问题的引入：

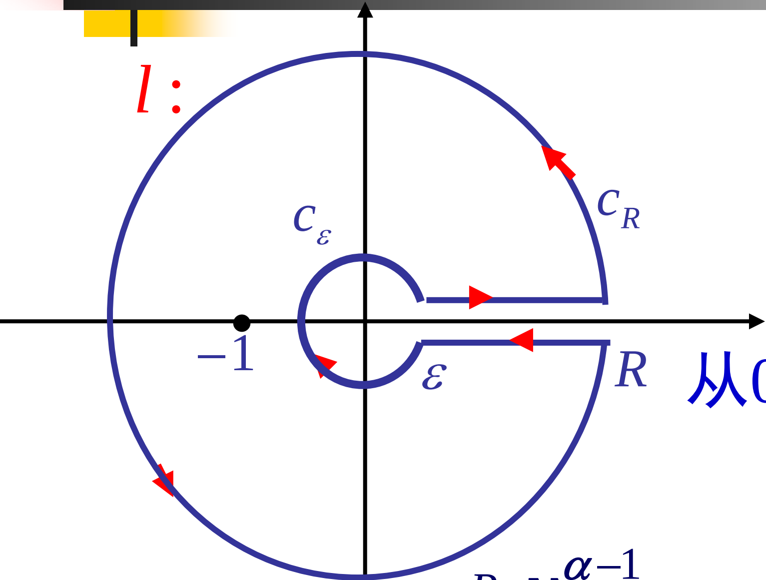
$$I=\int_0^{\infty}\frac{x^{\alpha-1}}{1+x}dx=?\quad (0<\alpha<1)$$

$$I=\int_0^{\infty}\frac{\ln x}{(1+x^2)^2}dx=?$$

§ 5. 4多值函数的积分

Integrating the multi-value functions

一、$\int_0^{\infty}\frac{x^{\alpha-1}}{1+x}dx,\ (0<\alpha<1)$



$$f(z)=\frac{z^{\alpha-1}}{1+z},\quad z^{\alpha-1}=\frac{1}{z^{1-\alpha}},0<\alpha<1$$

$f(z)$的支点:$0,\infty$;奇点:$z=-1$

从$0\to\infty$沿正实轴作割线，划出 单值区域

$$\oint_l f(z)dz=2\pi i\,res[f(z),-1]$$

$$\int_\varepsilon^R\frac{x^{\alpha-1}}{1+x}dx+\int_{C_R}\frac{z^{\alpha-1}}{1+z}dz+\int_R^\varepsilon\frac{(xe^{i2\pi})^{\alpha-1}}{1+xe^{i2\pi}}d(xe^{i2\pi})$$

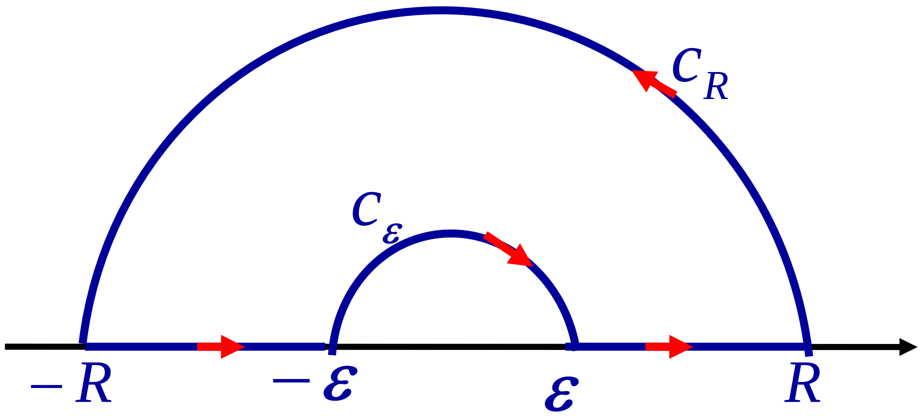
$$+\int_{C_\varepsilon}\frac{z^{\alpha-1}}{1+z}dz=2\pi i\,res[\frac{z^{\alpha-1}}{1+z},-1]\quad(1)$$

结论:

$$\boxed{\int_0^{\infty}\frac{x^{\alpha-1}}{1+x}dx=\frac{\pi}{\sin\pi\alpha}}$$

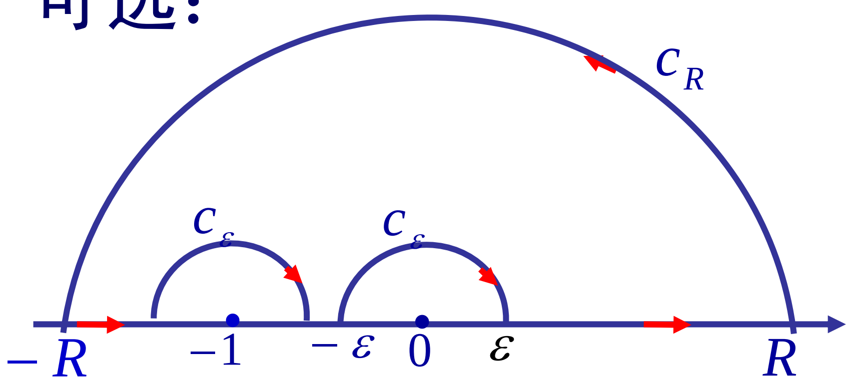
一、$\int_0^{\infty} \frac{x^{\alpha-1}}{1+x} dx, (0<\alpha<1)$

可否选如下围道?

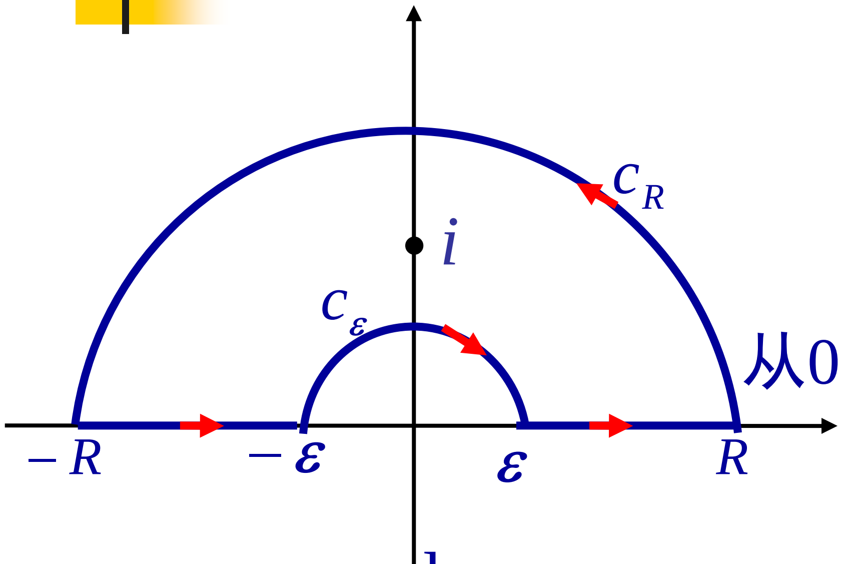


No!

可选:



二、$\int_0^\infty \frac{\ln x}{(1+x^2)^2}dx$



$$f(z)=\frac{\ln z}{(1+z^2)^2},$$

$f(z)$的支点：0, ∞；奇点：$z=\pm i$

从$0\to\infty$沿正实轴作割线，划出单值区域

$$\oint_l f(z)dz = 2\pi i\, res[f(z),i]$$

$$\int_\varepsilon^R \frac{\ln x}{(1+x^2)^2}dx+\int_{C_R}\frac{\ln z}{(1+z^2)^2}dz+\int_{-R}^{-\varepsilon}\frac{\ln x}{(1+x^2)^2}dx+$$

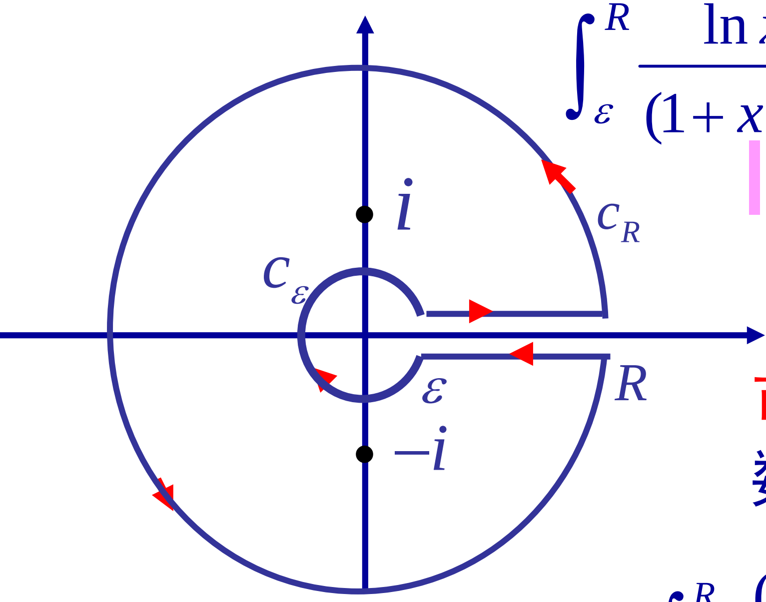
$$+\int_{C_\varepsilon}\frac{\ln z}{(1+z^2)^2}dz=2\pi i\, res[\frac{\ln z}{(1+z^2)^2},i] \qquad (2)$$

结论：$\boxed{\int_0^\infty \frac{\ln x}{(1+x^2)^2}dx=-\frac{\pi}{4}}$， $\boxed{\int_0^\infty \frac{1}{(1+x^2)^2}dx=\frac{\pi}{4}}$

二、$\int_0^{\infty} \frac{\ln x}{(1+x^2)^2} dx$

（1）选以上路径需否考虑幅角的变化？ No!

（2）可否选如图所示路径？



$$\int_\varepsilon^R \frac{\ln x}{(1+x^2)^2} dx + \int_{C_R} \frac{\ln z}{(1+z^2)^2} dz + \int_R^\varepsilon \frac{\ln(xe^{i2\pi})}{(1+x^2)^2} dx + \int_{C_\varepsilon} \frac{\ln z}{(1+z^2)^2} dz = 2\pi i \operatorname{res}[f(i) + f(-i)]$$

I　II　III　IV　V

可！但需考虑下列函数沿此路径积分：$f(z) = \frac{(\ln z)^2}{(1+z^2)^2}$

此时：

$$I + III = \int_\varepsilon^R \frac{(\ln x)^2}{(1+x^2)^2} dx - \int_\varepsilon^R \frac{(\ln x)^2 + i4\pi \ln x + (i2\pi)^2}{(1+x^2)^2} dx$$

$$= -i4\pi \int_\varepsilon^R \frac{\ln x}{(1+x^2)^2} dx + \int_\varepsilon^R \frac{4\pi^2}{(1+x^2)^2} dx$$

三、小结

$$\boxed{\int_0^{\infty}\frac{x^{\alpha-1}}{1+x}dx=\frac{\pi}{\sin\pi\alpha}}\qquad \boxed{\int_0^{\infty}\frac{\ln x}{(1+x^2)^2}dx=-\frac{\pi}{4},}$$

$$\int_0^{\infty}\frac{x^{\frac{1}{2}}}{1+x^2}dx=?\qquad \int_0^{\infty}\frac{\ln^3 x}{(1+x)^2(1+x^2)}dx=?,$$

答: $\dfrac{\pi}{\sqrt{2}},\quad -\dfrac{7}{128}\pi^4$

本节作业

习题5. 4:

1(1); 2(2); 4

Good-by!