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Functions of Bounded Variation

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0 Introduction

Since last semester, I have studied *Advanced Calculus* in Department of Mathematics, Tsing Hua University. In the course, *Principles of Mathematical Analysis* [2] was assigned to be our textbook and we learned the definition of the Riemann-Stieltjes integral in the book. However, the book only introduces the Riemann-Stieltjes integral of a function f with respect to a monotonically increasing function α. After some discussions with my advisor, I discovered that there is a more general definition of Riemann-Stieltjes integral, which does not have any requirements on α. Moreover, I found that the text *Mathematical Analysis*[1] written by Apostol, T.M. contains more details concerning the case when α is of bounded variation. This gives me the motivation to explore the properties of functions of bounded variation. In this article, it is separated into two parts, first of which contains some vital ideas of bounded variation functions. The remaining part includes some problems provided in the text *Mathematical Analysis.*

1 Propositions of Bounded Variation Functions

In the beginning of this section, we will introduce some concepts of functions of bounded variation. For brevity and clarity, we should introduce some notations and terminologies to avoid misunderstanding. We confine our attention to real-valued functions defined on bounded interval like $[a, b]$. Unless otherwise stated, $f, g, h, \ldots$ would stand for such functions mentioned above.

Definition 1 (partitions and refinements)**.** Let $[a, b]$ be a given bounded interval. A set of finite points

$$P = \{x_0, x_1, \ldots, x_n\},$$

which satisfies

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b,$$

is called a partition of $[a, b]$. We usually write

$$\Delta x_i = x_i - x_{i-1} \quad (i = 1, \ldots, n).$$

A partition P^* is called a refinement of P if $P^* \supset P$.

Definition 2 (functions of bounded variation)**.** Let f be a real-valued function defined on $[a, b]$, and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Denote

$$\Delta f_i = f(x_i) - f(x_{i-1}) \quad (i = 1, \ldots, n).$$

If there exists a positive number M such that

$$\sup \sum_{i=1}^{n} |\Delta f_i| \leq M,$$

where the supremum is taken among all partitions of $[a, b]$, then f is said to be of bounded variation on $[a, b]$, or briefly speaking, bounded variation function (BV function).

Now, some results immediately follow from the definition, as shown in the next two theorems.

Theorem 3. *If f is monotonic, then f is of bounded variation.*

Proof. For every partition of $[a,b]$ we have

$$\begin{cases} \Delta f_i \geq 0, \text{if } f \text{ is increasing} \\ \Delta f_i \leq 0, \text{if } f \text{ is decreasing} \end{cases}.$$

Hence, we get

$$\sum_{i=1}^{n} |\Delta f_i| = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \left| \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \right| = |f(b) - f(a)| .$$

This completes the proof. □

Theorem 4. *If f is continuous on $[a,b]$ and if f' exists and is bounded in (a,b), then f is of bounded variation.*

Proof. Since f' is bounded in (a,b), there is a positive number A such that $|f'(x)| \leq A$. For every partition of $[a,b]$, it follows by Mean Value Theorem that there exist $t_i \in (x_{i-1}, x_i)$, such that $f(x_i) - f(x_{i-1}) = f'(t_i)(x_i - x_{i-1})$. This gives

$$|\Delta f_i| \leq A(x_i - x_{i-1}),$$

and

$$\sum_{i=1}^{n} |\Delta f_i| \leq A(b-a),$$

which implies f is of bounded variation. □

In both cases discussed above, we have shown some sufficient conditions for functions to be of bounded variation if the functions feature the requirements. Next theorem demonstrates what necessary condition needs to be satisfied if a function is of bounded variation.

Theorem 5. *If f is a bounded variation function, then f is bounded.*

Proof. Since f is of bounded variation, there is a positive number M such that $\sum |\Delta f_i| \leq M$. To show f is bounded, we hope there is a positive number A such that $|f(x)| \leq A$ holds for every $x \in [a,b]$. Now, consider partition $P = \{a, x, b\}$(where $a < x \leq b$), the hypothesis give us

$$|f(x) - f(a)| + |f(b) - f(x)| \leq M.$$

This implies $|f(x) - f(a)| \leq M$, and it follows that

$$|f(x)| \leq |f(a)| + M. \tag{1}$$

Note that (1) also holds for $x = a$, thus it holds for all $x \in [a,b]$. □

After being acquainted with some bounded variation functions, we shall proceed to other more important topics and find characterizations of bounded variation functions. However, we have to introduce "total variation" of a bounded variation function in order to know more about this kind of functions.

Definition 6 (total variation)**.** Let f be a bounded variation function on $[a, b]$, and let P be a partition of $[a, b]$. We write

$$S(f;P) = \sum_{i=1}^{n} |\Delta f_i| \,.$$

The number

$$V_f(a,b) = \sup \{S(f;P) : P \text{ is a partition of } [a,b]\}$$

is called the total variation of f on the interval $[a, b]$. Sometimes, the notation will be shortened to V_f when there is no ambiguity.

It is worth to note that V_f must be finite, since f is of bounded variation. In the following discussions, we are going to study some properties of total variation as a function of f, in other words, we study how does $V_f(a, b)$ behave as f varies.

Theorem 7. *Let f, g be functions of bounded variation. Then so are their sum, difference, and product. Moreover, we have*

$$V_{f\pm g} \leq V_f + V_g \quad \textit{and} \quad V_{f\cdot g} \leq \|g\|_{\sup} \cdot V_f + \|f\|_{\sup} \cdot V_g,$$

where

$$\|\phi\|_{\sup} = \sup_{x\in[a,b]} |\phi(x)| \,,$$

for $\phi = f, g$.

Proof. Let a partition P of $[a, b]$ be given. Note that

$$\begin{aligned} &\sum_{i=1}^{n} |(f(x_i) \pm g(x_i)) - (f(x_{i-1}) \pm g(x_{i-1}))| \\ \leq\ &\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| \\ \leq\ &V_f + V_g. \end{aligned}$$

Hence, $V_f + V_g$ is an upper bound of $\sum |(f(x_i) \pm g(x_i)) - (f(x_{i-1}) \pm g(x_{i-1}))|$. This implies $f \pm g$ are of bounded variation and that $V_{f\pm g} \leq V_f + V_g$. Now, let $h(x) = f(x) \cdot g(x)$. Then, we have

$$\begin{aligned} |\Delta h_i| &= |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})| \\ &\leq |f(x_i)g(x_i) - f(x_{i-1})g(x_i)| + |f(x_{i-1})g(x_i) - f(x_{i-1})g(x_{i-1})| \\ &\leq \|g\|_{\sup} \cdot |\Delta f_i| + \|f\|_{\sup} \cdot |\Delta g_i|. \end{aligned}$$

We conclude that

$$\sum_{i=1}^{n} |\Delta h_i| \leq \sum_{i=1}^{n} \Big(\|g\|_{\sup} \cdot |\Delta f_i| + \|f\|_{\sup} \cdot |\Delta g_i| \Big) \leq \|g\|_{\sup} \cdot V_f + \|f\|_{\sup} \cdot V_g.$$

This gives that $\|g\|_{\sup} \cdot V_f + \|f\|_{\sup} \cdot V_g$ is an upper bound of $\sum |\Delta h_i|$ for all partitions. Therefore, $f \cdot g$ is of bounded variation and

$$V_{f\cdot g} \leq \|g\|_{\sup} \cdot V_f + \|f\|_{\sup} \cdot V_g.$$

The proof is completed. □

Remark. Theorem 7 shows that the set V of all functions of bounded variation on $[a, b]$ is a linear space. In fact, Theorem 11 (which we will discuss later) indicates that $V \subseteq S$ if S is any linear space which contains all monotonic functions on $[a, b]$.

Now, we wonder whether f/g is of bounded variation provided that both f and g are of bounded variation. However, if we do further observation on f/g, it is easy to see that f/g might not even bounded. Yet, if we assume that g is bounded away from zero, which literally means that the values of g would not be arbitrarily close to 0, then f/g is of bounded variation. Now, we write down this observation in a more precise and mathematical way.

Theorem 8. *Let f and g are of bounded variation. We assume that g is bounded away from zero, that is, there exists a positive number m such that $0 < m \leq |g(x)|$ for all $x \in [a, b]$. Then, f/g is of bounded variation. Moreover, we have*

$$V_{f/g} \leq \frac{V_f}{m} + \frac{\|f\|_{\sup} V_g}{m^2}.$$

Proof. Let a partition P of $[a, b]$ be given and let $h(x) = f(x)/g(x)$.

$$\begin{aligned}
|\Delta h_i| &= \left|\frac{f(x_i)}{g(x_i)} - \frac{f(x_{i-1})}{g(x_{i-1})}\right| = \left|\frac{f(x_i)g(x_{i-1}) - g(x_i)f(x_{i-1})}{g(x_i)g(x_{i-1})}\right| \\
&\leq \frac{\big(|f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_{i-1})| + |f(x_{i-1})g(x_{i-1}) - g(x_i)f(x_{i-1})|\big)}{|g(x_i)g(x_{i-1})|} \\
&\leq \frac{|\Delta f_i|}{|g(x_i)|} + \frac{\|f\|_{\sup}|\Delta g_i|}{m^2} \leq \frac{|\Delta f_i|}{m} + \frac{\|f\|_{\sup}|\Delta g_i|}{m^2}.
\end{aligned}$$

This implies

$$\sum_{i=1}^{k} |\Delta h_i| \leq \sum_{i=1}^{k} \left(\frac{|\Delta f_i|}{m} + \frac{\|f\|_{\sup}|\Delta g_i|}{m^2}\right) \leq \frac{V_f}{m} + \frac{\|f\|_{\sup} V_g}{m^2}.$$

This completes the proof. □

Next, we are going to study the properties of total variation $V_f(a, x)$ as a function of x. Before we start the discussion, we shall prove a theorem which is so called additive property of total variation.

Theorem 9 (additive property of total variation)**.** *Let f be of bounded varaition on $[a, b]$, and assume that $c \in (a, b)$. Then f is of bounded variation on $[a, c]$ and on $[c, b]$. Moreover, we have*

$$V_f(a, b) = V_f(a, c) + V_f(c, b).$$

Proof. Let P_1 and P_2 be partition of $[a, c]$ and $[c, b]$, respectively. Note that $P_0 = P_1 \cup P_2$ is a partition of $[a, b]$. We have

$$S(f; P_1) + S(f; P_2) = S(f; P_0) \leq V_f(a, b). \tag{2}$$

Now, if we fix the partition P_2, then we have

$$V_f(a, c) \leq V_f(a, b) - S(f; P_2)$$

by taking supremum on the left hand side. Taking supremum again on the left hand side of the inequality

$$S(f; P_2) \leq V_f(a, b) - V_f(a, c)$$

gives the conclusion that

$$V_f(a, c) + V_f(c, b) \leq V_f(a, b).$$

Note that for every partition P of $[a, b]$, we have $S(f; P) \leq S(f; P')$, where $P' = P \cup \{c\}$. Let $P_1 = P' \cap [a, c]$ and let $P_2 = P' \cap [c, b]$. Then,

$$S(f; P) \leq S(f; P') = S(f; P_1) + S(f; P_2) \leq V_f(a, c) + V_f(c, b).$$

Thus, $V_f(a, c) + V_f(c, b)$ is an upper bound of $\{S(f; P) : P \text{ is a partition of } [a, b]\}$. We conclude that $V_f(a, c) + V_f(c, b) \geq V_f(a, b)$. This completes the proof. □

Theorem 10. *Let f be a function of bounded variation on $[a, b]$, and let $V(x)$ denotes the function $V_f(a, x)$. ($V_f(a, a)$ is defined to be 0.) Then, both V and $V - f$ are monotonically increasing.*

Proof. If $a \leq x < y \leq b$, then Theorem 9 gives $V_f(a, x) + V_f(x, y) = V_f(a, y)$. This implies $V(y) - V(x) = V_f(x, y) \geq 0$, and therefore V is increasing. To prove $V - f$ is increasing, let $D(x) = V(x) - f(x)$ on $[a, b]$. If $a \leq x < y \leq b$, then

$$D(y) - D(x) = (V(y) - V(x)) - (f(y) - f(x)) = V_f(x, y) - (f(y) - f(x)).$$

But it follows from the definition of total variation that $V_f(x, y) \geq f(y) - f(x)$. Thus, we conclude that $D(y) \geq D(x)$ and D is increasing. □

Theorem 10 suggests what sufficient and necessary conditions need to be met for a function to be a bounded variation function.

Theorem 11. *Let f be a function defined on $[a, b]$. Then, f is of bounded variation if and only if f can be expressed as the difference of two increasing functions.*

Proof. If f is of bounded variation, then both V and $D = V - f$ are increasing (from Theorem 10). We have $f = V - D$ are the difference of two increasing function. Conversely, if f can be expressed as the difference of two increasing functions, then it follows from Theorem 3 and Theorem 7 that f is of bounded variation. □

2 Problems

Here are some problems provided in Apostol, T.M.'s book, most of which attracted my interest.

Problem 1. *A function f, defined on $[a,b]$, is said to satisfy a uniform Lipschitz condition of order $\alpha > 0$ on $[a,b]$ if there exists a constant $M > 0$ such that $|f(x) - f(y)| < M\,|x-y|^\alpha$ for all x and y in $[a,b]$.*

1. *If f is such a function, show that $\alpha > 1$ implies f is constant on $[a,b]$, whereas $\alpha = 1$ implies f is of bounded variation.*

2. *Give an example of a function f satisfying a uniform Lipschitz condition of order $\alpha < 1$ on $[a,b]$ such that f is not of bounded variation.*

3. *Give an example of a function g which is of bounded variation on $[a,b]$ but which satisfies no uniform Lipschitz condition on $[a,b]$.*

Solution.

1. Suppose f is a non-constant function satisfies a uniform Lipschitz condition of order $\alpha > 1$ on $[a,b]$. Since f is non-constant, there are two numbers $p < q$ in $[a,b]$ such that $f(p) \neq f(q)$. Let s and t denote the number $|p-q|$ and $|f(p) - f(q)|$, respectively. We consider a finite sequence $\{x_k\}_{k=0}^n$ explicitly defined by $x_k = p + ks/n$. The triangle inequality and the hypotheses of a uniform Lipschitz condition give us

$$\begin{aligned} |f(p) - f(q)| &\leq \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq \sum_{k=1}^{n} M\,|x_k - x_{k-1}|^\alpha \\ \Longrightarrow\ t &\leq n \cdot M \cdot \left(\frac{s}{n}\right)^\alpha. \end{aligned}$$

   The last inequality fails to be true whenever $n > (M/t)^{\frac{1}{\alpha-1}} \cdot s^{\frac{\alpha}{\alpha-1}}$. Hence, we conclude that a function f satisfies a uniform Lipschitz condition of order $\alpha > 1$ must be constant.

   Now, let g be a function satisfies a uniform Lipschitz condition of order 1, and let an arbitrary partition P be given. Note that

$$\sum_{i=1}^{n} |\Delta g_i| \leq \sum_{i=1}^{n} M^* \left(x_i - x_{i-1}\right) = M^*(b-a),$$

   which indicates g is of bounded variation. (M^* is used for the purpose of distinguishing itself from M.) Another approach see the subproblems *1* and *2* in Problem 2.

2. To solve this problem, we shall first prove some lemmas.

Lemma 1. *If f is a function defined on two closed intervals I, J with following properties:*

- *The intersection of I and J contains at most 1 point (which implies that the intersection point must be an endpoint of these intervals, if the intersection point exist).*
- *$f(I) \subset f(J)$.*
- *f satisfies a uniform Lipschitz condition of order α on both I and J.*

Then, f satisfies a uniform Lipschitz condition of order α on $I \cup J$.

Proof. Since f satisfies a uniform Lipschitz condition of order α on both I and J, there exist M_1, M_2 such that

$$|f(x) - f(y)| \le M_1 |x - y|^\alpha, \text{ for all } x, y \in I.$$
$$|f(x) - f(y)| \le M_2 |x - y|^\alpha, \text{ for all } x, y \in J.$$

Now, we claim that $|f(x) - f(y)| \le M|x - y|^\alpha$, for all $x, y \in I \cup J$, where $M = \max\{M_1, M_2\}$. It is easy to see that we only need to prove that $|f(x) - f(y)| \le M|x - y|^\alpha$ for all $x \in I$ and $y \in J$. From our hypothesis that $f(I) \subset f(J)$, there exists $x' \in J$ such that $f(x) = f(x')$. Note that

$$|f(x) - f(y)| = |f(x') - f(y)| \le M_2 |x' - y|^\alpha \le M_2 |x - y|^\alpha,$$

the last inequality holds from our first assumption. This proves the lemma. □

Lemma 2. *If x, y are two distinct numbers in $[0, 1]$, and if $\alpha \in (0, 1)$, then $|x^\alpha - y^\alpha| \le |x - y|^\alpha$.*

This implies that $f(x) = x^\alpha$ satisfies a uniform Lipschitz condition of order α.

Proof. Without loss of generality, we assume that $x > y$. For every $y \in [0, 1)$, we consider the function

$$\phi_y(x) = (x - y)^\alpha - x^\alpha + y^\alpha \qquad (x \ge y).$$

It is easy to see that $\phi_y(y) = 0$ and that

$$\frac{d\phi_y}{dx} = \alpha(x - y)^{\alpha - 1} - \alpha x^{\alpha - 1} \qquad (x > y).$$

Since $\alpha \in (0, 1)$, we have $d\phi_y(x)/dx > 0$ whenever $x > y$. Note that $\phi_y(x)$ is continuous at y, hence $\phi_y(x)$ is strictly increasing, we conclude that $\phi_y(x) > 0$ for $x > y$. This proves the lemma. □

Lemma 3. *Given $\epsilon > 0$. There exists a function f defined on $[a, b]$, such that:*

- *f satisfies a uniform Lipschitz condition of order α ($0 < \alpha < 1$).*

- $1 \le V_f(a,b) < \infty$ *and* $f(a) = f(b) = 0$.
- *There exists a positive number* $\delta < \epsilon$ *such that* $f([a,b]) = [0,\delta]$.

Proof. Let $l = b - a$ and let n be a positive integer such that

$$(2n)^{1-\alpha} \cdot l^\alpha \ge 1 \quad \text{and} \quad \left(\frac{l}{2n}\right)^\alpha < \epsilon.$$

Now, we divide $[a,b]$ into $2n$ intervals

$$I_k = \left[a + \frac{(k-1)l}{2n}, a + \frac{kl}{2n}\right] \quad (k = 1, 2, \dots, 2n).$$

We define f as follows:

$$f(x) = \begin{cases} \left(x - a - \dfrac{(m-1)l}{n}\right)^\alpha & \text{, if } x \in I_{2m-1} \\ \left(a + \dfrac{ml}{n} - x\right)^\alpha & \text{, if } x \in I_{2m} \end{cases} \qquad (m = 1, 2, \dots, n).$$

It is easy to see that f is increasing on every interval I_{2m-1} and it is decreasing on every interval I_{2m}. Moreover, we have $f(I_k) = [0, (l/2n)^\alpha]$ (this verify the third requirement of the function f) and f satisfies a uniform Lipschitz condition of order α on each interval I_k. It follows from Lemma 1 and Lemma 2 that f satisfies a uniform Lipschitz condition of order α on $[a,b]$. (In fact, we have $|f(x) - f(y)| \le |x - y|^\alpha$ from the proof of Lemma 1.) This completes the proof. □

After proving these lemmas, we can begin to prove the main problem. We construct a function f defined on $[0,1]$, satisfying following properties:

(a) $f(0) = 0$ and $f(1/n) = 0$, for all $n \in \mathbb{N}$.

(b) $|f(x) - f(y)| \le |x - y|^\alpha$, for all $x, y \in [1/(n+1), 1/n]$.

(c) $V_f(1/(n+1), 1/n) > 1$, for all $n \in \mathbb{N}$.

(d) $f([1/(n+2), 1/(n+1)]) \subset f([1/(n+1), 1/n])$.

This is possible by Lemma 3. We claim that f satisfies a uniform Lipschitz condition of order α on $[0,1]$ but is not of bounded variation. If f is of bounded variation, then it follows from Theorem 9 that

$$V_f(0,1) = V_f\left(0, \frac{1}{n}\right) + V_f\left(\frac{1}{n}, 1\right) > n - 1 \quad (n \in \mathbb{N}).$$

It contradicts the assumption that $V_f(0,1)$ is bounded. From the construction of f and Lemma 1, we have

$$|f(x) - f(y)| < |x - y|^\alpha, \text{ for all } x, y \in (0,1].$$

We also have $|f(x)| \le x^\alpha$. (It is easy to verify the construction in Lemma 3.) We conclude that f satisfies a uniform Lipschitz condition of order α on $[0,1]$. This shows f meets the requirements for the problem.

3. Define $g : [0, e^{-1}] \to [0, 1]$ as:

$$g(x) = \begin{cases} \sqrt{\dfrac{-1}{\ln x}} & , \text{ if } x \in (0, e^{-1}] \\ 0 & , \text{ if } x = 0 \end{cases}.$$

Now, note that the function g is continuous and monotonic, we conclude that g is of bounded variation from Theorem 3. Suppose g satisfies a uniform Lipschitz condition on $[0, e^{-1}]$ of order $0 < \alpha \leq 1$. Then, there exists $M > 0$ such that $|g(x) - g(y)| < M|x - y|^\alpha$ for all $x, y \in [0, e^{-1}]$. In particular,

$$\sqrt{\frac{-1}{\ln x}} < Mx^\alpha \tag{3}$$

holds for all $x \in (0, e^{-1}]$. This inequality is equivalent to

$$M^2 x^{2\alpha} \ln\left(\frac{1}{x}\right) > 1.$$

The substitution $x = e^{-t}$ turns (3) into

$$\frac{M^2 t}{e^{2\alpha t}} > 1, \; t \in [1, \infty). \tag{4}$$

However, it follows from L'Hospital's Rule that $\lim_{t\to\infty}(M^2 t)/e^{2\alpha t} = 0$, which contradicts (4). Thus, g satisfies no uniform Lipschitz condition on $[0, e^{-1}]$ and therefore g meets the requirements of the problem.

Remark. The inverse of the function constructed in the subproblem *3* in Problem 1 is $g^{-1} : [0, 1] \to [0, e^{-1}]$ defined by

$$g^{-1}(x) = \exp\left\{\left(\frac{-1}{x^2}\right)\right\}, \; x \in (0, 1], \text{ and } \quad g^{-1}(0) = 0.$$

This function $h = g^{-1}$ has some quite interesting properties:

1. The nth derivative of h at 0 exists. Moreover, $h^{(n)}(0) = 0$.
2. The Maclaurin series of h does not converge to h, although it converges everywhere on $\mathbb{R}$.

Problem 2. *A function f, defined on $[a, b]$, is said to be absolutely continuous, if for every $\epsilon > 0$ there is a $\delta > 0$ such that*

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon,$$

for every n ($n \in \mathbb{N}$) disjoint open subintervals (a_k, b_k) of $[a, b]$, the sum of whose length

$$\sum_{k=1}^{n} (b_k - a_k) < \delta.$$

Prove the following statements.

1. *Every absolutely continuous function on* $[a, b]$ *is continuous and of bounded variation.*

2. *If* f *satisfies a uniform Lipschitz condition of order* 1*, then* f *is absolutely continuous.*

Solution.

1. Let f be a function which is absolutely continuous on $[a, b]$. Given $\epsilon > 0$, then there exists a $\delta > 0$ satisfying the condition described above. If $x < y$ are two points in $[a, b]$ such that $y - x < \delta$, then $|f(x) - f(y)| < \epsilon$ (from the definition of absolutely continuous). This implies f is uniformly continuous and thus continuous.

   Now, we shall show that f is of bounded variation. Let a partition P of $[a, b]$ be given. Let N be a positive integer such that $b - a < N\delta$. Consider a refinement P' of P, where P' is defined as:

   $$P' = P \cup \left\{a + \frac{(b-a)}{N}, a + \frac{2(b-a)}{N}, \dots, a + \frac{(N-1)(b-a)}{N}\right\}.$$

   Let

   $$P_k = P' \cap \left[a + \frac{(k-1)(b-a)}{N}, a + \frac{k(b-a)}{N}\right] \quad (k = 1, 2, \dots, N).$$

   It is easy to see that $S(f; P_k) < \epsilon$ for each $k = 1, 2, \dots, N$ (from the definition of absolutely continuous). We conclude that

   $$S(f; P') = \sum_{k=1}^{N} S(f; P_k) < N\epsilon.$$

   Since P' is a refinement of P, we have $S(f, P) \leq S(f; P') < N\epsilon$, this shows f is of bounded variation.

2. Since f satisfies a uniform Lipschitz condition of order 1, there exists a constant $M > 0$ such that $|f(x) - f(y)| < M|x - y|$ for all x, y in $[a, b]$. Given $\epsilon > 0$. Choose $\delta = \epsilon/M$. Let (a_k, b_k) $(k = 1, 2, \dots, n)$ be n $(n \in \mathbb{N})$ disjoint open subintervals of $[a, b]$, the sum of whose length

   $$\sum_{k=1}^{n} (b_k - a_k) < \delta.$$

   Then, by some simple estimations, we get

   $$\sum_{k=1}^{n} |f(b_k) - f(a_k)| \leq \sum_{k=1}^{n} M(b_k - a_k) < M\delta = \epsilon.$$

   We conclude that f is absolutely continuous on $[a, b]$.

Remark. Although it seems that a function being of bounded variation and continuous would be more likely to be absolutely continuous. However, there exists a function which is of bounded variation and continuous but not absolutely continuous. Cantor function is an example of these kind of functions. [3]

Problem 3. *Show that a polynomial f is of bounded variation on every compact interval $[a, b]$. Describe a method for finding the total variation of f on $[a, b]$ if the zeros of the derivative f' are known.*

Solution. We first prove the case when $f_n(x) = x^n$ for all $n \in \mathbb{N}$. f_1 is monotonic on $[a, b]$ and thus is of bounded variation (by Theorem 3). Suppose the statement is true for all $n \leq k$, where k is a positive number. Then, $f_{k+1}(x) = f_k(x) \cdot f_1(x)$, hence f_{k+1} is of bounded variation (from Theorem 7). By induction, each $f_n(x) = x^n$ ($n \in \mathbb{N}$) is of bounded variation. It is easy to see that f is of bounded variation implies that cf is of bounded variation, where c is a real constant. (We could replace M to $|c| \cdot M$ in Definition 2 to show cf is of bounded variation.) Also, note that a constant function is of bounded variation. Now, for each polynomial f with degree n, we have

$$f(x) = \sum_{i=0}^{n} c_i x^i,$$

and therefore it is sum of finitely many functions of bounded variation. This implies that f is of bounded variation. (Theorem 7.)

If f' is zero, then f is constant and $V_f(a, b) = 0$. Suppose f' is not 0, and let $Z_{f'}$ be the set $\{t \in (a, b) : f'(t) = 0\}$. Since f' is a polynomial, $Z_{f'}$ is a finite set. Let X denote the set $Z_{f'} \cup \{a, b\}$. Because X is finite and contains at least 2 elements, we write $X = \{a = t_0 < t_1 < \cdots < t_m = b\}$. By Theorem 9, we have

$$V_f(a, b) = \sum_{i=1}^{m} V_f(t_{i-1}, t_i).$$

Now, note that the image under f' of (t_{i-1}, t_i) cannot contain both positive and negative numbers, otherwise the continuity of f' would indicate that there exists a number $s \in (t_{i-1}, t_i)$ such that $f'(s) = 0$ (Intermediate Value Theorem), which contradicts $s \notin Z_{f'}$. Hence, either $f'(x) > 0, \forall x \in (t_{i-1}, t_i)$ or $f'(x) < 0, \forall x \in (t_{i-1}, t_i)$ holds, thus f is monotonic on (t_{i-1}, t_i). We conclude that $V_f(t_{i-1}, t_i) = |f(t_{i-1}) - f(t_i)|$, therefore

$$V_f(a, b) = \sum_{i=1}^{m} |f(t_{i-1}) - f(t_i)|.$$

References

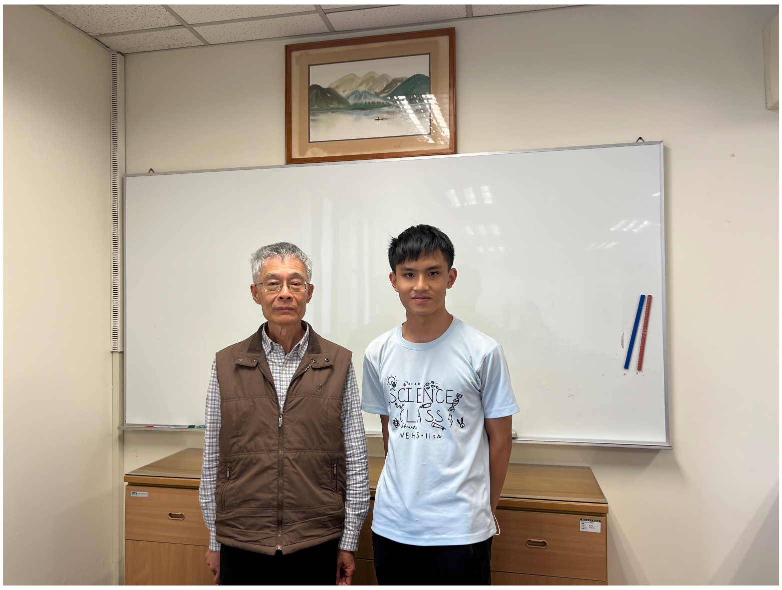
[1] T.M. Apostol. *Mathematical Analysis*. Addison-Wesley, 1974.

[2] Walter Rudin. *Principles of Mathematical Analysis*. Vol. 3. McGraw-Hill New York, 1964.

[3] Wikipedia. *Cantor function*. Last accessed 13 February 2022. URL: `https://en.wikipedia.org/wiki/Cantor_function`.

A 研究心得與研究照片

這年的專題研究，我學到不少新的東西，像是關於有界變差 (又稱有限變量，bounded variation) 函數的性質以及全變差 (total variation)，而這些概念爲建立黎曼積分理論的基礎。在這次研究的過程中，我很感謝我的指導教授：程守慶教授，指引我很多的研究方向，給我很多數學上的啓發，以及訓練我用英文書寫報告的能力。除此之外，我也很感謝清大給我的一切學習資源，包含微積分、高等微積分等課程，以及清大圖書館充裕的資源。



B 原創性比對

此篇報告經 Turnitin 所提供之原創性比對爲 51%，大致原因如下所列：

1. 在缺少大學數學的知識下，難做出新的「數學」，像是科展中的數學組的大部分內容不外乎爲一些「排列組合」或是高中的直角座標幾何等。我認爲我們有機會到清大找教授做專題，應趁此機會多學習大學數學，像是「高等微積分」、「拓樸」、「代數」，而做高中範圍內的題目對我來說意義不大。所以，我與教授討論出來此次專題的進行方式爲先讀特定的高等微積分參考書，接著寫成一份類似讀書心得的報告。(據我觀察，前幾屆的數學專題進行方式也大致上是這樣)

2. 此篇報告分爲兩大部分，前半部分主要列出了一些關於「有限變量 (bounded variation)」的性質與定義，主要參考 T.M. Apostol 所寫的書 [1]，我有試著我自己的話改寫，但是數學的文本中，很多時候會有慣用的字或是片語，甚至符號上可能也有專屬的代表意義，導致前半部分可能與 T.M. Apostol 的書有相似之處。而文章的後半部分則是我要去作答該書中的習題，而敘述基本上就是直接參考原書中的題目，但是解答是我自己給出的。事實上，那些題目最一開始我自己給出的解答不是那麼完美，甚至教授有針對其中幾題給我更好的解答後，我用我自己的話去重寫的。至於原創性比對的比例這麼高，我猜測是因爲那些題目早在 40 年前就出現了，難免在網路上被討論。