数学物理方法

Methods in Mathematical Physics

第十五章 贝塞尔函数

Bessel Function

武汉大学物理科学与技术学院

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Bessel Function

§15.2 贝塞尔函数的性质
Properties of Bessel Function

一、母函数关系式

$$e^{\frac{x}{2}(t-\frac{1}{t})}=\sum_{n=-\infty}^{\infty}J_n(x)t^n \qquad (1)$$

$$e^z=\sum_{k=0}^{\infty}\frac{1}{k!}z^k,\quad |z|<\infty$$

证明：$\because e^{\frac{x}{2}t}=\sum_{l=0}^{\infty}\frac{1}{l!}(\frac{x}{2}t)^l,\quad |t|<\infty$

$$e^{-\frac{x}{2t}}=\sum_{m=0}^{\infty}\frac{1}{m!}(-\frac{x}{2t})^m,\quad |t|>0$$

$$e^{\frac{x}{2}(t-\frac{1}{t})}=e^{\frac{x}{2}t}\cdot e^{-\frac{x}{2t}}=\sum_{l=0}^{\infty}\frac{1}{l!}(\frac{x}{2}t)^l\cdot\sum_{m=0}^{\infty}\frac{1}{m!}(-\frac{x}{2t})^m\cdot$$

$$=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m}{l!m!}(\frac{x}{2})^{l+m}t^{l-m}$$

一、母函数关系式

$$e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m}{l!m!}(\frac{x}{2})^{l+m}t^{l-m}$$

令 $l-m=n$, 则$l=m+n$

$$\to\sum_{l=0}^{\infty}\to\sum_{m+n=0}^{\infty}\to\sum_{n=-m}^{\infty}\to\sum_{n=-\infty}^{\infty}$$

$$e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty}\underline{\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+n)!m!}(\frac{x}{2})^{2m+n}}t^n$$

$$= \sum_{n=-\infty}^{\infty}J_n(x)t^n$$

$$\boxed{J_n(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(n+k)!}(\frac{x}{2})^{2k+n}}$$

一、母函数关系

$$f(z)=\sum_{k=-\infty}^{\infty}c_k(z-b)^k,\ c_k=\frac{1}{2\pi i}\oint_l\frac{f(z)}{(z-b)^{k+1}}dz$$

$$e^{\frac{x}{2}(t-\frac{1}{t})}=\sum_{n=-\infty}^{\infty}J_n(x)t^n \qquad (1)$$

问：1.$J_n(x)$的积分表示？

$$\boxed{J_n(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(x\sin\theta-n\theta)}d\theta}$$

$$\boxed{\text{或}\quad J_n(x)=\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta-n\theta)d\theta}$$

2.$J_n(x)$的微分式？

3.$J_\nu(x)$（$\nu\neq n$）有母函数关系吗？

$$J_\nu(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{2k+\nu}\qquad(4)$$

二、递推公式：

$$\begin{cases}\dfrac{d}{dx}[x^\nu J_\nu(x)]=x^\nu J_{\nu-1}(x) & (2)\\ \dfrac{d}{dx}[x^{-\nu} J_\nu(x)]=-x^{-\nu} J_{\nu+1}(x) & (3)\end{cases}$$

用途：（1）可派生出其他递推公式

$$xJ_\nu'(x)+\nu J_\nu(x)=xJ_{\nu-1}(x)\qquad(4)$$

$$xJ_\nu'(x)-\nu J_\nu(x)=-xJ_{\nu+1}(x)\qquad(5)$$

$$2J_\nu'(x)=J_{\nu-1}(x)-J_{\nu+1}(x)\qquad(6)$$

$$\frac{2\nu}{x}J_\nu(x)=J_{\nu-1}(x)+J_{\nu+1}(x)\qquad(7)$$

$\xleftarrow{?}\ \boxed{\nu=n}$

（2）只要查 $J_0(x)$ 和 $J_1(x)$ 表，可计算出任一 $J_\nu(x)$

二、递推公式：

$$2J'_\nu(x)=J_{\nu-1}(x)-J_{\nu+1}(x) \quad (6)$$

$$\begin{cases} \dfrac{d}{dx}[x^\nu J_\nu(x)] = x^\nu J_{\nu-1}(x) & (2) \\ \dfrac{d}{dx}[x^{-\nu} J_\nu(x)] = -x^{-\nu} J_{\nu+1}(x) & (3) \end{cases}$$

用途：（3）可用来计算含 $J_\nu(x)$ 的积分

例1： $\int_0^a x^3 J_0(x)dx = ?$ $\quad a^3 J_1(a) - 2a^2 J_2(a)$

例2： $\int J_1(x)dx = ?$ $\quad -J_0(x)+c$ $\quad J_1(x) = -J'_0(x)$

例3： $\int J_3(x)dx = ?$ $\quad -J_0(x)-2J_2(x)+c$

三、正交性

$$\boxed{\int_0^a \rho J_n(k_m^n\rho)J_n(k_l^n\rho)d\rho=\frac{a^2}{2}J_{n+1}^2(k_l^n a)\delta_{ml} \quad (8)}$$

证明：$\because \ \rho^2 R''(\rho)+\rho R'(\rho)+(k^2\rho^2-n^2)R(\rho)=0$

$$\rightarrow \frac{d}{d\rho}(\rho\frac{dR}{d\rho})+(k^2\rho \ -\frac{n^2}{\rho})R=0$$

$$\frac{d}{d\rho}[\rho\frac{dJ_n(k_m^n\rho)}{d\rho}]+[(k_m^n)^2\rho-\frac{n^2}{\rho}]J_n(k_m^n\rho)=0 \quad (9)$$

$$\frac{d}{d\rho}[\rho\frac{dJ_n(k_l^n\rho)}{d\rho}]+[(k_l^n)^2\rho-\frac{n^2}{\rho}]J_n(k_l^n\rho)=0 \quad (10)$$

$$J_n(k_m^n a)=0, m=1,2,\cdots,l,\cdots$$

三、正交性

证明：$$\frac{d}{d\rho}[\rho\frac{dJ_n(k_m^n\rho)}{d\rho}]+[(k_m^n)^2\rho-\frac{n^2}{\rho}]J_n(k_m^n\rho)=0 \quad (9)$$

$$\frac{d}{d\rho}[\rho\frac{dJ_n(k_l^n\rho)}{d\rho}]+[(k_l^n)^2\rho-\frac{n^2}{\rho}]J_n(k_l^n\rho)=0 \quad (10)$$

$$\int_0^a[(9)\cdot J_n(k_l^n\rho)-(10)\cdot J_n(k_m^n\rho)]d\rho:$$

$$[(k_m^n)^2-(k_l^n)^2]\int_0^a\rho J_n(k_m^n\rho)J_n(k_l^n\rho)d\rho$$

$$=\int_0^a \underline{J_n(k_m^n\rho)}\frac{d}{d\rho}\underline{[\rho\frac{dJ_n(k_l^n\rho)}{d\rho}]}d\rho-\int_0^a \underline{J_n(k_l^n\rho)}\frac{d}{d\rho}\underline{[\rho\frac{dJ_n(k_m^n\rho)}{d\rho}]}d\rho$$

$$=\rho J_n(k_m^n\rho)\frac{dJ_n(k_l^n\rho)}{d\rho}\Big|_0^a-\int_0^a\rho\frac{dJ_n(k_l^n\rho)}{d\rho}\frac{dJ_n(k_m^n\rho)}{d\rho}]d\rho$$

$$-\rho J_n(k_l^n\rho)\frac{dJ_n(k_m^n\rho)}{d\rho}\Big|_0^a+\int_0^a\rho\frac{dJ_n(k_m^n\rho)}{d\rho}\frac{dJ_n(k_l^n\rho)}{d\rho}]d\rho$$

三、正交性

$$xJ_\nu'(x)-\nu J_\nu(x)=-xJ_{\nu+1}(x)\quad(5)$$

$$[(k_m^n)^2-(k_l^n)^2]\int_0^a \rho J_n(k_m^n\rho)J_n(k_l^n\rho)d\rho$$

$$=\rho[J_n(k_m^n\rho)\frac{dJ_n(k_l^n\rho)}{d\rho}-J_n(k_l^n\rho)\frac{dJ_n(k_m^n\rho)}{d\rho}]\Big|_0^a$$

$$=0\quad(\because J_n(k_m^n a)=0, m=1,2,\cdots,l,\cdots)$$

1. 若 $m\neq l$: $\boxed{\int_0^a \rho J_n(k_m^n\rho)J_n(k_l^n\rho)d\rho=0}$

2. 若 $m=l$, 令 $m\to l$

$$\boxed{\int_0^a \rho J_n(k_m^n\rho)J_n(k_l^n\rho)d\rho}=\lim_{k_m^n\to k_l^n}\frac{aJ_n(k_m^n a)J_n'(k_l^n a)k_l^n}{(k_m^n)^2-(k_l^n)^2}$$

$$=\lim_{k_m^n\to k_l^n}\frac{a^2k_l^n J_n'(k_m^n a)J_n'(k_l^n a)}{2k_m^n}=\frac{a^2}{2}[J_n'(k_l^n a)]^2\boxed{=\frac{a^2}{2}J_{n+1}^2(k_l^n a)}$$

四、广义傅氏展开

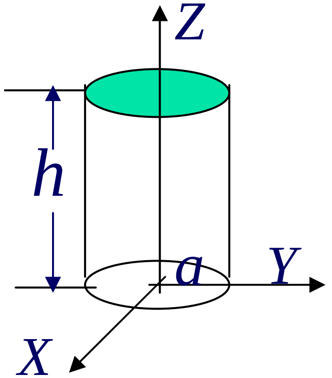
若$f(\rho)$在[0,a]上有连续的一阶导数，分段连续的二阶导数，且 $f(\rho)\big|_{\rho=0}\to$有界$,f(\rho)\big|_{\rho=a}=0$ 则

$$\boxed{f(\rho)=\sum_{m=1}^{\infty}c_m J_n(k_m^n\rho)}$$

$$\boxed{c_m=\frac{1}{\frac{a^2}{2}J_{n+1}^2(k_m^n a)}\int_0^a \rho f(\rho)J_n(k_m^n\rho)d\rho}$$

四、广义傅氏展开

例4：一半径为a高为h的均匀圆柱体，其下底和侧面保持温度为零度，上端温度为u_0，求柱内的稳定温度分布。



$$\begin{cases} \Delta u = 0,\ 0 \le \rho \le a, & (1) \\ u(a,z) = 0 & (2) \\ u(\rho,0) = 0 & (3) \\ u(\rho,h) = u_0 & (4) \end{cases}$$

解：

1. 令 $u(\rho,z) = R(\rho)Z(z)$

$$(1) \to \begin{cases} Z'' + \mu Z = 0 & (5) \\ \rho^2 R'' + \rho R' + (k^2\rho^2 - 0)R = 0 & (6) \end{cases}$$

$$(2) \to R(a) = 0 \quad (7); \qquad (3) \to Z(0) = 0 \quad (8)$$

2. 解本征值问题(6)(7)得

$$k^2 = -\mu = \left(\frac{x_m^0}{a}\right)^2,\ R_m(\rho) = J_0(k_m^0\rho),\ m = 1,2,\cdots$$

四、广义傅氏展开

解：3. 解方程（5）：

$$\begin{cases} Z'' + \mu Z = 0 & (5) \\ Z(0) = 0 & (8) \end{cases} \to Z_m(z) = c_m \sinh(k_m^0 z)$$

4. 叠加，定系数：

$$u(\rho, z) = \sum_{m=1}^{\infty} c_m \sinh(k_m^0 z) J_0(k_m^0 \rho)$$

$$\to \sum_{m=1}^{\infty} \underline{c_m \sinh(k_m^0 h)} J_0(k_m^0 \rho) = u_0$$

$$c_m = \frac{1}{\frac{a^2}{2} J_1^2(k_m^0 a) \sinh(k_m^0 h)} \int_0^a u_0 \rho J_0(k_m^0 \rho) d\rho$$

四、广义傅氏展开

$$\frac{d}{dx}[x^\nu J_\nu(x)] = x^\nu J_{\nu-1}(x) \qquad (2)$$

解：4. 叠加，定系数：

令 $x = k_m^0 \rho$

$$\int_0^a \rho J_0(k_m^0 \rho) d\rho = \frac{1}{(k_m^0)^2} \int_0^{(k_m^0 a)} x J_0(x) dx = \frac{a}{k_m^0} J_1(k_m^0 a)$$

问：$J_1(k_m^0 a) = 0$ ？

$$c_m = \frac{1}{\frac{a^2}{2} J_1^2(k_m^0 a) \sinh(k_m^0 h)} \int_0^a u_0 \rho J_0(k_m^0 \rho) d\rho$$

$$= \frac{2u_0}{(k_m^0 a) \sinh(k_m^0 h) J_1(k_m^0 a)}$$

$$u = \sum_{m=1}^{\infty} \frac{2u_0}{x_m^0} \frac{\sinh(k_m^0 z)}{\sinh(k_m^0 h)} \frac{J_0(k_m^0 \rho)}{J_1(k_m^0 a)}$$

五、小结（贝塞尔函数的性质）

（1）母函数关系式

$$e^{\frac{x}{2}(t-\frac{1}{t})}=\sum_{n=-\infty}^{\infty}J_n(x)t^n \qquad (1)$$

（2）递推公式：

$$\begin{cases}\dfrac{d}{dx}[x^\nu J_\nu(x)]=x^\nu J_{\nu-1}(x) & (2)\\ \dfrac{d}{dx}[x^{-\nu}J_\nu(x)]=-x^{-\nu}J_{\nu+1}(x) & (3)\end{cases}$$

（3）正交性

$$\int_0^a \rho J_n(k_m^n\rho)J_n(k_l^n\rho)d\rho=\frac{a^2}{2}J_{n+1}^2(k_l^n a)\delta_{ml} \qquad (8)$$

（4）广义傅氏展开

$$f(\rho)=\sum_{m=1}^{\infty}c_m J_n(k_m^n\rho)$$

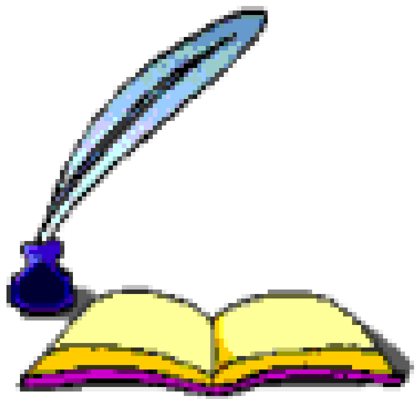
$$c_m=\frac{1}{\frac{a^2}{2}J_{n+1}^2(k_m^n a)}\int_0^a \rho f(\rho)J_n(k_m^n\rho)d\rho$$

本节作业

习题15.2 ： 2（3）（4）

7

8（1）



Good-by!