implementation of the algorithm. There are $2^6 = 64$ possibilities. We can store J as a vector with 6 entries, each of which is either 0 or 1.