Linear Algebra II

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1 Quotient and dual spaces

Definition 1 (Quotient spaces)**.** Let V be a vector space and let W be its subspace. Define a equivalence relation on V such that

$$v_1 \sim v_2 \text{ if } v_1 - v_2 \in W.$$

It is easy to verify that $\sim$ is indeed a equivalence relationship on V. For each $v_0 \in V$, define $[v_0] = \{v \in V : v \sim v_0\}$ the equivalence class of v_0. Then, $\{[v] : v \in V\}$ is called the quotient space V/W.

Remark. The quotient space V/W is equipped with a natural vector (linear) structure, namely,

$$\begin{cases} [v_1] + [v_2] = [v_1 + v_2] & \text{, for all } v_1, v_2 \in V \\ c[v_1] = [cv_1] & \text{, for all } v_1 \in V \text{ and } c \in \mathbb{F} \end{cases}.$$

Although it is crucial that we shall check these natural addition and scalar multiplication are "well-defined", we omitted here.

Definition 2 (Quotient maps)**.** There is a natural surjective map

$$\begin{aligned} \pi : V &\to V/W \\ v &\mapsto [v] \end{aligned},$$

which is called the quotient map. Moreover, it is a linear transformation.

Remark.

$$\begin{aligned} \ker \pi &= \{v \in V : \pi(v) = [0]\} \\ &= \{v \in V : [v] = [0]\} \\ &= \{v \in V : v - 0 \in W\} \\ &= W \ . \end{aligned}$$

Corollary. It follows from the dimension formula that $\dim_{\mathbb{F}} V/W = \dim_{\mathbb{F}} V - \dim_{\mathbb{F}} W$ whenever V is finite dimensional.

Here we give an alternative proof without using dimensional formula. Since V has finite dimension, let $\mathcal{B} = \{w_1, w_2, \ldots, w_s\}$ be a basis of W and extend $\mathcal{B}$ to $\mathcal{A} = \{w_1, w_2, \ldots, w_r\}$ a basis of V. We claim that $\{[w_{s+1}], \ldots, [w_s]\}$ is a basis of V/W. To see this, we shall show that:

1. $\{[w_{s+1}], \ldots, [w_r]\}$ generate V/W.
   Suppose $[v] \in V/W$. Let $v = \sum_{i=1}^{r} \alpha_i w_i$, then

$$[v] = \left[\sum_{i=s+1}^{r} \alpha_i w_i\right] = \sum_{i=s+1}^{r} \alpha_i [w_i] \quad .$$

2. $\{[w_{s+1}], \dots, [w_r]\}$ are linear independent over $\mathbb{F}$.
Suppose $\sum_{i=s+1}^{r} \alpha_i \cdot [w_i] = [0]$, for some $\alpha_i \in \mathbb{F}$. Then,

$$\begin{aligned} & \left[\sum_{i=s+1}^{r} \alpha_i w_i \right] = [0] \\ \iff & \sum_{i=s+1}^{r} \alpha_i w_i \in W \\ \iff & \sum_{i=s+1}^{r} \alpha_i w_i = \sum_{j=1}^{s} \beta_j w_j, \text{ for some } \beta_j \in \mathbb{F}. \end{aligned}$$

We conclude that α_i are all zeros, since $\mathcal{A}$ is a basis of V.

Discussions above show that $\dim_{\mathbb{F}} V/W = r - s = \dim_{\mathbb{F}} V - \dim_{\mathbb{F}} W$. Now, we shall study some property about the quotient space V/W. The next theorem characterize the quotient space V/W by the following universal property.

Theorem 3. *Let T be a linear transformation from V to U, such that $\ker T$ contain W, namely $W \subset \ker T$. Then, T factors through π uniquely. That is, there exists a unique linear transformation $S : V/W \to U$ such that*

$$T = S \circ \pi.$$

Proof. Define $S : V/W \to U$ by

$$S([v]) = T(v).$$

We first show that S is a well-defined map, namely, if $[v] = [v']$, then $T(v) = T(v')$. Note that $[v] = [v'] \implies v - v' \in W \subset \ker T$, we conclude $T(v) = T(v')$. By definition, S is a linear transformation and $S \circ \pi = T$. The uniqueness of such S follows from the surjectivity of π. □

Remark. The quotient space V/W with the quotient map π is the unique vector space satisfying the theorem. That is, if we are given $\pi' : V \to V'$ satisfying the property: for every linear transformation $T : V \to U$ with $W \subset \ker T$, there exists a unique $S' : V' \to U$ such that $S' \circ \pi' = T$. Then, $V' \simeq V/W$ uniquely.

Proof. From the assumptions, we have

$$\begin{cases} \exists!\ S : V/W \to V', \text{ such that } \pi' = S \circ \pi \\ \exists!\ S' : V' \to V/W, \text{ such that } \pi = S' \circ \pi' \end{cases}.$$

This shows $S \circ S' = \mathrm{Id}_{V'}$; $S' \circ S = \mathrm{Id}_{V/W}$ (using Theorem 3 again.) We conclude $V' \simeq V/W$ uniquely. □

Corollary. Let $T : V \to W$ be a linear transformation. Then,

$$V/\ker T \simeq \mathrm{Im}T.$$

Hence, $\dim_{\mathbb{F}} V/\ker T = \dim_{\mathbb{F}} \mathrm{Im}T$.

Proof. From Theorem 3, we have: there exists a unique $S : V/\ker T \to W$, such that $T = S \circ \pi$. It follows from the surjectivity of π that $\mathrm{Im}S = \mathrm{Im}T$. We claim that S is injective. Note that

$$\begin{aligned} \ker S &= \{[v] \in V/\ker T : S([v]) = 0\} \\ &= \{[v] \in V/\ker T : T(v) = 0\} \\ &= \{[v] \in V/\ker T : v \in \ker T\} \\ &= \{[0]\}. \end{aligned}$$

Thus, S is a bijection. This completes the proof. $\square$

Now, let $T : V \to V$ be a linear transformation and let $W \subset V$ be a T-invariant subspace. Then, T introduce a linear transformation $\widetilde{T}$ on V/W define by:

$$\begin{aligned} \widetilde{T} : V/W &\to V/W \\ [v] &\mapsto [T(v)] \end{aligned}.$$

This is a well-defined map since $[v] = [v'] \implies v - v' \in W$, then we have $T(v) - T(v') = T(v - v') \in W \implies [T(v)] = [T(v')]$

2 Inner product spaces

Definition 4 (inner product)**.** Let V be a vector space over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ is called an inner product if the following conditions are satisfied:

1. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$, for all $x, y, z \in V$.
2. $\langle cx, y \rangle = c \cdot \langle x, y \rangle$, for all $x, y \in V$ and $c \in \mathbb{F}$.
3. $\langle x, y \rangle = \overline{\langle y, x \rangle}$, for all $x, y \in V$.
4. $\langle x, x \rangle \geq 0$, for all $x \in V$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

We write $(V, \langle \cdot, \cdot \rangle)$ for a vector space V together with an inner product structure $\langle \cdot, \cdot \rangle$.

We could also define the concept of norm or length of a vector $v \in V$.

Definition 5 (norm)**.** For each $v \in V$, define the norm of v as $\|v\| = \langle v, v \rangle^{1/2}$.

Theorem 6 (Riesz representation Theorem in a finite dimensional space)**.** *Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Then,*

$$\begin{aligned} \Phi : V &\to \check{V} \\ v &\mapsto \Phi(v)(x) = \langle x, v \rangle \end{aligned}$$

is an isomorphism.

Proof. We first prove that Φ is injective. Note that

$$\ker \Phi = \{v \in V : \langle x, v \rangle = 0, \text{ for all } x \in V\} = \{0\}.$$

Since V is finite dimensional, we have $\dim_{\mathbb{F}} V = \dim_{\mathbb{F}} \check{V}$, thus Φ is an isomorphism. $\square$

In other words, inner product $\langle \cdot, \cdot \rangle$ identifies V with its dual space $\check{V}$ when V is finite dimensional. We now start study how to represent an inner product structure with a matrix. Suppose V is a finite dimensional vector space, and let $\mathcal{A} = \{v_1, v_2, \ldots, v_n\}$ be a basis of V. For any $x, y \in V$, there exist α_i, β_i such that

$$x = \sum_{i=1}^{n} \alpha_i \cdot v_i ; \quad y = \sum_{j=1}^{n} \beta_j \cdot v_j .$$

Then,

$$\langle x, y\rangle = \left\langle \sum_{i=1}^{n} \alpha_i \cdot v_i, \sum_{j=1}^{n} \beta_j \cdot v_j \right\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \overline{\beta_j} \langle v_i, v_j\rangle .$$

Hence, if we let

$$\Omega = \big(\langle v_i, v_j\rangle \big) \in M_n(\mathbb{F}),$$

we have

$$\langle x, y\rangle = \begin{pmatrix}\alpha_1 & \alpha_2 & \dots & \alpha_n\end{pmatrix} \cdot \Omega \cdot \begin{pmatrix}\overline{\beta_1}\\ \overline{\beta_2}\\ \vdots \\ \overline{\beta_n}\end{pmatrix}.$$

The matrix Ω is called the matrix of $\langle\ ,\ \rangle$ associated with $\mathcal{A}$.

Theorem 7 (change of basis)**.** *Let $\mathcal{B} = \{w_1, \dots, w_n\}$ be another basis of V. Assume that*

$$w_j = \sum_{i=1}^{n} a_{ij} v_i, \text{ for all } 1 \le j \le n.$$

Then,

$$\Omega' = A^{\mathrm{T}} \cdot \Omega \cdot \overline{A},$$

where Ω' is the matrix of $\langle\ ,\ \rangle$ associated with $\mathcal{B}$ and $A = (a_{ij})$.

Proof. Note that

$$\begin{aligned}\langle w_i, w_j\rangle &= \left\langle \sum_{k=1}^{n} a_{ki} v_k, \sum_{l=1}^{n} a_{lj} v_l \right\rangle \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n} a_{ki} \langle v_k, v_l\rangle \overline{a_{lj}} \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n} {a_{ik}}^{\mathrm{T}} \langle v_k, v_l\rangle \overline{a_{lj}},\end{aligned}$$

This proves the theorem. $\square$

Next, we shall ask whether we can define an inner product structure on V if we are given a matrix $\Omega \in M_n(\mathbb{F})$ and a basis $\mathcal{A}$ of V. The answer is no. In fact, the matrix can define an inner product structure on finite dimensional V if and only if it is positive definite. However,

Theorem 8. *If $\Omega = B \cdot B^*$ for some $B \in M_n(F)$ with $\det B \neq = 0$, then $\langle\ ,\ \rangle_{\Omega,\mathcal{A}}$ is an inner product for any choice of $\mathcal{A}$.*

Proof. Let $\mathcal{A} = \{v_1, v_2, \ldots, v_n\}$ be an arbitrary basis of V. It suffices to show the inner product defined by Ω satisfies the fourth axiom of Definition 4. If $x \in V$, then

$$x = \sum_{i=1}^{n} \alpha_i \cdot v_i, \text{ for some } \alpha_i \in \mathbb{F}.$$

We have

$$\begin{aligned}
\langle x, x \rangle_{\Omega,\mathcal{A}} &:= \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{pmatrix} \cdot \Omega \cdot \begin{pmatrix} \overline{\alpha_1} \\ \overline{\alpha_2} \\ \vdots \\ \overline{\alpha_n} \end{pmatrix} \\
&= \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{pmatrix} \cdot B \cdot B^* \cdot \begin{pmatrix} \overline{\alpha_1} \\ \overline{\alpha_2} \\ \vdots \\ \overline{\alpha_n} \end{pmatrix} \\
&= (yB) \cdot (yB)^*,
\end{aligned}$$

where $y = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{pmatrix}$ is a row vector. Write $yB = \begin{pmatrix} \beta_1 & \beta_2 & \ldots & \beta_n \end{pmatrix}$. We get

$$\langle x, x \rangle_{\Omega,\mathcal{A}} = \begin{pmatrix} \beta_1 & \beta_2 & \ldots & \beta_n \end{pmatrix} \cdot \begin{pmatrix} \overline{\beta_1} \\ \overline{\beta_2} \\ \vdots \\ \overline{\beta_n} \end{pmatrix} = \sum_{i=1}^{n} |\beta_i|^2 \geq 0,$$

and $\langle x, x \rangle_{\Omega,\mathcal{A}} = 0$ if and only if $y = 0$. From the assumption that $\det B \neq 0$, it follows $x = 0$ if $\langle x, x \rangle = 0$. $\square$