2019-2020《微积分 BII》期末考试

一、选择题(1-6 小题，每小题 3 分，共 18 分.)

1. 二元函数$f(x,y)$在点(x_0,y_0)的两个偏导数$f_x(x_0,y_0)$，$f_y(x_0,y_0)$存在，

则$f(x,y)$(　　)

(A)在该点可微　　(B)在该点连续

(C)在该点任意方向的方向导数都存在　　(D)以上都不对

2. 函数$f(x,y)=ax^2-y^2(a>0)$在(0,0)处(　　)

(A)不取极值　　(B)取极小值

(C)取极大值　　(D)是否取极值依赖于a

3. 若区域 D 为 $0\le y\le x^2$，$|x|\le 2$，则$\iint_D xy^2dxdy=$(　　)

(A) 0　　(B)$\frac{32}{3}$　　(C)$\frac{64}{3}$　　(D) 256

4. L 是曲线$y=x^2$上点(0,0)与(1,1)之间的一段弧，则$\int_L \sqrt{y}dS=$(　　)

(A)$\int_0^1\sqrt{1+(2x)^2}dx$　　(B)$\int_0^1 x\sqrt{1+(2x)^2}dx$

(C)　$\int_0^1 2x\sqrt{1+x^2}dx$　　(D)　$\int_0^1\sqrt{1+x^2}dx$

5. 下列级数是发散的为(　　)

(A)　$\sum_{n=1}^{\infty}\frac{\pi}{n^2}$　　(B)　$\sum_{n=1}^{\infty}sin\frac{\pi}{n^2}$　　(C)　$\sum_{n=1}^{\infty}cos\frac{\pi}{n^2}$

(D) $\sum_{n=1}^{\infty} \tan\frac{\pi}{n^2}$

6. 微分方程$y' + \frac{y}{x} = \frac{1}{x(x^2+1)}$ 的通解是（　）

(A) $arctanx + C$　　(B) $\frac{1}{x}(arctanx + C)$　　(C) $\frac{1}{x}arctanx + C$

(D) $arctanx + \frac{C}{x}$

二、填空题（共 6 小题，每小题 3 分，共 18 分）

7. 设$e^{\varepsilon} - xy\varepsilon = 0$，则$d\varepsilon =$ ________

8. $\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_0^{\sqrt{1-x^2-y^2}} dz =$________

9. 设 L：$x^2 + y^2 = 1$，取逆时针方向，则$\int_L -ydx + xdy =$ ________

10. 函数$z = xe^{2y}$在点 P(1, 0) 处沿着 P(1, 0) 到 Q(2, -1) 的方向导数为________

11. 设$\sum$ 为锥面$z = \sqrt{x^2+y^2}$介于$z = 0$ 和$z = 1$ 之间的部分，则$\iint_{\Sigma} (x^2+y^2)ds =$__

12. 已知 $f(x)$ 是以 2π 为周期的函数，在 $[-\pi,\pi)$ 上的表达式 $f(x) = \begin{cases} x+1, -\pi \le x < 0 \\ x-1, \quad 0 \le x < \pi \end{cases}$，以 S(x) 表示 f(x) 的傅立叶级数的和函数，则 S(3π) =________

三．解答题（共 7 小题）

13. (10 分) 求曲线$\begin{cases} x^2 + y^2 + z^2 = 6 \\ x + y + z = 0 \end{cases}$ 在 (1, -2, 1) 处切线和法平面方程

14. （10 分）设$z = f(xy^2, x^2y)$, 其中 f 具有二阶连续偏导数，求$\frac{\partial^2 z}{\partial x \partial y}$

15. （10 分）计算$\iint_{\Sigma} (x^3 + yz)dydz + (y^3 + xz)dzdx + dxdy$,

其中Σ 为锥面$z = \sqrt{x^2+y^2}$被 z=1 所截下方的部分下侧。

16. (10 分) 求幂级数$\sum_{n=1}^{\infty}\left(\frac{1}{2n+1}-1\right)x^{2n}$, 在区间 (-1, 1) 内的和函数

17. (10 分) 求常微分方程$y''-2y'+5y=xe^{x}$, 的通解

18. (8 分) 在椭球面$\frac{x^2}{5^2}+\frac{y^2}{3^2}+\frac{z^2}{2^2}=1$, 第一卦限上 P 点处作切平面，使切平面与三个坐标面所围四面体的体积最小，求 P 点坐标

19. (6 分) 判别$\sum_{n=1}^{\infty}\left[\frac{1}{n}-ln\frac{1+n}{n}\right]$的敛散性，并证明$\lim\limits_{n\to\infty}\frac{1+\frac{1}{2}+\circ\circ\circ+\frac{1}{n}}{\ln n}=1$

2019-2020《高等数学BII》参考答案

一、选择题

题号	1	2	3	4	5	6
答案	D	A	A	B	C	B

1. f_x, f_y连续$\Rightarrow f$可微$\Rightarrow \begin{cases} f连续 \Rightarrow \begin{cases} 极限存在 \\ 关于\ x,y\ 连续 \end{cases} \\ f_x, f_y\ 存在 \Rightarrow \begin{cases} 关于\ x,y\ 连续 \\ 沿坐标轴方向方向导数存在 \end{cases} \\ 沿任意方向导数存在 \end{cases}$

2. $f_x = 2ax, f_y = -2y, A = f_{xx} = 2a, B = f_{xy} = 0, C = f_{yy} = -2$

$AC - B^2 = -4a < 0$ 所以不取极值 所以选 A

3. xy^2是 x 的奇函数，D 关于 y 轴对称(边界曲线方程中$x \to -x$, 方程不变) 所以 I=0 所以选 A

4. $\int_L \sqrt{y}ds \Rightarrow \begin{cases} y = x^2 \Rightarrow y = |x| = x\ (0 \le x \le 1) \\ ds = \sqrt{1+y}dx = \sqrt{1+(2x)^2}dx \end{cases} \Rightarrow \int_0^1 x\sqrt{1+(2x)^2}dx$ 所以选 B

5. $\sum \frac{1}{n^2}$收敛$\Rightarrow \sum \frac{\pi}{n^2}$收敛，$\sin\frac{\pi}{n^2} \sim \frac{\pi}{n^2} \Rightarrow \sum sin\frac{\pi}{n^2}$收敛，$\tan\frac{\pi}{n^2} \sim \frac{\pi}{n^2} \Rightarrow \sum tan\frac{\pi}{n^2}$收敛

$\lim\limits_{n\to\infty} \cos\frac{\pi}{n^2} = \cos 0 = 1 \neq 0$ 所以$\sum \cos\frac{\pi}{n^2}$ 发散 所以选 C

6. $P(x) = -\frac{1}{x}$ $q(x) = \frac{1}{x(x^2+1)}$ $y = (\int q(x)e^{-\int P(x)dx}dx + C)e^{\int P(x)dx} =$

$(\int \frac{1}{x(x^2+1)}e^{\int \frac{1}{x}dx}dx + C)e^{-\int \frac{1}{x}dx} = (\int \frac{1}{x(x^2+1)}xdx + C)\frac{1}{x} = (\arctan x + C)\frac{1}{x}$ 选 B

二、填空题

7. $de^{\varepsilon} - d(xy\varepsilon) = 0 \quad \Rightarrow \quad e^{\varepsilon}d\varepsilon - xyd\varepsilon - x\varepsilon dy - y\varepsilon dx = 0 \quad \Rightarrow$

$d\varepsilon = \frac{y\varepsilon dx + x\varepsilon dy}{e^{\varepsilon} - xy} = \frac{y\varepsilon dx + x\varepsilon dy}{xy\varepsilon - xy} = \frac{\varepsilon}{x\varepsilon - x}dx + \frac{\varepsilon}{y\varepsilon - y}dy$

8. ①$z = 0, z = \sqrt{1-x^2-y^2}$(上半球面) D: $\begin{cases} 0 \le x \le 1 \\ 0 \le y \le \sqrt{1-x^2} \end{cases}$ $f = 1$

所以为$\frac{1}{8}$球的体积 $I = \frac{1}{8} \times \frac{4\pi}{3} \times 1^3 = \frac{\pi}{6}$

②$I = \int_0^1 dx \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}dy = \int_0^{\frac{\pi}{2}} d\theta \int_0^1 \sqrt{1-r^2}rdr$

$$= \frac{\pi}{2} \times -\frac{1}{2}\int_0^1 (1-r^2)^{\frac{1}{2}}d(1-r^2) = -\frac{\pi}{4} \times \frac{2}{3}(1-r^2)^{\frac{3}{2}}|_0^1 = \frac{\pi}{6}$$

9. $I = \iint\limits_D [1-(-1)]dxdy = 2 \times \pi \cdot 1^2 = 2\pi$

10. $\vec{l} = \overrightarrow{PQ} = (2-1,-1-0) = (1,-1)$ $\quad \vec{\vartheta} = (\frac{1}{\sqrt{1+1}}, \frac{-1}{\sqrt{2}}) = (\cos\alpha, \cos\beta)$

$$\nabla z = (z_x, z_y)_{(1,0)} = (e^{2y}, 2xe^{2y})_{(1,0)} = (1,2)$$

$$\frac{\partial z}{\partial l}\Big|_{(1,0)} = z_x \cos\alpha + z_y \cos\beta = 1 \times \frac{1}{\sqrt{2}} + 2 \times \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

11. $I = \iint_{Dxy} (x^2+y^2)\cdot\sqrt{2}dxdy = \int_0^{2\pi} d\theta \int_0^1 r^2 \cdot \sqrt{2} r dr = 2\pi \cdot \frac{\sqrt{2}r^4}{4}\Big|_0^1 = \frac{\sqrt{2}}{2}\pi$

$$ds = \sqrt{1+z_x^2+z_y^2}dxdy = \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}dxdy = \sqrt{2}dxdy$$

12. $S(3\pi) = S(3\pi-2\pi) = S(\pi) = \frac{f(\pi-0)+f(-\pi+0)}{2} = \frac{\pi-1+(-\pi+1)}{2} = 0$

三、解答题

13. $F = x^2+y^2+z^2-6 \quad F_x = 2x, F_y = 2y, F_z = 2z$

$$G = x+y+z \qquad G_x = 1, G_y = 1, G_z = 1$$

$$\vec{n} = \begin{vmatrix} i & j & k \\ F_x & F_y & F_z \\ G_x & G_y & G_z \end{vmatrix}_{(1,-2,1)} = \begin{vmatrix} i & j & k \\ 2 & -4 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 6(-1,0,1)$$

$\therefore$ 切线：$\frac{x-1}{-1} = \frac{y+2}{0} = \frac{z-1}{1}$，法平面：$-(x-1)+z-1=0$ 即 $x-z=0$

14. $\frac{\partial z}{\partial x} = f_1' \cdot y^2 + f_2' \cdot 2xy$

$$\frac{\partial^2 z}{\partial x \partial y} = 2y \cdot f_1' + y^2[f_{11}'' \cdot 2xy + f_{12}'' \cdot x^2] + 2x \cdot f_2' + 2xy[f_{21}'' \cdot 2xy + f_{22}'' \cdot x^2]$$

$$= 2yf_1' + 2xf_2' + 2xy^3 f_{11}'' + 5x^2y^2 f_{12}'' + 2x^3 y f_{22}''$$

15. $I=\Sigma+\Sigma_0-\iint_{\Sigma_0}=\iiint\ (3x^2+3y^2)dV-\iint_D\ dxdy=$
$\int_0^{2\pi}d\theta\int_0^1 rdr\int_r^1 3r^2dz-\pi$

$$=2\pi\int_0^1(3r^3-3r^4)dr-\pi=2\pi^3\left[\frac{r^4}{4}-\frac{r^5}{5}\right]\Big|_0^1-\pi=\frac{3\pi}{10}-\pi=-\frac{7\pi}{10}$$

16. $S(x)=\sum_{n=1}^{\infty}(\frac{1}{2n+1}-1)x^{2n}=\sum_{n=1}^{\infty}\frac{1}{2n+1}x^{2n}-\sum_{n=1}^{\infty}x^{2n}=S_1(x)-S_2(x)$

① $S_1(x)=\sum_{n=1}^{\infty}\frac{1}{2n+1}x^{2n}=\frac{1}{x}\sum_{n=1}^{\infty}\frac{x^{2n+1}}{2n+1}\ (x\neq 0)\quad g(x)=\sum_{n=1}^{\infty}\frac{x^{2n+1}}{2n+1}\ ,g(0)=0$

$$g'(x)=(\sum_{n=1}^{\infty}\frac{x^{2n+1}}{2n+1})'=\sum_{n=1}^{\infty}(\frac{x^{2n+1}}{2n+1})'=\sum_{n=1}^{\infty}\ x^{2n}\ =\frac{x^2}{1-x^2}$$

$$g(x)=g(0)+\int_0^x g'(x)\,dx$$

$$=\int_0^x\frac{x^2-1+1}{1-x^2}dx=\int_0^x(-1+\frac{1}{1-x^2})\,dx$$

$$=-x+\frac{1}{2}ln(\frac{1+x}{1-x})$$

$\therefore x\neq 0$ 时 $S_1(x)=-1+\frac{1}{2x}ln(\frac{1+x}{1-x})$

② $S_2(x)=\sum_{n=1}^{\infty}x^{2n}\ =\frac{x^2}{1-x^2}\qquad S(0)=0\quad |x|<1\ \rightarrow\ x\in(-1,0)\cup(0,1)$

所以$S(x)=\begin{cases}\frac{1}{2x}ln(\frac{1+x}{1-x})-1-\frac{x^2}{1-x^2}\\ 0\end{cases}$

$$\Rightarrow\begin{cases}\frac{\ln\left(\frac{1+x}{1-x}\right)}{2x}-\frac{1}{1-x^2} & x\in(-1,0)\cup(0,1)\\ 0 & x=0\end{cases}$$

17. ① $\lambda^2-2\lambda+5=0\Rightarrow(\lambda-1)^2+4=0\Rightarrow\lambda=1\pm 2i$

$\Rightarrow y=e^x\,(C_1\cos 2x+C_2\sin 2x)$

② $y^*=e^{ax}Q_m(x)\cdot x^k=e^x\cdot(ax+b)\quad Q_m(x)=ax+b$ （x 为一次多项式）

y^x代入原方程整理得：$Q''+(2\lambda+P_1)Q'+(\lambda^2+P_1\lambda+P_2)Q=x$

$$\begin{cases}Q=ax+b\\ Q'=a\\ Q''=0\end{cases}\qquad\begin{cases}\lambda=1\\ P_1=-2\\ P_2=5\end{cases}$$

代入得：$(1-2+5)(ax+b)=x\quad\therefore\begin{cases}4a=1\\4b=0\end{cases}\Rightarrow\begin{cases}a=\frac{1}{4}\\ b=0\end{cases}\quad\therefore y^*=\frac{x}{4}e^x$

$\therefore y=e^x\,(C_1\cos 2x+C_2\sin 2x)+\frac{x}{4}e^x$ 为非齐次通解

18. 设 $P(x,y,z)\quad\vec{n}=\left(\frac{2x}{5^2},\frac{2y}{4^2},\frac{2z}{3^2}\right)$ 切平面：$\frac{x}{25}(X-x)+\frac{y}{9}(Y-y)+\frac{z}{4}(Z-z)=0$

整理得：$\frac{x}{25}X+\frac{y}{9}Y+\frac{z}{4}Z-1=0$

$\therefore V=\frac{1}{6}\cdot\frac{25}{x}\cdot\frac{9}{y}\cdot\frac{4}{z}=\frac{150}{xyz}\qquad L=xyz+\lambda\left(\frac{x^2}{25}+\frac{y^2}{9}+\frac{z^2}{4}-1\right)$

$$\begin{cases} L_x = yz + \frac{2x}{25}\lambda = 0 \\ L_y = xz + \frac{2y}{9}\lambda = 0 \\ L_z = xy + \frac{2z}{4}\lambda = 0 \\ \frac{x^2}{25} + \frac{y^2}{9} + \frac{z^2}{4} = 1 \end{cases} \Rightarrow \begin{cases} xyz + \frac{2x^2}{25}\lambda \\ xyz + \frac{2y^2}{9}\lambda \\ xyz + \frac{2z^2}{4}\lambda \end{cases} \Rightarrow \begin{cases} \frac{2x^2}{25}\lambda = \frac{2y^2}{9}\lambda = \frac{2z^2}{4}\lambda \\ \frac{x^2}{25} = \frac{y^2}{9} = \frac{z^2}{4} \\ \frac{x^2}{25}\cdot 3 = 1 \Rightarrow x = \frac{5}{\sqrt{3}}, y = \frac{3}{\sqrt{3}}, z = \frac{2}{\sqrt{3}} \end{cases}$$

$\therefore P(\frac{5}{\sqrt{3}}, \frac{3}{\sqrt{3}}, \frac{2}{\sqrt{3}})$

19. ①$ln\left(\frac{1+n}{n}\right) = ln(\frac{1}{n}+1) = \frac{1}{n} - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)$

$\therefore \lim\limits_{n\to\infty} \frac{\frac{1}{n} - ln\left(\frac{1+n}{n}\right)}{\frac{1}{n^2}} = \lim\limits_{n\to\infty} \frac{\frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}} = \frac{1}{2} > 0$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$收敛 ∴原级数收敛

②$\therefore S_n = 1 + \frac{1}{2} + \circ\circ\circ + \frac{1}{n} - \left[\ln\frac{2}{1} + \ln\frac{3}{2} + \ldots + ln\left(\frac{1+n}{n}\right)\right]$

$$= 1 + \ldots + \frac{1}{n} - \ln(n+1) \to A(n \to \infty)$$

$\therefore \lim\limits_{n\to\infty} \frac{1 + \circ\circ\circ + \frac{1}{n}}{\ln n} = \lim\limits_{n\to\infty} \frac{1 + \ldots + \frac{1}{n} - \ln(n+1) + \ln(n+1)}{ln\, n} =$

$\lim\limits_{n\to\infty} \left(\frac{1 + \ldots + \frac{1}{n} - \ln(n+1)}{ln\, n} + \frac{\ln(n+1)}{ln\, n}\right)$

$$= 0 + \lim_{x\to\infty} \frac{\ln(x+1)}{ln\, x} = \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{x}} = 1$$