Math, Convex analysis#5

OSM Lab instructor, Jorge Barro

OSM Lab student, CHEN Anhua

Due Wednesday, July 21 at 8:00am

1. (7.1)
   Let $x, y \in conv(C)$, where $x = \sum_{i=1}^{n} a_i x_i$ and $y = \sum_{j=1}^{m} b_j y_j$. Then for $t \in [0,1]$, we define $tx + (1-t)y = \sum_{k}^{n+m} \theta c_k$, where $\theta_k = a_k, c_k = x_k$ if $k \leq n$ and $\theta_k = b_{k-n}, c_k = y_{k-n}$ if $k > n$. Since $c_k \in C$ and $t\sum_{i=1}^{n} + (1-t)\sum_{j=1}^{m} = 1$, $tx + (1-t)y \in conv(C) \implies conv(C)$ is a convex set.

2. (7.2)
   (i) If $x, y \in P = \{x \in V| < a, x >= b\}$, then $< a, x >= b, < a, y >= b$. $< a, \lambda x + (1-\lambda)y >= \lambda < a, x > + (1-\lambda) < a, y >= b$. Therefore, hyperplane is a convex set.
   (ii) If $x, y \in H = \{x \in V| < a, x > \leq b\}$, then $< a, x > \leq b, < a, y > \leq b$. $< a, \lambda x + (1-\lambda)y >= \lambda < a, x > + (1-\lambda) < a, y > \leq b$. Therefore, half space is a convex set.

3. (7.4)
   i). $||x-y||^2 = ||x-p+p-y||^2 =< (x-p+p-y), (x-p+p-y) >= ||x-p||^2 + ||p-y||^2 + 2 < x-p, p-y >$
   ii).If $< x-p, p-y > \geq 0$, it's easy to use i) to prove that $||x-y|| > ||x-p||$
   iii).This one could be easily proved by using the method in i) and substituting z by the convex combination of y and p.
   iv). If we let $z = \lambda y + (1-\lambda)p$, according to i), we have $||x-z||^2 = ||x-p||^2 + ||p-z||^2 + 2 < x-p, p-z > \implies 2 < x-p, p-z >= ||x-z||^2 - ||x-p||^2 - ||p-z||^2$. Then according to iii), it's equal to $2 < x-p, p-z >= 2\lambda < x-p, p-y > + \lambda^2||y-p||^2 - ||p-z||^2 = 2\lambda < x-p, p-y > + \lambda^2||y-p||^2 - ||\lambda(p-y)||^2 = 2\lambda < x-p, p-y >$. Therefore, by induction method, if we assume $< x-p, p-y > \geq 0$, then $< x-p, p-z > \geq 0$.

4. (7.6)
   Let $x, y \in \{x \in \mathbb{R}^n | f(x) \leq c\}$, since f is a convex function, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \leq \lambda c + (1-\lambda)c = c$. Therefore, $\lambda x + (1-\lambda)y \in \{x \in \mathbb{R}^n | f(x) \leq c\}$

5. (7.7)
   For $x, y \in C$, since f_i is convex function, $f(\lambda x + (1-\lambda)y) = \sum_{i=1}^{k} \lambda_i f_i(\lambda x +$

$(1-\lambda)y) \leq \sum_{i=1}^{k} \lambda_i(\lambda f_i(x) + (1-\lambda)f_i(y)) = \lambda f(x) + (1-\lambda)f(y)$. Therefore, the function is convex.

6. (7.13)
   If f is not constant, W.L.O.G., $f(a) < f(b)$ for $a < b$. For $c > b$, since the epigraph is also convex, $f(c) \leq f(a) + (c-a)\frac{f(b)-f(a)}{b-a}$. When c goes to inifinity, then f cannot be bounded above. Therefore f has to be a constant function.

7. (7.20)
   If f is convex, then $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$. Similarly for $-f$: $-f(\lambda x + (1-\lambda)y) \leq -\lambda f(x) - (1-\lambda)f(y)$. These two inequality implies that $f(\lambda x+(1-\lambda)y) = \lambda f(x)+(1-\lambda)f(y)$, indicating f is a linear function, therefore affine.

8. (7.21)
   Since $x \in \mathbb{R}^n$, the minimize x^* is an interior point. If x^* is the local minimizer of the first optimization problem, then $\phi^{'} f^{'}(x^*) = 0$, $f^{'}(x^*) = 0$ because $\phi(x)$ is a strictly increasing function. It's easy to prove the other direction since $f^{'}(x^*) = 0 \implies \phi^{'} f^{'}(x^*) = 0$