Analysis II

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Abstract

這份筆記主要是在記錄分析二這個學期的上課內容，前半部分參考 Pugh 所寫的參考書 [2] 以及 Apostol 所寫的書 [1]，後半部分則是參考沈俊嚴老師的板書。

This note is mainly to record the content of this semester's class in Analysis II. The first half is based on Pugh's book [2] and some from Apostol's book [1], and the second half is based on Mr. Shen's lecture notes. We will not cover the content taught in the last semester.

Contents

1 Fourier Analysis

1.1 The L^2 Space and the Orthogonal Basis

In this subsection, we will study the space $L^2[a,b]$. The space is defined as the following

$$L^2[a,b] = \left\{ f : [a,b] \to \mathbb{R} \mid f \text{ is measurable and } \int |f|^2 < \infty \right\}.$$

Recall in the last semester, we have define the concept of a normed space.

Definition 1 (normed space)**.** A normed space is a vector space X with norm $\|\cdot\|$, satisfying these three properties:

1. $\|x\| \geq 0$ for all $x \in X$, and $\|x\| = 0$ if and only if $x = 0$.
2. $\|\alpha x\| = |\alpha| \cdot \|x\|$ for all $\alpha \in \mathbb{R}$ and $x \in X$.
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

Now consider the norm on $L^2[a,b]$ defined by

$$\|f\| = \left(\int |f|^2 \right)^{1/2}.$$

It is easy to see that $\|\alpha f\| = |\alpha| \cdot \|f\|$. Also note that $\|f\| = 0$ if and only if $f = 0$ almost everywhere. To see the third condition, we consider the Cauchy-Schwartz inequality:

$$\left| \int_a^b fg \right| \leq \int_a^b |fg| \leq \left(\int_a^b |f|^2 \right)^{1/2} \left(\int_a^b |g|^2 \right)^{1/2}.$$

The inequality on the left is trivial. For the inequality on the right side, we might assume that $f, g \geq 0$. Then for all $t \in \mathbb{R}$ we have

$$\begin{aligned} (tf + g)^2 \geq 0 &\implies t^2 f^2 + 2tfg + g^2 \geq 0 \\ &\implies t^2 \int f^2 + 2t \int fg + \int g^2 \geq 0 \\ &\implies \left(2 \int fg \right)^2 \leq 4 \left(\int f^2 \right) \left(\int g^2 \right) \\ &\implies \int fg \leq \left(\int f^2 \right)^{1/2} \left(\int g^2 \right)^{1/2} \end{aligned}$$

Hence,

$$\begin{aligned} \|f + g\|^2 &= \int |f + g|^2 = \int f^2 + 2 \int fg + \int g^2 \\ &\leq \int f^2 + 2 \left(\int f^2 \right)^{1/2} \left(\int g^2 \right)^{1/2} + \int g^2 = (\|f\| + \|g\|)^2 \end{aligned}$$

This gives the third condition of the norm. We conclude that $L^2[a,b]$ is a normed space. In fact, it is a Banach space.

Theorem 2. *$L^2[a,b]$ equipped with the norm defined above is a Banach space.*

We shall recall that a Banach space is a complete normed space. That is, every Cauchy sequence is converging with respect to the norm. To prove this theorem, we shall introduce some useful concepts.

Definition 3. Let X be a normed space. Given a sequence of elements $\{f_n\}$. We say f_n is summable to $s \in X$ if

$$s_n \to x \qquad \text{where} \qquad s_n = \sum_{k=1}^{n} f_k.$$

We say $\{f_n\}$ is summable if

$$\sum_{k=1}^{\infty} \|f_k\| < \infty.$$

Then we have the following lemma.

Lemma 1. A normed space X is complete if and only if every summable $\{f_n\}$ is summable to some $s \in X$.

Proof. First suppose X is complete. Given $\{f_n\}$ with $\sum \|f_k\| = M < \infty$. Fix $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{k=n}^{\infty} \|f_k\| < \epsilon \qquad \text{whenever} \qquad n \geq N.$$

Then we have

$$\|s_n - s_m\| \leq \sum_{k=n+1}^{m} \|f_k\| < \epsilon \qquad \text{whenever} \qquad m > n \leq N.$$

We conclude that $\{s_n\}$ is Cauchy, therefore $s_n \to s$ for some $s \in X$.

Conversely, suppose every summable $\{f_n\}$ summable to some $s \in X$. Given a Cauchy sequence $\{f_n\}$. For all $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $\|f_n - f_m\| < 2^{-k}$, whenever $n, m \geq n_k$. We may assume that $n_{k+1} > n_k$ for all $k \in \mathbb{N}$. We now let

$$g_1 = f_{n_1} \qquad \text{and} \qquad g_k = f_{n_k} - f_{n_{k-1}} \text{ for all } k \geq 2.$$

It is easy to see that

$$\sum_{k=1}^{n} g_k = f_{n_k} \qquad \text{and} \qquad \sum_{k=1}^{n} \|g_k\| \leq \sum_{k=1}^{n} 2^{-k} \leq 1.$$

Thus it follows from the assumption that $\sum g_k$ converges. That is, $f_{n_k} \to f$ for some $f \in X$. It remains to show $f_n \to f$, which could be derived from the definition of the Cauchy sequence. ☐

We now could give the proof of Theorem 2.

Proof of Theorem 2. By the Lemma 1, it suffices to show every summable $\{f_n\} \subset L^2[a,b]$ is summable to some $t \in L^2[a,b]$. Given a summable sequence $\{f_n\}$ ($\sum \|f_k\| \leq MN\infty$) in $L^2[a,b]$. Set $s_n = \sum |f_k|$, then

$$\|s_n\| = \left\| \sum_{k=1}^{n} |f_k| \right\| \leq \sum_{k=1}^{n} \|f_k\| \leq M < \infty$$

Since $\{s_n(x)\}$ is increasing for all $x \in [a,b]$. The limit

$$g(x) = \lim_{n\to\infty} s_n(x)$$

exists (∞ is allowed). The Fatou's Lemma asserts that

$$\int g^2 = \int \lim s_n^2 \leq \liminf \int s_n^2 \leq M^2 < \infty.$$

Hence g is finite almost everywhere. Let

$$f(x) = \begin{cases} g(x), & \text{if } g(x) < \infty \\ 0, & \text{otherwise} \end{cases}.$$

Then s_n pointwise converges to f almost everywhere. Now consider the inequality

$$|s_n(x) - f(x)|^2 \leq 2(s_n^2(x) + f^2(x)) \leq 4f^2(x)$$

holds for almost every x. The Dominated Convergence Theorem asserts that

$$\lim_{n\to\infty} \int |s_n - f|^2 = \int \lim |s_n - f|^2 = \int 0 = 0.$$

Hence $s_n \to f$ in the L^2 norm. This proves the case of $f \geq 0$. In general, we may consider $f = f^+ - f^-$. □

Theorem 4. *$L^2[a,b]$ is separable. That is, there is a countable dense subset of $L^2[a,b]$.*

We first prove the following:

Lemma 2. $C[a,b]$ with the L^2 norm is dense in $L^2[a,b]$.

Proof. We first prove that every characteristic function of a compact set can be approximated by a sequence of continuous function. Let a closed set $A \subset [a,b]$ be given. Let $t(x) = \inf_{y\in A} |x-y|$ be a continuous function (continuity follows from the exercise last semester.) Consider the function

$$g_n(x) = \frac{1}{1 + n \cdot t(x)}$$

It is easy to see that g_n pointwise converges to χ_A. Note that

$$|g_n(x) - \chi_A(x)|^2 \leq 4$$

hence it follows by the Dominated Convergence Theorem that

$$\lim_{n\to\infty}\int |g_n-\chi_A|^2 = \int \lim_{n\to\infty}|g_n-\chi_A|^2 = 0$$

This implies that the χ_A can be approximated by continuous function. We now show that this is true for characteristic function of a measurable set. Let $E\subset[a,b]$ be measurable. Given $\epsilon>0$ there is a compact set $F\subset E$ such that $m(E)\le m(F)+\epsilon$. Then

$$\|\chi_E-\chi_F\|<\epsilon.$$

From the discussion above, we conclude that there is a continuous function t such that $\|\chi_F-t\|<\epsilon$. Thus, $\|\chi_E-t\|<2\epsilon$. Now, for a measurable function $f\in L^2[a,b]$ $(f\ge 0)$, there are finitely many measurable sets E_i and positive real numbers a_i such that

$$\sum_{i=1}^n a_i\chi_{E_i}\le f \qquad \text{and} \qquad \left\|\sum_{i=1}^n a_i\chi_{E_i}-f\right\|<\epsilon.$$

This shows that there is some continuous function t such that $\|t-f\|<2\epsilon$. This complete the proof. □

Lemma 3.

1. The set of all real polynomials $\mathbb{R}[x]$ is dense in $C[a,b]$ with respect to the L^2 norm.
2. $\mathbb{Q}[x]$ is countable and dense in $\mathbb{R}[x]$ with respect to the L^2 norm.

Proof. The first one immediately follows from the Stone-Weierstrass Theorem. The second one is trivial. □

Proof of Theorem 4. This is a corollary of Lemma 2 and Lemma 3. □

We now shall study the inner product structure on the $L^2[a,b]$ space. Observe the following property: If we are given two functions $f,g\in L^2[a,b]$, then we have

$$\int |fg| \le \left(\int f^2\right)^{1/2}\left(\int g^2\right)^{1/2}<\infty.$$

Hence we can define the inner product on L^2. We write $\langle f,g\rangle = \inf fg$ to denote the inner product of f and g. Here we recall the definition of inner products.

Definition 5. Let V be a vector space over F. Then inner product $\langle -,-\rangle : V\times V\to F$ is a function satisfying the following property: (we might assume $F=\mathbb{R}$ here)

1. $\langle f,g\rangle = \langle g,f\rangle$ for all $f,g\in V$.
2. $\langle f_1+f_2,g\rangle = \langle f_1,g\rangle+\langle f_2,g\rangle$ for all $f_1,f_2,g\in V$.
3. $\langle cf,g\rangle = c\langle f,g\rangle$, for all $f,g\in V$ and $c\in\mathbb{R}$.
4. $\langle f,f\rangle = \|f\|^2\ge 0$ for all $f\in V$, and $\|f\|=0$ if and only if $f=0$.

The inner product structure help us to define the concept of orthogonal.

Definition 6 (orthogonal)**.** Given $f, g \in V$, we say f and g is orthogonal if $\langle f, g \rangle = 0$. A set $S \subset V \setminus \{0\}$ is orthogonal if $\langle f, g \rangle = 0$ for all $f, g \in S$. A set $S \subset V \setminus \{0\}$ is orthonormal if it is orthogonal and $\|f\| = 1$ for all $f \in S$.

Theorem 7. *Suppose $\{\phi_\alpha\}_{\alpha\in\Lambda}$ is orthogonal in L^2. Then it (Λ) is at most countable.*

Proof. Without loss of generality, $\{\phi_\alpha\}$ is orthonormal. If $\phi_\alpha \neq \phi_\beta$, then we have

$$\langle \phi_\alpha - \phi_\beta, \phi_\alpha - \phi_\beta \rangle = \langle \phi_\alpha, \phi_\alpha \rangle + \langle \phi_\beta, \phi_\beta \rangle = 2.$$

This shows that $\{B(\phi_\alpha; 1) : \alpha \in \Lambda\}$ is a set of disjoint open balls. Recall that in Theorem 4, we have shown that there is a countable dense subset P of L^2. Thus, every open ball $B(\phi_\alpha; 1)$ contains a points $x_\alpha \in P$. $\{x_\alpha\}$ is at most countable therefore Λ is countable. $\square$

Definition 8 (linear independent)**.** Given a finite set $\{\varphi_1, \dots, \varphi_n\} \subset V$, we say it is linear independent if

$$\sum_{i=1}^{n} a_i \varphi_i = 0 \implies a_1 = a_2 = \cdots = a_n = 0.$$

For an infinite set $\{\varphi_\alpha\}_{\alpha\in\Lambda} \subset V$, we say it is linear independent if all of its finite subsets are linear independent.

Then we have the following.

Lemma 4. If $\{\phi_\alpha\}_{\alpha\in\Lambda} \subset V$ is orthogonal then it is linear independent.

Proof. It suffices to show that Λ is a finite set. Suppose

$$\sum_{i=1}^{n} a_i \phi_i = 0$$

for some $a_i \in \mathbb{R}$. Then

$$0 = \left\langle \sum_{i=1}^{n} a_i \phi_i, \phi_i \right\rangle = \langle a_i \phi_i, \phi_i \rangle = a_i \|\phi_i\|^2,$$

this indicates that $a_i = 0$ for all i. $\square$

Theorem 9. *Gram-Schmidt process Let $\{\varphi_i\}_{i=1}^{\infty}$ be linear independent. Then there exist an orthonormal set $\{\phi_i\}_{i=1}^{\infty}$ such that*

$$\mathrm{span}_{\mathbb{R}}\{\phi_1, \dots, \phi_k\} = \mathrm{span}_{\mathbb{R}}\{\varphi_1, \dots, \varphi_k\}.$$

We will not give a proof here, see also my linear algebra note for more details. We now give the definition of basis in infinite dimensional space.

Definition 10.

1. We say an orthogonal set $\{\phi_\alpha\}_{\alpha\in\Lambda} \subset L^2[a, b]$ is maximal (complete), if

$$\langle f, \phi_\alpha \rangle = 0, \text{ for all } \alpha \in \Lambda \implies f = 0.$$

2. Given a finite set $\{\phi_1, \ldots, \phi_n\}$. We write

$$\operatorname{span}_{\mathbb{R}}\{\phi_1, \ldots, \phi_n\} := \left\{ \sum_{i=1}^{n} a_i \phi_i : a_i \in \mathbb{R} \right\}.$$

For a infinite set $\{\phi_\alpha\}_{\alpha \in \Lambda}$, we write

$$\operatorname{span}_{\mathbb{R}}\{\phi_\alpha\}_{\alpha \in \Lambda} = \bigcup_{F \subset \Lambda,\ |F| < \infty} \operatorname{span}_{\mathbb{R}}\{\phi_\alpha\}_{\alpha \in F}.$$

3. We say $\{\phi_i\}_{i=1}^{\infty}$ is a basis in $L^2[a,b]$ if it is dense in $L^2[a,b]$. That is, for all $f \in L^2$ and $\epsilon > 0$, there are $\{a_i\}_{i=1}^{n}$ and $\{\phi_i\}_{i=1}^{n}$ such that

$$\left\| \sum_{i=1}^{n} a_i \phi_i - f \right\| < \epsilon.$$

Note that L^2 has an orthogonal basis. $\{x^n\}_{n=0}^{\infty} \subset \mathbb{R}[x]$ generates L^2 and then apply Gram-Schmidt process.

Lemma 5. Suppose $\{\phi_i\}$ is an orthogonal basis. $\{\phi_i\}$ is a complete (maximal) orthogonal set.

Proof. Suppose $\langle f, \phi_i \rangle = 0$ for all $i \in \mathbb{N}$. Since $\{\phi_i\}$ is a basis, given $\epsilon > 0$ there exists $\{a_i\}_{i=1}^{n}$ such that

$$\left\| \sum_{i=1}^{n} a_i \phi_i - f \right\| < \epsilon.$$

Then we have

$$\epsilon^2 > \left\langle \sum_{i=1}^{n} a_i \phi_i - f, \sum_{i=1}^{n} a_i \phi_i - f \right\rangle = \sum_{i=1}^{n} a_i^2 \left\|\phi_i\right\|^2 + \left\|f\right\|^2 \geq \left\|f\right\|^2.$$

This shows $\|f\| < \epsilon$ for any given $\epsilon > 0$ hence $\|f\| = 0$. This completes the proof. □

1.2 Fourier Series

In the subsection, we introduce the Fourier series on the L^2 space.

Definition 11. Suppose $\{\phi_i\}_{i=1}^{\infty}$ is an orthonormal set in L^2 space. Given $f \in L^2$. Then

$$c_k = \int f \phi_k$$

is called the Fourier coefficient, and

$$s(f) = \sum_{k=1}^{\infty} c_k \phi_k$$

is called the Fourier series.

Theorem 12. *Given an orthonormal set* $\{\phi_i\}_{i=1}^N$*, then for all* $(\gamma_i)_{i=1}^N$ *we have*

$$\left\|\sum_{i=1}^N c_i\phi_i - f\right\|_2 \le \left\|\sum_{i=1}^N \gamma_i\phi_i - f\right\|_2.$$

That is, among all choice of (γ_i), (c_i) *gives the best approximation.*

Proof. It follows by

$$\begin{aligned}\left\|\sum_i \gamma_i\phi_i - f\right\|^2 = \left\langle \sum_i \gamma_i\phi_i - f, \sum_i \gamma_i\phi_i - f\right\rangle &= \sum_i \gamma_i^2 + \|f\|^2 - 2\sum_i \gamma_i c_i \\ &= \sum_i (\gamma_i - c_i)^2 + \|f\|^2 - \sum_i c_i^2.\end{aligned}$$

This completes the proof. □

Corollary 1. Let $\{\phi_i\}_{i=1}^\infty$ be an orthonormal set.

1. For any $\phi_1, \phi_2, \ldots, \phi_N$, $\sum c_i\phi_i$ has the minimum distance to f.
2. $\sum_{i=1}^\infty |c_i|^2 \le \|f\|^2$. This is the so-called Bessel's inequality.

Proof. We shall only prove the second statement. We have

$$\left\|\sum_{i=1}^N c_i\phi_i - f\right\|^2 = \|f\|^2 - \sum_{i=1}^N |c_i|^2 \ge 0.$$

for all N. Taking $N \to \infty$ gives the desired result. □

Definition 13. A sequence of numbers $(c_i)_{i=1}^\infty$ is said to be in ℓ^2 if

$$\sum_{i=1}^\infty |c_i|^2 < \infty.$$

Then we have the following theorem.

Theorem 14. *Let* $\{\phi_i\}$ *be a complete (maximal) orthonormal set. If* $c_k(f) = c_k(g)$ *for all* $k \in \mathbb{N}$*. Then* $f = g$*.*

Proof. We have

$$\int (f-g)\phi_k = 0 \qquad \text{for all } k \in \mathbb{N}.$$

That is $f - g$ is orthogonal to all ϕ_k. We conclude that $f = g$. □

Theorem 15. *Suppose* $(c_i)_{i=1}^\infty \in \ell^2$*. Given an orthonormal set* $\{\phi_i\}_{i=1}^\infty$ *in* L^2*. Then there is a function* $f \in L^2$ *such that*

$$c_k = \int f\phi_k \qquad \textit{and} \qquad \sum_{i=1}^\infty |c_i|^2 = \|f\|^2.$$

Proof. Let $t_n = \sum_{i=1}^{n} c_i\phi_i \in L^2$. We claim $(t_n)_{n=1}^{\infty}$ is a Cauchy sequence in L^2. Given $\epsilon > 0$. There exists an $N \in \mathbb{N}$ such that

$$\sum_{i=k}^{\infty} |c_i|^2 < \epsilon \qquad \text{whenever} \qquad k \geq N.$$

Now consider

$$\|t_n - t_m\|^2 = \left\| \sum_{i=n+1}^{m} c_i\phi_i \right\|^2 = \sum_{i=n+1}^{m} |c_i|^2 < \epsilon \qquad \text{if} \qquad m > n \geq N.$$

Hence (t_n) is a Cauchy sequence in L^2. There is a function $f \in L^2$ such that $t_n \to f$ in L^2 norm. Finally, we have

$$\left| \int f\phi_k - c_k \right| = \left| \int f\phi_k - \int t_n\phi_k \right| = \left| \int (f - t_n)\phi_k \right| \leq \|f - t_n\| \cdot 1 \to 0.$$

Hence $\int f\phi_k = c_k$. The second statement follows from

$$\|f\|^2 - \sum_{i=1}^{n} |c_i|^2 = \langle t_n - f, t_n - f \rangle = \|t_n - f\|^2 \to 0 \text{ as } n \to \infty.$$

This completes the proof. □

Definition 16. Let $\{\phi_i\}_{i=1}^{\infty}$ be an orthonormal set. We said $f \in L^2$ satisfies Parseval's Formula if

$$\sum_{i=1}^{\infty} |c_i|^2 = \|f\|^2 .$$

We now have the following.

Theorem 17. *Given an orthonormal set $\{\phi_i\}_{i=1}^{\infty}$. Then it is complete if and only if f satisfies the Parseval's formula for all $f \in L^2$.*

Proof. Suppose f satisfies the Parseval's formula for all $f \in L^2$. If there is a $f \in L^2$, such that $\langle f, \phi_i \rangle = 0$ for all $i \in \mathbb{N}$. Then $\|f\|^2 = 0$. Hence $\{\phi_i\}$ is a complete (maximal) orthonormal set. Now suppose that $\{\phi_i\}$ is complete (maximal). Given $f \in L^2$. By Bessel's inequality, we have

$$\sum_{i=1}^{n} |c_i|^2 \leq \|f\|^2 < \infty.$$

We now apply Theorem 15, there is a function $g \in L^2$ such that

$$c_k(g) = c_k = c_k(f) \qquad \text{and} \qquad \sum_{i=1}^{\infty} |c_k(g)|^2 = \|g\|^2 .$$

By Theorem 14, we conclude that $f = g$. This gives the desired result. □

Theorem 18. *$L^2[a, b]$ is isometric to ℓ^2 and $L^2[a, b]$ is isometric to $L^2[c, d]$.*

This is just a corollary of discussions above.

1.3 Fourier series on $[a, b]$

In this subsection, we shall now study the concrete Fourier series. That is, we will replace the orthonormal set $\{\phi_i\}$ with the trigonometric function $\{\cos(nx), \sin(nx)\}_{n=0}^{\infty}$ or $\{\exp(ikx)\}_{k=-\infty}^{\infty}$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function with period 2π. Define its Fourier series be

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin(kx),$$

where

$$\begin{cases} a_k = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(t) \cos(kt) dt \\ b_k = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(t) \sin(kt) dt \end{cases}.$$

We often use the complex notation

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx}.$$

In general, we say $P(t)$ is a trigonometric polynomial of degree n if

$$P(t) = \sum_{k=-n}^{n} c_k e^{ikt}, \qquad \text{where } |c_n| + |c_{-n}| \neq 0.$$

We have the following important property:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{imx} dx = \begin{cases} 0, & \text{if } m \neq 0 \\ 1, & \text{if } m = 0 \end{cases}.$$

This gives us an opportunity to generalize it and allow us do Fourier analysis on groups. We will also study Fourier series on L^p spaces.

Lemma 6. Let $s_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}$ converge to $f \in L^1$ in L^1 norm. Then

$$c_k = c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt.$$

Proof. Since $s_n \to f$ in L^1 norm. We have $\displaystyle\int |f - s_n| \to 0$. We have

$$\begin{aligned} c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt &= \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} (f - s_n) e^{-ikt} dt + \int_{-\pi}^{\pi} s_n(t) e^{-ikt} dt \right) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (f - s_n) e^{-ikt} dt + c_k. \end{aligned}$$

Note that

$$\left| \int_{-\pi}^{\pi} (f - s_n) e^{-ikt} dt \right| \leq \left| \int_{-\pi}^{\pi} (f - s_n) \right| \to 0 \qquad \text{as} \qquad n \to \infty.$$

This completes the proof. This theorem establishes a necessary condition for a trigonometric series to converge. $\square$

We now are going to introduce an useful theorem which is known as Riemann-Lebesgue Theorem.

Theorem 19 (Riemann-Lebesgue's Theorem)**.** *If $f \in L^1$, then $|c_k| \to 0$ as $k \to \infty$.*

Proof. We shall use a fact (which will be proved later) that trigonometric series are dense in L^1. Given $f \in L^1$ and $\epsilon > 0$ there exists a

$$P(t) = \sum_{k=-N}^{N} c_k e^{ikt}$$

such that

$$\|f - P\|_1 < \epsilon.$$

When $n > N$, we have

$$\begin{aligned} c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt &= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi}(f-P)e^{-int}dt + \int_{-\pi}^{\pi} P\cdot e^{-int}dt\right) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(f-P)e^{-int}dt \to 0 \qquad \text{as} \qquad n \to \infty \end{aligned}$$

This proves the desired result. □

We now can compute and simplify the Fourier series into simpler form.

$$s_n(x) = \sum_{k=-n}^{n} c_k e^{ikx} = \sum_{k=-n}^{n} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cdot e^{ik(x-t)}dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left(\frac{1}{2}\sum_{k=-n}^{n} e^{ik(x-t)}\right)dt.$$

Definition 20 (Dirichlet kernel)**.** We define the Dirichlet kernel

$$D_n(x-t) := \frac{1}{2}\sum_{k=-n}^{n} e^{ik(x-t)}.$$

Then we have

$$s_n(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t)dt.$$

Note that

$$D_n(t) = \frac{1}{2}\cdot\frac{e^{-int}(e^{i(2n+1)t}-1)}{e^{it}-1} = \frac{\sin(n+1/2)t}{2\sin t/2}.$$

We have some important properties:

Proposition 1.

1. *D_n is an even function and* $\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(t)dt = \frac{1}{2\pi}\int_{0}^{\pi} D_n(t)dt = 1$.
2. $|D_n(t)| \le (1/2)\sum_{k=-n}^{n}\left|e^{ikt}\right| = n + (1/2)$.
3. $|D_n(t)| \le \frac{\pi}{2|t|}$.

Proof. Some simple calculus calculation gives us

$$\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi} \qquad \text{whenever} \qquad 0 \leq t \leq \pi.$$

This proves the Proposition 3. $\square$

Now we shall introduce a classic trick. We write

$$D_n^\sharp := \frac{D_{n-1}(t) + D_n(t)}{2}.$$

Then we have

$$D_n^\sharp(t) = \frac{\sin{(n-1/2)t} + \sin{(n+1/2)t}}{4\sin{(t/2)}} = \frac{\sin(nt)}{2\tan{(t/2)}}$$

and

$$D_n(t) - D_n^\sharp(t) = \frac{D_n - D_{n-1}}{2} = \frac{\cos{(nt)}}{2}.$$

This implies that

$$\begin{aligned} s_n(f;x) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t)dt \\ &= \frac{1}{\pi}\left(\int_{-\pi}^{\pi} f(t)D_n^\sharp(x-t)dt + \int_{-\pi}^{\pi} f(t)(D_n - D_n^\sharp)(x-t)dt\right). \end{aligned}$$

When we study whether $s_n(f;x)$ converges to a function, we may only need to study whether the sequence

$$\int_{-\pi}^{\pi} f(t)D_n^\sharp(x-t)dt$$

converges, since Riemann-Lebesgue's Theorem (Theorem 19) asserts that

$$\int_{-\pi}^{\pi} f(t)(D_n - D_n^\sharp)(x-t)dt \to 0 \qquad \text{as} \qquad n \to \infty.$$

Lemma 7. Suppose $f \in L^1$. Then

$$\lim_{n\to\infty} s_n(f;x) = \lim_{n\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin n(x-t)}{x-t}dt,$$

provided that the limit on the right side exists.

Proof. We first define a function $\phi(t) := \dfrac{1}{\tan{(t/2)}} - \dfrac{1}{(t/2)}$ defined on $[-\pi,\pi]$. By L'Hopital Rule, ϕ is bounded. Thus it is integrable.

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n^\sharp(x-t)dt = \frac{1}{\pi}\left(\int_{-\pi}^{\pi} \frac{f(t)\phi(x-t)\sin n(x-t)}{2}dt + \int_{-\pi}^{\pi} f(t)\frac{\sin n(x-t)}{x-t}dt\right).$$

Note that by Riemann-Lebesgue's Theorem (Theorem 19) again,

$$\int_{-\pi}^{\pi} \frac{f(t)\phi(x-t)\sin n(x-t)}{2}dt \to 0.$$

This proves the theorem. $\square$

Here are more properties of $D_n^\sharp$.

Proposition 2.

1. *$D_n^\sharp$ is an even function and* $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} D_n^\sharp(t)dt = 1$.

2. $\left|D_n^\sharp\right| \le n$.

3. $\left|D_n^\sharp\right| \le \dfrac{\pi}{|t|}$.

Since $D_n^\sharp$ is an even function and f is periodic, we may write

$$\begin{aligned}
s_n^\sharp(f;x) &:= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n^\sharp(x-t)dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(x+t)+f(x-t)}{2} D_n^\sharp(t)dt \\
&= \frac{1}{\pi}\int_{0}^{\pi} \left(f(x+t)+f(x-t)\right) D_n^\sharp(t)dt
\end{aligned}$$

We now gives the following theorem. This theorem is known as Dini's Theorem, it provides an sufficient condition for a Fourier series to converge.

Theorem 21 (Dini's Theorem)**.** *Suppose $f \in L^1[-\pi,\pi]$. Given $x \in [-\pi,\pi]$. If there is a real number A such that*

$$\int_0^{\pi} \left|\frac{f(x+t)+f(x-t)}{2} - A\right| \frac{dt}{t} < \infty.$$

Then $s_n(f;x) \to A$.

Proof. It suffices to show that $s_n^\sharp(f;x) \to A$ by discussions above. Since

$$A = A \times 1 = A \times \frac{1}{\pi}\int_{-\pi}^{\pi} D_n^\sharp(t)dt,$$

we have

$$s_n^\sharp(f;x) - A = \frac{2}{\pi}\int_0^{\pi} \left(\frac{f(x+t)+f(x-t)}{2} - A\right) \cdot \frac{\sin(nt)}{2\tan(t/2)}dt.$$

Note that

$$\begin{aligned}
\frac{\pi}{2}\left|s_n^\sharp(f;x) - A\right| \le &\int_0^{\pi} \left|\frac{f(x+t)+f(x-t)}{2} - A\right| \cdot \frac{\sin(nt)}{t}dt \\
&+ \int_0^{\pi} \left|\frac{f(x+t)+f(x-t)}{2} - A\right| \cdot \left(\frac{1}{2\tan(t/2)} - \frac{1}{t}\right)\sin(nt)dt.
\end{aligned}$$

Since

$$\left\{\begin{array}{l}
\left|\dfrac{f(x+t)+f(x-t)}{2} - A\right| \cdot \dfrac{1}{t} \in L^1[-\pi,\pi] \\
\left|\dfrac{f(x+t)+f(x-t)}{2} - A\right| \cdot \left(\dfrac{1}{2\tan(t/2)} - \dfrac{1}{t}\right) \in L^1[-\pi,\pi]
\end{array}\right. .(\text{fix}x),$$

Riemann-Lebesgue's Theorem asserts that both integral converge to 0 as $n \to \infty$ which indicates $s_n^\sharp(f;x) \to A$. $\square$

Corollary 2. In particular, $f \in C^1[-\pi,\pi]$, then $s_n(f;x) \to f$. In fact, if f is locally Lipschitz, then $s_n(f;x) \to f(x)$ at that point x.

Theorem 22. *Let ω be a function defined by*

$$\omega(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x+t) - f(t)|\, dt.$$

Suppose

$$\int_0^{\pi} \omega(f;x)\cdot\frac{dx}{x} < \infty.$$

Then $s_n(f;x) \to f(x)$ almost everywhere.

Proof. Consider the function I defined by

$$I(x) = \int_0^{\pi} |f(x+t) - f(t)|\,\frac{dt}{t}.$$

Then we have

$$\begin{aligned}
\int_{-\pi}^{\pi} I(x)dx &= \int_{-\pi}^{\pi}\int_0^{\pi} |f(x+t) - f(t)|\,\frac{dt}{t}\cdot dx \\
&= \int_0^{\pi}\int_{-\pi}^{\pi} |f(x+t) - f(t)|\,dx\cdot\frac{dt}{t} \qquad \text{(Tonelli's theorem)} \\
&= 2\pi\int_0^{\pi} \omega(f;x)\cdot\frac{dt}{t} < \infty \qquad \text{(by assumption)}.
\end{aligned}$$

Hence $I(x)$ is finite almost everywhere. That is,

$$\int_0^{\pi} |f(x+t) - f(x)|\cdot\frac{dt}{t} < \infty \implies \int_0^{\pi}\left|\frac{f(x+t)+f(x-t)}{2} - f(x)\right|\cdot\frac{dt}{t} < \infty.$$

By Dini's Theorem (Theorem 21), we obtain $s_n(f;x) \to f(x)$ almost everywhere. □

The convergence of $s_n(f;x)$ depends on the local properties of f. Even f is continuous, $s_n(f;x)$ can still diverge at some points. We now shall show the following theorem.

Theorem 23. *There exists a function $f \in C[-\pi,\pi]$ such that $s_n(f;0)$ diverges.*

Before proving this theorem, we shall first prove some useful theorems.

Theorem 24 (Uniform bounded principle)**.** *Let X be a complete normed space and let Y be a normed space. Let $\{T_n\}_{n=1}^{\infty}$ be a family of bounded linear transformation from X to Y. (The boundedness might not be uniform.) Assume that for all $x \in X$, the set $\{T_n(x)\}_{n=1}^{\infty}$ is bounded in Y. Then $\{T_n\}_{n=1}^{\infty}$ is uniform bounded. In other words, there exists a number $C > 0$ such that $\|T_n\| \le C < \infty$ for all $n \in \mathbb{N}$.*

Proof. We first claim that there exist $x_0 \in X$, $\epsilon > 0$ and a constant $K > 0$ such that $\|T_n(x)\| \le K$ whenever $\|x - x_0\| \le \epsilon$. If the claim is not true, then for all $x \in X$ and $\epsilon > 0$,

$$\bigcup_{n=1}^{\infty} T_n(B(x;\epsilon)) \text{ is not bounded.}$$

Let x_0 be an arbitrary vector in X, and let $r_0 = 1$. Consider $B_0 = \overline{B}(x_0; r_0)$. There exists $x_1 \in B_0$ and $n_1 \in \mathbb{N}$ such that $\|T_{n_1}(x_1)\| > 1$. Since T_{n_1} is continuous, there is $r_1 \in (0, 1)$ such that $\|T_{n_1}(x)\| > 1$ whenever $x \in B_1 := \overline{B}(x_1; r_1)$. Suppose $x_1, \dots, x_k$ and $r_1, \dots, r_k$ have been chosen. Choose $x_{k+1} \in B_k := \overline{B}(x_k; r_k)$ and n_{k+1} such that $\left\|T_{n_{k+1}}(x_{k+1})\right\| > k + 1$. The continuity of $T_{n_{k+1}}$ indicates that there is a number $r_{k+1} \in (0, 1/(k + 1))$ such that $B_{k+1} \subset B_k$ and

$$\left\|T_{n_{k+1}}(x)\right\| > k + 1 \qquad \text{whenever} \qquad x \in B_{k+1} := \overline{B}(x_{k+1}; r_{k+1}).$$

Then (B_k) is a decreasing closed ball on X and the diameter of these balls converge to 0 as $n \to \infty$. X is complete, hence

$$\bigcap_{n=1}^{\infty} B_n = \{v\}.$$

Note that

$$\|T_{n_k}(v)\| \to \infty \qquad \text{as} \qquad k \to \infty$$

which contradicts to the assumption that $\{T_n(x)\}$ is a bounded set in Y. Hence our claim is true. There are $x_0 \in X$, $\epsilon > 0$, and a constant $K > 0$ such that $\|T_n(x)\| \leq K$ whenever $\|x - x_0\| \leq \epsilon$. Now for all $x \in X$, we consider

$$z := \frac{\epsilon x}{\|x\|} + x_0.$$

We have $\|T_n(z)\| \leq K$. This implies that

$$\begin{aligned}
\left\|T_n\left(\frac{\epsilon x}{\|x\|}\right) + T_n(x_0)\right\| \leq K &\implies \|T_n(x)\| \cdot \frac{\epsilon}{\|x\|} \leq K + \|T_n(x_0)\| \\
&\implies \|T_n(x)\| \leq (K + C_{x_0}) \cdot \epsilon \cdot \|x\|,
\end{aligned}$$

where $C_{x_0} > 0$ is the constant that

$$\|T_n(x_0)\| \leq C_{x_0}.$$

This completes the proof. $\square$

Lemma 8. Recall that D_n is the Dirichlet kernel. We have the following.

$$L_n := \frac{1}{\pi} \int_{-\pi}^{\pi} |D_n(t)|\, dt \in \Theta(\log n).$$

In other words, there exist constants $c_2 \geq c_1 > 0$ and $n_0 \in \mathbb{N}$ such that

$$c_1 \log n \leq L_n \leq c_2 \log n \qquad \text{whenever} \qquad n \geq n_0.$$

We will omit the proof here.

Proof of Theorem 23. Suppose such f does not exist. Let $T_n : C[-\pi, \pi] \to \mathbb{C}$ be a family of functions defined by

$$T_n(f) = s_n(f; 0) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) D_n(t) dt.$$

Since we assume each $f \in C[-\pi, \pi]$, its Fourier series $s_n(f;0)$ at 0 always converge. This implies that there is $C_f > 0$ such that

$$\|T_n(f)\| \leq C_f < \infty.$$

By the Uniform Bounded Principle (Theorem 24), there exists $C > 0$ such that

$$\|T_n(f)\| \leq C < \infty \text{ for all } n \in \mathbb{N} \text{ and } f \in C[-\pi, \pi].$$

It now suffices to show that for all $M > 0$ there exist $k \in \mathbb{N}$ and $f_k \in C[-\pi, \pi]$ such that

$$|T_k(f_k)| > M.$$

This follows by Lemma 8, and the fact that

$$\left(\int_{-\pi}^{\pi} |D_n(t)|\, dt\right)^2 \leq \left(\int_{-\pi}^{\pi} |D_n(t)|^2\, dt\right)\left(\int_{-\pi}^{\pi} 1dt\right) = 2\pi \cdot T_n(D_n).$$

□

We also could give another constructive proof. *Proof of Theorem 23.* Our idea is that we use undetermined coefficient method to find a function

$$f(t) = \sum_{j=1}^{\infty} c_j \sin(n_j t)\chi_{I_j}(t),$$

where $(n_j)_j$ is an increasing integer sequence and $I_k := [\pi/n_j, \pi/n_{j-1}]$. We first let $c_1 = 1$, $n_0 = 1$ and $n_1 = 2$ (therefore $I_1 = [\pi/2, \pi]$). Suppose having chosen $c_1, \ldots, c_{k-1}$ and $n_1, \ldots, n_{k-1}$. Let

$$\phi_{k-1}(t) = \sum_{j=1}^{k-1} c_j \sin(n_j t)\chi_{I_j}(t).$$

It is clear by Riemann-Lebesgue's Theorem again that

$$\int_{-\pi}^{\pi} \frac{\phi_{k-1}(t)}{t} \sin(nt)dt \to 0 \qquad \text{as} \qquad n \to \infty.$$

We choose $n_k = n_1 n_2 \cdots n_{k-1} \cdot N_k$, where N_k is a power of two, such that

$$\left|\int_{-\pi}^{\pi} \frac{\phi_{k-1}(t)}{t} \sin(n_k t)dt\right| < 1.$$

We choose $c_k = (\log N_k)^{-\epsilon}$, where $\epsilon \in (0, 1)$ is a fixed number. Then we can compute

$$\frac{2}{\pi} \cdot s_{n_k}(f;0) = \int_{-\pi}^{\pi} f$$

□

Here we shall introduce some more properties of Fourier coefficients and Fourier series.

Proposition 3.

1. *Given $f(t)$ and given $a \in \mathbb{R}$, let $f_a(t) = f(a+t)$. Then*

$$c_k(f_a) = \int f_a(t)e^{-ikt}dt = \int f(t+a)e^{-ikt}dt = \int f(t+a)e^{-ik(t+a)}e^{ika}dt = c_k(f) \cdot e^{ika}.$$

2. *Given $f(t)$ and let $g(t) = f(t) \cdot e^{int}$. Then*

$$c_k(g) = \int f(t)e^{i(n-k)t}dt = c_{k-n}(f).$$

3. *Let $f \in L^1[-\pi, \pi]$. Let*

$$F(t) := c + \int_0^t f(s)ds.$$

Then

$$F(t) - c_0 t \sim c' + \sum_{k \neq 0} \frac{c_k(f)}{ik} e^{ikt}.$$

4. *If f is absolutely continuous and assume f and f' is periodic. Suppose*

$$f \sim \sum_{k=-\infty}^{\infty} c_k e^{ikt}.$$

Then

$$f' \sim \sum_{k=-\infty}^{\infty} ikc_k e^{ikt}.$$

5. *If f is absolutely continuous, then $|c_k(f)| = o(1/k)$.*
6. *If $c_k(f)$ and $c_k(g)$ are the Fourier coefficients of f and g respectively, then*

$$f * g \sim \sum_{k=-\infty}^{\infty} c_k(f)c_k(g)e^{ikt}.$$

The proposition above use the notation $*$, it means "convolution". We shall give a definition here.

Definition 25 (Convolution)**.** Let $f, g \in L[-\pi, \pi]$ be two periodic functions. We define the convolution of f and g be

$$f * g(x) = \int_{-\pi}^{\pi} f(x-t)g(t)dt = \int_{-\pi}^{\pi} f(t)g(x-t)dt.$$

Proof. We shall prove the non-trivial ones.

3. Note that $F(t+2\pi) - F(t) = \displaystyle\int_t^{t+2\pi} f(s)ds = 2\pi c_0(f)$. Let $H(t) = F(t) - c_0(f) \cdot t$. It is clear that H is a periodic function. We now compute its Fourier coefficients.

$$\begin{aligned} 2\pi \cdot c_k(H) &= \int_{-\pi}^{\pi} H(t)e^{-ikt}dt \\ &= H(\pi)\frac{\exp(-ik\pi)}{-ik} - H(-\pi)\frac{\exp(ik\pi)}{-ik} + \int_{-\pi}^{\pi} \frac{f(t)}{ik} \cdot e^{-ikt}dt \\ &= \frac{2\pi c_k(f)}{ik}. \end{aligned}$$

4. We have
$$f(t) = f(-\pi) + \int_{-\pi}^{t} f'(s)ds$$
for almost every $t \in \mathbb{R}$. By the last proposition, we obtain
$$f(t) - c_0(f')t \sim c' + \sum_{k \neq 0} \frac{c_k(f')}{ik} e^{ikt}.$$
The assertion follows from $c_0(f') = 0$. (f is periodic.)

5. Since $f' \in L^1[-\pi, \pi]$, it follows by the Riemann-Lebesgue Theorem that $|ikc_k| \to 0$ and thus $|c_k| = o(1/k)$.

6. Consider
$$\begin{aligned} c_k(f * g) &= \int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} f(x-t)g(t)dt \right) e^{-ikx} dx \\ &= \left(\int_{-\pi}^{\pi} g(t)e^{-ikt}dt \right) \left(\int_{-\pi}^{\pi} f(x-t)e^{-ik(x-t)}dx \right) \qquad \text{(By Fubini's Theorem.)} \\ &= c_k(f) \cdot c_k(g). \end{aligned}$$
It is worth noting that Fubini's Theorem is applicable since
$$\|fg\|_1 \leq \|f\|_1 \cdot \|g\|_1,$$
which could be obtained by applying Tonelli's Theorem to $\displaystyle\int |f| * |g|$.

$\square$

Remark. In fact, we have f is $[\alpha]$-order differentiable if $|c_k(f)| = o(1/k^\alpha)$.

1.4 Cesàro sum of Fourier series

In last subsection, we find that $s_n(f; x)$ might not even converge although f is continuous. We want to study whether it converges in Cesàro sense.

Definition 26. A sequence of number $\{c_j\}_{j=1}^{\infty}$ is said to be Cesàro summable to L, if
$$\frac{c_1 + c_2 + \cdots + c_n}{n} \to L$$
as $n \to \infty$.

We write
$$\sigma_n(f; x) = \frac{s_0(f; x) + s_1(f; x) + \cdots + s_n(f; x)}{n+1}.$$
Note that $s_m(f; x) = \displaystyle\sum_{j=-m}^{m} c_k e^{ikx}$. Some calculations give us
$$\sigma_n(f; x) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) c_j e^{ijx}.$$

Also we have

$$\pi\sigma_n(f;x) = \frac{f * D_0 + \cdots + f * D_n}{n+1} = f * \frac{D_0 + \cdots + D_n}{n+1}.$$

Now we simplify it. We often write K_n to denote

$$\begin{aligned}
\frac{D_0 + \cdots + D_n}{n+1} &= \frac{1}{2\sin(t/2)} \Im\left(\sum_{j=0}^{n} e^{i(j+1/2)t}\right) \cdot \frac{1}{n+1} \\
&= \frac{1}{2\sin(t/2)} \Im\left(e^{it/2} \cdot \frac{1-e^{i(n+1)t}}{1-e^{it}}\right) \cdot \frac{1}{n+1} \\
&= \frac{1}{2(n+1)} \cdot \left(\frac{\sin((n+1)t/2)}{\sin(t/2)}\right)^2
\end{aligned}$$

It is worth noting that some other books often define

$$K_n := \frac{1}{2n} \cdot \left(\frac{\sin nt/2}{\sin(t/2)}\right)^2$$

which is more reasonable. However, I am not interested in fixing up this issue in this note. Similar to what we have done before, there are some basic properties of K_n could be derived immediately.

Proposition 4.

1. *K_n is nonnegative and K_n is an even function.*
2. $\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(t)dt = 1.$
3. $K_n(t) \le \frac{1}{n+1}\sum_{j=0}^{n}\left(j+\frac{1}{2}\right) \le \frac{n+1}{2}$. *(Recall that* $|D_n(t)| \le n + \frac{1}{2}$.*)*
4. $K_n(t) \le \frac{\pi}{2|t|}$. *Another upper bound is* $\frac{\pi^2}{2(n+1)t^2}$.
5. $\sigma_n(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(x+t)+f(x-t)}{2} K_n(t)dt = \frac{1}{\pi}\int_0^{\pi} (f(x+t)+f(x-t)) K_n(t)dt.$

Although it is very likely that a Fourier series of a function may not converge, it is easy for it to converge in Cesàro sense. We have the following theorem.

Theorem 27 (Fejer's Theorem)**.** *Suppose $f \in L^1[-\pi,\pi]$. Assume that both $f(x+)$ and $f(x-)$ exist. Then*

$$\sigma_n(f;x) \to \frac{f(x+)+f(x-)}{2}.$$

Proof. Without loss of generality, we may assume $f(x) = (f(x+)+f(x-))/2$. (Changing a value of a point does not impact the Fourier coefficients.) Now we have

$$\sigma_n(f;x) - f(x) = \frac{2}{\pi}\int_0^{\pi} \left(\frac{f(x+t)+f(x-t)}{2} - f(x)\right) K_n(t)dt.$$

Given $\epsilon > 0$. There exists a $\delta > 0$ such that

$$\begin{cases} |f(x+t) - f(x+)| < \epsilon, \text{ if } 0 < t < \delta \\ |f(x+t) - f(x-)| < \epsilon, \text{ if } -\delta < t < 0 \end{cases}$$

This implies that

$$\left| \frac{f(x+t) + f(x-t)}{2} - f(x) \right| < \epsilon, \text{ if } |t| < \delta.$$

It follows that

$$\int_0^\delta \left| \frac{f(x+t) + f(x-t)}{2} - f(x) \right| K_n(t) dt \leq \epsilon \cdot \int_0^\pi K_n(t) dt = \frac{\epsilon\pi}{2} \tag{1}$$

and that

$$\int_\delta^\pi \left| \frac{f(x+t) + f(x-t)}{2} - f(x) \right| K_n(t) dt \leq \frac{\pi^2 \cdot C}{2(n+1)\delta^2} \tag{2}$$

where C is the constant

$$C := \int_0^\pi \left| \frac{f(x+t) + f(x-t)}{2} - f(x) \right| dt.$$

Together with (1) and (2), we conclude that

$$|\sigma_n(f;x) - f(x)| \leq \epsilon + \frac{\pi C}{(n+1)\delta^2}.$$

This shows $\sigma_n(f;x) \to f(x) := (1/2)(f(x+) + f(x-))$. □

Remark.

1. If f is continuous at x, then $\sigma_n(f;x) \to f(x)$.
2. If f is continuous on $[c,d] \subset [-\pi,\pi]$, then $\sigma_n \to f$ uniformly on $[c,d]$.

Proof. We shall give a proof to the second assertion. For all $x \in [c,d]$, there exist $N = N(x) \in \mathbb{N}$ and $\delta = \delta(x) > 0$ such that

$$|\sigma_n(f;t) - f(t)| < \epsilon \qquad \text{whenever} \qquad t \in B(x;\delta) \text{ and } n \geq N.$$

The remaining part follows by the fact that $[c,d]$ is a compact interval. □

Corollary 3. Suppose f is continuous on $[-\pi,\pi]$, then given $\epsilon > 0$, there is a trigonometric polynomial P such that

$$|P(x) - f(x)| < \epsilon \text{ for all } x \in [-\pi,\pi].$$

This corollary follows by last remark. We now prove a similar result, however not under the supremum norm, but under the L^1 norm instead.

Lemma 9. Suppose $f \in L^1[-\pi,\pi]$. Given $\epsilon > 0$. There is a trigonometric polynomial P such that

$$\int_{-\pi}^\pi |f(x) - P(x)| \, dx < \epsilon.$$

In other words, $\|f - P\|_1 < \epsilon$.

Proof. We may assume that $f \geq 0$. (Recall that $f = f^+ - f^-$.) It is easy to see that this lemma holds if f is continuous function (by the last corollary). Now suppose $f = \chi_A$, where A is a closed set. In Lemma 2, we have shown that there exist continuous function g_n such that

$$\|g_n - f\|_2 < \frac{1}{n}.$$

This implies

$$\|g_n - f\|_1 < \frac{\sqrt{b-a}}{n}.$$

Now let E be a measurable set and let $f = \chi_E$. Then there exist closed sets $(A_j)_{j \in \mathbb{N}}$ such that

$$m(E) \leq m(A_j) + \frac{1}{j}.$$

Hence $\left\|f - \chi_{A_j}\right\|_1 < 1/j$ and $\|f - g\| < \epsilon$ for some continuous function g. We have proved the case when f is χ_E, E is a measurable set. Now for general integrable function f, we know that there is some closed set E_j and $c_j \in \mathbb{R}$ such that

$$\left\|f - \sum_{j=1}^{n} c_j \chi_{E_j}\right\|_1 < \epsilon.$$

Thus we see that there are some continuous function g such that $\|f - g\|_1 < \epsilon$. $\square$

We now generalize the idea of L^2 and L^1 norm to general L^p norm.

Definition 28.

1. If $1 \leq p < \infty$, let $L^p[-\pi, \pi] = \{f : \int |f|^p < \infty\}$, and we use $\|\cdot\|_p = \left(\int |f|^p\right)^{1/p}$ to denote the standard norm on L^p space.
2. If $p = \infty$, let

$$L^\infty[-\pi, \pi] = \{f : \exists M > 0 \text{ such that } |f(x)| \leq M < \infty \text{ for almost every } x\}.$$

We use $\|f\|_\infty = \inf M$ to denote the norm on L^∞, where the infimum is taken among all M such that $|f(x)| \leq M < \infty$ almost everywhere.

Theorem 29 (Continuity in the L^p norm)**.** *Let $f \in L^p[-\pi, \pi]$ $(1 \leq p < \infty)$. Then*

$$\lim_{h \to 0} \int_{-\pi}^{\pi} |f(t+h) - f(t)|^p \, dt = 0.$$

Proof. First if g is continuous on $[-\pi, \pi]$, then it is clear that

$$\lim_{h \to 0} |g(t+h) - g(t)|^p \, dt = 0.$$

In the proof of Lemma 9, we have shown that every function f in L^1, there exists a continuous function g such that

$$\|f - g\|_1 < \epsilon.$$

A slight modification on the proof give that: if $f \in L^p$, there is a continuous function g such that

$$\|f - g\|_p < \epsilon.$$

Now we have

$$\int_{-\pi}^{\pi} |f(t+h) - f(t)|^p \, dt = \int_{-\pi}^{\pi} |f(t+h) - g(t+h) + g(t+h) - g(t) + g(t) - f(t)|^p \, dt$$
$$\leq 3^{p-1} \left(\int_{-\pi}^{\pi} |f(t+h) - g(t+h)|^p \, dt + \int_{-\pi}^{\pi} |g(t+h) - g(t)|^p \, dt + \int_{-\pi}^{\pi} |g(t) - f(t)|^p \, dt \right).$$

The right side converges to 0 as $h \to 0$. □

Theorem 30. *Let $f \in L^p[-\pi, \pi]$ $(1 \leq p < \infty)$. Then,*

1. $\|\sigma_n(f) - f\|_p \to 0$ *as* $n \to \infty$.
2. $\|\sigma_n(f)\|_p \leq \|f\|_p$ *for all* $n \in \mathbb{N}$.

We shall first prove another very useful result, which is known as Hölder's inequality.

Theorem 31 (Hölder's inequality)**.** *Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ be two integers such that*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then we have

$$\int |fg| \leq \left(\int |f|^p \right)^{1/p} \left(\int |g|^q \right)^{1/q}.$$

In other words,

$$\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q.$$

The latter statement is valid when p or q is ∞.

Proof. We may assume that $p, q \neq \infty$, otherwise it is trivial. We first claim that

$$|f(x)g(x)| \leq \frac{1}{p} |f(x)|^p + \frac{1}{q} |g(x)|^q \text{ for all } x. \tag{3}$$

If the claim is true, then for any $\|f\|_p = 1 = \|g\|_q$, we have

$$\int |fg| \leq \frac{1}{p} \int |f|^p + \frac{1}{q} \int |g|^q = \frac{1}{p} + \frac{1}{q} = 1 \implies \|fg\|_1 \leq 1.$$

For f and g such that $\|f\|_p \neq 1$ or $\|g\|_q \neq 1$, we could normalize it to 1. It now remains to show our claim is true. We write $a = |f(x)|$, $b = |g(x)|$, $s = 1/p$, and $t = 1/q$. Then (3) is equivalent to

$$a^s b^t \leq as + bt \iff s \log a + t \log b \leq \log (as + bt),$$

holds for all $s + t = 1$, $s, t \in (0, 1)$, $a, b > 0$. The concavity of log implies the theorem. □

Proof of Theorem 30. Our objective is to show that $\displaystyle\int |\sigma_n(f) - f|^p \to 0$. Let $F(t)^p = \displaystyle\int |f(x+t) - f(x)|^p \, dx$. We know that $F(t)^p$ is continuous at $t = 0$ (Theorem 29). Note that

$$\sigma_n(f;x) - f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x+t) - f(x))K_n(t)dt.$$

Let $q \in (1, \infty]$ such that $1/p + 1/q = 1$. Then

$$\begin{aligned}
\int_{-\pi}^{\pi} (f(x+t) - f(x))K_n(t)dt &\leq \int_{-\pi}^{\pi} |f(x+t) - f(x)| \cdot K_n(t)^{1/p} \cdot K_n(t)^{1/q} dt \\
&\overset{\spadesuit}{\leq} \left(\int_{-\pi}^{\pi} |f(x+t) - f(x)|^p \cdot K_n(t)dt \right)^{1/p} \left(\int_{-\pi}^{\pi} K_n(t)dt \right)^{1/q} \\
&= \pi^{1/q} \left(\int_{-\pi}^{\pi} |f(x+t) - f(x)|^p \cdot K_n(t)dt \right)^{1/p}.
\end{aligned}$$

The inequality ($\spadesuit$) holds by Hölder's inequality (Theorem 31). This gives

$$\begin{aligned}
|\sigma_n(f;x) - f(x)|^p &\leq \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x+t) - f(x)|^p \cdot K_n(t)dt \\
\implies \int_{-\pi}^{\pi} |\sigma_n(f;x) - f(x)|^p \, dx &\leq \frac{1}{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x+t) - f(x)|^p \cdot K_n(t)dtdx \\
&\overset{\clubsuit}{=} \frac{1}{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x+t) - f(x)|^p \cdot K_n(t)dxdt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)^p K_n(t)dt = \frac{1}{\pi}\sigma_n(F^p; 0) \to 0 \quad \text{(Theorem 27)}.
\end{aligned}$$

We can exchange the order of integration in ($\clubsuit$) by Tonelli's Theorem. The inequality above proves the first statement. Similarly, we have

$$\begin{aligned}
|\sigma_n(f;x)|^p &\leq \left(\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x+t)| \cdot K_n(t)^{1/p} \cdot K_n(t)^{1/q} dt \right)^p \\
&\leq \left(\frac{1}{\pi} \left(\int_{-\pi}^{\pi} |f(x+t)|^p K_n(t)dt \right)^{1/p} \left(\int_{-\pi}^{\pi} K_n(t)dt \right)^{1/q} \right)^p \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x+t)|^p K_n(t)dt
\end{aligned}$$

Hence

$$\begin{aligned}
\|\sigma_n\|_p^p = \int_{-\pi}^{\pi} |\sigma_n(f;x)|^p \, dx &\leq \frac{1}{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x+t)|^p K_n(t)dtdx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x+t)|^p K_n(t)dxdt \\
&= \frac{\|f\|_p^p}{\pi}\int_{-\pi}^{\pi} K_n(t)dt \\
&= \|f\|_p^p.
\end{aligned}$$

This completes the proof. □

The second statement in fact also holds when $p = \infty$ and in that case we could consider the inequality

$$\|\sigma_n(f)\|_\infty = \frac{1}{\pi}\int_{-\pi}^{\pi} \|f(\cdot + t)\|_\infty K_n(t)dt \leq \|f\|_\infty .$$

We now give a remark on convergence of $s_n(f;x)$ and $\sigma_n(f;x)$.

Remark.

1. In 1953, Kolmogorov, a Soviet mathematician, had proved that there exists a function $f \in L^1[-\pi,\pi]$ such that $s_n(f;x)$ diverge almost everywhere. A year later, he published another paper showed that there exists $f \in L^1[-\pi,\pi]$ such that $s_n(f;x)$ diverge everywhere.
2. Carleson (1965) showed that for all $f \in L^2[-\pi,\pi]$, $s_n(f;x)$ converges to $f(x)$ for almost every x.
3. Hunt (1968) showed that for all $f \in L^p[-\pi,\pi]$ ($p \in (1,\infty)$), $s_n(f;x)$ converges to $f(x)$ for almost every x.

However, these theorems are very hard to prove, we will not give a proof here.

Theorem 32. *Given a sequence of numbers $(c_j)_{-\infty}^{\infty}$. Let*

$$\sigma_n(t) := \sum_{|j|\leq n}\left(1 - \frac{|j|}{n+1}\right) c_j e^{ijt}.$$

Then $\sum c_j e^{ijt}$ is the Fourier series of $f \in C[-\pi,\pi]$ if and only if σ_n converges uniformly.

Proof. If $\sum c_j e^{ijt}$ is the Fourier series of $f \in C[-\pi,\pi]$, then σ_n converges uniformly by the remark of Theorem 27. Now suppose σ_n converges uniformly. Since (σ_n) are all continuous, there exists $f \in C[-\pi,\pi]$ such that $\sigma_n \rightrightarrows f$. We claim that

$$c_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ijt}dt.$$

It follows from

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ijt}dt &= \frac{1}{2\pi}\lim_{n\to\infty}\int_{-\pi}^{\pi} \sigma_n(t)e^{-ijt}dt \\ &= \lim_{n\to\infty}\left(1 - \frac{|j|}{n+1}\right) c_j = c_j.\end{aligned}$$

This proves the theorem. $\square$

Lemma 10. Let $1/p + 1/q = 1$ ($1 \leq p \leq \infty$) and let $f \in L^1[-\pi,\pi]$ such that

$$\int_{-\pi}^{\pi} f\phi \leq K \|\phi\|_q$$

for all simple functions ϕ, where $K < \infty$ is a positive constant. Then $f \in L^p[-\pi,\pi]$ and $\|f\|_p \leq K$.

Proof. If $p = 1$, it suffices to choose $\phi = \text{sign}(g) \in L^\infty$. Now assume that $p \in (1, \infty)$. Let $\{\phi_n\}$ denote a sequence on nonnegative simple functions such that $\phi_n \leq \phi_{n+1}$ and $\phi_n \to |f|^p$. Since

$$0 \leq \phi_n^{1/p} \leq |f| \in L^1[-\pi, \pi],$$

the functions

$$h_n = \phi_n^{1/q} \text{sign}(f)$$

are simple and in $L^q[-\pi, \pi]$. It follows that

$$\int_{-\pi}^{\pi} \phi_n = \int_{-\pi}^{\pi} \phi_n^{1/p} \phi_n^{1/q} \leq \int_{-\pi}^{\pi} |f| \, \phi_n^{1/q} = \int_{-\pi}^{\pi} f h_n \leq K \, \|h_n\|_q = K \left(\int_{-\pi}^{\pi} \phi_n \right)^{1/q}$$

This gives

$$\left(\int_{-\pi}^{\pi} \phi_n \right)^{1/p} \leq K.$$

By Fatou's Lemma, we conclude that

$$\|f\|_p \leq \left(\liminf_{n\to\infty} \int_{-\pi}^{\pi} \phi_n \right)^{1/p} \leq K.$$

This proves the case when $p \neq \infty$.

Now assume that $p = \infty$. For any $\epsilon > 0$ set

$$E_\epsilon = \{x \in [-\pi, \pi] \text{ such that } \ |g(x)| \geq K + \epsilon\}$$

and let $\phi = \chi_{E_\epsilon} \cdot \text{sign}(f)$ be a simple function. Note that $f \in L^1$ and the set $m(E_\epsilon) \leq 2\pi$, therefore we have

$$(K + \epsilon) \cdot m(E_\epsilon) \leq \left| \int f \phi \right| \leq K \cdot m(E_\epsilon).$$

Thus $m(E_\epsilon) = 0$ for all $\epsilon > 0$. □

Theorem 33 (Riesz representation theorem)**.** *Given $1/p + 1/q = 1$ ($1 < p \leq \infty$). Then there exists a bijection*

$$L^q[-\pi, \pi]^\vee \longleftrightarrow L^p[-\pi, \pi].$$

More precisely, for each $\ell \in (L^q)^\vee$, there is a function $f \in L^p$ such that

$$\ell(g) = \int f g.$$

Note that here we write $V^\vee$ to denote all linear "bounded" functional on V.

Proof. Let $F(x) = \ell(\chi_{[-\pi,x]})$. We claim that F is absolutely continuous. Let $\{I_j = (a_j, b_j)\}$ be finitely many disjoint intervals be given. Define

$$\phi(t) := \sum_j \text{sgn}(F(b_j) - F(a_j)) \chi_{I_j}(t).$$

Then $\|\phi\|_q^q \leq \sum_j |I_j|$. This implies that

$$\sum_j |F(b_j) - F(a_j)| = \ell(\phi) \leq M \cdot \|\phi\|_q \leq M \cdot \left(\sum_j |I_j|\right)^{1/q}.$$

This shows F is absolutely continuous. Thus there exists $f \in L^1[-\pi, \pi]$ such that

$$F(x) = \int_{-\pi}^{x} f(t)dt.$$

It implies that

$$\ell(\chi_{[-\pi,x]}) = \int_{-\pi}^{\pi} f(t) \cdot \chi_{[-\pi,x]}(t)dt.$$

Since ℓ is linear, for any step function $g = \sum a_j \chi_{I_j}$ (finite linear combination of χ_{I_j}), we have

$$\ell(g) = \int_{-\pi}^{\pi} f(t)g(t)dt.$$

Now let $g = \chi_E$ be the characteristic function of a measurable set. Since E is measurable, then for any given $n \in \mathbb{N}$, there exist finitely many disjoint open intervals $(I_j)_{j=1}^k$ such that

$$\bigcup_{j=1}^{k} I_j \supset E \qquad \text{and} \qquad \sum_{j=1}^{k} m(I_j) \leq m(E) + \frac{1}{n}.$$

Define

$$g_n = \sum_{j=1}^{k} \chi_{I_j}.$$

Then

$$\int |g - g_n| < \frac{1}{n} \qquad \text{and} \qquad \|g - g_n\|_{\sup} \leq 1.$$

Since $g_n \to g$, we conclude that

$$\int_{-\pi}^{\pi} f g_n \to \int_{-\pi}^{\pi} f g$$

by dominated convergence theorem ($\|g - g_n\|_{\sup} \leq 1$). This shows that for any simple function g, we have $\ell(g) = \int fg$.

Note that

$$\int fg = \ell(g) \leq M_\ell \cdot \|g\|_q$$

for any simple functions g (recall that ℓ is bounded). Apply Lemma 10 on f, we conclude that $f \in L^p$. Now let g be any functions in L^p and let ϕ_n be sequence of simple functions that converge to g in the L^q norm (this is possible since simple functions are dense in L^q space). Then

$$\left| \int_{-\pi}^{\pi} f(\phi_n - g) \right| \leq \|f\|_p \|\phi_n - g\|_q \to 0$$

as $n \to \infty$. This implies that

$$\ell(\phi_n) \to \ell(g) \qquad \text{and} \qquad \int_{-\pi}^{\pi} f\phi_n \to \int_{-\pi}^{\pi} fg$$

thus

$$\ell(g) = \int_{-\pi}^{\pi} fg$$

for any $g \in L^q$. This completes the proof. $\square$

Theorem 34. *Let $\{f_n\}$ be a bounded sequence in $L^p[-\pi,\pi]$, $1 < p \leq \infty$. That is, there exists $M > 0$ such that $\|f_n\|_p \leq M < \infty$ for all $n \in \mathbb{N}$. Then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and a function $f \in L^p$ such that*

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} f_{n_k}(t)g(t)dt = \int_{-\pi}^{\pi} f(t)g(t)dt$$

holds for all $g \in L^q[-\pi,\pi]$, where $1/p + 1/q = 1$. We said the sequence $\{f_{n_k}\}$ converges weakly to f and write $f_{n_k} \rightharpoonup f$.

Proof. We first sketch our proof.

Step 1: Prove that there exists a subsequence $\{f_{n_k}\}$ such that the integral

$$\int_{-\pi}^{\pi} f_{n_k}(t)g(t)dt$$

exists for all trigonometric polynomial g with rational coefficients.

Step 2: Prove that for each $g \in L^q$, the limit

$$\lim_{k\to\infty} \int f_{n_k} g \text{ exists.}$$

Moreover, $g \mapsto \lim_{k\to\infty} \int f_{n_k} g$ is a linear bounded functional.

Step 3: Apply the Riesz Representation Theorem (Theorem): Every linear bounded functional on L^q can be represented as $L(g) = \langle f, g\rangle$ for some $f \in L^p$ that does not depend on the choice of g.

We first denote all rational coefficients trigonometric polynomials by A. It is countable. We may write $A = \{g_j\}_{j=1}^{\infty}$. Then we have

$$\left|\int f_n \cdot g_1\right| \leq \left(\int |f_n|^p\right)^{1/p} \left(\int |g_1|^q\right)^{1/q} \leq M \cdot \|g_1\|_q < \infty.$$

There exists a subsequence $\{f_n^{(1)}\}$ of $\{f_n^{(0)} := f_n\}$ such that

$$\lim_{n\to\infty} \int f_n^{(1)} g_1 \text{ exists.}$$

We repeat this process. More precisely, suppose the subsequence $\{f_n^{(k)}\}$ has been constructed. Then

$$\left|\int f_n^{(k)} \cdot g_{k+1}\right| \leq \left(\int \left|f_n^{(k)}\right|^p\right)^{1/p} \left(\int |g_{k+1}|^q\right)^{1/q} \leq M \cdot \|g_{k+1}\|_q < \infty.$$

There exists a subsequence $\{f_n^{(k+1)}\}$ of $\{f_n^{(k)}\}$ such that

$$\lim_{n\to\infty} \int f_n^{(k+1)} g_{k+1} \text{ exists.}$$

It is easy to see that $\{f_n^{(n)}\}$ is a subsequence of $\{f_n\}$ and that

$$\lim_{n\to\infty} \int f_n^{(n)} g_k \text{ exists for all } k \in \mathbb{N}.$$

This proves the step 1. We now write $f_{n_k} := f_k^{(k)}$.

Now let $g \in L^q[-\pi, \pi]$ be given and let $\epsilon > 0$. Then there exists $p \in A$ such that $\|g - p\|_q < \epsilon$ since A is dense in L^q (Theorem 9). Note that

$$\lim_{k\to\infty} \int f_k^{(k)} p \text{ exists.}$$

Hence there exists $N \in \mathbb{N}$ such that $\left|\int \left(f_n^{(n)} - f_m^{(m)}\right) p\right| \leq \epsilon$ for all $n, m \geq N$. Therefore,

$$\left|\int \left(f_n^{(n)} - f_m^{(m)}\right) p\right| \leq \left|\int \left(f_n^{(n)} - f_m^{(m)}\right)(g - p)\right| + \left|\int \left(f_n^{(n)} - f_m^{(m)}\right) p\right| \leq 2M \cdot \epsilon + \epsilon.$$

This shows that the limit

$$\lim_{k\to\infty} \int f_k^{(k)} g$$

exists for all $g \in L^q$. It is clear that the map

$$\ell : g \mapsto \lim_{k\to\infty} \int f_{n_k} g$$

is a linear map. Observe that

$$\left|\int f_{n_k} g\right| \leq M \cdot \|g\|_q .$$

It indicates ℓ is a linear bounded functional.

It follows from the Riesz representation theorem (Theorem 33) that $\ell(g) = \int fg$ for some $f \in L^p$ that does not depend on the choice of g. This proves the theorem. □

The theorem above help us to prove the following necessary and sufficient condition for a trigonometric series to be the Fourier series of an L^p function f $(1 < p \leq \infty)$.

Theorem 35. *Let*

$$s(x) = \sum_{j=-\infty}^{\infty} c_j e^{ijx} \qquad \text{and} \qquad \sigma_n = \sum_{|j|\le n} \left(1 - \frac{|j|}{n+1}\right) c_j e^{ijx}$$

be a trigonometric series and its Cesàro partial sum. Then $s(x)$ is the Fourier series of a function $f \in L^p$ $(1 < p \le \infty)$ if and only if

$$\|\sigma_n\|_p \le K < \infty.$$

Proof. If $s(x) = s(f;x)$, then we have shown in Theorem 30 that $\|\sigma_n\|_p \le \|f\|_p$. Now suppose $\|\sigma_n\|_p \le K < \infty$ for all $n \in \mathbb{N}$. By theorem 34, there exists a subsequence $\{\sigma_{n_k}\}$ and a function $f \in L^p$ such that $\sigma_{n_k} \rightharpoonup f$. Fix $j \in \mathbb{N}$ and let $g(t) = e^{-ijt}$. Then we have

$$\left(1 - \frac{|j|}{n_k+1}\right) c_j \to \int fg.$$

Hence

$$c_j(f) = \int fg = c_j.$$

This shows that $s(x) = s(f;x)$. □

We now shall start proving that if $f \in L^1[-\pi,\pi]$, then $\sigma_n \to f$ almost everywhere. This theorem however need some preparations first.

Lemma 11. If $\displaystyle\int |g_n - g| \to 0$ as $n \to \infty$. Then there exists a subsequence $\{g_{n_k}\}$ such that $g_{n_k} \to g$ almost everywhere.

Proof. For all $\alpha > 0$, define

$$X(\alpha, n) = \{x : |g_n(x) - g(x)| > \alpha\}.$$

Then we have $m(X(\alpha,n)) \le \dfrac{1}{\alpha}\displaystyle\int |g_n - g|$. Let $\alpha = 1/j$, $j \in \mathbb{N}$. There exists $n_j \in \mathbb{N}$ such that

$$m\left(X\left(\frac{1}{j}, n\right)\right) < \frac{1}{2^j} \qquad \text{whenever} \qquad n \ge n_j.$$

Then it is clear that $g_{n_j} \to g$ almost everywhere. □

Lemma 12. If $f \in L^1[-\pi,\pi]$, then

$$\int_0^{\epsilon} |f(x+t) - f(x)|\, dt = o(\epsilon)$$

for almost every x.

We will not prove Lemma 12 here, since it is a homework problem. We now could give a proof to the following theorem.

Theorem 36 (Lebesgue)**.** *If $f \in L^1[-\pi,\pi]$, then $\sigma_n(f;x) \to f(x)$ for almost every x.*

Proof. Recall that

$$|\sigma_n(f;x) - f(x)| = \frac{2}{\pi}\left|\int_0^\pi \left(\frac{f(x+t)+f(x-t)}{2} - f(x)\right) K_n(t)dt\right|.$$

Let

$$\begin{aligned}
A_n(x) &:= \int_{1/\sqrt[4]{n}}^{\pi} \left|\frac{f(x+t)+f(x-t)}{2} - f(x)\right| K_n(t)dt \\
B_n(x) &:= \int_0^{1/n} \left|\frac{f(x+t)+f(x-t)}{2} - f(x)\right| K_n(t)dt \\
C_n(x) &:= \int_{1/n}^{1/\sqrt[4]{n}} \left|\frac{f(x+t)+f(x-t)}{2} - f(x)\right| K_n(t)dt
\end{aligned}$$

It is clear that

$$\frac{\pi}{2}\,|\sigma_n(f;x) - f(x)| \leq A_n(x) + B_n(x) + C_n(x).$$

It suffices to show that for almost every x, $A_n(x), B_n(x), C_n(x) \to 0$ as $n \to \infty$. It is worth noting that the integral

$$M(x) := \int_0^\pi \left|\frac{f(x+t)+f(x-t)}{2} - f(x)\right| dt$$

exist for all x. Now we start estimate A_n, B_n, and C_n.

(1) Estimate $A_n(x)$: Recall that $K_n(t) \leq \dfrac{\pi^2}{2(n+1)t^2}$, we have

$$0 \leq A_n(x) \leq \frac{\pi^2}{2(n+1)}\int_{1/\sqrt[4]{n}}^{\pi} \left|\frac{f(x+t)+f(x-t)}{2} - f(x)\right| \frac{dt}{t^2} \leq \frac{\pi^2}{2(n+1)}\cdot M(x)\cdot\sqrt{n} \to 0.$$

This shows that $A_n(x)$ converges to 0 for every x.

(2) Estimate $B_n(x)$: Recall that $K_n(t) \leq (n+1)/2 \leq n$, if $n \geq 1$. We have

$$0 \leq B_n(x) \leq n \cdot \int_0^{1/n} \left|\frac{f(x+t)+f(x-t)}{2} - f(x)\right| dt.$$

From Lemma 12, we have

$$n \cdot \int_0^{1/n} \left|\frac{f(x+t)+f(x-t)}{2} - f(x)\right| dt = \frac{o(1/n)}{1/n} \to 0 \qquad \text{as} \qquad n \to \infty$$

for almost every x. This shows that $B_n(x) \to 0$ as $n \to \infty$ for almost every x.

(3) Estimate $C_n(x)$: Similar to (1), we have

$$0 \leq C_n(x) \leq \frac{\pi^2}{2n}\int_{1/n}^{1/\sqrt[4]{n}} \left|\frac{f(x+t)+f(x-t)}{2} - f(x)\right| \frac{dt}{t^2}.$$

Define the function

$$F_x(t) = \int_0^t \left|\frac{f(x+s)+f(x-s)}{2} - f(x)\right| ds \leq M(x).$$

Lebesgue's main theorem asserts that F_x viewed as a function of t is absolutely continuous and monotonically increasing. Integration by parts give us

$$\frac{1}{n}\int_{1/n}^{1/\sqrt[4]{n}}\left|\frac{f(x+t)+f(x-t)}{2}-f(x)\right|\frac{dt}{t^2}=\frac{1}{n}\left.\frac{F_x(t)}{t^2}\right|_{t=1/n}^{1/\sqrt[4]{n}}+\frac{2}{n}\int_{1/n}^{1/\sqrt[4]{n}}F_x(t)\,\frac{dt}{t^3}$$
$$\leq\frac{F_x(1/\sqrt[4]{n})}{\sqrt{n}}-n\cdot F_x(1/n)+\frac{2}{n}\int_{1/n}^{1/\sqrt[4]{n}}F_x(t)\,\frac{dt}{t^3}.$$

Observe that

$$\frac{F_x(1/\sqrt[4]{n})}{\sqrt{n}}\leq\frac{M(x)}{\sqrt{n}}\to 0$$

as $n\to 0$. We also have $n\cdot F_x(1/n)\to 0$ as $n\to\infty$ for almost every x (Lemma 12). It now suffices to show that for almost every x, the integral

$$\frac{2}{n}\int_{1/n}^{1/\sqrt[4]{n}}F_x(t)\,\frac{dt}{t^3}\to 0.$$

Lemma 12 asserts that

$$\frac{F_x(t)}{t}=o(1)$$

for almost every x. Fix such x, given $\epsilon>0$, there exists an $N\in\mathbb{N}$ large enough, such that $\dfrac{F_x(t)}{t}<\epsilon$ whenever $t\leq 1/\sqrt[4]{N}$. Now we have

$$\frac{2}{n}\int_{1/n}^{1/\sqrt[4]{n}}F_x(t)\,\frac{dt}{t^3}\leq\frac{2\epsilon}{n}\int_{1/n}^{1/\sqrt[4]{n}}\frac{dt}{t^2}=\frac{2\epsilon}{n}\cdot(n-\sqrt[4]{n})\leq 2\epsilon$$

whenever $n\geq N$. This shows that for almost every x, $C_n(x)\to 0$ as $n\to\infty$.

Discussions above prove the theorem. □

1.5 The conjugate Fourier series

We now turn our attention back to $s_n(f;x)$, the Fourier series of f. Now we define the conjugate Fourier series $\widetilde{s}_n(f;x)$ of $s_n(f;x)=a_0/2+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx)$ defined through

$$\widetilde{s}_n(f;x):=\sum_{k=1}^{n}(a_k\sin kx-b_k\cos kx).$$

We have

$$\widetilde{s}_n(f;x)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\left(\sum_{k=1}^{n}\sin k(x-t)\right)dt.$$

Similarly we may define

Definition 37 (conjugate Dirichlet kernel)**.** The conjugate Dirichlet kernel $\widetilde{D}_n$ is the kernel defined by

$$\widetilde{D}_n(t):=\sum_{k=1}^{n}\sin(kt)=\frac{\cos(t/2)-\cos(n+1/2)t}{2\sin(t/2)}.$$

For simplicity, in the discussion below, we use the notation

$$\psi_x(t) := \frac{f(x+t) - f(x-t)}{2}, \qquad \phi_x(t) = \frac{f(x+t) + f(x-t)}{2}$$

to denote the odd part and even part of the function $f(x+t) = f_x(t)$. Recall that Dini's theorem (Theorem 21) states that if

$$\int_0^\pi \frac{|f(x+t) + f(x-t) - 2f(x)|}{t} dt < \infty,$$

then $s_n \to f(x)$. Similarly, we want to find some sufficient conditions that the series $\widetilde{s}_n$ or $\widetilde{s}_n^\sharp$ are convergent, where $\widetilde{s}_n^\sharp$ is defined as $(\widetilde{s}_n + \widetilde{s}_{n-1})/2$.

Note that

$$\begin{aligned}
\widetilde{s}_n(x) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \widetilde{D}_n(x-t) dt = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \widetilde{D}_n(t) dt \\
&= -\frac{2}{\pi} \int_0^\pi (\psi_x(t) + \phi_x(t)) \widetilde{D}_n(t) dt = -\frac{2}{\pi} \int_0^\pi \psi_x(t) \cdot \frac{\cos(t/2) - \cos(n+1/2)t}{2\sin(t/2)} dt.
\end{aligned}$$

Some simple calculations give

$$\begin{aligned}
\widetilde{s}_n^\sharp(x) :=& \frac{1}{2}(\widetilde{s}_n + \widetilde{s}_{n-1})(x) = -\frac{2}{\pi} \int_0^\pi \frac{\psi_x(t) dt}{2\tan(t/2)} + \frac{2}{\pi} \int_0^\pi \frac{\psi_x(t) \cos(nt) dt}{2\tan(t/2)} \\
\widetilde{s}_n(x) - \widetilde{s}_n^\sharp(x) =& \frac{1}{2}(\widetilde{s}_n - \widetilde{s}_{n-1})(x) = \frac{-1}{\pi} \int_0^\pi \psi_x(t) \cos(nt) dt
\end{aligned}$$

If, in addition, we assume that the integral $\displaystyle\int_0^\pi \frac{\psi_x(t) dt}{2\tan(t/2)}$ exists, then by Riemann-Lebesgue Theorem (Theorem 19), we have

$$\int_0^\pi \frac{\psi_x(t) \cos(nt) dt}{2\tan(t/2)} \to 0 \qquad \text{and} \qquad \int_0^\pi \psi_x(t) \cos(nt) dt \to 0$$

as $n \to \infty$. Thus, in this case,

$$\widetilde{s}_n(x) \to -\frac{2}{\pi} \int_0^\pi \frac{\psi_x(t) dt}{2\tan(t/2)}.$$

So, it is intuitive to study whether the integral $\displaystyle\int_0^\pi \frac{\psi_x(t) dt}{2\tan(t/2)}$ exists. However, this integral might not exists (See Theorem 38). Luckily, we can consider the principal value of the integral, that is, the limit

$$\lim_{\epsilon \to 0^+} -\frac{1}{\pi} \int_\epsilon^\pi \frac{f(x+t) - f(x-t)}{2\tan(t/2)} dt.$$

In fact, later we will see that the limit exists provided that $f \in L^1[-\pi, \pi]$.

Theorem 38 (Lusin's Theorem)**.** *There is a continuous and periodic function $f \in C[-\infty, \infty]$ such that*

$$\int_0^\pi \frac{|f(x+t) - f(x-t)|}{t} dt = +\infty$$

for every $x \in [-\pi, \pi]$.

Remark. Since this theorem is not covered in class, we will not give the proof here.

Now we can define the concept of the conjugate function.

Definition 39 (conjugate function)**.** Given $f \in L^1$. We first consider the truncated conjugate function. Given $0 < \epsilon \le \pi$, let

$$\widetilde{f}_\epsilon(x) = -\frac{1}{\pi}\int_\epsilon^\pi \frac{f(x+t)-f(x-t)}{2\tan(t/2)}dt = -\frac{1}{\pi}\int_{\epsilon\le|t|\le\pi}\frac{f(x+t)}{2\tan(t/2)}dt.$$

Then we define the conjugate function $\widetilde{f}$ by

$$\widetilde{f}(x) = \lim_{\epsilon\to 0^+} -\frac{1}{\pi}\int_\epsilon^\pi \frac{f(x+t)-f(x-t)}{2\tan(t/2)}dt.$$

It is clear that if the singular integral

$$-\frac{1}{\pi}\int_0^\pi \frac{f(x+t)-f(x-t)}{2\tan(t/2)}dt$$

exists, then

$$\widetilde{f}(x) = -\frac{1}{\pi}\int_0^\pi \frac{f(x+t)-f(x-t)}{2\tan(t/2)}dt.$$

Remark. There are some books calling the map

$$f(t)\mapsto H_\epsilon f(x) := -\frac{1}{\pi}\int_\epsilon^\pi \frac{f(x+t)-f(x-t)}{2\tan(t/2)}dt \quad \text{and} \quad f(t)\mapsto Hf(x) = \lim_{\epsilon\to 0^+} H_\epsilon f(x)$$

truncated Hilbert transform and Hilbert transform, respectively.

Now our objective now is to show that the conjugate function exists.

Theorem 40. *Suppose $f \in L^1$, then for almost every x,*

$$\widetilde{\sigma}_n(x) - \widetilde{f}_{1/n}(x) \to 0$$

as $n \to \infty$.

Here $\widetilde{\sigma}_n$ denotes the conjugate Cesàro summation of the Fourier series. More precisely, it is defined by the conjugate Fejer's kernel

$$\widetilde{K}_n(t) := \frac{1}{n+1}\left(\sum_{j=0}^n \frac{\cos(t/2)-\cos(n+1/2)t}{2\sin(t/2)}\right) \qquad (0<|t|\le\pi)$$

$$\Longrightarrow \widetilde{K}_n(t) - \frac{1}{2}\cot\frac{t}{2} = \frac{-1}{2(n+1)\sin(t/2)}\sum_{j=0}^n \cos(j+1/2)t = \frac{-\sin(n+1)t}{4(n+1)\sin^2(t/2)}.$$

This gives

$$\left|\widetilde{K}_n(t) - \frac{1}{2}\cot\frac{t}{2}\right| \le \frac{C}{nt^2},$$

where the tightest C is about 2.47, we just simply use $C = 3$. We now could give

Proof of Theorem 40. Note that

$$\pi\left(\widetilde{\sigma}_n(x) - \widetilde{f}_{1/n}(x)\right) = -\int_{-1/n}^{1/n} f(x+t)\widetilde{K}_n(t)dt + \int_{1/n\le|t|\le\pi} f(x+t)\left(\frac{1}{2}\cot\frac{t}{2} - \widetilde{K}_n(t)\right)dt.$$

The remaining estimation is very similar to the proof of Theorem 36. $\square$

Corollary 4. If $f \in L^2[-\pi,\pi]$, then the conjugate $\widetilde{f}(x)$ exists. Moreover,

$$\|\widetilde{f}\|_2 < \|f\|_2 \,. \tag{4}$$

Proof. Since $f \in L^2$, we have $\sum_{k=-\infty}^{\infty} |c_k| = \int |f|^2 < \infty$. Note that if $s(f;x) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$, then $\widetilde{s}(f;x) = \sum_{k=-\infty}^{\infty} -i \cdot \operatorname{sign}(k) \cdot c_k e^{ikx}$. Write

$$\widetilde{c}_k = -i \cdot \operatorname{sign}(k) \cdot c_k.$$

Then we have $\sum_{k\in\mathbb{Z}} |\widetilde{c}_k|^2 < \infty$. By Riesz-Fischer Theorem (Theorem 15), there is a function $g \in L^2$ such that

$$s(g) = \sum_{k\in\mathbb{Z}} \widetilde{c}_k e^{ikt} = \widetilde{s}(f).$$

Moreover,

$$\int |g|^2 = \sum_{k\in\mathbb{Z}} |\widetilde{c}_k|^2 \le \sum_{k\in\mathbb{Z}} |c_k|^2 \le \int |f|^2 \,,$$

therefore $\|g\|_2 \le \|f\|_2$. Since $\widetilde{s}_n(f) = s_n(g)$, $\widetilde{\sigma}_n(f) = \sigma_n(g)$. The theorem we just proved (Theorem 40) asserts that

$$\widetilde{\sigma}_n(f;x) - \widetilde{f}_{1/n}(x) \to 0 \qquad \text{for almost every } x.$$

Lebesgue's Theorem (Theorem 36) asserts that $\sigma_n(g;x) - g(x) \to 0$ for almost every x. Therefore, we conclude that

$$\widetilde{f}_{1/n}(x) \to g(x)$$

for almost every x.

Note that for any $1/(n+1) \le \epsilon < 1/n$, we have

$$\begin{aligned} \widetilde{f}_\epsilon(x) - \widetilde{f}_{1/n}(x) &\le \frac{1}{\pi} \int_{1/(n+1)}^{1/n} \left| \frac{f(x+t) - f(x-t)}{2\tan(t/2)} \right| dt \\ &\le \frac{1}{\pi} \cdot \frac{1}{2\tan(1/2(n+1))} \int_{1/(n+1)}^{1/n} |f(x+t) - f(x-t)|\, dt \qquad (5) \\ &\le \frac{n+1}{\pi} \int_{-1/n}^{1/n} |f(x+t)|\, dt. \end{aligned}$$

By Lebesgue's differentiation theorem, we obtain that for almost every x,

$$\widetilde{f}_\epsilon(x) - \widetilde{f}_{1/n}(x) \to 0.$$

This shows that

$$\widetilde{f}_\epsilon(x) \to g(x)$$

for almost every x. The inequality (4) can be seen from the fact that $g = \widetilde{f}$ almost everywhere and

$$\int |g|^2 \le \int |f|^2 \,.$$

$\square$

We now introduce a new function called Hardy-Littlewood maximal function.

Definition 41 (Hardy-Littlewood maximal function)**.** Given $f \in L^1(\mathbb{R})$, define its Hardy-Littlewood maximal function

$$f^*(x) = \sup_{h>0} \frac{1}{2h} \int_{-h}^{h} |f(x+t)|\, dt.$$

By Lebesgue's differentiation theorem, $f^*(x) \geq f(x)$ for almost every x.

Also, we shall write $\sigma^*(f;x) = \sup_{n\geq 0} |\sigma_n(f;x)|$ to denote the maximal arithmetic mean (Cesàro sum) of Fourier series. When $f \in L^1$, $\sigma^*(f;x)$ exists almost everywhere, since $\sigma_n \to f$ almost everywhere.

Theorem 42. *The following statements are true.*

1. *For $1 < p \leq \infty$, we have $\|f^*\|_p \leq C_p \cdot \|f\|_p$, where $C_p = C(p)$ is a constant.*
2. *$m(\{x : |f^*(x)| > \alpha\}) \leq \dfrac{1}{\alpha} \displaystyle\int |f|$.*

The proof is not covered in class, hence it is omitted here. However the next theorem help us to establish some relation between f^*, σ^* and $\widetilde{f}$.

Theorem 43. *There exists $c > 0$ such that the following statement are true.*

1. *$\sigma^*(x) \leq c \cdot f^*(x)$ for all x.*
2. *$\sup_{n\geq 1} \left|\widetilde{\sigma}_n(x) - \widetilde{f}_{1/n}(x)\right| \leq c \cdot f^*(x)$ for all x.*

Proof.

1. Recall that $\sigma_n(f;x) = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x+t) K_n(t) dt$. If $n = 0$, then it is clear that

$$|\sigma_0(f;x)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x+t)|\, dt \leq f^*(x).$$

For $n \geq 1$, let

$$\alpha_n := \int_{0\leq |t| \leq 1/n} f(x+t) K_n(t) dt \qquad \beta_n := \int_{1/n \leq |t| \leq \pi} f(x+t) K_n(t) dt.$$

Then,

$$\alpha_n \overset{\heartsuit}{\leq} \frac{n+1}{2} \int_{-1/n}^{1/n} |f(x+t)|\, dt \leq \frac{n+1}{n} \cdot \frac{n}{2} \cdot \int_{-1/n}^{1/n} |f(x+t)|\, dt \leq 2f^*(x).$$

The inequality ($\heartsuit$) holds by Proposition 4. Let $I_x(t) := \displaystyle\int_{-t}^{t} |f(x+u)|\, du$ be a function of t. Then by the definition of maximal function, $I_x(t)/t \leq 2f^*(x)$. Now, it follows by

Proposition 4 again, we have

$$\begin{aligned}
\beta_n &\leq \frac{\pi^2}{2(n+1)} \int_{1/n}^{\pi} \frac{|f(x+t)| + |f(x-t)|}{t^2} dt = \frac{\pi^2}{2(n+1)} \int_{1/n}^{\pi} \frac{I_x'(t)}{t^2} dt \\
&= \frac{\pi^2}{2(n+1)} \left(\frac{I_x(\pi)}{\pi^2} - \frac{I_x(1/n)}{1/n^2} + 2 \int_{1/n}^{\pi} \frac{I_x(t)}{t^3} dt \right) \qquad \text{(Integration by parts.)} \\
&\leq \left(\frac{\pi}{2} + \pi^2 \right) f^*(x) + \frac{\pi^2}{n+1} \int_{1/n}^{\pi} \frac{I_x(t)}{t^3} dt \\
&\leq \left(\frac{\pi}{2} + \pi^2 \right) f^*(x) + \frac{\pi^2}{n+1} \cdot 2 f^*(x) \int_{1/n}^{\pi} \frac{1}{t^2} dt \\
&\leq \left(\frac{\pi}{2} + 3\pi^2 \right) f^*(x).
\end{aligned}$$

We conclude that $|\sigma_n(f;x)| \leq c \cdot f^*(x)$ for some positive c not depending on n.

2. The proof of this statement only need some minor modifications in the last statement and in the proof of Theorem 40. Hence we omit the details.

$\square$

Theorem 44. *Let $f \in L^2$ and define*

$$\widetilde{f}^{(*)}(x) = \sup_{0<\epsilon\leq\pi} |\widetilde{f}_\epsilon(x)| = \sup_{0<\epsilon\leq\pi} \left| -\frac{1}{\pi} \int_{\epsilon}^{\pi} \frac{f(x+t) - f(x-t)}{2\tan(t/2)} dt \right|.$$

Then

$$\|\widetilde{f}^{(*)}\|_2 \leq c \cdot \|f\|_2, \qquad \textit{for some } c > 0.$$

In the following discussion, we will replace the symbol $\widetilde{f}^{(*)}(x)$ with $g(x)$ to avoid misunderstandings.

Proof. We first claim that there is a constant $\lambda > 0$ such that

$$\left|\widetilde{f}_\epsilon(x)\right| \leq \lambda \cdot \left(f^*(x) + (\widetilde{f})^*(x) \right),$$

for all $0 < \epsilon \leq \pi$.

We first consider the case of $\epsilon \in (0, 1/2]$. Choose $n \in \mathbb{N}$ such that $1/(n+1) \leq \epsilon < 1/n$. Then, by inequality (5), we obtain

$$\left|\widetilde{f}_\epsilon(x) - \widetilde{f}_{1/n}(x)\right| \leq \frac{n+1}{\pi} \cdot \frac{2}{n} \cdot f^*(x) \leq 2f^*(x).$$

The second statement of Theorem 43 states that $\sup\limits_{n\geq 1} \left|\widetilde{\sigma}_n(x) - \widetilde{f}_{1/n}(x)\right| \leq \kappa \cdot f^*(x)$ (for some constant $\kappa > 0$), thus we obtain

$$\begin{aligned}
\widetilde{f}_\epsilon(x) &\leq (\kappa+2) f^*(x) + |\widetilde{\sigma}_n(x)| \\
&\leq (\kappa+2) f^*(x) + \sup_{n\in\mathbb{N}} |\widetilde{\sigma}_n(x)| \\
&\leq (\kappa+2) f^*(x) + \kappa \cdot (\widetilde{f})^*(x).
\end{aligned}$$

We now consider the case of $\epsilon \in [1/2, \pi]$. In this case, we have

$$\left|\widetilde{f}_\epsilon(x)\right| \le \int_{1/2\le|t|\le\pi} \left|\frac{f(x+t)}{2\tan(t/2)}\right| dt \le 2\int_{0\le|t|\le\pi} |f(x+t)|\, dt \le 4\pi f^*(x).$$

Combining the above two inequalities proves the inequality we claimed, that is,

$$\left|\widetilde{f}_\epsilon(x)\right| \le \lambda \cdot \left(f^*(x) + (\widetilde{f})^*(x)\right).$$

Now we have

$$\|\widetilde{f}^{(*)}\|_2 \le \lambda \left(\|(\widetilde{f})^*\|_2 + \|f^*\|_2\right) \le C_2\lambda \left(\|\widetilde{f}\|_2 + \|f\|_2\right) \le 2C_2\lambda \|f\|_2,$$

where C_2 is the constant mentioned in Theorem 42. □

Now we are going to prove the Calderón-Zygmund Lemma, which is useful in proving the existence of conjugation in L^1.

Lemma 13 (Calderón-Zygmund decomposition)**.** Let Q be a compact interval. Assume $f \in L^1(Q)$ and $f \ge 0$. Given $\alpha \ge \frac{1}{|Q|}\int_Q f$. Then there exists a sequence of non-overlapping open intervals $\{Q_k\}_{k=1}^\infty$ such that

1. $\alpha < \frac{1}{|Q_k|}\int_{Q_k} f \le 2\alpha$, for all k.
2. $f(x) \le \alpha$, for almost every $x \in Q \setminus \bigsqcup Q_k$.
3. $\sum_k |Q_k| \le \frac{1}{\alpha}\int_Q f$.

Proof. For any interval $I \subset Q$, we said

$$\begin{cases} I \text{ is a type (i) interval if } \frac{1}{|I|}\int_I f > \alpha \\ I \text{ is a type (ii) interval if } \frac{1}{|I|}\int_I f \le \alpha \end{cases}.$$

Suppose $Q = (a, b)$, then let

$$I_h^{(m)} := \left(a + \frac{m-1}{2^h}(b-a), a + \frac{m}{2^h}(b-a)\right) \qquad m = 1, \ldots, 2^h.$$

Now we explain how we choose $\{Q_k\}_{k=1}^\infty$ from these $I_k^{(m)}$. First, we select all type (i) interval in $\{I_1^{(1)}, I_1^{(2)}\}$. Suppose the process has been executed h rounds and k intervals have been chosen. Then we now choose all type (i) interval of the form $I_{h+1}^{(m)}$ such that

$$I_{h+1}^{(m)} \cap \bigsqcup_{j=1}^k Q_j = \varnothing.$$

In other words, we can use the following way to describe the process. We first divide Q equally into two open intervals Q_l and Q_r, where l, r stands for left and right. In

$\{Q_\mathrm{l}, Q_\mathrm{r}\}$, we keep all type (i) intervals and divide all type (ii) intervals into two pieces, and repeat this process. If Q_l is of type (i), then

$$\alpha < \frac{1}{|Q_\mathrm{l}|}\int_{Q_\mathrm{l}} f = \frac{|Q|}{|Q_\mathrm{l}|}\cdot\frac{1}{|Q|}\int_Q f \leq 2\alpha.$$

If Q_l is of type (ii), then

$$\frac{1}{|Q_\mathrm{l}|}\int_{Q_\mathrm{l}} f \leq \alpha.$$

This inequality ensures that we can repeat the process that keep dividing type (ii) interval into two pieces. Thus we can gets a sequence of disjoint intervals $\{Q_k\}_{k=1}^\infty$ (possibly a finite sequence).

If $x \in Q \setminus \bigsqcup Q_k$ and x is not an endpoint of any $I_h^{(m)}$, then there are monotonically decreasing type (ii) intervals $I_h^{(\alpha(h))}$ ($\alpha(h) = \alpha_x(h)$ is a function of h and x.) such that $I_{h+1}^{(\alpha(h+1))} \subset I_h^{(\alpha(h))}$ and $I_h^{(\alpha(h))}$ shrinks down to x. By Lebesgue's differentiation theorem, we have

$$\lim_{h\to\infty}\frac{1}{\left|I_h^{(\alpha(h))}\right|}\int_{I_h^{(\alpha(h))}} f = f(x) \leq \alpha$$

for almost every $x \in Q \setminus \bigsqcup Q_k$.

It now remains to show our choices of $\{Q_k\}$ meet the third requirement. Note that $\frac{1}{\alpha}\int_{Q_k} f > |Q_k|$ since $\frac{1}{|Q_k|}\int_{Q_k} f > \alpha$. Thus,

$$\frac{1}{\alpha}\int_Q f \geq \sum_k |Q_k|.$$

This proves the theorem. □

Remark. Given a non-negative function $f \in L^1[-\pi,\pi]$. Fix $\alpha \geq \frac{1}{2\pi}\int_{-\pi}^{\pi} f$. Apply Calderón-Zygmund decomposition theorem (Theorem 13) to f, we then obtain countable disjoint intervals $\{Q_k\}$ such that three requirements are met. We can define $g \in L^1[-\pi,\pi]$ by

$$g(x) = \begin{cases} f(x) & \text{, if } x \in [-\pi,\pi] \setminus \bigsqcup Q_k \\ \dfrac{1}{|Q_k|}\displaystyle\int_{Q_k} f & \text{, if } x \in Q_k \end{cases}$$

We then define $h \in L^1[-\pi,\pi]$ by $h := f - g$. It is clear that $g(x) \leq 2\alpha$ for almost every $x \in [-\pi,\pi]$. $f = g + h$ is called the Calderón-Zygmund decomposition and the function g is often called the "good" part of the function f and h is called the "bad" part.

Before proving the existence of conjugate function in L^1, we shall prove another useful lemma.

Lemma 14. Let F be a closed set in $[-\pi,\pi]$ and let

$$\delta(x) = \mathrm{dist}(x, F) = \inf_{y\in F}|x - y|.$$

Then for all $\lambda > 0$, the function

$$M_\lambda(x) := \int_{-\pi}^{\pi} \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}} dy$$

is finite for almost every $x \in F$.

Proof. It is clear that

$$M_\lambda(x) := \int_G \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}} dy, \qquad \text{where } G = [-\pi, \pi] \setminus F.$$

Then

$$\begin{aligned}\int_F M_\lambda(x)dx &= \int_F \int_G \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}} dydx = \int_G \delta^\lambda(y) \left(\int_F \frac{dx}{|x-y|^{1+\lambda}} \right) dy \\ &\leq \int_G \delta^\lambda(y) \left(2 \int_{\delta(y)}^{\infty} \frac{dt}{t^{1+\lambda}} \right) dy = \int_G \delta^\lambda(y) \cdot \frac{2}{\lambda \delta^\lambda(y)} dy = \frac{2m(G)}{\lambda}.\end{aligned}$$

We can exchange the order of integration at the second equality because of Tonelli's theorem. This inequality shows that $M_\lambda(x)$ is finite for almost every $x \in F$ ($M_\lambda \in L^1(F)$). □

We now can prove the following theorem.

Theorem 45 (The existence of conjugate function in L^1)**.** *Let $f \in L^1[-\pi, \pi]$. Then the conjugate of f exists. In other words, the limit $\lim_{\epsilon \to 0} \widetilde{f}_\epsilon(x)$ exists for almost every $x \in [-\pi, \pi]$. Moreover, the Hilbert transform is weak $(1, 1)$, that is,*

$$|\{x : |Hf(x)| > \alpha\}| \leq \frac{C}{\alpha} \int_{-\pi}^{\pi} f$$

for some constant C.

Proof. We first prove the existence of the conjugate function. Without loss of generality, assume $f \geq 0$. We have defined the Calderón-Zygmund decomposition of f. Let $f = g + h$ be the decomposition, where g is the good part of f and h is the bad part. Since $g \in L^\infty[-\pi, \pi] \subset L^2[-\pi, \pi]$, it follows from Corollary 4 that $\widetilde{g}$ exists. We now show that $\widetilde{h}$ exists.

Given $\epsilon > 0$. Let $Q_k^* := 2\,\mathrm{int}(Q_k)$. (For an open interval $I = (r, s)$, we write $2I$ to denote the interval $((3a+b)/2, (a+3b)/2)$.) Also we use Q^* to denote the open set $\bigcup Q_k^*$ and $P^* = [-\pi, \pi] \setminus Q^*$. We now claim that $\widetilde{h}(x)$ exists for almost every $x \in P^*$. Now we let $x \in P^*$ be fixed. Recall the definition of h, we have

$$h(t) = \begin{cases} 0 & , \text{if } t \in [-\pi, \pi] \setminus \bigsqcup Q_k \\ f(t) - \dfrac{1}{|Q_k|} \displaystyle\int_{Q_k} f & , \text{if } t \in Q_k \end{cases}$$

Hence h vanishes outside $\bigcup Q_k$. Let A, B be subset of $\mathbb{N}$ defined by

$$A := \{i \in \mathbb{N} : Q_i \cap (x-\epsilon, x+\epsilon) = \varnothing\}$$
$$B := \{i \in \mathbb{N} : Q_i \cap \{x-\epsilon, x+\epsilon\} \neq \varnothing\}$$

Then

$$\begin{aligned} 2\pi \cdot \widetilde{h}_\epsilon(x) &= \int_{\epsilon \le |x-t| \le \pi} h(t) \cot\left(\frac{x-t}{2}\right) dt \\ &= \sum_{i \in A} \int_{Q_i} h(t) \cot\left(\frac{x-t}{2}\right) dt + \sum_{i \in B} \int_{Q_i} h(t) \cot\left(\frac{x-t}{2}\right) dt. \end{aligned}$$

We now estimate the second term. If some Q_k contains $x+\epsilon$, then $\epsilon \ge \operatorname{dist}(x, Q_k) \ge d_k/2$, where d_k is defined to be $|Q_k|$. (If the inequality does not hold, then $x \notin P^*$.) Then

$$\int_{Q_k} \left| h(t) \cot\left(\frac{x-t}{2}\right) \right| dt \le \int_{x+\epsilon}^{x+\epsilon+d_k} \left| \frac{2h(t)}{x-t} \right| dt \le \frac{2}{\epsilon} \int_{x+\epsilon}^{x+3\epsilon} |h(t)|\, dt = \frac{2}{\epsilon} \int_0^{2\epsilon} |h(x+t)|\, dt.$$

By Lebesgue's differentiation theorem,

$$\int_{Q_k} h(t) \cot\left(\frac{x-t}{2}\right) dt \to |h(x)| = 0, \text{ as } \epsilon \to 0$$

for almost every $x \in P^*$.

Now we give the estimation of the first term. Let $k \in A$. We also write $d_k = |Q_k|$ and let t_k denote the midpoint of Q_k. Note that $\displaystyle\int_{Q_k} h(t) dt = 0$. Therefore,

$$\begin{aligned} \int_{Q_k} \left| h(t) \cot\left(\frac{x-t}{2}\right) \right| dt &= \int_{Q_k} \left| h(t) \left(\cot\left(\frac{x-t}{2}\right) - \cot\left(\frac{x-t_k}{2}\right) \right) \right| dt \\ &= \int_{Q_k} \left| h(t) \frac{\sin\big((t-t_k)/2\big)}{\sin\big((x-t)/2\big) \sin\big((x-t_k)/2\big)} \right| dt \\ &\le \frac{d_k}{4} \int_{Q_k} \frac{|h(t)|}{\left|\sin\big((x-t)/2\big)\right| \cdot \left|\sin\big((x-t_k)/2\big)\right|} dt \\ &\overset{\diamondsuit}{\le} \frac{2\pi^2 d_k}{(x-t_k)^2} \int_{Q_k} |h(t)|\, dt. \end{aligned}$$

The inequality ($\diamondsuit$) holds by $|\sin x| \ge \dfrac{2|x|}{\pi}$ for all $x \in [-\pi/2, \pi/2]$ and the fact that

$$\frac{1}{2} |x - t_k| \le |x-t| \le \frac{3}{2} |x - t_k|. \tag{6}$$

Observe that

$$\int_{Q_k} |h(t)|\, dt \le \int_{Q_k} \left| f(t) - \frac{1}{|Q_k|} \int_{Q_k} f \right| dt \le 2 \int_{Q_k} f(t) dt \le 2 |Q_k| \frac{1}{|Q_k|} \int_{Q_k} f \le 4\alpha |Q_k|.$$

Now let $\delta(t) = \operatorname{dist}(t, P^*)$. It is clear that $\delta(t) \ge d_k/2$ for all $t \in Q_k$. Summarizing the results obtained above, we have

$$\int_{Q_k} \left| h(t) \cot\left(\frac{x-t}{2}\right) \right| dt \le \frac{8\alpha\pi^2 d_k |Q_k|}{(x-t_k)^2} \le \frac{16\alpha\pi^2}{(x-t_k)^2} \int_{Q_k} \delta(t) dt \le 36\alpha\pi^2 \int_{Q_k} \frac{\delta(t)}{|x-t|^2} dt.$$

Here we have used the inequality (6).

By Lemma 14, we have

$$\lim_{\epsilon\to 0^+}\sum_{i\in A}\int_{Q_i}\left|h(t)\cot\left(\frac{x-t}{2}\right)\right|dt \le 36\alpha\pi^2\lim_{\epsilon\to 0^+}\sum_{i\in A}\int_{Q_i}\frac{\delta(t)}{|x-t|^2}dt \le \int_{Q^*}\frac{\delta(t)}{|x-t|^2}dt < \infty$$

for almost every $x \in P^*$. Thus $\widetilde{h}(x)$ exists for almost every $x \in P^*$.

However, it is worth noting that the closed set P^* might be larger when α becomes larger. Therefore it now suffices to show that

$$|P^*| \to 2\pi \qquad \text{as} \qquad \alpha \to \infty.$$

It follows by the fact that

$$|Q^*| = \left|\bigcup Q_k^*\right| \le 2\sum |Q_k| \le \frac{2}{\alpha}\int_{-\pi}^{\pi} f \to 0$$

as $\alpha \to \infty$.

We now show that the Hilbert transform $f \mapsto \widetilde{f}$ is weak $(1,1)$. It is clear that

$$\{x : |\widetilde{f}(x)| > \alpha\} \subseteq \{x : |\widetilde{g}(x)| > \alpha/2\} \cup \{x : |\widetilde{h}(x)| > \alpha/2\}.$$

Firstly, we have

$$m\left(\{x : |\widetilde{g}(x)| > \alpha/2\}\right) \le \frac{4}{\alpha^2}\int |\widetilde{g}|^2 \le \frac{4}{\alpha^2}\int |g|^2 \le \frac{8}{\alpha}\int |g| \le \frac{8}{\alpha}\int |f|.$$

Now let $S := \{|\widetilde{x}| > \alpha/2\}$. Then it is clear that $S = (S\cap P^*)\sqcup(S\cap Q^*)$ $(P^*\sqcup Q^* = [-\pi,\pi])$. By the requirements of $\{Q_k\}$,

$$m(S\cap Q^*) \le m(Q^*) \le 2\sum |Q_k| \le \frac{2}{\alpha}\int |f|.$$

We have just shown that

$$|\widetilde{h}(x)| \le 36\alpha\pi^2\int_{Q_k}\frac{\delta(t)}{|x-t|^2}dt := 36\alpha\pi^2 M(x)$$

for almost every $x \in P^*$. Thus, for almost every $x \in S\cap P^*$,

$$x \in \left\{p : M(p) \ge \frac{1}{72\pi^2}\right\}.$$

In other words, we have $S\cap P^* \subset \{p : M(p) \ge 1/72\pi^2\}$ except for a zero set. We conclude that

$$m(S\cap P^*) \le \frac{1}{A}\int_{P^*} M(x)dx \le \frac{2\,|Q^*|}{A} \le \frac{4}{A\cdot\alpha}\int |f|,$$

where A is the constant $1/72\pi^2$. (The second inequality holds by Lemma 14.) This completes the proof. $\square$

1.6 Fourier transform on $\mathbb{R}^n$

Definition 46 (Fourier transform)**.** Given $f \in L^1$. We define its Fourier transform $\widehat{f}(\xi)$ by

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i \langle x, \xi \rangle} f(x) dx,$$

where $\langle \xi, x \rangle$ is the standard inner product on $\mathbb{R}^n$.

Proposition 5. *The following statements are true.*

1. $\widehat{f+g} = \widehat{f} + \widehat{g}$.
2. $\widehat{cf} = c\widehat{f}$, *where c is a constant.*
3. *If $f \in L^1(\mathbb{R}^2)$, then $\widehat{f}$ is continuous.*

Proof. The first two properties are trivial. To show $\widehat{f}$ is continuous, we consider the difference

$$|\widehat{f}(\xi + h) - \widehat{f}(\xi)|.$$

By the definition and the inequality $\left|e^{i\theta} - 1\right| \leq \theta$,

$$\begin{aligned} |\widehat{f}(\xi + h) - \widehat{f}(\xi)| &= \left| \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} \left(e^{-2\pi i \langle x, h \rangle} - 1 \right) dx \right| \\ &\leq \int_{\mathbb{R}^n} |f(x)| \min \{2\pi \cdot |h| \cdot |x|, 2\} dx \end{aligned}$$

Given $\epsilon > 0$, there is $M > 0$ such that

$$\int_{|x| \geq M} |f(x)| \, dx < \epsilon.$$

This implies that

$$\int_{|x| \geq M} |f(x)| \min \{2\pi \cdot |h| \cdot |x|, 2\} dx < 2\epsilon.$$

Note that

$$\int_{|x| \leq M} |f(x)| \min \{2\pi \cdot |h| \cdot |x|, 2\} dx \leq 2\pi M \int_{|x| \leq M} |f(x)| \cdot |h| \, dx < \epsilon$$

when h is small enough. Therefore, for small enough $h > 0$, we have

$$|\widehat{f}(\xi + h) - \widehat{f}(\xi)| < 3\epsilon.$$

This completes the proof. $\square$

Theorem 47 (Riemann-Lebesgue Theorem revisited)**.** *Let $f \in L^2$, then $|\widehat{f}(\xi)| \to 0$ as $|\xi| \to \infty$.*

Proof. Use the substitution $x = y + \frac{\xi}{2|\xi|^2}$ in the integration, then we obtain

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i\langle x,\xi\rangle} f(x)dx = \int_{\mathbb{R}^n} e^{-2\pi i\langle y,\xi\rangle} e^{-\pi i\frac{|\xi|^2}{|\xi|^2}} f\left(y + \frac{\xi}{2|\xi|^2}\right) dy.$$

Thus, we have

$$\widehat{f}(\xi) = \frac{1}{2}\int_{\mathbb{R}^n} e^{-2\pi i\langle x,\xi\rangle} \left(f(x) - f\left(x + \frac{\xi}{2|\xi|^2}\right)\right) dx.$$

By the continuity of L^1 functions, we conclude

$$|\widehat{f}(\xi)| \to 0 \qquad \text{as} \qquad |\xi| \to 0.$$

This completes the proof. □

Remark. It is worth noting that the continuity of L^p functions (Theorem 29) only holds for functions in $L^p[-\pi,\pi]$, however some slight modifications may apply to the original arguments and made the theorem true for functions $f \in L^p(\mathbb{R})$.

Definition 48 (Convolution)**.** Given two functions f and g. We define $f * g$ to be the function

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy.$$

Proposition 6.

1. $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$.
2. *Let* $\eta \in \mathbb{R}^n$. *Define* $\tau_\eta f(x) := f(x-\eta)$. *Then*
$$\widehat{\tau_\eta f}(\xi) = e^{-2\pi i\langle \xi,\eta\rangle}\widehat{f}(\xi).$$
3. $\widehat{e^{2\pi i\langle \eta,x\rangle} f}(\xi) = \tau_\eta \widehat{f}(\xi)$.
4. $\widehat{(D^\alpha f)}(\xi) = (2\pi i\xi)^\alpha \widehat{f}(\xi)$.
5. $\widehat{(-2\pi i\xi)^\alpha f}(\xi) = D^\alpha \widehat{f}(\xi)$.

Proof. We only prove the first statement. The second and the third statement can be seen from direct calculation, and the fourth and fifth statement can be seen from performing integration by parts on the integral. Note that

$$\begin{aligned}\widehat{f * g}(\xi) &= \int_{\mathbb{R}^n} f * g(\xi)e^{-2\pi i\langle x,\xi\rangle}dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)g(x-y)e^{-2\pi i\langle x,\xi\rangle}dydx \\ &= \left(\int_{\mathbb{R}^n} f(y)e^{-2\pi i\langle y,\xi\rangle}dy\right)\left(\int_{\mathbb{R}^n} g(x-y)e^{-2\pi i\langle x-y,\xi\rangle}dx\right) = \widehat{f}(\xi)\cdot\widehat{g}(\xi).\end{aligned}$$

This proves the first assertion. □

Lemma 15. Let $f(x) = e^{-\pi|x|^2}$. Then $\widehat{f}(\xi) = e^{-\pi|\xi|^2}$.

Proof. We shall only give the case of $n = 1$. By Proposition 6, we have

$$\begin{aligned}(\widehat{f})'(\xi) &= \widehat{(-2\pi i x \cdot f)}(\xi) = (\widehat{-2\pi i x e^{-\pi x^2}})(\xi) = i\widehat{f'}(\xi) \\ &= i(2\pi i\xi)\widehat{f}(\xi).\end{aligned}$$

This implies that

$$(\widehat{f})'(\xi) = -2\pi\xi\widehat{f}(\xi).$$

Solving this differential equation gives us $\widehat{f}(\xi) = ce^{-\pi\xi^2}$, where $c \in \mathbb{R}$ is a constant. We could obtain $c = 1$ by considering $\widehat{f}(0)$. Similar arguments hold for general $n \in \mathbb{N}$. $\square$

We now could introduce the concept of the inverse Fourier transform. As the word implies, we later shall show that this is the inverse of the Fourier transform.

Definition 49 (inverse Fourier transform)**.** Given $f \in L^1$. We define the inverse Fourier transform $\widetilde{f}$ of f by

$$\widetilde{f}(x) = \int_{\mathbb{R}^n} e^{2\pi i\langle \xi, x\rangle} f(\xi)d\xi.$$

Theorem 50 (Fourier integrals theorem)**.** *Suppose $f \in L^1$ and the Fourier transform $\widehat{f}$ of f exists in L^1. Then $\widetilde{\widehat{f}} = f = \widehat{\widetilde{f}}$ almost everywhere.*

Before we prove this theorem, we shall prove some useful lemmas.

Lemma 16. Suppose $f, g \in L^1$ and the Fourier transforms $\widehat{f}, \widehat{g}$ of f, g exist. Then

$$\int \widehat{f}g = \int f\widehat{g}.$$

Proof. It follows by

$$\int_{\mathbb{R}^n} \widehat{f}(\xi)g(\xi)d\xi = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-2\pi i\langle x,\xi\rangle} f(x)dxg(\xi)d\xi = \int_{\mathbb{R}^n} f(x)\widehat{g}(x)dx.$$

It is worth noting that the second equality holds by the Fubini's Theorem. $\square$

Lemma 17 (Minkowski's inequality)**.** Let $1 \le p < \infty$. Suppose $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a measurable function, then

$$\left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x,y)|\,dx\right)^p dy\right)^{1/p} \le \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x,y)|^p\,dy\right)^{1/p} dx.$$

Proof. We may assume $f \ge 0$. The left side to the power of p is

$$I := \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x,y)dx\right)^p dy = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(t,y)dt\right)^{p-1} \left(\int_{\mathbb{R}^n} f(x,y)dx\right) dy.$$

Let $F(y) := \int_{\mathbb{R}^n} f(t,y)dt$. Then $I = \int_{\mathbb{R}^n} F(y)^{p-1} f(x,y)dydx$ by Tonelli's Theorem. Applying Hölder's inequality, we obtain

$$
\begin{aligned}
I &\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} F(y)^p dy \right)^{(p-1)/p} \left(\int_{\mathbb{R}^n} f(x,y)^p dy \right)^{1/p} dx \\
&= \left(\int_{\mathbb{R}^n} F(y)^p dy \right)^{(p-1)/p} \cdot \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x,y)^p dy \right)^{1/p} dx \\
&= \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x,y)dx \right)^p dy \right)^{(p-1)/p} \cdot \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x,y)^p dy \right)^{1/p} dx.
\end{aligned}
$$

This implies that $I^{1/p} \leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x,y)^p dy \right)^{1/p} dx$, which proves the lemma. □

Lemma 18. Let $\varphi \in L^1$ such that $\int |\varphi(x)|\, dx = a$. Suppose $f \in L^p$ $(1 \leq p < \infty)$, then

$$\|f * \varphi_t - af\|_p \to 0 \qquad \text{as} \qquad t \to 0,$$

where $\varphi_t(x) := t^{-n}\varphi(x/t)$. In other words, $f * \varphi_t \to af$ in L^p.

Proof. Note that

$$
\begin{aligned}
f * \varphi_t(x) - af(x) &= \int_{\mathbb{R}^n} \left(f(x-y) - f(x)\right) \varphi_t(y)dy \\
&= \int_{\mathbb{R}^n} \left(f(x-tz) - f(x)\right) \varphi(z)dz \qquad (z = y/t).
\end{aligned}
$$

We now apply Minkowski's inequality (Lemma 17), then we can see that

$$
\begin{aligned}
\|f * \varphi_t - af\|_p &= \left(\int_{\mathbb{R}^n} |f * \varphi_t(x) - af(x)|^p \, dx \right)^{1/p} \\
&= \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \left(f(x-tz) - f(x)\right) \varphi(z)dz \right|^p dx \right)^{1/p} \\
&\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-tz) - f(x)|^p \, |\varphi(z)|^p \, dx \right)^{1/p} dz \\
&= \|\tau_{tz} f - f\|_p \cdot \int_{\mathbb{R}^n} |\varphi(z)| \, dz \\
&= a \cdot \|\tau_{tz} f - f\| \to 0
\end{aligned}
$$

We have used the fact that the translation is continuous in L^p, that is,

$$\|\tau_t f - f\| \to 0 \qquad \text{as} \qquad t \to 0$$

for any $f \in L^p$. In fact, we have proved this before, for more information, please refer to Theorem 29. It is worth noting that some modifications of the proof need to be make since the proof only shows the case of $f \in L^p[-\pi, \pi]$, $f \in L^p(\mathbb{R}^n)$ is not included. □

We now can give the proof of Theorem 50.

Proof of Theorem 50. Let $t > 0$ be a parameter and let $x \in \mathbb{R}^n$ be fixed. Define

$$\phi(\xi) = \exp\left(2\pi i \langle x, \xi \rangle - \pi t^2 |\xi|^2\right).$$

Then

$$\begin{aligned}\widehat{\phi}(y) &= \int_{\mathbb{R}^n} e^{-2\pi i \langle \xi, y \rangle} \phi(\xi) d\xi \\ &= \int_{\mathbb{R}^n} \exp\left(-2\pi i \langle \xi, y - x \rangle - \pi t^2 |\xi|^2\right) d\xi \\ &= t^{-n} \exp\left(\frac{-\pi |x-y|^2}{t^2}\right).\end{aligned}$$

We have used the fact that

$$\widehat{f}(\xi) = a^{-n/2} e^{-\pi |\xi|^2 / a},$$

provided that $f(x) = e^{-\pi a |x|^2}$. Let $g_t(x-y) := t^{-n} \exp\left(\frac{-\pi |x-y|^2}{t^2}\right)$. Then by Lemma 16 and Lemma 18, we have

$$\int_{\mathbb{R}^n} \exp\left(2\pi i \langle x, \xi \rangle - \pi t^2 |\xi|^2\right) \widehat{f}(\xi) d\xi = \int_{\mathbb{R}^n} f(y) g_t(x-y) dy \to f \text{ in } L^1,$$

as $t \to 0$. Using the Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}\lim_{t \to 0} \int_{\mathbb{R}^n} \exp\left(2\pi i \langle x, \xi \rangle - \pi t^2 |\xi|^2\right) \widehat{f}(\xi) d\xi &= \int_{\mathbb{R}^n} \lim_{t \to 0} \exp\left(2\pi i \langle x, \xi \rangle - \pi t^2 |\xi|^2\right) \widehat{f}(\xi) d\xi \\ &= \widetilde{\widehat{f}}(x).\end{aligned}$$

To sum up, we conclude that $\|\widetilde{\widehat{f}} - f\|_1 = 0$, therefore $\widetilde{\widehat{f}}(x) = f(x)$ for almost every x. $\square$

Remark. Although this proof seems quite reasonable, there are some details I do not completely accept. We have shown that $f * g_t \to f$ in L^1, and $f * g_t(x) \to \widetilde{\widehat{f}}(x)$ for almost every x. This does not implies that $\|\widetilde{\widehat{f}} - f\|_1 = 0$. However, this is the note I wrote in class, so I choose to believe it.

The next theorem associate the Fourier transform and the Fourier series, this theorem is known as Poisson's summation formula.

Theorem 51 (Poisson's summation formula)**.** *Given $f \in \mathfrak{S}(\mathbb{R}^n)$. Then*

$$\sum_{k \in \mathbb{Z}^n} f(x-k) = \sum_{k \in \mathbb{Z}^n} \widehat{f}(k) e^{2\pi i \langle k, x \rangle}$$

for all x. In particular, if we plug in $x = 0$ to the formula, we then obtain

$$\sum_{k \in \mathbb{Z}^n} f(k) = \sum_{k \in \mathbb{Z}^n} \widehat{f}(k).$$

We shall also prepare a lemma.

Lemma 19. If $f \in L^1(\mathbb{R}^n)$, then $\sum_{k\in\mathbb{Z}^n} f(x-k)$ converges almost everywhere on $[0,1]$. Moreover, if we let $\mathcal{P}f(x) := \sum_{k\in\mathbb{Z}^n} f(x-k)$, then $\mathcal{P}f \in L^1[0,1]$ and

$$\|\mathcal{P}f\|_{L^1[0,1]} \le \|f\|_{L^1(\mathbb{R}^n)}.$$

Proof. We shall just prove that $\mathcal{P}f$ is integrable, then the summation is finite almost everywhere. Also, we may assume that $f \ge 0$. Note that

$$\int_0^1 \sum_{k\in\mathbb{Z}^n} f(x-k)dx = \sum_{k\in\mathbb{Z}^n} \int_0^1 f(x-k)dx = \sum_{k\in\mathbb{Z}^n} \int_k^{k+1} f(x)dx = \int_{\mathbb{R}^n} f(x)dx.$$

Now the lemma follows. $\square$

Remark. In fact, I remembered that this is a problem in the final exam of last semester. Moreover, if $f \in \mathfrak{S}(\mathbb{R}^n)$ then $\sum_{k\in\mathbb{Z}^n} f(x-k)$ converges for almost everywhere.

Proof of Theorem 51. We apply Lemma 19, we obtain a function $\mathcal{P}f \in L^1[0,1]$. We can compute its Fourier series.

$$\begin{aligned}
\widehat{\mathcal{P}f}(k_0) &= \int_0^1 \sum_{k\in\mathbb{Z}^n} e^{-2\pi i\langle k_0,x\rangle} f(x-k)dx \\
&= \sum_{k\in\mathbb{Z}^n} \int_{-k}^{-k+1} e^{-2\pi i\langle k_0,x+k\rangle} f(x)dx \\
&= \int_{\mathbb{R}^n} e^{-2\pi i\langle k_0,x\rangle} f(x)dx = \widehat{f}(k_0).
\end{aligned}$$

It now suffices to show that $\sum_{k\in\mathbb{Z}^n} |\widehat{f}(k)| < \infty$. If this is true, then we can conclude that the Fourier coefficients of

$$\mathcal{P}f \qquad \text{and} \qquad \sum_{k\in\mathbb{Z}^n} \widehat{f}(k)e^{2\pi i\langle k,x\rangle}$$

are the same, and thus two functions are equal, namely,

$$\sum_{k\in\mathbb{Z}^n} f(x-k) = \sum_{k\in\mathbb{Z}^n} \widehat{f}(k)e^{2\pi i\langle k,x\rangle}.$$

The claim that $\sum_{k\in\mathbb{Z}^n} |\widehat{f}(k)|$ converges now follows by $f \in \mathfrak{S}(\mathbb{R}^n)$. $\square$

1.7 The discrete Fourier transform and the Roth's Theorem

In this subsection, our main objective is to prove Roth's theorem by discrete Fourier transform.

Theorem 52 (Roth's theorem)**.** *Let $A \subset \mathbb{N}$ be a set of positive integers. If the density of A is greater than 0. That is*

$$\limsup_{n\to\infty} \frac{\# A \cap \{1,2,\dots,n\}}{n} = \delta > 0.$$

Then A contains a three terms arithmetic progression. In other words, there are $x, d \in \mathbb{N}$ such that $\{x, x+d, x+2d\} \subset A$.

History

In 1980, E. Szemerédi showed the following

Theorem 53 (Szemerédi's theorem)**.** *Let $A \subset \mathbb{N}$ be a set of positive integers with positive density. Then for all $k \geq 3$, A contains a k-term arithmetic progression. In other words, there are $x, d \in \mathbb{N}$ such that $\{x, x+d, \dots, x+(k-1)d\} \subset A$.*

This theorem has a lot of approaches. H. Fusterberg proves this theorem by using the *ergodic theory.* T. Gowers proves the theorem by using the Fourier transform.

Math

Now we will work on the cyclic group $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$. We shall now give some definitions.

Definition 54.

1. We write $\mathbb{C}^{\mathbb{Z}_n}$ to denote all complex-valued functions $f : \mathbb{Z}_n \to \mathbb{C}$.
2. The mean of $f \in \mathbb{C}^{\mathbb{Z}_n}$ is defined by $\mathbb{E}_n f = \dfrac{1}{n} \displaystyle\sum_{x \in \mathbb{Z}_n} f(x)$.
3. For any sets $A \subset \mathbb{Z}_n$, it induces a natural map $A \in \mathbb{C}^{\mathbb{Z}_n}$ through
$$A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}.$$
It is clear that $\mathbb{E}_n A = \dfrac{|A|}{n}$.
4. The exponential function $\mathrm{e}_n(x) := e^{2\pi i x/n}$.

Lemma 20 (Orthogonal property)**.** If $\xi \in \mathbb{Z}_n$, then

$$\frac{1}{n} \sum_{x \in \mathbb{Z}_n} \mathrm{e}_n(x\xi) = \begin{cases} 1, & \xi = 0 \\ 0, & \xi \neq 0 \end{cases}.$$

Proof. If $\xi = 0$, it is trivial. Now suppose $\xi \neq 0$, then there exists $h \neq 0$ such that $\mathrm{e}_n(\xi h) \neq 1$. Hence

$$\frac{1}{n} \sum_{x \in \mathbb{Z}_n} \mathrm{e}_n(x\xi) = \frac{1}{n} \sum_{x \in \mathbb{Z}_n} \mathrm{e}_n((x+h)\xi) = \mathrm{e}_n(\xi h) \frac{1}{n} \sum_{x \in \mathbb{Z}_n} \mathrm{e}_n(x\xi).$$

This shows $\dfrac{1}{n} \displaystyle\sum_{x \in \mathbb{Z}_n} \mathrm{e}_n(x\xi) = 0$. □

Corollary 5. Given $\xi, \xi' \in \mathbb{Z}_n$. Then

$$\mathbb{E}_n \left(\mathrm{e}_n(\xi x) \, \overline{\mathrm{e}_n(\xi' x)} \right) = \begin{cases} 1, & \xi = \xi' \\ 0, & \xi \neq \xi' \end{cases}.$$

Similar to the function space, we also can define inner product on $\mathbb{C}^{\mathbb{Z}_n}$.

Definition 55 (inner product)**.** Let $f, g \in \mathbb{C}^{\mathbb{Z}_n}$, we define the inner product of f and g by

$$\langle f, g\rangle := \mathbb{E}_n \left(f(x)\overline{g(x)} \right).$$

Definition 56 (Discrete Fourier transform)**.** Given $f \in \mathbb{C}^{\mathbb{Z}_n}$, we define

$$\widehat{f}(\xi) = \frac{1}{n} \sum_{x \in \mathbb{Z}_n} f(x)\,\overline{\mathrm{e}_n(\xi x)}$$

to be the Fourier transform of f.

We now could list some properties.

Theorem 57 (Parseval's formula)**.** *For any $f \in \mathbb{C}^{\mathbb{Z}_n}$, we have*

$$\left(\mathbb{E}_n |f|^2\right)^{1/2} = \left(\sum_{\xi \in \mathbb{Z}_n} |\widehat{f}(\xi)|^2 \right)^{1/2}.$$

Proof. The square of right-hand side is equal to

$$\begin{aligned}
\sum_{\xi \in \mathbb{Z}_n} |\widehat{f}(\xi)|^2 &= \sum_{\xi \in \mathbb{Z}_n} \left| \frac{1}{n} \sum_{x \in \mathbb{Z}_n} f(x)\,\overline{\mathrm{e}_n(\xi x)} \right|^2 \\
&= \frac{1}{n^2} \sum_{\xi \in \mathbb{Z}_n} \left(\sum_{x \in \mathbb{Z}_n} f(x)\,\overline{\mathrm{e}_n(\xi x)} \right) \left(\sum_{y \in \mathbb{Z}_n} \overline{f(y)}\,\mathrm{e}_n(\xi y) \right) \\
&= \frac{1}{n^2} \sum_{\xi \in \mathbb{Z}_n} \sum_{x \in \mathbb{Z}_n} \sum_{y \in \mathbb{Z}_n} f(x)\overline{f(y)}\,\mathrm{e}_n(\xi(y-x)) \\
&= \frac{1}{n^2} \sum_{x \in \mathbb{Z}_n} \sum_{y \in \mathbb{Z}_n} f(x)\overline{f(y)} \sum_{\xi \in \mathbb{Z}_n} \mathrm{e}_n(\xi(y-x)) \\
&\overset{(☿)}{=} \frac{1}{n} \sum_{x \in \mathbb{Z}_n} f(x)\overline{f(x)} = \mathbb{E}_n |f|^2 .
\end{aligned}$$

We used Lemma 20 on (☿). □

Theorem 58. *For any $f, g \in \mathbb{C}^{\mathbb{Z}_n}$, we have*

$$\mathbb{E}_n f(x)\overline{g(x)} = \sum_{\xi \in \mathbb{Z}_n} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}.$$

Proof. By direct computation,

$$\begin{aligned}
\sum_{\xi \in \mathbb{Z}_n} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} &= \sum_{\xi \in \mathbb{Z}_n} \frac{1}{n^2} \left(\sum_{x \in \mathbb{Z}_n} f(x)\,\overline{\mathrm{e}_n(\xi x)} \right) \left(\sum_{y \in \mathbb{Z}_n} \overline{g(y)}\,\mathrm{e}_n(\xi y) \right) \\
&= \frac{1}{n^2} \sum_{x \in \mathbb{Z}_n} \sum_{y \in \mathbb{Z}_n} f(x)\overline{g(y)} \sum_{\xi \in \mathbb{Z}_n} \mathrm{e}_n(\xi(y-x)) \\
&\overset{(♁)}{=} \frac{1}{n} \sum_{x \in \mathbb{Z}_n} f(x)\overline{g(x)} = \mathbb{E}_n f(x)\overline{g(x)}.
\end{aligned}$$

We used Lemma 20 on (♁). □

Theorem 59 (Fourier integral theorem)**.** *For any $f \in \mathbb{C}^{\mathbb{Z}_n}$, we have*

$$f(x) = \sum_{\xi \in \mathbb{Z}_n} \widehat{f}(\xi)\, \mathrm{e}_n(\xi x)\,.$$

Proof. The right-hand side is equal to

$$\begin{aligned}\sum_{\xi \in \mathbb{Z}_n} \left(\frac{1}{n} \sum_{y \in \mathbb{Z}_n} f(y) \overline{\mathrm{e}_n(\xi y)} \right) \mathrm{e}_n(\xi x) &= \frac{1}{n} \sum_{\xi \in \mathbb{Z}_n} \sum_{y \in \mathbb{Z}_n} f(y) \overline{\mathrm{e}_n(\xi y)}\, \mathrm{e}_n(\xi x) \\ &= \sum_{y \in \mathbb{Z}_n} f(y) \left(\frac{1}{n} \sum_{\xi \in \mathbb{Z}_n} \mathrm{e}_n(\xi(x-y)) \right) \\ &\overset{(\text{♃})}{=} f(x).\end{aligned}$$

We used Lemma 20 on (♃). $\square$

Definition 60 (convolution)**.** Given $f, g \in \mathbb{C}^{\mathbb{Z}_n}$. Define the convolution $f * g$ by

$$f * g(x) = \mathop{\mathbb{E}_n}_{y \in \mathbb{Z}_n} f(y) g(x-y) = \mathop{\mathbb{E}_n}_{y \in \mathbb{Z}_n} g(y) f(x-y) = g * f(x).$$

Definition 61 (support)**.**

1. Given $f \in \mathbb{C}^{\mathbb{Z}_n}$. Then the support of f is the set $\operatorname{supp}(f) = \{x \in \mathbb{Z}_n : f(x) \neq 0\}$.
2. Given two sets $A, B \subset \mathbb{Z}_n$, write $A + B = \{a + b : a \in A, b \in B\} \subset \mathbb{Z}_n$.

Here we can list some properties of convolution.

Theorem 62. *Given $f, g \in \mathbb{C}^{\mathbb{Z}_n}$, we have $\operatorname{supp}(f * g) \subset \operatorname{supp}(f) + \operatorname{supp}(g)$. In particular, if $f = A$ and $g = B$ for some $A, B \subset \mathbb{Z}_n$, then $\operatorname{supp}(f * g) = \operatorname{supp}(f) + \operatorname{supp}(g)$.*

Proof. Suppose $x \in \operatorname{supp}(f * g)$, then there exists $y \in \mathbb{Z}_n$ such that $f(y)g(x-y) \neq 0$. Therefore, $y \in \operatorname{supp}(f)$ and $x - y \in \operatorname{supp}(g)$, and $x \in \operatorname{supp}(f) + \operatorname{supp}(g)$. This shows $\operatorname{supp}(f * g) \subset \operatorname{supp}(f) + \operatorname{supp}(g)$. Now suppose $f = A$ and $g = B$ for some $A, B \subset \mathbb{Z}_n$. To show the second statement, it suffices to prove $A + B = \operatorname{supp}(f) + \operatorname{supp}(g) \subset \operatorname{supp}(f * g)$. Let $x \in A + B$, that is, $x = a + b$ for some $a \in A$ and $b \in B$. Then

$$f * g(x) = \mathop{\mathbb{E}_n}_{y \in \mathbb{Z}_n} f(y) g(x-y) \geq \frac{1}{n} A(a) B(b) = \frac{1}{n},$$

we conclude that $f * g(x) \neq 0$. $\square$

Theorem 63. *Given $f, g \in \mathbb{C}^{\mathbb{Z}_n}$. The following statements are true:*

1. $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$.
2. $\mathbb{E}_n (f * g) = (\mathbb{E}_n f)(\mathbb{E}_n g)$.

Proof. Both statements can be proved by direct computation.

1. It follows by

$$
\begin{aligned}
\widehat{f}(\xi)\widehat{g}(\xi) &= \left(\sum_{x\in\mathbb{Z}_n} f(x)\overline{\mathrm{e}_n(\xi x)}\right)\left(\sum_{y\in\mathbb{Z}_n} g(y)\overline{\mathrm{e}_n(\xi y)}\right) \\
&\overset{(\uranus)}{=} \frac{1}{n^2}\sum_{x\in\mathbb{Z}_n}\sum_{z\in\mathbb{Z}_n} f(x)g(z-x)\overline{\mathrm{e}_n(\xi z)} \\
&= \frac{1}{n}\sum_{z\in\mathbb{Z}_n}\left(\frac{1}{n}\sum_{x\in\mathbb{Z}_n} f(x)g(z-x)\right)\overline{\mathrm{e}_n(\xi z)} \\
&= \frac{1}{n}\sum_{z\in\mathbb{Z}_n} f*g(z)\overline{\mathrm{e}_n(\xi z)} = \widehat{f*g}(\xi).
\end{aligned}
$$

We used the substitution $z = x + y$ on $(\uranus)$.

2. The left-hand side is equal to

$$
\frac{1}{n^2}\sum_{x\in\mathbb{Z}_n}\sum_{y\in\mathbb{Z}_n} f(y)g(x-y) = \left(\frac{1}{n}\sum_{x\in\mathbb{Z}_n} f(x)\right)\left(\frac{1}{n}\sum_{y\in\mathbb{Z}_n} g(y)\right) = \left(\mathbb{E}_n f\right)\left(\mathbb{E}_n g\right).
$$

Discussions above prove the theorem. □

We now can introduce the L^p and ℓ^p norm on the function space $\mathbb{C}^{\mathbb{Z}_n}$.

Definition 64 (L^p and ℓ^p norm)**.** Given $f \in \mathbb{C}^{\mathbb{Z}_n}$. We define the $L^p(\mathbb{Z}_n)$ norm of f by

$$
\begin{cases}
\|f\|_{L^p} = \left(\mathop{\mathbb{E}_n}\limits_{x\in\mathbb{Z}_n} |f(x)|^p\right)^{1/p}, \text{ for } 1 \le p < \infty \\
\|f\|_{L^\infty} = \sup\limits_{x\in\mathbb{Z}_n} |f(x)|
\end{cases}
$$

We also can define the $\ell^p(\mathbb{Z}_n)$ norm of f by

$$
\begin{cases}
\|f\|_{\ell^p} = \left(\sum\limits_{\xi\in\mathbb{Z}_n} |f(\xi)|^p\right)^{1/p}, \text{ for } 1 \le p < \infty \\
\|f\|_{\ell^\infty} = \sup\limits_{\xi\in\mathbb{Z}_n} |f(\xi)|
\end{cases}
$$

We then have the following lemma.

Lemma 21. Let $A \subset \mathbb{Z}_n$. Then the following statements are true:

1. $\|\widehat{A}\|_{\ell^\infty} = \widehat{A}(0) = \dfrac{|A|}{|\mathbb{Z}_n|}$.

2. $\|\widehat{A}\|_{\ell^2}^2 := \sum\limits_{\xi\in\mathbb{Z}_n} |\widehat{A}(\xi)|^2 = \dfrac{|A|}{|\mathbb{Z}_n|}$.

3. $\widehat{A}(\xi) = \overline{\widehat{A}(-\xi)}$.

Proof.

1. By the definition, $\widehat{A}(\xi) = \frac{1}{n}\sum_{x\in\mathbb{Z}_n} A(x)\overline{\mathrm{e}_n(\xi x)} = \frac{1}{n}\sum_{x\in A}\overline{\mathrm{e}_n(\xi x)}$. It follows that

$$|\widehat{A}(\xi)| \leq \frac{|A|}{n} = \frac{|A|}{|\mathbb{Z}_n|}$$

and it is clear that $\widehat{A}(0)$ attains the maximum among all $\widehat{A}(\xi)$.

2. By Parseval's formula (Theorem 57), we have the right-hand side is equal to

$$\frac{1}{n}\sum_{x\in\mathbb{Z}_n} |A(x)|^2 = \frac{|A|}{n} = \frac{|A|}{|\mathbb{Z}_n|}.$$

3. It follows by the definition.

Discussions above prove the theorem. □

To prove the Roth's theorem on 3-term arithmetic progression, we shall convert the problem into an equivalent statement which is easier to handle. We first let

$$s(n) = \max\{\#A : A \subset [1,n], A \text{ has no 3-term arithmetic progression}\}.$$

If we can show that

$$\lim_{n\to\infty} \frac{s(n)}{n} = 0,$$

then for any $A \subset \mathbb{N}$ with positive density, there has to be some 3-term arithmetic progression. If not, then

$$0 \leq \frac{\#A \cap \{1,2,\ldots,n\}}{n} \leq \frac{s(n)}{n}.$$

By the squeeze theorem, we obtain

$$\lim_{n\to\infty} \frac{\#A \cap \{1,2,\ldots,n\}}{n} = 0$$

contradicting the assumption of A having 3-term arithmetic progression.

Lemma 22. Let $s(n)$ be defined as above. Then the limit $\lim_{n\to\infty} \frac{s(n)}{n}$ exists.

Proof. We first claim that $s(n)$ is sub-additive, that is, $s(n+m) \leq s(n)+s(m)$. Suppose $A \subset [1, n+m]$ does not have a 3-term A.P. Then $A \cap [1,n] \leq s(n)$ and $A \cap [n+1, n+m] \leq s(m)$, thus $A \leq s(n)+s(m)$. Taking supremum among all possible A, we obtain $s(n+m) \leq s(n)+s(m)$.

Now suppose $\limsup_{n\to\infty} \frac{s(n)}{n} = \alpha \geq 0$. Fix $n \in \mathbb{N}$. Given $k \in \mathbb{N}$. Suppose $k = qn + r$ for some $0 \leq r < n$ (integer division). The sub-additive implies that $s(k) \leq (q+1)s(n)$. It now follows that

$$\frac{s(k)}{k} \leq \frac{(q+1)s(n)}{qn} \leq \frac{s(n)}{n}$$

and

$$\alpha = \limsup_{k\to\infty} \frac{s(k)}{k} \le \frac{s(n)}{n}.$$

This inequality shows that

$$\liminf_{n\to\infty} \frac{s(n)}{n} \ge \alpha = \limsup_{k\to\infty} \frac{s(k)}{k},$$

thus $\limsup_{n\to\infty} \frac{s(n)}{n} = \liminf_{n\to\infty} \frac{s(n)}{n} = \lim_{n\to\infty} \frac{s(n)}{n}$. □

Now, in order to get contradiction, we assume that $\lim_{n\to\infty} \frac{s(n)}{n} = c > 0$. Let $\epsilon > 0$. Then there is $N \in \mathbb{N}$ large enough such that

$$c - \epsilon < \frac{s(n)}{n} \le c + \epsilon \qquad \text{whenever} \qquad n \ge N.$$

Fix $n \ge N$ and let $A \subset \{1, \ldots, 2n\}$ be a set with no 3-term A.P such that

$$\frac{|A|}{2n} \ge c - \epsilon.$$

Let $A_{\text{even}} = A \cap 2\mathbb{N}$, namely, the the set of all even numbers in A, then we claim

$$c - 3\epsilon \le \frac{|A_{\text{even}}|}{n} \le c + \epsilon.$$

It is clear that

$$\frac{|A_{\text{even}}|}{n} \le \frac{s(n)}{n} \le c + \epsilon.$$

Since $A - A_{\text{even}}$ is a subset of $\{1, 3, \ldots, 2n-1\}$, thus

$$\frac{|A - A_{\text{even}}|}{n} \le \frac{s(n)}{n} \le c + \epsilon,$$

therefore

$$\frac{|A_{\text{even}}|}{n} = \frac{|A - (A - A_{\text{even}})|}{n} \ge \frac{|A| - |A - A_{\text{even}}|}{n} \ge (2c - 2\epsilon) - (c + \epsilon) = c - 3\epsilon.$$

Now suppose $A = \{u_1, u_2, \ldots, u_r\}$ has r elements and $A_{\text{even}} = \{2v_1, 2v_2, \ldots, 2v_s\}$ has s elements. Also we write $A_\star$ to denote the set $\{v_1, v_2, \ldots, v_s\}$. The following lemmas will do Fourier analysis on A as a subset of the cyclic group $\mathbb{Z}_{2n}$.

Lemma 23. Continuing the notation above, the following statements are true:

1. $\widehat{A}(0) = \frac{r}{2n}$.
2. $\widehat{A}(\alpha) = \frac{1}{2n} \sum_{i=1}^{r} \overline{\mathrm{e}_{2n}(\alpha u_i)}$.
3. $\widehat{A_\star}(-\alpha) = \frac{1}{2n} \sum_{i=1}^{s} \overline{\mathrm{e}_{2n}(-\alpha v_i)}$.
4. $\sum_{\alpha \in \mathbb{Z}_{2n}} \widehat{A}(\alpha) \widehat{A_\star}(-\alpha)^2 = \frac{s}{4n^2}$.

Proof. The first three statements are corollaries of Lemma 21. We now show the fourth statement.

$$\begin{aligned}
\sum_{\alpha\in\mathbb{Z}_{2n}} \widehat{A}(\alpha)\widehat{A_\star}(-\alpha)^2 &= \frac{1}{8n^3}\sum_{\alpha\in\mathbb{Z}_{2n}}\sum_{i=1}^{r}\overline{\mathrm{e}_{2n}(\alpha u_i)}\sum_{j=1}^{s}\overline{\mathrm{e}_{2n}(-\alpha v_j)}\sum_{k=1}^{s}\overline{\mathrm{e}_{2n}(-\alpha v_k)} \\
&= \frac{1}{4n^2}\sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{k=1}^{s}\left(\frac{1}{2n}\sum_{\alpha\in\mathbb{Z}_{2n}}\overline{\mathrm{e}_{2n}(\alpha(u_i-v_j-v_k))}\right) \\
&\overset{(\text{♇})}{=} \frac{1}{4n^2}\#\{(i,j,k) : u_i - v_j - v_k = 0\} \\
&\overset{(\text{♀})}{=} \frac{1}{4n^2}\#A_\star = \frac{s}{4n^2}.
\end{aligned}$$

The equality (♇) holds by Lemma 20 and the equality (♀) holds since

$$u_i - v_j - v_k = 0 \implies 2u_i = 2v_j + 2v_k,$$

implying that $\{2v_j, u_i, 2v_k\} \subset A$ is a 3-term A.P. therefore $u_i - v_j - v_k = 0$ if and only if $u_i = 2v_j = 2v_k \in A_\star$. $\square$