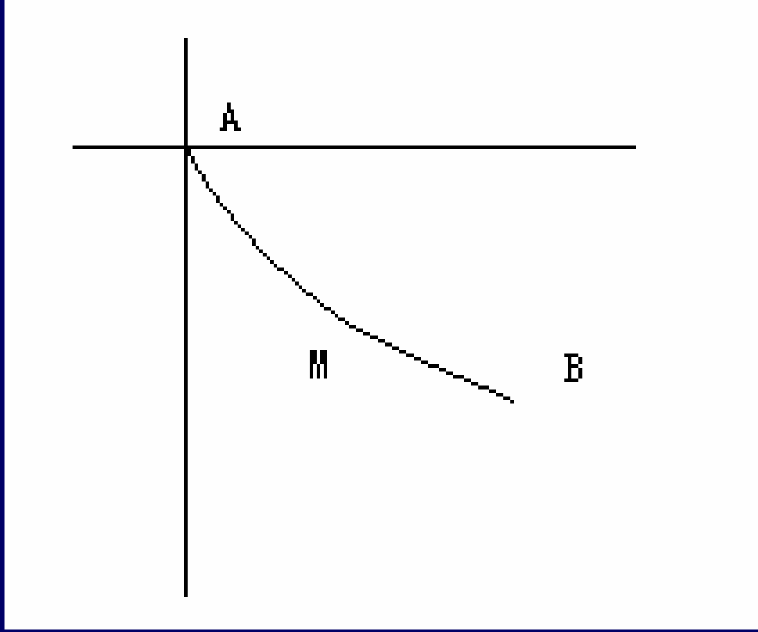
数学物理方法

第十一章　变分法

The Variational Method

物理科学与技术学院

§11.1 泛函和泛函的极值



一. 泛函

1696年,Basel大学Bernoulli提出，最速落径：

$$v=\frac{ds}{dt}=\sqrt{2gy}\quad \therefore \text{总的下降时间}$$

$$T=\int_{t_1(A)}^{t_2(B)}dt=\int_A^B\frac{ds}{v},\text{由于}ds=\sqrt{dx^2+dy^2}=\sqrt{1+(y')^2}\,dx$$

$$\therefore T=\int_A^B\frac{\sqrt{1+(y')^2}\,dx}{\sqrt{2gy}}$$

即 $T=T[y(x)]=\int_A^B\frac{\sqrt{1+(y')^2}\,dx}{\sqrt{2gy}},$ ——是函数的函数

一、泛函

1.定义:

泛函是函数的函数.记J=J[y(x)]

其中,J∈ B:复（实）；数集 y(x) ∈ C:复数集

注意:　(1)不同于普通函数

(2)不同于复合函数

(3)定义域:y(x)可取类

2.典型表达方式: $J[y(x)] = \int_a^b F(x, y, y')dx \to (1)$

$F(x, y, y')$泛函的核

3.最速落径问题:求泛函T[y(x)]的极小值问题

二. 泛函的极值

类似于上述的求极值问题, 如光学中费马原理, 力学中最小作用问题, 在物理中很多. 我们将会看到求泛函极值可归结为两种方法.

1. 求泛函极值方法

(1). 直接方法:从泛函直接求极值曲线

(2). 间接法:化为解微分方程-与变分问题有联系

为此, 先建立有关变分的概念

二、泛函的极值

2. 变分

(1) : 函数的变分 : 若 $y(x) \xrightarrow{\text{微变}} y(x) + t\eta(x)$, t 为小参数，

则记　$\delta(y) = t\eta(x)$　(2)　— 称 $t\eta(x)$ 为 $y(x)$ 的变分 .

注意 : δy 不同于 dy, dy 有一取极值过程 , δy 不取极限, 略去了高阶小量此时,

$$y'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \to \lim_{\Delta x \to 0} \frac{\Delta(y + t\eta)}{\Delta x} = y'(x) + t\eta'(x)$$

(2)变分与微分可交换次序 :

$$\delta(y') = t\eta'(x) = \frac{d}{dx}[t\eta(x)] = \frac{d}{dx}\delta(y) \to y'(x)\text{的变分}$$

即: $\delta(y') \equiv \dfrac{d}{dx}\delta(y)$

二、泛函的极值

(3) : 函数的变分

若(1)中$F\in C^2, y\in C^2$, 则当$y\to y+t\eta$

$$\Delta J = J[y(x)+t\eta]-J[y]$$

$$=\int_a^b [F(x,y+t\eta,y'+t\eta')-F(x,y,y')]dx$$

$$=\int_a^b [\frac{\partial F}{\partial y}t\eta+\frac{\partial F}{\partial y'}t\eta'+t\text{的高阶小量}]dx$$

记$\delta J=\int_a^b (\frac{\partial F}{\partial y}\delta y+\frac{\partial F}{\partial y'}\delta y')dx \to (4)$ —称为泛函$J[y(x)]$的第一次变分

三. 泛函极值的必要条件

设$J[y(x)]$的极值函数为 $y(x)$, 且有变分 $t\eta(x)$, 则

$J[y(x)+t\eta(x)]=\phi(t)$ (因为$y(x)$已经设定)

由一元函数取极值条件　，则J极值条件

$$\rightarrow \phi(t)\text{极值条件} \rightarrow \quad \frac{d\phi}{dt}\Big|_{t=0}=0$$

$$\frac{\partial J[y(x)+t\eta(x)]}{\partial t}\Big|_{t=0}=0, \text{即}, \int_a^b \frac{\partial F[x,y+t\eta,y'+t\eta',]}{\partial t}\Big|_{t=0}\,dx=0$$

$$\int_a^b [\frac{\partial F}{\partial(y+t\eta)}\eta+\frac{\partial F}{\partial(y'+t\eta')}\eta']\Big|_{t=0}\,dx=0,$$

即： $$\int_a^b [\frac{\partial F}{\partial y}\eta+\frac{\partial F}{\partial y'}\eta']dx=0 \rightarrow \int_a^b [\frac{\partial F}{\partial y}\delta y+\frac{\partial F}{\partial y'}\delta y']dx=0 \quad (5)$$

三. 泛函极值的必要条件

对照（4）: 极值条件

1. 泛函极值的问题是变分问题

$\delta J = 0$ 有

$$\because \quad \int_a^b \frac{\partial F}{\partial y'}\delta y' dx = \int_a^b \frac{\partial F}{\partial y'}\frac{d}{dx}(\delta y)dx$$

$$= \frac{\partial F}{\partial y'}\delta y \Big|_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y dx = -\int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y dx$$

代如（5）: $\displaystyle\int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y dx = 0$

$$\therefore \quad \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \rightarrow \boxed{Euler \text{ 方程}}$$

三. 泛函极值的必要条件

2. 泛函取极值的条件- $y(x)$满足 *Euler* 方程

(1)对于$J[y(x)]=\int_a^b F(x,y,y')dx$

有$\frac{\partial F}{\partial y}-\frac{d}{dx}(\frac{\partial F}{\partial y'})=0 \to$ 二阶常微分方程 $\Rightarrow y'\frac{\partial F}{\partial y'}-F=C$

(2) 对于$J[y_1(x),y_2(x)\cdots y_n(x)]$

$$=\int_a^b F(x;y_1,y_2\cdots y_n;y_1',y_2'\cdots y_n')dx$$

有$\frac{\partial F}{\partial y_i}-\frac{d}{dx}(\frac{\partial F}{\partial y_i'})=0,(i=0,1,2......n)$

三. 泛函极值的必要条件

(3)对于$J[y'(x),y'(x)\cdots y^{(n)}(x)]$

$$=\int_a^b F(x;y\ ;y',y''\cdots y^{(n)})dx$$

有$$\frac{\partial F}{\partial y}-\frac{d}{dx}(\frac{\partial F}{\partial y'})+\frac{d^2}{dx^2}(\frac{\partial F}{\partial y''})-\frac{d^3}{dx^3}(\frac{\partial F}{\partial y'''})=0$$

(4)多元函数$J[u(x,y,z)]=\int_a^b F(x,y,z;u;u_x,u_y,u_z)dxdydz$

$$\frac{\partial F}{\partial u}-\frac{\partial}{\partial x}(\frac{\partial F}{\partial u_x})-\frac{\partial}{\partial y}(\frac{\partial F}{\partial u_y})-\frac{\partial}{\partial z}(\frac{\partial F}{\partial u_z})=0$$

于是求泛函数极值问题$\rightarrow$解*Euler*方程问题

三. 泛函极值的必要条件

3. 若 F 不显含 x 则

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0 \rightarrow \quad y'\frac{\partial F}{\partial y'}-F=C$$

考虑 $\frac{d}{dx}\left[y'\frac{\partial F}{\partial y'}-F\right]=y''\frac{\partial F}{\partial y'}+y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial x}-\frac{\partial F}{\partial y}y'-\frac{\partial F}{\partial y'}y''$

$$=-y'\left[\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]-\frac{\partial F}{\partial x}$$

$\because F$不显含x,即$\frac{\partial F}{\partial x}=0\Rightarrow\frac{d}{dx}\left[y'\frac{\partial F}{\partial y'}-F\right]=0,$

$\therefore$ 积分可得$y'\frac{\partial F}{\partial y'}-F=C(B)$

三. 泛函极值的必要条件

例1.求最速落径

$$T[y(x)]=\int_{x_A}^{x_B}\frac{\sqrt{1+(y')^2}\,dx}{\sqrt{2gy}}$$

$$\therefore \delta\int_{x_A}^{x_B}\frac{\sqrt{1+(y')^2}\,dx}{\sqrt{2gy}}=0$$

$$F=\frac{\sqrt{1+(y')^2}\,dx}{\sqrt{2gy}}=\frac{1}{\sqrt{2g}}\underline{\frac{\sqrt{1+(y')^2}}{\sqrt{y}}}$$ 不显含x

三. 泛函极值的必要条件

于是*Euler*方程

$$y'\cdot\frac{\partial}{\partial y'}\frac{\sqrt{1+(y')^2}}{\sqrt{y}}-\frac{\sqrt{1+(y')^2}}{\sqrt{y}}=C$$

$$\frac{(y')^2}{\sqrt{[1+(y')^2]y}}-\frac{\sqrt{1+(y')^2}}{\sqrt{y}}=C$$

$$\Rightarrow\frac{(y')^4}{[1+(y')^2]y}+\frac{1+(y')^2}{y}-2\frac{y'^2}{y}=C^2$$

$$\frac{1}{[1+(y')^2]y}=C^2,\text{令}\frac{1}{C^2}=C_1$$

三. 泛函极值的必要条件

得 $y'=\dfrac{\sqrt{C_1-y}}{\sqrt{y}}, \rightarrow x-C_2=\displaystyle\int\dfrac{\sqrt{y}}{\sqrt{C_1-y}}dy$

令$y=C_1\sin^2\dfrac{\theta}{2}$,则$\begin{cases} x=\dfrac{C_1}{2}(\theta-\sin\theta)+C_2 \\ y=\dfrac{C_1}{2}(1-\cos\theta) \end{cases}$

是由半径为$\dfrac{C_1}{2}$的圆周上一固定点运动 产生的.

在图中x轴下方滚动.存在一条且仅一条通过 原点及点(x_B, y_B)的摆线.适当选择C_1和C_2可以给出这条摆线

四. 泛函的条件极值

$$\begin{cases} J[y(x)] = \int_a^b F(x, y, y')dx \\ \int_a^b G(x, y, y')dx = l \end{cases}, y(a) = y_0, y(b) = y_1$$

拉格朗日($Lagrange$)乘子法:

考虑: $\delta \int_a^b [F(x, y, y') + \lambda G(x, y, y')]dx = 0$

则 $\rightarrow \frac{\partial F}{\partial y} + \lambda \frac{\partial G}{\partial y} - \frac{d}{dx}[(\frac{\partial F}{\partial y'}) + \lambda \frac{d}{dx}(\frac{\partial G}{\partial y'})] = 0$

积分常数C_1, C_2和λ可由附加条件定出

四. 泛函的条件极值

例 2. $\begin{cases} J[y(x)] = \int_0^1 y'^2 dx \\ \int_0^1 y^2 dx = 1 \\ y(0) = 0, \quad y(1) = 1 \end{cases}$

考虑　$\delta \int_0^1 (y'^2 + \lambda y^2) dx = 0$

不显含x,也可推出一阶$Euler$方程,此处直接用二阶$Euler$

也不困难：　$2\lambda y - \frac{d}{dx}(2y') = 0$

四. 泛函的条件极值

即：$y'' - \lambda y = 0$

$y = C_1 e^{\sqrt{\lambda} x} + C_2 e^{-\sqrt{\lambda} x}$

由 $\begin{cases} y(0) = 0 \\ y(1) = 1 \end{cases}$ 得： $y_n = C_n \sin n\pi x (n = 1, 2 \cdots)$

再由 $\int_0^1 y^2 dx = 1$ 得：$C_n = \pm\sqrt{2}, \therefore y_n = \pm\sqrt{2} \sin n\pi x$

$$J[y(x)] = \int_0^1 [\frac{d}{dx} \pm \sqrt{2} \sin n\pi x]^2 dx = n^2 \pi^2$$

极小值为$J[y_1(x)] = \pi^2$

五. 里兹方法

1.对于$J[f(x)]$,

令 $y(x)=f(\varphi_1(x),\varphi_2(x)\ldots\varphi_n(x);C_1,C_2\ldots C_n)$

则 $J[f(x)]=\varphi(C_1,C_2\ldots C_n)$

于是当$\dfrac{\partial\varphi}{\partial C_i}=0$时,$J[f(x)]$取极值,$i=1,2,\ldots,n$

2.注意:

(1) f是近似解$\mathrm{f(x)}=\lim\limits_{n\to\infty} f(\varphi_1,\varphi_2\ldots\varphi_n;\mathrm{C}_1,\mathrm{C}_2\ldots\mathrm{C}_\mathrm{n})$

(2)适当选φ, f

注:一般为多项式三角式为它们的线性组合,满足边界条件

五. 里兹方法

3.例：求$J[f(x)]=\int_0^1 y'^2dx$(的极小值)　(1)

其中$\begin{cases}\int_0^1 y^2dx=1\\ y(0)=0,\ y(1)=0\end{cases}$　(2)

解：选$\varphi_n(x)=C_nx^n$

令$y(x)=x(x-1)(C_0+C_1x)=C_1x^3+(C_0-C_1)x^2-C_0x$　(3)

代入(1)：$\phi=J[y(x)]=\int_0^1[3C_1x^2+2(C_0-C_1)x-C_0]^2dx$

$$=\frac{1}{3}(C_0^2+C_0C_1+\frac{2}{5}C_1^2)\quad (4),$$

即　$\phi=\phi(C_0,C_1)$

五. 里兹方法

$$\text{代入}(2):\int_0^1 y^2dx=\frac{1}{30}(C_0^2+C_0C_1+\frac{2}{7}C_1^2)=1 \quad (5)$$

$$\text{记}\quad \psi=\frac{1}{30}(C_0^2+C_0C_1+\frac{2}{7}C_1^2)-1=0$$

由拉格朗日乘子法：若要求$y(x)=f(x_1,\ x_2\ldots x_n)$

在m个约束条件$:g_k(x_1,\ x_2\ldots x_n)=0,(k=1,2\ldots m)$

下的极值只需考虑 $F=y+\sum_{k=1}^{m}\lambda_k g_k$

使 $\begin{cases}\dfrac{\partial F}{\partial x_i}=0 \\ g_k(x_i)=0\end{cases}$ $i=1,2\ldots n$即可

五. 里兹方法

$$\therefore \text{应由}\begin{cases} \dfrac{\partial F}{\partial C_i} = \dfrac{\partial \phi}{\partial C_i} + \lambda \dfrac{\partial \psi}{\partial C_i} = 0 \\ \varphi = 0 \end{cases}, \text{联立消去}\lambda\text{来求}y$$

但在此不必,由(5): $C_0^2 + C_0 C_1 = 30 - \dfrac{2}{7} C_1^2 \quad (6)$

代入(4): $\phi = J[y(x)] = \dfrac{1}{3}(30 - \dfrac{2}{7}C_1^2 + \dfrac{2}{5}C_1^2)$

$= \dfrac{2}{3}(15 + \dfrac{2}{35}C_1^2), C_1 = 0$有极小值，此时 $\mathrm{J[f(x)]} = 10$

$C_1 = 0$代入(6): $C_0 = \pm\sqrt{30}$,代入(4)

$\therefore y(x) = \pm\sqrt{30}x(x-1) = \pm\sqrt{30}x(x-1)$

$= \pm\sqrt{30}[(x - \dfrac{1}{2})^2 - \dfrac{1}{4}]$

五. 里兹方法

$$(x-1/2)^2=\frac{1}{\pm\sqrt{30}}y+\frac{1}{4}=\pm 2\frac{1}{2\sqrt{30}}(y\pm\frac{\sqrt{30}}{4})$$

$$令 h=\frac{1}{2}, k=-(\pm\frac{\sqrt{30}}{4}), p=\frac{1}{2\sqrt{30}}$$

$$\Rightarrow (x-h)^2=\pm 2p(y-k)$$

$$顶点 (h,k); 焦点 (h,k\pm\frac{p}{2}); 准线\ y=-k\pm\frac{p}{2}$$

复习上次课:

泛函: $J[y(x)]=\int_a^b F(x,y,y')dx$

$$J[u(x,y,z)]=\iiint F(x,y,z;u;u_x,u_y,u_z)d\tau$$

泛函的变分: $\delta J=\int_a^b (\frac{\partial F}{\partial y}\delta y+\frac{\partial F}{\partial y'}\delta y')dx,$

泛函的极值条件: $\delta y=t\eta(x),\delta y'=\frac{d}{dx}\delta y$

$$\delta J=0 \Leftrightarrow Euler\text{方程}$$

复习上次课：

求泛函极值的方法：

间接方法：

对于J[y(x)]= $\int_a^b F(x,y,y')dx \rightarrow \frac{\partial F}{\partial y}-\frac{d}{dx}\frac{\partial F}{\partial y'}=0$

对于 $\int_a^b G(x,y,y')dx=l \rightarrow \frac{\partial F}{\partial y}+\lambda\frac{\partial G}{\partial y}-\frac{d}{dx}[(\frac{\partial F}{\partial y'})+\lambda\frac{d}{dx}(\frac{\partial G}{\partial y'})]=0$

直接方法：从泛函求极值直接方法

$\rightarrow$ 里兹($Ritz$)方法例

§8.2　变分法

一、变分法

1.思路

泛函极值　　$\leftarrow$ 泛函的$Euler$方程的解

$\uparrow$

直接方法求解

2.步骤:

(1)写出定解问题所对应的泛函

(2)用直接法（里兹法）求该泛函极值,此即写方程的解.