数学物理方法

Mathematical Methods in Physics

第八章 分离变量法

The Method of
Separation of Variables

武汉大学物理科学与技术学院

分离变量法习题课

*本章内容小结

*典型例题分析

一、正交曲线坐标系中的分离变量

二、齐次方程、齐次边界条件的定解问题

三、非齐次边界条件的定解问题

一、正交曲线坐标系中的分离变量

1、求圆环的狄氏问题

$$\begin{cases} \Delta u = 0 \;,\; (r_2 < r < r_1) & \langle 1\rangle \\ u(r_1,\theta) = \sin\theta & \langle 2\rangle \\ u(r_2,\theta) = 0 & \langle 3\rangle \end{cases}$$

①令：$u(r,\theta) = R(r)\Theta(\theta)$ <4>

<1>$\longrightarrow$ $$\begin{cases} \Theta''(\theta) + m^2\Theta = 0 & \langle 5\rangle \\ r^2R'' + rR' - m^2R = 0 & \langle 6\rangle \end{cases}$$

<3>$\longrightarrow$ $R(r_2) = 0$ <3>*

一、正交曲线坐标系中的分离变量

②解 $$\begin{cases} \Theta''(\theta) + m^2\Theta = 0 \\ \Theta(\theta + 2\pi) = \Theta(\theta) \end{cases}$$

得：$\boxed{\Theta(\theta) = A_m \cos m\theta + B_m \sin m\theta}$ $m = 0, 1, 2, \ldots$

③解 <6>：$$R_m(r) = \begin{cases} C_0 \ln r + D_0 & m = 0 \\ C_m r^m + D_m r^{-m} & m \neq 0 \end{cases}$$

$$\boxed{\begin{aligned} R_m(r) &\overset{<3>*}{=} C_0(\ln r - \ln r_2) \\ &\quad + C_m(r^m - r_2^{2m} r^{-m}) \end{aligned}}$$

一、正交曲线坐标系中的分离变量

④ $$u=\sum_{m=0}^{\infty}u_m(r,\theta)=\sum_{m=0}^{\infty}R_m(r)\Theta_m(\theta)$$

$$=\alpha_0(\ln r-\ln r_2)$$

$$+\sum_{m=1}^{\infty}\frac{r^{2m}-r_2^{2m}}{r^m}[\alpha_m\cos m\theta+\beta_m\sin m\theta]$$

由 $u(r_1,\theta)=\sin\theta$ 有： $\alpha_n=0;\quad \beta_n=0, n\neq 1$

$$\beta_1=\frac{r_1}{r_1^2-r_2^2}$$

$$\boxed{u(r,\theta)=\frac{r_1}{r_1^2-r_2^2}\frac{r^2-r_2^2}{r}\cdot\sin\theta}$$

一、正交曲线坐标系中的分离变量

2、求解扇形区域中的狄氏问题：

$$\begin{cases} \Delta u = 0 \ , \ (\rho < a \, , \alpha < \varphi < \beta) & \text{<1>} \\ u|_{\varphi=\alpha} = 0 \ , \ u|_{\varphi=\beta} = 0 & \text{<2>} \\ u|_{\rho=a} = f(\varphi) & \text{<3>} \end{cases}$$

①令 $u = R(\rho)\Phi(\varphi)$

则由式<1>，得： $n^2 = ?$

$$\begin{cases} \Phi'' + n^2\Phi = 0 & \text{<4>} \\ \rho^2 R'' + \rho R' - n^2 R = 0 & \text{<5>} \end{cases}$$

$n \neq 0,1,\cdots$

一、正交曲线坐标系中的分离变量

则由式<1>，得：
$$\begin{cases}\Phi''+\mu\Phi=0 & \langle 4\rangle\\ \rho^2R''+\rho R'-\mu R=0 & \langle 5\rangle\end{cases}$$

由<2>式，得：
$$\begin{cases}\Phi(\alpha)=0\\ \Phi(\beta)=0\end{cases}\qquad \langle 2\rangle^*$$

② 解：
$$\begin{cases}\Phi''(\varphi)+\mu\Phi=0\\ \Phi(\alpha)=0\ ,\ \Phi(\beta)=0\end{cases}$$
令：$\boxed{\theta=\varphi-\alpha}$

$$\rightarrow\begin{cases}\Phi''(\theta)+\mu\Phi=0\\ \Phi|_{\theta=0}=0\ ,\ \Phi|_{\theta=\beta-\alpha}=0\end{cases}$$

$$\boxed{\mu=[\frac{n\pi}{\beta-\alpha}]^2,\quad n=1,2,\cdots}$$

$$\boxed{\Phi(\varphi)=\Phi(\theta)=a_n\sin\frac{n\pi}{\beta-\alpha}\theta=a_n\sin\frac{n\pi}{\beta-\alpha}(\varphi-\alpha)}$$

一、正交曲线坐标系中的分离变量

③求解<5> $\rho^2 R'' + \rho R' - \mu R = 0 \qquad \mu = [\frac{n\pi}{\beta-\alpha}]^2$

因为$\rho < a$ $\quad R_n(\rho) = b\rho^{\sqrt{\mu}} = b_n \rho^{\frac{n\pi}{\beta-\alpha}}$

④ $$\boxed{u(\rho,\varphi) = \sum_{n=1}^{\infty} C_n \rho^{\frac{n\pi}{\beta-\alpha}} \sin\frac{n\pi}{\beta-\alpha}(\varphi-\alpha)}$$

由式<3>，得： $f(\varphi) = \sum_{n=1}^{\infty} C_n a^{\frac{n\pi}{\beta-\alpha}} \sin\frac{n\pi}{\beta-\alpha}(\varphi-\alpha)$

$$\boxed{\therefore\ C_n = a^{\frac{-n\pi}{\beta-\alpha}} \frac{2}{\beta-\alpha}\int_{\alpha}^{\beta} f(\varphi)\sin\frac{n\pi}{\beta-\alpha}(\varphi-\alpha)d\varphi}$$

一、正交曲线坐标系中的分离变量

3、在均匀电场E中，垂直于电场方向放入半径为a的无限长直圆柱电介质，介电常数为ε，求柱内外的电场。

【分析】 在介质圆柱未放入前：$\dfrac{\partial u}{\partial x}=-E \quad \to u=-E\rho\cos\varphi+c$

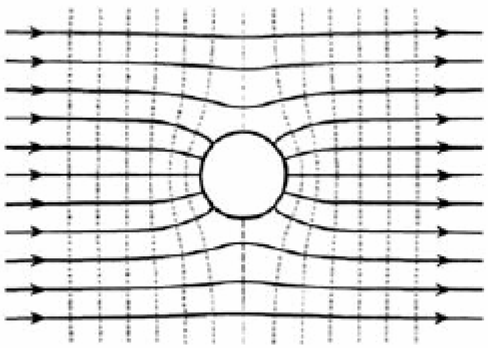
[设 $u(0,\varphi)=0$] $\quad \to u=-E\rho\cos\varphi$

在介质圆柱放入后： $u(\rho,\varphi)\big|_{\rho\to\infty}=-E\rho\cos\varphi$

【求解】

$$\begin{cases}\Delta u^{I}=0,\ 0<\rho<a \\ u^{I}\big|_{\rho=0}=\text{有限}\end{cases} \quad (1) \qquad \begin{cases}\Delta u^{II}=0,\ a<\rho<\infty \\ u^{II}\big|_{\rho\to\infty}=-E\rho\cos\varphi\end{cases} \quad (2)$$

$$\begin{cases}u^{I}\big|_{\rho=a}=u^{II}\big|_{\rho=a} & (3) \\ \varepsilon\dfrac{\partial u^{I}}{\partial\rho}\Big|_{\rho=a}=\dfrac{\partial u^{II}}{\partial\rho}\Big|_{\rho=a} & (4)\end{cases}$$



一、正交曲线坐标系中的分离变量

【求解】

$$u^{I}(\rho,\varphi)=\frac{A_0}{2}+\sum_{n=1}^{\infty}(A_n\cos n\varphi+B_n\sin n\varphi)\rho^{n} \qquad (5)$$

$$u^{II}(\rho,\varphi)=(C_0+D_0\ln\rho)A_0+\sum_{n=1}^{\infty}(A_n\cos n\varphi+B_n\sin n\varphi)(C_n\rho^{n}+D_n\rho^{-n})$$

$$=(\alpha_0+\beta_0\ln\rho)+\sum_{n=1}^{\infty}(\alpha_n\cos n\varphi+\beta_n\sin n\varphi)\rho^{n}$$

$$+\sum_{n=1}^{\infty}(\alpha_n'\cos n\varphi+\beta_n'\sin n\varphi)\rho^{-n}$$

$$u^{II}\Big|_{\rho\to\infty}=-E\rho\cos\varphi\to$$

$$\alpha_0=0,\beta_0=0;\ \alpha_n=0(n\neq 1),\beta_n=0;\quad \alpha_1\rho=-E\rho\ \to\alpha_1=-E$$

$$u^{II}(\rho,\varphi)=-E\rho\cos\varphi+\sum_{n=1}^{\infty}(\alpha_n'\cos n\varphi+\beta_n'\sin n\varphi)\rho^{-n} \qquad (6)$$

一、正交曲线坐标系中的分离变量

【求解】 $u^I\big|_{\rho=a}=u^{II}\big|_{\rho=a}$ (3) $\longrightarrow$

$$\frac{A_0}{2}+\sum_{n=1}^{\infty}(A_n a^n\cos n\varphi+B_n a^n\sin n\varphi)$$

$$=-Ea\cos\varphi+\sum_{n=1}^{\infty}\left(\frac{\alpha_n'}{a^n}\cos n\varphi+\frac{\beta_n'}{a^n}\sin n\varphi\right)$$

$$\longrightarrow\begin{cases} A_0=0 & (7)\\ A_1 a=-Ea+\dfrac{\alpha_1'}{a} & (8)\\ A_n a^n=\dfrac{\alpha_n'}{a^n} & (9)\\ B_n a^n=\dfrac{\beta_n'}{a^n} & (10)\end{cases}$$

一、正交曲线坐标系中的分离变量

【求解】

$$\varepsilon \frac{\partial u^{I}}{\partial \rho}\bigg|_{\rho=a} = \frac{\partial u^{II}}{\partial \rho}\bigg|_{\rho=a} \quad (4) \longrightarrow$$

$$\varepsilon \sum_{n=1}^{\infty} (A_n n a^{n-1} \cos n\varphi + B_n n a^{n-1} \sin n\varphi)$$

$$= -E\cos\varphi - \sum_{n=1}^{\infty} n \left(\frac{\alpha_n'}{a^{n+1}} \cos n\varphi + \frac{\beta_n'}{a^{n+1}} \sin n\varphi\right)$$

$$\longrightarrow \begin{cases} \varepsilon A_1 = -E - \dfrac{\alpha_1'}{a^2} & (11) \\ \varepsilon A_n a^{n-1} = -\dfrac{\alpha_n'}{a^{n+1}} & (12) \\ \varepsilon B_n a^{n-1} = -\dfrac{\beta_n'}{a^{n+1}} & (13) \end{cases}$$

一、正交曲线坐标系中的分离变量

【求解】

$$\begin{cases} A_0 = 0 & (7) \\ A_1 a = -Ea + \dfrac{\alpha_1'}{a} & (8) \\ A_n a^n = \dfrac{\alpha_n'}{a^n} & (9) \\ B_n a^n = \dfrac{\beta_n'}{a^n} & (10) \end{cases} \qquad \begin{cases} \varepsilon A_1 = -E - \dfrac{\alpha_1'}{a^2} & (11) \\ \varepsilon A_n a^{n-1} = -\dfrac{\alpha_n'}{a^{n+1}} & (12) \\ \varepsilon B_n a^{n-1} = -\dfrac{\beta_n'}{a^{n+1}} & (13) \end{cases}$$

$$\begin{cases} (8) \\ (11) \end{cases} \rightarrow \quad A_1 = -\frac{2E}{1+\varepsilon}, \alpha_1' = \frac{\varepsilon - 1}{\varepsilon + 1} a^2 E$$

$$\begin{cases} (9) \\ (12) \end{cases} \rightarrow \quad A_n = 0, \quad \alpha_n' = 0$$

$$\begin{cases} (10) \\ (13) \end{cases} \rightarrow \quad B_n = 0, \quad \beta_n' = 0$$

$\Rightarrow$

$$\boxed{u^I(\rho,\varphi) = -\frac{2E\rho}{1+\varepsilon}\cos\varphi}$$

$$\boxed{u^{II}(\rho,\varphi) = -(\rho - \frac{\varepsilon - 1}{\varepsilon + 1}\frac{a^2}{\rho})E\cos\varphi}$$

二、齐次问题

【公式】

$$\begin{cases} u_{tt} = a^2 u_{xx}, \quad 0 < x < l & (1) \\ u\big|_{x=0} = 0, \quad u\big|_{x=l} = 0 & (2) \\ u\big|_{t=0} = \varphi(x), \quad u_t\big|_{t=0} = \psi(x) & (3) \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\frac{n\pi a}{l}t + B_n \sin\frac{n\pi a}{l}t\right)\sin\frac{n\pi}{l}x \qquad (11)$$

$$A_n = \frac{2}{l}\int_0^l \varphi(\alpha)\sin\frac{n\pi}{l}\alpha d\alpha, \qquad B_n = \frac{2}{n\pi a}\int_0^l \psi(\alpha)\sin\frac{n\pi}{l}\alpha d\alpha \qquad (12)$$

二、齐次问题

1、求解
$$\begin{cases} u_{tt} = a^2 u_{xx}\ , \quad 0 < x < \pi\ , t > 0 \\ \left.\begin{array}{l} u(x,0) = 3\sin x \\ u_t(x,0) = 0 \end{array}\right\}, 0 \le x \le \pi \\ u(0,t) = u(\pi,t) = 0; \end{cases}$$

解: $u(x,t) = \sum_{n=1}^{\infty}(A_n \cos nat + B_n \sin nat)\sin nx$

$u(x,0) = 3\sin x \quad \to \sum_{n=1}^{\infty} A_n \sin nx = 3\sin x$

$\to A_1 = 3\ ,\ A_n = 0 \ \ (n \ne 1)$

$u_t(x,0) = 0 \quad \to \sum_{n=1}^{\infty} B_n na \sin nx = 0 \quad \to \boldsymbol{B_n} = 0$

$$\boxed{u(x,t) = 3\cos at \sin x}$$

二、齐次问题

2、求下列高维波动问题的解：

$$
\begin{cases}
u_{tt} = a^2 \Delta u \;,\; 0<x<1,\, 0<y<1,\, 0<z<1 & \langle 1 \rangle \\
u(x,y,z;0) = \sin \pi x \sin \pi y \sin \pi z & \langle 2 \rangle \\
u_t(x,y,z;0) = 0 & \langle 3 \rangle \\
u(0,y,z;t) = u(1,y,z;t) = 0 & \langle 4 \rangle \\
u(x,0,z;t) = u(x,1,z;t) = 0 & \langle 5 \rangle \\
u(x,y,0;t) = u(x,y,1;t) = 0 & \langle 6 \rangle
\end{cases}
$$

二、齐次问题

解： ① 令 $u(x,y,z;t)=X(x)Y(y)Z(z)T(t)$

$$\langle 1\rangle \rightarrow \begin{cases} T''-a^2\mu T=0 & \langle 7\rangle \\ X''-\alpha X=0 & \langle 8\rangle \\ Y''-\beta Y=0 & \langle 9\rangle \\ Z''-\gamma Z=0 & \langle 10\rangle \end{cases}$$

其中$(\mu=\alpha+\beta+\gamma)$

$$<4>\rightarrow \begin{cases} X(0)=0 \\ X(1)=0 \end{cases} \quad <4>'; \quad <5>\rightarrow \begin{cases} Y(0)=0 \\ Y(1)=0 \end{cases} \quad <5>'$$

$$<6>\rightarrow \begin{cases} Z(0)=0 \\ Z(1)=0 \end{cases} \quad <6>'$$

二、齐次问题

解：②

$$\text{解}\begin{cases} X''-\alpha X=0 \\ X(0)=0 \\ X(1)=0 \end{cases} \rightarrow \quad \begin{aligned} &X_m(x)=a_m\sin m\pi x\,, \\ &\alpha=-m^2\pi^2\ ,\ m=1,2,\ldots \end{aligned}$$

$$\text{解}\begin{cases} Y''-\beta Y=0 \\ Y(0)=0 \\ Y(1)=0 \end{cases} \rightarrow \quad \begin{aligned} &Y_n(y)=b_n\sin n\pi y\,, \\ &\beta=-n^2\pi^2\ ,\ n=1,2,\ldots \end{aligned}$$

$$\text{解}\begin{cases} Z''-\gamma Z=0 \\ Z(0)=0 \\ Z(1)=0 \end{cases} \rightarrow \quad \begin{aligned} &Z_l(y)=c_l\sin l\pi z\,, \\ &\gamma=-l^2\pi^2\ ,\ l=1,2,\ldots \end{aligned}$$

二、齐次问题

解：③ $T'' + a^2(n^2 + m^2 + l^2)\pi^2 T = 0$

$$T_{m,n,l}(t) = A'_{mnl}\cos\omega t + B'_{mnl}\sin\omega t$$

$$\omega^2 = a^2(n^2 + m^2 + l^2)\pi^2$$

④ $$u(x,y,z;t) = \sum_{m,n,l=1}^{\infty}(A_{mnl}\cos\omega t + B_{mnl}\sin\omega t)\cdot\sin m\pi x\sin n\pi y\sin l\pi z$$

$$\sum_{m,n,l=1}^{\infty} A_{mnl}\sin m\pi x\sin n\pi y\sin l\pi z = \sin\pi x\sin\pi y\sin\pi z$$

$$\to A_{111} = 1\ ,\ \ A_{mnl} = 0\,(m \neq 1, n \neq 1, l \neq 1)$$

$$\sum_{m,n,l=1}^{\infty} B_{mnl}\omega\sin m\pi x\sin n\pi y\sin l\pi z = 0 \ \to B_{mnl} = 0$$

$$\boxed{u(x,y,z;t) = \cos\sqrt{3}\pi at\sin\pi x\sin\pi y\sin\pi z}$$

二、齐次问题

3、求处于一维无限深势阱中的粒子状态。

解：① 令 $\psi(x,t)=\varphi(x)f(t)$

$$\begin{cases} i\hbar\dfrac{\partial\psi(x,t)}{\partial t}=-\dfrac{\hbar^2}{2\mu}\dfrac{\partial^2}{\partial x^2}\psi(x,t) \\ \psi(-a,t)=\psi(a,t)=0 \\ \psi(x,0)=\dfrac{1}{\sqrt{a}}\sin\dfrac{\pi}{a}(x+a) \end{cases} \rightarrow \begin{cases} i\hbar\dfrac{df}{dt}=E\cdot f \\ -\dfrac{\hbar^2}{2\mu}\dfrac{d^2\varphi}{dx^2}=E\varphi \end{cases}$$

$$③ \rightarrow \boxed{f(t)=b_n e^{-i\frac{E_n}{\hbar}t}}$$

$\varphi(-a)=\varphi(a)=0$ 记 $\dfrac{2\mu E}{\hbar^2}=\lambda$

② 解 $\begin{cases} \varphi''(x)+\lambda\varphi=0 \\ \varphi(-a)=\varphi(a)=0 \end{cases} \rightarrow \lambda_n=(\dfrac{n\pi}{2a})^2 \rightarrow E_n=\dfrac{n^2\pi^2\hbar^2}{8a^2\mu}$ $\boxed{\varphi_n(x)=C_n\sin\dfrac{n\pi}{2a}(x+a)}$

④ $$\boxed{\psi(x,t)=\frac{1}{\sqrt{a}}e^{-i\frac{E_2}{\hbar}t}\sin\frac{\pi}{a}(x+a)}$$

三、非齐次边界条件的定解问题

求解：$\begin{cases} u_t = Du_{xx} \\ u|_{x=0} = ct, u|_{x=l} = 0 \\ u|_{t=0} = 0 \end{cases}$

解：令 $u(x,t) = v(x,t) + w(x,t)$

$w(x,t) = ct(1 - \frac{x}{l})$

$$\rightarrow \begin{cases} v_t - Dv_{xx} = c(\frac{x}{l} - 1) \\ v|_{x=0} = 0, v|_{x=l} = 0 \\ v|_{t=0} = 0 \end{cases}$$

令 $v(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi x}{l}$

$$\rightarrow \begin{cases} T_n'(t) + D\frac{n^2\pi^2}{l^2} T_n(t) = f_n(t) \\ T_n(0) = 0 \end{cases}$$

$$f_n(t) = \frac{2}{l}\int_0^l c(\frac{\alpha}{l} - 1)\sin\frac{n\pi\alpha}{l} d\alpha = -\frac{2c}{n\pi}$$

$$\rightarrow T_n(t) = e^{-\frac{Dn^2\pi^2}{l^2}t}\left[\int (-\frac{2c}{n\pi}) e^{\frac{Dn^2\pi^2}{l^2}t} dt + c\right] = \boxed{-\frac{2cl^2}{n^3\pi^3 D}\left[1 - e^{-\frac{Dn^2\pi^2}{l^2}t}\right]}$$

再见！