Dynamic Programming: Preliminaries

OSM Bootcamp Chicago

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Preliminary topics

- Reminders, notation
- The curse of dimensionality
- (General state) Markov chains
- Nonlinear functional equations

References

- Stokey and Lucas (1989)
- Stachurski (2009)
- Lasota and Mackey. Chaos, Fractals and Noise (1998)

Reminder 1: Distributions

Let S be a nonempty set

A **distribution** ϕ on S is a function that assigns probabilities to subsets of S:

$$\phi(B) = \text{ probability mass assigned to } B \subset S$$

I'll often use notation such as

$$\int g(x)\phi(\mathrm{d}x)$$

Think of this as $\mathbb{E}\, g(X)$ when $X \sim \phi$

Example. If $S = \mathbb{R}$ and ϕ has a density f, then

$$\int g(x)\phi(\mathrm{d}x) = \int_{-\infty}^{\infty} g(x)f(x)\,\mathrm{d}x$$

Example. If $S = \mathbb{R}^2$ and ϕ has a density f, then

$$\int g(x)\phi(\mathrm{d}x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1, x_2)f(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2$$

Example. If $S = \{1, 2, \ldots, n\}$, then

$$\int g(x)\phi(\mathrm{d}x) = \sum_{i=1}^{n} g(x_i)\phi(x_i)$$

Reminder 2: Metric Spaces

Let $\mathscr{G}$ be a nonempty set and let ρ map $\mathscr{G} \times \mathscr{G}$ to $\mathbb{R}$

The pair $(\mathscr{G}, \rho)$ is called a **metric space** if, for any x, y, z in $\mathscr{G}$,

- $\rho(x, y) = 0$ if and only if $x = y$
- $\rho(x, y) = \rho(y, x)$
- $\rho(x, z) \leqslant \rho(x, y) + \rho(y, z)$

Example. $\mathscr{G} = \mathbb{R}^n$ and $\rho(x, y) = \|x - y\|$

Example. $S \subset \mathbb{R}^n$ and $\mathscr{C}$ is all continuous bounded functions from S to $\mathbb{R}$,

$$\rho(f,g) := \sup_{x \in S} |f(x) - g(x)|$$

The three axioms hold for $(\mathscr{C}, \rho)$

For example, if f, g and h are in $\mathscr{C}$ and $x \in S$, then

$$\begin{aligned}
|f(x) - h(x)| &= |f(x) - g(x) - (g(x) - h(x))| \\
&\leqslant |f(x) - g(x)| + |g(x) - h(x)| \\
&\leqslant \rho(f,g) + \rho(g,h)
\end{aligned}$$

$$\therefore \quad \rho(f,h) \leqslant \rho(f,g) + \rho(g,h)$$

Optimization and Computers

Some optimization problems are pretty easy

- All functions are differentiable
- Few choice variables (low dimensional)
- Concave (for max) or convex (for min)
- First order / tangency conditions relatively simple

Textbook examples often chosen to have this structure

In reality many problems don't have this structure

- Can't take derivatives
- No analytical solution for FOCs
- Many choice variables (high dimensional)
- Neither concave nor convex — local maxima and minima

Can Computers Save Us?

For any function we can always try brute force optimization

Here's an example for the following function

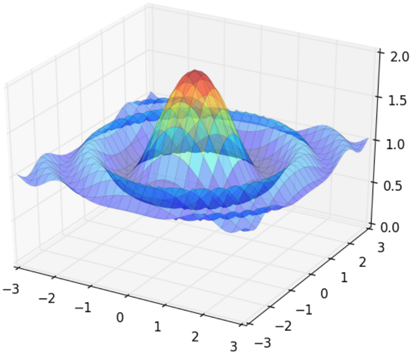


Figure: The function to maximize

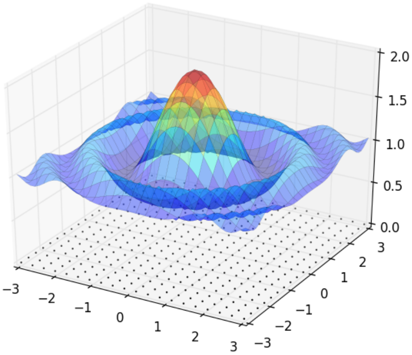


Figure: Grid of points to evaluate the function at

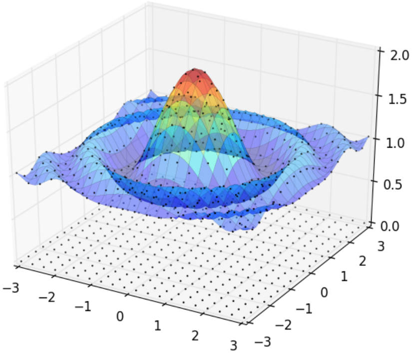


Figure: Evaluations

Grid size $= 20 \times 20 = 400$

Outcomes

- Number of function evaluations $= 400$
- Time taken = almost zero
- Maximal value recorded $= 1.951$
- True maximum $= 2$

Not bad and we can easily do better

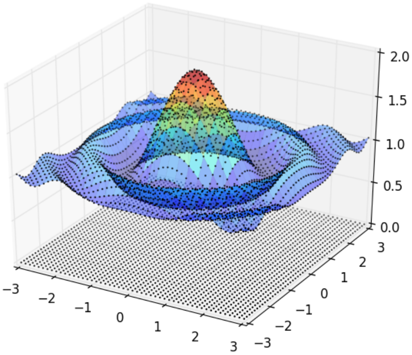


Figure: $50^2 = 2500$ evaluations

- Number of function evaluations $= 50^2$
- Time taken $= 400\ \mu$s
- Maximal value recorded $= 1.992$
- True maximum $= 2$

So why even study optimization?

The problem is mainly with larger numbers of choice variables

- 3 vars: $\max_{x_1,x_2,x_3} f(x_1,x_2,x_3)$
- 4 vars: $\max_{x_1,x_2,x_3,x_4} f(x_1,x_2,x_3,x_4)$
- $\cdots$

If we have 50 grid points per variable and

- 2 variables then evaluations $= 50^2 = 2500$
- 3 variables then evaluations $= 50^3 = 125,000$
- 4 variables then evaluations $= 50^4 = 6,250,000$
- 5 variables then evaluations $= 50^5 = 312,500,000$
- $\cdots$

Example. Recent study: Optimal placement of drinks across vending machines in Tokyo

Approximate dimensions of problem:

- Number of choices for each variable $= 2$
- Number of choice variables $= 1000$

Hence number of possibilities $= 2^{1000}$

How big is that?

```
In [10]: 2**1000
Out[10]:
1071508607186267320948425049060001810561404811705
5336074437503883703510511249361224931983788156958
5812759467291755314682518714528569231404359845
7757469857480393456777482423098542107460506237114
1877954182153046474983581941267398767559165543946
0770629145711964776865421676604298316526243868
37205668069376
```

Let's say my machine can evaluate about 1 billion possibilities per second

How long would that take?

```
In [16]: (2**1000 / 10**9) / 31556926  # In years
Out[16]:
339547840365144349278007955863635707280678989995
899349462539661933596146571733926965255861364854
060286985707326991591901311029244639453805988092
045933072657455119924381235072941549332310199388
301571394569707026437986448403352049168514244509
939816790601568621661265174170019913588941596
```

What about high performance computing?

- more powerful hardware
- faster CPUs
- GPUs
- vector processors
- cloud computing
- massively parallel supercomputers
- $\cdots$

Let's say speed up is 10^{12} (wildly optimistic)

```
In [19]: (2**1000 / 10**(9 + 12)) / 31556926
Out[19]:
3395478403651443492780079558636357072806789899958
```

For comparison:

```
In [20]: 5 * 10**9 # Expected lifespan of sun
Out[20]: 5000000000
```

Message: There are serious limits to computation

What's required is clever analysis

Exploit what information we have

- without information (oracle) we're stuck
- with information / structure we can do clever things

Examples later on...

Markov Chains on General Spaces

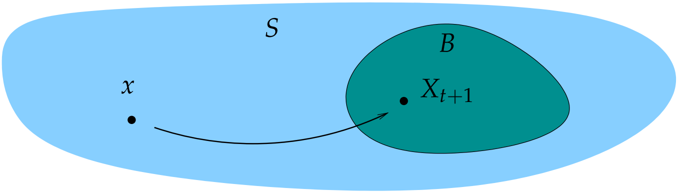
Let

- S be any set (called the **state space**)
- $P(x, \mathrm{d}y)$ be a **stochastic kernel** on S — a distribution over S for each $x \in S$

If $\{X_t\}$ is a stochastic process satisfying

$$P(x, B) = \mathbb{P}\{X_{t+1} \in B \mid X_t = x\}$$

then called a **Markov process** with stochastic kernel P



Example. Let $\{W_t\}$ be an IID sequence with distribution ϕ

Consider the stochastic difference equation

$$X_{t+1} = g(X_t, W_{t+1}) \quad \text{with} \quad X_0 = x_0$$

Each X_t takes values in S, a set of

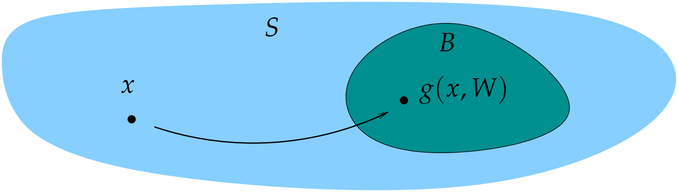
- vectors in $\mathbb{R}^n$, or
- scalars in $\mathbb{R}$, or
- something else...

This is a Markov process with stochastic kernel

$$P(x, B) = \phi\{w \in \mathbb{W} : g(x, w) \in B\}$$

Alternatively,

$$P(x, B) = \int \mathbb{1}\{g(x, w) \in B\}\,\phi(\mathrm{d}w)$$



Example. Let $S = \mathbb{R}$, let $\{W_t\} \overset{\text{IID}}{\sim} N(0,1)$ and let

$$X_{t+1} = aX_t + b + \sigma W_{t+1}$$

This is a **linear Gaussian** Markov process with kernel

$$P(x, \mathrm{d}y) := N(ax + b, \sigma^2)$$

That is, $P(x, B) = \int_B p(x, y)\, \mathrm{d}y$ where

$$p(x, y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{(y - ax - b)^2}{\sigma^2} \right\}$$

Example. Consider the Solow–Swan model

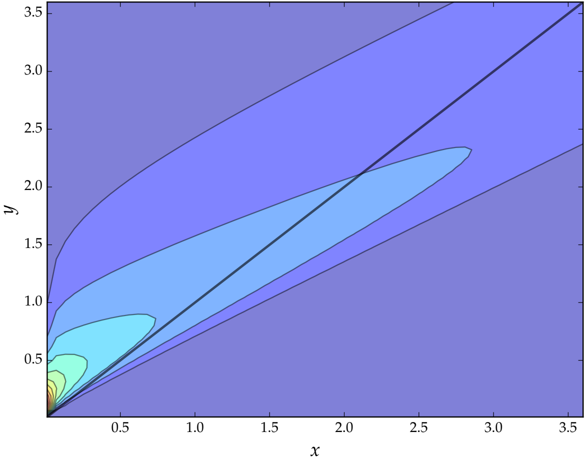
$$k_{t+1} = sf(k_t)W_{t+1} + (1-\delta)k_t \qquad \{W_t\}_{t \geqslant 1} \overset{\text{IID}}{\sim} \phi$$

Here

- k_t takes values in $S = (0, \infty)$
- $s, \delta \in (0,1)$ and $f(k) > 0$ when $k > 0$

The stochastic kernel is

$$P(k, B) = \phi \{ w \in \mathbb{W} \mid sf(k)w + (1-\delta)k \in B \}$$


3.5
3.0
2.5
2.0
1.5
1.0
0.5
y
0.5
1.0
1.5
2.0
2.5
3.0
3.5
x

Higher Order Kernels

Fix a Markov process $\{X_t\}$ with stochastic kernel P

Let P^n be the n-step kernel:

$$P^n(x, B) = \mathbb{P}\{X_{t+n} \in B \mid X_t = x\}$$

Fact. $\{P^k\}$ satisfies the **Chapman–Kolmogorov relation**

$$P^{n+k}(x, B) = \int P^k(x, \mathrm{d}z) P^n(z, B)$$

Example. Recall the process

$$X_{t+1} = g(X_t, W_{t+1}) \quad \text{with} \quad X_0 = x$$

The $n = 2$ kernel is

$$P^2(x, B) =$$

$$(\phi \times \phi) \{(w_1, w_2) \in \mathbb{W} \times \mathbb{W} \mid g(g(x, w_1), w_2) \in B\}$$

Higher order kernels

- $P^3(x, \mathrm{d}y) \stackrel{\mathscr{D}}{=} g(g(g(x, W_1), W_2), W_3)$
- etc.

Markov Operators

Given stochastic kernel P on S and $h \colon S \to \mathbb{R}$, let

$$(Ph)(x) = \int h(y) P(x, \mathrm{d}y) \qquad (x \in S)$$

Called the **Markov operator** corresponding to P

Example. If P corresponds to $X_{t+1} = g(X_t, W_{t+1})$, then

$$(Ph)(x) = \int h(g(x, w)) \phi(\mathrm{d}w)$$

Interpretations:

$$(Ph)(x) = \mathbb{E}\left[h(X_{t+1}) \mid X_t = x\right]$$

$$(P^n h)(x) = \mathbb{E}\left[h(X_{t+n}) \mid X_t = x\right]$$

Solving Equations

Discussion: When does this **vector equation** in $\mathbb{R}^n$ have a unique solution?

$$x = Ax + b$$

When does the **method of successive approximations** converge?

1. pick any $x_0 \in \mathbb{R}^n$
2. $x_{n+1} = Ax_n + b$

How else could we find a solution?

Discussion: Is there a unique $k \in (0,1)$ that solves

$$k = sk^{\alpha} + (1-\delta)k$$

Does $k_{n+1} = sk_n^{\alpha} + (1-\delta)k_n$ converge to the solution? When?

Is there a unique $(k_1, \ldots, k_d) \in (0,\infty)^d$ that solves

$$k_1 = s_1 \prod_{i=1}^{d} k_i^{\alpha_i} + (1-\delta)k_1$$

$$\vdots$$

$$k_d = s_d \prod_{i=1}^{d} k_i^{\alpha_i} + (1-\delta)k_d$$

Discussion: Consider the **asset price equation**

$$q_t = \beta \, \mathbb{E}_t [q_{t+1} + d_{t+1}]$$

Let $d_t = \delta(X_t)$ where $\{X_t\}$ is Markov $\sim P$

Guess a solution of the form $q_t = q(X_t)$ and rewrite as

$$q(X_t) = \beta \, \mathbb{E}_t [q(X_{t+1}) + \delta(X_{t+1})]$$

or as the **functional equation**

$$q(x) = \beta \int q(y) P(x, \mathrm{d}y) + \beta \int \delta(y) P(x, \mathrm{d}y) \qquad (x \in S)$$

- Unique solution? How to solve?

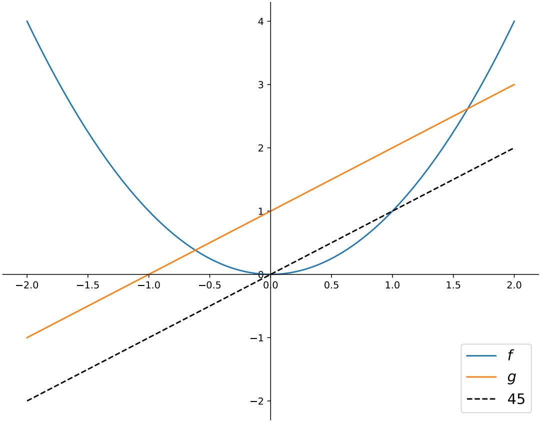
Fixed Points

Let $(\mathscr{G}, \rho)$ be a metric space and let $T \colon \mathscr{G} \to \mathscr{G}$

A **fixed point** of T is a point $x^* \in \mathscr{G}$ such that $Tx^* = x^*$

Examples.

- If $f(x) = x^2$ on $\mathbb{R}$, then 0 and 1 are fixed points
- If $g(x) = x + 1$ on $\mathbb{R}$, then g has no fixed points on $\mathbb{R}$


4
3
2
1
0
−1
−2
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
f
g
45

T is called a **contraction map** on $(\mathscr{G}, \rho)$ if

$$\exists\, \alpha < 1 \quad \text{such that} \quad \rho(Tx, Ty) \leqslant \alpha\rho(x, y), \quad \forall x, y \in \mathscr{G}$$

Example. $f(x) = \alpha x + b$ on metric space $(\mathbb{R}, |\cdot|)$ with $|\alpha| < 1$, since

$$|f(x) - f(y)| = |\alpha x - \alpha y| = |\alpha||x - y|$$

Fact. Every contraction T is continuous on $\mathscr{G}$

Proof: If $x_n \to x$ in $(\mathscr{G}, \rho)$, then

$$\rho(Tx_n, Tx) \leqslant \alpha\rho(x_n, x) \to 0$$

Fact. If T is a contraction map on $(\mathscr{G}, \rho)$ and $x \in \mathscr{G}$, then $\{T^k x\}$ is Cauchy

Sketch of proof: Along the trajectory $\{T^k x\}$ from x, we have

$$\begin{aligned} \rho(T^{k+1}x, T^k x) &\leqslant \alpha \rho(T^k x, T^{k-1}x) \\ &\leqslant \alpha^2 \rho(T^{k-1}x, T^{k-2}x) \\ &\;\vdots \\ &\leqslant \alpha^k \rho(Tx, x) \end{aligned}$$

Banach's Fixed Point Theorem

Theorem. If $(\mathscr{G}, \rho)$ is complete and T is a contraction, then T has a unique fixed point x^* in $\mathscr{G}$ and, for all $x \in \mathscr{G}$,

$$\lim_{k \to \infty} \rho(T^k x, x^*) = 0$$

Proof: Pick any $x \in \mathscr{G}$

The sequence $\{T^k x\}$ is Cauchy and hence converges to some x^*

The point x^* is a fixed point, since

$$Tx^* = T(\lim_k T^k x) = \lim_k T(T^k x) = \lim_k T^{k+1} x = x^*$$

Regarding uniqueness, if x^* and x^{**} are fixed points of T, then

$$\rho(x^*, x^{**}) = \rho(Tx^*, Tx^{**}) \leqslant \alpha\rho(x^*, x^{**})$$

$$\therefore \quad \rho(x^*, x^{**}) = 0$$

$$\therefore \quad x^* = x^{**}$$

Application: Asset Pricing

Recall the asset pricing equation

$$q(x) = \beta Pq(x) + \beta P\delta(x)$$

Let $\mathscr{C}$ be all continuous bounded functions on S with metric

$$\rho(f,g) := \sup_{x \in S} |f(x) - g(x)|$$

Let P have the **Feller property**, which is to say that

$$h \in \mathscr{C} \implies Ph \in \mathscr{C}$$

Let $\delta \in \mathscr{C}$ and let $\beta \in (0,1)$

Claim: The asset pricing equation

$$q(x) = \beta Pq(x) + \beta P\delta(x) \qquad (x \in S)$$

has a unique solution $q^* \in \mathscr{C}$

Remarks:

- We often write this as $q = \beta Pq + \beta P\delta$
- Equivalent: the operator $T \colon \mathscr{C} \to \mathscr{C}$ defined by

  $$Tq = \beta Pq + \beta P\delta$$

  has a unique fixed point in $\mathscr{C}$

To prove this we need to show that

1. $Tq = \beta Pq + \beta P\delta$ is in $\mathscr{C}$ when $q \in \mathscr{C}$
2. the pair $(\mathscr{C}, \rho)$ forms a complete metric space
3. T is a contraction map on $(\mathscr{C}, \rho)$

Here (1) follows from assumption and the proof of (2) is omitted

Regarding (3), fix any q, q' in $\mathscr{C}$ and any $x \in S$

We have,

$$
\begin{aligned}
|Tq(x) - Tq'(x)| &= \left|\beta P q(x) + \beta P\delta(x) - \beta P q'(x) - \beta P\delta(x)\right| \\
&= \beta \left| \int q(y) P(x, \mathrm{d}y) - \int q'(y) P(x, \mathrm{d}y) \right| \\
&= \beta \left| \int [q(y) - q'(y)] P(x, \mathrm{d}y) \right| \\
&\leqslant \beta \int |q(y) - q'(y)| P(x, \mathrm{d}y) \\
&\leqslant \beta \sup_y |q(y) - q'(y)| = \beta \rho(q, q')
\end{aligned}
$$

Taking the supremum with respect to x completes the proof