National Taiwan University
Department of Electrical Engineering
Algorithms, Spring 2024

Classes:
Prof. Shao-Hua Sun &
Prof. Hui-Ru Jiang

Homework #3

(on-line submission due: 9:10 AM, 9 May 2024; late submissions are not allowed)

TA: Yi-Chen Lin r12943096@ntu.edu.tw and Wei-Kai Liao r12921a23@ntu.edu.tw

Collaboration policy: You can discuss the problems with other students, but you must write the final answers by yourself. Please specify all of your collaborators (names and student id's) or resources (websites) for each problem. If you solve some problems by yourself, please also specify "no collaborators". Homework without collaborator specification will be penalized by 50%.

1. (10 pts) Suppose we have a set $S = \{a_1, a_2, ..., a_n\}$ of n proposed activities that wish to use a resource, such as a lecture hall, which can serve only one activity at a time. Each activity a_i has a start time s_i and a finish time f_i , where $0 \leq s_i < f_i < \infty$. If selected, activity a_i takes place during the half-open time interval $[si, fi)$. Activities a_i and a_j are compatible if the intervals $[si, fi)$ and $[si, fi)$ do not overlap. That is, a_i and a_j are compatible if $s_i \geq f_j$ or $s_j \geq f_i$. In the activity-selection problem, we wish to select a maximum-size subset of mutually compatible activities. Suppose that instead of always selecting the first activity to finish, we instead select the last activity to start that is compatible with all previously selected activities. Describe how this approach is a greedy algorithm, and prove that it yields an optimal solution.

2. (10 pts) Suppose you are given two sets A and B, each containing n positive integers. You can choose to reorder each set however you like. After reordering, let a_i be the ith element of set A, and let b_i be the ith element of set B. You then receive a payoff $\prod_{i=1}^{n} a_i^{b_i}$. Give an algorithm that will maximize your payoff. Prove that your algorithm maximizes the payoff, and state its running time.

3. (10 pts) Suppose that a data file contains a sequence of 8-bit characters such that all 256 characters are about equally common: the maximum character frequency is less than twice the minimum character frequency. Prove that Huffman coding in this case is no more efficient than using an ordinary 8-bit fixed-length code.

4. (10 pts) There are two types of professional wrestlers: "babyfaces" ("good guys") and "heels" ("bad guys"). Between any pair of professional wrestlers, there may or may not be a rivalry. Suppose we have n professional wrestlers and we have a list of r pairs of wrestlers for which there are rivalries. Give an $O(n+r)$-time algorithm that determines whether it is possible to designate some of the wrestlers as babyfaces and the remainder as heels such that each rivalry is between a babyface and a heel. If it is possible to perform such a designation, your algorithm should produce it.

5. (10 pts) Show that in an undirected graph, classifying an edge (u, v) as a tree edge or a back edge according to whether (u, v) or (v, u) is encountered first during the depth-first search is equivalent to classifying it according to the ordering of the four types in the classification scheme.

6. (10 pts) Give an algorithm that determines whether or not a given undirected graph $G = (V, E)$ contains a cycle. Your algorithm should run in $O(V)$ time, independent of $|E|$.

7. (10 pts) Give an $O(V + E)$-time algorithm to compute the component graph of a directed graph $G = (V, E)$. Make sure that there is at most one edge between two vertices in the component graph your algorithm produces.

8. (10 pts) Given a graph G and a minimum spanning tree T, suppose that we decrease the weight of one of the edges in T. Show that T is still a minimum spanning tree for G. More formally, let T be a minimum spanning tree for G with edge weights given by weight function w. Choose one edge $(x, y) \in T$ and a positive number k, and define the weight function w' by

$$w'(u,v) = \begin{cases} w(u,v) & \text{if } (u,v) \neq (x,y) \ , \\ w(x,y) - k & \text{if } (u,v) = (x,y) \ . \end{cases}$$

   Show that T is a minimum spanning tree for G with edge weights given by w'.

9. (10 pts) Suppose that all edge weights in a graph are integers in the range from 1 to $|V|$. How fast can you make Kruskal's algorithm run? What if the edge weights are integers in the range from 1 to W for some constant W?

10. (10 pts) Binary search of a sorted array takes logarithmic search time, but the time to insert a new element is linear in the size of the array. We can improve the time for insertion by keeping several sorted arrays. Specifically, suppose that we wish to support SEARCH and INSERT on a set of n elements. Let $k = \lceil \lg(n+1) \rceil$, and let the binary representation of n be $\langle n_{k-1}, n_{k-2}, \ldots, n_0 \rangle$. We have k sorted arrays $A_0, A_1, \ldots, A_{k-1}$, where for $i = 0, 1, \ldots, k-1$, the length of array A_i is 2^i. Each array is either full or empty, depending on whether $n_i = 1$ or $n_i = 0$, respectively. The total number of elements held in all k arrays is therefore $\sum_{i=0}^{k-1} n_i 2^i = n$. Although each individual array is sorted, elements in different arrays bear no particular relationship to each other.

    (a) Describe how to perform the SEARCH operation for this data structure. Analyze its worst-case running time.

    (b) Describe how to perform the INSERT operation. Analyze its worst-case and amortized running times.

    (c) Discuss how to implement DELETE.

Study and, in general, the pursuit of truth and beauty is a sphere of activity in which we are permitted to remain children all our lives. – A. Einstein