

Math, Problem Set #2, Probability Theory

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Due Wednesday, July 5 at 8:00am

1. (3.1)

Define $\langle x, y \rangle$ as $x \times y$ in a real inner product space. It's easy to show that $\langle x, y \rangle$ satisfies all the properties of an IPS.

(i) $\text{RHS} = \frac{1}{4}((x+y) \times (x+y) - (x-y) \times (x-y)) = x \times y = \text{LHS}$

(ii) $\text{RHS} = \frac{1}{2}(2x^2 + 2y^2) = x^2 + y^2 = \text{LHS}$

2. (3.2)

Let $x = a + bi$ and $y = c + di$, where $a, b, c, d \in \mathbb{R}$. $\text{RHS} = ac - bd + (bc + ad)i$

3. **Exercises from chapter.** Do the following exercises in Chapter 3 of **Humpherys and Jarvis** (forthcoming): 3.6, 3.8, 3.11, 3.12 (watch this movie [clip](#)), 3.16, 3.33, 3.36.

4. Construct examples of events A , B , and C , each of probability strictly between 0 and 1, such that

(a) $P(A \cap B) = P(A)P(B)$, $P(A \cap C) = P(A)P(C)$, $P(B \cap C) = P(B)P(C)$, but $P(A \cap B \cap C) \neq P(A)P(B)P(C)$.

(b) $P(A \cap B) = P(A)P(B)$, $P(A \cap C) = P(A)P(C)$, $P(A \cap B \cap C) = P(A)P(B)P(C)$, but $P(B \cap C) \neq P(B)P(C)$. (Hint: You can let Ω be a set of eight equally likely points.)

5. Prove that Benford's Law is, in fact, a well-defined discrete probability distribution.

6. A person tosses a fair coin until a tail appears for the first time. If the tail appears on the n th flip, the person wins 2^n dollars. Let the random variable X denote the player's winnings.

(a) (St. Petersburg paradox) Show that $E[X] = +\infty$.

(b) Suppose the agent has log utility. Calculate $E[\ln X]$.

7. (Siegel's paradox) Suppose the exchange rate between USD and CHF is 1:1. Both a U.S. investor and a Swiss investor believe that a year from now the exchange rate will be either 1.25 : 1 or 1 : 1.25, with each scenario having a probability of 0.5. Both investors want to maximize their wealth in their respective home currency (a year from now) by investing in a risk-free asset; the risk-free interest rates in the U.S. and in Switzerland are the same. Where should the two investors invest?
8. Consider a probability measure space with $\Omega = [0, 1]$.
 - (a) Construct a random variable X such that $E[X] < \infty$ but $E[X^2] = \infty$.
 - (b) Construct random variables X and Y such that $P(X > Y) > \frac{1}{2}$ but $E[X] < E[Y]$.
 - (c) Construct random variables X , Y , and Z such that $P(X > Y)P(Y > Z)P(X > Z) > 0$ and $E(X) = E(Y) = E(Z) = 0$.
9. Let the random variables X and Z be independent with $X \sim N(0, 1)$ and $P(Z = 1) = P(Z = -1) = \frac{1}{2}$. Define $Y = XZ$ as the product of X and Z . Prove or disprove each of the following statements.
 - (a) $Y \sim N(0, 1)$.
 - (b) $P(|X| = |Y|) = 1$.
 - (c) X and Y are not independent.
 - (d) $Cov[X, Y] = 0$.
 - (e) If X and Y are normally distributed random variables with $Cov[X, Y] = 0$, then X and Y must be independent.
10. Let the random variables X_i , $i = 1, 2, \dots, n$, be i.i.d. having the uniform distribution on $[0, 1]$, denoted $X_i \sim U[0, 1]$. Consider the random variables $m = \min\{X_1, X_2, \dots, X_n\}$ and $M = \max\{X_1, X_2, \dots, X_n\}$. For both random variables m and M , derive their respective cumulative distribution (cdf), probability density function (pdf), and expected value.
11. You want to simulate a dynamic economy (e.g., an OLG model) with two possible states in each period, a "good" state and a "bad" state. In each period, the probability of both shocks is $\frac{1}{2}$. Across periods the shocks are independent. Answer the following questions using the Central Limit Theorem and the Chebyshev Inequality.
 - (a) What is the probability that the number of good states over 1000 periods differs from 500 by at most 2%?
 - (b) Over how many periods do you need to simulate the economy to have a probability of at least 0.99 that the proportion of good states differs from $\frac{1}{2}$ by less than 1%?

12. If $E[X] < 0$ and $\theta \neq 0$ is such that $E[e^{\theta X}] = 1$, prove that $\theta > 0$.

References

Humpherys, Jeffrey and Tyler J. Jarvis, “Foundations of Applied Mathematics, Volume II: Algorithm Design and Optimization,” forthcoming. SIAM, Philadelphia, PA.