Math, Inner Product Space#3

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1. (4.2)
   D should take the form of $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$
   $p_D(z) = det(zI - D) = z^3$. If $p_D(z) = 0$, it indicates that eigenvalue of D is 0 with the algebric multiplicity of 3. Also $\mathcal{N}(0I - D) = \mathcal{N}(-D) = span\{[x, 0, 0]^T\}$. The geometric multiplicity is 1, since $dim(\mathcal{N}(-D)) = 1$.

2. (4.4)
   Using 4.3, we could show if $(tr(A)^2 - 4det(A))$ is non-negative, then the matrix only got real eigenvalues, otherwise, it only gets imaginary eigenvalues.
   i. If the matrix is Hermitian, the $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$, then $(tr(A)^2 - 4det(A)) = (a-d)^2 + 4b^2 \geq= 0$
   ii. If the matrix is skew-Hermitian, then $\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$. Therefore, $(tr(A)^2 - 4det(A)) = -4b^2 < 0$ for $b \neq 0$

3. (4.6)
   Let $A_{n\times n}$ be an upper triangular matrix. $det(\lambda I - A) = 0 \implies \prod_{i=1}^{n}(\lambda_i - a_{ii}) = 0$. Therefore, the eigenvalues of matrix A are its diagonal elements.

4. (4.8)
   i. To prove $\{sin(x), cos(x), sin(2x), cos(2x)\}$ is the basis for V, we need to prove they are linearly independent. This is equivalent to prove that for $\forall x \in \mathbb{R}, asin(x) + bcos(x) + csin(2x) + dcos(2x) = 0 only when a = b = c = d = 0$. When $x = 0, b + d = 0$. When $x = \pi, -b + d = 0$. Therefore, $b = d = 0$. Also, when $x = \frac{\pi}{2}, a - d = 0 \implies a = 0$. Then $\forall x \in \mathbb{R}, dcos(2x) = 0 \implies d = 0$. This completes the proof that S is a basis for V.
   ii. $(asin(x)+bcos(x)+csin(2x)+dcos(2x))' = (-b)sin(x)+acos(x)+(-2d)sin(2x)+ 2ccos(x)$. So the matrix representation of D is $\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$
   iii.

5. (4.13)
Let $B = P^{-1}AP$, where B is a diagonal matrix. Since P is a non-singular matrix, it indicates that A is diagonalizable. According to the Theorem 4.3.7, A is also semisimple. Then columns of P are just the eigenvectors of A and B is a diagonal marix with eigenvalues on its diagonal. $\lambda = 1$ and 0.4. Therefore, we let $P = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$

6. (4.15)
We will use Theorem 4.3.7 and Proposition 4.3.10 to prove this. According to Theorem 4.3.7, A is semisimple, therefore diagonalizable. We could write $D = P^{-1}AP$ where D is a diagonalizable matrix with eigenvalues of A on its diagonal. And columns of P form the eigenspace of A. According to Proposition 4.3.10, we know that for $i = 1, 2, \ldots n, D^i = P^{-1}A^iP)$. Also, becasue P is a linear operator, we have $P^{-1}f(A)P = a_0I + a_1P^{-1}AP + \ldots + a_nP^{-1}A^nP = a_0I + a_1D + \ldots + a_nD^n$. Since we know that $D^i = \begin{bmatrix} \lambda_1^i & & \\ & \ddots & \\ & & \lambda_n^i \end{bmatrix}$, Then

$$P^{-1}f(A)P = \begin{bmatrix} \sum_{i=0}^n a_i\lambda_1^i & & \\ & \ddots & \\ & & \sum_{i=0}^n a_i\lambda_n^i \end{bmatrix} = \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix}$$

7. (4.16)
i. We have in 4.13 that $D = P^{-1}AP, where D = \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix}$ and $P = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$.
According to proposition 4.3.10, $A^n = PD^nP^{-1}$. So $\lim_{n\to\infty} A^n = \lim_{n\to\infty} PD^nP^{-1} = P(\lim_{n\to\infty} D^n)P^{-1} = P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$
ii. It doesn't matter.
iii. Since the eigenvalues of A are 1 and 0.4, according to Theorem 4.3.12, the eigenvalues are $3 + 5 * 1 + 1^3 = 9$ and $3 + 5 * 0.4 + 0.4^3 = 5.064$.

8. (4.18)
Since the characteristic polynomial of A and A^T is the same, therefore, A and A^T have the same eigenvalues. If λ is a eigenvalue of A, it's Also an eigenvalue of A^T. Therefore, $\exists$non-zero x, such that $A^TX = \lambda x \implies x^TA = \lambda x^T$.

9. (4.20)
Since A is Hermitian, $B = U^HAU \implies B^H = (U^HAU)^H = (U^HA^HU) = (U^HAU) = B$.

10. (4.24)
    i. If A is Hermitian,

11. (4.31)
    i. $||A||_2 = \sup_{x\neq 0} \frac{||Ax||_2}{||x||_2}$
    $= \sup_{x\neq 0} \frac{||U\Sigma V^H x||_2}{||x||_2}$
    $= \sup_{x\neq 0} \frac{||\Sigma V^H x||_2}{||x||_2}$ (invariant under multiplication by orthonormal matrix)
    $= \sup_{x\neq 0} \frac{||\Sigma y||_2}{||Vy||_2}$ (let $y = V^H x$)
    $= \sup_{x\neq 0} \frac{||\Sigma y||_2}{||y||_2}$ (invariant under multiplication by orthonormal matrix)
    $= \sup_{x\neq 0} \frac{(\sum_{i=1}^n |\sigma_i y_i|^2)^{\frac{1}{2}}}{(\sum_{i=1}^n |y_i|^2)^{\frac{1}{2}}}$
    We let y $= [1, 0, \ldots 0]^H$, then $||A||_2 = \sigma_1$
    ii. Since, based on singular value decomposition, $A^{-1} = (U\Sigma V^H)^{-1} = V\Sigma^{-1}U^H$. The diagonal elements of Σ^{-1} are $\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \ldots, \frac{1}{\sigma_n}$, the biggest of which will be $\frac{1}{\sigma_n}$. Therefore, according to the result of i., we know that $||A^{-1}||_2 = \frac{1}{\sigma_n}$.
    iii. It's easy to prove that $||A||_2^H = ||A||_2^T = ||A||_2$ since the Σ of their SVD is the same. $A^H A = V\Sigma^H U^H U\Sigma V^H = V\Sigma^H \Sigma V^H$. Since the diagonal elements of $\Sigma^H\Sigma$ are $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$. By using the results of i., we could know that $||A^H A|| = \sigma_1^2$.
    iv. $||UAV||_2^2 = ||UAVV^H A^H U^H||_2 = ||UAA^H U^H||_2 = ||AA^H U^H||_2 = ||(AA^H U^H)^H||_2 = ||UAA^H||_2 = ||AA^H||_2 = ||A||_2^2$

12. (4.32)
    i. $||UAV||_F = (tr((UAV)^H UAV))^{\frac{1}{2}} = (tr(V^H A^H AV))^{\frac{1}{2}} = (tr(VV^H A^H A))^{\frac{1}{2}} = (tr(A^H A))^{\frac{1}{2}} = ||A||_F$
    ii. Accoding to SVD Theorem, let $A = U\Sigma V^H$, so $||A||_F = tr(A^H A)^{\frac{1}{2}} = tr(V\Sigma^H \Sigma V^H)^{\frac{1}{2}} = tr(V^H V\Sigma^H\Sigma)^{\frac{1}{2}} = tr(\Sigma^H\Sigma)^{\frac{1}{2}} = (\sum_{i=1}^r \sigma_i)^{\frac{1}{2}}$

13. (4.33)
    By 4.31, we know that $||A||_2 = \sigma_1$. According to SVD Theorem, let $A = U\Sigma V^H$. Let $x = \sum_{i=1}^n a_i v_i = Va$, where we expressed x in the orthonormal basis of V. Similarly, we let $y = Ub$. Since $x^H x, y^H y = 1 \implies a^H V^H V a = 1 \implies a^H a = 1$ and $b^H b = 1$.
    $\sup_{||x||_2=1,||y||_2=1} |y^H Ax|$
    $= \sup_{||a||=1,||b||=1} |b^H U^H U\Sigma V^H V a|$
    $= \sup_{||a||=1,||b||=1} |b^H \Sigma a|$
    $= \sup_{||a||=1,||b||=1} |\sum_{i=1}^n \sigma_i a_i b_i|$
    $= \sigma_1$ (According to Cauchy Schwartz inequality)
    $= ||A||_2$

14. (4.36)

15. (4.38)
Let $A = U_1\Sigma_1V_1^H$
i. $AA^\dagger A$
$= U_1\Sigma_1V_1^H V_1\Sigma_1^{-1}U_1^H U_1\Sigma_1V_1^H$
$= U_1\Sigma_1V_1^H = A$
ii. $A^\dagger AA^\dagger$
$= V_1\Sigma_1^{-1}U_1^H U_1\Sigma_1V_1^H V_1\Sigma_1^{-1}U_1^H$
$= V_1\Sigma_1^{-1}U_1^H = A^\dagger$
iii.$(AA^\dagger)^H$
$= (V_1\Sigma_1^{-1}U_1^H)^H(U_1\Sigma_1V_1^H)^H$
$= U_1\Sigma_1V_1^H V_1\Sigma_1^{-1}U_1^H$
$= AA^\dagger$
iv.$(A^\dagger A)^H$
$= (U_1\Sigma_1V_1^H)^H(V_1\Sigma_1^{-1}U_1^H)^H$
$= V_1\Sigma_1^{-1}U_1^H U_1\Sigma_1V_1^H$
$= A^\dagger A$