HW1_Q3_Solution

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Question 3:

In the smallest branch of the smallest bank, the number of customers in the queue (waiting customers), is a random variable $Q \in \{0, 1, 2\}$. You cannot have more than 2 customers waiting in the queue, because the've been downsizing and the branch is really small.

The distribution of Q is dependent on a parameter θ.

$$Q = \begin{cases} 0 & \text{w.p. } 4\theta^2 \\ 1 & \text{w.p. } 4\theta - 8\theta^2 \\ 2 & \text{w.p. } 1 - 4\theta + 4\theta^2 \end{cases}$$

The bank's headquarters randomly sampled the queue during five independent times. The results were $\{0, 1, 0, 0, 0\}$ customers in the queue.

Answer the following questions:

1. Find an unbiased estimator $\hat{\Theta}$ for the parameter θ for a sample of size $n = 5$. What is $\hat{\Theta}$ based on the current sample? (you should get 0.45)
2. Find an unbiased estimator for the expected number of customers waiting in the queue based on a sample of size $n = 5$. What is the estimate of the expected number of customers, based on the current sample? (0.2)
3. Find an estimator for θ in the maximum likelihood estimation method. (0.45)

Question 3 - Solution:

1.

We need to find $\hat{\Theta}$ such that $E(\hat{\theta}) = \theta$. It is known that $E(\overline{X}) = E(X_i)$ so:

$$E(\overline{X}) = E(X_i) = 0 \cdot 4\theta^2 + 1 \cdot \left(4\theta - 8\theta^2\right) + 2 \cdot \left(1 - 4\theta + 4\theta^2\right) = 2 - 4\theta \ \rightarrow$$

$$\theta = \frac{2 - E(\bar{X})}{4} = E(\frac{2 - \bar{X}}{4}) \rightarrow$$

$$\hat{\theta} = \frac{2 - \bar{X}}{4}$$

2.

We need to find $\hat{u}$ such that $E(\hat{u}) = E(X)$ We know that $E(\overline{X}) = E(X)$, in the context of our sample - $\overline{X} = 0.2$

3.

Let's form the likelihood function -

$$\begin{aligned} &L(\theta) = P(X_1 = 0) \cdot P(X_2 = 1) \cdot P(X_3 = 0) \cdot P(X_4 = 0) \cdot P(X_5 = 0) \\ &= [4\theta(1 - 2\theta)] \cdot \left[4\theta^2\right]^4 = 1024\theta^9 - 2048\theta^{10} \\ &\rightarrow L'(\theta) = 9216\theta^8 - 20480\theta^9 = 0 \rightarrow \hat{\theta} = 0.45 \end{aligned}$$

Let's verify it's a maxima -

```r
library(ggplot2)
```

```
## Registered S3 methods overwritten by 'ggplot2':
##   method         from
##   [.quosures     rlang
##   c.quosures     rlang
##   print.quosures rlang
```

```r
fun.1 <- function(x) 1024*(x^9) - 2048*(x^10)
set.seed(1492)
ggplot(data.frame(x = seq(from = -0.3, to = 0.5, by = 0.001)), aes(x)) +
  stat_function(fun = fun.1, colour = "red") + ggtitle("y = 1024x^9 -2048x^10") + theme(plot.title = el
```