数学物理方法

Methods in Mathematical Physics

第十章 格林函数法

Method of Green's Function

武汉大学物理科学与技术学院

第十章 习题课

一、δ函数及其在物理上的应用

二、用电像法求格林函数

三、用格林函数法求泊松方程的狄氏问题

一、δ函数及其在物理上的应用

1. $$\begin{cases} \delta(x-x_0)=\begin{cases} 0 & x\neq x_0 \\ \infty & x=x_0 \end{cases} \\ \int_{-\infty}^{\infty}\delta(x-x_0)dx=1 \end{cases}$$

2. $$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0)$$

3. $$\int_{-\infty}^{\infty} f(x)\delta^{(n)}(x-x_0)dx = (-1)^n f^{(n)}(x_0)$$

4. $$\delta[\varphi(x)]=\sum_{i=1}^{n}\frac{\delta(x-x_i)}{|\varphi'(x_i)|},\text{其中}\ \varphi(x_i)=0$$

一、δ函数及其在物理上的应用

1. $H(x)=\begin{cases}0, & x<0\\ 1, & x>0\end{cases}$ 证明：$\delta(x)=\dfrac{d\,H(x)}{d\,x}$

设$\varphi(x)$为任意一无穷次可微且在无穷区域等于零的函数，则

$$\int_{-\infty}^{\infty}\frac{dH}{dx}\varphi(x)dx=\varphi(x)H(x)\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}H(x)\varphi'(x)dx$$

$$=-\int_{0}^{\infty}\varphi'(x)dx=\varphi(0)$$

$$\int_{-\infty}^{\infty}\delta(x)\varphi(x)dx=\varphi(0)$$

$$\boxed{\delta(x)=\frac{d\,H(x)}{d\,x}}$$

一、δ函数及其在物理上的应用

2、证：$\begin{cases} \delta(x^2-a^2)=\dfrac{1}{2|a|}[\delta(x-a)+\delta(x-a)] \\ \delta(x^2)=\dfrac{\delta(x)}{|x|} \end{cases}$

(1) $\because \delta[\varphi(x)]=\sum_{k=1}^{n}\dfrac{\delta(x-x_k)}{|\varphi'(x_k)|}$ 其中$\varphi(x_k)=0, k=1,2,\cdots,n$

令$\varphi(x)=x^2-a^2$ 则$\varphi'(x)=2x$,

且由$x^2-a^2=0$有$x_1=a, x_2=-a$

故有$\delta(x^2-a^2)=\delta[\varphi(x)]=\sum_{k=1}^{2}\dfrac{\delta(x-x_k)}{|2x_k|}$

$$\delta(x^2-a^2)=\frac{1}{2|a|}[\delta(x-a)+\delta(x+a)]$$

一、δ函数及其在物理上的应用

$$(2)\because \int_{-\infty}^{\infty} f(x)\frac{1}{2|a|}[\delta(x-a)+\delta(x+a)]dx=\frac{1}{2|a|}[f(a)+f(-a)]$$

$$\int_{-\infty}^{\infty} f(x)\frac{1}{2|x|}[\delta(x-a)+\delta(x+a)]dx$$

$$=f(a)\frac{1}{2|a|}+f(-a)\cdot\frac{1}{2|-a|}=\frac{1}{2|a|}[f(a)+f(-a)]$$

$$\therefore \delta(x^2-a^2)=\frac{1}{2|a|}[\delta(x-a)+\delta(x+a)]=\frac{1}{2|x|}[\delta(x-a)+\delta(x+a)]$$

在上式中取$a=0$ $\quad \delta(x^2)=\dfrac{\delta(x)}{|x|}$

一、δ函数及其在物理上的应用

3.$F[\delta(x)]=?$，$\delta(x)$的积分表式？

$F[\cos ax]=?,\quad F[\sin ax]=?$

$$F[\delta(x)]=1\quad \therefore \delta(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega x}d\omega \quad \boxed{\rightarrow \int_{-\infty}^{\infty}e^{i\omega x}d\omega=2\pi\delta(x)}$$

$$F[\cos ax]=\int_{-\infty}^{\infty}\frac{e^{iax}+e^{-iax}}{2}e^{-i\omega x}dx=\frac{1}{2}[\int_{-\infty}^{\infty}e^{i(a-\omega)x}dx+\int_{-\infty}^{\infty}e^{-i(a+\omega)x}dx]$$

$$\rightarrow \int_{-\infty}^{\infty}e^{i(a-\omega)x}dx=2\pi\delta(a-\omega),\int_{-\infty}^{\infty}e^{-i(a+\omega)x}dx=2\pi\delta(a+\omega),$$

$$\boxed{\therefore F[\cos ax]=\pi[\delta(a-\omega)+\delta(a+\omega)]}$$

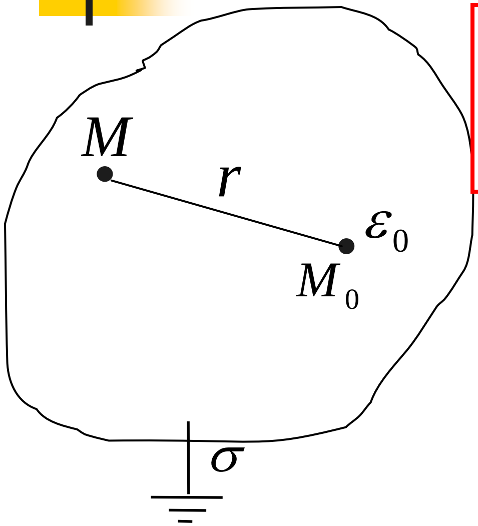
$$\boxed{F[\sin ax]=i\pi[\delta(a+\omega)-\delta(a-\omega)]}$$

思考： $F[e^{iax}]=\int_{-\infty}^{\infty}e^{iax}e^{-i\omega x}dx=\int_{-\infty}^{\infty}e^{i(a-\omega)x}dx=2\pi\delta(a-\omega)\rightarrow$

$F[\cos x]+iF[\sin x]=2\pi\delta(a-\omega)\rightarrow F[\cos x]=2\pi\delta(a-\omega),\ F[\sin x]=0$

是否正确？

二、用电像法求狄氏格林函数



$$G(M, M_0) = \frac{1}{4\pi r} + g$$

$$\begin{cases} \Delta g = 0, M \in \tau \\ g|_\sigma = -\frac{1}{4\pi r}|_\sigma \end{cases}$$

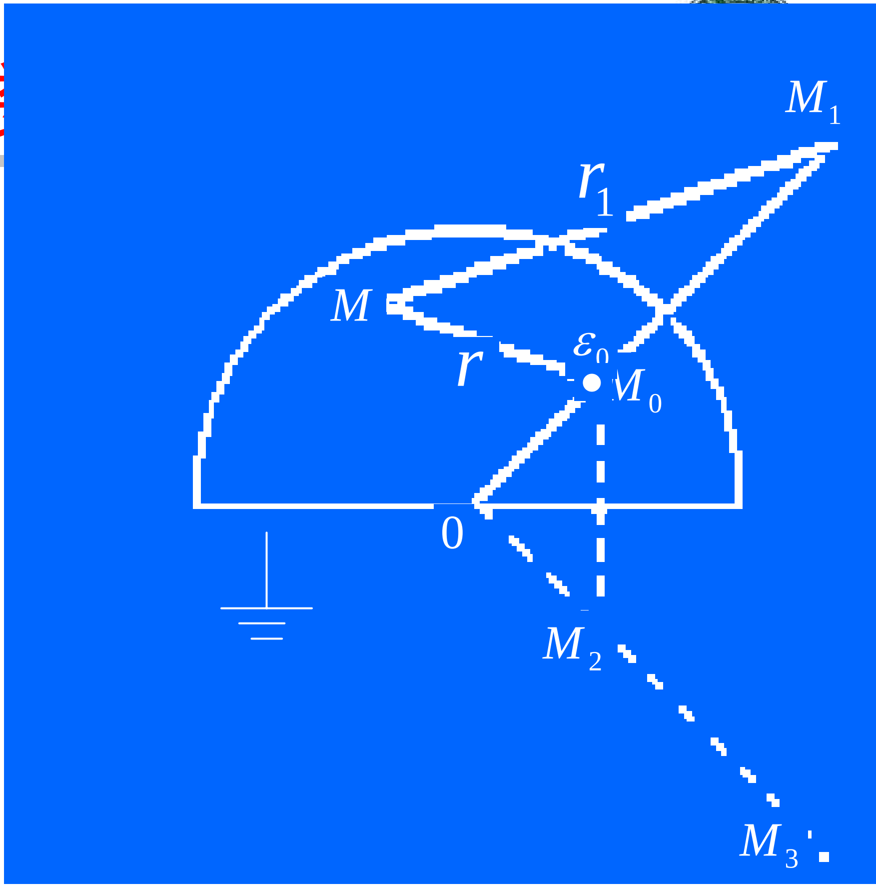
$$G = \frac{1}{2\pi}\ln\frac{1}{r} + g$$

$$\begin{cases} \Delta g = 0, M \in \sigma \\ g|_l = -\frac{1}{2\pi}\ln\frac{1}{r}|_l \end{cases}$$

求G $\rightarrow$ 求M点电位$\rightarrow$求感应电荷产生的电位

用电像法求：在像点放一虚构的点电荷，来等效的代替边界面上的感应电荷所产生的电位。

二、用电像法求格林函



1. 求上半球域的G

$$G=\frac{1}{4\pi r}+g$$

$$\begin{cases}\Delta g=0\ ,\ \ M\in\tau\\ g\,|_{\sigma}=-\dfrac{1}{4\pi r}\,|_{\sigma}\end{cases}$$

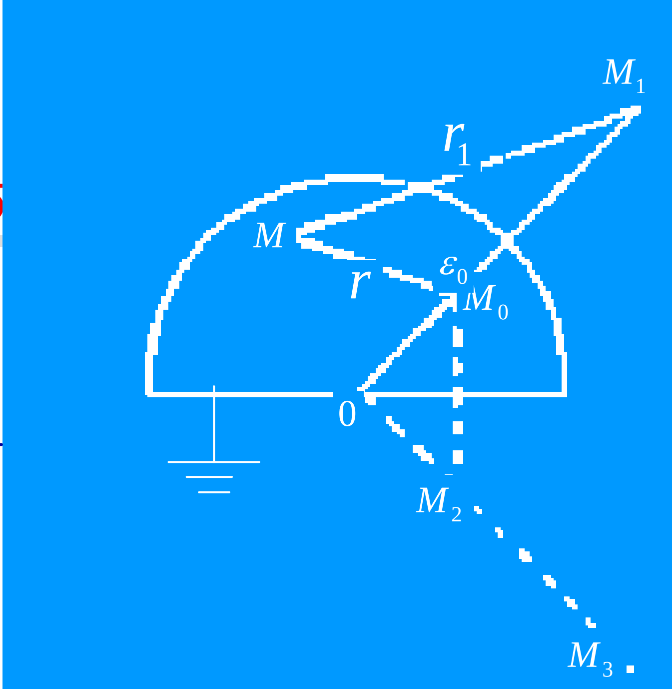
(1)确定象点:设$M_0(\rho_0,\theta_0,\varphi_0)$

则$M_1(\dfrac{a^2}{\rho_0},\theta_0,\varphi_0),M_2(\rho_0,-\theta_0,\varphi_0),M_3(\dfrac{a^2}{\rho_0},-\theta_0,\varphi_0),$

二、用电像法求

$$\frac{\rho_0}{a}=\frac{a}{\rho_1}=\frac{r}{r_1}$$

§ 5



(2)确定g及q_i :在M_i点放置$-q_i$则

$$\Delta\left(\sum_{i=1}^{3}\left(\frac{1}{4\pi\varepsilon_0}\frac{q_i}{r_i}\right)\right)=0, M\in\tau \quad g=\sum_{i=1}^{3}\frac{1}{4\pi\varepsilon_0}\frac{q_i}{r_i}$$

而由 $\displaystyle \frac{1}{4\pi\varepsilon_0}\frac{q_i}{r_i}\Big|_\sigma=-\frac{1}{4\pi r}\Big|_\sigma$

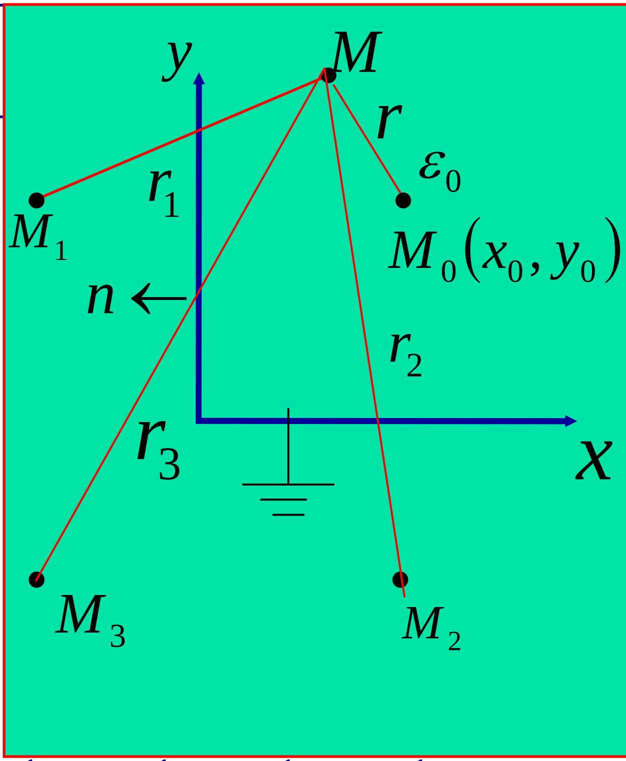
有: $\displaystyle \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1}\Big|_{\rho=a}=\frac{q_1}{4\pi\varepsilon_0\dfrac{a}{\rho_0}r}\Big|_{\rho=a}=-\frac{1}{4\pi r}\Big|_{\rho=a} \quad \therefore q_1=-\varepsilon_0\frac{a}{\rho_0}$

$$\frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_2}\Big|_\sigma=-\frac{1}{4\pi r}\Big|_\sigma \ (\because \sigma: r_2=r)\ \therefore q_2=-\varepsilon_0 \text{ 显然}, q_3=\varepsilon_0\frac{a}{\rho_0}$$

$$\boxed{G=\frac{1}{4\pi}\left[\left(\frac{1}{r}-\frac{a/\rho_0}{r_1}\right)-\left(\frac{1}{r_2}-\frac{a/\rho_0}{r_3}\right)\right]}$$

二、用电像法求格林函数

2、求四分之一平面的狄氏格林函数



设$M_0(x_0,y_0)$点置有电荷ε_0，则有

象点：$M_1(-x_0,y_0), M_2(x_0,-y_0), M_3(-x_0,-y_0)$

电象：$-\varepsilon_0 \qquad -\varepsilon_0 \qquad \varepsilon_0$

$$g_i: \quad -\frac{1}{2\pi}\ln\frac{1}{r_1}, \quad -\frac{1}{2\pi}\ln\frac{1}{r_2}, \quad \frac{1}{2\pi}\ln\frac{1}{r_3}$$

$$G=\frac{1}{2\pi}\ln\frac{1}{r}+g$$

$$\begin{cases}\Delta g=0, M\in\sigma \\ g|_l=-\frac{1}{2\pi}\ln\frac{1}{r}|_l\end{cases}$$

故$G=\frac{1}{2\pi}\ln\frac{1}{r}-\frac{1}{2\pi}\ln\frac{1}{r_1}-\frac{1}{2\pi}\ln\frac{1}{r_2}+\frac{1}{2\pi}\ln\frac{1}{r_3}$

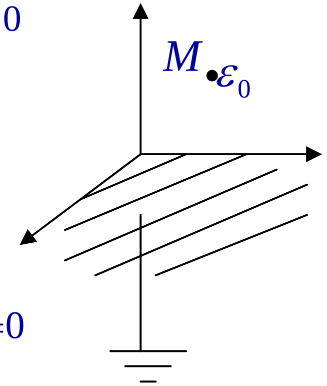
$$\boxed{G=\frac{1}{2\pi}\ln\frac{r_1r_2}{rr_3}}$$

三、求泊松方程的狄氏问题

1、求上半空间的狄氏问题

$$\begin{cases}\Delta u = 0, \ \ z > 0 \\ u|_{z=0} = f(x,y)\end{cases} \rightarrow u(M) = -\iint_\sigma f(M_0)\frac{\partial G}{\partial n_0}dx_0 dy_0$$

$$\begin{cases}\Delta G = -\delta(M - M_0) \\ G|_{z=0} = 0\end{cases} \Longrightarrow \begin{cases} G = \dfrac{1}{4\pi r} + g \\ \Delta g = 0, z > 0 \\ g|_{z=0} = -\dfrac{1}{4\pi r}|_{z=0}\end{cases}$$



(1)在$M_1(x,y,-z)$放$-q$,则$\Delta(\dfrac{-q}{4\pi\varepsilon_0 r_1}) = 0$ ， $z > 0$

使$\dfrac{-q}{4\pi\varepsilon_0 r_1}|_{z=0} = -\dfrac{\varepsilon_0}{4\pi\varepsilon_0 r}|_{z=0}$ 则 $g = -\dfrac{q}{4\pi\varepsilon_0 r_1}$

$$-q = -\varepsilon_0 \qquad \boxed{\therefore G = \frac{1}{4\pi r} - \frac{1}{4\pi r_1}}$$

三、求泊松方程的狄氏问

问: 下半空间狄氏问题?

G有无差别? $\frac{\partial G}{\partial n}$有无差别?

$$(2)\ \frac{\partial G}{\partial n}=\frac{\partial G}{\partial(-z)}=-\frac{1}{4\pi}\left[\frac{\partial}{\partial z}\frac{1}{r}-\frac{\partial}{\partial z}\frac{1}{r_1}\right]$$

$$\frac{\partial G}{\partial n}=\frac{\partial G}{\partial(-z)}=\frac{1}{4\pi}\frac{(z-z_0)}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}}$$

$$-\frac{1}{4\pi}\frac{(z+z_0)}{[(x-x_0)^2+(y-y_0)^2+(z+z_0)^2]^{3/2}}$$

$$\frac{\partial G}{\partial n}\Big|_{z=0}=\frac{1}{2\pi}\frac{-z_0}{[(x-x_0)^2+(y-y_0)^2+z_0^{\,2}]^{3/2}}$$

$$u(M)=\frac{z}{2\pi}\iint_{-\infty}^{\infty}f(x_0,y_0)\frac{1}{[(x-x_0)^2+(y-y_0)^2+z^2]^{3/2}}dx_0dy_0$$

三、求泊松方程的狄氏问题

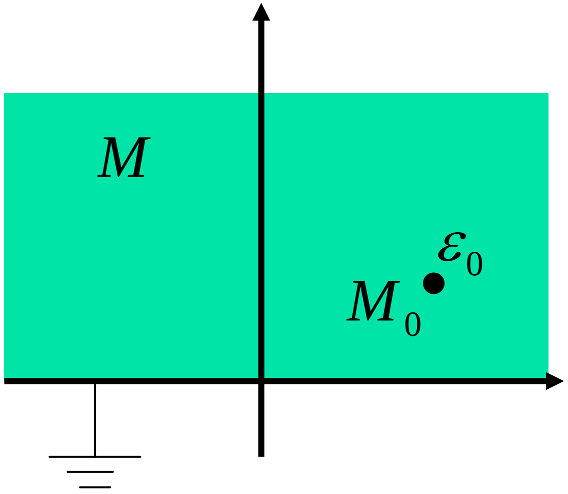
2、求上半平面的狄氏问题：

$$\begin{cases} \Delta u = 0 \;, y > 0 \\ u\big|_{y=0} = f(x) \end{cases} \qquad u(x,y) = -\int_{-\infty}^{\infty} f(x_0)\frac{\partial G}{\partial n_0}dx_0$$

$$G = \frac{1}{2\pi}\ln\frac{1}{r} - \frac{1}{2\pi}\ln\frac{1}{r_1} = \frac{1}{2\pi}\ln\frac{r_1}{r}$$

$$r = \sqrt{(x-x_0)^2+(y-y_0)^2},$$

$$r_1 = \sqrt{(x-x_0)^2+(y+y_0)^2}$$



三、求泊松方程的狄氏

$$u(x,y)=-\int_{-\infty}^{\infty} f(x_0)\frac{\partial G}{\partial n_0}dx_0$$

$$G=\frac{1}{2\pi}\ln\frac{r_1}{r}=\frac{1}{4\pi}\left\{\ln\left[(x-x_0)^2+(y+y_0)^2\right]-\ln\left[(x-x_0)^2+(y-y_0)^2\right]\right\}$$

$$\frac{\partial G}{\partial n}=\frac{\partial G}{\partial(-y)}=-\frac{\partial G}{\partial y}$$

$$\therefore \frac{\partial G}{\partial y}\Big|_{y=0}=\frac{1}{4\pi}\left[\frac{2(y+y_0)}{(x-x_0)^2+(y+y_0)^2}-\frac{2(y-y_0)}{(x-x_0)^2+(y-y_0)^2}\right]_{y=0}$$

$$=\frac{y_0}{\pi}\left[\frac{1}{(x-x_0)^2+y_0^2}\right]$$

$$u(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(x_0)}{(x_0-x)^2+y^2}dx_0$$

三、求泊松方程的狄氏问题

3、用格林函数法重新求解

$$\begin{cases}\Delta_2 u = 0, & \rho < a \\ u|_{\rho=a} = A\cos\varphi\end{cases}$$

解:由圆域的狄氏积分公式有

$$u(\rho,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \frac{A\cos\varphi_0 (a^2-\rho^2)}{a^2+\rho^2-2a\rho\cos(\varphi-\varphi_0)} d\varphi_0$$

将 $\dfrac{\text{分子}/a^2}{\text{分母}/a^2}$,并记 $\varepsilon = \dfrac{\rho}{a}$

于是有:$u(\rho,\varphi) = \dfrac{(1-\varepsilon^2)A}{2\pi} \underbrace{\int_0^{2\pi} \dfrac{\cos\varphi_0}{1-2\varepsilon\cos(\varphi-\varphi_0)+\varepsilon^2} d\varphi_0}_{I}$

三、求泊松方程的狄氏问题

$$\text{令} z=e^{i\varphi_0}, \text{则} \cos\varphi_0=\frac{1}{2}\left(z+\frac{1}{z}\right)=\frac{z^2+1}{2z}$$

$$\cos(\varphi-\varphi_0)=\frac{1}{2}\left[e^{i(\varphi-\varphi_0)}+e^{-i(\varphi-\varphi_0)}\right]$$

$$=\frac{1}{2}\left[e^{i\varphi}e^{-i\varphi_0}+e^{-i\varphi}e^{i\varphi_0}\right]=\frac{1}{2}\left[e^{i\varphi}\frac{1}{z}+e^{-i\varphi}z\right], \quad d\varphi_0=\frac{dz}{iz}$$

$$\therefore I=\int_0^{2\pi}\frac{\cos\varphi_0}{1-2\varepsilon\cos(\varphi-\varphi_0)+\varepsilon^2}d\varphi_0$$

$$=\frac{-1}{2i}\oint_{|z|=1}\underbrace{\frac{1+z^2}{z[\varepsilon e^{-i\varphi}z^2-(1+\varepsilon^2)z+\varepsilon e^{i\varphi}]}}_{f(z)}dz$$

三、求泊松方程的狄氏问题

$$奇点 z=\begin{cases}\dfrac{(1+\varepsilon^2)\pm\sqrt{(1+\varepsilon)^2-4\varepsilon^2}}{2e^{-i\varphi}\varepsilon} \\ 0\end{cases}=\begin{cases}\dfrac{e^{i\varphi}}{\varepsilon} & >1 \\ \varepsilon e^{i\varphi} & <1 \\ 0 & <1\end{cases}$$

$$\therefore \sum_{k=1}^{2} resf(z_k)=resf(0)+resf(\varepsilon e^{i\varphi})$$

$$=\frac{1}{\varepsilon}e^{-i\varphi}+\frac{1+\varepsilon^2 e^{2i\varphi}}{\varepsilon e^{i\varphi}(\varepsilon^2-1)}=\frac{2\varepsilon}{\varepsilon^2-1}\cos\varphi$$

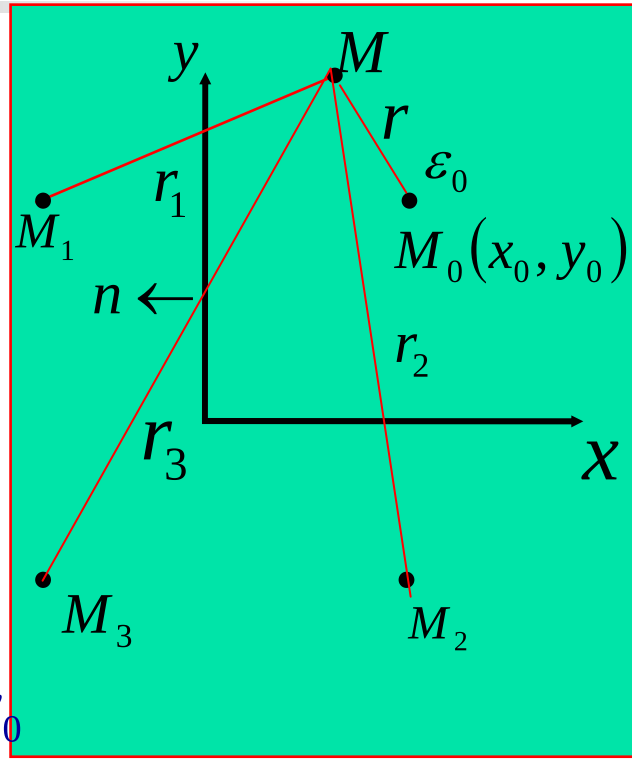
$$\therefore I=-\frac{1}{2i}\cdot 2\pi i\cdot\frac{2\varepsilon}{\varepsilon^2-1}\cos\varphi=\frac{2\pi\varepsilon}{1-\varepsilon^2}\cos\varphi$$

$$\boxed{u(\rho,\varphi)}=\frac{\left[1-(\rho/a)^2\right]}{2\pi}A\frac{2\pi\,\rho/a}{1-(\rho/a)^2}\cos\varphi\ \boxed{=\frac{A}{a}\rho\cos\varphi}$$

三、求泊松方程的狄氏问题

2、求四分之一平面的狄氏问题：

$$\begin{cases} u_{xx}+u_{yy}=0\ ,0\le x<\infty,0\le y<\infty \\ u(0,y)=f(y)\quad (0\le y<\infty) \\ u(x,0)=0 \end{cases}$$



$$u(x,y)=-\int_l f(M_0)\frac{\partial G}{\partial n_0}dl_0$$

$$=-\int_0^{\infty}0\cdot\frac{\partial G}{\partial(-y_0)}dx_0-\int_0^{\infty}f(y_0)\frac{\partial G}{\partial(-x_0)}dy_0$$

$$u(x,y)=\int_0^{\infty}f(y_0)\frac{\partial G}{\partial x_0}dy_0$$

$$\boxed{G=\frac{1}{2\pi}\ln\frac{r_1r_2}{rr_3}}$$

三、求泊松方程的狄氏问题

$$G=\frac{1}{2\pi}\ln\frac{r_1 r_2}{r r_3}=\frac{1}{4\pi}\left\{\ln\left[(x+x_0)^2+(y-y_0)^2\right]+\left[(x-x_0)^2+(y+y_0)^2\right]\right.$$
$$\left.-\ln\left[(x-x_0)^2+(y-y_0)^2\right]-\left[(x+x_0)^2+(y+y_0)^2\right]\right\}$$

$$\therefore \frac{\partial G}{\partial x}\Big|_{x=0}=\frac{1}{4\pi}\Big[\frac{2(x+x_0)}{(x+x_0)^2+(y-y_0)^2}+\frac{2(x-x_0)}{(x-x_0)^2+(y+y_0)^2}$$
$$-\frac{2(x-x_0)}{(x-x_0)^2+(y-y_0)^2}-\frac{2(x+x_0)}{(x+x_0)^2+(y+y_0)^2}\Big]_{x=0}$$

$$=\frac{x_0}{\pi}\left[\frac{1}{x_0^2+(y-y_0)^2}-\frac{1}{x_0^2+(y+y_0)^2}\right]$$

$$\boxed{u(x,y)=\frac{x}{\pi}\int_0^{\infty}\Big[\frac{1}{x^2+(y_0-y)^2}-\frac{1}{x^2+(y_0+y)^2}\Big]f(y_0)dy_0}$$

Good-by!