数学物理方法

Methods in Mathematical Physics

第十二章 非线性方程

Nonlinear Equations

武汉大学物理科学与技术学院

第十二章 非线性方程

Nonlinear Equations

引 言

Intrduction

一、为何要引入非线性方程

1、线性，只是理想的状况，初步的近似，大多事物的本来面目，却是非线性的。

2、现代科学技术的发展，使非线性科学占有极其重要的地位。

二、何谓非线性方程

方程中含有未知函数和未知函数偏导数的的高次项的方程

如，KdV： $$u_\tau + u_\xi + 12uu_\xi + u_{\xi\xi\xi} = 0 \quad (1)$$

三、非线性方程的特点：

引　言

1、不满足叠加原理

2、其定解问题一般不满足适定性；

3、没有普遍的理论和求解方法，大多都不能求得其解析解；

4、其解对初始条件具有敏感性（如，下页蝴蝶效应）；

5、其求解途经为：

（1）解依赖于自变量的

- 幂组合 $\begin{cases} u = u(\xi) \\ \xi = x^{\alpha} t^{\beta} \end{cases}$
- 线性组合 $\begin{cases} u = u(\xi) \\ \xi = x + at \end{cases}$

将偏微方程化为常微方程求解。

（2）通过自变量或函数变换，将非线性方程化为线性方程解

三、非线性方程的特点：

- 大气对流的洛仑兹（Lorenz）方程为

$$\begin{cases} \dot{x}(t) = -10x + 10y \\ \dot{y}(t) = 28x - y - xz \\ \dot{z}(t) = \dfrac{8}{3}x + xy \end{cases}$$

三、非线性方程的特点：

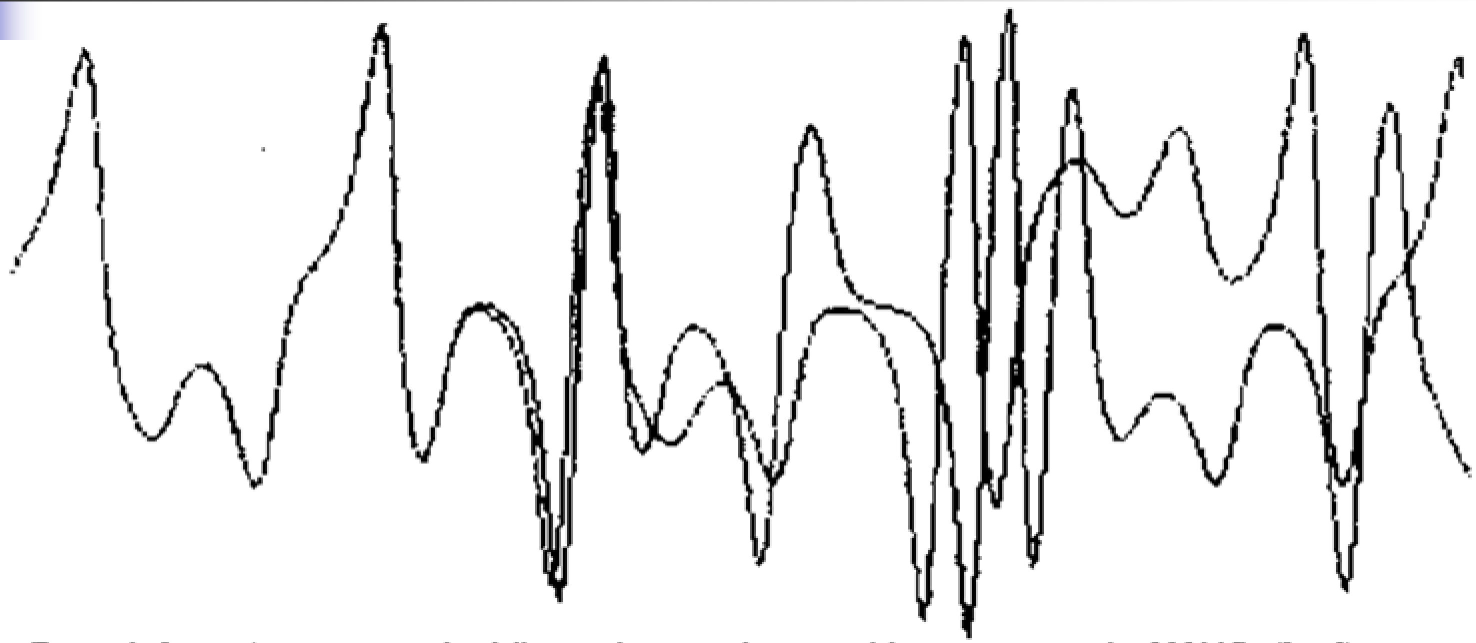

Figure 1: Lorenz's experiment: the difference between the start of these curves is only .000127. (Ian Stewart, Does God Play Dice? The Mathematics of Chaos, pg. 141)

0.506127~0.506

第十二章 非线性方程
Nonlinear Equations

§12.1 非线性方程的某些初等解法

Some primary solving methods For nonliner equations

一、Kirchhoff 变换

对于 $\nabla\cdot[G(u)\nabla u]=0 \quad (1)$

$$\boxed{\text{若选}\quad w=\int_{u_0}^{u}G(\xi)d\xi \quad (2)}$$ — Kirchhoff 变换

则 $(1)\to\Delta w=0$

证明： 令 $w=w(u)$，使 $\nabla w=G(u)\nabla u$ ，

则 $(1)\to\Delta w=0$

此时，$\dfrac{dw}{du}\nabla u=G(u)\nabla u\to\dfrac{dw}{du}=G(u)$

即：$w=\int_{u_0}^{u}G(\xi)d\xi \quad (2)$

一、Kirchhoff 变换

对于 $\nabla\cdot[G(u)\nabla u]=0 \quad (1)$

$$\boxed{\text{若选}\quad w=\int_{u_0}^{u}G(\xi)d\xi \quad (2)}$$ — **Kirchhoff 变换**

则 $(1)\to\Delta w=0$

例题：求解：$\begin{cases}\nabla\cdot[u^2\nabla u]=0\\ u\big|_{y=0}=Ax\end{cases} \quad (3)$

选：$w=\int_0^u\xi^2d\xi=\dfrac{1}{3}u^3$， 则$(3)\to\begin{cases}\Delta w=0\\ w\big|_{y=0}=\dfrac{1}{3}A^3x^3\end{cases}$

二、Cole-Hopf变换

对于 $u_t + uu_x = \delta u_{xx}$ (4) —Burgers方程

若作变换 $\boxed{u = -2\delta \dfrac{\partial \ln v}{\partial x} \quad (5)}$ — Cole-Hopf变换

则 (4) $\rightarrow v_t = \lambda v_{xx}$

证明： 1、求解(4)即要求解

$$\psi_t + \frac{1}{2}\psi_x^2 = \delta \psi_{xx} \quad (6)$$

其中， $\psi_x = u$ (7)

二、Cole-Hopf变换

证明：$\because (4) \rightarrow \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}(\delta u_x - \dfrac{u^2}{2})$

由全微分存在的充要条件有：

$$d\psi = udx + (\delta u_x - \frac{u^2}{2})dt$$

其中 $\begin{cases} \underline{\psi_x = u} \\ \psi_t = \delta u_x - \dfrac{u^2}{2} = \delta \psi_{xx} - \dfrac{1}{2}\psi_x^2 \end{cases}$

即（4）$\rightarrow \underline{\psi_t + \dfrac{1}{2}\psi_x^2 = \delta \psi_{xx} \quad (8)}$

二、C0le-Hopf变换

2、引入变换 $v=g(\psi)$,

使 (8) $\rightarrow v_t=\delta v_{xx}$ (9)

令 $v=g(\psi)$, 则 $v_t=g'(\psi)\psi_t$, $v_x=g'(\psi)\psi_x$

$$v_{xx}=g''(\psi)(\psi_x)^2+g'(\psi)\psi_{xx}$$

于是 (9) $\rightarrow \psi_t-\delta\dfrac{g''(\psi)}{g'(\psi)}(\psi_x)^2=\delta\psi_{xx}$ (10)

对比(8)和(10): $-\delta\dfrac{g''(\psi)}{g'(\psi)}=\dfrac{1}{2}\rightarrow g''(\psi)=-\dfrac{1}{2\delta}g'(\psi)$

$$\rightarrow g(\psi)=c_1e^{-\frac{1}{2\delta}\psi}+c_2$$

二、C0le-Hopf 变换

$$\to g(\psi) = c_1 e^{-\frac{1}{2\delta}\psi} + c_2$$

取$c_1 = 1, c_2 = 0$ $\quad \to g(\psi) = e^{-\frac{1}{2\delta}\psi} = v$

$$\to \psi = -2\delta \ln v,$$

3、结论：

故作变换 $u = \psi_x = -2\delta \dfrac{\partial \ln v}{\partial x}$

$$u_t + uu_x = \delta u_{xx} \quad (4) \quad \to v_t = \lambda v_{xx}$$

三、相似变换

对于 $$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}[G(u)\frac{\partial u}{\partial x}] \quad (11)$$

作变换
$$\begin{cases} u=u(\xi) & (12) \\ \xi=\dfrac{x}{\sqrt{t}} & (13) \end{cases}$$

Boltzman变换

$$(11)\rightarrow\frac{d}{d\xi}[G(u)\frac{du}{d\xi}]+\frac{\xi}{2}\frac{du}{d\xi}=0$$

而称：
$$\begin{cases} u=u(\xi) & (12) \\ \xi=x^{\alpha}t^{\beta} & (14) \end{cases}$$

—自型解

相似变换

三、相似变换

证明： 对于 $\dfrac{\partial u}{\partial t}=\dfrac{\partial}{\partial x}[G(u)\dfrac{\partial u}{\partial x}]$ (11)

令 $\begin{cases} u=u(\xi) & (12) \\ \xi = x^{\alpha}t^{\beta} & (14) \end{cases}$ 则

$$\frac{\partial u}{\partial t}=\frac{du}{d\xi}\frac{\partial \xi}{\partial t}=\beta\, x^{\alpha}t^{\beta-1}u'(\xi), \qquad \frac{\partial u}{\partial x}=\alpha\, x^{\alpha-1}t^{\beta}u'(\xi)$$

$$\frac{\partial}{\partial x}[G(u)\frac{\partial u}{\partial x}]=G(u)\frac{\partial}{\partial x}\frac{\partial u}{\partial x}+(\frac{\partial u}{\partial x})^2G'(u)$$

$$=\alpha(\alpha-1)\,x^{\alpha-2}t^{\beta}G(u)u'(\xi)+\alpha^2\,x^{2(\alpha-1)}t^{2\beta}\frac{d}{d\xi}[G(u)u'(\xi)]$$

三、相似变换

$$(11) \to \beta \frac{x^2}{t} \xi \frac{du}{d\xi} = \alpha(\alpha-1)\xi G(u)\frac{du}{d\xi} + \alpha^2\xi^2 \frac{d}{d\xi}[G(u)\frac{du}{d\xi}] \quad (11)'$$

选 $\dfrac{x^2}{t} = g(\xi) = \xi^2$， 即 $\xi = \dfrac{x}{\sqrt{t}}$，

亦即（14）中选 $\alpha = 1, \beta = -\dfrac{1}{2}$

$$(11)' \to -\frac{\xi}{2}\frac{du}{d\xi} = \frac{d}{d\xi}[G(u)\frac{du}{d\xi}]$$

$$(11) \to \frac{d}{d\xi}[G(u)\frac{du}{d\xi}] + \frac{\xi}{2}\frac{du}{d\xi} = 0$$

四、行波解：

对于 $$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}[u^n\frac{\partial u}{\partial x}] \quad (15)$$

令 $$\boxed{\begin{cases} u=g(\xi) & (16) \\ \xi=x+at & (17) \end{cases}}$$ —行波解

则 $$\frac{\partial u}{\partial t}=a\frac{dg}{d\xi},\quad \frac{\partial u}{\partial x}=\frac{dg}{d\xi}$$

$$(15)\rightarrow a\frac{dg}{d\xi}=\frac{d}{d\xi}[g^n\frac{dg}{d\xi}]$$

$$\boxed{\rightarrow u=g(\xi)=\{n[a(x+at)+c]\}^{\frac{1}{n}}}$$

五、端迹变换

$$\begin{cases} u_1\dfrac{\partial F}{\partial x}+u_2\dfrac{\partial F}{\partial y}+u_3\dfrac{\partial E}{\partial x}+u_4\dfrac{\partial E}{\partial y}=0 \\ v_1\dfrac{\partial F}{\partial x}+v_2\dfrac{\partial F}{\partial y}+v_3\dfrac{\partial E}{\partial x}+v_4\dfrac{\partial E}{\partial y}=0 \end{cases} \quad (16) \qquad \begin{aligned} u_i&=u_i(F,E)\\ v_i&=v_i(F,E)\\ i&=1,2,\cdots \end{aligned}$$

$x=x(F,E),y=(F,E)$ —端迹变换

$$\rightarrow\begin{cases} u_1\dfrac{\partial y}{\partial E}-u_2\dfrac{\partial x}{\partial E}-u_3\dfrac{\partial y}{\partial F}+u_4\dfrac{\partial x}{\partial F}=0 \\ v_1\dfrac{\partial y}{\partial E}-v_2\dfrac{\partial x}{\partial E}-v_3\dfrac{\partial y}{\partial F}+v_4\dfrac{\partial x}{\partial F}=0 \end{cases}$$

五、端迹变换

$$\begin{cases} u_1\dfrac{\partial F}{\partial x}+u_2\dfrac{\partial F}{\partial y}+u_3\dfrac{\partial E}{\partial x}+u_4\dfrac{\partial E}{\partial y}=0 \\ v_1\dfrac{\partial F}{\partial x}+v_2\dfrac{\partial F}{\partial y}+v_3\dfrac{\partial E}{\partial x}+v_4\dfrac{\partial E}{\partial y}=0 \end{cases} \quad (16)$$

证明： 令 $J=\dfrac{\partial(F,E)}{\partial(x,y)}=F_xE_y-E_xF_Y\neq 0 \quad (17)$

则可引入 $\begin{cases} x=x(F,E) \\ y=y(F,E) \end{cases} \quad (18)$

$$\frac{\partial}{\partial x}(18):\quad \begin{cases} 1=x_FF_x+x_EE_x \\ 0=y_FF_x+y_EE_x \end{cases} \rightarrow$$

五、端迹变换

$$\rightarrow\begin{cases} F_x = \dfrac{y_E}{x_E y_E - x_E y_F} = \dfrac{y_E}{j_E} = J y_E \\ E_x = -J y_F \end{cases} \quad (19)$$

$$j = \frac{\partial(x, y)}{\partial(F, E)} = J^{-1} \neq 0$$

$$\frac{\partial}{\partial y}(18): \quad \begin{cases} F_y = -J x_E \\ E_y = J x_F \end{cases} \quad (20)$$

(19),(20)代入（16）：

$$\rightarrow\begin{cases} u_1 \dfrac{\partial y}{\partial E} - u_2 \dfrac{\partial x}{\partial E} - u_3 \dfrac{\partial y}{\partial F} + u_4 \dfrac{\partial x}{\partial F} = 0 \\ v_1 \dfrac{\partial y}{\partial E} - v_2 \dfrac{\partial x}{\partial E} - v_3 \dfrac{\partial y}{\partial F} + v_4 \dfrac{\partial x}{\partial F} = 0 \end{cases}$$

六、小结

1. $\nabla\cdot[G(u)\nabla u]=0 \quad \rightarrow \Delta w=0$

若选 $w=\int_{u_0}^{u}G(\xi)d\xi$ — Kirchhoff 变换

2. 对于 $u_t+uu_x=\delta u_{xx} \quad \rightarrow v_t=\lambda v_{xx}$

若作变换 $u=-2\delta\dfrac{\partial\ln v}{\partial x}$ — Cole-Hopf变换

3. $\dfrac{\partial u}{\partial t}=\dfrac{\partial}{\partial x}[G(u)\dfrac{\partial u}{\partial x}] \quad \rightarrow \dfrac{d}{d\xi}[G(u)\dfrac{du}{d\xi}]+\dfrac{\xi}{2}\dfrac{du}{d\xi}=0$

若令 $u=u(\xi),\xi=\dfrac{x}{\sqrt{t}}$ —相似变换

4. $\dfrac{\partial u}{\partial t}=\dfrac{\partial}{\partial x}[u^n\dfrac{\partial u}{\partial x}] \quad \rightarrow u=g(\xi)=\{n[a(x+at)+c]\}^{\frac{1}{n}}$

$[u=g(\xi),\xi=x+at]$—行波解

六、小结

5. $$\begin{cases} u_1\dfrac{\partial F}{\partial x}+u_2\dfrac{\partial F}{\partial y}+u_3\dfrac{\partial E}{\partial x}+u_4\dfrac{\partial E}{\partial y}=0 \\ v_1\dfrac{\partial F}{\partial x}+v_2\dfrac{\partial F}{\partial y}+v_3\dfrac{\partial E}{\partial x}+v_4\dfrac{\partial E}{\partial y}=0 \end{cases}$$

$x=x(F,E), y=(F,E)$—端迹变换

$$\rightarrow\begin{cases} u_1\dfrac{\partial y}{\partial E}-u_2\dfrac{\partial x}{\partial E}-u_3\dfrac{\partial y}{\partial F}+u_4\dfrac{\partial x}{\partial F}=0 \\ v_1\dfrac{\partial y}{\partial E}-v_2\dfrac{\partial x}{\partial E}-v_3\dfrac{\partial y}{\partial F}+v_4\dfrac{\partial x}{\partial F}=0 \end{cases}$$

Good-by!