Chapter 7. Moving Beyond Linearity

1. a) By construction, for all $x \le \xi$:

$$f(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3$$
$$= f_1(x)$$

Then we have: $a_1 = \beta_0$, $b_1 = \beta_1$, $c_1 = \beta_2$, $d_1 = \beta_3$

b) For all $x > \xi$:

$$f(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 (x-\xi)^3$$
$$= \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 (x^3 - 3\xi x^2 + 3\xi^2 x - \xi^3)$$
$$= (\beta_0 - \beta_4 \xi^3) + (\beta_1 + 3\beta_4 \xi^2) x + (\beta_2 - 3\beta_4 \xi) x^2 + (\beta_3 + \beta_4) x^3$$
$$= a_2 + b_2 x + c_2 x^2 + d_2 x^3$$
$$= f_2(x)$$

Then we have: $a_2 = \beta_0 - \beta_4 \xi^3$, $b_2 = \beta_1 + 3\beta_4 \xi^2$
$c_2 = \beta_2 - 3\beta_4 \xi$, $d_2 = \beta_3 + \beta_4$

c) Plug in ξ at $f_1(x)$: $f_1(x) = \beta_0 + \beta_1 \xi + \beta_2 \xi^2 + \beta_3 \xi^3$
Plug in ξ at $f_2(x)$: $f_2(x) = \beta_0 - \beta_4 \xi^3 + (\beta_1 + 3\beta_4 \xi^2)\xi + (\beta_2 - 3\beta_4 \xi)\xi^2 + (\beta_3 + \beta_4)\xi^3$

$$= \beta_0 + \beta_1 \xi + \beta_2 \xi^2 + \beta_3 \xi^3 + (-\beta_4 \xi^3 + 3\beta_4 \xi^3 - 3\beta_4 \xi^3 + \beta_4 \xi^3)$$

$$= \beta_0 + \beta_1 \xi + \beta_2 \xi^2 + \beta_3 \xi^3$$

Hence $f_1(\xi) = f_2(\xi)$

d) $f_1'(x) = \beta_1 + 2\beta_2 x + 3\beta_3 x^2$
Plug in ξ, we get: $f_1'(\xi) = \beta_1 + 2\beta_2 \xi + 3\beta_3 \xi^2$

$f_2'(x) = (\beta_1 + 3\beta_4 \xi^2) + 2(\beta_2 - 3\beta_4 \xi) x + 3(\beta_3 + \beta_4) x^2$
Plug in ξ, we get: $f_2'(\xi) = \beta_1 + 3\beta_4 \xi^2 + 2(\beta_2 - 3\beta_4 \xi)\xi + 3(\beta_3 + \beta_4)\xi^2$

$$= \beta_1 + 2\beta_2 \xi + 3\beta_3 \xi^2 + (3\beta_4 \xi^2 - 6\beta_4 \xi^2 + 3\beta_4 \xi^2)$$
$$= \beta_1 + 2\beta_2 \xi + 3\beta_3 \xi^2$$

Hence $f_1'(\xi) = f_2'(\xi)$.

e) $f_1''(x) = 2\beta_2 + 6\beta_3 x$

Plug in ξ : $f_1''(\xi) = 2\beta_2 + 6\beta_3 \xi$

$f_2''(x) = 2(\beta_2 - 3\beta_4\xi) + 6(\beta_3 + \beta_4)x$

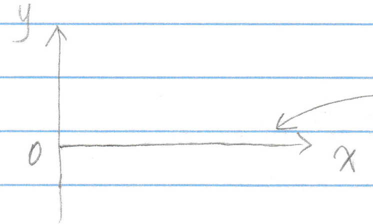
Plug in ξ :

$$
\begin{aligned}
f_2''(\xi) &= 2(\beta_2 - 3\beta_4\xi) + 6(\beta_3 + \beta_4)\xi \\
&= 2\beta_2 + 6\beta_3\xi + (-6\beta_4\xi + 6\beta_4\xi) \\
&= 2\beta_2 + 6\beta_3\xi
\end{aligned}
$$

Hence $f_1''(\xi) = f_2''(\xi)$

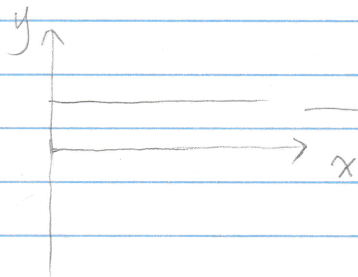
2.

a)



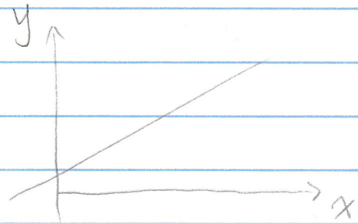
The fitted line overlaps with the x Axis.
The fitted line is $y=0$.

b)



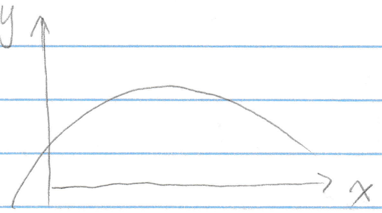
The fitted line is $g(x_i) = \mathbb{E}(y)$, $\forall i$

c)



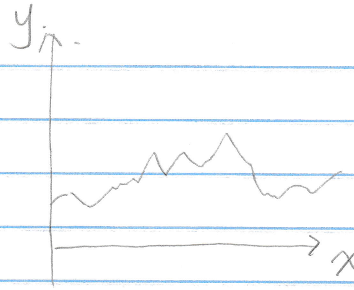
The fitted line is of the form $g(x) = ax + b$.

d)



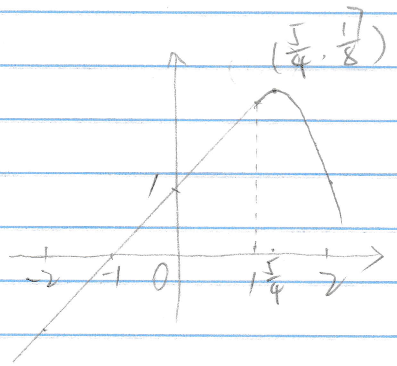
The fitted line is of the form $g(x) = ax^2 + bx + c$.

e)



$\longrightarrow$ The fitted line is the interpolating spline.

3.

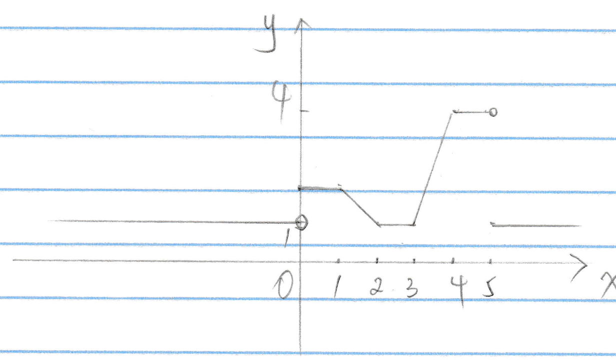


$$\begin{cases} Y = 1 + X & x < 1 \\ Y = 1 + X - 2(X-1)^2 & x \geq 1 \end{cases}$$

4 The function:

$$y = \begin{cases} 1 & x < 0 \\ 2 & 0 \leq x < 1 \\ 3 - x & 1 \leq x < 2 \\ 1 & 2 \leq x < 3 \\ 3x - 8 & 3 \leq x < 4 \\ 4 & 4 \leq x < 5 \\ 1 & x \geq 5 \end{cases}$$

The plot:



5

a) $\hat{g}_2$ has smaller trainning RSS

b) $\hat{g}_1$ has smaller test RSS

c) In this case, $\hat{g}_1 = \hat{g}_2$ and they have the same trainning and test RSS