



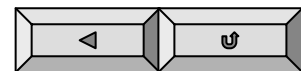
# 数学物理方法

Methods in Mathematical Physics

## 第九章 积分变换法

The Method of Integral Transforms

武汉大学物理科学与技术学院





# 傅氏变换习题课

- 一、傅氏变换或傅氏逆变换
- 二、傅氏变换的有关性质及其应用
- 三、傅氏变换法



# 一、傅氏变换或傅氏逆变换

傅氏变换习题课

1. 求  $F\left[\frac{\sin ax}{x}\right] = ? \quad a > 0$

$$F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$\begin{aligned} \text{解: } F\left[\frac{\sin ax}{x}\right] &= \int_{-\infty}^{\infty} \frac{e^{iax} - e^{-iax}}{2ix} e^{-i\omega x} dx \\ &= \int_0^{\infty} \frac{\sin(a-\omega)x}{x} dx + \int_0^{\infty} \frac{\sin(a+\omega)x}{x} dx \end{aligned}$$

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2} \rightarrow$$

$$\int_0^{\infty} \frac{\sin Ax}{x} dx = \begin{cases} \frac{\pi}{2}, & A > 0 \\ -\frac{\pi}{2}, & A < 0 \end{cases}$$



# 一、傅氏变换或傅氏逆变换

傅氏变换习题课

$$F\left[\frac{\sin ax}{x}\right] = \int_0^{\infty} \frac{\sin(a-\omega)x}{x} dx + \int_0^{\infty} \frac{\sin(a+\omega)x}{x} dx$$

(1) 若  $a > |\omega|$ , 则有  $a - \omega > 0, a + \omega > 0$

$$\int_0^{\infty} \frac{\sin(a-\omega)x}{x} dx = \int_0^{\infty} \frac{\sin(a+\omega)x}{x} dx = \frac{\pi}{2} \rightarrow F\left[\frac{\sin ax}{x}\right] = \pi$$

(2) 若  $a = |\omega|$ , ① 则当  $\omega > 0$  时有:  $\omega = a$ ,

$$\int_0^{\infty} \frac{\sin(a-\omega)x}{x} dx = 0, \int_0^{\infty} \frac{\sin(a+\omega)x}{x} dx = \frac{\pi}{2} \rightarrow F\left[\frac{\sin ax}{x}\right] = \frac{\pi}{2}$$

② 则当  $\omega < 0$  时有:  $\omega = -a$ ,

$$\int_0^{\infty} \frac{\sin(a-\omega)x}{x} dx = \frac{\pi}{2}, \int_0^{\infty} \frac{\sin(a+\omega)x}{x} dx = 0 \rightarrow F\left[\frac{\sin ax}{x}\right] = \frac{\pi}{2}$$



# 一、傅氏变换或傅氏逆变换

傅氏变换习题课

$$F\left[\frac{\sin ax}{x}\right] = \int_0^{\infty} \frac{\sin(a-\omega)x}{x} dx + \int_0^{\infty} \frac{\sin(a+\omega)x}{x} dx$$

(3) 若  $a < |\omega|$ , 则 ① 当  $\omega > 0$  时,  $a - \omega < 0, a + \omega > 0$

$$\int_0^{\infty} \frac{\sin(a-\omega)x}{x} dx = -\frac{\pi}{2}, \quad \int_0^{\infty} \frac{\sin(a+\omega)x}{x} dx = \frac{\pi}{2} \rightarrow F\left[\frac{\sin ax}{x}\right] = 0$$

② 则当  $\omega < 0$  时,  $a - \omega > 0, a + \omega < 0$

$$\rightarrow F\left[\frac{\sin ax}{x}\right] = 0$$

(1), (2), (3)  $\rightarrow$

$$F\left[\frac{\sin ax}{x}\right] = \begin{cases} \pi, & a > |\omega| \\ \frac{\pi}{2}, & a = |\omega| \\ 0, & a < |\omega| \end{cases}$$



# 一、傅氏变换或傅氏逆变换

傅氏变换习题课

2. 在量子力学中一维体系中的一个状态在坐标表象中的波函数为

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} c(p) e^{i\frac{p}{\hbar}x} dp$$

求该状态在动量表象中的波函数  $c(p)$

$$\text{解: } \psi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\pi\hbar}{\sqrt{2\pi\hbar}} c(p) e^{i\frac{p}{\hbar}x} d\left(\frac{p}{\hbar}\right)$$

$$\rightarrow \sqrt{2\pi\hbar} c(p) = \int_{-\infty}^{\infty} \psi(x) e^{-i\frac{p}{\hbar}x} dx$$

$$\rightarrow c(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) e^{-i\frac{p}{\hbar}x} dx$$



# 一、傅氏变换或傅氏逆变换

傅氏变换习题课

4. 设  $r = \sqrt{x^2 + y^2 + z^2} = |\vec{r}|,$

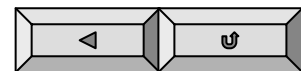
$$\omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = |\vec{\omega}|$$

证明 : (1)  $F\left[\frac{1}{r}\right] = \frac{4\pi}{\omega^2};$  (2)  $F\left[\frac{1}{r}e^{-\mu r}\right] = \frac{4\pi}{\omega^2 + \mu^2} (\mu > 0)$

提示 :  $F^{-1}\left[\frac{4\pi}{\omega^2}\right] = \frac{1}{(2\pi)^3} \int \int \int_{-\infty}^{\infty} \frac{4\pi}{\omega^2} e^{i\vec{\omega} \cdot \vec{r}} d\vec{\omega}$

$$= \frac{4\pi}{(2\pi)^3} \int_0^{\infty} \int_0^{2\pi} \int_0^{\pi} \frac{1}{\omega^2} e^{i\omega r \cos \theta} \omega^2 \sin \theta d\theta d\varphi d\omega$$

$$\boxed{x = \cos \theta} = \frac{1}{\pi} \int_0^{\infty} \int_{-1}^1 e^{i\omega r x} dx d\omega$$





## 二、有关性质及其应用

傅氏变换习题课

已知: 
$$\int_{-\infty}^{\infty} \frac{f(\xi) d\xi}{(x - \xi)^2 + a^2} = \frac{1}{x^2 + b^2}, 0 < a < b$$

求  $f(x) = ?$

解: 
$$\int_{-\infty}^{\infty} \frac{f(\xi)}{(x - \xi)^2 + a^2} d\xi = f(x) * \frac{1}{x^2 + a^2},$$

$$\rightarrow F \left[ \int_{-\infty}^{\infty} \frac{f(\xi)}{(x - \xi)^2 + a^2} d\xi \right] = F \left[ f(x) * \frac{1}{x^2 + a^2} \right]$$

$$F[f(x)] \cdot F \left[ \frac{1}{x^2 + a^2} \right] = F \left[ \frac{1}{x^2 + b^2} \right]$$

$$\rightarrow F[f(x)] = F \left[ \frac{1}{x^2 + b^2} \right] / F \left[ \frac{1}{x^2 + a^2} \right]$$





## 二、有关性质及其应用

傅氏变换习题课

$$F\left[\frac{1}{x^2 + a^2}\right] = \int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{x^2 + a^2} dx = 2 \int_0^{\infty} \frac{\cos \omega x}{x^2 + a^2} dx$$

$$\frac{1}{z^2 + a^2} \xrightarrow{z \rightarrow \infty} 0, \text{奇点: } \pm ia$$

$$(1) \text{ 若 } \omega > 0, \quad F\left[\frac{1}{x^2 + a^2}\right] = 2\pi i \operatorname{res} \frac{e^{i\omega z}}{z^2 + a^2} \Big|_{z=ia} = \frac{\pi}{a} e^{-\omega a}$$

$$(2) \text{ 若 } \omega < 0, \quad F\left[\frac{1}{x^2 + a^2}\right] = -2\pi i \operatorname{res} \frac{e^{i\omega z}}{z^2 + a^2} \Big|_{z=-ia} = \frac{\pi}{a} e^{\omega a}$$

$$(1), (2) \rightarrow F\left[\frac{1}{x^2 + a^2}\right] = \frac{\pi}{a} e^{-|\omega|a}, \quad F\left[\frac{1}{x^2 + b^2}\right] = \frac{\pi}{b} e^{-|\omega|b}$$

$$F[f(x)] = \frac{a}{b} e^{-|\omega|(b-a)}$$

$$\rightarrow f(x) = \frac{a(b-a)}{\pi b} \frac{1}{x^2 + (b-a)^2}$$



### 三、傅氏变换法

#### 傅氏变换习题课

#### 1. 求解上半平面的狄氏问题

$$\begin{cases} \Delta u = 0, y > 0 & (1) \\ u|_{y=0} = f(x) & (2), \quad \lim_{x^2+y^2 \rightarrow \infty} u = 0 & (3) \end{cases}$$

解: (1) 记  $F[u(x, y)] = \tilde{u}(\omega, y)$ ,  $F[f(x)] = \tilde{f}(\omega)$

$$(2) \begin{cases} \frac{d^2 \tilde{u}}{dy^2} - \omega^2 \tilde{u}(\omega, y) = 0 \\ \tilde{u}(\omega, 0) = \tilde{f}(\omega) \\ \tilde{u}(\omega, y)|_{y \rightarrow \pm\infty} \rightarrow 0 \end{cases} \rightarrow \tilde{u}(\omega, y) = \tilde{f}(\omega) e^{-|\omega|y}$$

$$\begin{aligned} (3) u(x, y) &= F^{-1}[\tilde{u}(\omega, y)] = F^{-1}[\tilde{f}(\omega) e^{-|\omega|y}] \\ &= F^{-1}F[f(x) * F^{-1}[e^{-|\omega|y}]] \end{aligned}$$



### 三、傅氏变换法

#### 傅氏变换习题课

#### 1. 求解上半平面的狄氏问题

$$\begin{cases} \Delta u = 0, y > 0 & (1) \\ u|_{x=0} = f(x) & (2), \quad \lim_{x^2+y^2 \rightarrow \infty} u = 0 & (3) \end{cases}$$

$$(3) u(x, y) = F^{-1}[\tilde{u}(\omega, y)] = F^{-1}F[f(x) * F^{-1}[e^{-|\omega|y}]]$$

$$F^{-1}[e^{-|\omega|y}] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|\omega|y} e^{i\omega x} d\omega = \frac{y}{\pi} \frac{1}{x^2 + y^2}$$

$$\text{或: } \because F\left[\frac{1}{x^2 + b^2}\right] = \frac{\pi}{b} e^{-|\omega|b} (\text{见上题}) \rightarrow F^{-1}[e^{-|\omega|y}] = \frac{y}{\pi} \frac{1}{x^2 + y^2}$$

$$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{(x - \xi)^2 + y^2} d\xi$$



### 三、傅氏变换法

傅氏变换习题课

$$2、\begin{cases} u_{tt} + a^2 u_{xxxx} = 0, -\infty < x < \infty, t > 0 & (1) \\ u(x, 0) = \varphi(x) & (2) \\ u_t(x, 0) = a\psi''(x) & (3) \end{cases}$$

解：(1) 记  $F[u(x, t)] = \tilde{u}(\omega, t)$ ,  $F[\varphi(x)] = \tilde{\varphi}(\omega)$ ,  $F[\psi(x)] = \tilde{\psi}(\omega)$

$$\text{则} \begin{cases} \frac{d^2 \tilde{u}(\omega, t)}{dt^2} + a^2 \omega^4 \tilde{u}(\omega, t) = 0 & (4) \\ \tilde{u}(\omega, 0) = \tilde{\varphi}(\omega) & (5) \\ \tilde{u}_t(\omega, 0) = -a\omega^2 \tilde{\psi}(\omega) & (6) \end{cases}$$

(2) 求  $\tilde{u}(\omega, t)$  :

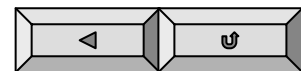
$$\tilde{u}(\omega, t) = \tilde{\varphi}(\omega) \cos a\omega^2 t - \tilde{\psi}(\omega) \sin a\omega^2 t$$



### 三、傅氏变换法

### 傅氏变换习题课

$$\begin{aligned}(3) \quad u(x, t) &= F^{-1}[\tilde{u}(\omega, t)] \\&= F^{-1}[\tilde{\varphi}(\omega)\cos a\omega^2 t] - F^{-1}[\tilde{\psi}(\omega)\sin a\omega^2 t] \\&= F^{-1}F[\varphi(x) * F^{-1}[\cos a\omega^2 t]] - F^{-1}F[\psi(x) * F^{-1}[\sin a\omega^2 t]] \\F^{-1}[e^{ia\omega^2 t}] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ia\omega^2 t} e^{i\omega x} d\omega \\&= \frac{1}{2\pi} e^{-i\frac{x^2}{4at}} \int_{-\infty}^{\infty} e^{iat\left(\omega + \frac{x}{2at}\right)^2} d\omega \\&= \frac{1}{2\pi} \frac{e^{-i\frac{x^2}{4at}}}{\sqrt{at}} \int_{-\infty}^{\infty} e^{i\xi^2} d\xi = \frac{1}{\pi\sqrt{at}} e^{-i\frac{x^2}{4at}} e^{i\frac{\pi}{4}} \frac{\sqrt{\pi}}{2}\end{aligned}$$





### 三、傅氏变换法

傅氏变换习题课

$$F^{-1}\left[e^{ia\omega^2 t}\right] = \frac{1}{\pi\sqrt{at}} e^{-i\frac{x^2}{4at}} e^{i\frac{\pi}{4}} \frac{\sqrt{\pi}}{2}$$

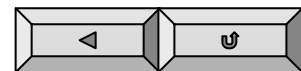
$$F^{-1}\left[\cos a\omega^2 t + i\sin a\omega^2 t\right] = \frac{1}{2\sqrt{\pi at}} \left[ \cos\left(\frac{\pi}{4} - \frac{x^2}{4at}\right) + i\sin\left(\frac{\pi}{4} - \frac{x^2}{4at}\right) \right]$$

$$(3) \quad u(x, t) = F^{-1}[\tilde{u}(\omega, t)]$$

$$= F^{-1}F\left[\varphi(x) * F^{-1}[\cos a\omega^2 t]\right] + F^{-1}F\left[\psi(x) * F^{-1}[\sin a\omega^2 t]\right]$$

$$\rightarrow u(x, t) = \frac{1}{2\sqrt{\pi at}} \int_{-\infty}^{\infty} \varphi(x - \xi) \cos\left(\frac{\pi}{4} - \frac{\xi^2}{4at}\right) d\xi$$

$$- \frac{1}{2\sqrt{\pi at}} \int_{-\infty}^{\infty} \psi(x - \xi) \sin\left(\frac{\pi}{4} - \frac{\xi^2}{4at}\right) d\xi$$





### 三、傅氏变换法

傅氏变换习题课

3、无限长的杆沿杆长方向有温差，设其热导系数为1，初始温度分布为  $\cos x$  求此热导问题。

$$\begin{cases} u_t = u_{xx}, -\infty < x < \infty \\ u(x, 0) = \cos x \end{cases}$$

解：记  $F[u(x, t)] = \tilde{u}(\omega, t)$ ，则

$$\begin{cases} \frac{d\tilde{u}}{dt} - (i\omega)^2 \tilde{u}(\omega, t) = 0 \\ \tilde{u}(\omega, 0) = \pi[\delta(1 - \omega) + \delta(1 + \omega)] \end{cases} \rightarrow \tilde{u}(\omega, t) = \pi[\delta(1 - \omega) + \delta(1 + \omega)]e^{-\omega^2 t}$$

$$u(x, t) = F^{-1}[\tilde{u}(\omega, t)] = \frac{1}{2\pi} \pi \int_{-\infty}^{\infty} e^{-\omega^2 t} [\delta(\omega - 1) + \delta(\omega + 1)] e^{i\omega x} d\omega$$

$$u(x, t) = \frac{1}{2} [e^{-t} e^{ix} + e^{-t} e^{-ix}] = e^{-t} \cos x$$



### 三、傅氏变换法

傅氏变换习题课

又解: 
$$\begin{cases} u_t = u_{xx}, & -\infty < x < \infty \\ u(x, 0) = \cos x \end{cases}$$

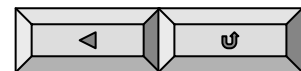
记  $F[u(x, t)] = \tilde{u}(\omega, t), \quad F[\cos x] = \tilde{\varphi}(\omega)$

$$\begin{cases} \frac{d\tilde{u}}{dt} - (i\omega)^2 \tilde{u}(\omega, t) = 0 \\ \tilde{u}(\omega, 0) = \tilde{\varphi}(\omega) \end{cases} \rightarrow \tilde{u}(\omega, t) = \tilde{\varphi}(\omega) e^{-\omega^2 t}$$

$$u(x, t) = F^{-1}[\tilde{u}(\omega, t)] = F^{-1}[\tilde{\varphi}(\omega) e^{-\omega^2 t}] = \cos(x) * F^{-1}[e^{-\omega^2 t}]$$

$$F^{-1}[e^{-\omega^2 t}] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\omega^2 t} e^{i\omega x} d\omega = \frac{1}{\pi} \int_0^{\infty} e^{-\omega^2 t} \cos \omega x d\omega$$

$$= \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \quad u(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{\xi^2}{4t}} \cos(\xi - x) d\xi$$







### 三、傅氏变换法

傅氏变换习题课

又解: 
$$\begin{cases} u_t = u_{xx}, -\infty < x < \infty \\ u(x, 0) = \cos x \end{cases}$$

$$u(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{\xi^2}{4t}} \cos(\xi - x) d\xi$$

$$= \frac{1}{\sqrt{\pi t}} \cos x \int_0^{\infty} e^{-\frac{\xi^2}{4t}} \cos \xi d\xi$$

$$= \frac{1}{\sqrt{\pi t}} \cos x \frac{1}{2} e^{-t} \sqrt{\pi t}$$

$$u(x, t) = e^{-t} \cos x$$

再见！

