Math, Problem Set #1, Probability Theory

OSM Lab instructor, Karl Schmedders

OSM Lab student, CHEN Anhua (Peter)

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1. **Exercises from chapter.** Do the following exercises in Chapter 3 of Humpherys and Jarvis (forthcoming): 3.6, 3.8, 3.11, 3.12 (watch this movie clip), 3.16, 3.33, 3.36.

(3.6)

2. Construct examples of events A, B, and C, each of probability strictly between 0 and 1, such that

   (a) $P(A \cap B) = P(A)P(B)$, $P(A \cap C) = P(A)P(C)$, $P(B \cap C) = P(B)P(C)$, but $P(A \cap B \cap C) \neq P(A)P(B)P(C)$.
   Let $\Omega = \{1,2,3,4,5,6,7,8\}$, each event has a probability of $\frac{1}{8}$ to happen
   $A = \{1,2,3,4\}$, $B = \{1,2,5,6\}$, $C = \{1,2,7,8\}$
   $P(A \cap B) = \frac{1}{4} = P(A)P(B)$
   $P(A \cap C) = \frac{1}{4} = P(A)P(C)$
   $P(B \cap C) = \frac{1}{4} = P(B)P(C)$
   However $P(A \cap B \cap C) = \frac{1}{4} \neq P(A)P(B)P(C) = \frac{1}{8}$

   (b) $P(A \cap B) = P(A)P(B)$, $P(A \cap C) = P(A)P(C)$, $P(A \cap B \cap C) = P(A)P(B)P(C)$, but $P(B \cap C) \neq P(B)P(C)$. (Hint: You can let Ω be a set of eight equally likely points.)

   Let $\Omega = \{1,2,3,4,5,6,7,8\}$, each event has a probability of $\frac{1}{8}$ to happen
   $A = \{1,2,3,4\}$, $B = \{1,2,5,6\}$, $C = \{1,3,7,8\}$
   $P(A \cap B) = \frac{1}{4} = P(A)P(B)$
   $P(A \cap C) = \frac{1}{4} = P(A)P(C)$
   $P(A \cap B \cap C) = \frac{1}{8} = P(A)P(B)P(C)$
   However $P(B \cap C) = \frac{1}{8} \neq P(B)P(C)$

3. Prove that Benford's Law is, in fact, a well-defined discrete probability distribution.

1) The lower bound of $log(1+\frac{1}{d}) = log1 = 0$ and bounded above by $log2 < 1$

2) $\sum_{d=1}^{9} log(1+\frac{1}{d}) = 1$

3) For finite additivity additivity, we get $P(\cup_{d=1}^{\infty} E_d) = \sum_{d=1}^{\infty} P(E_d)$

4. A person tosses a fair coin until a tail appears for the first time. If the tail appears on the nth flip, the person wins 2^n dollars. Let the random variable X denote the player's winnings.

   (a) (St. Petersburg paradox) Show that $E[X] = +\infty$.

   $E(x) = \sum_{n=1}^{\infty} 0.5^n 2^n$
   $= +\infty$

   (b) Suppose the agent has log utility. Calculate $E[\ln X]$.

   $E[lnX] = \sum_{n=1}^{\infty} 0.5^n ln(2^n)$
   $= ln2 \sum_{n=1}^{\infty} \frac{n}{2^n}$
   $= 2ln2$

5. (Siegel's paradox) Suppose the exchange rate between USD and CHF is 1:1. Both a U.S. investor and a Swiss investor believe that a year from now the exchange rate will be either $1.25 : 1$ or $1 : 1.25$, with each scenario having a probability of 0.5. Both investors want to maximize their wealth in their respective home currency (a year from now) by investing in a risk-free asset; the risk-free interest rates in the U.S. and in Switzerland are the same. Where should the two investors invest?

   If the investor investor invest in their own currency, only get the risk-free interest rate: $1 + r_f$. If they invest in foreign currency, they will get: $0.5 \times (1 + r_f) \times 1.25 + 0.5 \times (1 + r_f) \times \frac{1}{1.25} = 1.025 \times (1 + r_f)$. Therefore, they should all invest in the foreign currency.

6. Consider a probability measure space with $\Omega = [0, 1]$.

   (a) Construct a random variable X such that $E[X] < \infty$ but $E[X^2] = \infty$.

   (b) Construct random variables X and Y such that $P(X > Y) > \frac{1}{2}$ but $E[X] < E[Y]$.

   (c) Construct random variables X, Y, and Z such that
   $P(X > Y)P(Y > Z)P(X > Z) > 0$ and $E(X) = E(Y) = E(Z) = 0$.

7. Let the random variables X and Z be independent with $X \sim N(0, 1)$ and $P(Z = 1) = P(Z = -1) = \frac{1}{2}$. Define $Y = XZ$ as the product of X and Z. Prove or disprove each of the following statements.

   (a) $Y \sim N(0, 1)$.
   It's true. In order to prove X and Y have same distribution, we just need to make sure their PDF or CDF are the same

$P(Y < x)$
$= P(Z = 1)P(X < x) + P(Z = -1)P(X > -x)$
$= P(X < x)$ since normal distribution is symmetrical around mean

(b) $P(|X| = |Y|) = 1$.
It's true. Proof is omitted.

(c) X and Y are not independent.
it's true because $P(X < a|Y < a) = \frac{1}{2}$, when $a <= 0$ while $P(x < a) < \frac{1}{2}$
Therefore, $P(X < a|Y < a) \neq P(x < a)$, indicating X and Y are not independent.

(d) $Cov[X, Y] = 0$.
it's true.
$cov[X, Y] = cov[X, XZ]$
$= E[XXZ] - E[X]E[XZ]$
$= E[XXZ] - E[Z]E[X]E[X]$ since X and E are independent and $E[Z] = 0$
$= E[XY]$
$= cov[X, Y] - E[X]E[Y]$
$= 0$

(e) If X and Y are normally distributed random variables with $Cov[X, Y] = 0$, then X and Y must be dependent.
it's false. (a) to (d) provide an example where X and Y are not independent.

8. Let the random variables X_i, $i = 1, 2, \ldots, n$, be i.i.d. having the uniform distribution on $[0, 1]$, denoted $X_i \sim U[0, 1]$. Consider the random variables $m = \min\{X_1, X_2, \ldots, X_n\}$ and $M = \max\{X_1, X_2, \ldots, X_n\}$. For both random variables m and M, derive their respective cumulative distribution (cdf), probability density function (pdf), and expected value.

(a) m
CDF:
$P(m \leq x)$
$= 1 - P(m \geq x)$
$= 1 - P(\text{All Xs are larger than x})$
$= 1 - (1 - x)^n$
PDF:
$n(1 - x)^{n-1}$
Expected value:

$$E(m) = \int_0^1 n(1-x)^{n-1}dx = \frac{1}{n+1}$$

(b) M
CDF:
$P(M \leq x)$
$= P(\text{All Xs are smaller than x})$
$= (x)^n$
PDF:
$n(x)^{n-1}$
Expected value:

$$E(M) = \int_0^1 n(x)^{n-1}dx = \frac{n}{n+1}$$

9. You want to simulate a dynamic economy (e.g., an OLG model) with two possible states in each period, a "good" state and a "bad" state. In each period, the probability of both shocks is $\frac{1}{2}$. Across periods the shocks are independent. Answer the following questions using the Central Limit Theorem and the Chebyshev Inequality.

   (a) What is the probability that the number of good states over 1000 periods differs from 500 by at most 2%?
   We define
   $$X = \begin{cases} 1 & \text{state is good} \\ 0 & \text{state is bad} \end{cases} \tag{1}$$
   By Central Limit Theorem:
   $P(|\sum_{n=1}^{1000} X_n - 500| \leq 10)$
   $= \Phi(\frac{10}{0.5\sqrt{1000}}) - \Phi(\frac{-10}{0.5\sqrt{1000}})$
   $= 0.47$

   (b) Over how many periods do you need to simulate the economy to have a probability of at least 0.99 that the proportion of good states differs from $\frac{1}{2}$ by less than 1%?
   According to the Chebyshev inequality:
   $P(|X - 0.5| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2}$
   $\varepsilon = 0.005$ and $\sigma^2 = 0.25$
   Therefore, in order to fufill the condition that $P(|X - 0.5| \geq \varepsilon) \leq 0.01$, we could let $\frac{\sigma^2}{n\varepsilon^2} = 0.01$
   We will get $n = 1000000$.

10. If $E[X] < 0$ and $\theta \neq 0$ is such that $E[e^{\theta X}] = 1$, prove that $\theta > 0$.
Proof:
we will use Jensen inequality to prove. Let $f(x) = e^{\theta x}$. Since the second order derivative of $f(x)$ is always positive, $f(x)$ is a convex function. According to

Jensen's inequality, $E[e^{\theta x}] \geq e^{\theta E[x]}$. Since $E[e^{\theta X}] = 1$, $e^{\theta E[x]} \leq 1$. This means $\theta E[x] \leq 0$ and $\theta \neq 0$. Since $E[x] < 0$, $\theta > 0$.

References

Humpherys, Jeffrey and Tyler J. Jarvis, "Foundations of Applied Mathematics, Volume II: Algorithm Design and Optimization," forthcoming. SIAM, Philadelphia, PA.