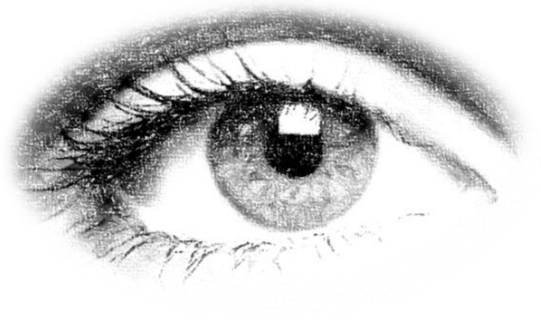

Projective Geometry

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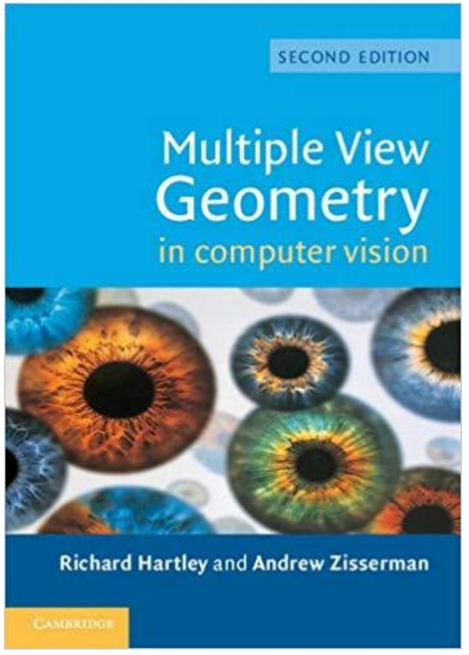
Outline

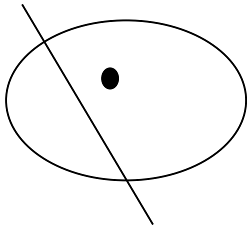
- Projective 2D geometry
- Projective 3D geometry



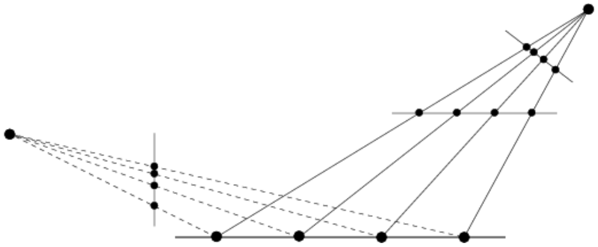
[Slides credit: Marc Pollefeys]

Projective 2D Geometry

- Points, lines & conics
- Transformations & invariants



- 1D projective geometry and the cross-ratio



Homogeneous Coordinates

- Homogeneous representation of lines

$$ax + by + c = 0 \qquad (a,b,c)^{\mathsf{T}}$$

$$(ka)x + (kb)y + kc = 0, \forall k \neq 0 \qquad (a,b,c)^{\mathsf{T}} \sim k(a,b,c)^{\mathsf{T}}$$

equivalence class of vectors, any vector is representative

- Homogeneous representation of points

$\mathrm{x} = (x, y)^{\mathsf{T}}$ on $\mathrm{l} = (a,b,c)^{\mathsf{T}}$ if and only if $ax + by + c = 0$

$$(x,y,1)(a,b,c)^{\mathsf{T}} = (x,y,1)\mathrm{l} = 0 \qquad (x, y, 1)^{\mathsf{T}} \sim k(x, y, 1)^{\mathsf{T}}, \forall k \neq 0$$

The point x lies on the line l if and only if $\mathrm{x}^{\mathsf{T}}\mathrm{l}=\mathrm{l}^{\mathsf{T}}\mathrm{x}=0$

Homogeneous coordinates $(x_1, x_2, x_3)^{\mathsf{T}}$ but only 2DOF

Inhomogeneous coordinates $(x, y)^{\mathsf{T}}$

The point x=$(x_1, x_2, x_3)^{\mathrm{T}}$ represent the point $(x_1/x_3, x_2/x_3)^{\mathrm{T}}$ in $\mathbb{R}^2$

Points and Lines

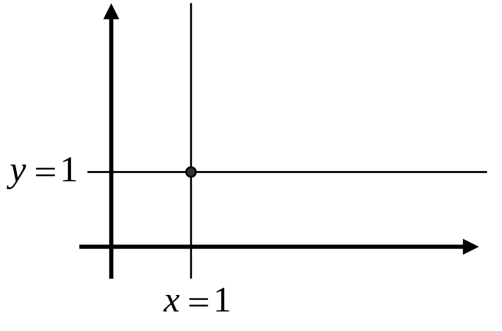
- Intersections of lines

  The intersection of two lines $\mathbf{l}$ and $\mathbf{l}'$ is $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$

- Line joining two points

  The line through two points $\mathbf{x}$ and $\mathbf{x}'$ is $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$

Example

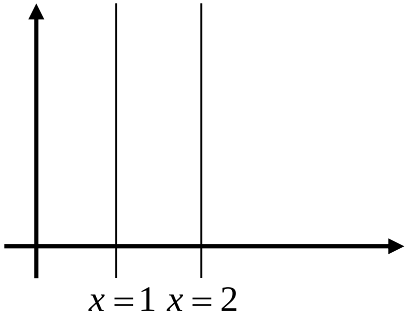


Ideal Points and the Line at Infinity

- Intersections of parallel lines

$\mathrm{l} = (a, b, c)^{\mathsf{T}}$ and $\mathrm{l}' = (a, b, c')^{\mathsf{T}}$ $\qquad$ $\mathrm{l} \times \mathrm{l}' = (b, -a, 0)^{\mathsf{T}}$

Example



$(b, -a)$ tangent vector (line's direction)

(a, b) normal direction

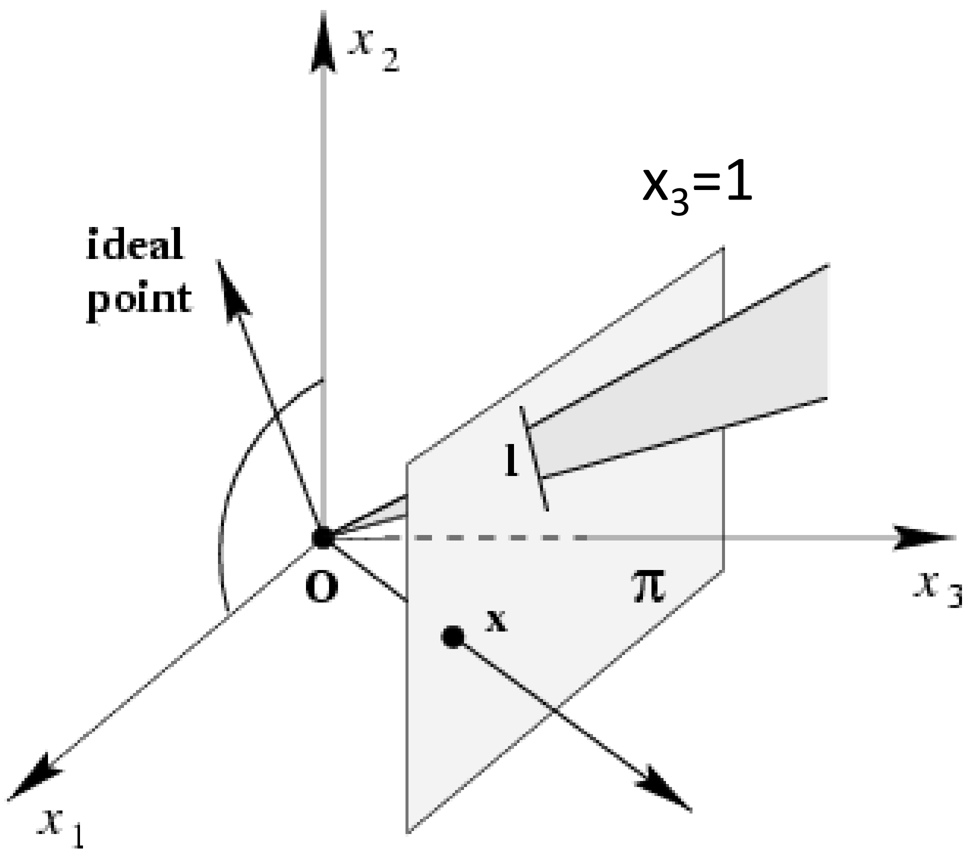
Ideal points $\qquad (x_1, x_2, 0)^{\mathsf{T}}$

Line at infinity $\qquad \mathrm{l}_\infty = (0, 0, 1)^{\mathsf{T}}$

$\mathbf{P}^2 = \mathbf{R}^2 \cup \mathrm{l}_\infty$

Note that in $\mathbf{P}^2$ there is no distinction between ideal points and others

A Model for the Projective Plane



exactly one line through two points

exaclty one point at intersection of two lines

Duality

$$\mathbf{x} \longleftrightarrow \mathbf{l}$$

$$\mathbf{x}^{\mathsf{T}}\mathbf{l} = 0 \longleftrightarrow \mathbf{l}^{\mathsf{T}}\mathbf{x} = 0$$

$$\mathbf{x} = \mathbf{l} \times \mathbf{l}' \longleftrightarrow \mathbf{l} = \mathbf{x} \times \mathbf{x}'$$

- Duality principle:

  To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the role of points and lines in the original theorem

Conics

Curve described by 2nd-degree equation in the plane

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

or *homogenized* $x \mapsto x_1/x_3,\ y \mapsto x_2/x_3$

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

or in matrix form

$$\mathrm{x}^{\mathsf{T}}\mathbf{C}\,\mathrm{x} = 0 \quad \text{with} \quad \mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

symmetric

5DOF: $\{a : b : c : d : e : f\}$

Five Points Define a Conic

For each point the conic passes through

$$a x_i^2 + b x_i y_i + c y_i^2 + d x_i + e y_i + f = 0$$

or

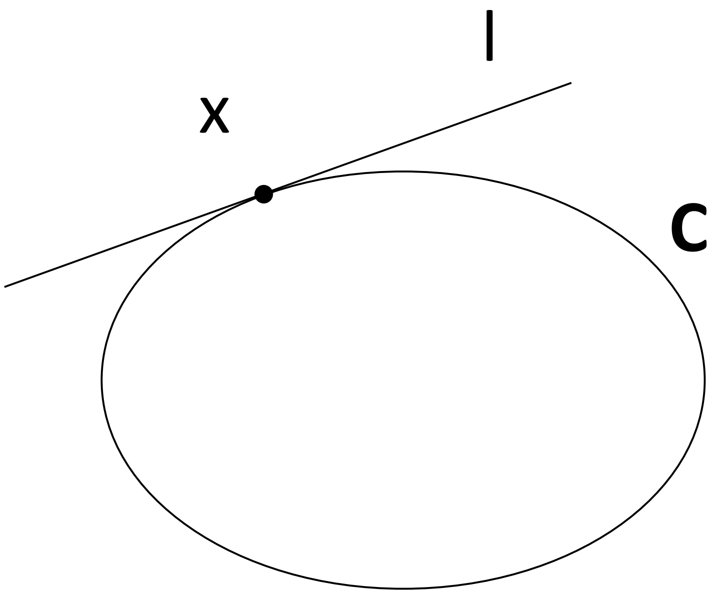
$$\left(x_i^2, x_i y_i, y_i^2, x_i, y_i, f\right)\mathbf{c} = 0 \qquad \mathbf{c} = \left(a, b, c, d, e, f\right)^{\mathsf{T}}$$

stacking constraints yields

$$\begin{bmatrix} x_1^2 & x_1 y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2 y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3 y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4 y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5 y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix}\mathbf{c} = 0$$

Tangent Lines to Conics

The line l tangent to **C** at point x on **C** is given by l=**C**x

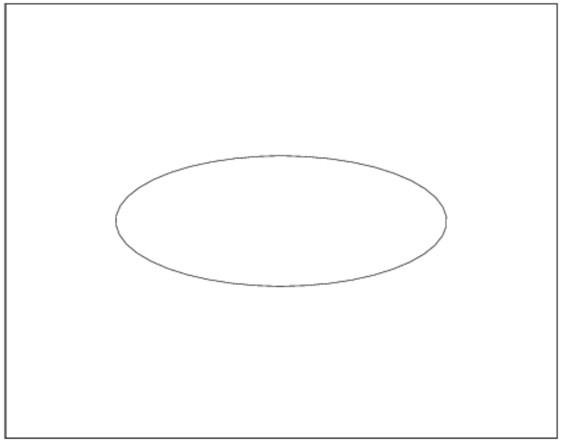


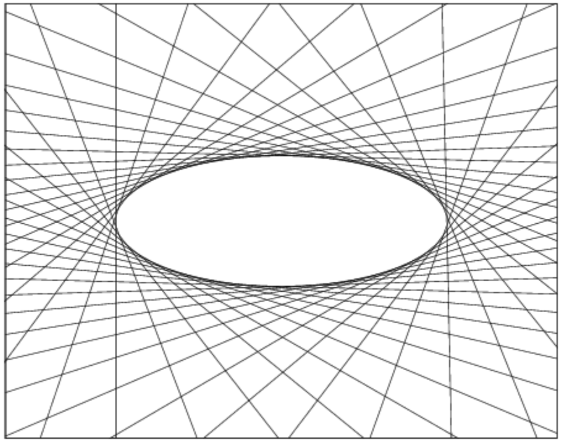
Dual Conics

A line tangent to the conic **C** satisfies $l^{\mathsf{T}}\mathbf{C}^{*}l=0$

In general (**C** full rank): $\mathbf{C}^{*}=\mathbf{C}^{-1}$

Dual conics = line conics = conic envelopes





Projective Transformations

Definition:

A *projectivity* is an invertible mapping h from P^2 to itself such that three points x_1, x_2, x_3 lie on the same line if and only if $h(x_1), h(x_2), h(x_3)$ do.

Theorem:

A mapping $h: P^2 \rightarrow P^2$ is a projectivity if and only if there exist a non-singular 3x3 matrix **H** such that for any point in P^2 reprented by a vector x it is true that $h(x) = \mathbf{H}x$

Definition: Projective transformation

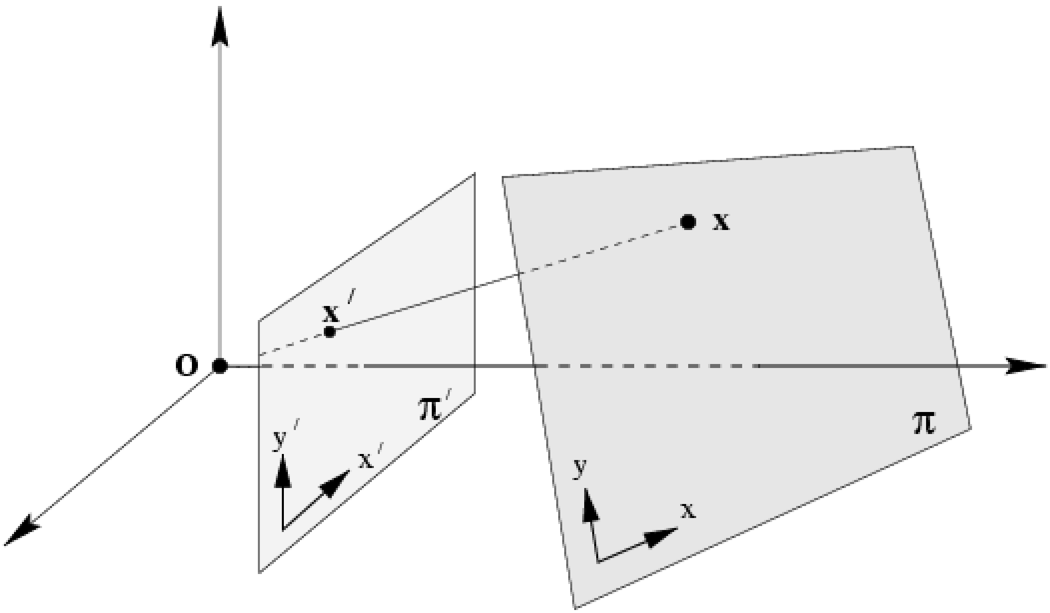
$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or $\mathbf{x}' = \mathbf{H}\,\mathbf{x}$

8DOF

projectivity=collineation=projective transformation=homography

Mapping between Planes



central projection may be expressed by x'=Hx

(application of theorem)

Removing Projective Distortion

select four points in a plane with know coordinates

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \qquad y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

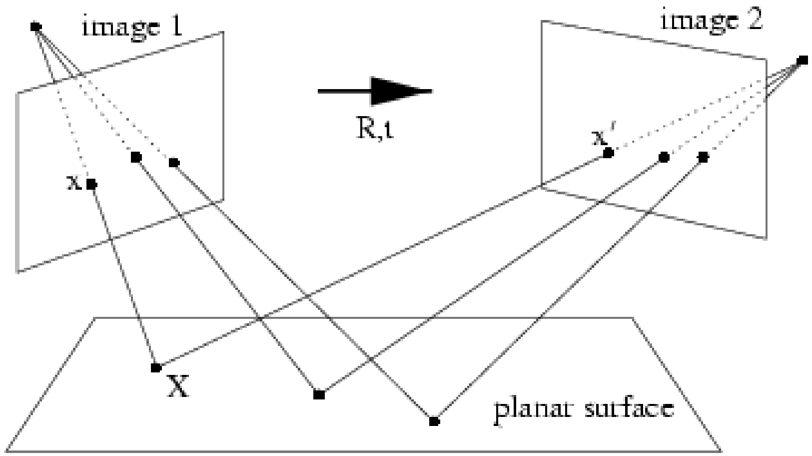
$$x'\left(h_{31}x + h_{32}y + h_{33}\right) = h_{11}x + h_{12}y + h_{13}$$
$$y'\left(h_{31}x + h_{32}y + h_{33}\right) = h_{21}x + h_{22}y + h_{23}$$

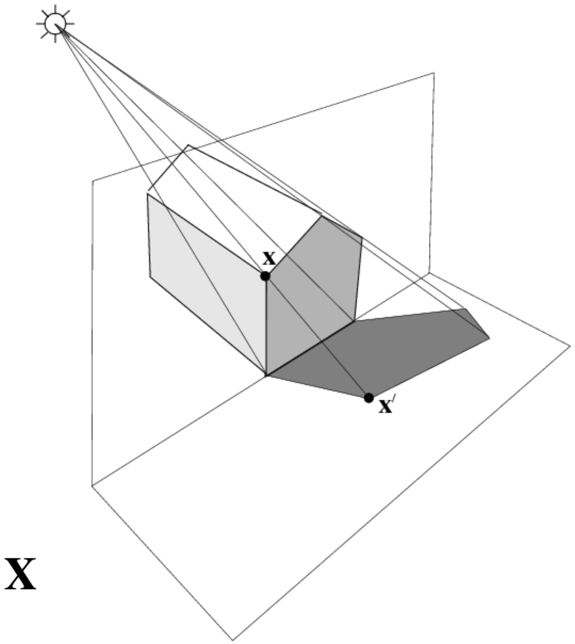
(linear in h_{ij})

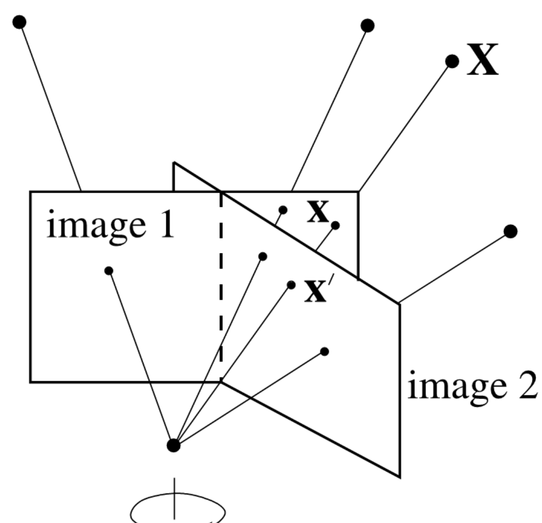
(2 constraints/point, 8DOF $\Rightarrow$ 4 points needed)

Remark: no calibration at all necessary

More Application Examples







Transformation of Lines and Conics

For a point transformation

$$\mathrm{x}' = \mathbf{H}\,\mathrm{x}$$

Transformation for lines

$$\mathrm{l}' = \mathbf{H}^{-\mathsf{T}}\,\mathrm{l}$$

Transformation for conics

$$\mathbf{C}' = \mathbf{H}^{-\mathsf{T}}\mathbf{C}\mathbf{H}^{-1}$$

Transformation for dual conics

$$\mathbf{C}'^{*} = \mathbf{H}\mathbf{C}^{*}\mathbf{H}^{\mathsf{T}}$$

A Hierarchy of Transformations

- Projective linear group
- Affine group (last row (0,0,1))
- Euclidean group (upper left 2x2 orthogonal)
- Oriented Euclidean group (upper left 2x2 det 1)

Alternative, characterize transformation in terms of elements or quantities that are preserved or *invariant*

e.g. Euclidean transformations leave distances unchanged

Similarity

Affine

Projective

Class I: Isometries

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \varepsilon\cos\theta & -\sin\theta & t_x \\ \varepsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad \varepsilon = \pm 1$$

orientation preserving: $\varepsilon = 1$ **(Euclidean transform)**
orientation reversing: $\varepsilon = -1$

$$\mathrm{x}' = \mathbf{H}_E\, \mathrm{x} = \begin{bmatrix} \mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \mathrm{x} \qquad \mathbf{R}^\mathsf{T}\mathbf{R} = \mathbf{I}$$

3DOF (1 rotation, 2 translation), can be computed from 2 point correspondences

special cases: pure rotation, pure translation

Invariants: length, angle, area

Class II: Similarities

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\mathrm{x}' = \mathbf{H}_S\ \mathrm{x} = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix} \mathrm{x} \qquad \mathbf{R}^{\mathsf{T}}\mathbf{R} = \mathbf{I}$$

4DOF (1 scale, 1 rotation, 2 translation),
can be computed from 2 point correspondences

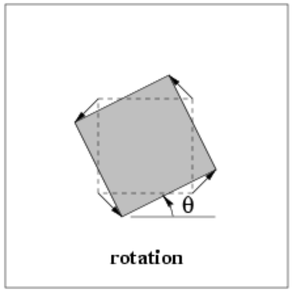
also know as *equi-form* (shape preserving)

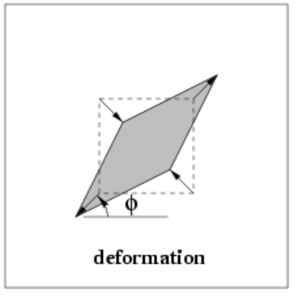
Invariants: ratios of length, angle, ratios of areas, parallel lines

Class III: Affine Transformations

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\mathrm{x}' = \mathbf{H}_A\,\mathrm{x} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix} \mathrm{x}$$





$$\mathbf{A} = \mathbf{R}(\theta)\mathbf{R}(-\phi)\mathbf{D}\mathbf{R}(\phi) \qquad \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

6DOF (2 scale, 2 rotation, 2 translation),

can be computed from 3 point correspondences

non-isotropic scaling! (2DOF: scale ratio and orientation)

Invariants: parallel lines, ratios of parallel lengths, ratios of areas

Class VI: Projective Transformations

$$\mathrm{x}' = \mathbf{H}_P\, \mathrm{x} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^{\mathsf{T}} & v \end{bmatrix} \mathrm{x} \qquad \mathrm{v} = (v_1, v_2)^{\mathsf{T}}$$

8DOF (2 scale, 2 rotation, 2 translation, 2 line at infinity)
can be computed from 4 point correspondences

Action non-homogeneous over the plane

Invariants: cross-ratio of four points on a line, (ratio of ratio)

Action of Affinities and Projectivities on Line at Infinity

$$\begin{bmatrix} \mathbf{A} & \mathrm{t} \\ 0^{\mathsf{T}} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}$$

Line at infinity stays at infinity, but points move along line

$$\begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^{\mathsf{T}} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v_1 x_1 + v_2 x_2 \end{pmatrix}$$

Line at infinity becomes finite, allows to observe vanishing points, horizon

Decomposition of Projective Transformations

$$\mathbf{H} = \mathbf{H}_S \mathbf{H}_A \mathbf{H}_P = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K} & 0 \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix}$$

S: similarity

A: Affine

P: Projective

$$\mathbf{A} = s\mathbf{RK} + \mathrm{tv}^\mathsf{T}$$

decomposition unique (if chosen s>0)

$\mathbf{K}$ upper-triangular, $\det \mathbf{K} = 1$

Example:

$$\mathbf{H} = \begin{bmatrix} 1.707 & 0.586 & 1.0 \\ 2.707 & 8.242 & 2.0 \\ 1.0 & 2.0 & 1.0 \end{bmatrix}$$

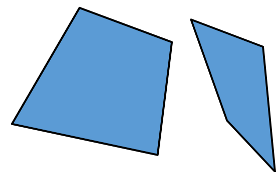
$$\mathbf{H} = \begin{bmatrix} 2\cos 45^\circ & -2\sin 45^\circ & 1.0 \\ 2\sin 45^\circ & 2\cos 45^\circ & 2.0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Summary of Transformations

Invariant Properties

Projective
8dof

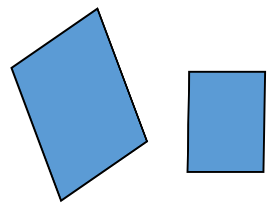
$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$



Concurrency, collinearity, order of contact (intersection, tangency, inflection, etc.), cross ratio

Affine
6dof

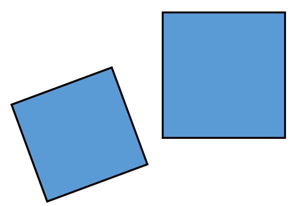
$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Parallellism, ratio of areas, ratio of lengths on parallel lines (e.g midpoints), linear combinations of vectors (centroids).
The line at infinity l_∞

Similarity
4dof

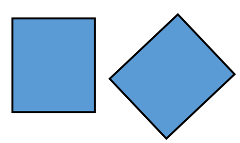
$$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Ratios of lengths, angles.
The circular points I,J

Euclidean
3dof

$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

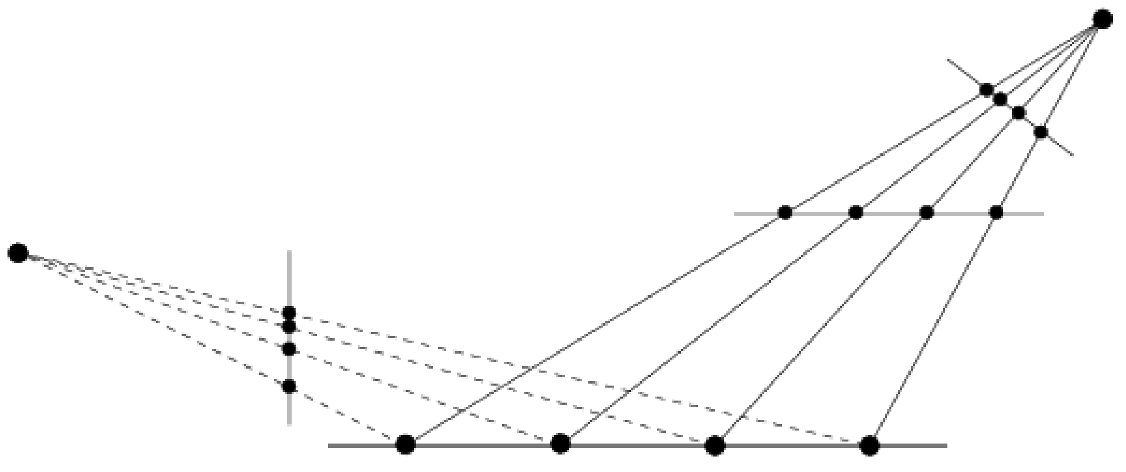


lengths, areas.

Cross Ratio

$$Cross(\bar{\mathrm{x}}_1, \bar{\mathrm{x}}_2, \bar{\mathrm{x}}_3, \bar{\mathrm{x}}_4) = \frac{|\bar{\mathrm{x}}_1, \bar{\mathrm{x}}_2||\bar{\mathrm{x}}_3, \bar{\mathrm{x}}_4|}{|\bar{\mathrm{x}}_1, \bar{\mathrm{x}}_3||\bar{\mathrm{x}}_2, \bar{\mathrm{x}}_4|} \qquad |\bar{\mathrm{x}}_i, \bar{\mathrm{x}}_j| = \det\begin{bmatrix} x_{i1} & x_{j1} \\ x_{i2} & x_{j2} \end{bmatrix}$$

Invariant under projective transformations



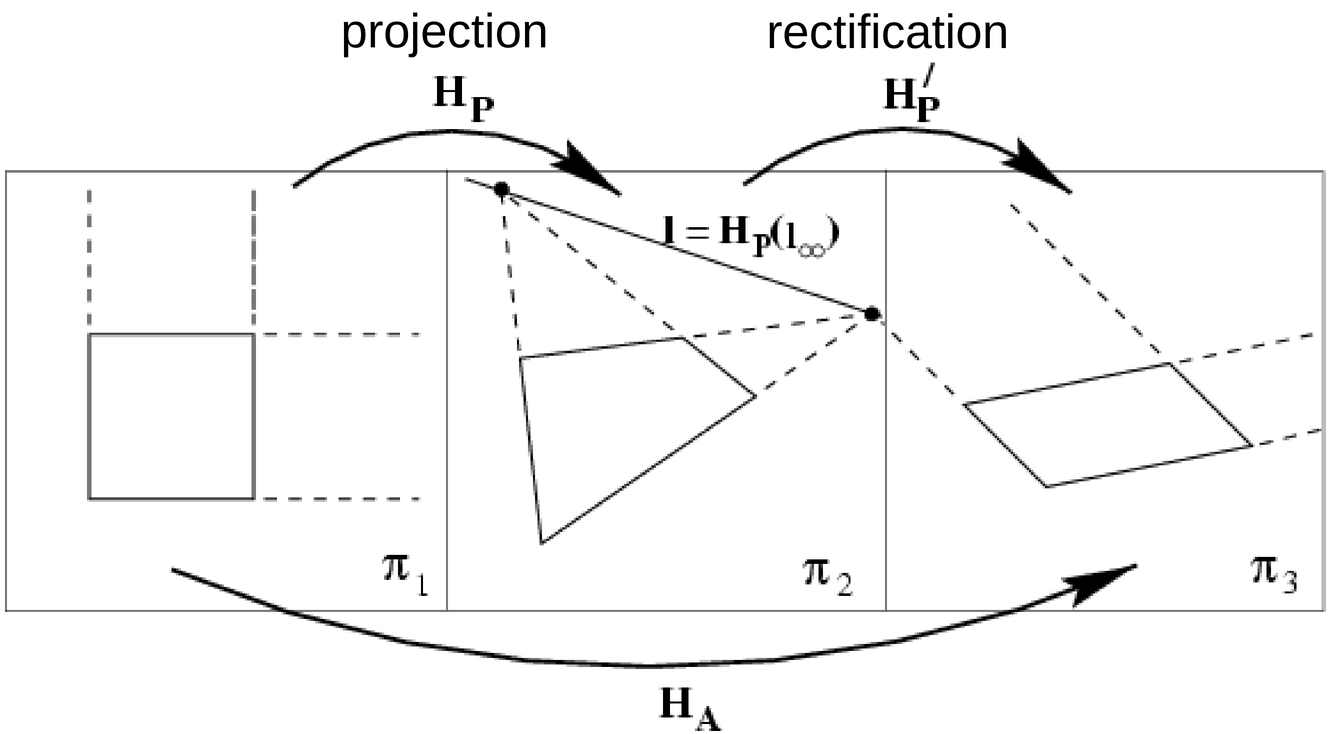
The Line at Infinity

$$\mathbf{l}'_\infty = \mathtt{H}_\mathtt{A}^{-\mathsf{T}}\mathbf{l}_\infty = \begin{bmatrix} \mathtt{A}^{-\mathsf{T}} & \mathbf{0} \\ -\mathbf{t}^\mathsf{T}\mathtt{A}^{-\mathsf{T}} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{l}_\infty$$

The line at infinity l_∞ is a fixed line under a projective transformation H if and only if H is an affinity

Note: not fixed pointwise

Affine Properties from Images

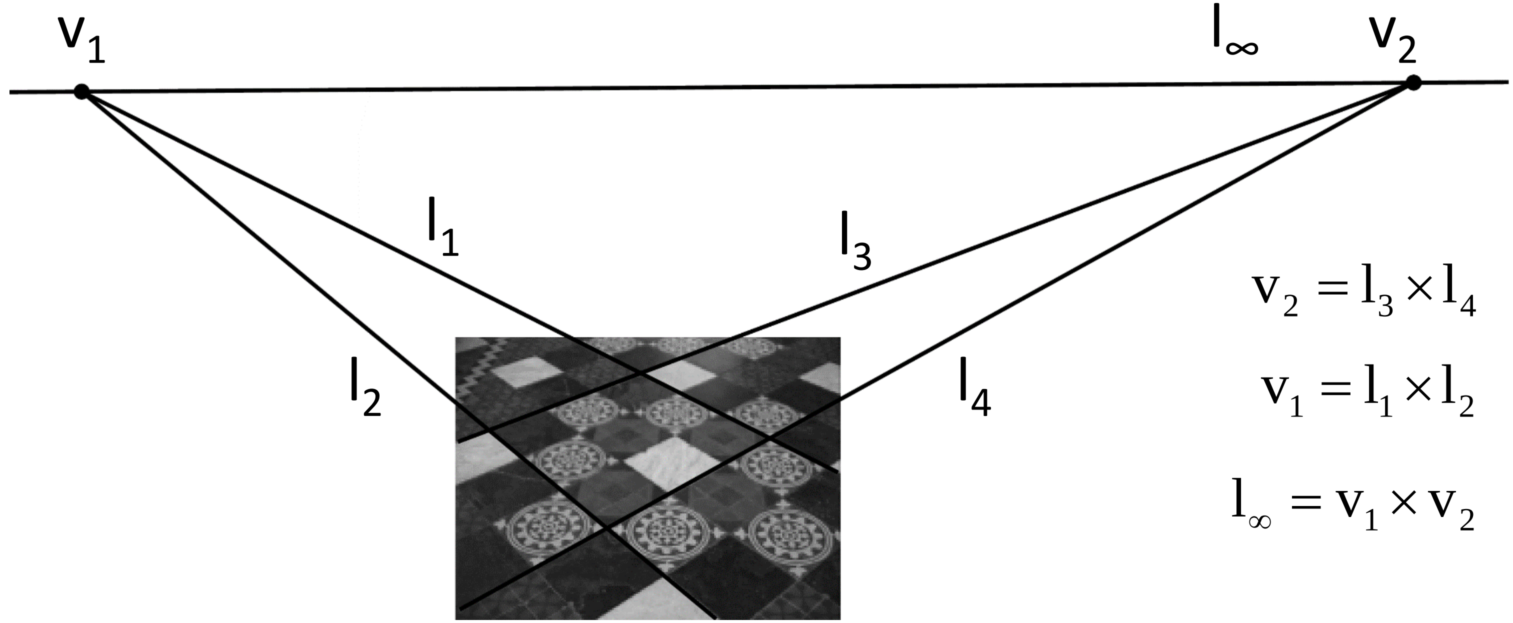


$$H'_P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}$$

$$\mathrm{l} = \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}^{\mathrm{T}}, l_3 \neq 0$$

$$H'^{-\mathrm{T}}_P (l_1, l_2, l_3)^{\mathrm{T}} = (0,0,1)^{\mathrm{T}} = l_\infty$$

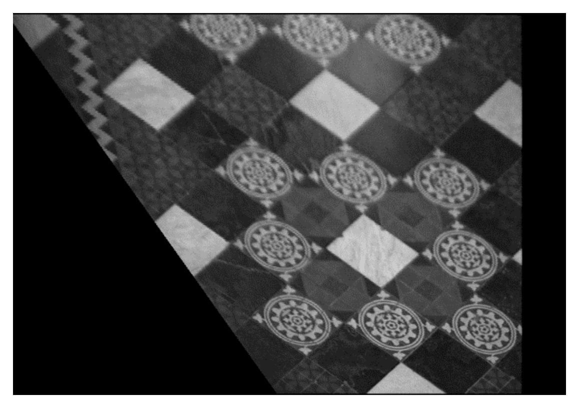
Affine Rectification



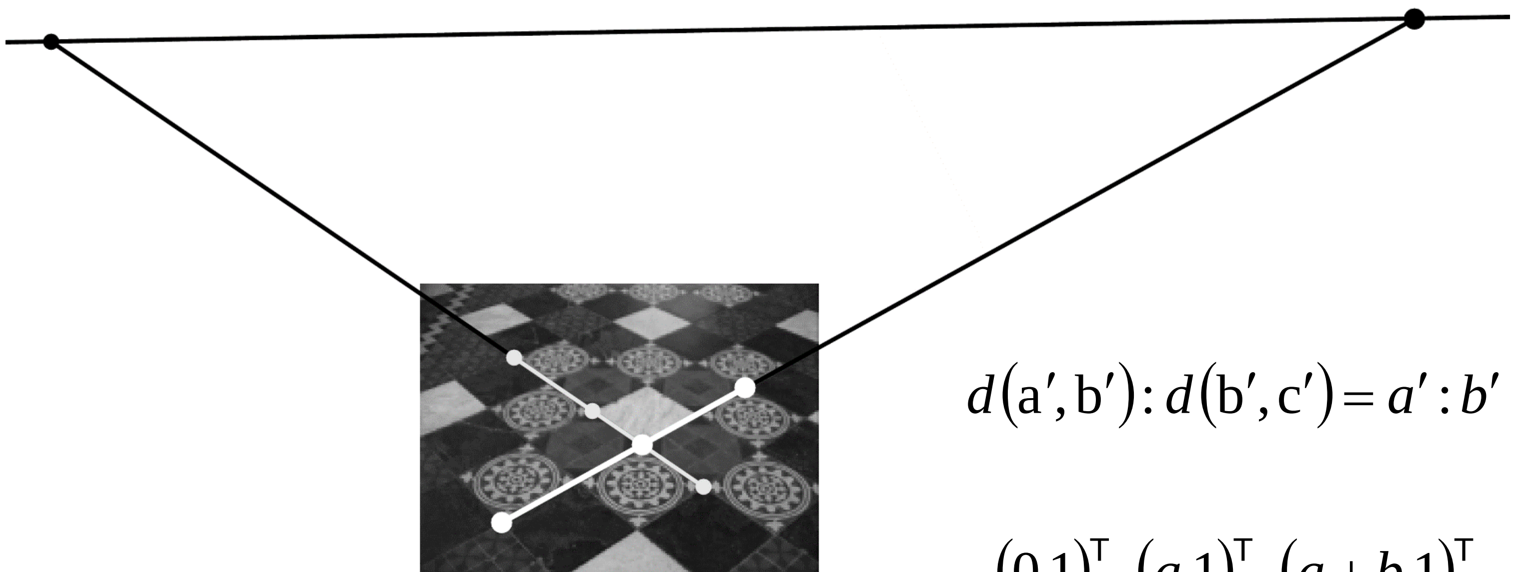
$$v_2 = l_3 \times l_4$$

$$v_1 = l_1 \times l_2$$

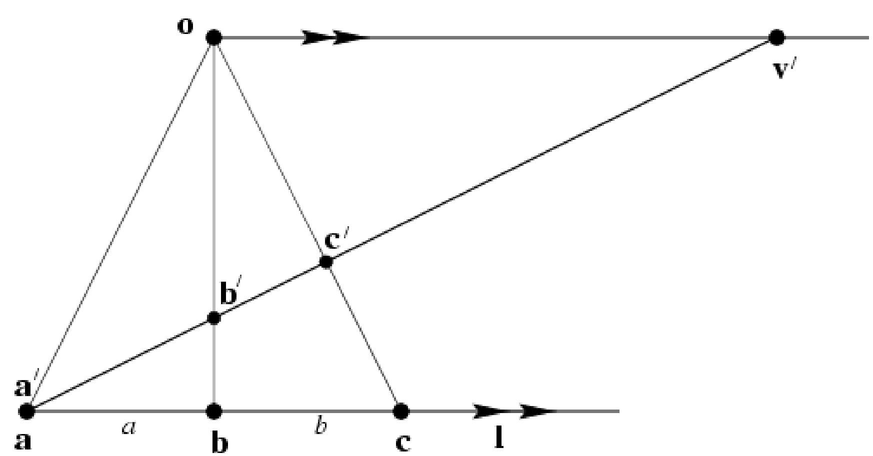
$$l_\infty = v_1 \times v_2$$



Distance Ratios



$$d(\mathrm{a}', \mathrm{b}') : d(\mathrm{b}', \mathrm{c}') = a' : b'$$



$$(0,1)^{\top}, (a,1)^{\top}, (a+b,1)^{\top}$$

$$\Big\downarrow \mathbf{H}$$

$$\mathrm{a}', \mathrm{b}', \mathrm{c}'$$

$$\mathrm{v}' = \mathbf{H}(1,0)^{\top}$$

The Circular Points

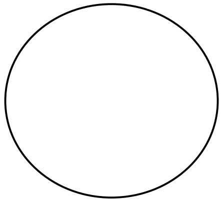
$$\mathrm{I} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \qquad \mathrm{J} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

$$\mathrm{I}' = \mathbf{H}_S\,\mathrm{I} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = se^{i\theta} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \mathrm{I}$$

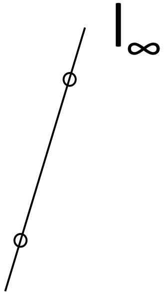
The circular points I, J are fixed points under the projective transformation **H** iff **H** is a similarity

The Circular Points

"circular points"



$$x_1^2 + x_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$
$$x_3 = 0$$



$$x_1^2 + x_2^2 = 0$$

$$\mathrm{I} = (1, i, 0)^\mathsf{T}$$

$$\mathrm{J} = (1, -i, 0)^\mathsf{T}$$

Algebraically, encodes orthogonal directions

$$\mathrm{I} = (1,0,0)^\mathsf{T} + i(0,1,0)^\mathsf{T}$$

Conic Dual to the Circular Points

$$\mathbf{C}_\infty^* = \mathrm{IJ}^\mathsf{T} + \mathrm{JI}^\mathsf{T} \qquad \mathbf{C}_\infty^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{C}_\infty^* = \mathbf{H}_S \mathbf{C}_\infty^* \mathbf{H}_S^\mathsf{T}$$

The dual conic $\mathbf{C}_\infty^*$ is fixed conic under the projective transformation **H** iff **H** is a similarity

Angles

Euclidean: $\mathrm{l} = (l_1, l_2, l_3)^{\mathsf{T}} \quad \mathrm{m} = (m_1, m_2, m_3)^{\mathsf{T}}$

$$\cos\theta = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}$$

Projective: $$\cos\theta = \frac{\mathrm{l}^{\mathsf{T}} \mathbf{C}_\infty^* \mathrm{m}}{\sqrt{(\mathrm{l}^{\mathsf{T}} \mathbf{C}_\infty^* \mathrm{l})(\mathrm{m}^{\mathsf{T}} \mathbf{C}_\infty^* \mathrm{m})}}$$

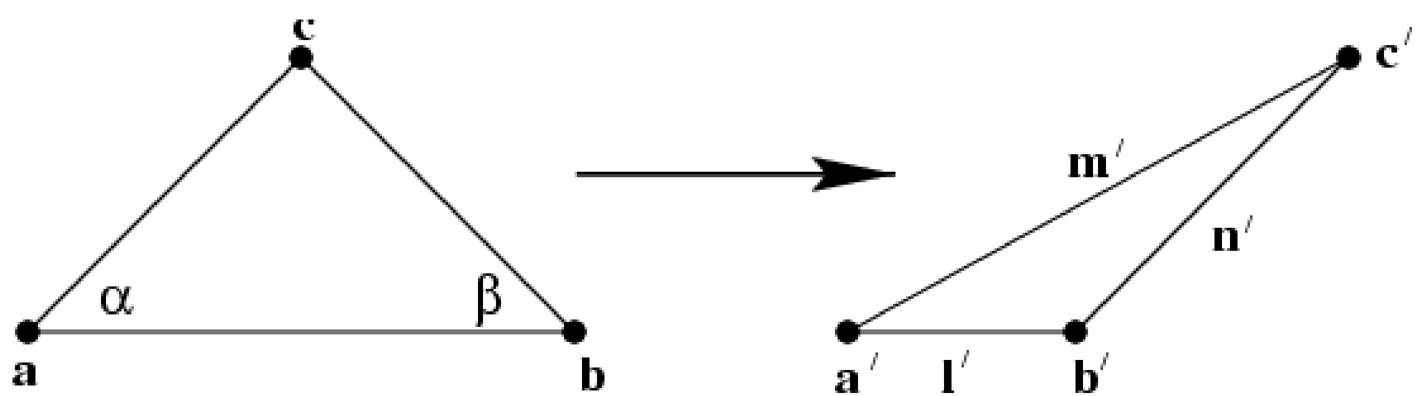
(This equation is Invariant to projective transform)

$\mathrm{l}^{\mathsf{T}} \mathbf{C}_\infty^* \mathrm{m} = 0$ If orthogonal

Length Ratios

$$\frac{d(b,c)}{d(a,c)} = \frac{\sin\alpha}{\sin\beta}$$

$\cos\alpha$ and $\cos\beta$ can be derived with the equations in the previous page



Metric Properties from Images

$$\mathbf{C}_\infty^*{}' = (\mathbf{H}_P\mathbf{H}_A\mathbf{H}_S)\mathbf{C}_\infty^*(\mathbf{H}_P\mathbf{H}_A\mathbf{H}_S)^\mathsf{T}$$

$$= (\mathbf{H}_P\mathbf{H}_A)\mathbf{H}_S\mathbf{C}_\infty^*\mathbf{H}_S^\mathsf{T}(\mathbf{H}_P\mathbf{H}_A)^\mathsf{T}$$

$$= (\mathbf{H}_P\mathbf{H}_A)\mathbf{C}_\infty^*(\mathbf{H}_P\mathbf{H}_A)^\mathsf{T}$$

$$= \begin{bmatrix} \mathbf{K}\mathbf{K}^\mathsf{T} & \mathbf{K}^\mathsf{T}\mathrm{v} \\ \mathrm{v}^\mathsf{T}\mathbf{K} & \mathrm{v}^\mathsf{T}\mathrm{v} \end{bmatrix}$$

Rectifying transformation from SVD

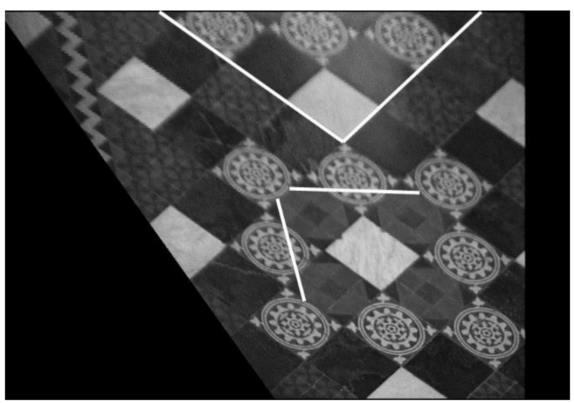
$$\mathbf{C}_\infty^*{}' = \mathbf{U}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{U}^\mathsf{T} \qquad \mathbf{H} = \mathbf{U}$$

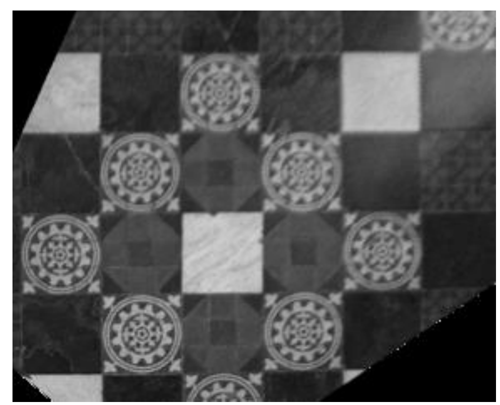
Metric from Affine

Suppose an image has been affinely rectified (**v**=0)

$$\begin{pmatrix} l'_1 & l'_2 & l'_3 \end{pmatrix} \begin{bmatrix} \mathbf{KK}^\mathsf{T} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} m'_1 \\ m'_2 \\ m'_3 \end{pmatrix} = 0$$

$$(l'_1 m'_1, l'_1 m'_2 + l'_2 m'_1, l'_2 m'_2)(k_{11}^2 + k_{12}^2, k_{22}k_{12}, k_{22}^2)^\mathrm{T} = 0$$





Metric from Projective

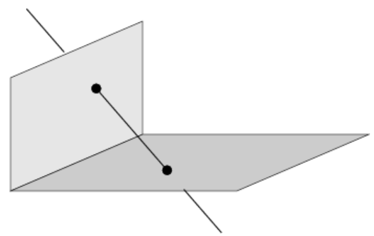
$$\mathrm{l}^{\mathsf{T}}\,\mathbf{C}_{\infty}^{*}\,\mathrm{m}=0 \qquad \begin{pmatrix} l_1' & l_2' & l_3' \end{pmatrix}\begin{bmatrix} \mathbf{K}\mathbf{K}^{\mathsf{T}} & \mathbf{K}^{\mathsf{T}}\mathrm{v} \\ \mathrm{v}^{\mathsf{T}}\mathbf{K} & \mathrm{v}^{\mathsf{T}}\mathrm{v} \end{bmatrix}\begin{pmatrix} m_1' \\ m_2' \\ m_3' \end{pmatrix}=0$$

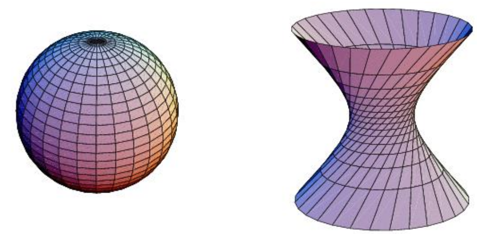
$$\left(l_1'm_1', 0.5\left(l_1'm_2'+l_2'm_1'\right), l_2'm_2', 0.5\left(l_1'm_3'+l_3'm_1'\right), 0.5\left(l_2'm_3'+l_3'm_2'\right), l_3'm_3'\right)\mathrm{c}=0$$

$$\mathbf{c}=(a,b,c,d,e,f)^{\mathrm{T}}$$

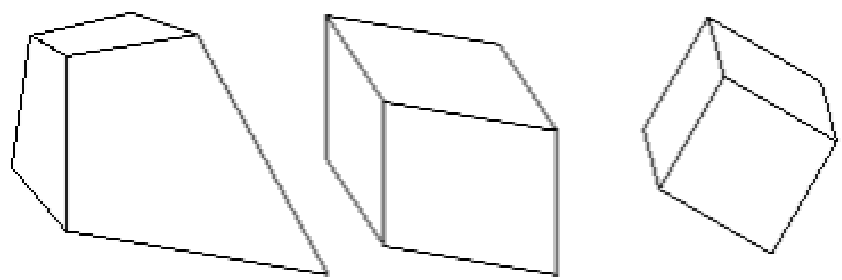
Projective 3D Geometry

- Points, lines, planes and quadrics

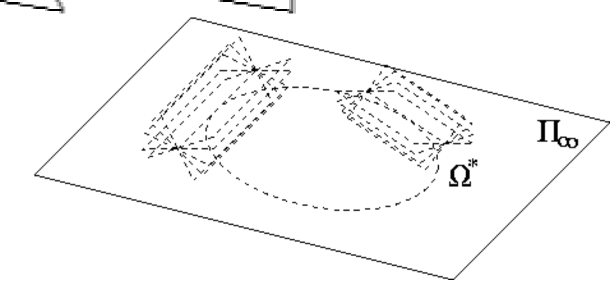




- Transformations



- Π_∞, ω_∞ and Ω_∞



3D Points

3D point

$\left(X, Y, Z\right)^{\mathsf{T}}$ in $\mathbf{R}^3$

$\mathrm{X} = \left(X_1, X_2, X_3, X_4\right)^{\mathsf{T}}$ in $\mathbf{P}^3$

$$\mathrm{X} = \left(\frac{X_1}{X_4}, \frac{X_2}{X_4}, \frac{X_3}{X_4}, 1\right)^{\mathsf{T}} = \left(X, Y, Z, 1\right)^{\mathsf{T}} \qquad \left(X_4 \neq 0\right)$$

projective transformation

$$\mathrm{X}' = \mathbf{H}\,\mathrm{X} \qquad \text{(4x4-1=15 dof)}$$

Dual: points $\leftrightarrow$ planes, lines $\leftrightarrow$ lines

Planes

3D plane

$$\pi_1 X + \pi_2 Y + \pi_3 Z + \pi_4 = 0$$

$$\pi_1 X_1 + \pi_2 X_2 + \pi_3 X_3 + \pi_4 X_4 = 0$$

$$\pi^{\mathsf{T}} \mathrm{X} = 0$$

Transformation

$$\mathrm{X}' = \mathbf{H}\,\mathrm{X}$$

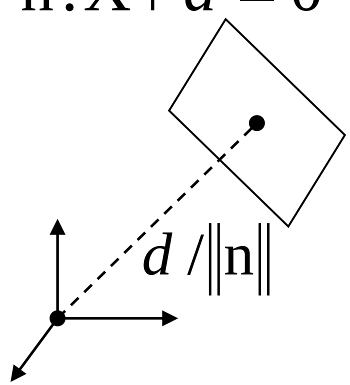
$$\pi' = \mathbf{H}^{-\mathsf{T}}\,\pi$$

Euclidean representation

$$\mathrm{n}.\tilde{\mathrm{X}} + d = 0$$

$$\mathrm{n} = (\pi_1, \pi_2, \pi_3)^{\mathsf{T}} \qquad \tilde{\mathrm{X}} = (X, Y, Z)^{\mathsf{T}}$$

$$\pi_4 = d \qquad X_4 = 1$$



Planes from Points

Solve π from $X_1^{\mathsf{T}}\pi = 0$, $X_2^{\mathsf{T}}\pi = 0$ and $X_3^{\mathsf{T}}\pi = 0$

$$\begin{bmatrix} X_1^{\mathsf{T}} \\ X_2^{\mathsf{T}} \\ X_3^{\mathsf{T}} \end{bmatrix} \pi = 0$$

(solve $\boldsymbol{\pi}$ as right nullspace of $\begin{bmatrix} X_1^{\mathsf{T}} \\ X_2^{\mathsf{T}} \\ X_3^{\mathsf{T}} \end{bmatrix}$)

Or implicitly from coplanarity condition

$$\det\begin{bmatrix} X_1 & (X_1)_1 & (X_2)_1 & (X_3)_1 \\ X_2 & (X_1)_2 & (X_2)_2 & (X_3)_2 \\ X_3 & (X_1)_3 & (X_2)_3 & (X_3)_3 \\ X_4 & (X_1)_4 & (X_2)_4 & (X_3)_4 \end{bmatrix} = 0$$

$$\det\left[X\ X_1\ X_2\ X_3\right] = 0$$

$$X_1 D_{234} - X_2 D_{134} + X_3 D_{124} - X_4 D_{123} = 0$$

$$\pi = \left(D_{234}, -D_{134}, D_{124}, -D_{123}\right)^{\mathsf{T}}$$

Points from Planes

Solve X from $\pi_1^{\mathsf{T}}\mathrm{X}=0,\ \pi_2^{\mathsf{T}}\mathrm{X}=0$ and $\pi_3^{\mathsf{T}}\mathrm{X}=0$

$$\begin{bmatrix}\pi_1^{\mathsf{T}}\\ \pi_2^{\mathsf{T}}\\ \pi_3^{\mathsf{T}}\end{bmatrix}\mathrm{X}=0$$ (solve X as right nullspace of $\begin{bmatrix}\pi_1^{\mathsf{T}}\\ \pi_2^{\mathsf{T}}\\ \pi_3^{\mathsf{T}}\end{bmatrix}$)

Points and Planes

- Projective transformation

Under the point transformation $\boldsymbol{X}' = H\boldsymbol{X}$, a plane transforms as $\boldsymbol{\pi}' = H^{-T}\boldsymbol{\pi}$

- Parametrized points on a plane

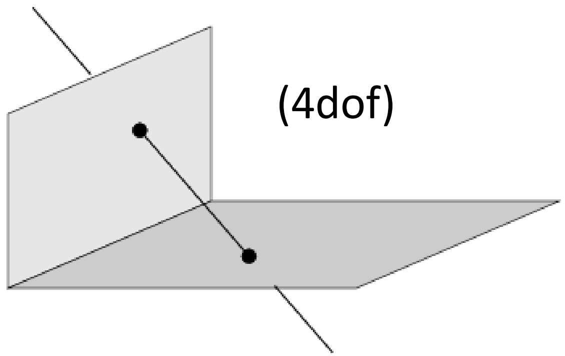
Representing a plane $\pi = (a, b, c, d)^{\mathsf{T}}$ by its span

$\mathrm{X} = \mathbf{M}\,\mathrm{x}$ x is a 3-vector parameter (a point on the projective plane)

$\pi^{\mathsf{T}}\mathbf{M} = 0$

M is not unique $\mathbf{M} = \begin{bmatrix} \mathrm{p} \\ \mathrm{I}_{3\mathrm{x}3} \end{bmatrix}$ $p = \left(-\frac{b}{a}, -\frac{c}{a}, -\frac{d}{a} \right)$

Lines



(4dof)

Defined as the join of two points A, B

$$W = \begin{bmatrix} A^{\mathsf{T}} \\ B^{\mathsf{T}} \end{bmatrix} \qquad \lambda A + \mu B$$

(Dual) Defined as the intersection of two planes P, Q

$$W^{*} = \begin{bmatrix} P^{\mathsf{T}} \\ Q^{\mathsf{T}} \end{bmatrix} \qquad \lambda P + \mu Q$$

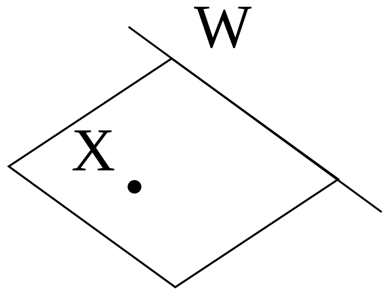
$$W^{*} W^{\mathsf{T}} = W W^{*\mathsf{T}} = 0_{2\times 2}$$

Example: X-axis

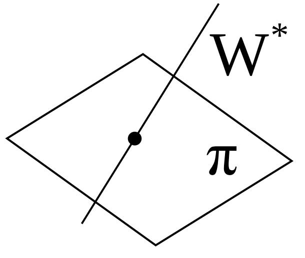
$$W = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad W^{*} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Points, Lines and Planes

$$\mathbf{M} = \begin{bmatrix} W \\ X^{\mathsf{T}} \end{bmatrix} \qquad \mathbf{M}\pi = 0$$



$$\mathbf{M} = \begin{bmatrix} W^* \\ \pi^{\mathsf{T}} \end{bmatrix} \qquad \mathbf{M}X = 0$$



Plücker Matrices

Plücker matrix (4x4 skew-symmetric homogeneous matrix)

$$l_{ij} = A_i B_j - B_i A_j$$

$$\mathrm{L} = \mathrm{AB}^{\mathsf{T}} - \mathrm{BA}^{\mathsf{T}}$$

1. L has rank 2 $\quad \mathrm{LW}^{*\mathsf{T}} = 0_{4\times 2}$
2. 4dof
3. generalization of $\mathrm{l} = \mathrm{x} \times \mathrm{y}$
4. L independent of choice A and B
5. Transformation $\quad \mathrm{L}' = \mathrm{HLH}^{\mathsf{T}}$

<u>Example</u>: X-axis

$$\mathrm{L} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Plücker Matrices

Dual Plücker matrix L^*

$$\mathrm{L}^* = \mathrm{P}\mathrm{Q}^\mathsf{T} - \mathrm{Q}\mathrm{P}^\mathsf{T}$$

$$\mathrm{L}^{*'} = \mathrm{H}^{-\mathsf{T}}\mathrm{L}\mathrm{H}^{-1}$$

Correspondence

$$l_{12} : l_{13} : l_{14} : l_{23} : l_{42} : l_{34} = l^*_{34} : l^*_{42} : l^*_{23} : l^*_{14} : l^*_{13} : l^*_{12}$$

Join and incidence

$\pi = \mathrm{L}^*\mathrm{X}$ (plane through point and line)

$\mathrm{L}^*\mathrm{X} = 0$ (point on line)

$\mathrm{X} = \mathrm{L}\pi$ (intersection point of plane and line)

$\mathrm{L}\pi = 0$ (line in plane)

$[\mathrm{L}_1, \mathrm{L}_2, \ldots]\pi = 0$ (coplanar lines)

Quadrics and Dual Quadrics

$X^{T}QX = 0$ (Q : 4x4 symmetric matrix)

$$Q = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \\ \circ & \circ & \circ & \bullet \end{bmatrix}$$

1. 9 d.o.f.
2. in general 9 points define quadric
3. det Q=0 ↔ degenerate quadric
4. Polar plane $\pi = QX$
5. (plane ∩ quadric)=conic $C = M^{T}QM$ $\pi : X = Mx$
6. transformation $Q' = H^{-T}QH^{-1}$

Q^{*}: dual quadric, equations on planes

$\pi^{T}Q^{*}\pi = 0$

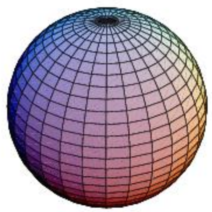
1. relation to quadric $Q^{*} = Q^{-1}$ (non-degenerate)
2. transformation $Q'^{*} = HQ^{*}H^{T}$

Quadric Classification

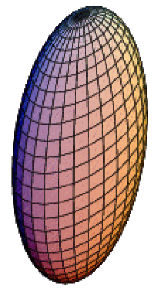
Rank	Sign.	Diagonal	Equation	Realization
4	4	(1,1,1,1)	$X^2+ Y^2+ Z^2+1=0$	No real points
	2	(1,1,1,-1)	$X^2+ Y^2+ Z^2=1$	Sphere
	0	(1,1,-1,-1)	$X^2+ Y^2= Z^2+1$	Hyperboloid (1S)
3	3	(1,1,1,0)	$X^2+ Y^2+ Z^2=0$	Single point
	1	(1,1,-1,0)	$X^2+ Y^2= Z^2$	Cone
2	2	(1,1,0,0)	$X^2+ Y^2= 0$	Single line
	0	(1,-1,0,0)	$X^2= Y^2$	Two planes
1	1	(1,0,0,0)	$X^2=0$	Single plane

Quadric Classification

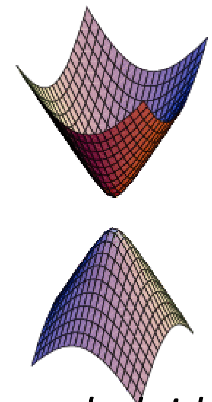
Projectively equivalent to *sphere:*



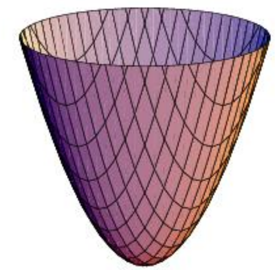
sphere



ellipsoid

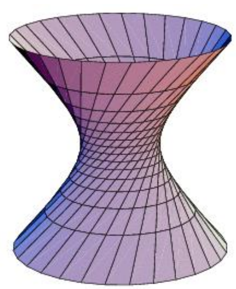


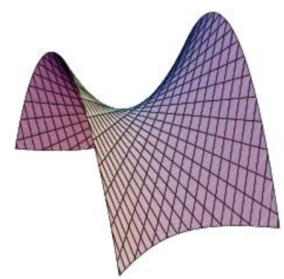
hyperboloid of two sheets



paraboloid

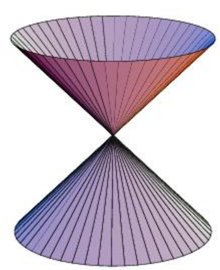
Ruled quadrics: (contain straight line)



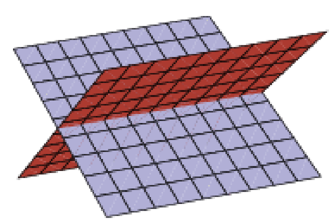


hyperboloids of one sheet

Degenerate ruled quadrics:

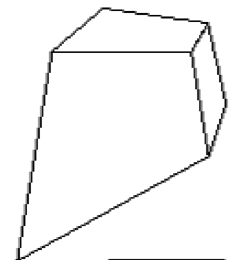
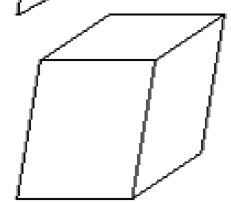
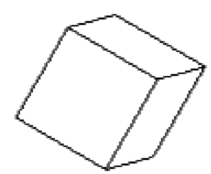
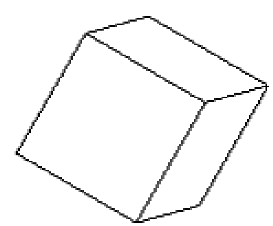


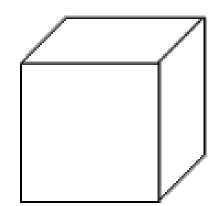
cone



two planes

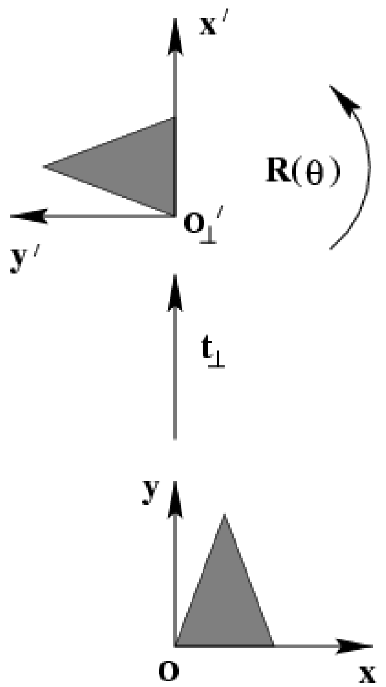
Hierarchy of Transformations

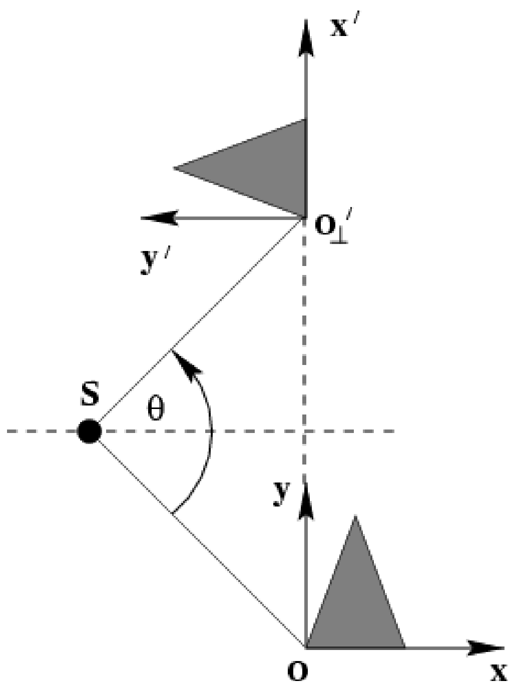
				Invariant Properties
	Projective 15dof	$\begin{bmatrix} A & t \\ v^T & v \end{bmatrix}$		Intersection and tangency
5 for affine scaling	Affine 12dof	$\begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix}$		Parallellism of planes, Volume ratios, centroids, **The plane at infinity π_∞**
3 for rotation 3 for translation 1 for isotropic scaling	Similarity 7dof	$\begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix}$		**The absolute conic Ω_∞**
	Euclidean 6dof	$\begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$		Volume



Screw Decomposition

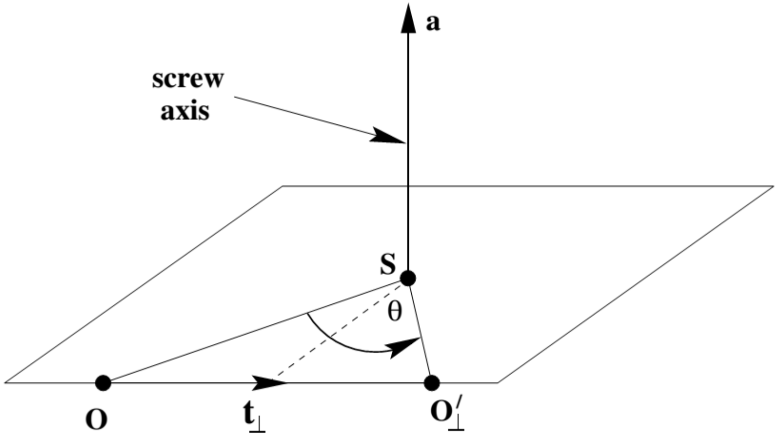
Any particular translation and rotation is equivalent to a rotation about a screw axis and a translation along the screw axis.

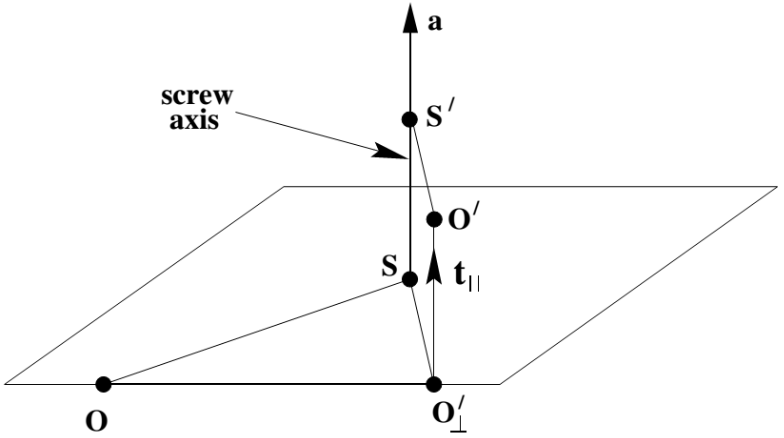




Screw Decomposition

Any particular translation and rotation is equivalent to a rotation about a screw axis and a translation along the screw axis.





screw axis // rotation axis

$$t = t_{//} + t_{\perp}$$

The Plane at Infinity

$$\pi'_{\infty} = \mathbf{H}_A^{-\mathsf{T}} \pi_{\infty} = \begin{bmatrix} \mathbf{A}^{-\mathsf{T}} & 0 \\ -\mathbf{A}\,\mathrm{t} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \pi_{\infty}$$

The plane at infinity π_{∞} is a fixed plane under a projective transformation H iff H is an affinity

1. canonical position $\pi_{\infty} = (0,0,0,1)^{\mathsf{T}}$
2. contains directions $\mathrm{D} = (X_1, X_2, X_3, 0)^{\mathsf{T}}$
3. two planes are parallel $\Leftrightarrow$ line of intersection in π_{∞}
4. line // line (or plane) $\Leftrightarrow$ point of intersection in π_{∞}