数学物理方法

Methods of Mathematical Physics

第七章 行波法

travelling wave method

武 汉 大 学

物理科学与技术学院

习题课

一、一维公式的应用

二、用行波法求解定解问题

*三、三维波动问题的求解

内容小结

1、
$$\begin{cases} u_{tt}=a^2u_{xx}\ ,\ -\infty<x<\infty & (1) \\ u|_{t=0}=\varphi(x),\ -\infty<x<\infty & (2) \\ u_t\ |_{t=0}=\psi(x),\ -\infty<x<\infty & (3) \end{cases}$$

的解为

$$\boxed{u(x,t)=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\alpha)d\alpha \qquad (6)}$$

（1）的通解为 $\boxed{u(x,t)=f_1(x+at)+f_2(x-at)}$

2、行波法：引入坐标变换简化方程；

先求通解，再求特解。

内容小结

3、对于一般强迫振动：
$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t) \\ u|_{t=0} = \varphi(x) \\ u_t|_{t=0} = \psi(x) \end{cases}$$

令 $u = u^I + u^{II}$， 使：

$$u^I: \begin{cases} u_{tt}^I - a^2 u_{xx}^I = 0 \\ u^I|_{t=0} = \varphi(x) \\ u_t^I|_{t=0} = \psi(x) \end{cases}$$

$$u^{II}: \begin{cases} u_{tt}^{II} - a^2 u_{xx}^{II} = f(x,t) \\ u^{II}|_{t=0} = 0 \\ u_t^{II}|_{t=0} = 0 \end{cases}$$

$$u = \frac{1}{2}[\varphi(x+at)+\varphi(x-at)]$$
$$+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\alpha)d\alpha$$
$$+\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\alpha,\tau)d\alpha d\tau$$

内容小结

4、一般三维无源波动问题：

$$\begin{cases} u_{tt} - a^2 \Delta u = 0 \\ u\big|_{t=0} = \varphi(M) \\ u_t\big|_{t=0} = \psi(M) \end{cases}$$

可以推得其解为：

$$\boxed{u(M,t) = \frac{1}{4\pi a}\left[\frac{\partial}{\partial t}\iint_{s_{at}^M} \frac{\varphi(M')}{at} ds + \iint_{s_{at}^M} \frac{\psi(M')}{at} ds\right]}$$

—泊松（Poisson）公式

其中， s_{at}^M –以M为中心at为半径的球面；

$M' = M'(x', y', z')$ –球面s_{at}^M上的点；

一、一维公式的应用

行波法习题课

求解
$$\begin{cases} u_{xx} - u_{yy} = 1 \\ u(x,0) = \sin x \\ u_y(x,0) = x \end{cases}$$

解
$$\begin{cases} u^I_{yy} - u^I_{xx} = 0 \\ u^I(x,0) = \sin x \\ u^I_y(x,0) = x \end{cases} \qquad \begin{cases} u^{II}_{yy} - u^{II}_{xx} = -1 \\ u^{II}(x,0) = 0 \\ u^{II}_y(x,0) = 0 \end{cases}$$

$$u^I(x,t) = \sin x \cos y + xy \qquad u^{II}(x,t) = -\frac{1}{2}y^2$$

$$\boxed{u(x,t) = \sin x \cos y + xy - \frac{1}{2}y^2}$$

二、用行波法求解定解问题

1、求解半无界弦的自由运动

$$\begin{cases} u_{tt} = a^2 u_{xx} \ , & (1) \\ u|_{t=0} = \varphi(x), \ 0 \le x < \infty & (2) \\ u_t|_{t=0} = \psi(x), \ 0 \le x < \infty & (3) \\ u(0,t) = 0 & (4) \end{cases}$$

$$f_1(x) = \frac{1}{2}\varphi(x) + \frac{1}{2a}\int_{x_0}^{x} \psi(\alpha)d\alpha + \frac{c}{2} \quad (6)$$

$$f_2(x) = \frac{1}{2}\varphi(x) + \frac{1}{2a}\int_{x}^{x_0} \psi(\alpha)d\alpha - \frac{c}{2} \quad (7)$$

$$x \ge 0$$

$$f_1(at) + f_2(-at) = 0 \quad at > 0 \ (8)$$

(1) $x - at \ge 0$:

$$u(x,t) = \frac{1}{2}[\varphi(x+at) + \varphi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\alpha)d\alpha$$

端点影响未传到。

(2) $x - at < 0$:

$$u(x,t) = \frac{1}{2}[\varphi(x+at) - \underbrace{\varphi(at-x)]}_{\text{反射波}} + \frac{1}{2a}\int_{at-x}^{x+at} \psi(\alpha)d\alpha$$

端点影响已传到

二、用行波法求解定解问题

2、用行波法求解

$$\begin{cases} u_{xx}+2u_{xy}-3u_{yy}=0 & (1) \\ u(x,0)=3x^2 & (2) \\ u_y(x,0)=0 & (3) \end{cases}$$

令 $\begin{cases} \xi = 3x-y \\ \eta = x+y \end{cases}$

方程（1）的通解为

$$u(x,y)=f_1(3x-y)+f_2(x+y) \quad (4)$$

$f_1(3x)=\frac{3}{4}(3x^2-c)$ 令 $3x=X$, 则 $f_1(X)=\frac{3}{4}(\frac{X^2}{3}-c)$

$f_2(x)=\frac{3}{4}(x^2+c)$

$$\boxed{u(x,y)=3x^2+y^2}$$

二、用行波法求解定解问题

3、求解定解问题

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial x \partial y} = x^2 y \quad x>1, y>0 & (1) \\
u(x,0) = x^2 \ , & (2) \\
u(1,y) = \cos y \ , & (3)
\end{cases}
$$

方程（1）的通解为

$$u(x,y) = \frac{x^3 y^2}{6} + f(x) + g(y) \quad (4)$$

$$\boxed{u(x,y) = \frac{x^3 y^2}{6} + x^2 + \cos y - \frac{y^2}{6} - 1}$$

二、用行波法求解定解问题

4、求解弦振动方程的古沙问题

$$\begin{cases} u_{tt} = u_{xx} \quad (-t < x < t, t > 0) & (1) \\ u(x,-x) = \varphi(x) \quad (x \le 0) & (2) \\ u(x,x) = \psi(x) \quad (x \ge 0) & (3) \\ \varphi(0) = \psi(0) & (4) \end{cases}$$

通解： $u(x,y) = f_1(x+t) + f_2(x-t) \quad (5)$

$$f_2(y) = \varphi(\frac{y}{2}) - f_1(0), \quad y \le 0$$

$$f_1(y) = \psi(\frac{y}{2}) - f_2(0), \quad y \ge 0$$

$$\boxed{u(x,t) = \psi(\frac{x+t}{2}) + \varphi(\frac{x-t}{2}) + \varphi(0)}$$

*三、某些三维波动问题的求解

行波法
习题课

1、求解

常规解法—泊松公式：

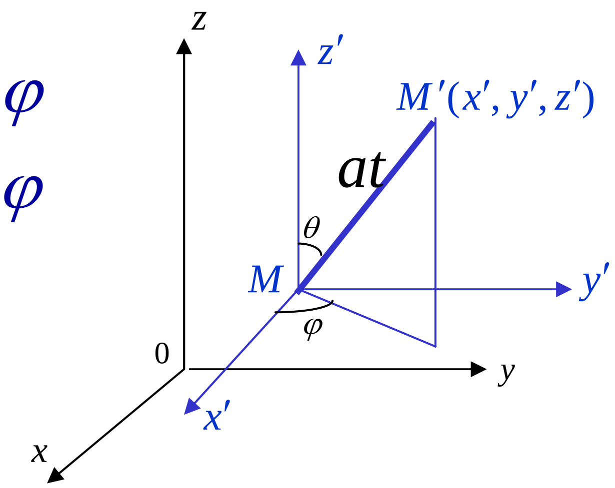
$$
\begin{cases}
u_{tt} = a^2 \Delta u \\
u|_{t=0} = x^3 + y^2 z \\
u_t|_{t=0} = 0
\end{cases}
$$

$$u(M,t)=\frac{1}{4\pi a}\left[\frac{\partial}{\partial t}\iint_{s_{at}^{M}}\frac{\varphi(M')}{at}ds+\iint_{s_{at}^{M}}\frac{\psi(M')}{at}ds\right]$$

$$x' = x + at\sin\theta\cos\varphi$$

$$y' = y + at\sin\theta\sin\varphi$$

$$z' = z + at\cos\theta$$



$$\varphi(M') = x'^3 + y'^2 z'$$

$$= (x + at\sin\theta\cos\varphi)^3 + (y + at\sin\theta\sin\varphi)^2 (z + at\cos\theta)$$

$$= x^3 + 3x^2 at\sin\theta\cos\varphi + 3x(at)^2\sin^2\theta\cos^2\varphi + (at)^3\sin^3\theta\cos^3\varphi$$

$$+ (y^2 + 2yat\sin\theta\sin\varphi + (at)^2\sin^2\theta\sin^2\varphi)(z + at\cos\theta)$$

*三、某些三维波动问题的求解

1、求解

常规解法—泊松公式:

$$\varphi(M')=x^3+3x^2at\sin\theta\cos\varphi+3x(at)^2\sin^2\theta\cos^2\varphi$$
$$+(at)^3\sin^3\theta\cos^3\varphi+y^2z+y^2at\cos\theta+2yzat\sin\theta\sin\varphi$$
$$+2y(at)^2\sin\theta\cos\theta\sin\varphi+z(at)^2\sin^2\theta\sin^2\varphi$$
$$+(at)^3\sin^2\theta\cos\theta\sin^2\varphi$$

$$u(M,t)=\frac{1}{4\pi a}\frac{\partial}{\partial t}\iint_{S_{at}^M}\frac{\varphi(M')}{at}ds=\frac{1}{4\pi a}\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{\pi}\varphi(M')at\sin\theta d\theta d\varphi$$

$$\boxed{u(M,t)=x^3+3a^2t^2x+zy^2+a^2t^2z}$$

*三、某些三维波动问题的求解

行波法
习题课

1、求解

法二：令 $u = u^{I} + u^{II}$

$$\begin{cases} u_{tt} = a^2 \Delta u \\ u|_{t=0} = x^3 + y^2 z \\ u_t|_{t=0} = 0 \end{cases} \rightarrow \begin{cases} u^{I}_{tt} = a^2 u^{I}_{xx} \\ u^{I}|_{t=0} = x^3 + \\ u^{I}_t|_{t=0} = 0 \end{cases} \begin{cases} u^{II}_{tt} = a^2 u^{II}_{yy} \\ u^{II}|_{t=0} = y^2 z \\ u^{II}_t|_{t=0} = 0 \end{cases}$$

$$\boxed{u(x,t) = \frac{1}{2}[\varphi(x+at) + \varphi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\alpha)d\alpha}$$

$$u^{I} = \frac{1}{2}[(x+at)^3 + (x-at)^3] = x^3 + 3x(at)^2$$

$$u^{II} = \frac{1}{2}[(y+at)^2 z + (y-at)^2 z] = zy^2 + z(at)^2$$

$$\boxed{u(M,t) = x^3 + 3a^2t^2x + zy^2 + a^2t^2z}$$

*三、某些三维波动问题的求解

2、求解
$$\begin{cases} u_{tt} = a^2 \Delta u, r > 0, t > 0 \\ u|_{t=0} = \varphi(r) \\ u_t|_{t=0} = \psi(r) \end{cases} \qquad (1)$$

因为在球对称的球坐标系中有：

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru) \qquad \text{令 } v = ru$$

（1）化为：
$$\begin{cases} v_{tt} = a^2 v_{rr}, r > 0, t > 0 \\ v|_{t=0} = r\varphi(r) \\ v_t|_{t=0} = r\psi(r) \end{cases} \qquad (2)$$

$$\boxed{u(r,t) = \frac{1}{2r}[(r+at)\varphi(r+at) + (r-at)\varphi(r-at)] + \frac{1}{2ar}\int_{r-at}^{r+at} \alpha\psi(\alpha)d\alpha}$$

Good-by!