数学物理方法

Methods in Mathematical Physics

第九章　积分变换法

The Method of Integral Transforms

武汉大学物理科学与技术学院

傅氏变换习题课

一、傅氏变换或傅氏逆变换

二、傅氏变换的有关性质及其应用

三、傅氏变换法

一、傅氏变换或傅氏逆变换

1. 求 $F\left[\frac{\sin ax}{x}\right]=?\quad a>0$

$$\boxed{F[f(x)]=\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx}$$

解：$F\left[\frac{\sin ax}{x}\right]=\int_{-\infty}^{\infty}\frac{e^{iax}-e^{-iax}}{2ix}e^{-i\omega x}dx$

$$=\int_{0}^{\infty}\frac{\sin(a-\omega)x}{x}dx+\int_{0}^{\infty}\frac{\sin(a+\omega)}{x}dx$$

$$\boxed{\int_{0}^{\infty}\frac{\sin x}{x}dx=\frac{\pi}{2}\rightarrow}\qquad \int_{0}^{\infty}\frac{\sin Ax}{x}dx=\begin{cases}\frac{\pi}{2}, & A>0\\ -\frac{\pi}{2}, & A<0\end{cases}$$

一、傅氏变换或傅氏逆变换

$$F\left[\frac{\sin ax}{x}\right]=\int_0^{\infty}\frac{\sin(a-\omega)x}{x}dx+\int_0^{\infty}\frac{\sin(a+\omega)x}{x}dx$$

(1) 若 $a>|\omega|$, 则有 $a-\omega>0, a+\omega>0$

$$\int_0^{\infty}\frac{\sin(a-\omega)x}{x}dx=\int_0^{\infty}\frac{\sin(a+\omega)x}{x}dx=\frac{\pi}{2}\longrightarrow \boxed{F\left[\frac{\sin ax}{x}\right]=\pi}$$

(2) 若 $a=|\omega|$, ①则当 $\omega>0$ 时有：$\omega=a$,

$$\int_0^{\infty}\frac{\sin(a-\omega)x}{x}dx=0,\int_0^{\infty}\frac{\sin(a+\omega)x}{x}dx=\frac{\pi}{2}\longrightarrow \boxed{F\left[\frac{\sin ax}{x}\right]=\frac{\pi}{2}}$$

②则当 $\omega<0$ 时有：$\omega=-a$,

$$\int_0^{\infty}\frac{\sin(a-\omega)x}{x}dx=\frac{\pi}{2},\int_0^{\infty}\frac{\sin(a+\omega)x}{x}dx=0\longrightarrow \boxed{F\left[\frac{\sin ax}{x}\right]=\frac{\pi}{2}}$$

一、傅氏变换或傅氏逆变换

$$F\left[\frac{\sin ax}{x}\right]=\int_0^{\infty}\frac{\sin(a-\omega)x}{x}dx+\int_0^{\infty}\frac{\sin(a+\omega)x}{x}dx$$

(3) 若 $a<|\omega|$, 则 ①当 $\omega>0$ 时, $a-\omega<0, a+\omega>0$

$$\int_0^{\infty}\frac{\sin(a-\omega)x}{x}dx=-\frac{\pi}{2},\ \int_0^{\infty}\frac{\sin(a+\omega)x}{x}dx=\frac{\pi}{2}\rightarrow \boxed{F\left[\frac{\sin ax}{x}\right]=0}$$

②则当 $\omega<0$ 时, $a-\omega>0, a+\omega<0$

$$\rightarrow \boxed{F\left[\frac{\sin ax}{x}\right]=0}$$

$$(1),(2),(3)\rightarrow \boxed{F\left[\frac{\sin ax}{x}\right]=\begin{cases}\pi, & a>|\omega|\\ \dfrac{\pi}{2}, & a=|\omega|\\ 0, & a<|\omega|\end{cases}}$$

一、傅氏变换或傅氏逆变换

2. 在量子力学中一维体系中的一个状态在坐标表象中的波函数为

$$\psi(x)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}c(p)e^{i\frac{p}{\hbar}x}dp$$

求该状态在动量表象中的波函数 $c(p)$

解：$\psi(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\pi\hbar}{\sqrt{2\pi\hbar}}c(p)e^{i\frac{p}{\hbar}x}d\left(\frac{p}{\hbar}\right)$

$$\rightarrow\sqrt{2\pi\hbar}c(p)=\int_{-\infty}^{\infty}\psi(x)e^{-i\frac{p}{\hbar}x}dx$$

$$\rightarrow\boxed{c(p)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\psi(x)e^{-i\frac{p}{\hbar}x}dx}$$

一、傅氏变换或傅氏逆变换

4.设$r=\sqrt{x^2+y^2+z^2}=|\vec{r}|$,

$$\omega=\sqrt{\omega_1^2+\omega_2^2+\omega_3^2}=|\vec{\omega}|$$

证明 :(1) $F\left[\frac{1}{r}\right]=\frac{4\pi}{\omega^2}$; (2) $F\left[\frac{1}{r}e^{-\mu r}\right]=\frac{4\pi}{\omega^2+\mu^2}(\mu>0)$

提示 : $F^{-1}\left[\frac{4\pi}{\omega^2}\right]=\frac{1}{(2\pi)^3}\int\int\int_{-\infty}^{\infty}\frac{4\pi}{\omega^2}e^{i\vec{\omega}\cdot\vec{r}}d\vec{\omega}$

$$=\frac{4\pi}{(2\pi)^3}\int_0^{\infty}\int_0^{2\pi}\int_0^{\pi}\frac{1}{\omega^2}e^{i\omega r\cos\theta}\omega^2\sin\theta d\theta d\varphi d\omega$$

$\boxed{x=\cos\theta}$

$$=\frac{1}{\pi}\int_0^{\infty}\int_{-1}^{1}e^{i\omega rx}dxd\omega$$

二、有关性质及其应用

已知: $\int_{-\infty}^{\infty}\frac{f(\xi)d\xi}{(x-\xi)^2+a^2}=\frac{1}{x^2+b^2}, 0<a<b$

求 $f(x)=?$

解: $\int_{-\infty}^{\infty}\frac{f(\xi)}{(x-\xi)^2+a^2}d\xi=f(x)*\frac{1}{x^2+a^2},$

$$\to F\left[\int_{-\infty}^{\infty}\frac{f(\xi)}{(x-\xi)^2+a^2}d\xi\right]=F\left[f(x)*\frac{1}{x^2+a^2}\right]$$

$$F[f(x)]\cdot F\left[\frac{1}{x^2+a^2}\right]=F\left[\frac{1}{x^2+b^2}\right]$$

$$\to F[f(x)]=F\left[\frac{1}{x^2+b^2}\right]\Big/F\left[\frac{1}{x^2+a^2}\right]$$

二、有关性质及其应用

$$F\left[\frac{1}{x^2+a^2}\right]=\int_{-\infty}^{\infty}\frac{e^{-i\omega x}}{x^2+a^2}dx=2\int_0^{\infty}\frac{\cos\omega x}{x^2+a^2}dx$$

$$\frac{1}{z^2+a^2}\xrightarrow{z\to\infty}0,\text{奇点}:\pm ia$$

$$(1)\ \text{若}\ \omega>0,\quad F\left[\frac{1}{x^2+a^2}\right]=2\pi i\,\mathrm{res}\frac{e^{i\omega z}}{z^2+a^2}\Big|_{z=ia}=\frac{\pi}{a}e^{-\omega a}$$

$$(2)\ \text{若}\ \omega<0,\quad F\left[\frac{1}{x^2+a^2}\right]=-2\pi i\,\mathrm{res}\frac{e^{i\omega z}}{z^2+a^2}\Big|_{z=-ia}=\frac{\pi}{a}e^{\omega a}$$

$$(1),(2)\to F\left[\frac{1}{x^2+a^2}\right]=\frac{\pi}{a}e^{-|\omega|a},\quad F\left[\frac{1}{x^2+b^2}\right]=\frac{\pi}{b}e^{-|\omega|b}$$

$$F[f(x)]=\frac{a}{b}e^{-|\omega|(b-a)}$$

$$\boxed{\to f(x)=\frac{a(b-a)}{\pi b}\frac{1}{x^2+(b-a)^2}}$$

三、傅氏变换法

1. 求解上半平面的狄氏问题

$$\begin{cases} \Delta u = 0, y > 0 \quad (1) \\ u\big|_{y=0} = f(x) \ (2), \ \lim\limits_{x^2+y^2\to\infty} u = 0 \quad (3) \end{cases}$$

解：(1) 记 $F[u(x,y)] = \tilde{u}(\omega, y)$, $\quad F[f(x)] = \tilde{f}(\omega)$

$$(2) \begin{cases} \dfrac{d^2\tilde{u}}{dy^2} - \omega^2\tilde{u}(\omega, y) = 0 \\ \tilde{u}(\omega,0) = \tilde{f}(\omega) \\ \tilde{u}(\omega, y)\big|_{y\to\pm\infty} \to 0 \end{cases} \quad \to \tilde{u}(\omega, y) = \tilde{f}(\omega)e^{-|\omega|y}$$

$$(3)\ u(x,y) = F^{-1}[\tilde{u}(\omega, y)] = F^{-1}\left[\tilde{f}(\omega)e^{-|\omega|y}\right]$$

$$= F^{-1}F\left[f(x) * F^{-1}\left[e^{-|\omega|y}\right]\right]$$

三、傅氏变换法

1. 求解上半平面的狄氏问题

$$\begin{cases} \Delta u = 0, y > 0 \quad (1) \\ u\Big|_{x=0} = f(x) \quad (2), \quad \lim\limits_{x^2+y^2\to\infty} u = 0 \quad (3) \end{cases}$$

$$(3)\, u(x,y) = F^{-1}[\tilde{u}(\omega,y)] = F^{-1}F\left[f(x) * F^{-1}\left[e^{-|\omega|y}\right]\right]$$

$$F^{-1}\left[e^{-|\omega|y}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|\omega|y} e^{i\omega x} d\omega = \frac{y}{\pi}\frac{1}{x^2+y^2}$$

或：$\because F\left[\dfrac{1}{x^2+b^2}\right] = \dfrac{\pi}{b} e^{-|\omega|b}$（见上题）$\to F^{-1}\left[e^{-|\omega|y}\right] = \dfrac{y}{\pi}\dfrac{1}{x^2+y^2}$

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)}{(x-\xi)^2+y^2} d\xi$$

三、傅氏变换法

2、$$\begin{cases} u_{tt}+a^2u_{xxxx}=0,-\infty<x<\infty,t>0 & (1)\\ u(x,0)=\varphi(x) & (2)\\ u_t(x,0)=a\psi''(x) & (3)\end{cases}$$

解：(1) 记 $F[u(x,t)]=\tilde{u}(\omega,t),F[\varphi(x)]=\tilde{\varphi}(\omega),F[\psi(x)]=\tilde{\psi}(\omega)$

则 $$\begin{cases} \dfrac{d^2\tilde{u}(\omega,t)}{dt^2}+a^2\omega^4\tilde{u}(\omega,t)=0 & (4)\\ \tilde{u}(\omega,0)=\tilde{\varphi}(\omega) & (5)\\ \tilde{u}_t(\omega,0)=-a\omega^2\tilde{\psi}(\omega) & (6)\end{cases}$$

（2）求 $\tilde{u}(\omega,t)$：

$$\tilde{u}(\omega,t)=\tilde{\varphi}(\omega)\cos a\omega^2t-\tilde{\psi}(\omega)\sin a\omega^2t$$

三、傅氏变换法

(3) $u(x,t) = F^{-1}[\tilde{u}(\omega,t)]$

$$= F^{-1}[\tilde{\varphi}(\omega)\cos a\omega^2 t] - F^{-1}[\tilde{\psi}(\omega)\sin a\omega^2 t]$$

$$= F^{-1}F[\varphi(x) * F^{-1}[\cos a\omega^2 t]] - F^{-1}F[\psi(x) * F^{-1}[\sin a\omega^2 t]]$$

$$F^{-1}\left[e^{ia\omega^2 t}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ia\omega^2 t} e^{i\omega x}\, d\omega$$

$$= \frac{1}{2\pi} e^{-i\frac{x^2}{4at}} \int_{-\infty}^{\infty} e^{iat\left(\omega + \frac{x}{2at}\right)^2} d\omega$$

$$= \frac{1}{2\pi} \frac{e^{-i\frac{x^2}{4at}}}{\sqrt{at}} \int_{-\infty}^{\infty} e^{i\xi^2} d\xi = \frac{1}{\pi\sqrt{at}} e^{-i\frac{x^2}{4at}} e^{i\frac{\pi}{4}} \frac{\sqrt{\pi}}{2}$$

三、傅氏变换法

$$F^{-1}\left[e^{ia\omega^2 t}\right]=\frac{1}{\pi\sqrt{at}}e^{-i\frac{x^2}{4at}}e^{i\frac{\pi}{4}}\frac{\sqrt{\pi}}{2}$$

$$F^{-1}\left[\cos a\omega^2 t+i\sin a\omega^2 t\right]=\frac{1}{2\sqrt{\pi at}}\left[\cos\left(\frac{\pi}{4}-\frac{x^2}{4at}\right)+i\sin\left(\frac{\pi}{4}-\frac{x^2}{4at}\right)\right]$$

$$(3)\quad u(x,t)=F^{-1}\left[\tilde{u}(\omega,t)\right]$$

$$=F^{-1}F\left[\varphi(x)*F^{-1}\left[\cos a\omega^2 t\right]\right]+F^{-1}F\left[\psi(x)*F^{-1}\left[\sin a\omega^2 t\right]\right]$$

$$\boxed{\rightarrow u(x,t)=\frac{1}{2\sqrt{\pi at}}\int_{-\infty}^{\infty}\varphi(x-\xi)\cos\left(\frac{\pi}{4}-\frac{\xi^2}{4at}\right)d\xi}$$

$$\boxed{-\frac{1}{2\sqrt{\pi at}}\int_{-\infty}^{\infty}\psi(x-\xi)\sin\left(\frac{\pi}{4}-\frac{\xi^2}{4at}\right)d\xi}$$

三、傅氏变换法

3、无限长的杆沿杆长方向有温差，设其热导系数为1，初始温度分布为 $\cos x$ 求此热导问题。

$$\begin{cases} u_t = u_{xx}, -\infty < x < \infty \\ u(x,0) = \cos x \end{cases}$$

解：记 $F[u(x,t)] = \tilde{u}(\omega,t)$，则

$$\begin{cases} \dfrac{d\tilde{u}}{dt} - (i\omega)^2 \tilde{u}(\omega,t) = 0 \\ \tilde{u}(\omega,0) = \pi[\delta(1-\omega) + \delta(1+\omega)] \end{cases} \to \tilde{u}(\omega,t) = \pi[\delta(1-\omega)+\delta(1+\omega)]e^{-\omega^2 t}$$

$$u(x,t) = F^{-1}[\tilde{u}(\omega,t)] = \frac{1}{2\pi}\pi\int_{-\infty}^{\infty} e^{-\omega^2 t}[\delta(\omega-1)+\delta(\omega+1)]e^{i\omega x}d\omega$$

$$\boxed{u(x,t) = \frac{1}{2}[e^{-t}e^{ix} + e^{-t}e^{-ix}] = e^{-t}\cos x}$$

三、傅氏变换法

又解: $\begin{cases} u_t = u_{xx}, -\infty < x < \infty \\ u(x,0) = \cos x \end{cases}$

记 $F[u(x,t)] = \tilde{u}(\omega,t),\quad F[\cos x] = \tilde{\varphi}(\omega)$

$$\begin{cases} \dfrac{d\tilde{u}}{dt} - (i\omega)^2 \tilde{u}(\omega,t) = 0 \\ \tilde{u}(\omega,0) = \tilde{\varphi}(\omega) \end{cases} \to \tilde{u}(\omega,t) = \tilde{\varphi}(\omega) e^{-\omega^2 t}$$

$$u(x,t) = F^{-1}[\tilde{u}(\omega,t)] = F^{-1}[\tilde{\varphi}(\omega)e^{-\omega^2 t}] = \cos(x) * F^{-1}[e^{-\omega^2 t}]$$

$$F^{-1}[e^{-\omega^2 t}] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\omega^2 t} e^{i\omega x} d\omega = \frac{1}{\pi}\int_0^{\infty} e^{-\omega^2 t}\cos\omega x d\omega$$

$$= \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \qquad u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{\xi^2}{4t}} \cos(\xi - x) d\xi$$

三、傅氏变换法

又解：$\begin{cases} u_t = u_{xx}, -\infty < x < \infty \\ u(x,0) = \cos x \end{cases}$

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{\xi^2}{4t}} \cos(\xi - x) d\xi$$

$$= \frac{1}{\sqrt{\pi t}} \cos x \int_0^{\infty} e^{-\frac{\xi^2}{4t}} \cos \xi d\xi$$

$$= \frac{1}{\sqrt{\pi t}} \cos x \frac{1}{2} e^{-t} \sqrt{\pi t}$$

$$\boxed{u(x,t) = e^{-t} \cos x}$$

再见！