数学物理方法特训讲义

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——*What is a derivative really？Answer: a limit.*

——*What is an integral really？Answer: a limit.*

——*What is an infinite series* $a_1+a_2+a_3+\cdots$ *really？Answer: a limit.*

This leads to:

——*What is a limit? Answer: a number.*

And finally, the last question:

——*What is a number？*

——*Augustin Cauchy*（1789-1846）——

Q1:What is a number？

Algebra: *a solution of algebraic equation*

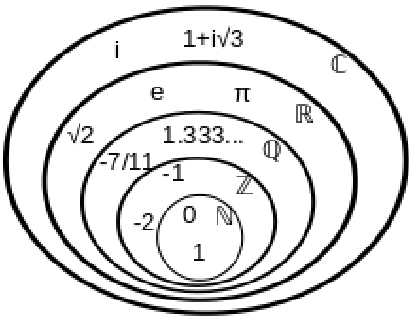
$x-1=0 \Leftrightarrow$ 正整数$\mathbb{N}$（natural number）

$x+1=0 \Leftrightarrow$ 整数$\mathbb{Z}$（integer）

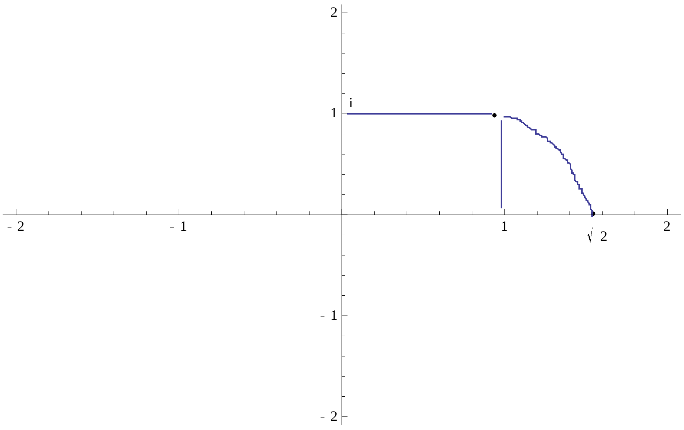
$2x+1=0 \Leftrightarrow$ 有理数$\mathbb{Q}$（rational number）

$x^2-2=0 \Leftrightarrow$ 实数$\mathbb{R}$（real number）

$x^2+1=0 \Leftrightarrow$ 复数$\mathbb{C}$（complex number）



Geometry: *a point in an Argand diagram*



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Q2: How to understand imaginary unit i?

Algebra: *"1"* Geometry: *"rotation"*

Q3: Do you know an important identity $\frac{dz}{z} = id\theta$ around the unit circle?

Analytic algebra: $z = e^{i\theta}, dz = ie^{i\theta}d\theta \Rightarrow \frac{dz}{z} = id\theta$ Geometry: Need images to illustrate

Q4: What is a function?

Algebra: *a mapping f:* $x \to y$ Geometry: *a curve, surface*

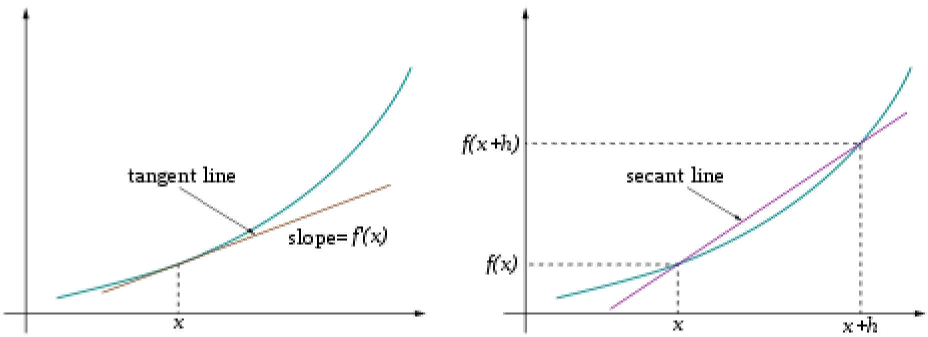
A complex function: $z = x + iy$ $z \to w, w = f(z)$ $w = u + iv$

Example $f(z) = z^2 = (x+iy)^2 = \underline{(x^2 - y^2)} + i\,\underline{(2xy)}$

In general $w = f(z) = u(x,y) + iv(x,y)$

Q5: What is a derivative really?

Algebra: *a limit* $\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x)$ Geometry：*The slope of a curve @x*



Q6: What is an analytic function?（什么是解析函数）

We have known $f(z) = u(x,y) + iv(x,y)$

① What about derivative?

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = f'(z)$$

Cauchy-Riemann condition (equation, relation)（柯西-黎曼条件）

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$$\boxed{\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}}$$

$(\Delta y=0,\Delta z=\Delta x\to 0)$

$$f'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}=\lim_{\Delta x\to 0}\frac{u(x+\Delta x,y)-u(x,y)+i[v(x+\Delta x,y)-v(x,y)]}{\Delta x}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial y}$$

$(\Delta x=0,\Delta z=\Delta y\to 0)$

$$f'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}=\lim_{\Delta y\to 0}\frac{u(x,y+\Delta y)-u(x,y)+i[v(x,y+\Delta y)-v(x,y)]}{i\Delta y}=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

Cauchy-Riemann 方程是函数可导的必要条件，但不是充分条件!

充要条件：

若 $w=f(z)=u(x,y)+iv(x,y)$ 的实部 $u(x,y)$和虚部 $v(x,y)$均可微，且满足 Cauchy-Riemann 条件，则函数 $f(z)$ 可导。

解析函数

如果函数 $f(z)$ 在 z_0 及其邻域内处处可导，那么称 $f(z)$ 在 z_0 点处解析。如果 $f(z)$ 在区域 D 内每一 点 解 析，那么称 $f(z)$ 在 D 内解析，或称 $f(z)$ 是 D 内的一个解析函数。

Remark：

解析函数这一重要概念是与区域密切联系的。如果说函数 $f(z)$ 在某点 z_0 解析，其意义是指在 z_0 点及其邻域内可导。如果 $f(z)$ 在 z_0 点不解析，那么称 z_0 点为 $f(z)$ 的奇点。

函数解析的充要条件一：该区域内可导的充要条件处处成立

$$\boxed{f'(z)=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}=\frac{\partial v}{\partial y}-i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial y}}$$

Example:已知 $u(x,y)=x^2-y^2$,求解析函数 $f(z)$

$$u_x=2x,u_y=-2y$$

$$f'(z)=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}=2x+2iy=2z\Rightarrow f(z)=z^2+C$$

② What about integral? 函数解析的充要条件二：该区域内满足柯西定理

$$\int_C f(z)dz = \int_C (u+iv)(dx+idy) = \int_C (udx-vdy) + i\int_C (vdx+udy)$$

An important formula

$$\oint_C \frac{dz}{z} = 2\pi i$$

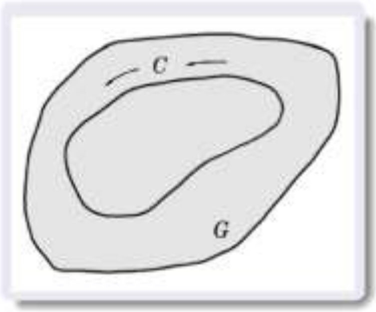
$$\oint_C (z-z_0)^m dz = \begin{cases} 2\pi i & (m=-1) \\ 0 & (m \neq -1 \quad and \quad integer) \end{cases}$$

单连通区域的Cauchy定理

如果函数$f(z)$在单连通区域$\overline{G}$中解析，则沿$\overline{G}$中任何一个分段光滑的闭合围道C有

$$\oint_C f(z)\,\mathrm{d}z = 0$$

这里的C也可以是$\overline{G}$的边界



Key Point

Cauchy's Theorem

This is perhaps the most important theorem in the area of complex analysis. The theorem states that if $f(z)$ is analytic everywhere within a simply-connected region then:

$$\oint_C f(z)\mathrm{d}z = 0$$

for every simple closed path C lying in the region.

Cauchy's Integral Formula

If $f(z)$ is analytic inside and on the boundary C of a simply-connected region then for any point z_0 inside C,

$$\oint_C \frac{f(z)}{z-z_0}\mathrm{d}z = 2\pi \mathrm{i}\ f(z_0).$$

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③ What about series? 函数解析的充要条件三：该区域内可展开为泰勒级数

$$f(z)=\sum_{k=0}^{\infty}a_k(z-z_0)^k=a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots+a_k(z-z_0)^k+\cdots \underset{a_k=\frac{f^{(k)}(z_0)}{k!}}{\overset{|z-z_0|<R}{\rightleftarrows}} \text{解析函数} f(z)$$

Taylor series（☆Q: What is e？Answer: a limit & a series）

★An important view: $f(x)$ can be regarded as an *infinite-dimension vector* $\vec{V}=(f_1, f_2, f_3,\cdots, f_n,\cdots)$

Example: $\vec{r}=r_x\vec{i}+r_y\vec{j}+r_z\vec{k}$ Basis: $\{\vec{i},\vec{j},\vec{k}\}$

The most important is $\begin{cases}\vec{i}\cdot\vec{i}=\vec{j}\cdot\vec{j}=\vec{k}\cdot\vec{k}=1(normal)\\ \vec{i}\cdot\vec{j}=\vec{j}\cdot\vec{k}=\vec{k}\cdot\vec{i}=0(orthogonal)\end{cases}$

Similarly: $f(x)=e^x=a_0+a_1x+a_2x^2+\cdots a_nx^n+\cdots$ Basis: $\{1,x,x^2,x^3,\cdots,x^n,\cdots\}$

But the basis $\{1,x,x^2,x^3,\cdots,x^n,\cdots\}$ is not orthonormal!(正交归一)

To determine the coefficients, we can differentiate $f(x)$ and then set $x=0$:

$$f(0)=e^0=a_0=1, f'(0)=e^0=a_1=1, f''(0)=e^0=2\times1a_2=1,$$

$$f'''(0)=e^0=3\times2\times1a_3=1,\cdots, f^{(n)}(0)=e^0=n\times(n-1)\times\cdots\times3\times2\times1a_n=1,\cdots$$

So, basically

$$a_0=1, a_1=1, a_2=\frac{1}{2\times1}=\frac{1}{2!}, a_3=\frac{1}{3\times2\times1}=\frac{1}{3!},\cdots, a_n=\frac{1}{n!},\cdots$$

Finally,

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots\quad -\infty<x<\infty \qquad e^z=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots+\frac{z^n}{n!}+\cdots\quad |z|<\infty$$

Let $x=1$,then $e=1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}+\cdots$（series） or $e=\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n$（limit）

In the same way, we can get

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots\quad -\infty<x<\infty \qquad \sin z=\sum_{k=0}^{\infty}(-1)^k\frac{z^{2k+1}}{(2k+1)!}\quad (|z|<\infty)$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots\quad -\infty<x<\infty \qquad \cos z=\sum_{k=0}^{\infty}(-1)^k\frac{z^{2k}}{(2k)!}\quad (|z|<\infty)$$

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☆*Q: What is Euler formula ? Answer:* $e^{i\theta}=\cos\theta+i\sin\theta$

Proof: We know this $e^{z}=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots+\frac{z^n}{n!}+\cdots$

Set $z=i\theta$, $e^{i\theta}=1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\cdots+\frac{(i\theta)^n}{n!}+\cdots$

Since $i^0=1, i^1=i, i^2=-1, i^3=-i, i^4=1,\cdots$ （$i^0+i^1+i^2+i^3=0$）

$$e^{i\theta}=\left[1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\cdots+(-1)^n\frac{\theta^{2n}}{(2n)!}+\cdots\right]+i\left[\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots+(-1)^{n-1}\frac{\theta^{2n-1}}{(2n-1)!}+\cdots\right]=\cos\theta+i\sin\theta$$

So, we can attain *Euler formula* $e^{i\theta}=\cos\theta+i\sin\theta$

Specially $$e^{i\pi}=\cos\pi+i\sin\pi=-1\Leftrightarrow e^{i\pi}+1=0$$

Unbelievable Remarkable!

Applications（应用）

- Compound-angle identities: $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$$e^{i(\alpha-\beta)}=\cos(\alpha-\beta)+i\sin(\alpha-\beta)$$

$$LHS=e^{i(\alpha-\beta)}=e^{i\alpha}\cdot e^{-i\beta}=(\cos\alpha+i\sin\alpha)(\cos\beta-i\sin\beta)$$
$$=\cos\alpha\cos\beta+\sin\alpha\sin\beta+i(\sin\alpha\cos\beta-\cos\alpha\sin\beta)$$

Then, real part:

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

Imaginary part:

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

- Double-angle identities:

$$e^{2i\alpha}=\cos2\alpha+i\sin2\alpha$$

$$LHS=e^{i\alpha}\cdot e^{i\alpha}=(\cos\alpha+i\sin\alpha)^2=\cos^2\alpha+2i\sin\alpha\cos\alpha-\sin^2\alpha$$

Then, real part: $\cos2\alpha=\cos^2\alpha-\sin^2\alpha$ imaginary part: $\sin2\alpha=2\sin\alpha\cos\alpha$

数学物理方法特训讲义—复变函数

<u>Q: What is Fourier series?</u>

$$f\left(x\right)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\cos\frac{k\pi x}{L}+b_k\sin\frac{k\pi x}{L}\right)$$

Specially, we make the functions 2π periodic

$$f\left(x\right)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\cos kx+b_k\sin kx\right)$$

The most important thing is how to identify the coefficients.（比较系数）

We choose $\{1,\sin x,\sin 2x,\cdots,\sin kx,\cos x,\cos 2x,\cdots,\cos kx\}$ as basis（orthogonal），

$$\int_{-\pi}^{\pi}\cos nx\cos mxdx=0,\ \int_{-\pi}^{\pi}\sin nx\sin mxdx=0,\quad n\neq m\quad \int_{-\pi}^{\pi}\cos nx\sin mxdx=0$$

To identify the coefficients, we can integrate $f(x)$ from $-\pi$ *to* π*:*

$$\int_{-\pi}^{\pi}f(x)dx=\int_{-\pi}^{\pi}\frac{a_0}{2}dx+\underbrace{\sum_{k=1}^{\infty}\int_{-\pi}^{\pi}\left(a_k\cos kx+b_k\sin kx\right)dx}_{0}$$

So, it's trivial.
$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$

To identify a_n, we multiply cosnx both sides and integrate $f(x)$ cosnx from –π to π:

$$\int_{-\pi}^{\pi}f(x)\cos nxdx=\underbrace{\int_{-\pi}^{\pi}\frac{a_0}{2}\cos nxdx}_{0}+\underbrace{\sum_{k=1}^{\infty}\int_{-\pi}^{\pi}a_k\cos kx\cos nxdx}_{a_k\delta_{kn}}+\underbrace{\sum_{k=1}^{\infty}\int_{-\pi}^{\pi}b_k\sin kx\cos nxdx}_{0}$$

So, we can attain
$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos kxdx$$

Similarly, we can attain
$$b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin kxdx$$

In general,
$$a_k=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{k\pi x}{L}dx\qquad b_k=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{k\pi x}{L}dx$$

$$[a_k]=\left[\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{k\pi x}{L}dx\right]=L^{-1}[f]L=[f]$$

$$[b_k]=\left[\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{k\pi x}{L}dx\right]=L^{-1}[f]L=[f]$$

By dimensional analysis, $a_k\quad b_k$ are PROJECTIONS!（投影）

Orthogonal functions and Fourier series!

Define the inner product of two functions *f* and *g*:

$$\langle f,g\rangle_w = \int_a^b f(x)g(x)w(x)dx$$

In simple cases, weight function w(x) = 1.   Orthonormal

$$\int_a^b f(x)g(x)w(x)dx = 0$$

$$<\varphi_n,\varphi_m> = \int_{x_0}^{x_1} \varphi_n(x)\varphi_m(x)\ dx = \delta_{nm}.$$

$$f(x) = \sum_{n=0}^{\infty} \alpha_n\varphi_n(x). \qquad f(x) \simeq \sum_{n=1}^{N} \alpha_n\varphi_n(x).$$

$$\begin{aligned}
f(x)\varphi_k(x) &= \sum_{n=0}^{\infty} \alpha_n\varphi_n(x)\varphi_k(x), \\
\int_{x_0}^{x_1} f(x)\varphi_k(x)\ dx &= \int_{x_0}^{x_1}\sum_{n=0}^{\infty} \alpha_n\varphi_n(x)\varphi_k(x)\ dx, \\
&= \sum_{n=0}^{\infty} \alpha_n \underbrace{\int_{x_0}^{x_1} \varphi_n(x)\varphi_k(x)\ dx}_{\delta_{nk}}, \\
&= \sum_{n=0}^{\infty} \alpha_n\delta_{nk}, \\
&= \alpha_0\underbrace{\delta_{0k}}_{=0} + \alpha_1\underbrace{\delta_{1k}}_{=0} + \ldots + \alpha_k\underbrace{\delta_{kk}}_{=1} + \ldots + \alpha_\infty\underbrace{\delta_{\infty k}}_{=0}, \\
&= \alpha_k.
\end{aligned}$$

So trading k and n

$$\alpha_n = \int_{x_0}^{x_1} f(x)\varphi_n(x)\ dx.$$

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<u>Q:How to calculate the infinite series</u> $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}+\cdots$ *?*

Proof

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}+\cdots=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}+\cdots+\frac{1}{n^2}+\cdots$$

$$<1+\underbrace{\frac{1}{2^2}+\frac{1}{2^2}}_{\frac{1}{2}}+\underbrace{\frac{1}{4^2}+\frac{1}{4^2}+\frac{1}{4^2}+\frac{1}{4^2}}_{\frac{1}{4}}+\cdots=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=\frac{1}{1-\frac{1}{2}}=2$$

So, $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}+\cdots$ is convergent!

★Infinite geometric series:

$$\underline{\underline{\sum_{n=1}^{\infty}q^n=\frac{1}{1-q}}} \qquad \text{Or} \qquad \underline{\underline{\sum_{n=1}^{\infty}(-1)^n q^n=\frac{1}{1+q}}}$$

Consider,

$$\sin z=z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}\cdots+(-1)^n\frac{z^{2n+1}}{(2n+1)!}+\cdots=0 \quad z=0,\pm\pi,\pm2\pi,\pm3\pi,\cdots$$

Divided by z,

$$\frac{\sin z}{z}=1-\frac{z^2}{3!}+\frac{z^4}{5!}-\frac{z^6}{7!}\cdots+(-1)^n\frac{z^{2n}}{(2n+1)!}+\cdots=0 \quad z=\pm\pi,\pm2\pi,\pm3\pi,\cdots$$

We can get

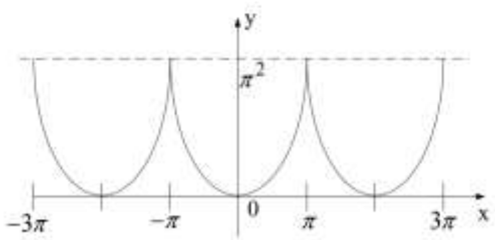
$$\frac{\sin z}{z}=1-\frac{z^2}{3!}+\frac{z^4}{5!}-\frac{z^6}{7!}\cdots+(-1)^n\frac{z^{2n}}{(2n+1)!}+\cdots=\left(1-\frac{z^2}{\pi^2}\right)\left(1-\frac{z^2}{4\pi^2}\right)\left(1-\frac{z^2}{9\pi^2}\right)\cdots$$

To compare the coefficient of z^2, we can attain

$$-\frac{1}{3!}=-\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\cdots\right)\Rightarrow 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}+\cdots=\frac{\pi^2}{6}$$

★求 $f(x)=x^2(-\pi\le x\le\pi)$ 的傅里叶级数

$$\boxed{f(x)=x^2\sim\frac{\pi^2}{3}-4\left(\cos x-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^3}+\cdots\right)}$$



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Algebraic equation

$$ax^2+bx+c=a(x-x_1)(x-x_2)=0$$

Equation coefficients:

$$a=a, b=-a(x_1+x_2), c=ax_1x_2$$

$$x_1+x_2=-\frac{b}{a}, x_1x_2=\frac{c}{a}$$

$$x_i=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \quad (i=1,2)$$

Another way

$$ax^2+bx+c=c\left(\frac{a}{c}x^2+\frac{b}{c}x+1\right)$$

$$=c\left[\frac{1}{x_1x_2}x^2+\frac{-(x_1+x_2)}{x_1x_2}x+1\right]$$

$$=c\frac{x^2-(x_1+x_2)x+x_1x_2}{x_1x_2}$$

$$=c\frac{(x-x_1)(x-x_2)}{x_1x_2}$$

$$=c\left(1-\frac{x}{x_1}\right)\left(1-\frac{x}{x_2}\right)$$

$$a=\frac{c}{x_1x_2}, b=-c\left(\frac{1}{x_1}+\frac{1}{x_2}\right), c=c$$

$$ax^3+bx^2+cx+d=a(x-x_1)(x-x_2)(x-x_3)=0$$

Equation coefficients:

$$a=a, b=-a(x_1+x_2+x_3),$$
$$c=a(x_1x_2+x_2x_3+x_1x_3), d=-ax_1x_2x_3$$

Another way:

$$ax^3+bx^2+cx+d=d\left(\frac{a}{d}x^3+\frac{b}{d}x^2+\frac{c}{d}x+1\right)$$

$$=d\left[\begin{array}{l}\frac{1}{-x_1x_2x_3}x^3+\left(\frac{1}{x_1x_2}+\frac{1}{x_2x_3}+\frac{1}{x_1x_3}\right)x^2\\ -\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right)x+1\end{array}\right]$$

$$=-d\frac{(x-x_1)(x-x_2)(x-x_3)}{x_1x_2x_3}$$

$$=d\left(1-\frac{x}{x_1}\right)\left(1-\frac{x}{x_2}\right)\left(1-\frac{x}{x_3}\right)$$

$$a=-\frac{d}{x_1x_2x_3}, b=d\left(\frac{1}{x_1x_2}+\frac{1}{x_2x_3}+\frac{1}{x_1x_3}\right),$$

$$c=-d\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right), d=d$$

If $a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\cdots+a_2x^2+a_1x+a_0=0$ has root $x_1,x_2,\cdots,x_{n-1},x_n$

Then we can write down

$$a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\cdots+a_2x^2+a_1x+a_0=a_n(x-x_1)(x-x_2)\cdots(x-x_{n-1})(x-x_n)=0$$

Equation coefficient

$$\boxed{a_{n-1}=-a_n(x_1+x_2+\cdots+x_{n-1}+x_n)}$$

Another way

$$a_0+a_1x+a_2x^2+\cdots+a_{n-1}x^{n-1}+a_nx^n=a_0\left(1-\frac{x}{x_1}\right)\left(1-\frac{x}{x_2}\right)\cdots\left(1-\frac{x}{x_{n-1}}\right)\left(1-\frac{x}{x_n}\right)=0$$

Similarly, the equation coefficient

$$\boxed{a_1 = -a_0\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_{n-1}}+\frac{1}{x_n}\right)}$$

If $b_0 - b_1x^2 + b_2x^4 - \cdots + b_{n-1}x^{2n-2} + b_nx^{2n} = 0$ has root $\pm\beta_1, \pm\beta_2, \cdots, \pm\beta_{n-1}, \pm\beta_n$

Some mathematical tricks,

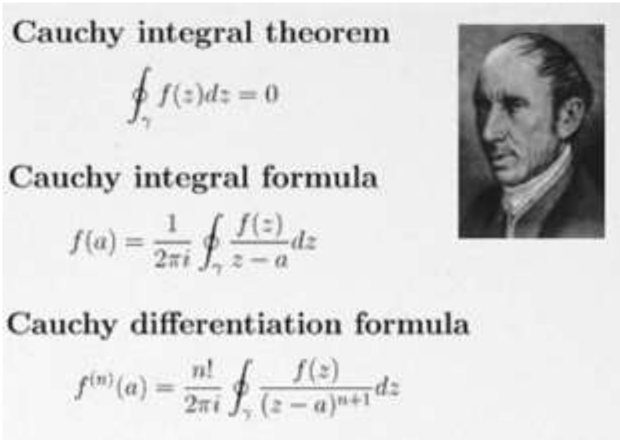
$$b_0 - b_1x^2 + b_2x^4 - \cdots + b_{n-1}x^{2n-2} + b_nx^{2n} = b_0\left(1-\frac{x^2}{\beta_1{}^2}\right)\left(1-\frac{x^2}{\beta_2{}^2}\right)\cdots\left(1-\frac{x^2}{\beta_{n-1}{}^2}\right)\left(1-\frac{x^2}{\beta_n{}^2}\right) = 0$$

Q7: What should we do with singularities？

① What about derivative?

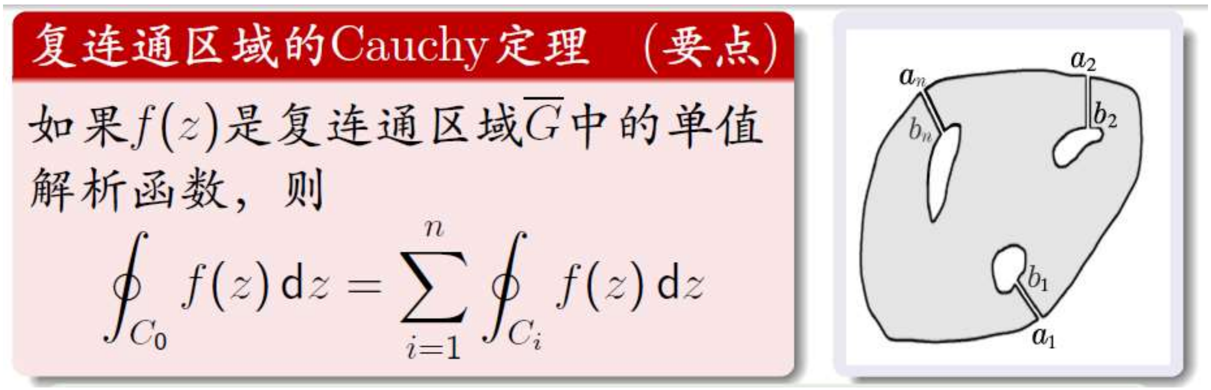
柯西-黎曼条件：

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$



② What about integral?

复连通区域柯西定理：



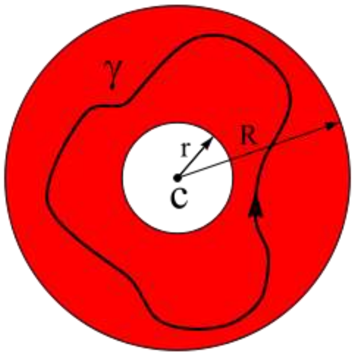
③ What about series?

Laurent series $\sum_{k=-\infty}^{\infty} c_k(z-b)^k = \sum_{k=-\infty}^{-1} c_k(z-b)^k + \sum_{k=0}^{\infty} c_k(z-b)^k$

Unlike the Taylor series which expresses f(z) as a series of terms with non-negative powers of z, a Laurent series includes terms with negative powers！

数学物理方法特训讲义—复变函数

Laurent series& Residue theorem



An analytic function $f(z)$ whose Laurent series is given by

$$f(z)=\sum_{n=-\infty}^{n=+\infty} a_n (z-z_0)^n$$

can be integrated term by term using a closed contour γ encircling z_0,

$$\int_\gamma f(z)\,dz = \sum_{n=-\infty}^{\infty} a_n \int_\gamma (z-z_0)^n\,dz$$

$$= \sum_{n=-\infty}^{-2} a_n \int_\gamma (z-z_0)^n\,dz + a_{-1}\int_\gamma \frac{dz}{z-z_0} + \sum_{n=0}^{\infty} a_n \int_\gamma (z-z_0)^n\,dz.$$

The Cauchy integral theorem requires that the first and last terms vanish, so we have

$$\int_\gamma f(z)\,dz = a_{-1}\int_\gamma \frac{dz}{z-z_0},$$

where a_{-1} is the complex residue. Using the contour $z=\gamma(t)=e^{it}+z_0$ gives

$$\int_\gamma \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{i\,e^{it}\,dt}{e^{it}} = 2\pi i,$$

so we have

$$\underline{\underline{\int_\gamma f(z)\,dz = 2\pi i\,a_{-1}.}}$$

If the contour γ encloses multiple poles, then the theorem gives the general result

$$\underline{\underline{\int_\gamma f(z)\,dz = 2\pi i \sum_{a\in A} \operatorname*{Res}_{z=a_i} f(z),}}$$

where A is the set of poles contained inside the contour.

★This amazing theorem therefore says that the value of a contour integral for any contour in the complex plane depends only on the properties of a few very special points inside the contour.