数学物理方法

Mathematical Methods in Physics

第八章 分离变量法

The Method of Separation of Variables

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问题的引入：

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t),\ 0 < x < l,\ t > 0 & (1) \\ u\big|_{x=0} = 0,\quad u\big|_{x=l} = 0 & (2) \\ u\big|_{t=0} = \varphi(x),\quad u_t\big|_{t=0} = \psi(x) & (3) \end{cases}$$

§ 8. 2

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t),\ 0 < x < l,\ t > 0 & (1) \\ u\big|_{x=0} = g(t),\quad u\big|_{x=l} = h(t) & (2) \\ u\big|_{t=0} = \varphi(x),\quad u_t\big|_{t=0} = \psi(x) & (3) \end{cases}$$

$u(x,t) = ?$

§ 8. 3 非齐次边界条件的处理

Inhomogeneous boundary Conditions

一、定解问题：

$$
\begin{cases}
u_{tt} - a^2 u_{xx} = 0, 0 < x < l, t > 0 & (1) \\
u|_{x=0} = g(t), u|_{x=l} = h(t) & (2) \\
u|_{t=0} = \varphi(x), u_t|_{t=0} = \psi(x) & (3)
\end{cases}
$$

二、求解

1、思路： 若令 $u(x,t) = X(x)T(t)$

$$
(2) \rightarrow \begin{cases} X(0)T(t) = g(t) \\ X(l)T(t) = h(t) \end{cases} \rightarrow \begin{cases} X(0) = g(t)/T(t) \\ X(l) = h(t)/T(t) \end{cases}
$$

能否使边界条件齐次化？ 无法确定其值

二、求解

2、求解

（1）边界条件齐次化：

令 $$u(x,t)=v(x,t)+w(x,t) \qquad (4)$$

使 $$\begin{cases} w|_{x=0}=u|_{x=0}=g(t) & (5) \\ w|_{x=l}=u|_{x=l}=h(t) & (6) \end{cases}$$

（2）确定辅助函数 $w(x,t)$

令 $$w(x,t)=A(t)x+B(t)$$

则 $$\boxed{w(x,t)=\frac{h(t)-g(t)}{l}x+g(t) \qquad (7)}$$

二、求解

2、求解

（3）求解 v(x, t) 的定解问题

$$(1)-(3)\rightarrow\begin{cases} v_{tt}-a^2v_{xx}=-(w_{tt}-a^2w_{xx}) & (8)\\ v|_{x=0}=0,\ v|_{x=l}=0 & (9)\\ \left.\begin{aligned} &v|_{t=0}=\varphi(x)-w(x,0)\\ &v_t|_{t=0}=\psi(x)-w_t(x,0)\end{aligned}\right\} & (10)\end{cases}$$

§8.1，§8.2 → $v(x,t)$

(4) 定解问题（1）-（3）的解

$$\boxed{u(x,t)=v(x,t)+\frac{h(t)-g(t)}{l}x+g(t)}$$

三、小结

1、以上介绍的方法也适用于带有其他非齐次边界条件的定解问题，其基本做法是：

① 作变换　令 $u(x,t)=v(x,t)+w(x,t)$

选择 $w(x,t)=A(t)x+B(t)$

或 $w(x,t)=A(t)x^2+B(t)x$【当两端均为第2类 非齐次边界条件时。】

确定 $A(t)$，$B(t)$ 使关于 $v(x,t)$ 的的边界条件齐次化

② 解关于v(x, t)的定解问题，从而最后求得：

$$u(x,t)=v(x,t)+w(x,t)$$

2、边界条件的齐次化，一般将导致方程的非齐次化

四、例题

试研究一端固定，一端作周期运动 $\sin\omega t$ 的弦运动。

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0 \ , \ 0 < x < l & (1) \\ u(0,t) = 0 \ , \ u(l,t) = \sin\omega t & (2) \\ u(x,0) = 0 \ , \ u_t(x,0) = 0 \ , \ 0 \le x \le l & (3) \end{cases}$$

① 令 $$u(x,t) = v(x,t) + w(x,t) \quad (4)$$

选 $$w(x,t) = \frac{\sin\omega t}{l} x + 0 = \frac{x}{l}\sin\omega t \quad (5)$$

$$\rightarrow \begin{cases} v_{tt} - a^2 v_{xx} = \dfrac{\omega^2}{l} x \sin\omega t & (6) \\ v(0,t) = v(l,t) = 0 & (7) \\ v(x,0) = 0 \ , \ v_t(x,0) = -\dfrac{\omega}{l} x & (8) \end{cases}$$

四、例题

$$\rightarrow \begin{cases} v_{tt} - a^2 v_{xx} = \dfrac{\omega^2}{l} x \sin \omega t & (6) \\ v(0,t) = v(l,t) = 0 & (7) \\ v(x,0) = 0 \;,\; v_t(x,0) = -\dfrac{\omega}{l} x & (8) \end{cases}$$

② 令 $$v(x,t) = v^I(x,t) + v^{II}(x,t) \qquad (9)$$

$$\begin{cases} v^I{}_{tt} - a^2 v^I{}_{xx} = 0 \\ v^I(0,t) = v^I(l,t) = 0 \qquad (10) \\ v^I(x,0) = 0 \;,\; v^I{}_t(x,0) = -\dfrac{\omega}{l} x \end{cases} \qquad \begin{cases} v^{II}{}_{tt} - a^2 v^{II}{}_{xx} = \dfrac{\omega^2}{l} x \sin \omega t \\ v^{II}(0,t) = v^{II}(l,t) = 0 \qquad (11) \\ v^{II}(x,0) = v^{II}{}_t(x,0) = 0 \end{cases}$$

四、例题

③ 求解

$$\begin{cases} v^I{}_{tt} - a^2 v^I{}_{xx} = 0 \\ v^I(0,t) = v^I(l,t) = 0 \qquad\qquad (10) \\ v^I(x,0) = 0\ ,\ v^I{}_t(x,0) = -\dfrac{\omega}{l}x \end{cases}$$

$$v^I(x,t) = \sum_{n=1}^{\infty}[A_n \cos\frac{n\pi a}{l}t + B_n \sin\frac{n\pi a}{l}t]\sin\frac{n\pi}{l}x$$

$$A_n = 0,\ B_n = \frac{2}{n\pi a}\int_0^l -\frac{\omega}{l}\alpha \sin\frac{n\pi\alpha}{l}d\alpha = \frac{2\omega l(-1)^n}{(n\pi)^2 a}$$

$$\boxed{v^I(x,t) = \frac{2\omega l}{\pi^2 a}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin\frac{n\pi a}{l}t\sin\frac{n\pi}{l}x \quad (12)}$$

四、例题

④ 令 $v^{II}(x,t)=\sum_{n=1}^{\infty}T_n(t)\sin\frac{n\pi}{l}x$

$$(11)\rightarrow\begin{cases}\sum_{n=1}^{\infty}\left[T_n''(t)+\frac{(an\pi)^2}{l^2}T_n(t)\right]\sin\frac{n\pi x}{l}=\frac{\omega^2}{l}x\sin\omega t\\ \sum_{n=1}^{\infty}T_n(0)\sin\frac{n\pi x}{l}=0\\ \sum_{n=1}^{\infty}T_n'(0)\sin\frac{n\pi x}{l}=0\end{cases}$$

$$\begin{cases}T_n''(t)+\frac{a^2n^2\pi^2}{l^2}T_n(t)=f_n(t)\\ T_n(0)=0\\ T_n'(0)=0\end{cases}\qquad(13)$$

$$f_n(t)=\frac{2}{l}\int_0^l\frac{\omega^2}{l}\alpha\sin\omega t\sin\frac{n\pi\alpha}{l}d\alpha$$

$$=\frac{2\omega^2}{n\pi}\sin\omega t(-1)^n\qquad(14)$$

四、例题

$$T_n(t)=\frac{l}{n\pi a}\int_0^t f_n(\tau)\sin\frac{n\pi a}{l}(t-\tau)d\tau$$

$$\omega_n=\frac{n\pi a}{l}$$

$$T_n(t)=\frac{l}{n\pi a}\int_0^t \frac{2\omega^2(-1)^{n+1}}{n\pi}\sin\omega\tau\sin\frac{n\pi a(t-\tau)}{l}d\tau$$

$$=\frac{\omega^2 l(-1)^{n+1}}{a(n\pi)^2}\left[\frac{\sin\omega_n t+\sin\omega t}{\omega_n+\omega}-\frac{\sin\omega_n t-\sin\omega t}{\omega_n-\omega}\right]\quad(15)$$

$$v^{II}(x,t)=\sum_{n=1}^{\infty}T_n(t)\sin\frac{n\pi}{l}x$$

$$w(x,t)=\frac{x}{l}\sin\omega t\quad(5)$$

$$v^{I}(x,t)=\frac{2\omega l}{\pi^2 a}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin\frac{n\pi a}{l}t\sin\frac{n\pi}{l}x\quad(12)$$

$$u(x,t)=v^{I}(x,t)+v^{II}(x,t)+w(x,t)$$

五、思考

$$\begin{cases} u_{xx} + u_{yy} = 0, \quad 0 < x < a, 0 < y < b \\ u(0,y) = f_1(y), u(a,y) = f_2(y) \\ u(x,0) = g_1(x), u(x,b) = g_2(x) \end{cases}$$

如何将边界条件齐次化?

例:

$$\begin{cases} u_{xx} + u_{yy} = 0, \quad 0 < x < a, 0 < y < b & (1) \\ u(0,y) = b - y, u(a,y) = 0 & (2) \\ u(x,0) = h\sin\dfrac{\pi}{a}x, u(x,b) = 0 & (3) \end{cases}$$

五、思考

$$\begin{cases} u_{xx} + u_{yy} = 0, \quad 0 < x < a, 0 < y < b & (1) \\ u(0, y) = b - y, u(a, y) = 0 & (2) \\ u(x,0) = h\sin\dfrac{\pi}{a}x, u(x,b) = 0 & (3) \end{cases}$$

法一： 令 $u(x,t) = v(x,t) + w(x,t)$

选 $w(x,t) = \dfrac{0-(b-y)}{a}x + (b-y)$

$$(1)-(3) \to \begin{cases} v_{xx} + v_{yy} = 0 \\ v(0, y) = 0, v(a, y) = 0 \\ v(x,0) = h\sin\dfrac{\pi}{a}x - b(1-\dfrac{x}{a}), v(x,b) = 0 \end{cases}$$

→ §8.1

五、思考

$$\begin{cases} u_{xx}+u_{yy}=0, \quad 0<x<a, 0<y<b & (1) \\ u(0,y)=b-y, u(a,y)=0 & (2) \\ u(x,0)=h\sin\dfrac{\pi}{a}x, u(x,b)=0 & (3) \end{cases}$$

法二： 令 $u(x,t)=v(x,t)+w(x,t)$

选 $$w(x,t)=\frac{0-h\sin\dfrac{\pi}{a}x}{b}+h\sin\frac{\pi}{a}x$$

$$(1)-(3)\rightarrow\begin{cases} v_{xx}+v_{yy}=0 \\ v(0,y)=b-y, v(a,y)=0 \\ v(x,0)=0, v(x,b)=0 \end{cases}$$

→ §8.1

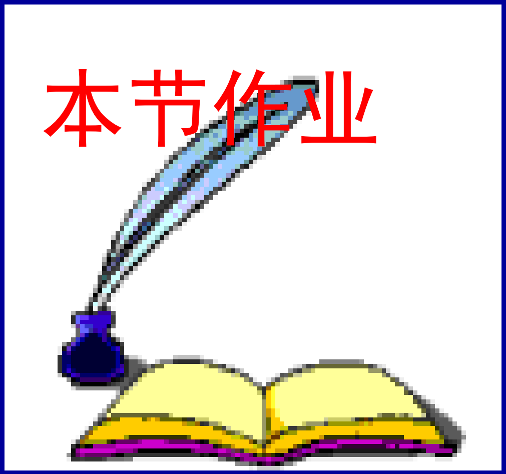
五、思考

$$\begin{cases} u_{xx}+u_{yy}=0, \quad 0<x<a, 0<y<b & (1) \\ u(0,y)=b-y, u(a,y)=0 & (2) \\ u(x,0)=h\sin\frac{\pi}{a}x, u(x,b)=0 & (3) \end{cases}$$

法三： 令 $u(x,t)=v(x,t)+w(x,t)$

$$\begin{cases} v_{xx}+v_{yy}=0, \\ v(0,y)=b-y, v(a,y)=0 \\ v(x,0)=0, v(x,b)=0 \end{cases} \qquad \begin{cases} w_{xx}+w_{yy}=0, \\ w(0,y)=0, w(a,y)=0 \\ w(x,0)=h\sin\frac{\pi}{a}x, w(x,b)=0 \end{cases}$$

§8.1



习题 8.3: 1；3