数学物理方法

Methods in Mathematical Physics

第九章　积分变换法

The Method of Integral Transforms

武汉大学物理科学与技术学院

第九章　积分变换法

§9.2 傅里叶变换法

The Method of FourierTransforms

一、波动问题

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, -\infty < x < \infty, t > 0 & (1) \\ u\big|_{t=0} = \varphi(x) & (2) \\ u_t\big|_{t=0} = \psi(x) & (3) \end{cases}$$

1、对定解问题各项施行傅氏变换

记 $\int_{-\infty}^{\infty} u(x,t) e^{-i\omega x} dx = \tilde{u}(\omega,t)$, $\int_{-\infty}^{\infty} \varphi(x) e^{-i\omega x} dx = \tilde{\varphi}(\omega)$

$\int_{-\infty}^{\infty} \psi(x) e^{-i\omega x} dx = \tilde{\psi}(\omega)$,

则

$$\begin{cases} \dfrac{d^2\tilde{u}(\omega,t)}{dt^2} + a^2\omega^2\tilde{u}(\omega,t) = 0 & (4) \\ \tilde{u}(\omega,0) = \tilde{\varphi}(\omega) & (5) \\ \tilde{u}_t(\omega,0) = \tilde{\psi}(\omega) & (6) \end{cases}$$

一、波动问题

2、解常微分方程定解问题（4）—（6）得

$$\tilde{u}(\omega,t)=\tilde{\varphi}(\omega)\cos a\omega t+\frac{1}{a\omega}\tilde{\psi}(\omega)\sin a\omega t$$

3、求傅氏逆变换

$$u(x,t)=F^{-1}[\tilde{u}(\omega,t)]=F^{-1}[\tilde{\varphi}(\omega)\cos a\omega t]+F^{-1}\left[\frac{\tilde{\psi}(\omega)}{a\omega}\sin a\omega t\right]$$

$$\boxed{u(x,t)=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\alpha)d\alpha}$$

附：上节例题结果：

(2) 已知 $F[\varphi(x)]=G(\omega)$

$$F^{-1}[G(\omega)\cos a\omega t]=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]$$

$$F^{-1}\left[\frac{G(\omega)}{a\omega}\sin a\omega t\right]=\frac{1}{2a}\int_{x-at}^{x+at}\varphi(\xi)d\xi$$

二. 输运问题

$$\begin{cases} u_t - a^2 u_{xx} = f(x,t) \quad ,-\infty < x < \infty, t > 0 & (7) \\ u(x,0) = \varphi(x) & (8) \end{cases}$$

(1) 对定解问题各项施行傅氏变换

记 $\int_{-\infty}^{\infty} u(x,t) e^{-i\omega x} dx = \tilde{u}(\omega,t)$, $\int_{-\infty}^{\infty} f(x,t) e^{-i\omega x} dx = \tilde{f}(\omega,t)$,

$\int_{-\infty}^{\infty} \varphi(x) e^{-i\omega x} dx = \tilde{\varphi}(\omega)$,

则

$$\begin{cases} \dfrac{d\tilde{u}}{dt} + a^2\omega^2\tilde{u} = \tilde{f}(\omega,t) & (9) \\ \tilde{u}(\omega,0) = \tilde{\varphi}(\omega) & (10) \end{cases}$$

$$F^{-1}\left[e^{-a^2\omega^2 t}\right]=\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2 t}}$$

二. 输运问题

(2) 解常微分方程定解问题（9）-（10）得

$$\tilde{u}(\omega,t)=e^{-a^2\omega^2 t}\tilde{\varphi}(\omega)+\int_0^t \tilde{f}(\omega,\tau)e^{-a^2\omega^2(t-\tau)}d\tau$$

(3) 求傅氏逆变换

$$u(x,t)=F^{-1}\left[\tilde{\varphi}(\omega)e^{-a^2\omega^2 t}\right]+\int_0^t F^{-1}\left[\tilde{f}(\omega,\tau)e^{-a^2\omega^2(t-\tau)}\right]d\tau$$

$$=F^{-1}F[\varphi(x)*F^{-1}(e^{-a^2\omega^2 t})]+\int_0^t F^{-1}F[f(x,\tau)*F^{-1}(e^{-a^2\omega^2(t-\tau)})]d\tau$$

$$=\varphi(x)*\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2 t}}+\int_0^t f(x,\tau)*\frac{1}{2a\sqrt{\pi(t-\tau)}}e^{-\frac{x^2}{4a^2(t-\tau)}}d\tau$$

$$\boxed{\begin{aligned}u(x,t)=&\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}\varphi(\xi)e^{-\frac{(\xi-x)^2}{4a^2 t}}d\xi\\&+\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{1}{\sqrt{t-\tau}}\int_{-\infty}^{\infty}f(\xi,\tau)e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}d\xi d\tau\end{aligned}}$$

二. 输运问题

2. $\begin{cases} u_t - a^2 \Delta u = 0 \\ u\big|_{t=0} = \varphi(\vec{r}) \end{cases}$

$$F^{-1}\left[e^{-a^2\omega^2 t}\right] = F^{-1}\left[e^{-a^2\left(\omega^2{}_1+\omega^2{}_2+\omega^2{}_3\right)t}\right] = \left(\frac{1}{2a\sqrt{\pi t}}\right)^3 e^{\frac{x^2+y^2+z^2}{-4a^2 t}}$$

记 $F\left[u(\vec{r},t)\right] = \iiint_{-\infty}^{\infty} u(\vec{r},t) e^{-i\vec{\omega}\cdot\vec{r}} d\vec{r} = \tilde{u}(\vec{\omega},t)$

$$F\left[\varphi(\vec{r})\right] = \iiint_{-\infty}^{\infty} \varphi(\vec{r}) e^{-i\vec{\omega}\cdot\vec{r}} d\vec{r} = \tilde{\varphi}(\vec{\omega})$$

$$F[\Delta u] = \iiint_{-\infty}^{\infty} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) e^{-i\vec{\omega}\cdot\vec{r}} d\vec{r} = -\omega^2 \tilde{u}(\vec{\omega},t)$$

$$\begin{cases} \dfrac{d\tilde{u}(\vec{\omega},t)}{dt} + a^2\omega^2 \tilde{u}(\vec{\omega},t) = 0 \\ \tilde{u}(\vec{\omega},0) = \tilde{\varphi}(\vec{\omega}) \end{cases} \qquad \tilde{u}(\vec{\omega},t) = \tilde{\varphi}(\vec{\omega}) e^{-a^2\omega^2 t}$$

$$\Longrightarrow \boxed{u(\vec{r},t) = \frac{1}{8a^3(\pi t)^{3/2}} \iiint_{-\infty}^{\infty} \varphi(\vec{r}_1) e^{-\frac{|\vec{r}-\vec{r}_1|^2}{4a^2 t}} d\vec{r}_1}$$

二. 输运问题

附：
$$F[\Delta u]=\iiint_{-\infty}^{\infty}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)e^{-i\vec{\omega}\cdot\vec{r}}d\vec{r}$$
$$=\iiint_{-\infty}^{\infty}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)e^{-i(\omega_1 x+\omega_2 y+\omega_3 z)}dxdydz$$
$$=\iint\left[\int_{-\infty}^{\infty}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)e^{-i\omega_1 x}dx\right]e^{-i\omega_2 y}dye^{-i\omega_3 z}dz$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[(i\omega_1)^2\tilde{u}(\omega_1,y,z)+\frac{\partial^2\tilde{u}(\omega_1,y,z)}{\partial y^2}+\frac{\partial^2\tilde{u}(\omega_1,y,z)}{\partial z^2}\right]e^{-i\omega_2 y}dye^{-i\omega_3 z}dz$$
$$=\int_{-\infty}^{\infty}\left[-\omega_1{}^2\tilde{u}(\omega_1,\omega_2,z)-\omega_2{}^2\tilde{u}(\omega_1,\omega_2,z)+\frac{\partial^2\tilde{u}(\omega_1,\omega_2,z)}{\partial z^2}\right]e^{-i\omega_3 z}dz$$
$$=-\tilde{u}(\omega_1,\omega_2,\omega_3)[\omega_1{}^2+\omega_2{}^2+\omega_3{}^2]=-\tilde{u}(\omega_1,\omega_2,\omega_3)\omega^2=-\omega^2\tilde{u}(\vec{\omega})$$

三、稳定问题

$$\Delta u = -\frac{1}{\varepsilon_0}\rho(x,y,z)$$

记 $u(x,y,z)=u(\vec{r})$

$$\frac{1}{\varepsilon_0}\rho(x,y,z)=f(\vec{r})$$

$$\int\int\int_{-\infty}^{\infty}u(\vec{r})e^{-i\vec{\omega}\cdot\vec{r}}d\vec{r}=\tilde{u}(\vec{\omega})$$

$$\int\int\int_{\infty}^{\infty}f(\vec{r})e^{-i\vec{\omega}\cdot\vec{r}}d\vec{r}=\tilde{f}(\vec{\omega})$$

$$\omega^2\tilde{u}(\vec{\omega})=f(\vec{\omega})$$

$$\tilde{u}(\vec{\omega})=\frac{\tilde{f}(\vec{\omega})}{\omega^2}=F[f(\vec{r})]\cdot F\left[F^{-1}\left[\frac{1}{\omega^2}\right]\right]=F\left[F^{-1}\left[\frac{1}{\omega^2}\right]*f(\vec{r})\right]$$

$$\because F\left[\frac{1}{r}\right]=\frac{4\pi}{\omega^2},\therefore F^{-1}\left[\frac{1}{\omega^2}\right]=\frac{1}{4\pi r}=\frac{1}{4\pi|\vec{r}|}$$

$$\therefore u(\vec{r})=F^{-1}[\tilde{u}(\vec{\omega})]$$

$$\boxed{u(\vec{r})=\frac{1}{4\pi}\int\int\int_{-\infty}^{\infty}\frac{f(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d\vec{r}\,'}$$

四、小结

1、傅氏变换法精神：

对方程中各项施行傅氏变换，从而将偏微分方程化为常微分方程求解。

2、傅氏变换法的解题步骤：

（1）对方程中各项选择适当变量施行傅氏变换

（2）解像函数的常微分方程的定解问题

（3）求逆变换得原定解问题的解

四、小结

3、由傅氏变换法得到的三种典型定解问题的解：

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, -\infty < x < \infty, t > 0 \\ u\big|_{t=0} = \varphi(x) \\ u_t\big|_{t=0} = \psi(x) \end{cases}$$

$$\boxed{u(x,t) = \frac{1}{2}[\varphi(x+at) + \varphi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\alpha)d\alpha}$$

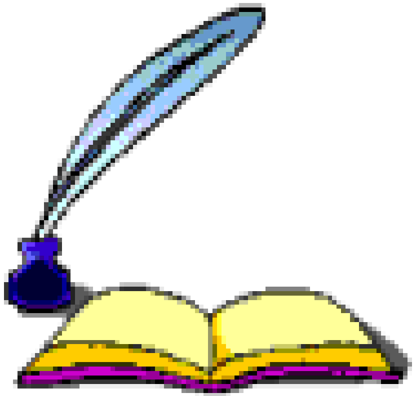
$$\begin{cases} u_t - a^2 u_{xx} = f(x,t) \\ u(x,0) = \varphi(x) \end{cases}$$

$$\boxed{\begin{aligned} u(x,t) = &\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}\varphi(\xi)e^{-\frac{(\xi-x)^2}{4a^2 t}}d\xi \\ &+ \frac{1}{2a\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t-\tau}}\int_{-\infty}^{\infty} f(\xi,\tau)e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}d\xi d\tau \end{aligned}}$$

$$\Delta u = -\frac{1}{\varepsilon_0}\rho(x,y,z)$$

$$\boxed{u(\vec{r}) = \frac{1}{4\pi}\iiint_{-\infty}^{\infty}\frac{f(\vec{r}')}{|\vec{r}-\vec{r}'|}d\vec{r}'}$$

本节作业



习题 9.2：

4；　6（2）

Good-by!