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Interpreting Factor Models

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ABSTRACT

We argue that tests of reduced-form factor models and horse races between "characteristics" and "covariances" cannot discriminate between alternative models of investor beliefs. Since asset returns have substantial commonality, absence of near-arbitrage opportunities implies that the stochastic discount factor can be represented as a function of a few dominant sources of return variation. As long as some arbitrageurs are present, this conclusion applies even in an economy in which all cross-sectional variation in expected returns is caused by sentiment. Sentiment-investor demand results in substantial mispricing only if arbitrageurs are exposed to factor risk when taking the other side of these trades.

REDUCED-FORM FACTOR MODELS ARE ubiquitous in empirical asset pricing. In these models, the stochastic discount factor (SDF) is represented as a function of a small number of portfolio returns. In equity market research, models such as the three-factor SDF of Fama and French (1993) and various extensions are popular with academics and practitioners alike. These models are reduced-form because they are not derived from assumptions about investor beliefs, preferences, and technology that prescribe which factors should appear in the SDF. What interpretation should one give such a model if it works well empirically?

That there exists a factor representation of the SDF is almost a tautology.[1] The economic content of the factor-model evidence lies in the fact that not only do covariances with the factors explain the cross section of expected returns, but the factors also account for a substantial share of the time-series comovement of stock returns. As a consequence, an investor who wants to benefit from the

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[1] If the law of one price (LOP) holds, one can always construct a single-factor or multifactor representation of the SDF in which the factors are linear combinations of asset payoffs (Hansen and Jagannathan (1991)). Thus, the mere fact that a low-dimensional factor model "works" has no economic content beyond the LOP.

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expected return spread between, say, value and growth stocks or recent winner and loser stocks must invariably take on substantial factor risk exposure.

Researchers often interpret the evidence that expected return spreads are associated with exposures to volatile common factors as a distinct feature of "rational," as opposed to "behavioral," asset pricing models. For example, Cochrane (2011, p. 1075) writes,

> Behavioral ideas—narrow framing, salience of recent experience, and so forth—are good at generating anomalous prices and mean returns in individual assets or small groups. They do not [...] naturally generate covariance. For example, "extrapolation" generates the slight autocorrelation in returns that lies behind momentum. But why should all the momentum stocks then rise and fall together the next month, just as if they are exposed to a pervasive, systematic risk?

In a similar vein, Daniel and Titman (1997) and Brennan, Chordia, and Subrahmanyam (1998) suggest that one can test for the relevance of behavioral effects on asset prices by looking for a component of expected return variation associated with stock characteristics (such as value/growth, momentum, etc.) that is orthogonal to factor covariances. This view that behavioral effects on asset prices are distinct from and orthogonal to common factor covariances is pervasive in the literature.[2]

Contrary to this standard interpretation, we argue that there is no such clear distinction between factor pricing and behavioral asset pricing. If sentiment—which we use as a catch-all term for distorted beliefs, liquidity demands, or other distortions—affects asset prices, the resulting expected return spreads between assets should be explained by common factor covariances in a similar way as in standard rational expectations asset pricing models. The reason is that the existence of a relatively small number of arbitrageurs should be sufficient to ensure that near-arbitrage opportunities—that is, trading strategies that earn extremely high Sharpe ratios (SRs)—do not exist. To take up Cochrane's example, if stocks with momentum did not rise and fall together next month to a considerable extent, the expected return spread between winner and loser stocks would not exist in the first place because arbitrageurs would have picked this low-hanging fruit. Arbitrageurs neutralize components of sentiment-driven asset demand that are not aligned with

[2] For example, Brennan, Chordia, and Subrahmanyam (1998, p. 346) describe the reduced-form factor model studies of Fama and French as follows: "... Fama and French (FF) (1992a, b, 1993b, 1996) have provided evidence for the continuing validity of the rational pricing paradigm." The standard interpretation of factor pricing as distinct from models of mispricing also appears in more recent work. To provide one example, Hou, Karolyi, and Kho (2011, p. 2528) write that "Some believe that the premiums associated with these characteristics represent compensation for pervasive extra-market risk factors, in the spirit of a multifactor version of Merton's (1973) Intertemporal Capital Asset Pricing Model (ICAPM) or Ross's (1976) Arbitrage Pricing Theory (APT) (Fama and French (1993, 1996), Davis, Fama, and French (2000)), whereas others attribute them to inefficiencies in the way markets incorporate information into prices (Lakonishok, Shleifer, and Vishny (1994), Daniel and Titman (1997), Daniel, Titman, and Wei (2001))."

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common factor covariances, but are reluctant to trade aggressively against components that would expose them to factor risk. Only in the latter case can the sentiment-driven demand have a substantial impact on expected returns. These conclusions apply not only to the equity factor models that we focus on here, but also to no-arbitrage bond pricing models and currency factor models.

We start by analyzing implications of the absence of near-arbitrage opportunities for the reduced-form factor structure of the SDF. For typical sets of assets and portfolios, the covariance matrix of returns is dominated by a small number of factors. These empirical facts combined with the absence of near-arbitrage opportunities imply that the SDF can be represented to a good approximation as a function of these few dominant factors.[3] This conclusion also applies to models with sentiment-driven investors, as long as arbitrageurs eliminate the most extreme forms of mispricing.

If this reasoning is correct, then it should be possible to obtain a low-dimensional factor representation of the SDF based purely on information from the covariance matrix of returns. We show that a factor model with a small number of principal component (PC) factors does about as well as popular reduced-form factor models in explaining the cross section of expected returns on anomaly portfolios. Thus, there does not seem to be anything special about the construction of the reduced-form factors proposed in the literature—purely statistical factors do just as well. For typical test asset portfolios, their return covariance structure essentially dictates that the first few PC factors must explain the cross section of expected returns.[4] Otherwise, near-arbitrage opportunities would exist.

Tests of characteristics versus covariances, like those pioneered in Daniel and Titman (1997), look for variation in expected returns that is orthogonal to factor covariances. Ex-post and in sample such orthogonal variation always exists, perhaps even with statistical significance according to conventional criteria. It is an open question, however, whether such near-arbitrage opportunities are a robust and persistent feature of the cross section of stock returns. To address this question, we perform a pseudo out-of-sample exercise. Splitting the sample period into two subsamples, we extract the PCs from the covariance matrix of returns in one subperiod and then use the portfolio weights implied by the first subsample PCs to construct factors out of sample in the second subsample. While factors beyond the first few PCs contribute substantially to

[3] This notion of the absence of near-arbitrage is closely related to the interpretation of the arbitrage pricing theory (APT) in Ross (1976). When discussing the empirical implementation of the APT in a finite-asset economy, Ross suggests bounding the maximum squared SR of any arbitrage portfolio at twice the squared SR of the market portfolio. However, our interpretation of APT-type models differs from some of the literature. For example, Fama and French (1996) regard the APT as a rational pricing model. We disagree with this narrow interpretation, as the APT is just a reduced-form factor model.

[4] The number of factors depends heavily on the underlying space of test assets. For instance, for the Fama-French 5×5 size and book-to-market (B/M) sorted portfolios, there are three dominant factors. For payoff spaces with weaker factor structure, the number of dominant factors is larger.

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the maximum SR in sample, PCs beyond the first few no longer add to the SR out of sample. Thus, in-sample deviations from low-dimensional factor pricing do not appear to reliably persist out of sample.

It would be wrong, however, to jump from the evidence that expected returns line up with common factor covariances to the conclusion that the idea of sentiment-driven asset prices can be rejected. To show this, we build a model of a multiasset market in which fully rational risk-averse investors (arbitrageurs) trade with investors whose asset demands are based on distorted beliefs about the true distribution of returns (sentiment investors). We make two plausible assumptions. First, the covariance matrix of asset cash flows features a few dominant factors that drive most of the stocks' covariances. Second, sentiment investors cannot take extreme positions that would require substantial leverage or extensive use of short-selling. In this model, all cross-sectional variation in expected returns is caused by distorted beliefs and yet a low-dimensional factor model explains the cross section of expected returns. To the extent that sentiment-investor demand is orthogonal to covariances with the dominant factors, arbitrageurs elastically accommodate this demand and take the other side with minimal price concessions. Only sentiment-investor demand that is aligned with covariances with the dominant factors affects prices because it is risky for arbitrageurs to take the other side. As a result, the SDF in this economy can be represented to a good approximation as a function of the first few PCs, even though all deviations of expected returns from the CAPM are caused by sentiment. Therefore, the fact that a low-dimensional factor model holds is consistent with behavioral explanations just as much as it is with rational explanations.

This model makes clear that empirical horse races between covariances with reduced-form factors and stock characteristics that are meant to proxy for mispricing or sentiment-investor demand (as, for example, in Daniel and Titman (1997), Brennan, Chordia, and Subrahmanyam (1998), Davis, Fama, and French (2000), and Daniel, Titman, and Wei (2001)) set the bar too high for behavioral models; even in a world in which belief distortions affect asset prices, expected returns should line up with common factor covariances. Tests of factor models with ad-hoc macroeconomic factors (as, for example, in Chen, Roll, and Ross (1986), Cochrane (1996), Li, Vassalou, and Xing (2006), and Liu and Zhang (2008)) are not more informative either. As shown in Nawalkha (1997) (see also Shanken (1992), Reisman (1992), and Lewellen, Nagel, and Shanken (2010)), if K dominant factors drive return variation and the SDF can be represented as a linear combination of these K factors, then the SDF can be represented, equivalently, by a linear combination of any K macroeconomic variables with possibly very weak correlation with the K factors.

Relatedly, theoretical models that derive relationships between firm characteristics and expected returns, taking as given an arbitrary SDF, do not shed light on the rationality of investor beliefs. Models such as Berk, Green, and Naik (1999), Johnson (2002), Liu, Whited, and Zhang (2009), or Liu and Zhang (2014) apply equally to our sentiment-investor economy as they do to an economy in which the representative investor has rational expectations.

These models show how firm investment decisions are aligned with expected returns in equilibrium, according to *firms*' first-order conditions. But these models do not speak to the question of under which types of beliefs—rational or otherwise—*investors* align their marginal utilities with asset returns through their first-order conditions.

The observational equivalence between behavioral and rational asset pricing with regard to factor pricing also applies, albeit to a lesser degree, to partial equilibrium intertemporal capital asset pricing models (ICAPMs) in the tradition of Merton (1973). In the ICAPM, the SDF is derived from the first-order condition of an investor who holds the market portfolio and faces exogenous time-varying investment opportunities. This leaves open the question of how to endogenously generate the time variation in investment opportunities in a way that is internally consistent with the investor's choice to hold the market portfolio. We show that time-varying investor sentiment is one possibility. If sentiment-investor asset demands in excess of market portfolio weights have a single-factor structure and are mean-reverting around zero, then arbitrageurs' first-order condition implies an ICAPM that resembles the one in Campbell (1993) and Campbell and Vuolteenaho (2004a) whereby arbitrageurs demand risk compensation both for cash-flow beta ("bad beta") and discount-rate beta ("good beta") exposure.

How can we differentiate between rational and behavioral explanations for returns? We argue that the only way to answer this question is to develop and test structural models with explicit assumptions about beliefs and preferences. Such models deliver testable predictions about the factors that should be in the SDF and the probability distributions under which a model-implied SDF prices assets. Of course, a test of a specific model of preferences and beliefs cannot yield generic conclusions about the validity of rational or behavioral approaches to asset pricing, but an empirical examination of reduced-form factor models does not help circumvent this fundamental problem.

On the theoretical side, our work is related to Daniel, Hirshleifer, and Subrahmanyam (2001). Their model also includes sentiment-driven investors trading against arbitrageurs. In contrast to our model, however, sentiment investors' position size is not constrained. As a consequence, for idiosyncratic belief distortions, both sentiment traders (mistakenly) and arbitrageurs (correctly) perceive a near-arbitrage opportunity and take huge offsetting bets against each other. With such unbounded position sizes, even idiosyncratic belief distortions can have a substantial effect on prices and dominant factor covariances do not fully explain the cross section of expected returns. We deviate from their setup because it seems plausible that sentiment-investor position sizes and leverage are bounded.

On the empirical side, our paper is related to Stambaugh and Yuan (2016). They construct "mispricing factors" to explain a large number of anomalies. Our model of sentiment-driven asset prices explains why such mispricing factors work in explaining the cross section of expected returns. Empirically, our factor construction based on PCs is different, as the construction uses only the covariance matrix of returns and not stock characteristics or expected returns.

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Kogan and Tian (2015) conduct a factor-mining exercise based on the factors constructed by sorting on characteristics. They find that such factors are not robust in explaining the cross section of expected returns out of sample. While we obtain similar results for higher order PC factors, we find that the first few PC factors are actually robustly related to the cross section of expected returns out of sample.

The rest of the paper is organized as follows. In Section I, we describe the portfolio returns data that we use in this study. In Section II, we lay out implications of the absence of near-arbitrage opportunities and we report the empirical results on the factor pricing with PC factors. Section III presents the model in which fully rational risk-averse arbitrageurs trade with sentiment investors. Section IV develops a model with time-varying investor sentiment, which results in ICAPM-type hedging demands. Section V concludes.

I. Portfolio Returns

To analyze the role of factor models empirically, we use two sets of portfolio returns. First, we use 15 anomaly long-short strategies defined as in Novy-Marx and Velikov (2016) and the underlying 30 portfolios from the long and short sides of these strategies. This set of returns captures many of the most prominent features of the cross section of stock returns discovered over the past few decades. We recreate the anomaly portfolios and use daily returns on these strategies for all estimation. Second, for comparison, we use the 5×5 size and B/M sorted portfolios of Fama and French (1993).[5]

Table I provides descriptive statistics for the anomaly long-short portfolios. Mean returns on long-short strategies range from 0.69% to 16.31% per year. Annualized squared SRs, shown in the second column, range from 0.01 to 0.83. Since these long-short strategies have low correlation with the market factor, these squared SRs are roughly equal to the incremental squared SR that the strategy would contribute if added to the market portfolio.

The factor structure of returns plays an important role in our subsequent analysis. To prepare the stage, we analyze the commonality in these anomaly strategy returns. We perform an eigenvalue decomposition of the covariance matrix of the 30 underlying portfolio returns and extract the PCs, ordered from the PC with the highest eigenvalue (which explains most of the comovement of returns) to the one with the lowest. We then run a time-series regression of each long-short strategy return on the first, on the first and the second, etc., up to a regression on PCs one to five. The last five columns in Table I report

[5] We thank Ken French for making these returns available on his website. Monthly anomaly returns are available on Robert Novy-Marx's website. For the anomaly strategies in Novy-Marx and Velikov (2016), we use those strategies that can be constructed since 1963, are not classified as high-turnover strategies, and are not largely redundant. Based on this latter exclusion criterion, we eliminate monthly imbalanced net issuance (and use only annually imbalanced net issuance). We also exclude the gross margin and asset turnover strategies that are subsumed, in terms of their ability to generate variation in expected returns, by the gross profitability strategy, as shown in Novy-Marx (2013).

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Table I

Anomalies: Returns and Principal Component Factors

The sample period is July 1966 to December 2015. The anomaly long-short strategy daily returns are as defined in Novy-Marx and Velikov (2016). Average returns and squared Sharpe ratios are reported in annualized terms. Mean returns and squared Sharpe ratios are calculated for 15 long-short anomaly strategies. Principal component factors are extracted from returns on the 30 portfolios underlying the long and short sides of these strategies.

	Mean Return	PC Factor-Model R^2					
		SR^2	PC1	PC1-2	PC1-3	PC1-4	PC1-5
Size	1.29	0.01	0.08	0.11	0.60	0.64	0.69
Gross Profitability	4.56	0.17	0.03	0.05	0.13	0.16	0.50
Value	5.71	0.19	0.02	0.02	0.48	0.67	0.67
ValProf	8.45	0.63	0.08	0.10	0.34	0.38	0.46
Accruals	4.63	0.20	0.00	0.00	0.00	0.01	0.01
Net Issuance (rebal.-A)	9.06	0.83	0.15	0.26	0.27	0.38	0.40
Asset Growth	4.31	0.14	0.07	0.09	0.22	0.44	0.46
Investment	5.41	0.24	0.06	0.07	0.13	0.18	0.20
Piotroski's F-score	0.69	0.02	0.02	0.07	0.15	0.15	0.16
ValMomProf	9.59	0.43	0.01	0.44	0.63	0.70	0.80
ValMom	5.71	0.15	0.03	0.35	0.73	0.73	0.73
Idiosyncratic Volatility	9.25	0.20	0.34	0.55	0.69	0.92	0.94
Momentum	16.31	0.61	0.01	0.72	0.72	0.91	0.92
Long Run Reversals	4.22	0.09	0.01	0.01	0.40	0.52	0.58
Beta Arbitrage	7.12	0.24	0.14	0.33	0.33	0.46	0.75

the R^2s from these regressions. Since we are looking at long-short portfolio returns that are roughly market-neutral, the first PC does not explain much of the time-series variation of returns. With the first and second PCs combined, the explanatory power in terms of R^2 ranges from zero for the accruals strategy to 0.72 for the momentum strategy. Once the first five PCs are included in the regression, the explanatory power is more uniform (except for accruals), with R^2 values ranging from 0.16 for the F-score strategy to 0.94 for the idiosyncratic volatility strategy, with most strategies having an R^2 above 0.5. Thus, a substantial portion of the time-series variation in returns of these anomaly portfolios can be traced to a few common factors.

For the second set of returns from the size-B/M portfolios, it is well known from Fama and French (1993) that three factors—the excess return on the value-weighted market index (MKT), the small minus large stock factor (SMB), and the high minus low B/M factor (HML)—explain more than 90% of the time-series variation of returns. While SMB and HML are constructed in a rather special way from a smaller set of six size-B/M portfolios, we obtain essentially similar factors from the first three PCs of the 5×5 size-B/M portfolio returns.

The first PC is, to a good approximation, a level factor that puts equal weight on all 25 portfolios. The first two PCs that remain after removing the level factor are essentially the SMB and HML factors. Figure 1 plots the eigenvectors. PC1, shown on the left, has positive weights on small stocks and negative weights on

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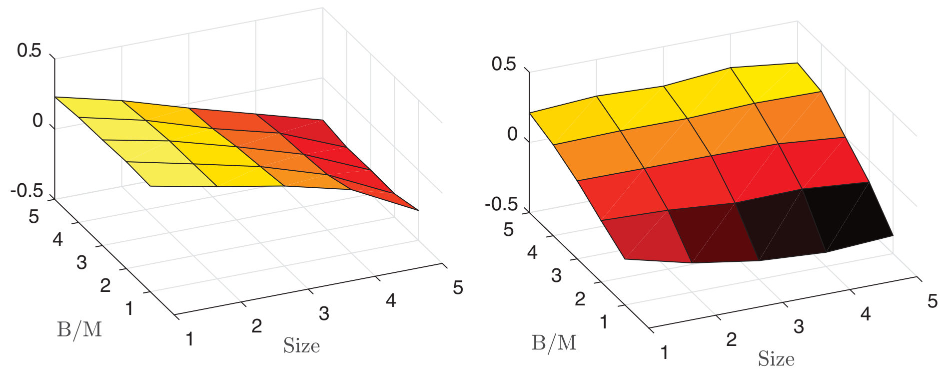

Figure 1. Eigenvectors. The figures plot eigenvector weights corresponding to the second (left) and third (right) principal components of the 25 size-B/M portfolio returns. (Color figure can be viewed at wileyonlinelibrary.com)

large stocks, that is, it is similar to SMB. PC2, shown on the right, has positive weights on high-B/M stocks and negative weights on low-B/M stocks, that is, it is similar to HML. This shows that the Fama and French (1993) factors are not special in any way; rather, they simply summarize in a succinct fashion cross-sectional variation in the size-B/M portfolio returns, similar to the first three PCs.[6]

II. Factor Pricing and the Absence of Near-Arbitrage

In this section, we start by showing that if we have assets with a few dominant factors that drive much of the covariances of returns (i.e., small number of factors with large eigenvalues), then those factors must explain asset returns. Otherwise near-arbitrage opportunities would arise, which would be implausible even if one entertains the possibility that prices could be influenced substantially by the subjective beliefs of sentiment investors.

Consider an economy with discrete time $t = 0, 1, 2, \ldots$. There are N assets in the economy, indexed by $i = 1, \ldots, N$, with a vector of returns in excess of the risk-free rate, R. Let $\mu \equiv E[R]$ and denote the covariance matrix of excess returns by Γ.

Assume that the LOP holds. The LOP is equivalent to the existence of an SDF M such that $\mathrm{E}[MR] = 0$. Note that, while $\mathrm{E}[\cdot]$ represents the objective expectations of the econometrician, there is no presumption here that $\mathrm{E}[\cdot]$ also represents the subjective expectations of investors. Thus, the LOP does not embody an assumption about beliefs and hence about the rationality of investors (apart from ruling out beliefs that violate the LOP).

[6] A related observation appears in Lewellen, Nagel, and Shanken (2010), who note that three factors formed as linear combinations of the 25 size-B/M portfolio returns with random weights often explain the cross section of expected returns on these portfolios about as well as the Fama and French (1993) factors do.

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Now consider the minimum-variance SDF in the span of excess returns, constructed as in Hansen and Jagannathan (1991) as

$$M = 1 - \mu' \Gamma^{-1}(R - \mu). \tag{1}$$

Since we work with excess returns, the SDF can be scaled by an arbitrary constant, and we normalize it to have $\mathrm{E}[M] = 1$. The variance of the SDF,

$$\mathrm{Var}(M) = \mu' \Gamma^{-1} \mu, \tag{2}$$

equals the maximum squared SR achievable from the N assets.

Now define the absence of near-arbitrage as the absence of extremely high-SR opportunities (under objective probabilities) as in Cochrane and Saá-Requejo (2000). Ross (1976) also proposes a bound on the squared SR for an empirical implementation of his APT in a finite-asset economy. He suggests ruling out squared SR greater than two times the squared SR of the market portfolio. Such a bound on the maximum squared SR is equivalent, via (2), to an upper bound on the variance of the SDF M that resides in the span of excess returns.

Our perspective on this issue differs from some of the extant literature. For example, MacKinlay (1995) suggests that the SR should be (asymptotically) bounded under "risk-based" theories of the cross section of stock returns, but stay unbounded under alternative hypotheses that include "market irrationality." A similar logic underlies the characteristics versus covariances tests in Daniel and Titman (1997) and Brennan, Chordia, and Subrahmanyam (1998). However, ruling out extremely high-SR opportunities implies only weak restrictions on investor beliefs and preferences, with plenty of room for "irrationality" to affect asset prices. Even in a world in which many investors' beliefs deviate from rational expectations, near-arbitrage opportunities should not exist as long as some investors ("arbitrageurs") with sufficient risk-bearing capacity have beliefs that are close to objective beliefs. We can then think of the pricing equation $\mathrm{E}[MR] = 0$ as the first-order condition of arbitrageurs' optimization problem and hence of the SDF as representing the marginal utility of the arbitrageur. The model in Section III shows that extremely high volatility of M can occur only if the wealth of arbitrageurs in the economy is small and the sentiment investors they are trading against take huge concentrated bets on certain types of risk. Given sufficient arbitrageur wealth and reasonable constraints on sentiment investors' trading, the variance of M is bounded from above.

We now show that the absence of near-arbitrage opportunities implies that one can represent the SDF as a function of the dominant factors driving return variation. Consider the eigendecomposition of the excess returns covariance matrix

$$\Gamma = Q \Lambda Q', \qquad \text{with} \qquad Q = (q_1, \ldots, q_N), \tag{3}$$

where λ_i are the diagonal elements of Λ. Assume that the first PC is a level factor, that is, $q_1 = \frac{1}{\sqrt{N}}\iota$, where ι is a conformable vector of ones. This implies

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that $q_k'\iota = 0$ for $k > 1$, that is, the remaining PCs are long-short portfolios. In Appendix A, we show that

$$\begin{aligned}\mathrm{Var}(M) &= (\mu' q_1)^2 \lambda_1^{-1} + \mu' Q_z \Lambda_z^{-1} Q_z' \mu \\ &= \frac{\mu_m^2}{\sigma_m^2} + N\overline{\mathrm{Var}}(\mu_i) \sum_{k=2}^{N} \frac{\overline{\mathrm{Corr}}(\mu_i, q_{ki})^2}{\lambda_k},\end{aligned} \tag{4}$$

where the z subscripts stand for matrices with the first PC removed, $\mu_m = \frac{1}{\sqrt{N}} q_1' \mu$, $\sigma_m^2 = \frac{\lambda_1}{N}$, and $\overline{\mathrm{Var}}(.)$ and $\overline{\mathrm{Corr}}(.)$ denote cross-sectional variance and correlation, respectively. According to this expression for SDF variance, a strong factor structure in test asset returns combined with significant cross-sectional variation in average returns implies that expected returns must line up with the first few (high eigenvalue) PCs, as otherwise $\mathrm{Var}(M)$ would be very high and near-arbitrage opportunities would exist. To see this, note that the sum of the squared correlations of μ_i and q_{ki} is always equal to one. But the magnitude of the sum weighted by the inverse λ_k depends on which of the PCs the vector μ lines up with. If it lines up with high-λ_k PCs, then the sum is much lower than if it lines up with low-λ_k PCs. For typical test assets, eigenvalues decay rapidly beyond the first few PCs. In this case, a high correlation of μ_i with a low-eigenvalue q_{ki} would lead to an enormous maximum SR.

This argument relies only on the absence of near-arbitrage opportunities. Therefore, the result obtains for both rational and behavioral classes of models. Indeed, the logic applies to any set of test assets as long as this set has a strong factor structure. We use the anomaly portfolios in our analysis because they exhibit these properties and are well studied in the recent literature. If, on the other hand, a set of test assets exhibits only weak factor structure, equation (4) shows that these assets cannot have much cross-sectional dispersion in expected returns without violating reasonable bounds on the maximum variance of the SDF (or SR).

We have maintained so far that expected returns must line up with the first few PCs, as otherwise high-SR opportunities would arise. We now provide empirical support for this assertion. We quantify these relationships using our two sets of test assets by asking, counterfactually, what the maximum SR of the test assets would be if expected returns did not line up, as they do in the data, with the first few (high eigenvalue) PCs, but instead were also correlated with the higher order PCs. To do so, we go back to equation (4). We assume that μ_i is correlated with the first K PCs, while the correlation with the remaining PCs is exactly zero. For simplicity of exposition, we further assume that all nonzero correlations are equal.[7] We set $\overline{\mathrm{Var}}(\mu_i)$ and Λ equal to their sample values.

Figure 2 presents the results. Panel A shows the counterfactual squared SR for the 30 anomaly portfolios. If expected returns of these portfolios lined up

[7] Since the sum of all squared correlations must add up to one, each squared correlation is $1/K$.

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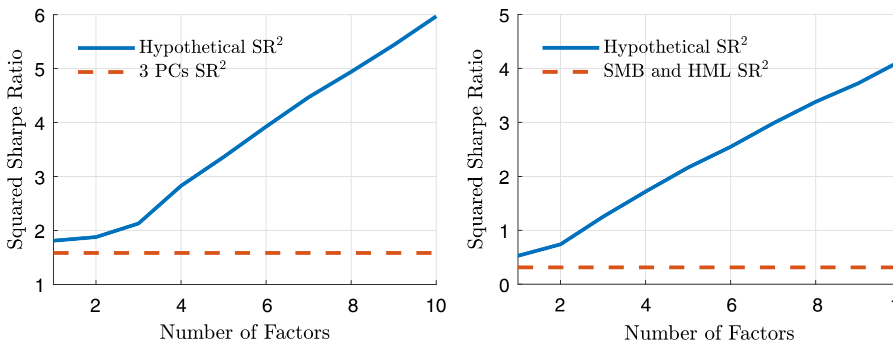

Panel A: 30 anomaly portfolios (in excess of level factor)

Panel B: 5×5 size-B/M portfolios (in excess of level factor)

Figure 2. Hypothetical Sharpe ratios. The figures show hypothetical Sharpe ratios if expected returns line up with first K (high eigenvalue excluding PC1) principal components. Panel A uses 30 anomaly portfolios as test assets. Panel B focuses on 25 size-B/M portfolios. (Color figure can be viewed at wileyonlinelibrary.com)

equally with the first two PCs (excluding the level factor) but not the higher order ones, the squared SR would be around two. The squared SR of the first three PCs is plotted as the dashed line in the figure for comparison. If expected returns lined up instead equally with the first 10 PCs, the squared SR would be almost six. Panel B shows a similar analysis for the 5×5 size-B/M portfolios. Here, the counterfactual squared SR also increases rapidly with K. If expected returns lined up equally with (only) the first two PCs (excluding the level factor), the squared SR would be higher than the sum of the squared SRs of SMB and HML (SMB and HML do not price the cross section of the 5×5 size-B/M portfolios perfectly). However, if expected returns were correlated equally with the first 10 PCs, the squared SR would exceed four.

We can look at the data from another perspective. Fixing $\mathrm{Var}(M)$ and $\overline{\mathrm{Var}}(\mu_i)$, we can bound the minimum cross-sectional R^2 of a factor model based on K PCs, $\sum_2^K \overline{\mathrm{Corr}}(\mu_i, q_{ki})^2$. Figure 3 presents the results. The plots show that reasonable bounds on the maximum squared SR imply that a low-dimensional factor model constructed from high-eigenvalue PCs will provide a good approximation to expected returns. Looking at Panel A (30 anomaly portfolios), a bound of $\mathrm{SR}^2 < 2$ implies that a three-factor model will deliver greater than 90% cross-sectional R^2. The strength of this argument, of course, depends heavily on the underlying space of test assets. For payoff spaces with weaker factor structure or lower variation in expected returns, the number of factors in the SDF (the value of K) could be higher for any given bound on $\mathrm{Var}(M)$. This is seen in Panel B, which presents the results for the Fama-French 25 size-B/M portfolios. Since this set of assets has significantly lower $\overline{\mathrm{Var}}(\mu_i)$ than the anomaly portfolios, but similar factor structure (eigenvalues of the covariance matrix), the bounds are looser.

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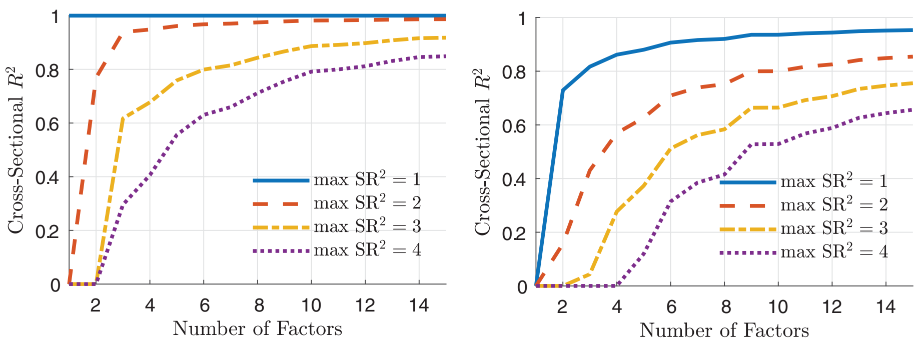

Panel A: 30 anomaly portfolios (in excess of level factor)

Panel B: 5 × 5 size-B/M portfolios (in excess of level factor)

Figure 3. Minimum cross-sectional R^2. The figures plot minimum cross-sectional R^2s from a K-factor model (high eigenvalue excluding PC1) for various bounds on Var(M). Panel A uses 30 anomaly portfolios as test assets. Panel B focuses on 25 size-B/M portfolios. (Color figure can be viewed at wileyonlinelibrary.com)

A. Principal Components as Reduced-Form Factors: Pricing Performance

Based on the no-near-arbitrage logic developed above, it should not require a judicious construction of factor portfolios to find a reduced-form SDF representation—brute statistical force should do. We already show earlier in Figure 1 that the first three PCs of the 5 × 5 size-B/M portfolios are similar to the three Fama-French factors. We now investigate the *pricing* performance of PC factor models. Our focus therefore is on how well covariances with factors explain expected returns rather than how well factors explain variances. This is an important distinction. It is perfectly possible for a factor to be important in explaining return variance but to play no role in pricing.

Table II shows that the first few PCs do a good job capturing cross-sectional variation in expected returns of the anomaly portfolios. We run time-series regressions of the 15 long-short anomaly excess returns on the PC factors extracted from 30 underlying portfolio returns. The upper panel in Table II reports the pricing errors, that is, the intercepts or alphas, from these regressions. The raw mean excess return (in % per year) is shown in the first column, and alphas for specifications with an increasing number of PC factors are shown in the second to sixth columns. With just the first PC (PC1, roughly the market) as a single factor, the SDF does not fit well. Alphas reach magnitudes up to 17% per year. Adding PC2 and PC3 to the factor model drastically shrinks the pricing errors. With five factors, the maximum (absolute) alpha is six.

The bottom panel reports the (ex-post) maximum squared SR of the anomaly portfolios (4.23) and the maximum squared SR of the PC factors. With five factors, the highest SR combination of the factors achieves a squared SR of 1.77. This is still considerably below the maximum squared SR of the anomaly

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Table II

Explaining Anomalies with Principal Component Factors

The sample period is July 1966 to December 2015. The anomaly long-short strategy daily returns are as defined in Novy-Marx and Velikov (2016). Average returns, factor-model alphas, and squared Sharpe ratios are reported in annualized terms. Mean returns and alphas are calculated for 15 long-short anomaly strategies. Maximum squared Sharpe ratios and principal component factors are extracted from returns on the 30 portfolios underlying the long and short sides of these strategies.

	Mean Return	PC Factor-Model Alphas				
		PC1	PC1-2	PC1-3	PC1-4	PC1-5
Size	1.29	2.47	4.89	2.07	3.29	4.08
Gross Profitability	4.56	3.96	2.33	3.37	4.33	2.48
Value	5.71	6.28	6.63	3.80	0.97	1.16
ValProf	8.45	9.45	8.02	6.34	5.33	4.43
Accruals	4.63	4.57	3.94	3.98	4.25	4.39
Net Issuance (rebal.-A)	9.06	10.30	6.55	6.87	5.31	4.87
Asset Growth	4.31	5.31	3.42	2.07	−0.64	−0.27
Investment	5.41	6.31	5.25	4.38	3.16	3.57
Piotroski's F-score	0.69	0.94	−0.32	0.17	0.21	0.12
ValMomProf	9.59	10.17	−0.39	−2.44	−0.59	−1.95
ValMom	5.71	6.51	−2.95	−5.86	−5.66	−5.83
Idiosyncratic Volatility	9.25	13.13	2.52	5.00	0.08	−0.74
Momentum	16.31	17.12	−2.41	−2.34	2.12	2.57
Long Run Reversals	4.22	4.76	4.69	1.92	−0.53	−1.48
Beta Arbitrage	7.12	8.87	1.85	1.57	−1.00	1.23

	max SR^2	PC Factors' Max. Squared SR				
		PC1	PC1-2	PC1-3	PC1-4	PC1-5
All anomalies	4.23	0.11	1.34	1.45	1.69	1.77
χ^2 p-value for zero pricing errors		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
For comparison:						
25 size-B/M	3.66	0.23	0.24	0.69	0.95	1.00
χ^2 p-value for zero pricing errors		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
MKT, SMB, and HML	0.64	–	–	–	–	–

portfolios, and the p-values from a χ^2-test of the zero-pricing-error null hypothesis rejects at a high level of confidence. However, it is important to realize that this pricing performance of the PC1-5 factor model is actually better than the performance of the Fama and French (1993) model—which is typically regarded as a successful factor model—in pricing the 5×5 size-B/M portfolios. As the table shows, the maximum squared SR of the 5×5 size-B/M portfolios is 3.66. But the squared SR of MKT, SMB, and HML is only 0.64. A combination of the first three PCs (PC1-3) of the size-B/M portfolios (including the level factor) has a squared SR of 0.69 and gets slightly closer to the mean-variance frontier than the Fama-French factors. Thus, while the PC factor models and the Fama-French factor model are statistically rejected at a high level of confidence, the fact that the Fama-French model is typically viewed as successful in explaining the size-B/M portfolio returns suggests that one should also view the PC1-3

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factor model as successful. In terms of the distance to the mean-variance frontier, the PC1-5 factor model for the anomalies in the upper panel is even better at explaining the cross section of anomaly returns than the Fama-French model in explaining the size-B/M portfolio returns.

Overall, this analysis shows that one can construct reduced-form factor models simply from the PCs of the return covariance matrix. There is nothing special, for example, about the construction of the Fama-French factors. Intended or not, these factors are similar to the first three PCs of the size-B/M portfolios and they perform similarly well in explaining the cross section of average returns of those portfolios.

While reduced-form factor model tests cannot help differentiate between competing models of investor beliefs, summarizing the cross section of expected returns in a parsimonious fashion with a factor-SDF can still be useful to quantify the extent of the challenge that theoretical asset pricing models face. If the dominant factors earn substantial SRs, then a rational expectations model would require large technological shocks (as in Bansal and Yaron (2004)) or high effective risk aversion (e.g., due to habits as in Campbell and Cochrane (1999)). A behavioral model such as the one in Section III would require substantial belief distortions and a significant wealth share of sentiment investors.

B. Characteristics versus Covariances: In Sample and Out of Sample

Daniel and Titman (1997) and Brennan, Chordia, and Subrahmanyam (1998) propose tests that look for expected return variation that is correlated with firm *characteristics* (e.g., B/M), but not with reduced-form factor model *covariances*. Framed in reference to our analysis above, this would mean looking for cross-sectional variation in expected returns that is orthogonal to the first few PCs—which implies that it must be variation that lines up with some of the higher order PCs. The underlying assumption behind these tests is that "irrational" pricing effects should manifest as mispricing that is orthogonal to covariances with the first few PCs.

From the evidence in Table II that the ex-post squared SR obtainable from the first few PCs falls short (by a substantial margin) of the ex-post squared SR of the test assets, one might be tempted to conclude that (i) there is convincing evidence for mispricing orthogonal to factor covariances, and (ii) therefore the approach of looking for mispricing unrelated to factor covariances is a useful way to test behavioral asset pricing models. After all, at least ex-post, average returns appear to line up with components of characteristics that are orthogonal to factor covariances.

We think that this conclusion is not warranted. First, there is certainly substantial sampling error in the ex-post squared SR. Of course, the χ^2-test in Table II takes the sampling error into account and still rejects the low-dimensional factor models.[8] However, there are additional reasons to suspect

[8] Bootstrap simulation, which accounts for the significant nonnormality of daily returns, leads to similar rejection of the factor models, albeit with higher p-values.

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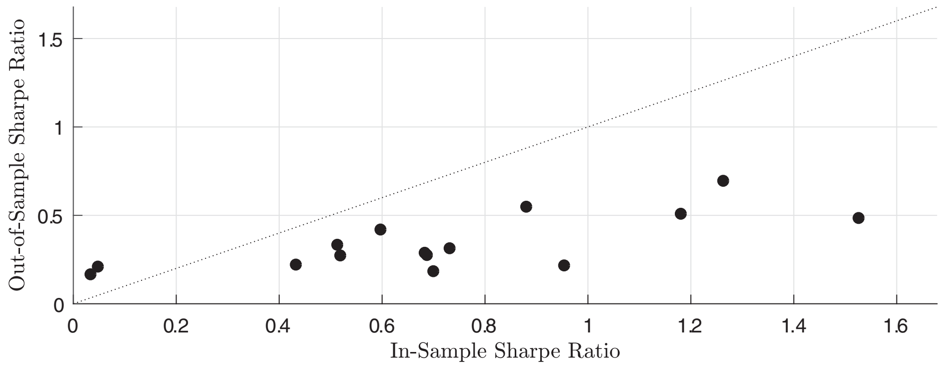

Figure 4. In-sample and out-of-sample Sharpe ratios. The figure shows in-sample and out-of-sample Sharpe ratios of 15 anomaly long-short strategies. The sample period is split into two halves. In-sample Sharpe ratios are those in the first subperiod. Out-of-sample Sharpe ratios are those in the second subperiod. Sharpe ratios are annualized.

that high ex-post SRs are not robust indicators of persistent near-arbitrage opportunities. Short-lived near-arbitrage opportunities might exist for a while before being recognized and eliminated by arbitrageurs. Data-snooping biases (Lo and MacKinlay (1990)) further overstate in-sample SRs.

To shed light on this robustness issue, we perform a pseudo out-of-sample analysis. We split our sample period into two halves, treating the first half as our in-sample period and the second half as our out-of-sample period. We start with a univariate perspective with the 15 anomaly long-short portfolios. Figure 4 plots in-sample SRs (in the first subperiod) on the x-axis and the out-of-sample SRs on the y-axis. The figure shows that there is generally a substantial deterioration. The out-of-sample SRs are, on average, less than half as large as the in-sample values and almost all of them are lower in the out-of-sample period. Furthermore, the strategies that hold up best are those that have relatively low in-sample SRs. This is a first indication that high in-sample SRs do not readily lead to high out-of-sample SRs.

Our pseudo out-of-sample evidence is consistent with recent work by McLean and Pontiff (2016) that examines the *true* out-of-sample performance of a large number of cross-sectional return predictors that appeared in the academic literature in recent decades. They find a substantial decay in returns from the researchers' in-sample period to the out-of-sample period after the publication of the academic study. Most relevant for our purposes is their finding that the predictors with higher in-sample t-statistics are the ones that experience the greatest decay.[9]

In Figure 5, Panel A, we consider all 30 portfolios underlying the 15 long-short strategies jointly. Focusing first on the full sample, we look at the

[9] In private correspondence, Jeff Pontiff provided us with estimation results showing that a stronger decay is also present for predictors with high in-sample SR. We thank Jeff for sending us those results.

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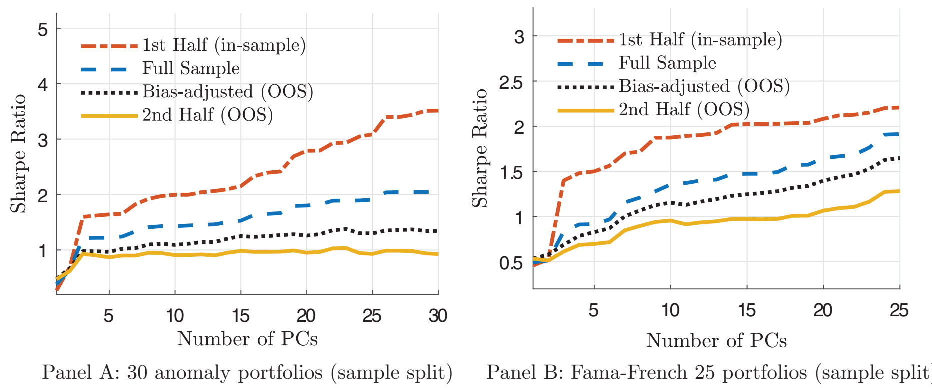

Figure 5. In-sample and out-of-sample maximum Sharpe ratios. The figures plot in-sample and out-of-sample maximum Sharpe ratios (annualized) of the first K principal components (including the level factor) of 30 anomaly long and short portfolio returns (Panel A) and 25 size-B/M portfolios (Panel B). We split the sample period into two halves and then extract principal components in the first subperiod and calculate the Sharpe ratio-maximizing combination of the first K principal components using sample means and covariances. We then apply the portfolio weights implied by this combination in the out-of-sample period (second subperiod). Full-sample Sharpe ratios (blue dashed line) are shown for comparison. The black dotted line represents the mean-bias-adjusted out-of-sample Sharpe ratio with bias calculated from a bootstrap simulation. (Color figure can be viewed at wileyonlinelibrary.com)

maximum SR that can be obtained from a combination of the first K PCs (including the level factor). The dashed blue line shows a significant increase in SR beyond the first few PCs, graphically demonstrating the rejection of low-dimensional factor models (formally shown in Table II). However, this pattern may be spurious. Data-snooping biases and transient near-arbitrage opportunities can inflate the full-sample SRs. Such biases should manifest as very high SRs in the early part of the data followed by significantly lower SRs later. We observe exactly this pattern when looking at the red (first half in-sample) and yellow (second half out-of-sample) lines. In sample (first half) with $K = 3$, the maximum SR is around 1.6, but increasing K further raises the SR to nearly four for $K = 30$. However, out of sample, the picture looks different. For each K, we now take the asset weights that yield the maximum SR from the first K PCs in the first subperiod, and we apply these weights to returns from the second subperiod.[10] The solid yellow line in the figure shows the resulting out-of-sample SR. Not surprisingly, the SRs are lower out of sample. Most importantly, it makes virtually no difference whether one picks $K = 3$ or $K = 30$—the out-of-sample SR is about the same and stays mostly around one. Hence, while the higher order PCs add substantially to the SR in sample, they provide no incremental improvement of the SR in the out-of-sample period. Thus, whatever these higher order PCs were picking up in

[10] Results are similar if we recalculate optimal weights each date using data available up to that point (both expanding and rolling windows).

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the in-sample period is not a robust feature of the cross section of expected returns.

One potential concern with these analyses is that mean-variance optimal portfolios constructed using in-sample estimates of means and covariances naturally perform poorly out of sample. If so, the diminished SR contribution of low-eigenvalue PCs in the out-of-sample analysis could be an artifact of an inefficient method of constructing portfolios. It is important to realize, however, that our PC-based method is already "robust" by design and thus avoids the usual pitfalls of naïve mean-variance optimization. A major problem with mean-variance optimization is that eigenvalues of the sample covariance matrix are too "extreme" in the sense that large-sample eigenvalues are biased upward and small ones biased downward relative to the true eigenvalues. Using these incorrect eigenvalues results in portfolios with extreme weights, due to the inversion of the tiny eigenvalues. Ignoring the small-eigenvalue PC portfolios is, thus, a form of regularization well suited to the out-of-sample portfolio problem.[11]

To the extent that any concerns remain, we address them in three ways. First, our covariance estimates throughout the paper are calculated from daily returns. This largely eliminates any downward bias in out-of-sample SRs due to sampling error in covariances, which could be substantial if we used monthly data instead. Second, we solve for optimal portfolio weights using the realized second-half covariance matrix and obtain similar out-of-sample performance (not shown). Finally, we use a bootstrap simulation to quantify the remaining bias resulting from uncertainty in estimated means.[12] The black dotted line shows that the mean-bias-adjusted out-of-sample SR is only moderately higher than the unadjusted value. Importantly, the conclusion that SRs do not increase significantly beyond the first few PCs is unchanged. In Panel B, we repeat the analysis for the 5×5 size-B/M portfolios and their PC factors. The results are similar.

In summary, the empirical evidence suggests that reduced-form factor models with a few PC factors provide a good approximation of the SDF, as one would expect if near-arbitrage opportunities do not exist. However, as we discuss in the rest of the paper, this fact tells us little about the "rationality" of investors and the degree to which "behavioral" effects influence asset prices.

[11] The closest counterparts to our procedure in the linear regression setting are principal component regression (PCR—a procedure that replaces explanatory variables in a regression with their first few PCs) and ridge regression (a continuous version of PCR—a procedure that continuously shrinks small PCs of explanatory variables relatively more than high PCs). See Friedman, Hastie, and Tibshirani (2001) for more details.

[12] We randomly sample (without replacement) half of the returns to extract PCs and calculate the SR-maximizing combination of the first K PCs in the subsample. We then apply the portfolio weights implied by this combination in the out-of-sample period (remainder of the data). The procedure is repeated 1,000 times. We calculate bias as the mean of the bootstrap distribution minus the full-sample value.

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III. Factor Pricing in Economies with Sentiment Investors

We now show that the mere absence of near-arbitrage opportunities has limited economic content. We model a multiasset market in which fully rational risk-averse investors (arbitrageurs) trade with investors whose asset demands are driven by distorted beliefs (sentiment investors).

Consider an IID economy with discrete time $t = 0, 1, 2, \ldots$. There are N stocks in the economy, indexed by $i = 1, \ldots, N$. The supply of each stock is normalized to $1/N$ shares. A risk-free bond is available in perfectly elastic supply at a gross interest rate of $R_F > 1$. Stock i earns time-t dividends of D_{it} per share. We collect the individual-stock dividends in the column vector D_t and assume that $D_t \sim \mathcal{N}(0, \Gamma)$.

We assume that the covariance matrix of asset cash flows Γ features a few dominant factors that drive most of the stocks' covariances. This assumption is consistent with empirical evidence in Ball, Sadka, and Sadka (2009), who show that there is strong factor structure in fundamentals. Since prices are constant in our IID case, the covariance matrix of returns equals the covariance matrix of dividends, Γ. Therefore, even with belief distortions, returns inherit a strong factor structure. Consider further the eigenvalue decomposition of the covariance matrix $\Gamma = Q\Lambda Q'$. Assume that the first PC is a level factor, with identical constant value for each element of the corresponding eigenvector $q_1 = \iota N^{-1/2}$. Then the variance of returns on the market portfolio is

$$\sigma_m^2 = \mathrm{Var}(R_{m,t+1}) = N^{-2}\iota' q_1 q_1' \iota \lambda_1 = N^{-1}\lambda_1. \tag{5}$$

By construction, all other PCs are long-short portfolios, that is, $\iota' q_k = 0$ for $k > 1$.

There are two groups of investors in this economy. The first group comprises competitive rational arbitrageurs in measure $1 - \theta$. The representative arbitrageur has exponential utility with absolute risk aversion a. In this IID economy, the optimal strategy for the arbitrageur is to maximize next-period wealth, that is,

$$\max_y \mathrm{E}[-\exp(-aW_{t+1})] \tag{6}$$

$$\text{s.t.} \quad W_{t+1} = (W_t - C_t)R_F + y' R_{t+1}, \tag{7}$$

where $R_{t+1} \equiv P_{t+1} + D_{t+1} - P_t R_F$ is a vector of dollar excess returns. From arbitrageurs' first-order condition and budget constraint, we obtain their asset demand

$$y_t = \frac{1}{a}\Gamma^{-1}\mathrm{E}[R_{t+1}], \tag{8}$$

where expectations are taken under the objective measure.

The second group comprises sentiment investors who have biased expectations—in making their decisions these investors use state probabilities that differ from the objective probabilities used by arbitrageurs. Sentiment

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investors are present in measure θ. Like arbitrageurs, they have exponential utility with absolute risk aversion a and they face a similar budget constraint, but they have an additional sentiment-driven component to their demand, δ. Their risky-asset demand vector is thus given by

$$x_t = \frac{1}{a}\Gamma^{-1}\mathrm{E}[R_{t+1}] + \delta, \tag{9}$$

where we assume that $\delta'\iota = 0$. The first term is the rational component of demand, which is equivalent to the arbitrageur's demand. The second term is sentiment investors' excess demand, δ, which is driven by their behavioral biases or misperceptions of the true distribution of returns. This misperception is only cross-sectional; there is no misperception of the market portfolio return distribution since $\delta'\iota = 0$.

If δ were completely unrestricted, prices could be arbitrarily strongly distorted even if arbitrageurs are present. Unbounded δ would imply that sentiment investors can take unbounded portfolio positions, including high levels of leverage and unbounded short sales. This is not plausible. Extensive short selling and high leverage is presumably more likely for arbitrageurs than for less sophisticated sentiment-driven investors. For this reason, we constrain sentiment investors' "extra" demand due to the belief distortion to

$$\delta'\delta \leq 1. \tag{10}$$

This constraint is a key difference between our model and models like that in Daniel, Hirshleifer, and Subrahmanyam (2001). In their model, no such constraint is imposed. As a consequence, when sentiment investors (wrongly) perceive a near-arbitrage opportunity, they are willing to take an extremely levered bet on this perceived opportunity. Arbitrageurs, in turn, are equally willing to take a bet in the opposite direction to exploit the actual near-arbitrage opportunity generated by the sentiment-investor demand. Since sentiment investors are as aggressive in pursuing their perceived opportunity as arbitrageurs are in pursuing theirs, mispricing can be big even for "idiosyncratic" risks. Imposing the constraint (10) prevents sentiment investors from taking such extreme positions, which is arguably realistic. By limiting the cross-sectional sum of squared deviations from rational weights in this way, the maximum deviation that we allow in an individual stock is approximately one that results in a portfolio weight of ± 1 in one stock and $1/N \pm 1/N$ in all others. [13] Thus, the constraint still allows sentiment investors to have rather substantial portfolio tilts, but it prevents the most extreme ones.

Market clearing,

$$\theta\delta + \frac{1}{a}\Gamma^{-1}\mathrm{E}[R_{t+1}] = \frac{1}{N}\iota, \tag{11}$$

[13] In equilibrium, a representative rational investor with objective expectations would hold the market portfolio with weights $1/N$. Deviating to a weight of one in a single stock and zero in the remaining $N-1$ stocks therefore implies a sum of squared deviations of $(1-1/N)^2 + (N-1)/N^2 = 1 - 1/N \approx 1$ and exactly zero mean deviation.

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implies

$$\mathrm{E}[R_{t+1}] - \mu_m \iota = -a\theta\Gamma\delta, \tag{12}$$

where $\mu_m \equiv (1/N)\iota'\mathrm{E}[R_{t+1}]$ and we use the fact that, due to the presence of the level factor, ι is an eigenvector of Γ and so $\Gamma^{-1}\iota = \frac{1}{\lambda_1}\iota = \frac{1}{N\sigma_m^2}\iota$. Moreover, we use $\mu_m = a\sigma_m^2$. After substituting into arbitrageurs' optimal demand, we get

$$y = \frac{1}{N}\iota - \theta\delta. \tag{13}$$

Consequently, we obtain the SDF

$$\begin{aligned} M_{t+1} &= 1 - a\,(R - \mathrm{E}[R])'\,y \\ &= 1 - a[R_{m,t+1} - \mu_m] + a(R_{t+1} - \mathrm{E}[R_{t+1}])'\theta\delta, \end{aligned} \tag{14}$$

and the SDF variance

$$\mathrm{Var}(M) = a^2\sigma_m^2 + a^2\theta^2\delta'\Gamma\delta. \tag{15}$$

The effect of δ on the factor structure and the volatility of the SDF depends on how δ lines up with the PCs. To characterize the correlation of δ with the PCs, we express δ as a linear combination of PCs,

$$\delta = \boldsymbol{Q}\beta, \tag{16}$$

with $\beta_1 = 0$. Note that $\delta'\delta = \beta'\boldsymbol{Q}'\boldsymbol{Q}\beta = \beta'\beta$, so the constraint (10) can be expressed in terms of β:

$$\beta'\beta \leq 1. \tag{17}$$

A. Dimensionality of the SDF

All deviations from the CAPM in the cross section of expected returns in our model are caused by sentiment. If the share of sentiment investors were zero, the CAPM would hold. However, as we now show, for sentiment investors' belief distortions to generate a cross section of expected stock returns with SRs comparable to what is found in empirical data, the SDF must have a low-dimensional factor representation.

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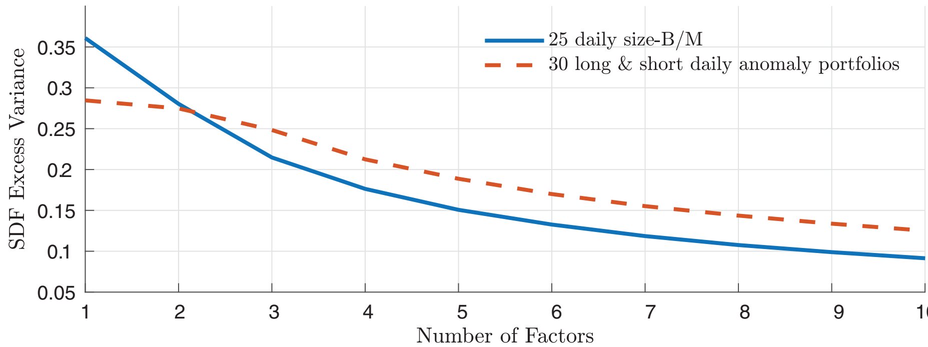

Figure 6. SDF excess variance. The plot shows SDF excess variance, $V(\beta)$, achieved when sentiment-investor demands $\delta = Q\beta$ line up equally with the first K PCs (excluding the level factor). The blue solid curve corresponds to 5×5 size-B/M portfolios; the red dashed curve is based on 30 anomaly long and short portfolios. (Color figure can be viewed at wileyonlinelibrary.com)

We combine (16) and (15) to obtain excess SDF variance, expressed, for comparison, as a fraction of the SDF variance accounted for by the market factor:

$$V(\beta) \equiv \frac{\text{Var}(M) - a^2\sigma_m^2}{a^2\sigma_m^2} = \frac{\theta^2}{\sigma_m^2}\delta'\Gamma\delta = \kappa^2 \sum_{k=2}^{N} \beta_k^2 \lambda_k, \tag{18}$$

where $\kappa \equiv \frac{\theta}{\sigma_m}$. From equation (18), we see that SDF excess variance is linear in the eigenvalues of the covariance matrix, with weights β_k^2. For the sentiment-driven demand component δ to have a large impact on SDF variance and hence the maximum SR, the β_k corresponding to high eigenvalues must have a large absolute value. This means that δ must line up primarily with the high-eigenvalue (volatile) PCs of asset returns. The constraint (17) implies that if β lined up with some of the low-eigenvalue PCs instead, the loadings on high-eigenvalue PCs would be substantially reduced and hence the variance of the SDF would be low. As a consequence, either the SDF can be approximated well by a low-dimensional factor model with the first few PCs as factors, or the SDF cannot be volatile and hence SRs only very small.

We now assess this claim quantitatively. Figure 6 illustrates this with data based on the covariance matrix of actual portfolios used as Γ and with $\theta = 0.5$. We consider two sets of portfolios: (i) 25 size-B/M portfolios and (ii) 30 anomaly portfolios underlying the long and short positions in the 15 anomalies in Table I. Returns are in excess of the level factor. We set β to have equal weight on the first K PCs, and to zero on the rest. Thus, a low K implies that the SDF

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has a low-dimensional factor representation in terms of the PCs, while a high K implies that it has a high-dimensional representation in which the high-eigenvalue PCs are not sufficient to represent the SDF. Equation (18) provides the excess variance of the SDF in each case.

Figure 6 plots the result with K on the horizontal axis. For both sets of portfolios, a substantial SDF excess variance can be achieved only if δ lines up with the first few (high-eigenvalue) PCs and hence the SDF is driven by a small number of PC factors. If K is high, so that δ also lines up with low-eigenvalue PCs, then the limited amount of variation in δ permitted by the constraint (10) is neutralized to a large extent by arbitrageurs. This is because arbitrageurs find it attractive to trade against sentiment demand if doing so does not require taking on risk exposure to high-eigenvalue PCs.

Similarly to what we argue in Section II, the value of K depends on the payoff space being considered. If the underlying test assets had a weaker factor structure than in the examples we analyzed, sentiment could line up with a larger number of PCs; the "sentiment-based" SDF would contain a larger number of factors and could still exhibit high excess variance.

Crucially, our analysis does not directly require there to be a strong factor structure in (biased) beliefs.[14] Instead, the model implies that arbitrageurs' activities will ensure that only belief components that are aligned with loadings on major common factors can have substantial price effects. Other belief components may exist, but they should not affect prices much in equilibrium.

In summary, if the SDF can be represented by a low-dimensional factor model with the first few PCs as factors, this does not necessarily imply that pricing is "rational." Even in an economy in which all deviations from the CAPM are caused by sentiment, one would still expect the SDF to have such a low-dimensional factor representation because only sentiment-driven demand that lines up with the main sources of return comovement should have substantial price impact when arbitrageurs are present in the market. Our analysis shows that one could avoid this conclusion only if sentiment investors could take huge leverage and short positions (which would violate constraint (10)) or if arbitrage capital was largely absent. Neither of these two alternatives seems plausible.

B. Characteristics versus Covariances

Our model sheds further light on the meaning of characteristics versus covariances tests as in Daniel and Titman (1997), Brennan, Chordia, and Subrahmanyam (1998), and Davis, Fama, and French (2000). As noted in Section II.B, the underlying assumption behind these tests is that "irrational" pricing effects should manifest as mispricing that is orthogonal to covariances with the

[14] Equation (16) is simply a tautology: since the matrix of eigenvectors of dividends (returns) Q is of full rank, it forms a basis in $\mathbb{R}^N$ and hence we can "project" δ onto Q as β. Our analysis merely shows that, unless the first few elements of such projection (β) are "large," sentiment-driven demands have negligible effect on equilibrium prices. If the first few elements do turn out to be large, the remaining loadings must be "small" due to the constraint in equation (17).

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first few PCs (which implies that mispricing must instead be correlated with low-eigenvalue PCs).

To apply our model to this question, we can think of the belief distortion δ as being associated with certain stock characteristics. For example, elements of δ could be high for growth stocks with low B/M due to overextrapolation of recent growth rates or for stocks with low prior 12-month returns due to underreaction to news. We examine whether it is possible for a substantial part of cross-sectional variation in expected returns to be orthogonal to covariances with the first few PCs.

Equilibrium expected returns in our model are given by (12) and hence cross-sectional variance in expected returns is

$$\begin{aligned}\frac{1}{N}(\mathrm{E}[R_{t+1}] - \mu_m \iota)'(\mathrm{E}[R_{t+1}] - \mu_m \iota) &= a^2\theta^2\delta'\Gamma'\Gamma\delta \\ &= a^2\theta^2\beta'\Lambda^2\beta. \qquad (19)\end{aligned}$$

The cross-sectional variance in expected returns explained by the first K PCs is

$$a^2\theta^2\sum_{k=2}^{K}\beta_k^2\lambda_k^2. \qquad (20)$$

We set $\theta = 0.5$ and take the covariance matrix from empirically observed portfolio returns using two sets of portfolios: the 25 size-B/M portfolios (with $K = 2$) and the 30 anomaly portfolios (with $K = 3$), both in excess of the level factor. For any choice of β, we can compute the proportion of cross-sectional variation in expected returns explained by the first K PCs, that is, the ratio of (20) to (19), and the ratio of (the upper bound of) cross-sectional variance in expected returns, (19), to squared expected excess market returns. Depending on the choice of the elements of the β vector, various combinations of cross-sectional expected return variance and the share explained by the first K PCs are possible. We search over these combinations by varying the elements of β subject to the constraint (17). In Figure 7, we plot the right envelope, that is, the maximal cross-sectional expected return variation for a given level of share explained by the first K PCs.[15]

As Figure 7 shows, it is not possible to generate much cross-sectional variation in expected returns without having the first two PCs of size-B/M portfolios (in excess of the level factor) and three PCs of the 30 anomaly portfolios explain almost all of the cross-sectional variation in expected returns of their respective portfolios. For comparison, the ratio of cross-sectional variation in expected returns and the squared market excess return is around 0.17 for the 5×5 size-B/M portfolios and slightly below 0.47 for the anomaly portfolios (depicted with dashed vertical lines in the figure). To achieve these levels of cross-sectional variation in expected returns, virtually all expected return variation has to be aligned with loadings on the first few PCs.

[15] Appendix B provides more details on the construction of Figure 7.

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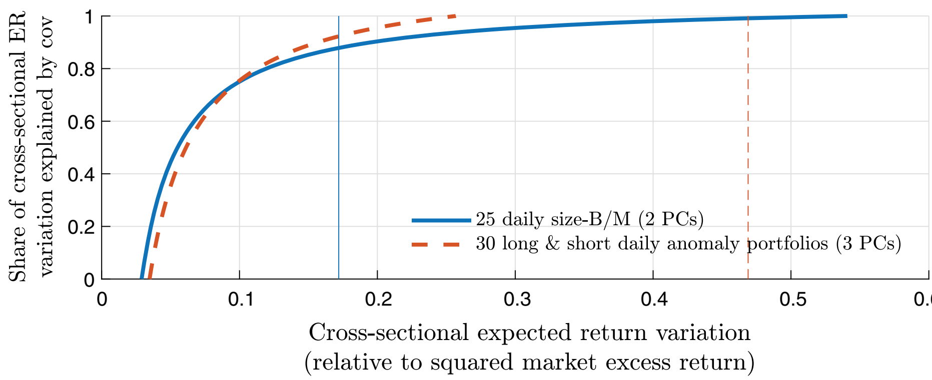

Figure 7. Characteristics versus covariances. Cross-sectional variation in expected returns explained by the first two principal components for 5 × 5 size-B/M portfolios and three principal components for anomaly long and short portfolios. Portfolio returns are represented in excess of the level factor. Vertical lines depict in-sample estimates of the ratio of cross-sectional variation in expected returns and the squared market excess return for the two sets of portfolios. (Color figure can be viewed at wileyonlinelibrary.com)

Thus, despite the fact that all deviations from the CAPM in this model are due to belief distortions, a horse race between characteristics and covariances as in Daniel and Titman (1997) cannot discriminate between rational and sentiment-driven theories of the cross section of expected returns. Covariances and expected returns are almost perfectly correlated in this model—if they were not, near-arbitrage opportunities would arise, which would not be consistent with the presence of some rational investors in the model.

C. Analyst Forecast Bias as a Proxy for Investor Misperceptions

The preceding model and analysis show that loadings on a few large PCs should "explain" the cross section of expected returns even in "behavioral" models, not just in "rational" models. To devise tests that are more informative about investor beliefs, researchers must exploit additional predictions of the model that relate returns to other data such as macroeconomic variables, information on portfolio holdings, or data on investor beliefs.

Here, we briefly explore whether data on beliefs are broadly consistent with the model in Section III. Following La Porta (1996) and Engelberg, McLean, and Pontiff (2018), we use analyst forecasts as proxy for sentiment-investor expectations. Under this assumption, we can shed light on the connection between belief distortions and factor loadings that is implied by our model.[16]

The prediction of the model in Section III is that belief distortions should "line up" with the large PC factors in returns as otherwise they would not have much of an effect on equilibrium asset prices. We now provide some suggestive evidence in favor of this mechanism.

[16] We thank an anonymous referee for suggesting this analysis of analyst forecasts.

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Following DellaVigna and Pollet (2007), we define the scaled earnings surprise (forecast error) of firm i in quarter q as

$$s_{i,q} = \frac{EPS_{i,q} - \widehat{EPS}_{i,q}}{P_{i,q}}, \tag{21}$$

where $\widehat{EPS}_{i,q}$ is the consensus (mean) forecast from I/B/E/S, $EPS_{i,q}$ is the realized earnings per share, and $P_{i,q}$ is the (split-adjusted) share price five trading days prior to the earnings announcement.[17] If analyst forecasts satisfied rational expectations, the mean surprise should be zero. Defining bias as the difference between the analyst forecast and the rational expectations forecast, the law of iterated expectations implies $bias = -\mathrm{E}[s_{i,q}]$. To explore bias in the cross section, we form calendar-time "portfolios" of earnings surprises for each anomaly.

As a concrete example, consider portfolios sorted on size (equity market capitalization). For each calendar quarter and size decile, we calculate the value-weighted average surprise. We base portfolio assignments on information known prior to the current quarter and use beginning-of-quarter market capitalization when computing value weights. Since $s_{i,q}$ is a per-share measure scaled by price, the value-weighted average has an interpretation of a buy-and-hold portfolio "surprise." Finally, we compute *bias* as the negative of the time-series average of the quarterly portfolio surprise. Positive bias indicates analyst "optimism," whereas negative bias reflects "pessimism."

If biased beliefs are responsible for CAPM return anomalies, we should find that portfolios with large α have, on average, pessimistic analyst forecasts and vice versa. Table III presents the results (we normalize the ordering to go from low to high CAPM α as one reads from left to right). The H-L column conceptually maps to the distortion, δ, in (9). The pattern in analyst forecast bias aligns with the model prediction: bias is systematically more negative for high-α portfolios. This analysis is related to the finding in Engelberg, McLean, and Pontiff (2018) that anomaly returns are an order of magnitude higher on earnings days and that anomaly signals predict analyst forecast errors in a way that suggests anomalous returns result from biased expectations.

Our model has an additional implication. If the bound on total bias in (10) is "tight," then behavioral distortions must line up with the large eigenvectors of returns to have any impact on equilibrium prices and expected returns. Letting $\tilde{\alpha}$ be the CAPM error of the base assets, equation (12) can be transformed as

$$\alpha \propto -\Lambda\beta, \tag{22}$$

where $\alpha \equiv Q'\tilde{\alpha}$ is the CAPM error of PC portfolios. Since Λ is diagonal, equation (22) can be read equation-by-equation as $\alpha_i \propto -\lambda_i\beta_i$. This implies that α "lines up" in the cross section with $-\Lambda\beta$. A tight bound implies that the first few elements of β are "large" (in magnitude) and the remaining β_j are "small."

[17] We use the I/B/E/S unadjusted Surprise History file, which includes announcements from 1993 to 2014. Results are similar if we use the unadjusted Summary Statistics file.

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Table III

Anomalies: Analyst Forecast Bias

We define earnings surprises relative to analyst forecasts as in DellaVigna and Pollet (2007). We value-weight the stock-level surprises by beginning-of-quarter equity market capitalization to construct calendar quarter portfolio-level surprises. The table reports the negative of the time-series average of the quarterly portfolio-level surprise series. Columns P1 and P10 correspond to short and long ends of long-short strategies (column H-L), respectively. We include intermediate portfolios in columns P4 and P7 (portfolios 4 and 7, respectively). For Piotroski's F-score, stocks are sorted into two portfolios due to discreteness of the underlying characteristic variable. t-statistics are Newey and West (1987) with four lags (one year). The sample period is 1993Q1 to 2014Q4.

	Low α (P1)	P4	P7	High α (P10)	H-L (δ)	t(δ)
Size	−5.0	−6.7	−5.9	−4.7	0.3	0.3
Gross Profitability	−6.4	−7.9	−5.2	−3.8	2.6	3.0
Value	−3.8	−5.9	−9.5	−9.0	−5.2	4.4
ValProf	−5.1	−4.8	−6.5	−7.7	−2.6	3.0
Accruals	−4.1	−5.5	−5.2	−6.6	−2.5	3.1
Net Issuance (rebal.-A)	−5.6	−6.2	−6.1	−6.0	−0.4	0.4
Asset Growth	−4.0	−4.7	−6.2	−8.1	−4.2	4.5
Investment	−4.3	−4.8	−5.4	−9.9	−5.6	4.6
ValMomProf	−3.8	−5.3	−5.0	−9.6	−5.8	6.2
ValMom	−2.8	−5.2	−6.8	−12.1	−9.3	8.4
Idiosyncratic Volatility	−6.1	−6.2	−6.2	−3.7	2.4	1.5
Momentum	−2.9	−4.8	−5.8	−9.0	−6.1	3.2
Long Run Reversals	−4.6	−4.8	−7.0	−14.2	−9.6	4.3
Beta Arbitrage	−5.9	−6.6	−6.3	−3.0	2.9	3.7
Piotroski's F-score	−5.8	–	–	−5.0	0.8	2.0
Mean	−4.6	−5.7	−6.2	−7.7	−3.1	7.0
Median	−4.5	−5.4	−6.1	−7.9	−3.4	–

Table IV presents estimates of β (behavioral bias, δ, rotated into PC space). For ease of interpretation, all PC portfolios are normalized to have positive expected return (α). The first column shows that most of the large-magnitude (significant) β_i are negative, consistent with positive αs. The third column presents $\frac{\beta_i^2}{\beta'\beta}$, the fraction of "total bias" accounted for by each PC. The first PC accounts for 53% of the total bias, as predicted if the bound on $\beta'\beta$ is tight. Importantly, the model predicts that equilibrium CAPM α arises not necessarily for PCs with large β_i, but rather for those with large $\lambda_i\beta_i$ (where λ_i is the variance of returns). The fourth and fifth columns give $-\lambda_i\beta_i$ and $\mathrm{E}[R_i]$, which have a cross-sectional correlation of 0.8, consistent with the proportionality prediction of equation (22).

Finally, the last column shows the proportion of cross-sectional variance in expected returns, $\beta'\Lambda^2\beta$ (see Section III.B), accounted for by each PC. According to the model, this is the partial-R^2 in a regression of individual-asset CAPM α on PC factor loadings. The estimates imply that loadings on the first PC should "explain" 90% of the anomaly returns, suggesting essentially a single-factor model of the cross section (plus a level factor). The strong lining up of belief distortions, δ, with the large eigenvectors of return covariances, Q, suggests

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Table IV

Forecast Bias in Principal Component Portfolios

The table uses principal component portfolios based on 15 long-short anomaly strategies. β is $Q'\delta$, where Q is the eigenvector matrix computed from return covariances, $\Gamma = Q'\Lambda Q$, and δ is the analyst forecast bias from Table III. Mean returns are computed using the full sample of daily data. t-statistics are Newey and West (1987) with four lags (one year). The sample period is 1993Q1 to 2014Q4.

PC	β	t (β)	% of $\beta'\beta$	$-\lambda\beta$	E (R)	% of $\beta'\Lambda^2\beta$
1	−12.6	8.5	52.9	11.4	10.8	90.1
2	2.7	1.3	2.4	−1.6	5.1	1.7
3	−5.6	4.5	10.3	2.3	0.2	3.5
4	1.3	1.3	0.6	−0.4	2.3	0.1
5	−7.9	6.7	20.8	2.3	2.7	3.7
6	−2.5	3.5	2.1	0.6	1.7	0.2
7	2.0	1.6	1.3	−0.4	0.8	0.1
8	4.6	3.5	7.0	−0.8	1.3	0.4
9	−1.6	2.2	0.8	0.2	0.5	0.0
10	−1.0	0.9	0.3	0.1	1.0	0.0
11	0.8	0.7	0.2	−0.1	1.0	0.0
12	−0.1	0.1	0.0	0.0	0.4	0.0
13	1.3	1.8	0.5	−0.1	0.7	0.0
14	0.2	0.3	0.0	−0.0	2.8	0.0
15	1.4	2.4	0.7	−0.1	0.2	0.0

that the model's mechanism is more than just a theoretical possibility and is plausibly important in generating the observed patterns in expected returns.

D. Investment-Based Expected Stock Returns

So far our focus has been on the interpretation of empirical reduced-form factor models. A related literature uses reduced-form specifications of the SDF in models of firm decisions with the goal of deriving predictions about the cross section of stock returns. Our critique that reduced-form factor models have little to say about the beliefs and preferences of investors applies to these models as well.

The models in this literature feature firms that make optimal investment decisions. These models generate the prediction that stock characteristics such as the B/M ratio, firm size, investment, and profitability should be correlated with expected returns. We discuss two classes of such models. In the first class, firms continuously adjust investment, subject to adjustment costs. One recent example is Lin and Zhang (2013). In the second class, firms are presented with randomly arriving investment opportunities that differ in systematic risk. Firms can either take or reject an arriving project. A prominent example of a model of this kind is Berk, Green, and Naik (1999) (BGN).

Our focus is on whether these models have anything to say about why investors price some stocks to have higher expected returns than others. These

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models are often presented as rational theories of the cross section of expected returns that are contrasted with behavioral theories in which some investors are not fully rational. [18] However, a common feature of these models is that firms optimize taking as given a generic SDF that is not restricted further. Existence of such a generic SDF requires nothing more than the absence of arbitrage opportunities. These models therefore make essentially no assumption about investor preferences and beliefs. As a consequence, they cannot deliver conclusions about investor preferences or beliefs. As our analysis above shows, it is perfectly possible to have an economy in which all cross-sectional variation in expected returns is caused by sentiment, and yet an SDF not only exists but also has a low-dimensional structure in which the first few PCs drive SDF variation, similar to many popular reduced-form factor models. For this reason, models that focus on firm optimization, taking a generic SDF as given, cannot answer questions about investor rationality.

To illustrate, consider a model of firm investment similar to the one in Lin and Zhang (2013). Firms operate in an IID economy, and they take the SDF as given when making real investment decisions. At each point in time, a firm has a one-period investment opportunity. For an investment I_t, the firm will make profit Π_{t+1} per unit invested. The firm faces quadratic adjustment costs and the investment depreciates fully after one period. The full-depreciation assumption is not necessary for what we want to show, but it simplifies the exposition. To reduce clutter, we also drop the i subscripts for each firm.

Every period, the firm has the objective

$$\max_{I_t} -I_t - \frac{c}{2}I_t^2 + \mathrm{E}[M_{t+1}\Pi_{t+1}I_t]. \tag{23}$$

The SDF that appears in this objective function is not restricted further. Hence, the SDF could be, for example, the SDF (14) from our earlier example economy in which all cross-sectional variation in expected returns is due to sentiment. Taking this SDF as given, we get the firm's first-order condition

$$I_t = \frac{1}{c}\left(\mathrm{E}[M_{t+1}\Pi_{t+1}] - 1\right) \tag{24}$$

$$= \frac{1}{c}\left(\mathrm{E}[M_{t+1}]\mathrm{E}[\Pi_{t+1}] + \mathrm{Cov}(M_{t+1}, \Pi_{t+1}) - 1\right). \tag{25}$$

[18] To provide a few examples, BGN (p. 1553) motivate their analysis by pointing to these competing explanations and note that "these competing explanations are difficult to evaluate without models that explicitly tie the characteristics of interest to risks and risk premia"; Daniel, Hirshleifer, and Subrahmanyam (2001) cite BGN as a "rational model of value/growth effects"; Grinblatt and Moskowitz (2004) include BGN among "rational risk-based explanations" of past returns-related cross-sectional predictability patters; and Johnson (2002) builds a related model based on a reduced-form SDF in a paper entitled "Rational Momentum Effects."

Since the economy features IID shocks, I_t is constant over time, that is, we can write $I_t = I$. The firm's cash flow net of (recurring) investment each period is

$$D_{t+1} = I\Pi_{t+1} - \frac{c}{2}I^2 - I. \tag{26}$$

If we let Π_{t+1} be normally distributed, this fits into our earlier framework as the cash-flow generating process (with a slight modification to allow for positive average cash flow and heterogeneous expected profitability across firms)

$$I = \frac{1}{c}\left(\mathrm{E}[M_{t+1}]\mathrm{E}[\Pi_{t+1}] + \frac{1}{I}\mathrm{Cov}(M_{t+1}, D_{t+1}) - 1\right), \tag{27}$$

where M_{t+1} is the SDF (14) that reflects sentiment-investor demand.

Thus, a firm with high $\mathrm{E}[\Pi_{t+1}]$ (relative to other firms) must have either high investment or a strongly negative $\mathrm{Cov}(M_{t+1}, D_{t+1})$ (which implies a high expected return). Similarly, a firm with high I must have either high profitability or a not very strongly negative $\mathrm{Cov}(M_{t+1}, D_{t+1})$ (which implies a low expected return). Taken together, I and $\mathrm{E}[\Pi_{t+1}]$ should explain cross-sectional variation in $\mathrm{Cov}(M_{t+1}, D_{t+1})$ and hence in expected returns.

These relationships arise because firms align their investment decisions with the SDF and the expected return—which is their cost of capital—that they face in the market. From the viewpoint of a firm in this type of model, whether or not cross-sectional variation in expected returns is caused by sentiment is irrelevant. The implications for firm investment and for the relation between expected returns, investment, and profitability are observationally equivalent. Thus, the empirical evidence in Fama and French (2006), Hou, Xue, and Zhang (2014), and Novy-Marx (2013) that investment and profitability are related, cross-sectionally, to expected stock returns is to be expected in a model in which firms optimize. Moreover, as long as the firm optimizes, the Euler equation $\mathrm{E}[M_{t+1}R_{t+1}] = 1$ also holds for the firm's investment return, as in Liu, Whited, and Zhang (2009), again irrespective of whether *investors* are rational or have distorted beliefs.

Testing whether empirical relationships between expected returns, investment, and profitability exist in the data amounts to testing models of firm decision-making, not models of how investors price assets. Thus, evidence on these empirical relationships does not help resolve the question of how to specify investor beliefs and preferences. Only models that make assumptions about these beliefs and preferences—which result in restrictions on the SDF—can deliver testable predictions that potentially help discriminate between competing models of how investors price assets.

For example, if one couples a model of firm investment with a standard rational-expectations consumption Euler equation on the investor side (e.g., as in Gomes, Kogan, and Zhang (2003)), then the model delivers testable predictions about the identity of the risk factor in the SDF: covariances with consumption growth should explain the cross section of expected returns. In this example, modeling of firm investment can provide insights into the relationship between firm characteristics and choices and the systematic consumption

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risk of the firm, but the firm-investment side of the model does not provide any predictions about the nature of the risks investors care about and what the prices of those risks are.

Turning to the second class of models, we focus on the version of BGN with constant interest rates, which is sufficient to produce the key predictions of their model. BGN assume the existence of a generic SDF M that is not restricted further apart from an auxiliary assumption that M is log-normal. Hence, this SDF could represent, for example, one that arises in an economy in which sentiment causes all cross-sectional variation in expected returns, as in our earlier example economy. All of their conclusions about the relationships between expected returns, firms' B/M ratios, and firm size would arise in this model irrespective of the specification of investor beliefs and preferences (rational, behavioral, or otherwise).

Firms in their model are presented with randomly arriving and dying investment projects that all have the same expected profitability and scale, but differ randomly in the covariance of their cash-flow shocks ε_i with the SDF. Projects with very negative $\gamma_i = \mathrm{Cov}(\varepsilon_i, M)$ have a high expected return, that is, a high cost of capital, and are rejected, while projects with less negative γ_i are taken on by the firm. Again, it is important to keep in mind that γ_i is a covariance with a generic SDF. Other than the existence of such an SDF, nothing has been assumed that would imply γ_i has to represent "rationally priced" risk. Each firm also has an (identical) stock of growth options from the future arrival of new investment projects. Since expected profitability is assumed to be constant in this model, and since we are working with the constant-interest rate version, the value of these growth options is simply the value of a risk-free bond. At a given point in time, the firm's return covariance with M is then determined by the number of projects, n_t, the firm has taken on in the past that are still alive (relative to a constant stock of riskless growth options) and by the aggregated γ_i of the still-alive projects, which we denote by γ_t. Since expected excess returns are equal to the negative of the covariance with M, it follows that

$$\mathrm{E}[R_{t+1}] = f(n_t, \gamma_t) \tag{28}$$

for some function $f(.)$. As BGN show (see their equation (45)), this leads to a linear relationship between expected returns, the B/M ratio, and market value,

$$\mathrm{E}[R_{t+1}] = a_0 + a_1(B_t/MV_t) + a_2(1/MV_t), \tag{29}$$

where B_t/MV_t depends positively on n_t (as having more ongoing projects reduces the weight on riskless growth options) and positively on γ_i (as a higher expected return lowers market value), while $1/MV_t$ depends negatively on n_t (as taking on more projects raises market value) and positively on γ_i.

Nowhere in this derivation is there any assumption that would restrict investor preferences and beliefs further than asserting the existence of an SDF. Thus, if BGN's model of firm decision-making is correct, the conclusions that expected returns are linear in B/MV and $1/MV$, as in (29), would apply in any

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world in which an SDF exists, even if all cross-sectional variation in expected returns is caused by sentiment (as in our model in Section III). Thus, in terms of investor beliefs and preferences, the BGN model is as much a "behavioral" model as it is a "rational" model.

IV. Factor Pricing in Economies with Sentiment Investors: Dynamic Case

In this section, we show that the observational equivalence between "behavioral" and "rational" asset pricing with regard to factor pricing also applies, albeit to a lesser degree, to partial equilibrium ICAPM models in the tradition of Merton (1973). To demonstrate this, we specify and solve a dynamic model with time-varying investor sentiment.

We model the economy in a discrete-time and infinite-horizon framework. The setup is an extension of the IID model in Section III to the dynamic case in which sentiment demand is time-varying. Like in the previous setup, there are N stocks, $i = 1, \ldots, N$, each in supply of $1/N$ shares, with per-period dividends $D_t \sim \mathcal{N}(0, \Gamma)$. The risk-free one-period bond is in perfectly elastic supply at a constant interest rate of r_F. Define the gross interest rate as $R_F = 1 + r_F$, and assume that there exists a measure $(1-\theta)$ of arbitrageurs. We model the asset demands of sentiment investors as exogenous and equal to the equilibrium demand in the static model (see (13) and the market-clearing condition), but additionally subject to an IID stochastic shock. For sentiment investors, we therefore have

$$x_t = \frac{1}{N}\iota + (1-\theta)\delta\xi_t. \tag{30}$$

Market clearing implies the following demand for arbitrageurs:

$$y_t = \frac{1}{N}\iota - \theta\delta\xi_t, \tag{31}$$

where $\xi_{t+1} \sim \mathcal{N}(0, \omega^2)$ is a time-varying component (scalar) of arbitrageurs' demand and δ gives the direction of sentiment distortion. We assume that δ has a level component, $\delta'\iota \neq 0$, so that market discount rates are not constant. The setup effectively assumes a single time-series factor in sentiment investors' demand.[19]

We solve for prices consistent with these equilibrium demands. Arbitrageurs maximize their life-time exponential utility

$$J_t(W_t, \xi_t) = \max_{(C_s, y_s), s \geq t} \mathrm{E}_t\left[-\sum_{s=t}^{\infty} \beta^s \exp(-\alpha C_s)\right], \tag{32}$$

[19] Campbell and Kyle (1993) present a related single-asset model in which ξ follows an Ornstein-Uhlenbeck process.

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where the maximization is subject to

$$W_{t+1} = (W_t - C_t)R_F + y_t' R_{t+1}, \tag{33}$$

with $R_{t+1} \equiv P_{t+1} - R_F P_t + D_{t+1}$.

We define the market portfolio as $R_{M,t+1} \equiv \frac{1}{N}\iota' R_{t+1}$. We guess that prices are linear in ξ_t and the log value function is quadratic in ξ_t,

$$P_t = a_0 + a_1 \xi_t, \tag{34}$$

$$J_t(W_t, \xi_t) = -\beta^t \exp(-\gamma W_t - b_0 - b_1 \xi_t - b_2 \xi_t^2). \tag{35}$$

In Appendix C, we show how to solve for the constants a_i and b_i and establish that equilibrium expected returns are given by

$$\mathrm{E}(R_t) = \gamma \mathrm{Cov}\left(R_t, R_{M,t}\right) + \tilde{\delta}_{DR} \mathrm{Cov}\left(R_t, \mathrm{E}_t\left[R_{M,t+1}\right]\right), \tag{36}$$

where $\gamma = \alpha \frac{r_F}{R_F} > 0$ and $\tilde{\delta}_{DR} > 0$ are two positive prices of risk (see Appendix C, case 2).

Thus, we get an ICAPM similar to Campbell (1993, equation (25)) or a "bad beta, good beta" specification as in Campbell and Vuolteenaho (2004b). The presence of sentiment traders is indirectly reflected in $\mathrm{Cov}(R_t, \mathrm{E}_t[R_{M,t+1}])$. This covariance shrinks to zero as θ (the mass of sentiment traders) goes to zero.

In summary, the analysis shows that time-varying investor sentiment can give rise to an ICAPM-like SDF. As in our static model in the previous section, this model is "behavioral" and "risk-based" at the same time. Deviations from the static CAPM are caused by sentiment, but from the viewpoint of arbitrageurs, time-varying sentiment generates hedging demands because it makes their investment opportunities time-varying. When evaluating how aggressively to accommodate sentiment-investor demand in a particular stock, arbitrageurs consider the covariance of the stock's return with the sentiment-driven investment opportunity state variable. As a result, expected returns reflect this state-variable risk.

V. Conclusions

Reduced-form factor models are useful to provide a parsimonious summary of the cross section of asset returns. Yet, their success or failure in explaining the cross section of asset returns does not help answer the question of whether asset pricing is "rational." As we show, even if all cross-sectional variation in expected returns is driven by belief distortions on the part of some investors, a low-dimensional SDF with the first few PCs of returns as factors should still explain asset prices. This only requires that near-arbitrage opportunities be absent. For the same reason, tests that look for stock characteristics that capture expected return variation in the cross section that is orthogonal to common factor covariances are unlikely to be of much help either. As a result,

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tests of reduced-form factor models cannot shed light on questions regarding the "rationality" of investors.

Indeed, the framing of the question concerning investor rationality is unhelpfully imprecise in the first place. The arbitrageurs in our model are perfectly rational. From their viewpoint, expected returns are consistent with the risk premia that they require as compensation for tilting their portfolio weights away from the market portfolio. But it is the sentiment-investor demand that arbitrageurs accommodate that causes these risk premia. Thus, there is no dichotomy between "risk-based" and "behavioral" asset pricing in this model.

The only way to better understand investor beliefs is to develop and test structural asset pricing models with specific assumptions about these beliefs and preferences that deliver predictions about the factors that should be in the SDF and the probability distribution under which this SDF prices assets. While we discuss these issues in the context of equity markets research, similar conclusions apply to reduced-form no-arbitrage models in bond and currency market research.

Recognizing that factor covariances should explain cross-sectional variation in expected returns even in a model of sentiment-driven asset prices should also be useful for the development of models that meet the Cochrane (2011) challenge presented in the introduction of our paper. The answer to his question could be that some components of sentiment-driven asset demands are aligned with covariances with important common factors, while some are orthogonal to these factor covariances. Trading by arbitrageurs largely eliminates the effects of the orthogonal components of asset demand, but those that are correlated with common factor exposures survive because arbitrageurs are not willing to accommodate these demands without compensation for the factor risk exposure.

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Appendix A: The Absence of Near-Arbitrage

In this appendix, we present the derivation of the SDF variance. Define

$$\omega_m = \frac{1}{\sqrt{N}} q_1, \tag{A1}$$

$$\mu_m = \omega_m', \mu \tag{A2}$$

$$\sigma_m^2 = \omega_m' \Gamma \omega_m. \tag{A3}$$

The last definition implies that $\sigma_m^2 = \frac{\lambda_1}{N}$. Then,

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$$\mathrm{Var}(M) = \frac{\mu_m^2}{\sigma_m^2} + (\mu - \mu_m)' Q_z \Lambda_z^{-1} Q_z' (\mu - \mu_m). \tag{A4}$$

Let

$$\omega_k = \frac{1}{\sqrt{N}} q_k, \tag{A5}$$

$$\mu_k = \omega_k' \mu, \tag{A6}$$

$$\sigma_k^2 = \omega_k' \Gamma \omega_k. \tag{A7}$$

The last definition implies that $\sigma_k^2 = \frac{\lambda_k}{N}$, which is decreasing from second-order to higher order PCs proportional to the eigenvalues. We refer to R_k as the return on the zero-investment portfolio associated with the kth PC.

We now have that

$$\mathrm{Var}(M) = \mu_m^2/\sigma_m^2 + \sum_{k=2}^{N} N^2 \frac{\overline{\mathrm{Cov}}(\mu_i, q_{ki})^2}{\lambda_k} \tag{A8}$$

$$= \mu_m^2/\sigma_m^2 + \overline{\mathrm{Var}}(\mu_i) \sum_{k=2}^{N} \frac{\overline{Corr}(\mu_i, q_{ki})^2}{\sigma_k^2}. \tag{A9}$$

Covariance is a cross-sectional covariance, and for the second line, we use the fact that q_{ki} is mean zero and has variance N^{-1}. The sum of the squared correlations is equal to one. But the sum weighted by the inverse σ_k^2 depends on which of the PCs μ lines up with. If it lines up with high-σ_k^2 PCs, then the sum is much lower than if it lines up with low-σ_k^2 PCs. Thus, if expected returns line up with low-eigenvalue PCs, we get a much higher SR.

Appendix B: Characteristics versus Covariances

In this appendix, we provide additional details on the construction of Figure 7, which plots the right envelope of the set generated by all βs that satisfy restriction (17). To construct this right envelope, we put all of the weight of β onto two eigenvectors: the eigenvector associated with the highest eigenvalue (first eigenvector) and the $(K+1)$th eigenvector, that is, the eigenvector associated with the highest PC from the remaining $N-K$ PCs not used in equation (20). We then vary weights on these two components in a way that satisfies (17). For each set of weights, we compute the ratio of (20) to (19), and the ratio of (the upper bound of) the cross-sectional variance in expected returns, (19), to squared expected excess market returns.

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Appendix C: Dynamic Model

In this appendix, we solve a more general case of the model by assuming that sentiment-investor demand follows an AR(1): $\xi_{t+1} \sim \mathcal{N}(\hat{\xi}_t, \omega^2)$, with the mean of ξ_{t+1} given by

$$\hat{\xi}_t \equiv \mu + \phi \xi_t. \tag{C1}$$

The model can be easily specialized to the case considered in Section IV by setting $\mu = \phi = 0$.

The Bellman equation for the arbitrageurs' problem is given by

$$J_t(W_t, \xi_t) = \max_{C_t, y_t} \left\{ -\beta^t \exp(-\alpha C_t) + \mathrm{E}_t \left[J_{t+1}(W_{t+1}, \xi_{t+1}) \right] \right\}. \tag{C2}$$

We guess that equilibrium prices and arbitrageurs' value function are given by:

$$P_t = a_0 + a_1 \xi_t, \tag{C3}$$

$$J_t(W_t, \xi_t) = -\beta^t \exp(-\gamma W_t - b_0 - b_1 \xi_t - b_2 \xi_t^2), \tag{C4}$$

where a_0 and a_1 are vectors of constants and γ, b_0, b_1, and b_2 are scalars. Note that, based on this guess, realized returns are

$$R_{t+1} = D_{t+1} + a_1(\xi_{t+1} - \xi_t) - r_F a_0 - r_F a_1 \xi_t. \tag{C5}$$

Substituting (C5) into (C4), we have that

$$\begin{aligned} \mathrm{E}_t \left[J_{t+1} \right] = & - \beta^{t+1} \exp \left(-\gamma \left[(W_t - C_t) R_F - y_t' (r_F a_0 + R_F a_1 \xi_t) \right] - b_0 \right) \\ & \times \mathrm{E}_t \left[\exp \left(-\gamma y_t' D_{t+1} \right) \right] \\ & \times \mathrm{E}_t \left[\exp \left(- \left(b_1 + \gamma y_t' a_1 \right) \xi_{t+1} - b_2 \xi_{t+1}^2 \right) \right], \end{aligned} \tag{C6}$$

where

$$\mathrm{E}_t \left[\exp \left(-\gamma y_t' D_{t+1} \right) \right] = \exp \left(\frac{1}{2} \gamma^2 y_t' \Gamma y_t \right) \tag{C7}$$

since D_{t+1} is normally distributed. We can rewrite the last expectation in (C6) as

$$\mathrm{E}_t \left[\exp \left(- \left(b_1 + \gamma y_t' a_1 \right) \xi_{t+1} - b_2 \xi_{t+1}^2 \right) \right] = \exp \left(b_2 \lambda_t^2 \right) \times \mathrm{E}_t \left[\exp \left(-b_2 \omega^2 \zeta_{t+1}^2 \right) \right], \tag{C8}$$

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where $\zeta_{t+1} = \frac{\xi_{t+1}+\lambda_t}{\omega} \sim \mathbb{N}(\hat{\zeta}_t, 1)$, $\hat{\zeta}_t = \frac{\hat{\xi}_t+\lambda_t}{\omega}$, and $\lambda_t = \frac{b_1+\gamma y_t' a_1}{2b_2}$. Using the moment-generating function of a noncentral χ^2 distribution, we compute the expectation as[20]

$$\mathrm{E}_t\left[\exp\left(-b_2\omega^2\zeta_{t+1}^2\right)\right] = \frac{1}{\sqrt{1+2b_2\omega^2}}\exp\left(\frac{-b_2\omega^2\hat{\zeta}_t^2}{1+2b_2\omega^2}\right). \tag{C9}$$

The first-order condition for y_t yields

$$0 = \gamma(r_F a_0 + R_F a_1 \xi_t) + \gamma^2\Gamma y_t - \frac{1}{1+2b_2\omega^2}\left(\frac{b_1+\gamma y_t' a_1}{2b_2} + \hat{\xi}_t\right)\gamma a_1 + \frac{b_1+\gamma y_t' a_1}{2b_2}\gamma a_1. \tag{C10}$$

We can compute a_0 and a_1 by plugging in the market-clearing condition for y_t in equation (31) and applying the method of undetermined coefficients (since market clearing has to hold for any value of ξ_t). We omit these calculations for brevity.

Going back to (C6), we can write

$$\mathrm{E}_t\left[J_{t+1}(W_{t+1}, \xi_{t+1})\right] = -\beta^{t+1}\exp\left(-\gamma(W_t - C_t)R_F + \hat{\lambda}_0 + \hat{\lambda}_1\xi_t + \hat{\lambda}_2\xi_t^2\right), \tag{C11}$$

where $\hat{\lambda}_0$, $\hat{\lambda}_1$, and $\hat{\lambda}_2$ can be easily solved for by plugging in equations (C7), (C8), (C9), and (31) into (C6). We now evaluate the first-order condition for consumption,

$$U'(C_t) = \frac{\partial \mathrm{E}_t\left[J_{t+1}(W_{t+1}, \xi_{t+1})\right]}{\partial C_t}. \tag{C12}$$

After taking logs, we obtain

$$\log\alpha - \alpha C_t = \log(\beta\gamma R_F) - \gamma R_F(W_t - C_t) + \hat{\lambda}_0 + \hat{\lambda}_1\xi_t + \hat{\lambda}_2\xi_t^2, \tag{C13}$$

which we can use to solve for consumption:

$$C_t = \frac{\gamma R_F}{\alpha+\gamma R_F}W_t + \frac{1}{\alpha+\gamma R_F}\left[\log\left(\frac{\alpha}{\beta\gamma R_F}\right) - \hat{\lambda}_0 - \hat{\lambda}_1\xi_t - \hat{\lambda}_2\xi_t^2\right]. \tag{C14}$$

[20] We assume that the technical condition $-b_2\omega^2 < 0.5$ is satisfied.

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Substituting this solution into the Bellman equation, and using the fact that $1 - \gamma R_F/(\alpha + \gamma R_F) = \alpha/(\alpha + \gamma R_F)$, we get

$$
\begin{aligned}
&\exp(-\gamma W_t - b_0 - b_1\xi_t - b_2\xi_t^2)\\
&= \exp(-\alpha C_t) + \beta \exp[-\gamma(W_t - C_t)R_F + \hat{\lambda}_0 + \hat{\lambda}_1\xi_t + \hat{\lambda}_2\xi_t^2]\\
&= \exp\left(-\frac{\alpha\gamma R_F}{\alpha+\gamma R_F}W_t\right)\exp\left\{-\frac{\alpha}{\alpha+\gamma R_F}\log\left(\frac{\alpha}{\beta\gamma R_F}\right)\right\}\\
&\quad \exp\left\{\frac{\alpha}{\alpha+\gamma R_F}\left(\hat{\lambda}_0 + \hat{\lambda}_1\xi_t + \hat{\lambda}_2\xi_t^2\right)\right\} + \beta\exp\left(-\frac{\alpha\gamma R_F}{\alpha+\gamma R_F}W_t\right)\\
&\quad \exp\left\{\frac{\gamma R_F}{\alpha+\gamma R_F}\log\left(\frac{\alpha}{\beta\gamma R_F}\right)\right\}\exp\left\{\frac{\alpha}{\alpha+\gamma R_F}\left(\hat{\lambda}_0 + \hat{\lambda}_1\xi_t + \hat{\lambda}_2\xi_t^2\right)\right\}\\
&= \left(\frac{\alpha}{\beta\gamma R_F}\right)^{-\frac{\alpha}{\alpha+\gamma R_F}}\left(1+\beta\left(\frac{\alpha}{\beta\gamma R_F}\right)\right)\exp\left\{\frac{\alpha}{\alpha+\gamma R_F}\hat{\lambda}_0\right\}\\
&\quad \times \exp\left(-\frac{\alpha\gamma R_F}{\alpha+\gamma R_F}W_t\right)\exp\left\{\frac{\alpha}{\alpha+\gamma R_F}\left(\hat{\lambda}_1\xi_t + \hat{\lambda}_2\xi_t^2\right)\right\}. \qquad \text{(C15)}
\end{aligned}
$$

Comparing coefficients, we obtain

$$\gamma = \frac{\alpha\gamma R_F}{\alpha+\gamma R_F} = \alpha\frac{r_F}{R_F}, \tag{C16}$$

$$b_1 = -\frac{\alpha}{\alpha+\gamma R_F}\hat{\lambda}_1 = -\frac{\hat{\lambda}_1}{R_F}, \tag{C17}$$

$$b_2 = -\frac{\alpha}{\alpha+\gamma R_F}\hat{\lambda}_2 = -\frac{\hat{\lambda}_2}{R_F}, \tag{C18}$$

and can solve along similar lines for b_0.

Asset Pricing. Having solved for these coefficients, we can now look at asset pricing. Expected returns are given by

$$\mathrm{E}_t\left(R_{t+1}\right) = -r_F a_0 - R_F a_1 \xi_t + a_1\hat{\xi}_t. \tag{C19}$$

Combining equations (C10) and (C19) yields

$$\mathrm{E}_t\left(R_{t+1}\right) = a_1\hat{\xi}_t + \gamma\Gamma y_t - \frac{1}{1+2b_2\omega^2}\left(\frac{b_1+\gamma y_t' a_1}{2b_2} + \hat{\xi}_t\right)a_1 + \frac{b_1+\gamma y_t' a_1}{2b_2}a_1 \tag{C20}$$

$$= \gamma\Gamma y_t + \frac{\gamma}{1+2b_2\omega^2}\omega^2 a_1 a_1' y_t + \frac{b_1+2b_2\hat{\xi}_t}{1+2b_2\omega^2}\omega^2 a_1. \tag{C21}$$

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Note that the following identities hold:

$$\mathrm{Cov}_t\left(D_{t+1}, R_{A,t+1}\right) = \Gamma y_t, \tag{C22}$$

$$\mathrm{Cov}_t\left(R_{t+1}, \xi_{t+1}\right) = \omega^2 a_1, \tag{C23}$$

where $R_{A,t+1} = R'_{t+1} y_t$ is the return on arbitrageurs' investment portfolio. We plug these identities into equation (C21) to obtain the pricing equation

$$\mathrm{E}_t\left(R_{t+1}\right) = \delta_{CF} \times \mathrm{Cov}_t\left(D_{t+1}, R_{A,t+1}\right) + \delta_{\xi,t} \times \mathrm{Cov}_t\left(R_{t+1}, \xi_{t+1}\right), \tag{C24}$$

where $\delta_{CF} = \gamma$ and $\delta_{\xi,t} = \frac{\gamma a_1' y_t}{1+2b_2\omega^2} + \frac{b_1+2b_2\hat{\xi}_t}{1+2b_2\omega^2}$.

Case 1. Sentiment investors' demands are IID with nonzero mean, that is, $\hat{\xi}_t = \mu$, $\phi = 0$. In this case, we can easily condition down (C24):

$$\mathrm{E}(R_t) = \delta_{CF} \times \mathrm{Cov}\left(D_t, R_{A,t}\right) + \delta_\xi \times \mathrm{Cov}\left(R_t, \xi_t\right), \tag{C25}$$

where we use the fact that $E_{t-1}[D_t] = 0$ and $\mathrm{E}_{t-1}[\xi_t]$ are constant. All prices of risk in this case are constant, δ_{CF} is as above, and $\delta_\xi = \frac{\gamma a_1'(\frac{1}{N}\iota - \theta\mu\delta)}{1\, +\, 2b_2\omega^2} + \frac{b_1\, +\, 2\mu b_2}{1\, +\, 2b_2\omega^2}$.

Case 2. Sentiment investors' demands are IID with zero mean, that is, $\mu = \phi = 0$. This is the case described in Section IV. Specialize (C21) and condition down as follows:

$$\mathrm{E}(R_t) = \gamma\Gamma\frac{1}{N}\iota + \left(\frac{\gamma\frac{1}{N}a_1'\iota + b_1}{1+2b_2\omega^2}\right) \times \omega^2 a_1 \tag{C26}$$

$$= \gamma\mathrm{Cov}\left(D_t, R_{M,t}\right) + \left(\frac{\gamma\frac{1}{N}a_1'\iota + b_1}{1+2b_2\omega^2}\right)\mathrm{Cov}\left(R_t, \xi_t\right). \tag{C27}$$

Here, we use the fact that $\mathrm{E}(\xi_t) = 0$ (Case 2) and $R_{A,t+1} = R_{M,t+1} - \theta\xi_t R_{\delta,t+1}$, where $R_{\delta,t+1} = R'_{t+1}\delta$ is the return on a portfolio driven by sentiment-demand distortions.

Note that

$$\mathrm{Cov}\left(R_t, R_{M,t}\right) = \left[\mathrm{Cov}\left(D_t, R_{M,t}\right) + \frac{1}{N}a_1'\iota \times \omega^2 a_1\right] + R_F^2\frac{1}{N}a_1'\iota \times \omega^2 a_1, \tag{C28}$$

where the term in the square brackets reflects the expectation of conditional covariance and the last term is the covariance of conditional expectations. Plugging this into (C27) yields

$$\mathrm{E}(R_t) = \gamma\mathrm{Cov}\left(R_t, R_{M,t}\right) + \tilde{\delta}_\xi\mathrm{Cov}\left(R_t, \xi_t\right), \tag{C29}$$

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where

$$\tilde{\delta}_\xi = \frac{b_1 + \gamma \frac{1}{N} a_1' \iota - \gamma \left(1 + R_F^2\right) \left(1 + 2b_2\omega^2\right) \frac{1}{N} a_1' \iota}{1 + 2b_2\omega^2} \tag{C30}$$

$$= \frac{b_1}{1 + 2b_2\omega^2} - \gamma R_F^2 \frac{1}{N} a_1' \iota - \frac{2b_2\omega^2}{1 + 2b_2\omega^2} \gamma \frac{1}{N} a_1' \iota. \tag{C31}$$

Without loss of generality, assume that $\delta' \iota > 0$, that is, positive shocks to ξ_t correspond to positive sentiment demand for the market portfolio. One can also show that the price of the aggregate market portfolio is increasing in sentiment demand, $a_1' \iota > 0$, and arbitrageurs' value function is decreasing in the amount of aggregate sentiment (increasing in aggregate market expected returns), $b_1 < 0$, $b_2 > 0$ (see, for instance, Campbell and Kyle (1993), Kim and Omberg (1996), Campbell and Viceira (1999)). These facts together imply that $\tilde{\delta}_\xi < 0$.

Finally, note that

$$\mathrm{Cov}\left(R_t, \mathrm{E}_t R_{M,t+1}\right) = -R_F \frac{1}{N} a_1' \iota \times \mathrm{Cov}\left(R_t, \xi_t\right). \tag{C32}$$

We can therefore rewrite (C29) as

$$\mathrm{E}(R_t) = \gamma \mathrm{Cov}\left(R_t, R_{M,t}\right) + \tilde{\delta}_{DR} \mathrm{Cov}\left(R_t, \mathrm{E}_t R_{M,t+1}\right), \tag{C33}$$

where $\tilde{\delta}_{DR} = -\tilde{\delta}_\xi (R_F \frac{1}{N} a_1' \iota)^{-1} > 0$.

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