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INVENTORY HEDGING AND OPTION MARKET MAKING

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In this paper, we develop an inventory-based approach to analyze the option market making activity. Indeed, we formulate and analytically solve the price-setting problem of a monopolistic option market maker facing exogenous public supply and demand first on a single exercise price (the "single option economy") and next on multiple exercise prices (the "multi-options economy"). While in the "single option economy" the familiar result that market maker inventory and price level are inversely related holds, the same is not necessarily true in the "multi-options economy". Additionally, we examine under which theoretical condition hedging is totally effective (i.e., the variance of the market maker hedged position is zero). Last but not least, our model is fully consistent with actual option market making practices, which consist in trading hedge portfolios to reduce risk. As such, our approach can be considered as a bridge between market microstructure and standard option pricing literature.

Keywords: Option market making; inventory model.

1. Introduction

A distinctive feature of option markets is that market makers seldom maintain "naked" option positions. Rather, they systematically hedge their inventory exposure by trading the underlying security. In this paper, we theoretically establish that such hedging practices have a substantial impact on two determinant aspects of option market making activity: bid-ask spreads setting and optimal inventory control.

While there has been substantial amount of research to uncover the economic rationale behind the existence of a bid-ask spread on common stock, so far, con-

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siderably less effort has been devoted to a better understanding of option spreads. To some extent, this is not surprising since, as we shall soon argue, the models that have been developed to explain common stock spreads do not easily transfer to option markets.

Indeed, there are essentially two strands of literature analyzing spreads on common stock. Historically, inventory models first analyzed the spread as a reward earned by the market maker for providing liquidity when public supply and demand cannot instantaneously clear. In this literature, the market maker typically starts with a given amount of stock and cash. He is also facing exogenous supply and demand for the risky security and accordingly sets a bid-ask spread in order to maximize his expected profit utility. In this literature, an important concern is the probability of "failure" (i.e., either running out of cash or out of stock). Indeed, Garman [12] shows that no matter how the market maker sets the bid-ask spread, there is still a positive probability of "failure".

It is easy to see that this approach cannot be easily adapted to option markets. Indeed, it is well documented that option market makers make intensive use of the concepts that have been developed since the introduction of the Black and Scholes [5] model. More specifically, it is usual for a market maker who sells an option to immediately trade the so-called replicating portfolio. For instance writing a call to the public will entail purchasing a specific amount of underlying stock while purchasing a call will result in a sale of stock. Intuitively then, it is clear that the previous strategy will adjust levels of stocks and cash so that the "failure" would be less of a concern. From an empirical perspective, it is also noticeable that early research by Ho and Macris [17] who analyzed the book of an AMEX options specialist hinted at the fact that the inventory management may be heavily affected by hedging practices.

The second strand of literature essentially focuses on the existence of asymmetrically informed traders in the marketplace. In this literature, the market maker is typically facing two groups of traders: informed traders who know when the risky security is not adequately priced and noise traders who trade for exogenous reasons. Consequently, the market maker will impose the bid-ask spread in order to recoup losses incurred when trading with the informed.

While there has been some attempts to adapt the former concept to option markets (see [2, 7, 11]), again it is not clear that the asymmetric information approach is consistent with actual option market making practices. Glosten [14] for instance conjectures that: "In the case of options and futures markets, one can make a reasonable argument that asymmetric information is not a large problem". His argument essentially relies on replication concepts that are specific to derivatives securities valuation.

Indeed, when the option market maker writes an under-priced call option, then he will in the same time purchase the underlying stock. Consequently, if the reason for the under-pricing of the call option is the under-pricing of the underlying stock then, by trading the replicating portfolio, the market maker will have hedged away part of the risk. One could further argue that the underlying stock price is not the only determinant of the call price. Indeed some traders could also possess specific information about stock volatility. Even in this case however, it is conceivable that our argument would still hold since it is well known that options can be used to hedge other options so that the market maker could (perhaps only partially) hedge away this extra risk.

Additionally, empirical research investigating asymmetric information as a key determinant of option spreads is mixed. Indeed, while Easley, O'Hara and Srivinas [11] find significant evidence of adverse selection in CBOE option quotes, other studies by Vijh [26], George and Longstaff [13] and Cho and Engle [9] fail to reach the same conclusion.

Overall, it is our belief that existing microstructure paradigms are not well suited to analyze option market making activities because they do not fully take into account the specificity of these derivative securities. In this paper, we propose instead an approach that explicitly incorporates the standard practice of trading replicating portfolios to hedge risk away. By adding hedging considerations to the conventional inventory model, we derive market maker's optimum option quote setting and inventory control policies.

For our analysis to hold, we assume that, given an arbitrage option-pricing model, the option market maker attempts to optimize the cost of replication. We also assume that, in the spirit of arbitrage models, the underlying stock transactions take place within an exogenous trading range, which can, at times, be assimilated to a posted bid-ask spread. To this trading range on the stock correspond arbitrage bounds prices on a call option price. These bounds represent the largest possible bid-ask spread that could be imposed on the call option by an option market maker. However, if this market maker is able to trade within the stock posted bid-ask spread, then he will also be able to quote a tighter option bid-ask spread. We further formulate and solve the quote setting problem in a standard expected utility maximization framework and linear public supply and demand for options. We first analyze a single option economy and next a multi-options economy. In particular, we show that introducing additional options affects both the equilibrium bid-ask spreads and the net market maker inventory positions. Our theoretical results confirm the optimality of the popular spread trading strategies typically implemented by option market makers.

The paper is organized as follows. In Sec. 2, we introduce the fundamental structure of the model and discuss the trading mechanism. In Sec. 3, we formulate and solve for the equilibrium bid-ask spread given exogenous linear supply and demand in a single option economy. We also compare our approach with relevant theoretical literature. In Sec. 4, we discuss the market maker problem when several exogenous supply and demand curves corresponding to options with different exercise prices are involved. We further comment on appropriate empirical implications. In Sec. 5, we discuss issues related to the extension of our framework to a multi-period economy. Section 6 concludes the paper. All proof's technical details are relegated to the Appendix.

2. The Model

2.1. *The securities*

We consider a single period economy with two primary securities, a stock and a risk free bond. The stock (which we will also refer to as the underlying security) trades for at least the minimum price S^L and at most the maximum price S^U. So, S^L and S^U simply define a trading range in which all stock transactions must take place. In the spirit of arbitrage option pricing models where the underlying price process is usually assumed, the existence and size of this trading zone are treated as exogenous. Of course, when the stock is traded through a posted bid-ask spread it is sometimes possible to assimilate trading range and posted bid-ask spread. This assimilation however is not necessarily warranted.

Roll [24] for instance, argues that the effective spread may often differ from the posted bid-ask spread. This could be either because of untimely updating of the posted quotes, or because the stock market maker would be willing to provide a discount to large traders if they know that their trade is not information-based. In short, we simply assume that the stock transaction price lies within a specific trading range (possibly the posted bid-ask spread). We further specify the minimum price S^L and the maximum price S^U as follows:

$$S^L \equiv S(1-\tau)\,, \tag{2.1a}$$

$$S^U \equiv S(1+\tau)\,, \tag{2.1b}$$

$$\text{and} \quad 0 \leq \tau < 1\,.$$

In (2.1), we interpret τ as a maximum transaction cost incurred by stock traders. At time 1, all agents in the economy agree that the stock price can only take two values (or state of the world) with subjective probabilities strictly between 0 and 1:

$$S_u = uS\,, \tag{2.2a}$$

$$S_d = dS\,. \tag{2.2b}$$

The risk-free bond pays \$1 in any state of the world at time 1 and trades for \$1 at time 0. This is without loss of generality since it is always possible to choose the bond as the numeraire and scale accordingly other prices. This partial equilibrium constitutes a trivial extension of the traditional arbitrage based option valuation model (see [8]) where the primary securities processes are exogenous and agents are only expected to agree on the terminal state space.

To preclude dominant trading strategies (for a formal definition of dominant trading strategies, see [23, p. 5] , it is natural to impose the following condition:

$$0 < d < (1 - \tau) < (1 + \tau) < u\,. \tag{2.3}$$

Expression (2.3) simply says that the stock return down and up magnitudes must exceed the bid and ask transaction costs magnitudes.

In addition to the primary securities, there is a market for options written on the stock. In this section, we restrict ourselves to a single call option with exercise price K whose terminal payoff in each state of the world is as follows:

$$C_u(K) = \mathrm{Max}(S_u - K, 0)\,, \tag{2.4a}$$

$$C_d(K) = \mathrm{Max}(S_d - K, 0)\,, \tag{2.4b}$$

where $\mathrm{Max}(x, 0) = x$ if $x > 0$ and 0 otherwise.

The following proposition states that the option must trade within a specific price interval hence providing lower and upper bounds for the call price.

Proposition 2.1. *A call option with maturity date* 1 *and exercise price* K *must trade within the interval with* $[C^L, C^U]$ *with*

$$C^L = sS^L + b\,, \tag{2.5a}$$

$$C^U = sS^U + b\,, \tag{2.5b}$$

$$s = \frac{C_u(K) - C_d(K)}{S_u - S_d}\,, \tag{2.5c}$$

$$b = C_u(K) - sS_u\,. \tag{2.5d}$$

Proof. The proof is a simple no-arbitrage argument and can be found in [23, pp. 24–27] for instance. In the option pricing literature, s, the quantity of stock that must be traded to replicate the call terminal payoff, is usually referred to as the hedge ratio or delta. Merton [21, pp. 432–440] implements a similar framework to value a call option. Indeed, he designates C^L and C^U as the most "conservatives" bounds on the option price. Furthermore, Merton points out the fact that, since the market maker is in the business of purchasing and writing options at the same time, he will only hedge his net exposure, hence reducing his hedging cost. In this paper, we make Merton's intuitive argument precise. □

For any stock price $\tilde{S}$ traded between S^L and S^U, the price of the replicating (or hedging) portfolio is given by:

$$\tilde{C} = s\tilde{S} + b\,. \tag{2.6}$$

So, Proposition 2.1 also implies that $C^L \leq \tilde{C} \leq C^U$. Our model crucially relies on this last point. Indeed, because the option maker is able to purchase the stock at a price $\tilde{S} \leq S^U$ or sell it at a price $\tilde{S} \geq S^L$, he can also "better" his option quotes (with respect to the no-arbitrage bounds C^L and C^U). Of course, this is essentially possible because the option market maker clients (whom we shall refer to as "the public") are not able to trade the stock within the quoted bid-ask spread. This assumed asymmetry between the option market maker and the public underlies this paper's results. At this point, a couple of comments are in order.

First, it is important to observe that we are not assuming that the option market maker is an informed trader (i.e., that he would have superior knowledge of the fundamental value of the traded stock). Instead, we are assuming that the option market maker has some bargaining power that allows him to trade inside the posted stock bid-ask spread. This is consistent with Hasbrouck [16] who reports that a substantial number of inside the quotes trades on the NYSE result from large traders being able to bargain for better deals. The public on the over hand is more likely to trade at the stock bid-ask spread posted by the specialist.

Second, our model is a partial equilibrium approach in the sense that, very much in the spirit of arbitrage-free option pricing models, the option market maker treats the spread on the underlying stock as an exogenous input. This is not contrary to the well-documented empirical evidence (by [16] and others) that the stock spread is certainly affected by informed trading.

2.2. *The trading mechanism*

We now specify the trading mechanism on the option market. Specifically, we assume that there is a single market maker who posts a quoted price C^A at which he commits to sell the call at time 0 and a quoted price C^B at which he commits to purchase the call at time 0. The quantities bought and sold by the market maker (respectively Q^B and Q^A) at the quoted prices are determined by the exogenous demand and supply schedules of the public. For concretness, we impose the following linear demand and supply schedules (for $\alpha \geq 0$ and $\beta \geq 0$):

$$Q^A = I^A - \alpha C^A \,, \tag{2.7a}$$

$$\text{and} \quad Q^A = 0 \text{ if } C^A > C^U \,, \tag{2.7b}$$

$$Q^B = I^B + \beta C^B \,, \tag{2.7c}$$

$$\text{and} \quad Q^B = 0 \text{ if } C^B < C^L \,. \tag{2.7d}$$

Observe that inequalities (2.7b) and (2.7d) express the fact that no trader will sell the call below the minimum arbitrage bound C^L or purchase it above the maximum arbitrage bound C^U. Hence these functions are consistent with Proposition 2.1. In addition, we impose the following assumption:

Assumption 2.1. *The intercepts of the supply and demand functions are given by*:

$$I^A = \alpha C^U \,, \tag{2.8a}$$

$$I^B = -\beta C^L \,. \tag{2.8b}$$

Assumption 2.1 implies that:

$$Q^U \equiv I^A - \alpha C^U = 0 \,, \tag{2.9a}$$

$$Q^L \equiv I^B + \beta C^L = 0 \,. \tag{2.9b}$$

Such an assumption is natural since, if the posted quotes were such that $C^B = C^L$ and $C^A = C^U$ then, the public may as well trade the replicating portfolio. So, under Assumption 2.1, the option market maker only has a viable economic role if he proposes bid and ask quotes C^B and C^A such that $C^L < C^B$ and $C^A < C^U$.

Finally, after having bought and sold the quantities Q^B and Q^A at prices C^B and C^A, the option market maker proceeds to hedge his net option position, $(Q^B - Q^A)$, by trading the hedging portfolio at the price $\tilde{C}$ (with $\tilde{C} = s\tilde{S}+b$ and $S^L \leq \tilde{S} \leq S^U$). Hence, in this paper, market making risk only derives from the fact that, when the quotes C^B and C^A are posted, the actual price $\tilde{C}$ at which the hedge will be completed is unknown.

Hence, by assumption, all the option trades happen either at the bid price C^B or at the ask price C^A of the option market maker. The empirical evidence broadly supports this modeling choice. In their study of S&P 500 index options, Cho and Engle [9] report that 75% of at-the-money options and 81% of out-of the-money options are traded at the market maker bid-ask quotes.

Next, we assume that, even though the market maker does not know the exact cost of replication $\tilde{C}$, he does know the distribution of the traded stock price $\tilde{S}$ (hence the distribution of $\tilde{C}$). For practical purposes, the distribution of $\tilde{S}$ is restricted as follows:

Assumption 2.2. *The traded stock price $\tilde{S}$ is normally distributed with $\tilde{S} = \bar{S} + \tilde{\varepsilon}$ where $\bar{S} = pS^L + (1-p)S^U$ (for $0 \leq p \leq 1$) and $\tilde{\varepsilon}$ has mean 0 and standard deviation σ.*

The former assumption allows model tractability but it also has limitations since, if $\tilde{S}$ is normally distributed then, $\Pr ob(\tilde{S} > S^U) > 0$ and $\Pr ob(\tilde{S} < S^L) > 0$. However, the magnitude of these probabilities clearly depends on the magnitude of σ. Indeed, assume for concretness that $p = 0.5$ and $\sigma = \frac{\tau S}{n}$ (for $n \geq 1$). Then

$$\tilde{S} = 0.5S(1-\tau) + 0.5S(1+\tau) = S \,,$$

hence

$$\Pr ob(S^L < \tilde{S} < S^U) = \Pr ob(\bar{S} - \tau\bar{S} < \tilde{S} < \bar{S} + \tau\bar{S}) = \Pr ob(\bar{S} - n\sigma < \tilde{S} < \bar{S} + n\sigma) \,,$$

so that $\Pr ob(\tilde{S} > S^U) = \Pr ob(\tilde{S} < S^L) = 1 - \Phi(n) \to 0$ as $n \to \infty$, where $\Phi(n)$ is the standard normal distribution function. Of course, the previous parameterization is consistent only if the market maker stock trades distribution is symmetric around the its mean $\bar{S}$. In the next section, we formulate and solve the optimal price setting problem.

3. The Single Option Economy

3.1. *Defining an equilibrium*

In order to determine the optimal call option bid-ask spread, we first specify the profit function of the market maker. We denote this function by $\tilde{\Pi}(C^B, C^A)$ and express it as follows:

$$\tilde{\Pi}(C^B, C^A) = [Q^A(C^A - \tilde{C})] - [Q^B(C^B - \tilde{C})]\,. \tag{3.1}$$

Inspection of Eq. (3.1) reveals that $\tilde{\Pi}(C^B, C^A)$ is a random quantity whose value depends on $\tilde{C}$, the random price at which the hedging portfolio will be traded. The first bracketed term represents the profit realized from selling the quantity Q^A for the price C^A (with $C^A < C^U$) minus the cost of hedging the written calls at $\tilde{C}$. The second bracketed term represents the cost incurred from purchasing the quantity Q^B at the bid price C^B (with $C^B > C^L$) plus the profit of hedging the purchased calls at $\tilde{C}$. Next, we specify the market maker objective function as follows:

$$\mathrm{Max}_{C^B, C^A} E[U(\tilde{\Pi}(C^B, C^A))]\,, \tag{3.2}$$

where U is the market maker Von-Neumann and Morgenstern utility function and $E(\bullet)$ denotes the expectation operator with respect of the distribution of $\tilde{C}$. For concreteness, we assume that the market maker's utility function U is the popular negative exponential so that:

$$U(\lambda\Pi) = -\exp(-\lambda\Pi)\,, \tag{3.3}$$

where "exp" denotes the exponential function and $\lambda, \Pi > 0$. We now solve the minimization problem in (3.2) for the negative exponential utility function and $\tilde{C}$ normally distributed. Standard results on exponential of normal distributions imply that solving (3.2) is simply equivalent to solving:

$$\mathrm{Max}_{C^B, C^A} E(\tilde{\Pi}(C^B, C^A)) - \frac{1}{2}\lambda \mathrm{Var}(\tilde{\Pi}(C^B, C^A))\,, \tag{3.4}$$

where $\mathrm{Var}(\bullet)$ denotes the variance operator with respect to the distribution of $\tilde{C}$. In the following proposition, we establish the existence of a unique equilibrium and further examine some of its properties.

Proposition 3.1. *There exists a unique set of bid ask quotes* (C^B, C^A) *solution of the maximization problem in* (3.4). *Explicitly, we obtain*:

$$C^B = \frac{C^L + \bar{C} + \lambda v^2(\beta C^L + \frac{1}{2}\alpha(C^U + C^L))}{2 + \lambda v^2(\alpha + \beta)}\,, \tag{3.5a}$$

$$C^A = \frac{C^U + \bar{C} + \lambda v^2(\alpha C^U + \frac{1}{2}\beta(C^U + C^L))}{2 + \lambda v^2(\alpha + \beta)}\,. \tag{3.5b}$$

Proof. The proof results from the usual first order conditions applied to the objective function in (3.4) (see details in Appendix A). □

Adding (A.4a) to (A.4b) shows that:

$$C^A - C^B = \frac{1}{2}(C^U - C^L)\,. \tag{3.6}$$

Hence, the dollar size of the quoted option spread is independent of α and β the elasticity coefficients. It is easy to observe that:

$$C^A - C^B = s\tau S\,. \tag{3.7}$$

In words, the quoted option spread is a linear function of s, the option delta, and τ, half of the underlying stock spread. The empirical evidence regarding the plausibility of this result is mixed. In his study of CBOE equity options, Vijh [26] reports that dollar option spreads are nearly equal to their underlying stock spreads. On the other hand, George and Longstaff [13] report dollar option spreads significantly varying with option moneyness and hence delta. We further characterize the equilibrium by showing that under a specific condition on the supply and demand elasticity coefficients α and β, the variance of the market maker profit is zero.

Lemma 3.1. *If $\alpha = \beta$ (the respective slopes of the supply and demand function are equal), then there is no market making risk* $(\mathrm{Var}(\tilde{\Pi}(C^B, C^A)) = 0)$.

Lemma 3.1 confirms the natural intuition that, if the slope coefficients of the public's supply and demand functions are equal, then there is no hedging risk for the option market maker. Indeed, one can readily check that, if this case and in this case only, the quantities of calls bought and sold by the market maker are strictly equal so that there is no need to trade the replicating portfolio.

3.2. *Numerical illustration*

In order to provide a better grasp of the theoretical results, we plot the market maker net inventory position and the option bid-ask spread for some parameters specific values. More specifically, we set $S = 100$, $u = \exp(0.3)$, $d = u^{-1}$ to describe the stock price process. In this example, we focus on a call with exercise price $K = 101$. We also assume that that the stock transaction cost zone is about 1% (or $\tau = 0.005$). In addition, we set $p = 0.5$ so that $\bar{S} = S$ and $\sigma = \frac{\tau S}{3}$. This parameterization implies that the option market maker expects to trade at the mid-point of the stock trading range and that 99.86% of the traded prices occur in the range $[\bar{S}(1-\tau), \bar{S}(1+\tau)]$. Finally, we set the coefficient risk aversion coefficient to $\lambda = 5$ (additional sensitivity analysis is available upon request and shows that the results are not substantially impacted by this parameter). Next, we define the market maker inventory as follows:

$$Z = Q^B - Q^A = \beta(C^B - C^L) - \alpha(C^U - C^A)\,, \tag{3.8}$$

where the last equality in (3.7) results from (2.7) and (2.8). Hence, $Z > 0$ ($Z < 0$) means that the market maker net position is long (short).

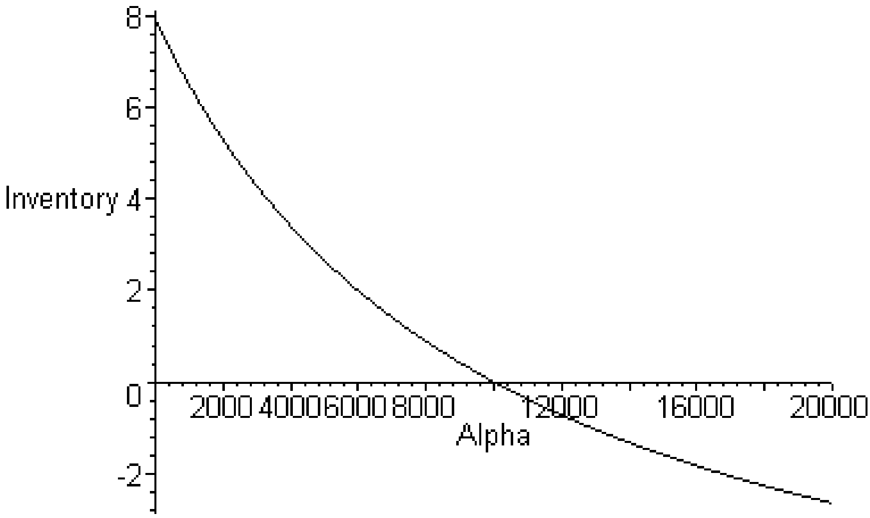

Fig. 1. Market maker net inventory position Z ($K = 101$) for fixed supply slope $\beta = 10{,}000$ and varying demand slope α.

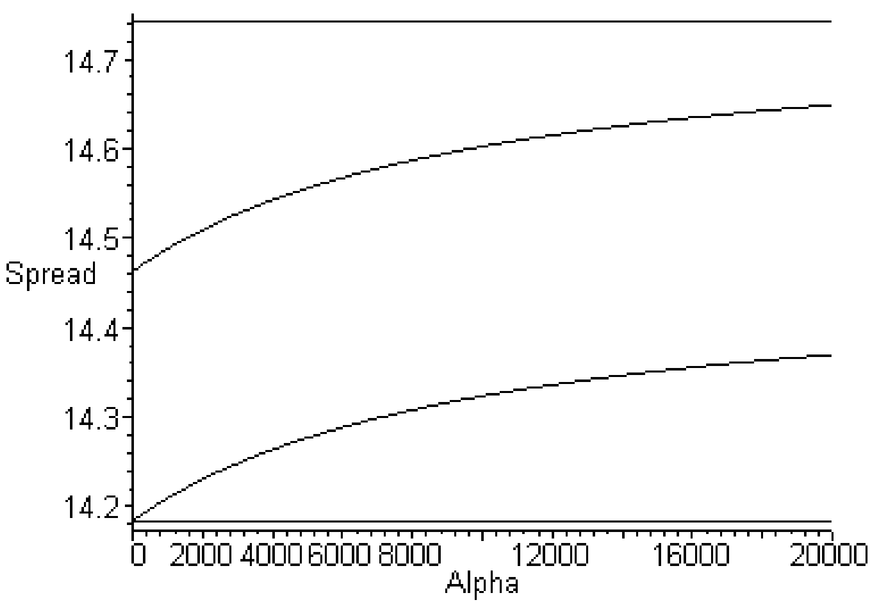

Fig. 2. Bid-ask spread ($K = 101$) for fixed supply slope $\beta = 10{,}000$ and varying demand slope α. The upper and lower horizontal lines represent C^U and C^L, respectively the upper and lower bounds on the call price.

Figure 1 displays the market maker inventory position for a fixed supply slope $\beta = 10,000$ and a demand slope α varying from 0 to 20,000. Figure 2 displays the call bid-ask spread for the same fixed supply slope $\beta = 10,000$ and the demand slope coefficient α varying from 0 to 20,000. On this plot, the upper and lower horizontal lines represent the no-arbitrage bounds C^U and C^L.

Inside the no-arbitrage bounds, Fig. 2 shows that the spread is shifting upward as the slope coefficient α is increasing. It is also important to observe that the market maker inventory position and the price level (as measured by the mid-point of the bid-ask quotes) are inversely related. Such a feature is typical of inventory models (see [25] for instance).

Figure 3 instead displays a market scenario where the demand slope coefficient α = 10,000 is fixed and the supply slope coefficient β varies from 0 to 20,000. Finally, Fig. 4 displays the call bid-ask spread for this fixed demand and varying supply slopes scenario. As in Fig. 2, the upper and lower horizontal lines represent the no-arbitrage bounds C^U and C^L. This time however, the spread is shifting downward as the slope coefficient β is increasing. So, when his inventory position is positive and large, the option market maker lowers his bid-ask spread. Again, this implies an inverse relation between the net inventory position and the price level.

3.3. *Summary*

In this section, we summarize relevant theoretical research on stock inventory models and contrast our findings with previous results. There are essentially two paradigms that reach conclusions related to ours.

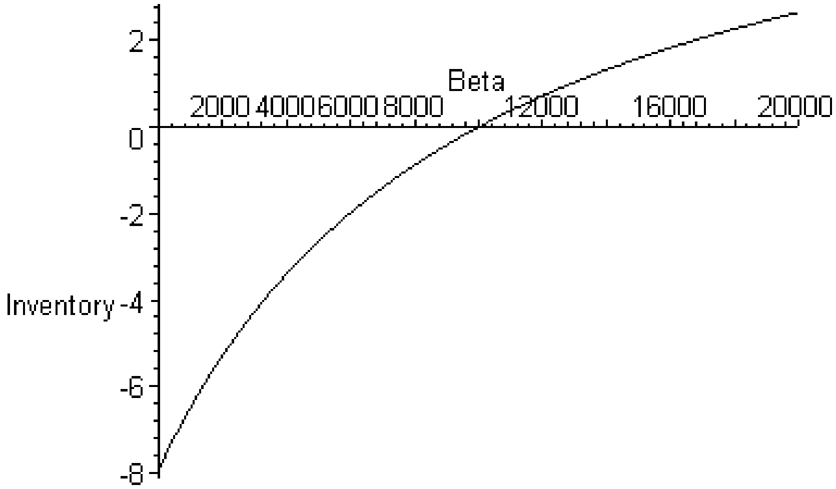

Fig. 3. Market maker net inventory position $Z(K = 101)$ for fixed demand slope α = 10,000 and varying supply slope β.

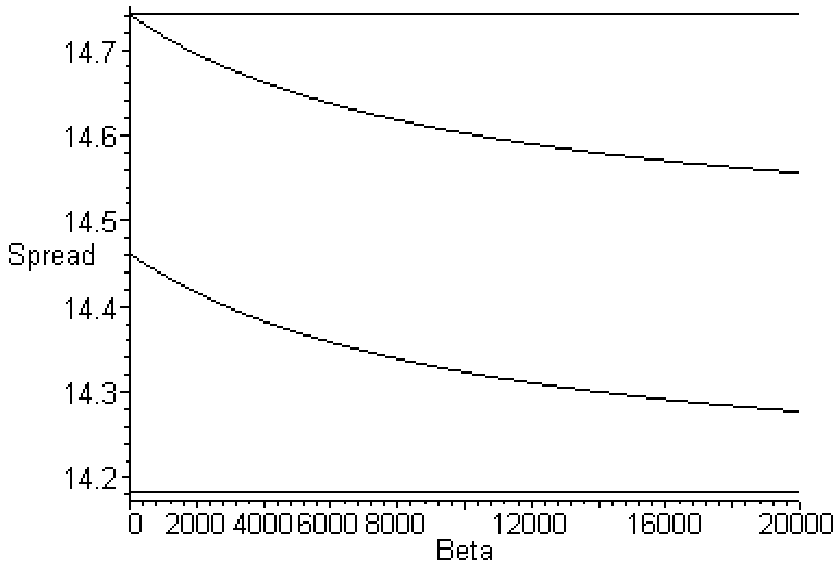

Fig. 4. Bid-ask spread ($K = 101$) for fixed demand slope $\alpha = 10{,}000$ and varying supply slope β. The upper and lower horizontal lines represent C^U and C^L, respectively the upper and lower bounds on the call price.

Amihud and Mendelson [1980] (extending Garman [12]) model a risk-neutral monopolist market maker who sets the bid-ask spread to maximize expected profit when facing stochastic stock order flow. They find that, as his inventory increases, the market maker will lower both his bid and ask quotes. They also find that there exists an optimum inventory position for the market maker.

Stoll [25] instead models the bid-ask spread as a cost incurred by traders to compensate a risk-averse market maker from deviating from his optimal security allocation (in the sense of standard portfolio theory). Finally, Ho and Stoll [18] extend Stoll [25] to a multi-period continuous time setting where the market maker is facing both stochastic order flow and returns variability. The empirical implications of the latter two models are similar (i.e., optimal bid-ask spread, independence of the bid-ask spread size from the inventory level and existence of a "preferred" inventory level).

Our work is related to the inventory models surveyed in that we model a risk-averse option market maker facing a stochastic order flow. In our model however, risk derives from the inability of the option market maker to hedge perfectly when trading the underlying asset. Like in previous literature three main results obtain. First, there exists a positive optimal option bid-ask spread. Second, the inventory affects the placement of the spread but not its size. Third, there exist an optimal or "preferred" inventory level for the option market maker.

4. The Multi-Options Economy

4.1. *The multi-options trading mechanism*

In this section, we address the more general problem of setting optimal bid-ask spreads when several options with different exercise prices are available for trading. The solution of this problem is an extension of the framework previously developed in Sec. 3. However, this is a rather important extension, since it is well known that spread trading is a typical option market making strategy (see [3, Chapter 7] for extensive details). Spread trading usually refers to the widespread practice of simultaneously purchasing and writing options with different exercise prices. Intuitively, spread trading can potentially reduce the risks associated with using a misspecified option-pricing model. This is essentially because only relative pricing errors are relevant to spread traders.

In this paper, we provide an alternative motivation for spread trading. Indeed, we analyze this practice in the context of imbalance between the public net supply and demand across strike prices. We further solve for that the optimal market maker quote setting strategy. An important practical implication is that, if the public is essentially demanding options on all strikes and the market maker is essentially supplying these options, then it is never possible to cancel market making risk through delta-hedging. Indeed, since financial institutions are essentially net option writers (see [15]), the former analysis provides the appropriate framework for assessing their market making risk.

To keep the exposition transparent, we only derive a two-option equilibrium in this paper. A further extension to a larger spectrum of exercise prices is conceptually straightforward albeit computationally tedious. Hence, we consider an option market where two calls with exercise prices K_1 and K_2 (with $K_1 < K_2$) are traded. The terminal payoffs for each call K_i ($i = 1, 2$) in each state of the world is as follows:

$$C_u(K_i) = \mathrm{Max}(S_u - K_i, 0)\,, \tag{4.1a}$$

$$C_d(K_i) = \mathrm{Max}(S_d - K_i, 0)\,. \tag{4.1b}$$

From Proposition 2.1, we know that call K_i must trade within the interval $\lfloor C_i^L, C_i^U \rfloor$ where:

$$C_i^L = s_i S^L + b_i\,, \tag{4.2a}$$

$$C_i^U = s_i S^U + b_i\,. \tag{4.2b}$$

$$s_i = \frac{C_u(K_i) - C_d(K_i)}{S_u - S_d}\,, \tag{4.3a}$$

$$b_i = C_u(K_i) - s_i S_u\,. \tag{4.3b}$$

Next, we specify the demand and supply functions for exercise prices K_i. We denote by Q_i^A the public demand on call K_i and by Q_i^B the public supply on call K_i. For concreteness, we assume the following linear demand and supply schedules (for $\alpha_i \geq 0$, $\beta_i \geq 0$):

$$Q_i^A = I_i^A - \alpha_i C_i^A \,, \tag{4.4a}$$

$$Q_i^A = 0 \text{ if } C_i^A > C_i^U \,, \tag{4.4b}$$

$$Q_i^B = I_i^B + \beta_i C_i^B \,, \tag{4.4c}$$

$$Q_i^B = 0 \text{ if } C_i^B < C_i^U \,. \tag{4.4d}$$

In addition, to enforce the economic viability of the option market maker, we impose the following assumption.

Assumption 4.1. *The intercepts of the supply and demand functions are given by*:

$$I_i^A = \alpha_i C_i^U \,, \tag{4.5a}$$

$$I_i^B = -\beta_i C_i^L \,. \tag{4.5b}$$

Assumption 4.1 mirrors Assumption 2.1 in the single option economy. It implies that there will be no option transactions if the market maker posts quotes such that $C_i^B = C_i^L$, $C_i^A = C_i^U$ since then, the public may just as well trade the replicating portfolio.

4.2. *The multi-option equilibrium*

In order to solve the optimal quote setting problem, we express the option market making profit function as:

$$\begin{aligned}\tilde{\Pi}(C_1^B, C_1^A, C_2^B, C_2^A) = {} & [Q_1^A(C_1^A - \tilde{C}_1) - Q_1^B(C_1^B - \tilde{C}_1)] \\ & + [Q_2^A(C_2^A - \tilde{C}_2) - Q_2^B(C_2^B - \tilde{C}_2)] \,.\end{aligned} \tag{4.6}$$

The interpretation of expression (4.6) is analogous to the single option's profit function. Finally, we formulate the market maker maximization problem as:

$$\mathrm{Max}_{C_1^B, C_1^A, C_2^B, C_2^A} E[U(\tilde{\Pi}(C_1^B, C_1^A, C_2^B, C_2^A))] \,. \tag{4.7}$$

Using the negative exponential utility function coupled with the normality assumption, we specify the objective function in (4.7) as:

$$\mathrm{Max}_{C_1^B, C_1^A, C_2^B, C_2^A} E(\tilde{\Pi}(C_1^B, C_1^A, C_2^B, C_2^A)) - \frac{1}{2}\lambda \mathrm{Var}(\tilde{\Pi}(C_1^B, C_1^A, C_2^B, C_2^A)) \,. \tag{4.8}$$

The following proposition asserts the existence of a unique equilibrium to the market maker quote setting problem.

Proposition 4.1. *There exists a unique set of bid and ask quotes* $(C_1^B, C_1^A, C_2^B, C_2^A)$ *solutions of problem* (4.8). *They are given by*:

$$C_1^B = \frac{\bar{C}_1 + C_1^L(1+\lambda v_1^2\beta_1) + \frac{1}{2}\lambda\sigma^2(s_1^2\alpha_1(C_1^U + C_1^L) + s_1 s_2((\alpha_2 C_2^U + \beta_2 C_2^L) - m_2(\alpha_2+\beta_2)) + s_2^2(\alpha_2+\beta_2)(m_1 + C_1^L))}{2 + \lambda v^2(s_1^2(\alpha_1+\beta_1) + s_2^2(\alpha_2+\beta_2))}, \quad (4.9a)$$

$$C_1^A = \frac{\bar{C}_1 + C_1^U(1+\lambda v_1^2\alpha_1) + \frac{1}{2}\lambda\sigma^2(s_1^2\beta_1(C_1^U + C_1^L) + s_1 s_2((\alpha_2 C_2^U + \beta_2 C_2^L) - m_2(\alpha_2+\beta_2)) + s_2^2(\alpha_2+\beta_2)(m_1 + C_1^U))}{2 + \lambda v^2(s_1^2(\alpha_1+\beta_1) + s_2^2(\alpha_2+\beta_2))}, \quad (4.9b)$$

$$C_2^B = \frac{\bar{C}_2 + C_2^L(1+\lambda v_2^2\beta_2) + \frac{1}{2}\lambda\sigma^2(s_2^2\alpha_2(C_2^U + C_2^L) + s_1 s_2((\alpha_1 C_1^U + \beta_1 C_1^L) - m_1(\alpha_1+\beta_1)) + s_1^2(\alpha_1+\beta_1)(m_2 + C_2^L))}{2 + \lambda v^2(s_1^2(\alpha_1+\beta_1) + s_2^2(\alpha_2+\beta_2))}, \quad (4.9c)$$

$$C_2^A = \frac{\bar{C}_2 + C_2^U(1+\lambda v_2^2\alpha_2) + \frac{1}{2}\lambda\sigma^2(s_2^2\beta_2(C_2^U + C_2^L) + s_1 s_2((\alpha_1 C_1^U + \beta_1 C_1^L) - m_1(\alpha_1+\beta_1)) + s_1^2(\alpha_1+\beta_1)(m_2 + C_2^U))}{2 + \lambda v^2(s_1^2(\alpha_1+\beta_1) + s_2^2(\alpha_2+\beta_2))}. \quad (4.9d)$$

Proof. This proof is similar to the proof of Proposition 3.1. The first order conditions generate four linear equations with four unknowns (see Appendix B for details). Adding (B.4a) to (B.4b) and (B.4c) to (B.4d) shows that for each call K_i $(i = 1, 2,)$:

$$C_i^A - C_i^B = \frac{1}{2}(C_i^U - C_i^L)\,. \tag{4.10}$$

As in the single option economy, the size of the option spreads only depends on the upper and lower no-arbitrage bounds. In the following lemma, we establish which relationship between the elasticity coefficients α_1, α_2, β_1 and β_2 must hold in order to obtain zero variance on the market making profit function. □

Lemma 4.1. *If* $\alpha_1 = \frac{s_1\beta_1(C_1^U - C_1^L) - s_2(\alpha_2 - \beta_2)(C_2^U - C_2^L)}{s_1(C_1^U - C_1^L)}$ *then, there is no market making risk* $(\mathrm{Var}(\tilde{\Pi}(C_1^B, C_1^A, C_2^B, C_2^B)) = 0)$.

At this stage, we are able to examine more precisely the scenario where the public supply functions are all absolutely inelastic (i.e., $\beta_1 = 0$ and $\beta_2 = 0$) so that the option market maker provides all the liquidity. Indeed from Lemma 4.1, we obtain $\alpha_1 = \frac{-s_2\alpha_2(C_2^U - C_2^L)}{s_1(C_1^U - C_1^L)} < 0$ (if $\alpha_2 > 0$). But, since this is ruled out by assumption $(\alpha_1 \geq 0)$, the implication is that market-making risk can never be zero in this specific scenario $(\mathrm{Var}(\tilde{\Pi}(C_1^B, C_1^A, C_2^B, C_2^B)) > 0)$.

4.3. *Numerical illustration*

We now turn a numerical illustration of the previous theoretical developments. Since we wish to contrast with the single option economy's numerical results, we maintain all the assumptions of Sec. 2.2 but those necessary to consider a second exercise price. More explicitly, two calls with respective exercise prices $K_1 = 99$ and $K_2 = 101$ are traded. We assume that the fixed demand and supply slopes for call K_1 are given by $\alpha_1 = 10{,}000$ and $\beta_1 = 0$ (hence absolutely inelastic public's supply on call K_1). As in Sec. 2.2, call K_2 supply's coefficient is set at $\beta_2 = 10{,}000$. We define the market maker net inventory position on call K_i $(i = 1, 2)$ as follows:

$$Z_i = Q_i^B - Q_i^A = \beta_1(C_i^B - C_i^L) - \alpha_i(C_i^U - C_i^A)\,. \tag{4.11}$$

In Figs. 5 and 6, we graph the market maker net inventory positions for a demand coefficient α_2 varying from 0 to 20,000. Figures 7 and 8 display the bid-ask spreads on each call when α_2 varies from 0 to 20,000. On these last two plots, the upper and lower horizontal lines represent the no-arbitrage bounds on each call. Finally, Fig. 9 displays the bid-ask spread mid-point quote for call K_2 in the single and the two options economy (explicitly the lower curve represents $\frac{1}{2}(C^B + C^A)$ and the upper curve $\frac{1}{2}(C_2^B + C_2^A)$). Altogether, a cursory observation of these plots allows us to make two important points.

First, while the market maker net inventory position on call K_2 decreases, his net inventory position on call K_1 instead increases as α_2 varies from 0 to 20,000. This

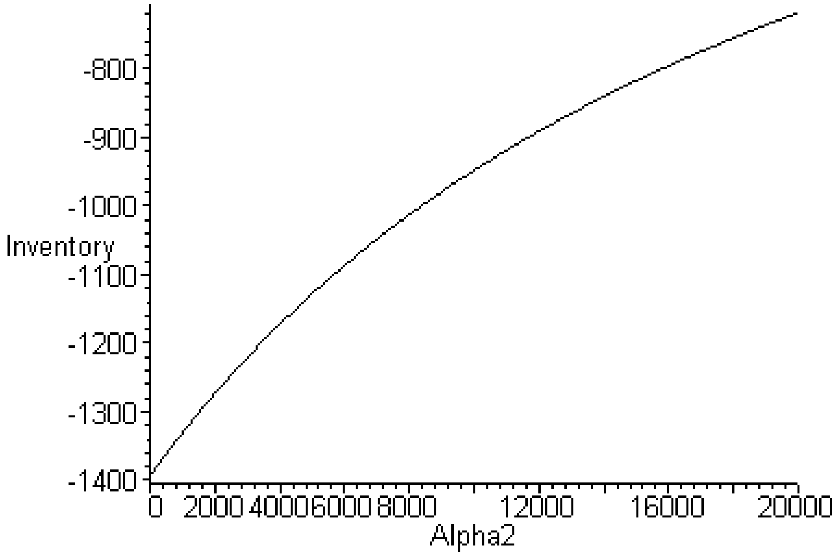

Fig. 5. Market maker net inventory position on Z_1 call K_1 ($K_1 = 99$) for fixed demand slope $\alpha_1 = 10{,}000$, fixed supply slope $\beta_1 = 0$ and fixed supply slope $\beta_2 = 10{,}000$. The demand slope α_2 on call K_2 ($K_2 = 101$) is varying.

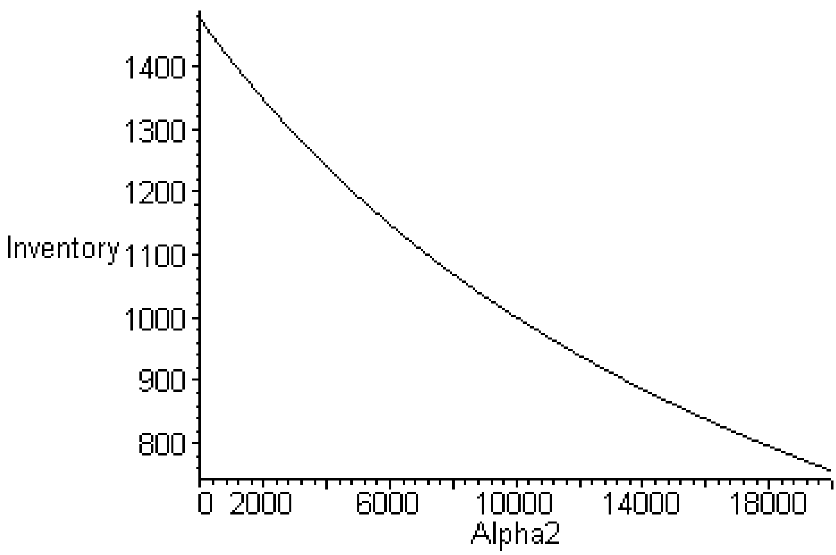

Fig. 6. Market maker net inventory position Z_2 on call K_2 ($K_2 = 101$) for fixed demand slope $\alpha_1 = 10{,}000$, fixed supply slope $\beta_1 = 0$ and fixed supply slope $\beta_2 = 10{,}000$. The demand slope α_2 on call K_2 ($K_2 = 101$) is varying.

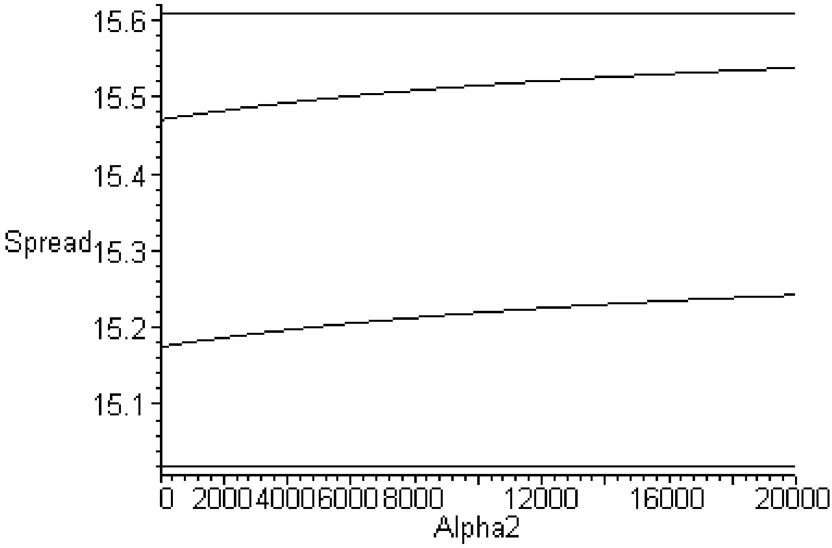

Fig. 7. Bid-ask spread of call K_1 ($K_1 = 99$) for fixed demand slope $\alpha_1 = 10{,}000$, fixed supply slope $\beta_1 = 0$ and fixed supply slope $\beta_2 = 10{,}000$. The demand slope α_2 on call K_2 ($K_2 = 101$) is varying. The upper and lower horizontal lines represent C_1^U and C_1^L, respectively the upper and lower bounds on call K_1 price.

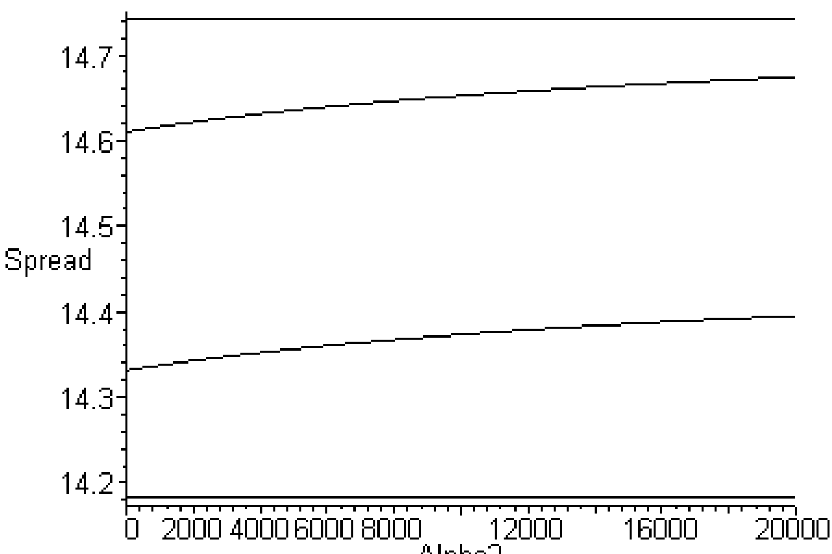

Fig. 8. Bid-ask spread of call K_2 ($K_2 = 101$) for fixed demand slope $\alpha_1 = 10{,}000$, fixed supply slope $\beta_1 = 0$ and fixed supply slope $\beta_2 = 10{,}000$. The demand slope α_2 on call K_2 ($K_2 = 101$) is varying. The upper and lower horizontal lines represent C_2^U and C_2^L, respectively the upper and lower bounds on call K_2 price.

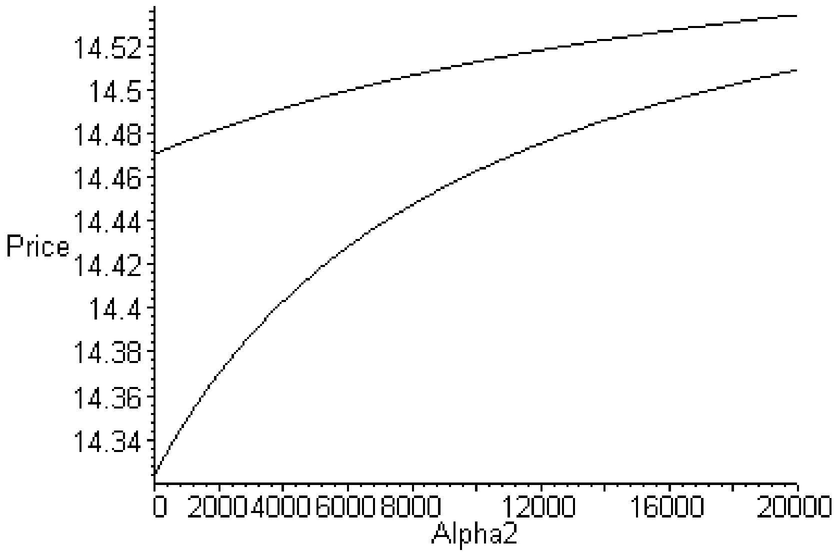

Fig. 9. The lower curve plots the bid-ask spread mid-point of call K_2 ($K_2 = 101$) for fixed supply slope $\beta_2 = 10{,}000$ and varying demand slope α_2 in a "single option economy". The upper curve plots the bid-ask spread mid-point of the same call K_2 ($K_2 = 101$) for fixed demand slope $\alpha_1 = 10{,}000$, fixed supply slope $\beta_1 = 0$ and fixed supply slope $\beta_2 = 10{,}000$ in a "multi-options economy" ($K_1 = 99$). The demand slope α_2 on call K_2 ($K_2 = 101$) is varying.

is a clear manifestation of the fact that the option market maker is implementing a spread trading strategy to take advantage of the relative imbalance in supply and demand slope coefficients across strikes. In addition, we also observe that the magnitude of his inventory position on call K_2 is considerably larger than in the single option market.

Second, Figs. 7, 8 and 9 all point out to the fact that the bid-ask spreads are less sensitive as α_2 varies from 0 to 20,000. In particular, Fig. 9 clearly shows that, for call K_2, the spread mid-point in the two options economy is always above the same in the single option economy. In other words, the price impact of the demand elasticity α_2 variation is dampened by the introduction of call K_1 whose demand elasticity α_1 is strongly positive. In addition, and in contrast to the single option economy in particular and stock inventory models in general, we also observe a positive relationship between the bid-ask spread and the net inventory for call K_1 as α_2 varies from 0 to 20,000.

4.4. *Empirical implications*

Recent empirical research (see [10] for instance) has established that option-pricing models based on the assumption that the underlying stock follows a one-dimensional diffusion are typically rejected. Using intra-day data, Bakshi, Cao and Chen [4] investigate which option patterns are inconsistent with the one-dimensional class models. As a result, they uncover two broad types of violations. First, call prices do not always move in the same direction as the underlying stock (violations of the "monotonicity property"). Second, option prices are not perfectly correlated with each other (violations of the "perfect correlation property").

They further investigate whether a two-dimensional stochastic volatility model (SV model) like Heston's [19] could account better for the documented patterns. They conclude that, while the SV model has superior ability to explain violations of the "perfect correlation property", it cannot effectively account for violations of the "monotonicity property". The authors attribute this failure to the fact that, intra-day, volatility does not vary enough. They also suggest that microstructure factors may help to account for the observed patterns.

Like the SV model, our microstructure framework implies that options are not redundant securities. In addition, the numerical illustration provided in this section shows that, for a given underlying stock bid-ask spread, our model predicts that the mid-point of the option spread will vary according to the supply and demand curve on each strike. Hence, our model has the potential to account for violations of the "monotonicity property". On the other hand, under current specifications, it will not be effective in explaining "perfect correlation property" violations since, as the appendix shows, there is perfect correlation across strikes under the current specifications. However, we conjecture that introducing stochastic supply and demand functions will allow more flexibility. This is left for future research.

5. Multi-Periods Issues

In this section we formulate and discuss issues associated with the extension of our single-period model to a multi-period economy. This is necessarily an important extension if we wish to study the inter-temporal patterns of inventories and option spreads. In the remainder of the section, we provide a general solution to the option quote-setting problem based on the recursive dynamic programming approach. The numerical implementation of this solution however involves (as in Secs. 3 and 4), the specification of a parametric utility function. In addition, we argue that, when the random costs of option hedging portfolios are independent across time and states of the world then, the inter-temporal problem reduces to a sequence of static optimization problems.

We consider an economy where the securities (risk-free bond, stock and options) are traded at t discrete times ($0 \leq t \leq T < \infty$). As previously, the risk-free bond pays \$1 in any state of the world and trades for \$1 at time 0. The stock price follows a standard binomial multiplicative process starting from S_0 in state $\omega_0 \equiv 1$ at time 0 (see [23, pp. 72–106] for details). At time 1, all traders agree that the stock price process can move either "up" or "down" to the state $\omega_1 = \{u, d\}$. Next, at time 2, the process jumps to the state $\omega_2 = \{uu, ud, du, dd\}$. The state space process further evolves in this fashion until terminal time T. In each state, the stock price

S_{t,ω_t} is specified as follows.

$$S_{t,\omega_t} = \begin{cases} uS_{t,\omega_{t-1}} & \text{if} \quad \omega_t = u\omega_{t-1} \\ dS_{t,\omega_{t-1}} & \phantom{\text{if}} \quad \omega_t = d\omega_{t-1} \end{cases}, \tag{5.1}$$

for $1 \le t \le T$. It is well known that the arbitrage-free price of a call with exercise price K_i $(1 \le i \le N)$ is given by:

$$C_{i,T,\omega_T} = \mathrm{Max}(S_{T,\omega_T} - K_i, 0)\,, \tag{5.2a}$$

$$C_{i,t,\omega_t} = \pi C_{i,t+1,u\omega_t} + (1-\pi)C_{i,t+1,d\omega_t}\,, \tag{5.2b}$$

and $\pi = \frac{1-d}{u-d}$ is the so-called martingale probability (see [23, p. 104]), for all $t(0 \le t \le T-1)$.

Next, as in Sec. 2, we assume that there exist sets of trading ranges $[S^L_{\omega_t}, S^U_{\omega_t}]$ in which the stock price may be traded by the option market maker. Specifically, we impose:

$$S^L_{t,\omega_t} \equiv S_{\omega_t}(1-\tau)\,, \tag{5.3a}$$

$$S^U_{t,\omega_t} \equiv S_{\omega_t}(1+\tau)\,. \tag{5.3b}$$

If we assimilate this trading range to a quoted bid-ask spread, then this is equivalent to assuming a constant percentage spread on the underlying stock. Consequently, the lower and upper price bounds $\lfloor C^L_{\omega_t}, C^U_{\omega_t} \rfloor$ for a call with exercise price K_i $(1 \le i \le N)$ are given by:

$$C^L_{i,t,\omega_t} = s_{i,t,\omega_t} S^L_{i,t,\omega_t} + b_{i,t,\omega_t}\,, \tag{5.4a}$$

$$C^U_{i,t,\omega_t} = s_{i,t,\omega_t} S^U_{i,t,\omega_t} + b_{i,t,\omega_t}\,. \tag{5.4b}$$

$$s_{i,t,\omega_t} = \frac{C_{i,t,u\omega_t} - C_{i,t,d\omega_t}}{S_{t,u\omega_t} - S_{t,d\omega_t}}\,, \tag{5.5a}$$

$$b_{i,t,\omega_t} = C_{i,t,u\omega_t} - s_{i,t,\omega_t} S_{t,u\omega_t}\,. \tag{5.5b}$$

In addition, for all strike prices K_i $(1 \le i \le N)$ and all time period $t(0 \le t \le T-1)$, we specify the supply and demand functions as follows:

$$Q^A_{i,t,\omega_t} = I^A_{i,t,\omega_t} - \alpha_{i,t,\omega_t} C^A_{i,t,\omega_t}\,, \tag{5.6a}$$

$$Q^A_{i,t,\omega_t} = 0 \text{ if } C^A_{i,\omega_t} > C^U_{i,t,\omega_t}\,, \tag{5.6b}$$

$$Q^B_{i,t,\omega_t} = I^B_{i,t,\omega_t} + \beta_{i,t,\omega_t} C^B_{i,t,\omega_t}\,, \tag{5.6c}$$

$$Q^B_{i,t,\omega_t} = 0 \text{ if } C^B_{i,t,\omega_t} < C^U_{i,t,\omega_t}\,. \tag{5.6d}$$

To ensure economic viability of the market, as in Assumption 4.1, we impose:

$$I^A_{i,t,\omega_t} = \alpha_{i,t,\omega_t} C^U_{i,t,\omega_t}\,, \tag{5.7a}$$

$$I^B_{i,t,\omega_t} = -\beta_{i,t,\omega_t} C^L_{i,t,\omega_t}\,. \tag{5.7b}$$

Finally, we use the standard dynamic programming approach to determine the optimum quotes C^B_{i,t,ω_t} and C^A_{i,t,ω_t} that the market maker must set for all strike prices $K_i (1 \leq i \leq N)$. First, for all $t (1 \leq t \leq T-1)$, we specify the so-called derived utility of profit J_{t,ω_t} as follows:

$$J_{t,\omega_t} = \mathrm{Max}_{C^B_{i,t,\omega_t}, C^B_{i,t,\omega_t}} E_{t,\omega_t}(U(\Pi_{t,\omega_t}))\,, \tag{5.8}$$

and

$$\Pi_{t,\omega_t} = \sum_{i=1}^{N} Q^A_{i,t,\omega_t}(C^A_{i,t,\omega_t} - \tilde{C}_{i,t,\omega_t}) - \sum_{i=1}^{N} Q^B_{i,t,\omega_t}(C^B_{i,t,\omega_t} - \tilde{C}_{i,t,\omega_t})\,,$$

where U is a time-separable, additive Von-Neumann and Morgenstern utility function, $\tilde{C}_{i,t,\omega_t} = s_{i,t,\omega_t}\tilde{S}_{i,t,\omega_t} + b_{i,t,\omega_t}$ is the random cost of call K_i hedging portfolio and $E_{t,\omega_t}(\bullet)$ is the conditional expectation operator at time t and in state ω_t. Next, we iterate one time step backward and solve the following recursive maximization problem:

$$\begin{aligned}&\mathrm{Max}_{C^B_{i,t-1,\omega_{t-1}} C^A_{i,t-1,\omega_{t-1}}} E_{t-1,\omega_{t-1}}(U(\Pi_{t-1,\omega_{t-1}}) + \pi J_{t,u\omega_{t-1}} \\ &\quad + (1-\pi) J_{t,d\omega_{t-1}})\,,\end{aligned} \tag{5.9}$$

where

$$\begin{aligned}\Pi_{t-1,\omega_{t-1}} &= \sum_{i=1}^{N} Q^A_{i,t-1,\omega_{t-1}}(C^A_{i,t-1,\omega_{t-1}} - \tilde{C}_{i,t-1,\omega_{t-1}}) \\ &\quad - \sum_{i=1}^{N} Q^B_{i,t-1,\omega_{t-1}}(C^B_{i,t-1,\omega_{t-1}} - \tilde{C}_{i,t-1,\omega_{t-1}})\end{aligned}$$

and $\pi = \frac{1-d}{u-d}$ is the martingale probability.

Subsequently, the process described by expressions (5.8) and (5.9) is iterated until time 0 to obtain the optimum quotes C^B_{i,t,ω_t} and C^A_{i,t,ω_t} for all $t (0 \leq t \leq T-1)$.

Observe that, in its full generality, the dynamic programming solution depends on the inter-temporal joint distribution of the hedging portfolios $\tilde{C}_{i,t,\omega_t}$. If the former constitutes an independent random vector then, solving the dynamic programming problem is equivalent to solving a sequence of single-period problems (as in Sec. 3 and 4). However, if we wish to study the time series patterns of options quotes and market maker inventories then, we probably need to introduce a richer correlation structure. This is left for future research.

6. Conclusion

In this paper, we analyze the option market maker activity in a very different perspective than in the existing market microstructure literature. Indeed, we argue that option market making risk is essentially a hedging risk. Our analysis is consistent with actual option market making practices. Moreover, we show that, when a cross-section of exercise prices is available, the quote-setting problem is more complex since the market maker must take into account the imbalance between supply and demand elasticity coefficients across strikes.

Our approach produces several potentially important testable implications. First, we show that the dollar size of the option spreads depends only on the upper and lower no-arbitrage bounds on the option prices (but is independent of the supply and demand elasticity coefficients). Second, we show that, in a multi-option economy, the market maker optimal net inventory position is a spread position when there is imbalance between supply and demand elasticity coefficients across strikes. In addition, we conjecture that a richer spectrum of exercise prices available will induce larger net inventory positions. Recent empirical work by Cho and Engle [9] broadly supports these theoretical results. Indeed, these researchers informally put forward a new market microstructure theory, which they call "derivative hedge theory". Our paper instead rigorously derives such a theory.

There are several additional issues that could be addressed to extend this research. First, one could consider a model where the public supply and demand functions are stochastic when the market maker sets option quotes. Practically, this would amount to a market where additional traders compete with the market maker to provide liquidity. Hence, for a posted option spread the market maker may for instance purchase or sell fewer options than he was expecting at the time he posted his quotes. A second important application would be to use this theoretical structure in a risk management perspective. Indeed, since we are able to compute the expected profit as well as the standard deviation of the market maker profit, then, we should also be able to provide an accurate assessment of the amount of capital needed to run the option market making activity.

Appendix A

Using expressions (2.7) and (2.8), we write the profit function as:

$$\tilde{\Pi}(C^B, C^A) = -\alpha(C^A - C^U)(C^A - \tilde{C}) + \beta(C^B - C^L)(\tilde{C} - C^B)\,. \tag{A.1}$$

Then, we have:

$$E(\tilde{\Pi}(C^B, C^A)) = -\alpha(C^A - C^U)(C^A - \bar{C}) + \beta(C^B - C^L)(\bar{C} - C^B)\,, \tag{A.2}$$

and

$$\mathrm{Var}(\tilde{\Pi}(C^B C^A)) = \sigma^2(\alpha(C^A - C^U) + \beta(C^B - C^L))^2\,. \tag{A.3}$$

From expression (2.6), we immediately obtain that the replicating portfolio price $\tilde{C}$ is normally distributed with mean $\bar{C} = s(pS^L + (1-p)S^U) + b$ and standard deviation $v = s\sigma$.

Substituting into the objective function (3.4), and applying first order conditions leads to the following system of two linear equations in C^B and C^A:

$$\bar{C} - 2C^B + C^L - \lambda(\alpha(C^A - C^U) + \beta(C^B - C^L))v^2 = 0\,, \tag{A.4a}$$

$$2C^A - \bar{C} - C^U + \lambda(\alpha(C^A - C^U) + \beta(C^B - C^L))v^2 = 0\,. \tag{A.4b}$$

Solving this linear system yields the expressions for equilibrium prices in (3.5a) and (3.5b).

Appendix B

Using expressions (4.3), (4.4) and (4.5), we write the profit function as:

$$\begin{aligned}\tilde{\Pi}(C_i^B, C_1^A, C_2^B, C_2^A) &= -\alpha_1(C_1^A - C_1^U)(C_1^A - \tilde{C}_1) + \beta_1(C_1^B - C_1^L)(\tilde{C}_1 - C_1^L)\\ &\quad - \alpha_2(C_2^A - C_2^U)(C_2^A - \tilde{C}_2) + \beta_2(C_2^B - C_2^L)(\tilde{C}_2 - C_2^L)\,.\end{aligned} \tag{B.1}$$

Next, we have:

$$\begin{aligned}E(\tilde{\Pi}(C_1^B, C_1^A, C_2^B, C_2^A)) &= -\alpha_1(C_1^A - C_1^U)(C_1^A - \bar{C}_1) + \beta_1(C_1^B - C_1^L)(\bar{C}_1 - C_1^L)\\ &\quad - \alpha_2(C_2^A - C_2^U)(C_2^A - \bar{C}_2) + \beta_2(C_2^B - C_2^L)(\bar{C}_2 - C_2^L)\,,\end{aligned} \tag{B.2}$$

and

$$\begin{aligned}\mathrm{Var}(\tilde{\Pi}(C_1^B, C_1^A, C_2^B, C_2^A)) &= (\alpha_1(C_1^A - C_1^U) + \beta_1(C_1^B - C_1^L))^2\mathrm{Var}(\tilde{C}_1)\\ &\quad + (\alpha_2(C_2^A - C_2^U) + \beta_2(C_2^B - C_2^L))^2\mathrm{Var}(\tilde{C}_2)\\ &\quad + 2(\alpha_1(C_1^A - C_1^U) + \beta_1(C_1^B - C_1^L))(\alpha_2(C_2^A - C_2^U)\\ &\quad + \beta_2(C_2^B - C_2^L))\mathrm{Cov}(\tilde{C}_1, \tilde{C}_2)\,.\end{aligned} \tag{B.3}$$

Applying (2.6) to call K_1 and K_2, it is clear that the replicating portfolios $\tilde{C}_1$ and $\tilde{C}_2$ are normally distributed with respective mean $\bar{C}_1 = s_1(pS^B + (1-p)S^A) + b_1(\bar{C}_2 = s_2(pS^B + (1-p)S^A) + b_2)$, respective standard deviation $v_1 = s_1\sigma(v_2 = s_2\sigma)$ and correlation coefficient $\rho_{1,2} = 1$.

Substituting into the objective function (4.7), and applying first order conditions leads to the following system of four linear equations in C_1^B, C_1^A, C_2^B and C_2^A:

$$\begin{aligned}&\bar{C}_1 - 2C_1^B + C_1^L - \lambda\sigma^2(s_1^2(\alpha_1(C_1^U - C_1^A) + \beta_1(C_1^L - C_1^B))\\ &\qquad + s_1s_2((\alpha_2(C_2^U - C_2^A) + \beta_2(C_2^L - C_2^B)))) = 0\,,\end{aligned} \tag{B.4a}$$

$$\begin{aligned}&2C_1^A - \bar{C}_1 - C_1^U + \lambda\sigma^2(s_1^2(\alpha_1(C_1^U - C_1^A) + \beta_1(C_1^L - C_1^B))\\ &\qquad + s_1s_2((\alpha_2(C_2^U - C_2^A) + \beta_2(C_2^L - C_2^B)))) = 0\,,\end{aligned} \tag{B.4b}$$

$$\begin{aligned}&\bar{C}_2 - 2C_2^B + C_2^L - \lambda\sigma^2(s_2^2(\alpha_2(C_2^U - C_2^A) + \beta_2(C_2^L - C_2^B))\\ &\qquad + s_1s_2((\alpha_1(C_1^U - C_1^A) + \beta_1(C_1^L - C_1^B)))) = 0\,,\end{aligned} \tag{B.4c}$$

$$2C_2^A - \bar{C}_2 - C_2^U + \lambda\sigma^2(s_2^2(\alpha_2(C_2^U - C_2^A) + \beta_2(C_2^L - C_2^B)) + s_1 s_2((\alpha_1(C_1^U - C_1^A) + \beta_1(C_1^L - C_1^B)))) = 0\,. \quad \text{(B.4d)}$$

Solving this linear system yields the expressions for equilibrium prices in (4.9).

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