数学物理方法

Methods in Mathematical Physics

第十章 格林函数法

Method of Green's Function

武汉大学物理科学与技术学院

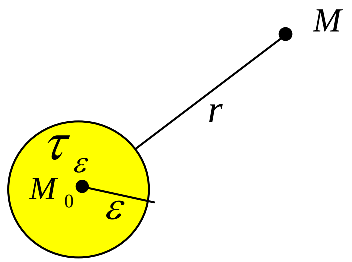
第十章 格林函数法

§10.3- §10.4 格林函数

Green’s Functions

一、泊松方程的格林函数

1、三维： $\Delta G = -\delta(M - M_0)$



$$\Delta G = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right) = -\delta(r)$$

(1) 若 $r \neq 0$： $G = -C_1\dfrac{1}{r}$

(2) 若 $r = 0$： 考虑： $\iiint_{\tau_\varepsilon} \Delta G dv = -\iiint_{\tau_\varepsilon} \delta(r) dv = -1$

又 $\iiint_{\tau_\varepsilon} \Delta G dv = \iiint_{\tau_\varepsilon} \nabla \cdot \nabla G dv = \iint_{\sigma_\varepsilon} \nabla G \cdot d\vec{\sigma} = C_1 4\pi$

$$\rightarrow C_1 = -\frac{1}{4\pi},$$

(1),(2) $\rightarrow$ $\boxed{G = \dfrac{1}{4\pi r}}$ —泊松方程格林函数。

一、泊松方程的格林函数

2、二维：$\Delta G = -\delta(M - M_0)$

$$\Delta G = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) = -\delta(r),\quad r = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

$$\iint_\sigma \nabla\cdot\nabla u\, d\sigma = \int_l \nabla u \cdot d\vec{l}$$

$$\boxed{G(M, M_0) = \frac{1}{2\pi}\ln\frac{1}{r}}$$

二、狄氏格林函数

1、三维：$\begin{cases}\Delta G=-\delta(x-x_0,y-y_0,z-z_0),M\in\tau\\ G|_\sigma=0\end{cases}$

令 $G(M,M_0)=F(M,M_0)+g(M,M_0)$

使 $\Delta F(M,M_0)=-\delta(M-M_0)\qquad M\in\tau$

则 $\boxed{G(M,M_0)=\dfrac{1}{4\pi r}+g}$ —狄氏格林函数

$$\boxed{\begin{cases}\Delta g=0,M\in\tau\\ g|_\sigma=-\dfrac{1}{4\pi r}|_\sigma\end{cases}}$$

二、狄氏格林函数

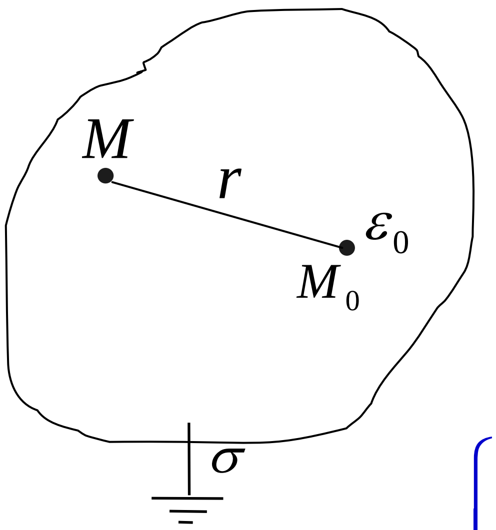
2、二维：
$$\begin{cases}\Delta G = -\delta(x - x_0, y - y_0) \\ G|_l = 0\end{cases}$$

类似 $G = \dfrac{1}{2\pi}\ln\dfrac{1}{r} + g$ —狄氏格林函数

$$\begin{cases}\Delta g = 0, M \in \sigma \\ g|_l = -\dfrac{1}{2\pi}\ln\dfrac{1}{r}|_l\end{cases}$$

二、狄氏格林函数

3、狄氏格林函数的物理意义：



$$\begin{cases}\Delta G = -\delta(M - M_0,)\,, M \in \tau \\ G|_{\sigma} = 0\end{cases}$$

$$\boxed{G(M, M_0) = \frac{1}{4\pi r} + g}$$

$$\boxed{\begin{cases}\Delta g = 0, M \in \tau \\ g|_{\sigma} = -\frac{1}{4\pi r}|_{\sigma}\end{cases}}$$

$G - M$点电位 $\begin{cases} \varepsilon_0 \text{提供}: \frac{1}{4\pi\varepsilon_0}\frac{\varepsilon_0}{r} = \frac{1}{4\pi r} \\ \text{感应电荷提供}v: \begin{cases}\Delta v = 0, M \in \tau \\ v|_{\sigma} = -\frac{1}{4\pi r}|_{\sigma}\end{cases}\end{cases}$

$$\boxed{\therefore v = g}$$

二、狄氏格林函数

3、狄氏格林函数的物理意义：

求$G\rightarrow$ 求M点电位 $\rightarrow$求感应电荷产生的电位

对于三维：

$$\text{即求:}\begin{cases}\Delta g=0, M\in\tau \\ g|_{\sigma}=-\dfrac{1}{4\pi r}|_{\sigma}\end{cases}$$

对于二维：

$$\text{即求:}\begin{cases}\Delta g=0, M\in\sigma \\ g|_{l}=-\dfrac{1}{2\pi}\ln\dfrac{1}{r}|_{l}\end{cases}$$

三、用电像法求狄氏格林函数

1、问题的引入：

求解球内的狄氏问题：$\begin{cases} \Delta u = 0\ ,\ \rho < a \\ u|_{\rho=a} = f(M) \end{cases}$

解：$$u(M) = -\iint_\sigma f(M_0)\frac{\partial G}{\partial n_0}d\sigma_0$$

$$G(M, M_0) = \frac{1}{4\pi r} + g\ ;\ \begin{cases} \Delta g = 0,\ M \in \rho < a \\ g|_{\rho=a} = -\dfrac{1}{4\pi r}\Big|_{\rho=a} \end{cases}$$

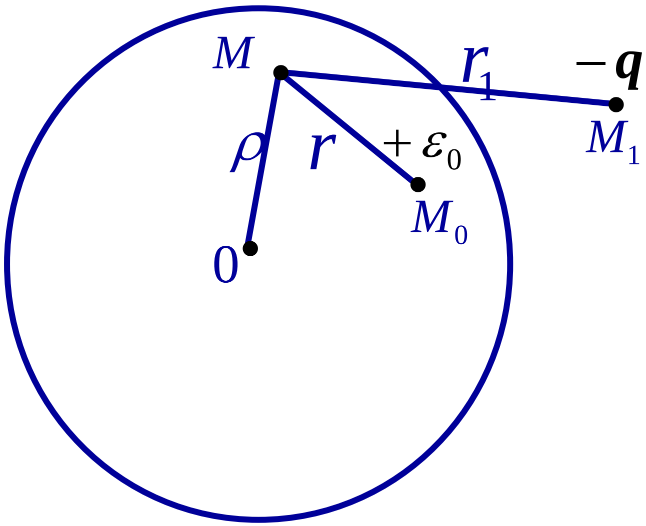
求u $\rightarrow$ 求G $\rightarrow$ 求M点电位 $\boxed{\rightarrow \text{求感应电荷产生的电位}\ \ g}$

三、用电像法求狄氏格林函数

2、用电像法求g：

(1) 分析：若能在σ外的某点M_1放一适当的负q,则

$$\Delta\left(\frac{-q}{4\pi\varepsilon_0 r_1}\right)=0,\ \ M\in\rho<a$$



使： $-\frac{q}{4\pi\varepsilon_0 r_1}\Big|_{\rho=a}=-\frac{1}{4\pi r}\Big|_{\rho=a}$

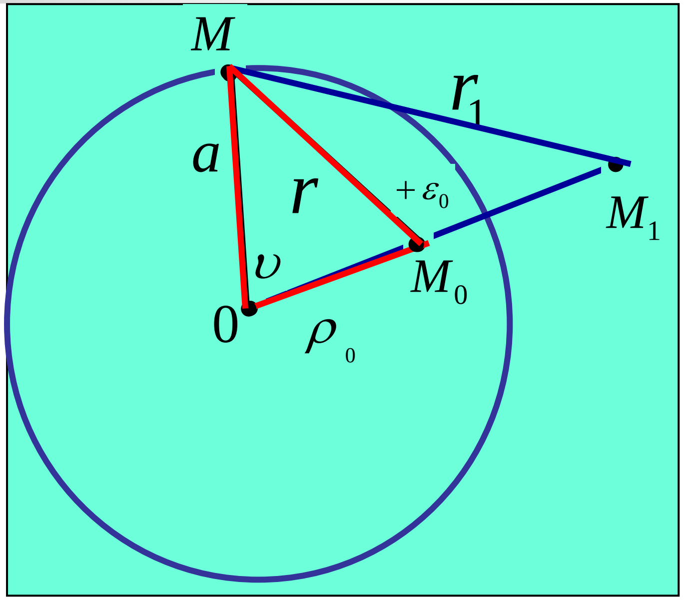
则 $g=-\frac{q}{4\pi\varepsilon_0 r_1}$

$\therefore$求$g\rightarrow a)$确定M_1的位置；$b)$确定q大小问题

三、用电像法求狄氏格林函数

2、用电像法求g：

（2）求球域的G



a) $r_1 = ?$记$|OM_0| = \rho_0$，$|OM_1| = \rho_1$

使 $\rho_0 \cdot \rho_1 = a^2$ 即$\boxed{\dfrac{\rho_0}{a} = \dfrac{a}{\rho_1}}$

则称M_1为M_0关于球面$\rho = a$的像

b) $q = ?$ $\quad \dfrac{1}{r}\Big|_\sigma = ?$ $\quad \because \Delta OM_0M \backsim \Delta OM_1M$

$$\therefore \frac{\rho_0}{a} = \frac{a}{\rho_1} = \frac{r}{r_1}, \quad \text{即} \boxed{\frac{1}{r}\Big|_{\rho=a} = \frac{a/\rho_0}{r_1}\Big|_{\rho=a}}$$

三、用电像法求狄氏格林函数

2、用电像法求g：

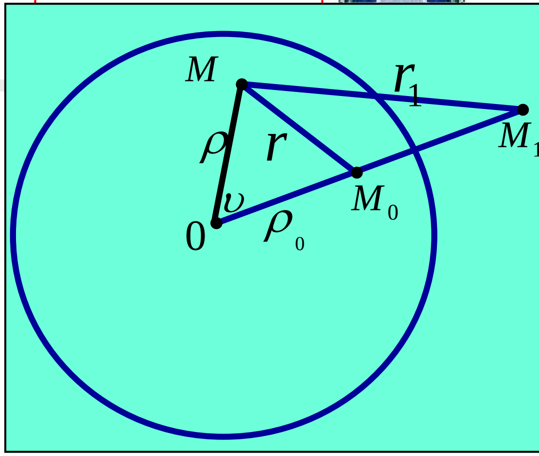
（2）求球域的G

$$g=\frac{-\varepsilon_0 a/\rho_0}{4\pi\varepsilon_0 r_1}=\frac{-a/\rho_0}{4\pi r_1} \qquad q=\frac{\varepsilon_0 a}{\rho_0}$$

$$\boxed{G=\frac{1}{4\pi r}-\frac{a/\rho_0}{4\pi r_1}} \qquad -q=-\frac{\varepsilon_0 a}{\rho_0}$$ 是ε_0的电像

(3) 电像法：这种在像点放一虚构的点电荷，来等效代替边界面上的感应电荷所产生的电位的方法称之为电像法

三、用电像法求狄氏格林函数



3、求 $u(M)$

$$\frac{\partial G}{\partial n}=\frac{1}{4\pi}\left[\frac{\partial}{\partial\rho}\left(\frac{1}{r}\right)-\frac{a}{\rho_0}\frac{\partial}{\partial\rho}\left(\frac{1}{r_1}\right)\right]$$

$$r=\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos\gamma}\qquad r_1=\sqrt{\rho^2+\rho_1^2-2\rho\rho_1\cos\gamma}$$

$$\frac{\partial}{\partial\rho}\frac{1}{r}=\frac{\rho_0^2-\rho^2-r^2}{2\rho r^3}\rightarrow\frac{\partial}{\partial\rho}\left(\frac{1}{r}\right)_{\rho=a}=\frac{\rho_0^2-a^2-r^2}{2ar^3}$$

类似有: $$\frac{a}{\rho_0}\frac{\partial}{\partial\rho}\left(\frac{1}{r_1}\right)_{\rho=a}=\frac{a}{\rho_0}\frac{\rho_1^2-a^2-r_1^2}{2ar_1^3}=\frac{a^2-r^2-\rho_0^2}{2ar^3}$$

$$\left.\frac{\partial G}{\partial n}\right|_{\rho=a}=\frac{1}{4\pi a}\frac{\rho_0^2-a^2}{r^3}$$

三、用电像法求狄氏格林函数

3、求 $u(M)$

$$u(M)=-\iint_{\sigma} f(M_0)\frac{\partial G}{\partial n_0}d\sigma_0$$

$$=\frac{1}{4\pi a}\int_0^{2\pi}\int_0^{\pi} f(\theta_0,\varphi_0)\frac{a^2-\rho^2}{\left(a^2+\rho^2-2a\rho\cos\gamma\right)^{\frac{3}{2}}}a^2\sin\theta_0 d\theta_0 d\varphi_0$$

$$\boxed{u(M)=\frac{a}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(\theta_0,\varphi_0)\frac{a^2-\rho^2}{\left(a^2+\rho^2-2a\rho\cos\gamma\right)^{\frac{3}{2}}}\sin\theta_0 d\theta_0 d\varphi_0}$$

—球的泊松积分公式

其中，$\cos\gamma=\sin\theta\sin\theta_0\cos(\varphi-\varphi_0)+\cos\theta\cos\theta_0$

四、注释

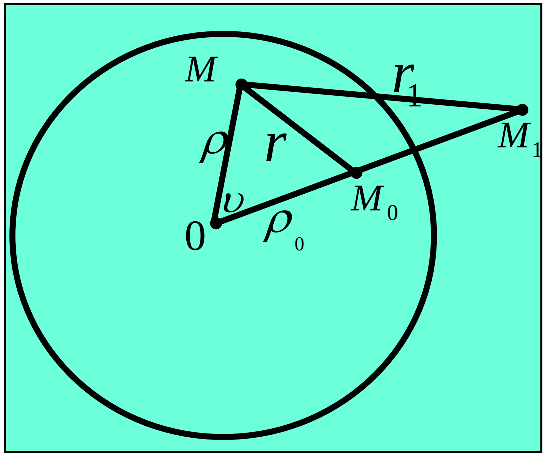
1、$\cos\gamma = ?$

设 $\vec{I}$ 为 $\vec{OM}$ 方向单位向量, $\vec{I}_0$ 为 $\vec{OM}_0$ 方向单位向量

则 $\vec{I} = x\vec{i} + y\vec{j} + z\vec{k} = \sin\theta\cos\varphi\,\vec{i} + \sin\theta\sin\varphi\,\vec{j} + \cos\theta\,\vec{k}$

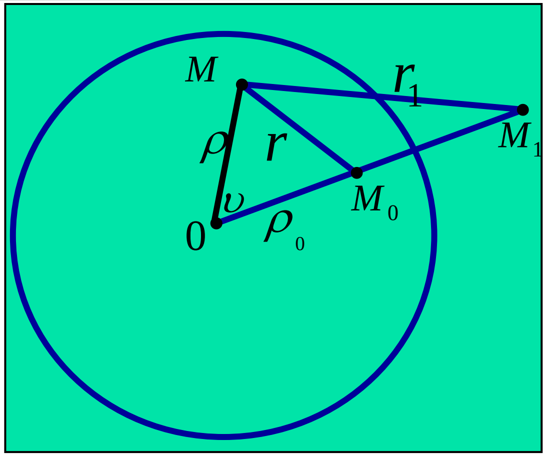
$$\vec{I}_0 = x_0\vec{i} + y_0\vec{j} + z_0\vec{k} = \sin\theta_0\cos\varphi_0\vec{i} + \sin\theta_0\sin\varphi_0\vec{j} + \cos\theta_0\vec{k}$$

$$\begin{aligned}\therefore \vec{I}\cdot\vec{I}_0 &= \left|\vec{I}\cdot\vec{I}_0\right|\cos\gamma = \cos\gamma \\ &= \sin\theta\cos\varphi\sin\theta_0\cos\varphi_0 \\ &+ \sin\theta\sin\varphi\sin\theta_0\sin\varphi_0 \\ &+ \cos\theta\cos\theta_0\end{aligned}$$



四、注释

2、对于球外的狄氏问题：



$$\begin{cases} \Delta u_1 = 0 \ , \ \rho > a \\ u_1 |_{\rho=a} = f(M) \end{cases}$$

$$u_1(M) = -u(M)$$

$$\boxed{u_1(M) = \frac{-a}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(\theta_0,\varphi_0)\frac{a^2-\rho^2}{\left(a^2+\rho^2-2a\rho\cos\gamma\right)^{\frac{3}{2}}}\sin\theta_0 d\theta_0 d\varphi_0}$$

四、注释

3、对于圆内的狄氏问题：

$$\begin{cases}\Delta u = 0\ ,\ \rho < a \\ u|_{\rho=a} = f(\varphi)\end{cases}$$

$$G = \frac{1}{2\pi}\ln\frac{1}{r} + g$$

$$\begin{cases}\Delta g = 0,\ M \in \sigma \\ g|_{\rho=a} = -\frac{1}{2\pi}\ln\frac{1}{r}|_{\rho=a}\end{cases}$$

$$\boxed{u(M) = \frac{1}{2\pi}\int_0^{2\pi} f(\varphi_0)\frac{a^2-\rho^2}{a^2+\rho^2-2a\rho\cos(\varphi-\varphi_0)}d\varphi_0}$$

4、对于圆外的狄氏问题：

$$\begin{cases}\Delta u_1 = 0\ ,\ \rho > a \\ u_1|_{\rho=a} = f(\varphi)\end{cases}$$

$$u_1(M) = -u(M)$$

$$\boxed{u(M) = \frac{1}{2\pi}\int_0^{2\pi} f(\varphi_0)\frac{\rho^2-a^2}{a^2+\rho^2-2a\rho\cos(\varphi-\varphi_0)}d\varphi_0}$$

五、小结

$$\Delta G = -\delta(M - M_0) \rightarrow \quad \boxed{G = \frac{1}{4\pi r}}$$

$$\begin{cases} \Delta G = -\delta(M - M_0), M \in \tau \\ G|_\sigma = 0 \end{cases} \rightarrow \boxed{G(M, M_0) = \frac{1}{4\pi r} + g}$$

$$\rho < a: \quad \boxed{G = \frac{1}{4\pi r} - \frac{a/\rho_0}{4\pi r_1}}$$

$$\boxed{\begin{cases} \Delta g = 0 \\ g|_\sigma = -\frac{1}{4\pi r}|_\sigma \end{cases}}$$

$$\begin{cases} \Delta u = 0, \quad \rho < a \\ u|_{\rho=a} = f(M) \end{cases}$$

$$\boxed{u(M) = \frac{a}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(\theta_0, \varphi_0)\frac{a^2 - \rho^2}{\left(a^2 + \rho^2 - 2a\rho\cos\gamma\right)^{\frac{3}{2}}}\sin\theta_0 d\theta_0 d\varphi_0}$$

本节作业

习题 10.3：1；

习题 10.4：1；2 ；3

Good-by!