数学物理方法

Mathematical Methods in Physics

第八章 分离变量法

The Method of Separation of Variables

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问题的引入：

令 $u(x,t)=X(x)T(t)$

$$u(x,t)\big|_{x=0}=f(t) \longrightarrow X(0)=f(t)/T(t)$$

令 $u(x,t)=X(x)Y(y)$

$$XY\big|_{\sqrt{x^2+y^2}=a}=0$$

$$u\big|_{\rho=a}=0$$

$$R(a)\Phi(\varphi)=0 \longrightarrow R(a)=0$$

令 $u(\rho,\varphi)=R(\rho)\Phi(\varphi)$

可见：能否分离变量

1）与边界条件的齐次与否有关；

2）与坐标系的选取有关；

§8.4 正交曲线坐标系

Orthogonal Curvilineae Coordinates

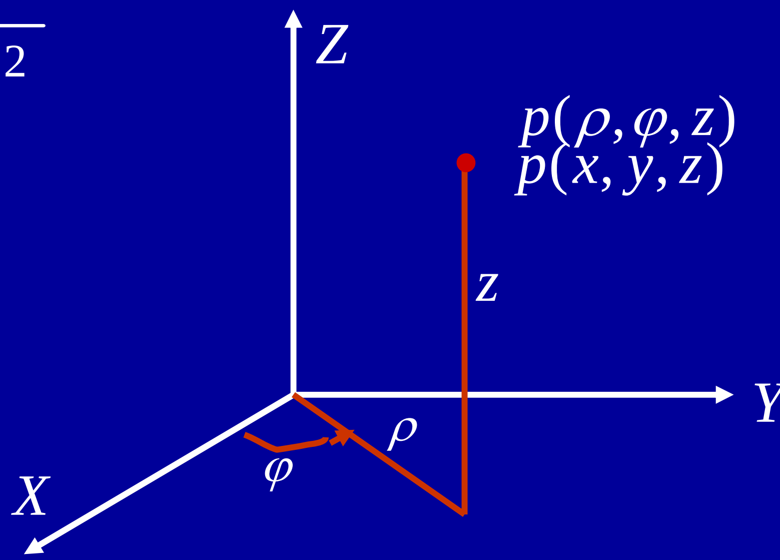
一、正交曲线坐标系：

由三族互相正交的曲面而定义的坐标系。

1、柱坐标（ρ，φ，z)：

$$0 \le \rho < \infty \;, -\infty < \varphi < \infty \;, -\infty < z < \infty$$

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases} \qquad \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \varphi = tg^{-1}\dfrac{y}{x} \\ z = z \end{cases}$$

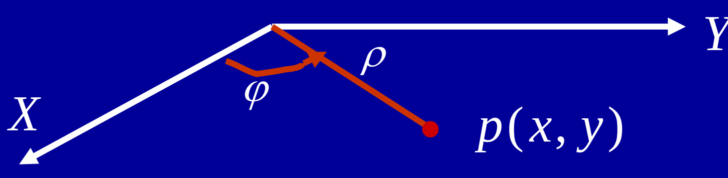


一、正交曲线坐标系：

由三族互相正交的曲面而定义的坐标系。

2、极坐标（ρ，φ)：$0\le\rho<\infty$，$-\infty<\varphi<\infty$，

$$\begin{cases} x=\rho\cos\varphi \\ y=\rho\sin\varphi \end{cases} \quad \begin{cases} \rho=\sqrt{x^2+y^2} \\ \varphi=tg^{-1}\dfrac{y}{x} \end{cases}$$



一、正交曲线坐标系：

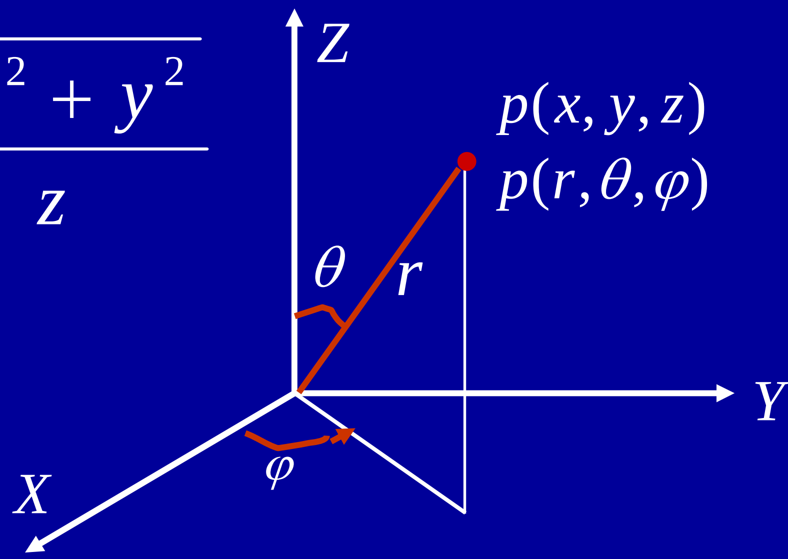
由三族互相正交的曲面而定义的坐标系。

3、球坐标系 (r, θ, φ)：

$$0 \le r < \infty\ ,\ 0 < \theta < \pi\ ,\ -\infty < \varphi < \infty$$

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = tg^{-1}\dfrac{\sqrt{x^2+y^2}}{z} \\ \varphi = tg^{-1}\dfrac{y}{x} \end{cases}$$



二、坐标系的选择：　：

应该选择坐标系，使所研究问题的边界面和一个或坐标面重合。

边界：	长方形	球	圆柱	圆锥
坐标：	“直”	“球”	“柱”	“球”

三、正交曲线坐标系中的Δu

1、在柱坐标系中

$$\frac{\partial u}{\partial \rho}=\frac{\partial u}{\partial x}\cos\varphi+\frac{\partial u}{\partial y}\sin\varphi \qquad (1)$$

$$\frac{\partial^2 u}{\partial \rho^2}=\frac{\partial^2 u}{\partial x^2}\cos^2\varphi+2\frac{\partial^2 u}{\partial x\partial y}\sin\varphi\cos\varphi+\frac{\partial^2 u}{\partial y^2}\sin^2\varphi \qquad (2)$$

$$\frac{\partial u}{\partial \varphi}=-\frac{\partial u}{\partial x}\rho\sin\varphi+\frac{\partial u}{\partial y}\rho\cos\varphi \qquad (3)$$

$$\frac{\partial^2 u}{\partial \varphi^2}=\frac{\partial^2 u}{\partial x^2}\rho^2\sin^2\varphi-2\frac{\partial^2 u}{\partial x\partial y}\rho^2\sin\varphi\cos\varphi+$$

$$+\frac{\partial^2 u}{\partial y^2}\rho^2\cos^2\varphi-\left(\frac{\partial u}{\partial x}\cos\varphi+\frac{\partial u}{\partial y}\sin\varphi\right)\rho \qquad (4)$$

三、正交曲线坐标系中的Δu

1、在柱坐标系中

$$\boxed{\Delta u=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^{2}}\frac{\partial^{2}u}{\partial\varphi^{2}}+\frac{\partial^{2}u}{\partial z^{2}}}$$

2、在极坐标系中

$$\boxed{\Delta u=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^{2}}\frac{\partial^{2}u}{\partial\varphi^{2}}}$$

3、在球坐标系中

$$\boxed{\Delta u=\frac{1}{r^{2}}\frac{\partial}{\partial r}\left(r^{2}\frac{\partial u}{\partial r}\right)+\frac{1}{r^{2}\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^{2}\sin^{2}\theta}\frac{\partial^{2}u}{\partial\varphi^{2}}}$$

§8.5 正交曲线坐标系中的分离变量

Separation of Variables in Terms of Orthogonal Curvilineae Coordinates

一、$\Delta u+\lambda u=0(\Delta u=0)$ 在物理学中的重要地位

令 $u(x,y,z;t)=T(t)v(x,y,z)$

$$u_{tt}=a^2\Delta u \rightarrow \begin{cases} T''+a^2\lambda T=0 \\ \underline{\Delta v+\lambda v=0} \end{cases}$$

$$u_t=D\Delta u \rightarrow \begin{cases} T'+\lambda DT=0 \\ \underline{\Delta v+\lambda v=0} \end{cases}$$

$$\underline{\Delta u=0}$$

二、柱坐标系中亥姆霍兹方程的分离变量

1. $\Delta u + \lambda u = 0 \rightarrow ?$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{\partial^2 u}{\partial z^2}+\lambda u=0$$

令 $u(\rho,\varphi,z)=R(\rho)\Phi(\varphi)Z(z)$

$$\rightarrow\begin{cases} Z''+\mu Z=0 \\ \Phi''+n^2\Phi=0 \\ \rho^2R''+\rho R'+(k^2\rho^2-n^2)R=0 \end{cases}$$

本征值: μ, n^2, k^2

$\downarrow \quad \downarrow \quad \downarrow$

本征函数: Z, Φ, R

设 $\lambda-\mu\geq 0$

记 $\lambda-\mu=k^2$

$x=k\rho$, $y(x)=R(\rho)$

$$\boxed{x^2y''+xy'+(x^2-n^2)y=0}$$

n阶*Bessel eq.*

二、柱坐标系中亥姆霍兹方程的分离变量

$$2.\ \Delta u = 0 \to ?$$

$$\Delta u + \lambda u = 0$$

令 $u(\rho,\varphi,z) = R(\rho)\Phi(\varphi)Z(z)$

$$\lambda = 0$$

$$\Delta u = 0 \to \begin{cases} Z'' + \mu Z = 0 \\ \Phi'' + n^2\Phi = 0 \\ \rho^2 R'' + \rho R' + (k^2\rho^2 - n^2)R = 0 \quad (-\mu = k^2) \end{cases}$$

$$x = k\rho\ ,\ y(x) = R(\rho)$$

$$\boxed{x^2 y'' + xy' + (x^2 - n^2)y = 0}$$

二、柱坐标系中亥姆霍兹方程的分离变量

3、例：一个半径为a薄圆盘，上下两面绝热。若已知圆盘边温度，求圆盘上的稳定温度分布。

$$\begin{cases} \Delta u = 0 \ , \ \rho < a \\ u|_{\rho=a} = f(\varphi) \end{cases}$$

(1) 令 $u(\rho,\varphi) = R(\rho)\Phi(\varphi) \rightarrow \begin{cases} \Phi'' + n^2\Phi = 0 \\ \rho^2 R''(\rho) + \rho R'(\rho) - n^2 R = 0 \end{cases}$

(2) 解 $\begin{cases} \Phi'' + n^2\Phi = 0 \\ \Phi(\varphi + 2\pi) = \Phi(\varphi) \end{cases}$ $n^2, \ n = 0,1,2,\cdots$

$$\boxed{\Phi_n(\varphi) = A_n' \cos n\varphi + B_n' \sin n\varphi}$$

(3) 解 $\begin{cases} \rho^2 R''(\rho) + \rho R'(\rho) - n^2 R = 0 \\ R(\rho)\big|_{\rho=0} \rightarrow 有限 \end{cases}$ $\boxed{\rightarrow R_n(\rho) = C_n \rho^n}$

二、柱坐标系中亥姆霍兹方程的分离变量

3、例： $\begin{cases} \Delta u = 0 \ , \ \rho < a \\ u|_{\rho=a} = f(\varphi) \end{cases}$

(4) $u(\rho,\varphi) = \sum_{n=0}^{\infty} \rho^n (A_n \cos n\varphi + B_n \sin n\varphi)$

$$\sum_{n=0}^{\infty} (A_n a^n \cos n\varphi + B_n a^n \sin n\varphi) = f(\varphi)$$

$$\alpha_0 = A_0 a^0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\, d\varphi, \alpha_n = A_n a^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\varphi)\cos\varphi\, d\varphi$$

$$\beta_n = B_n a^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\varphi)\sin\varphi\, d\varphi$$

$$\boxed{u(\rho,\varphi) = \sum_{n=0}^{\infty} \left(\frac{\rho}{a}\right)^n (\alpha_n \cos n\varphi + \beta_n \sin n\varphi)}$$

三、球坐标系中亥姆霍兹方程的分离变量：

1、$\Delta u+\lambda u=0\rightarrow$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial u}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial u^2}{\partial\varphi^2}+\lambda u=0\ (1)$$

令 $u(r,\theta,\varphi)=R(r)y(\theta,\varphi)$

$k^2=\lambda$

（1）$\rightarrow$
$$\begin{cases} r^2\dfrac{d^2R}{dr^2}+2r\dfrac{dR}{dr}+[k^2r^2-l(l+1)]R=0\ (2) \\ \dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}(\sin\theta\dfrac{\partial y}{\partial\theta})+\dfrac{1}{\sin^2\theta}\dfrac{\partial^2 y}{\partial\varphi^2}+l(l+1)y=0\ (3)\end{cases}$$

$$y(\theta,\varphi)=\Theta(\theta)\Phi(\varphi)$$

（3）$\longrightarrow$
$$\begin{cases}\Phi''+m^2\Phi=0\ ,\ m=0,1,2,.. & (4) \\ \dfrac{1}{\sin\theta}\dfrac{d}{d\theta}(\sin\theta\dfrac{d\Theta}{d\theta})+[l(l+1)-\dfrac{m^2}{\sin^2\theta}]\Theta=0\ (5)\end{cases}$$

三、球坐标系中亥姆霍兹方程的分离变量：

1、$\Delta u+\lambda u=0 \to \quad u(r,\theta,\varphi)=R(r)\Theta(\theta)\Phi(\varphi)$

$$(\lambda=k^2)\quad \begin{cases} r^2\dfrac{d^2R}{dr^2}+2r\dfrac{dR}{dr}+[k^2r^2-l(l+1)]R=0 & (2) \\ \Phi''+m^2\Phi=0\ ,\quad m=0,1,2,.. & (4) \\ \dfrac{1}{\sin\theta}\dfrac{d}{d\theta}(\sin\theta\dfrac{d\Theta}{d\theta})+[l(l+1)-\dfrac{m^2}{\sin^2\theta}]\Theta=0 & (5) \end{cases}$$

本征值： $l(l+1);\quad m^2;\quad l(l+1),m^2$

$\downarrow \quad \downarrow \quad \downarrow$

本征函数： $R(r),\quad \Phi,\quad \Theta$

令：$x=kr\ ,\ y(x)/\sqrt{x}=R(r)$

$$\boxed{x^2y''+xy'+[x^2-(l+\frac{1}{2})^2]y=0\ (2)'}$$

$x=\cos\theta\ ,\ y(x)=\Theta(\theta)$

$$\boxed{(1-x^2)y''-2xy'+[l(l+1)-\frac{m^2}{1-x^2}]y=0\ (5)'}$$

$(2),(2)'$ – 球*Bessel eq.*;$(5),(5)'$ – 缔合*Legender eq.*

三、球坐标系中亥姆霍兹方程的分离变量：

2、$\Delta u = 0 \to ?$

$\Delta u + \lambda u = 0$ 令 $u(r,\theta,\varphi) = R(r)\Theta(\theta)\Phi(\varphi)$

$\downarrow \lambda = 0$

$$\Delta u = 0 \to \begin{cases} \Phi'' + m^2\Phi = 0 \ , \quad m = 0,1,2,\ldots & (4) \\ \dfrac{1}{\sin\theta}\dfrac{d}{d\theta}(\sin\theta\dfrac{d\Theta}{d\theta}) + [l(l+1) - \dfrac{m^2}{\sin^2\theta}]\Theta = 0 & (5) \\ r^2\dfrac{d^2R}{dr^2} + 2r\dfrac{dR}{dr} - l(l+1)]R = 0 & (6) \end{cases}$$

—Euler方程

$x = \cos\theta \ , \ y(x) = \Theta(\theta)$

$$\boxed{(1-x^2)y'' - 2xy' + [l(l+1) - \frac{m^2}{1-x^2}]y = 0 \ (5)'}$$

—缔合Legendre方程

四、小结：

1、在柱坐标系中

$$\boxed{\Delta u=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial u}{\partial\rho})+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{\partial^2 u}{\partial z^2}}$$

2、在极坐标系中

$$\boxed{\Delta u=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial u}{\partial\rho})+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}}$$

3、在球坐标系中

$$\boxed{\Delta u=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial u}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}}$$

四、小结：

4、在柱坐标系中　令　$u(\rho,\varphi,z)=R(\rho)\Phi(\varphi)Z(z)$

$$\Delta u+\lambda u=0$$
$$\Delta u=0$$

$$\rightarrow\begin{cases} Z''+\mu Z=0 \\ \Phi''+n^2\Phi=0 \qquad 0\le k^2=\begin{cases}\lambda-\mu \\ -\mu\end{cases} \\ \rho^2R''+\rho R'+(k^2\rho^2-n^2)R=0 \end{cases}$$

$x=k\rho\ ,\ y(x)=R(\rho)$

$$\boxed{x^2y''+xy'+(x^2-n^2)y=0}$$

本征值：　$\mu,\quad n,\quad k^2$

$\qquad\quad\downarrow\quad\ \downarrow\quad\ \downarrow$

本征函数：$Z,\quad \Phi,\quad R$

四、小结：

$$x^2y''+xy'+[x^2-(l+\frac{1}{2})^2]y=0\,(2)'$$

5、在球坐标系中令 $u(r,\theta,\varphi)=R(r)\Theta(\theta)\Phi(\varphi)$

$$\Delta u+\lambda u=0\rightarrow\begin{cases} r^2\dfrac{d^2R}{dr^2}+2r\dfrac{dR}{dr}+[k^2r^2-l(l+1)]R=0 \quad (\lambda=k^2)\\ \Phi''+m^2\Phi=0\ ,\ m=0,1,2,.. \\ \dfrac{1}{\sin\theta}\dfrac{d}{d\theta}(\sin\theta\dfrac{d\Theta}{d\theta})+[l(l+1)-\dfrac{m^2}{\sin^2\theta}]\Theta=0 \end{cases}$$

$$\Delta u=0\rightarrow\begin{cases} \dfrac{1}{\sin\theta}\dfrac{d}{d\theta}(\sin\theta\dfrac{d\Theta}{d\theta})+[l(l+1)-\dfrac{m^2}{\sin^2\theta}]\Theta=0 \\ r^2\dfrac{d^2R}{dr^2}+2r\dfrac{dR}{dr}-l(l+1)]R=0 \end{cases}$$

本征值： $l(l+1),\quad m^2,\quad l(l+1),m^2$

$\downarrow \quad \downarrow \quad \downarrow$

本征函数： $R(r),\quad \Phi,\quad \Theta$

$$(1-x^2)y''-2xy'+[l(l+1)-\frac{m^2}{1-x^2}]y=0\,(5)'$$

习题 8.5: 1; 6; 12

再见！