数学物理方法特训讲义——微分方程

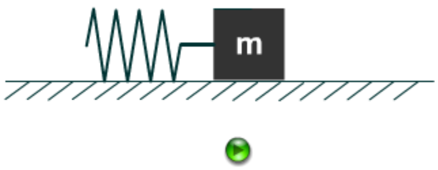
Differential equations (any equation containing derivation)

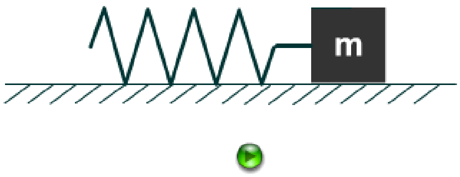
Q1 How to describe a dynamics system? （如何描述一个动力学系统）

① Catch the degrees of freedom（DOF）

② Write down the equation of motion（EOM）

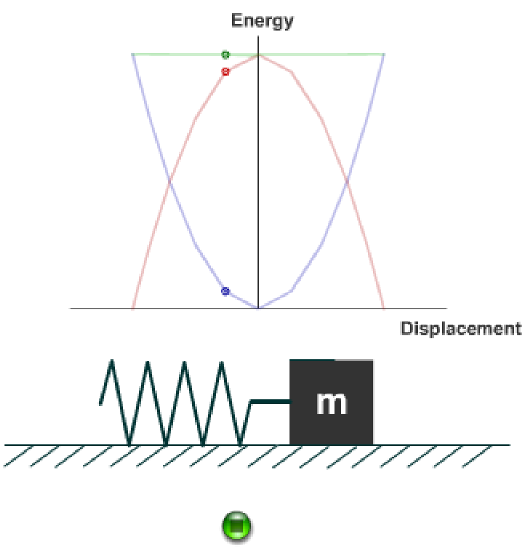
Example （simple harmonic motion）

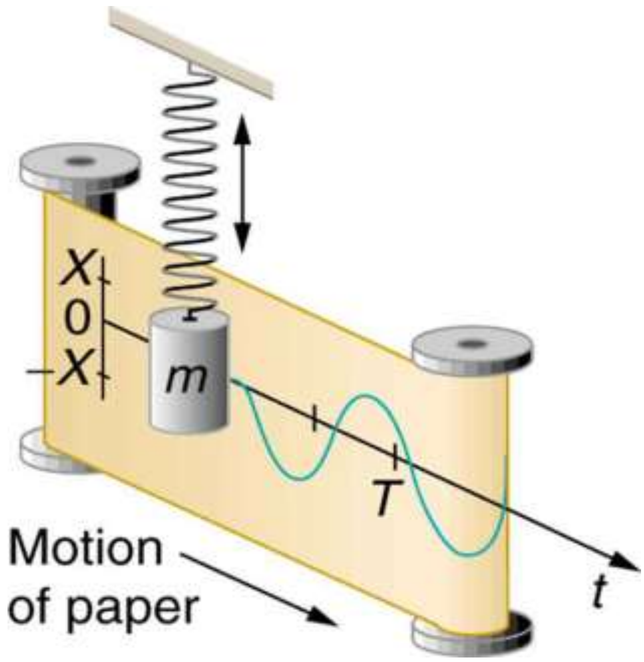




Hooke's law & Newton’s 2nd law

$$\vec{f} = kx\vec{i} = m\ddot{x}\vec{i}$$





Newton’s dot notation: $kx + m\ddot{x} = 0$

Leibniz’s notation: $kx + m\dfrac{d^2x}{dt^2} = 0 \Rightarrow v(t) = \dfrac{dx}{dt} = -A\omega_0 \sin(\omega_0 t + \varphi)$

Lagrange's prime notation: $kx(t) + mx''(t) = 0 \Rightarrow v(t) = x'(t) = -A\omega_0 \sin(\omega_0 t + \varphi)$

Euler's subscript notation: $kx_t + mx_{tt} = 0 \Rightarrow v(t) = x_t = -A\omega_0 \sin(\omega_0 t + \varphi)$

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① Ordinary Differential Equation (ODE):

Differentiation with respect to one independent variable

Particle view→ODE $$kx + m\frac{d^2x}{dt^2} = 0$$

② Partial Differential Equation (PDE):

Differentiation with respect to two or more independent variables

Field view（wave）→PDE $$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{d^2 u}{dx^2}$$

波动方程: $$u_{tt} - a^2 u_{xx} = f(x,t)$$

扩散传导方程: $$u_t - a^2 u_{xx} = (f \ , x$$

稳定场方程: $$\Delta u = f(\bar{r})$$

初始条件:说明物理现象初始状态的条件　　　　边界条件:说明边界上的约束情况的条件

Q2 How to solve a differential equation?

$$PDE \to ODE \to \begin{cases} Algebraic\ equation\left(complex\ variables\right) \\ Calculus\left(derivative, integral, series\right) \\ Guess\left(e^{\lambda x}, \sin\omega x, \cos\omega x\right) \end{cases}$$

$$m\ddot{x} + kx = 0$$

By dimensional analysis, we introduce $\omega_0{}^2 \equiv \frac{k}{m}$ (angular frequency)

Try $$x_1(t) = c_1 \sin\omega_0 t, x_2(t) = c_2 \cos\omega_0 t$$

$$x(t) = x_1(t) + x_2(t) = c_1 \sin\omega_0 t + c_2 \cos\omega_0 t = A\cos(\omega_0 t + \varphi)$$

$$\dot{x} = v(t) = -A\omega_0 \sin(\omega_0 t + \varphi) \qquad \ddot{x} = a(t) = -A\omega_0{}^2 \cos(\omega_0 t + \varphi)$$

Check

$$m\left[-A\omega_0{}^2\cos(\omega_0 t + \varphi)\right] + k\left[A\cos(\omega_0 t + \varphi)\right] = m\left[-A\frac{k}{m}\cos(\omega_0 t + \varphi)\right] + k\left[A\cos(\omega_0 t + \varphi)\right] = 0$$

What about energy？

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 = \text{const.}$$

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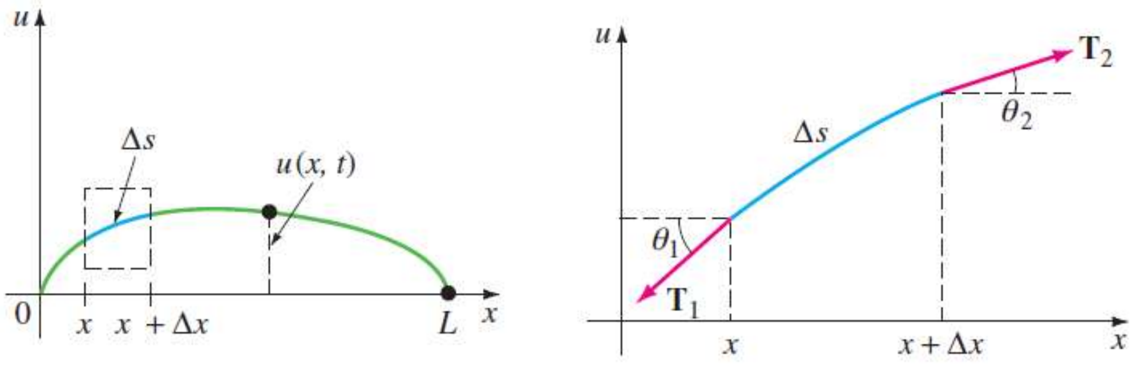
Method 1: separation of variables

Example

$$\begin{cases} u_{tt}=a^2u_{xx}, & 0<x<l,t>0 \quad (1) \\ u(0,t)=0,u(l,t)=0, & t\geq 0 \quad (2) \\ u(x,0)=\varphi(x),u_t(x,0)=\psi(x), & 0\leq x\leq l \quad (3) \end{cases}$$



(a) Segment of string

(b) Enlargement of segment

解：假设$u(x,t)=T(t)X(x)$,代入方程$u_{tt}=a^2u_{xx}$得到

$$\frac{T''(t)}{a^2T(t)}=\frac{X''(x)}{X(x)}=-\lambda$$

由边界条件可知

$$\begin{cases} X''(x)+\lambda X(x)=0, & 0<x<l \\ X(0)=0, X(t)=0 \end{cases}$$

特征值：$\lambda_n=\left(\frac{n\pi}{l}\right)^2$ 特征函数：$X_n(x)=\sin\frac{n\pi x}{l}, \quad n=1,2,\cdots$

将特征值代入方程:

$$T_n''(t)+a^2\lambda_n^2T_n(t)=0$$

其通解是:

$$T_n(t)=C_n\cos\frac{n\pi at}{l}+D_n\sin\frac{n\pi at}{l}, \quad n=1,2,\cdots$$

把解$u_n(x,t)=T_n(t)X_n(x)$叠加得到

$$u(x,t)=\sum_{n=1}^{\infty}T_n(t)X_n(x)=\sum_{n=1}^{\infty}\left(C_n\cos\frac{n\pi at}{l}+D_n\sin\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}$$

利用初始条件得到

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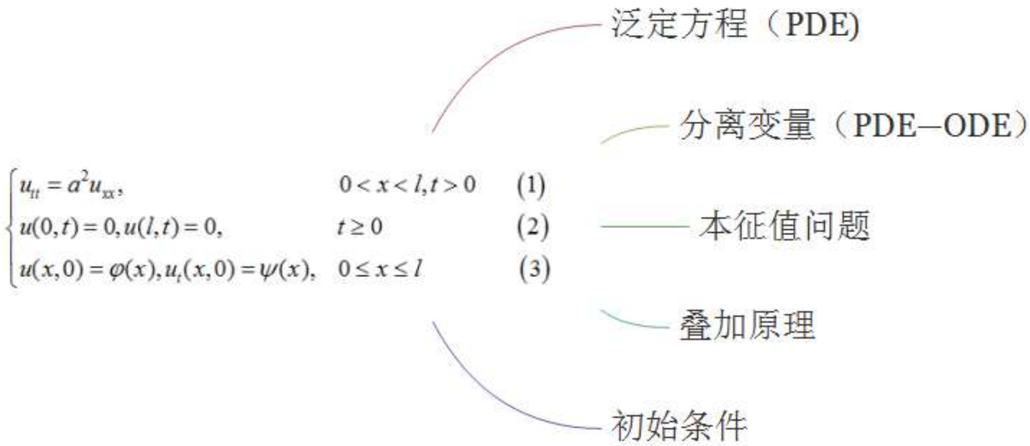
$$\begin{cases} \varphi(x)=u(x,0)=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi x}{l}, & 0<x<l \\ \psi(x)=u_t(x,0)=\sum_{n=1}^{\infty}D_n\frac{n\pi a}{l}\sin\frac{n\pi x}{l}, & 0<x<l \end{cases}$$

利用 Fourier 级数的展开公式得到

$$\begin{cases} C_n=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{n\pi x}{l}dx, & n=1,2,\cdots \\ D_n=\frac{2}{n\pi a}\int_0^l\psi(x)\cos\frac{n\pi x}{l}dx, & n=1,2,\cdots \end{cases}$$

$$u_n(x,t)=\left(C_n\cos\frac{n\pi at}{l}+D_n\sin\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}=A_n\cos(\omega_n t-\theta_n)\sin\frac{n\pi x}{l}$$



物理问题更重要的是讨论，望大家课上多思考！此外，实际情况多为非齐次方程，需要求特解 y_p、见表格

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x+7$	$Ax+B$
3. $3x^2-2$	Ax^2+Bx+C
4. x^3-x+1	Ax^3+Bx^2+Cx+E
5. $\sin 4x$	$A\cos 4x+B\sin 4x$
6. $\cos 4x$	$A\cos 4x+B\sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x-2)e^{5x}$	$(Ax+B)e^{5x}$
9. x^2e^{5x}	$(Ax^2+Bx+C)e^{5x}$
10. $e^{3x}\sin 4x$	$Ae^{3x}\cos 4x+Be^{3x}\sin 4x$
11. $5x^2\sin 4x$	$(Ax^2+Bx+C)\cos 4x+(Ex^2+Fx+G)\sin 4x$
12. $xe^{3x}\cos 4x$	$(Ax+B)e^{3x}\cos 4x+(Cx+E)e^{3x}\sin 4x$

Examle2
$$\begin{cases} u_{tt} = a^2 u_{xx} + Ax \quad 0 < x < l, t > 0 \\ u(x,0) = 0, u_t(x,0) = 0 \\ u(0,t) = 0, u(l,t) = 0 \end{cases}$$

解：该方程所对应的齐次方程是第一类边界条件，于是可设

$$U(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi}{l} x$$

$$Ax = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi}{l} x$$

其中 $f_n(t) = \frac{2}{l} \int_0^l Ax \sin \frac{n\pi}{l} x$

$$\therefore \quad f_n(t) = \frac{2}{l} \cdot \left(-\frac{l}{n\pi}\right) \int_0^l Ax \, d\cos \frac{n\pi}{l} x = -\frac{2}{n\pi} \left[Ax \cos \frac{n\pi}{l} x \Big|_0^l - \int_0^l A \cos \frac{n\pi}{l} x \, dx \right] = -\frac{2Al}{n\pi} (-1)^n$$

代入方程和初始条件有：

$$\begin{cases} \sum_{n=1}^{\infty} T_n''(t) \sin \frac{n\pi}{l} x = \sum_{n=1}^{\infty} (-1) \left(\frac{n\pi a}{l}\right)^2 \sin \frac{n\pi}{l} x + \sum_{n=1}^{\infty} \frac{2Al}{n\pi} (-1)^{n+1} \sin \frac{n\pi}{l} x \\ \sum_{n=1}^{\infty} T_n(0) \sin \frac{n\pi}{l} x = 0, \sum_{n=1}^{\infty} T_n'(0) \sin \frac{n\pi}{l} x = 0 \end{cases}$$

整理得：

$$\begin{cases} T_n''(t) + \left(\frac{n\pi a}{l}\right)^2 T_n(t) = \frac{2Al}{n\pi} (-1)^{n+1} \\ T_n(0) = 0, T_n'(0) = 0 \end{cases}$$

解得

$$T_n(t) = C_n \cos \frac{n\pi a}{l} t + D_n \sin \frac{n\pi a}{l} t + \frac{2Al^3}{a^2\pi^3} \frac{(-1)^{n+1}}{n^3}$$

将 $T_n(0) = 0$ 代入上式得

$$C_n = -\frac{2Al^3}{a^2\pi^3} \frac{(-1)^{n+1}}{n^3} = \frac{2Al^3}{a^2\pi^3} \frac{(-1)^n}{n^3}$$

将 $T_n'(0) = (0)$ 代入得 $D_n = 0$

$$T_n(t) = \frac{2Al^3}{a^2\pi^3} \frac{(-1)^n}{n^3} \cos \frac{n\pi a}{l} t + \frac{2Al^3}{a^2\pi^3} \frac{(-1)^{n+1}}{n^3}$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2Al^3}{a^2\pi^3} \frac{(-1)^{n+1}}{n^3} \sin \frac{n\pi}{l} x + \sum_{n=1}^{\infty} \frac{2Al^3}{a^2\pi^3} \frac{(-1)^n}{n^3} \cos \frac{n\pi a}{l} t \sin \frac{n\pi}{l} x$$

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Examle3
$$\begin{cases} u_t - a^2 u_{xx} = A\sin\omega t & 0<x<l, t>0 \\ u(0,t) = u_x(l,t) = 0 \\ u(x,0) = 0 \end{cases}$$

解：该问题对应的齐次方程的特征函数为

$$X_n(x) = C_n \sin\frac{n+\frac{1}{2}}{l}\pi x \qquad n = 0,1,2,......$$

故令

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t)\sin\frac{n+\frac{1}{2}}{l}\pi x,\quad A\sin\omega t = \sum_{n=0}^{\infty} f_n(t)\sin\frac{n+\frac{1}{2}}{l}\pi x$$

代入方程和初始条件有：

$$\begin{cases} T_n{}'(t) + \dfrac{(n+\frac{1}{2})^2\pi^2 a^2}{l^2} T_n(t) = f_n(t) \\ T_n(0) = 0 \end{cases}$$

其中 $f_n(t) = \frac{2}{l}\int_0^l A\sin\omega t \sin\frac{n+\frac{1}{2}}{l}\pi x\, dx = \frac{2A\sin\omega t}{(n+\frac{1}{2})\pi}$

证 $\beta = \frac{(n+\frac{1}{2})^2\pi^2 a^2}{l^2}$ 则有：

$$T_n(t) = \frac{2A}{(n+\frac{1}{2})\pi}\left[\frac{\beta\sin\omega t - \omega\cos\omega t}{\beta^2+\omega^2} + c e^{-\beta t}\right]$$

将 $T_n(0)=0$ 代入上式得 $C = \frac{1}{\beta^2+\omega^2}$

$$u(x,t) = \sum_{n=0}^{\infty}\frac{2A}{(n+\frac{1}{2})\pi}\sin\frac{(n+\frac{1}{2})\pi}{l}x\cdot\frac{\left[\frac{(n+\frac{1}{2})\pi a}{l}\right]^2\sin\omega t - \omega\cos\omega t + \omega e^{-\left(\frac{(n+\frac{1}{2})}{l}\pi a\right)^2 t}}{\left[\frac{(n+\frac{1}{2})\pi a}{l}\right]^2 + \omega^2}$$

Method 2: integral transform methods