数学物理方法

Mathematical Methods in Physics

第五章　留数定理

The theorem of residues

武汉大学

物理科学与技术学院

问题的引入：

阻尼振动：$\int_0^{\infty} \frac{\sin x}{x} = ?$

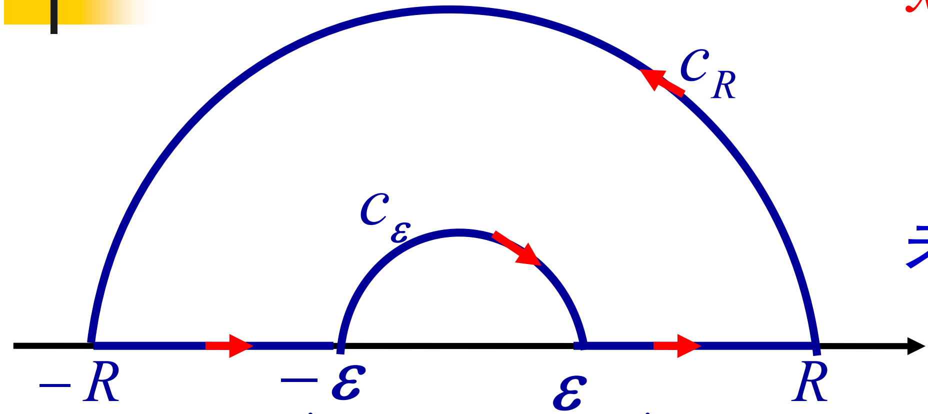
光学衍射：$\int_0^{\infty} \sin x^2 = ?$

热传导问题：$\int_0^{\infty} e^{-ax^2} \cos bx dx = ?$

§ 5. 3物理问题中的几个积分

Several integrals in Physical problems

一、Dirichlel积分 $\int_0^{\infty}\frac{\sin x}{x}dx$

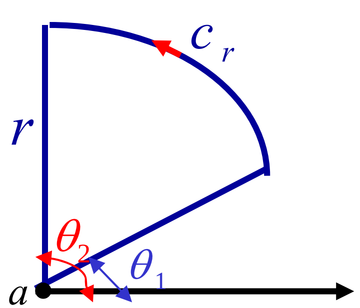


思路：

考虑$\frac{e^{iz}}{z}$沿如图所示围道积分。

$$\int_\varepsilon^R \frac{e^{ix}}{x}dx+\int_{c_R}\frac{e^{iz}}{z}dz+\int_{-R}^{-\varepsilon}\frac{e^{ix}}{x}dx+\int_{c_\varepsilon}\frac{e^{iz}}{z}dz=0 \quad (1)$$

附：小弧引理



若$f(z)$在$c_r: z-a=re^{i\theta},\theta_1\le\theta\le\theta_2$上连续

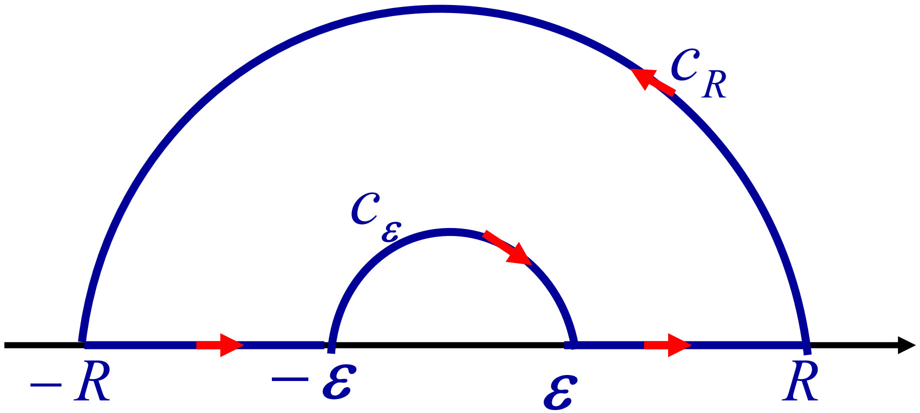
且$\lim\limits_{r\to 0}(z-a)f(z)=\lambda$,则

$$\boxed{\lim_{r\to 0}\int_{c_r} f(z)dz=i(\theta_2-\theta_1)\lambda}$$

$$=i(\theta_2-\theta_1)res\, f(a)$$

当a为单极点时

一、Dirichlel积分 $\int_0^{\infty}\frac{\sin x}{x}dx$



结论：$\boxed{\int_0^{\infty}\frac{\sin x}{x}dx=\frac{\pi}{2}}$

$$\int_0^{\infty}\frac{\sin ax}{x}dx=?$$

答：

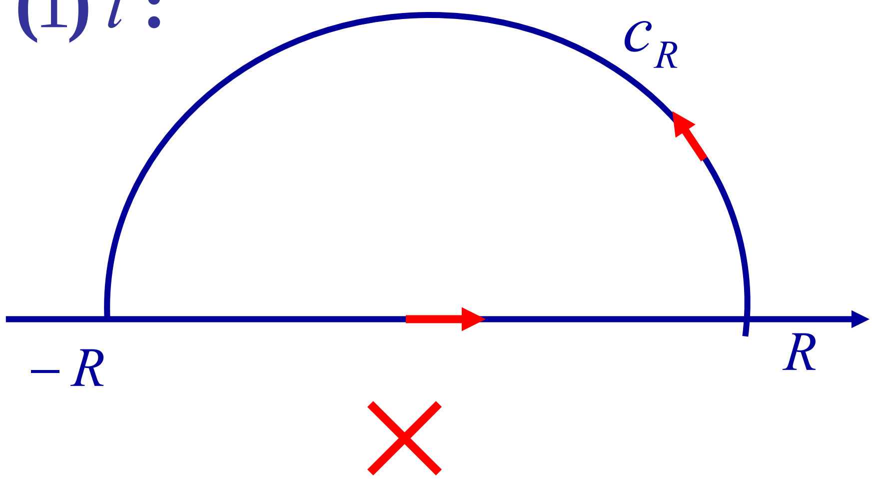
$$\int_0^{\infty}\frac{\sin ax}{x}dx=\begin{cases}\frac{\pi}{2}, a>0\\ -\frac{\pi}{2}, a<0\end{cases}$$

$$\int_{-\infty}^{\infty}\frac{\sin^3 x}{x^3}dx=?$$

答：$\frac{3}{4}\pi$

二、Fresnel积分 $\int_0^{\infty}\begin{Bmatrix}\sin x^2\\ \cos x^2\end{Bmatrix}dx$

(1) l :



思路:

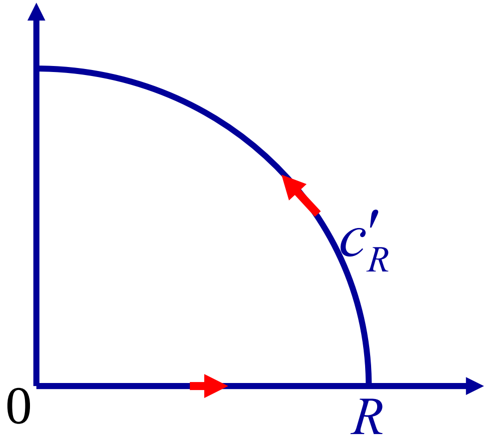
考虑 $F(z)=e^{iz^2}$ 沿如图所示 l 的积分,

$$\oint_l e^{iz^2}dz=\int_{-R}^{R}e^{ix^2}dx+\int_{c_R}e^{iz^2}dz=0 \quad (2)$$

$$\left|\oint_{c_R}e^{iz^2}dz\right|\le\int_0^{\pi}e^{-R^2\sin 2\theta}Rd\theta\xrightarrow{R\to\infty}\infty$$

二、Fresnel积分 $\int_0^{\infty}\begin{Bmatrix}\sin x^2\\ \cos x^2\end{Bmatrix}dx$

(2) l:



修改路径:

考虑 $F(z)=e^{iz^2}$ 沿

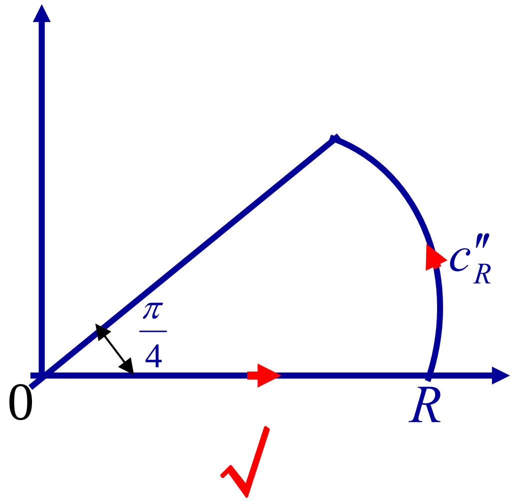
如图所示 l 的积分,

$$\oint_l e^{iz^2}dz=\int_0^R e^{ix^2}dx+\int_{C_R'} e^{iz^2}dz+\int_R^0 e^{i(iy)^2}d(iy)=0 \quad (3)$$

$$\int_0^{\infty}\cos x^2dx=\int_0^{\infty}\sin x^2dx=?$$

二、Fresnel积分 $\int_0^\infty \begin{Bmatrix} \sin x^2 \\ \cos x^2 \end{Bmatrix} dx$

(3) l:



再修改路径:

考虑 $F(z)=e^{iz^2}$ 沿如图所示 l 的积分,

$$\oint_l e^{iz^2}dz=\int_0^R e^{ix^2}dx+\int_{c''_R} e^{iz^2}dz+\int_R^0 e^{i(xe^{i\frac{\pi}{4}})^2}d(xe^{i\frac{\pi}{4}})=0 \quad (4)$$

结论:

$$\boxed{\int_0^\infty \cos x^2 dx=\int_0^\infty \sin x^2 dx=\frac{\sqrt{2\pi}}{4}}$$

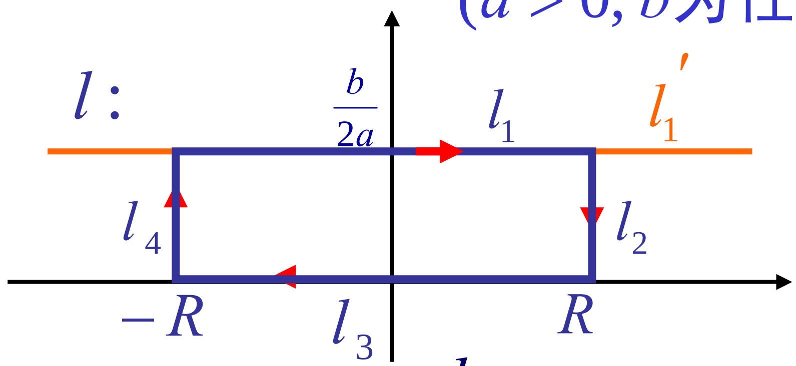
三、热传导问题积分 $\int_0^{\infty} e^{-ax^2}\cos bx dx$

$(a>0, b$为任意实数$)$

$$\int_0^{\infty} e^{-ax^2}\cos bx dx$$

$$=\frac{1}{2}e^{-\frac{b^2}{4a}}\int_{-\infty}^{\infty} e^{-a\left(x+\frac{ib}{2a}\right)^2}dx$$

$$=\frac{1}{2}e^{-\frac{b^2}{4a}}\int_{l_1'} e^{-az^2}dz$$



l_1': $z=x+i\frac{b}{2a}, x:-\infty\to\infty$

$$\int_l e^{-az^2}dz=\int_{l_1} e^{-az^2}dz+\int_{\frac{b}{2a}}^{0} e^{-a(R+iy)^2}d(iy)+$$

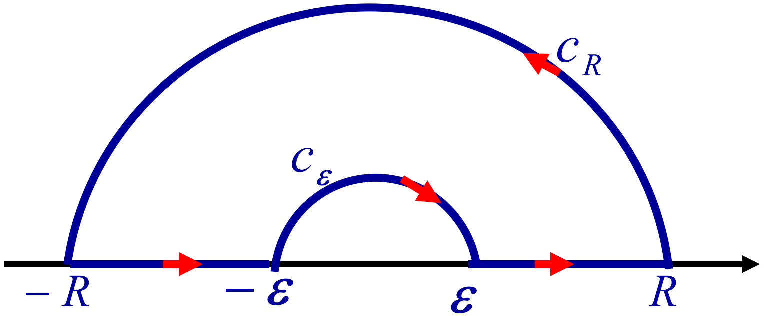
$$+\int_{R}^{-R} e^{-ax^2}dx+\int_{0}^{\frac{b}{2a}} e^{-a(-R+iy)^2}d(iy)=0$$

结论：

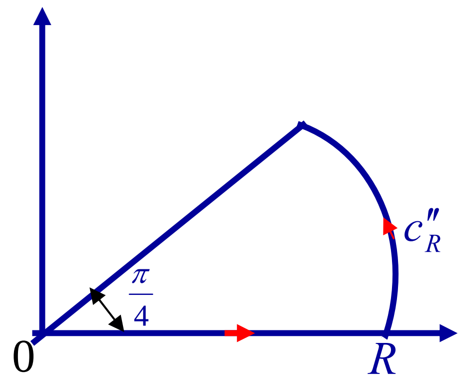
$$\boxed{\int_0^{\infty} e^{-ax^2}\cos bx dx=\frac{1}{2}e^{-\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}}\quad(a>0)}$$

四、小结

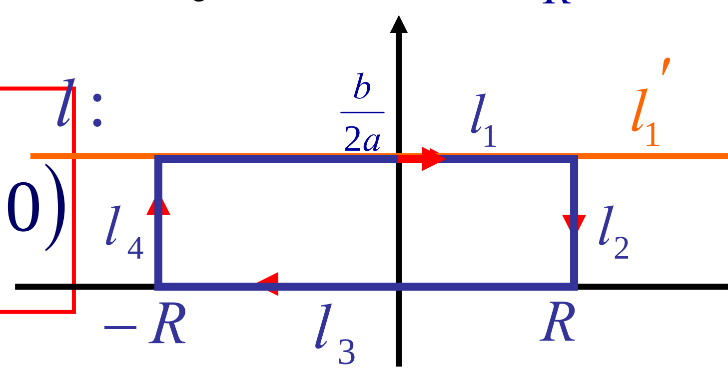
$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$



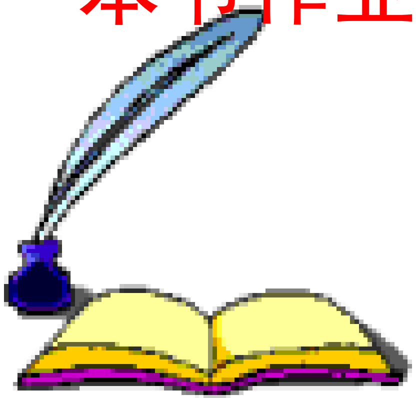
$$\int_0^{\infty} \cos x^2 dx = \int_0^{\infty} \sin x^2 dx = \frac{\sqrt{2\pi}}{4}$$



$$\int_0^{\infty} e^{-ax^2} \cos bx dx = \frac{1}{2} e^{-\frac{b^2}{4a}} \sqrt{\frac{\pi}{a}} \quad (a > 0)$$



本节作业

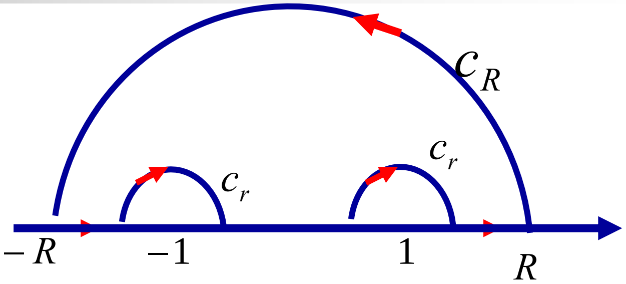


习题5.3:

3(2) ; 4(2);5(2)

问题的引入:

由前：(4) $\int_{-\infty}^{\infty} \frac{1}{1-x^2} dx = ?$



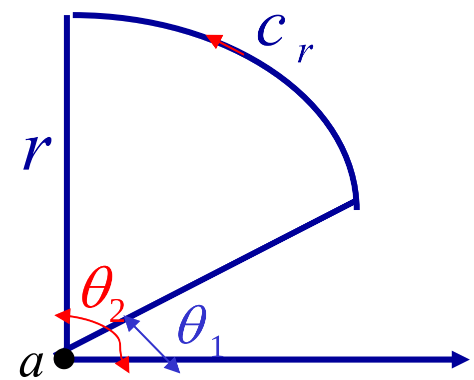
$$\lim_{r\to 0} \int_{c_r} f(z) dz = ?$$

小弧引理:

若$f(z)$在$c_r : z-a = re^{i\theta}, \theta_1 \le \theta \le \theta_2$上连续

且$\lim\limits_{r\to 0} (z-a) f(z) = \lambda$,则

$$\boxed{\lim_{r\to 0} \int_{c_r} f(z) dz = i(\theta_2 - \theta_1)\lambda}$$



$$= i(\theta_2 - \theta_1) res\, f(a)$$

当a为$f(z)$的单极点时

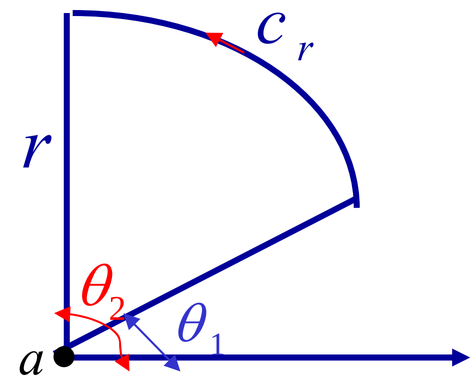
附：小弧引理证明

若$f(z)$在$c_r: z-a=re^{i\theta},\theta_1\le\theta\le\theta_2$上连续

且$\lim\limits_{r\to0}(z-a)f(z)=\lambda$,则

$$\lim_{r\to0}\int_{c_r}f(z)dz=i(\theta_2-\theta_1)\lambda$$
$$=i(\theta_2-\theta_1)res\,f(a)$$

当a为单极点时



证: $$\because\int_{c_r}\frac{dz}{z-a}=\int_{\theta_1}^{\theta_2}\frac{rie^{i\theta}}{re^{i\theta}}d\theta=i(\theta_2-\theta_1)$$

$$\therefore\left|\int_{c_r}f(z)dz-i(\theta_2-\theta_1)\lambda\right|=\left|\int_{c_r}\frac{f(z)(z-a)-\lambda}{z-a}dz\right|$$

$$\le\frac{\max|f(z)(z-a)-\lambda|}{r}r(\theta_2-\theta_1)<\varepsilon'(\theta_2-\theta_1)=\varepsilon$$

Good-by!