CSC165: Problem Set 3

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1. (a) **Define P(n):** "$d_{2n-1} \leq \sqrt{2n-1}$", where $n \in \mathbb{Z}^+$. We want to prove that $\forall$n $\in \mathbb{Z}^+$, P(n). We are going to use proof by induction on n to prove this statement

<u>Base Case:</u> Let n = 1. Then we have that $d_{2\times 1-1} = d_1 = \frac{1}{d_0} = \frac{1}{1} = 1$. This conclusion comes from the way we defined d_n. We also notice that $\sqrt{2n-1} = \sqrt{1} = 1$. Therefore $1 \leq 1$ So, $d_{2n-1} \leq \sqrt{2n-1}$ for the base case, as required.

<u>Induction Hypothesis:</u> Let $k \in \mathbb{Z}^+$. We assume P(k), i.e, $d_{2k-1} \leq \sqrt{2k-1}$. We want to show P(k+1), i.e. that $d_{2(k+1)-1} \leq \sqrt{2(k+1)-1}$. That is, we want to show $d_{2k+1} \leq \sqrt{2k+1}$.
We notice by the definition of d_n that $d_{2k+1} = \frac{2k+1}{d_{2k}}$ and that $d_{2k} = \frac{2k}{d_{2k-1}}$. This is true because both subscripts are greater than 0(because k is a positive integer, $2k+1 > 0$ and $2k-1 > 0$).
Combining these two and because we assume P(k) and both sides are greater than zero for every n in the positive integers, we notice that:

$$\begin{aligned} d_{2k+1} &= \frac{(2k+1)\cdot d_{2k-1}}{2k} \\ &\leq \frac{(2k+1)\cdot\sqrt{2k-1}}{2k} \; (\mathit{Because\ from\ our\ inductive\ hypothesis\ ,we\ know}\ d_{2k-1} \leq \sqrt{2k-1}) \end{aligned}$$

Then, we can calculate:

$$\begin{aligned} d_{2k+1} &\leq \frac{(2k+1)\cdot\sqrt{2k-1}}{2k} \\ &\leq \sqrt{\frac{(2k+1)^2\cdot(2k-1)}{4k^2}} \\ &\leq \sqrt{\frac{(2k+1)\cdot(2k+1)\cdot(2k-1)}{4k^2}} \\ &\leq \sqrt{\frac{(2k+1)\cdot(4k^2-1)}{4k^2}} \; (\mathit{Because}\ (2k+1)(2k-1) = 4k^2-1) \\ &\leq \sqrt{\frac{(2k+1)\cdot(4k^2-1)}{4k^2-1}} \; (\mathit{Since}\ k \in \mathbb{Z}^+\ , 4k^2 \geq 4k^2-1\ \mathit{and\ they\ are\ both\ greater\ than}\ 0, \mathit{so}\ \frac{1}{4k^2} \leq \frac{1}{4k^2-1}) \\ &\leq \sqrt{(2k+1)} \end{aligned}$$

Therefore we have showed that P(k+1) holds which leads to conclude that $\forall n \in \mathbb{Z}^+$, P(n).

(b) *Proof.* Let us start by defining the predicate:
P(n): "$d_{2n} > \sqrt{2n}$", where $n \in \mathbb{N}$.
We will prove, $\forall n \in \mathbb{N}, P(n)$ by induction on n.

Base Case: Let n = 0. Then by definition, we know $d_{2n} = d_0 = \sqrt{0} = 1$.
And $\sqrt{2n} = \sqrt{2 \times 0} = \sqrt{0} = 0$.
So, $d_{2n} > \sqrt{2n}$.
Hence, P(0) holds.

Inductive Step: Let $k \in \mathbb{N}$ and assume P(k), i.e, that $d_{2k} > \sqrt{2k}$. We want to show that P(k+1), i.e, $d_{2(k+1)} > \sqrt{2(k+1)}$.
First, by definition of d_n, we have:

$$d_{2(k+1)} = \frac{2(k+1)}{d_{2(k+1)-1}} \qquad (\textit{Because } k \in \mathbb{N}, \textit{ so } 2(k+1) \textit{ is always greater than } 0)$$
$$d_{2(k+1)} = \frac{2(k+1)}{d_{2k+1}}$$
$$d_{2(k+1)} = \frac{2k+2}{d_{2k+1}}$$

And since $k \in \mathbb{N}$, then $k+1 \in \mathbb{Z}^+$, so from the conclusion of (a), we know that $d_{2(k+1)-1} \leq \sqrt{2(k+1)-1}$. That is, $d_{2k+1} \leq \sqrt{2k+1}$. And also by the definition of d_n, we know that, since $k \in \mathbb{N}$, then $2k+1 > 0$ and $d_{2k+1} > 0$. So, we get $\frac{1}{d_{2k+1}} \geq \frac{1}{\sqrt{2k+1}}$.
And we know that $d_{2(k+1)} = \frac{2k+2}{d_{2k+1}}$
So, $d_{2(k+1)} \geq \frac{2k+2}{\sqrt{2k+1}}$ (since $\frac{1}{d_{2k+1}} \geq \frac{1}{\sqrt{2k+1}}$)
So $d_{2(k+1)} > \frac{2k+1}{\sqrt{2k+1}}$ (Because $2k+2 > 2k+1$)
$d_{2(k+1)} > \frac{(2k+1)\sqrt{2k+1}}{\sqrt{2k+1}\sqrt{2k+1}}$
$d_{2(k+1)} > \frac{(2k+1)\sqrt{2k+1}}{2k+1}$
$d_{2(k+1)} > \sqrt{2k+1}$
Therefore, P(k+1) also holds. So we completed the proof for $\forall n \in \mathbb{N}, P(n)$. □

2. (a) (i): The decimal value of the balanced ternary number $(T011T)_{bt}$ is:
$(-1) \times 3^4 + 0 \times 3^3 + 1 \times 3^2 + 1 \times 3^1 + (-1) \times 3^0$
$= -81 + 0 + 9 + 3 - 1$
$= -70$
(ii): Because we know that:

$$\begin{aligned} 210 &= 243 - 27 - 9 + 3 \\ &= 1 \times 3^5 - 1 \times 3^3 - 1 \times 3^2 + 1 \times 3^1 \\ &= 1 \times 3^5 + 0 \times 3^4 + (-1) \times 3^3 + (-1) \times 3^2 + 1 \times 3^1 + 0 \times 3^0 \end{aligned}$$

So, by the definition of balanced ternary, we know that 210 can be represented as $(10TT10)_{bt}$.

(b) *Proof.* We will Use proof by induction for this proof

Define Predicate:
$P(n)$: "$6 \mid 3^n - 3$",where $n \in \mathbb{Z}^+$

We want to show $\forall n \in \mathbb{Z}^+, P(n)$

Base case:
Let $n = 1$. We want to prove $P(1)$.
$3^n - 3 = 3^1 - 3 = 3 - 3 = 0$

We will show $6 \mid 0$. That is, $\exists k \in \mathbb{Z}, 0 = 6k$.
Let $k = 0$. Then we have $0 = 6 \cdot 0 = 6k$. So, $6 \mid 3^n - 3$. Base case holds.

Inductive Step:
We want to show $\forall k \in \mathbb{N}, P(k) \Rightarrow P(k+1)$.
Let $k \in \mathbb{N}$ and assume $P(k)$ is true. We want to show $P(k+1)$. That is, $\exists t \in \mathbb{Z}, 3^{k+1} - 3 = 6t$.

From inductive hypothesis we know, $6 \mid 3^k - 3$. So we know there exist $t_0 \in \mathbb{Z}$ such that $3^k - 3 = 6 \cdot t_0$.
Let $t = 3 \cdot t_0 + 1$. Then,

$$\begin{aligned}
3^k - 3 &= 6t_0 \\
3(3^k - 3) &= 6t_0 \cdot 3 \\
3^{k+1} - 9 &= 18 \cdot t_0 \\
3^{k+1} - 9 + 6 &= 18t_0 + 6 \\
3^{k+1} - 3 &= 18t_0 + 6 \\
3^{k+1} - 3 &= 6(3t_0 + 1) \\
3^{k+1} - 3 &= 6t
\end{aligned}$$

Hence, $6 \mid 3^{k+1} - 3$. So, $P(k+1)$ holds.
We have shown $\forall n \in \mathbb{Z}^+, 6 \mid 3^n - 3$. The proof is completed.

$\square$

(c) *Proof.* We will Use proof by induction on n for this proof

Define Predicate:
$P(n)$: "$\forall x \in \mathbb{N}, \left(\exists d_0, d_1, \cdots, d_{n-2}, d_{n-1} \in \{0,1\}, x = \sum_{i=0}^{n-1} d_i \cdot 3^i\right) \Longrightarrow \left(6 \nmid x - 2 \wedge 6 \nmid x - 5\right)$", where $n \in \mathbb{Z}^+$.
Also, we say a positive integer x is n-positively balanced if and only if $\exists d_0, d_1, \cdots, d_{n-2}, d_{n-1} \in \{0,1\}$ such that x can be expressed as $(d_{n-1}d_{n-2}\cdots d_0)_{bt}$. In this case, $x = \sum_{i=0}^{n-1} d_i \cdot 3^i$.

Base case:
Let n = 1. We want to show P(1) is true.
That is, Let $x \in \mathbb{N}$, assume x is 1-digit positively balanced. We want to show $6 \nmid x - 2 \wedge 6 \nmid x - 5$.
Since x is 1-digit positively balanced, $x = (0)_{bt}$ or $x = (1)_{bt}$. We are going to show that base case holds in two parts.

Part 1: $x = (0)_{bt}$
$x = (0)_{bt} = 0 \cdot 3^0 = 0$
If $x = 0$, then $\frac{x-2}{6} = \frac{0-2}{6} = -\frac{2}{6} \notin \mathbb{Z}$
Also, $\frac{x-5}{6} = \frac{0-5}{6} = -\frac{5}{6} \notin \mathbb{Z}$
So, $6 \nmid x - 2 \wedge 6 \nmid x - 5$

Part 2: $x = (1)_{bt}$
$x = (1)_{bt} = 1 \cdot 3^0 = 1$
If $x = 1$, then $\frac{x-2}{6} = \frac{1-2}{6} = -\frac{1}{6} \notin \mathbb{Z}$
Also, $\frac{x-5}{6} = \frac{1-5}{6} = -\frac{4}{6} \notin \mathbb{Z}$
So, $6 \nmid x - 2 \wedge 6 \nmid x - 5$

Therefore we have shown the base case P(1) holds.

Inductive Step:
Let $k \in \mathbb{Z}^+$. We want to show $P(k) \Longrightarrow P(k+1)$ for all $k \in \mathbb{Z}^+$.

Assume $P(k)$ is true. That is, $\forall x \in \mathbb{N}, (x \text{ is k-digit positively balanced}) \Rightarrow 6 \nmid x-2 \wedge 6 \nmid x-5$.
We want to show $P(k+1)$. That is,
$\forall x \in \mathbb{N}, (x \text{ is (k+1)-digit positively balanced}) \Rightarrow 6 \nmid x-2 \wedge 6 \nmid x-5$.

Let $y \in \mathbb{N}$ and assume y is $k+1$ digit positively balanced which can be expressed as $(d_k d_{k-1} \cdots d_0)_{bt}$ where $d_i \in \{0,1\}$

We want to show that $6 \nmid y-2 \wedge 6 \nmid y-5$.

Also, let $t = (d_{k-1} d_{k-2} \cdots d_0)_{bt}$, by definition of positively balanced, we know that t is k-digit positively balanced.

Then we know from inductive hypothesis that $\forall x \in \mathbb{N}, (x \text{ is k-digit positively balanced}) \Rightarrow 6 \nmid x-2 \wedge 6 \nmid x-5$, and since from our assumption t is k-digit positively balanced, so $6 \nmid t-2 \wedge 6 \nmid t-5$.

And since $t = (d_{k-1} d_{k-2} \cdots d_0)_{bt}$, and $y = (d_k d_{k-1} \cdots d_0)_{bt}$, so from the defeinition of balanced ternary and positively balanced, we know that d_k is 0 or 1. And $y = t + d_k \times 3^k$
We will use prove $6 \nmid y-2 \wedge 6 \nmid y-5$ by cases depending on the value of d_k; $d_k = 0$ and $d_k = 1$.

Case 1: $d_k = 0$
Then, $y = t + 0 \times 3^k = t$
From our induction hypothesis, we know that $6 \nmid t-2 \wedge 6 \nmid t-5$.
Therefore, $6 \nmid y-2 \wedge 6 \nmid y-5$.

Case 2: $d_k = 1$
Then, $y = t + 1 \times 3^k = t + 3^k$
we are going to show that $6 \nmid y-2 \wedge 6 \nmid y-5$.

Part 1: Prove $6 \nmid y-2$
From b), we know since $k \in \mathbb{Z}^+$, then $6 \mid 3^k - 3$. So, $\exists m \in \mathbb{Z}, 3^k - 3 = 6m$.

$$\begin{aligned}
\text{Then,} \quad y-2 &= t + 3^k - 2 \\
y-2 &= t + (3^k - 3) + 1 \\
y-2 &= t + 6m + 1 \\
\frac{y-2}{6} &= \frac{t+1}{6} + m \qquad (1)
\end{aligned}$$

We will show that $\frac{y-2}{6} \notin \mathbb{Z}$:
From Quotient Remainder Theorem (QRT), we know since $t \in \mathbb{Z}$ and $6 \in \mathbb{Z}^+$, there exist $q, r \in \mathbb{Z}$ such that $t = 6q + r$ and $0 \le r < 6$ where r is unique. We will show $r \neq 5$ by contradiction.

Assume $r = 5$, then $t = 6 \cdot q + 5$.
We will show that $6 \mid t-5$, i.e, $i \in \mathbb{Z}, t-5 = 6 \times i$.
Let i = q. Then, we have:
$t - 5 = 6q + 5 - 5 = 6q = 6i$
So, $6 \mid t-5$ and this is contradiction to our previous conclusion $6 \nmid t-5$..
Therefore, $r \neq 5$.

Now, back to (1),
$$\begin{aligned}\frac{y-2}{6} &= \frac{t+1}{6} + m\\ &= \frac{6q+r+1}{6} + m\\ &= \frac{r+1}{6} + q + m\end{aligned}$$

Since $r \in \mathbb{Z}, 0 \leq r < 6$ and $r \neq 5$, r can only be 0, 1, 2, 3, 4. But we know that:

$$\begin{aligned}\frac{0+1}{6} &= \frac{1}{6} \notin \mathbb{Z}\\ \frac{1+1}{6} &= \frac{1}{3} \notin \mathbb{Z}\\ \frac{2+1}{6} &= \frac{1}{2} \notin \mathbb{Z}\\ \frac{3+1}{6} &= \frac{2}{3} \notin \mathbb{Z}\\ \frac{4+1}{6} &= \frac{5}{6} \notin \mathbb{Z}\end{aligned}$$

So, there is no integer r that satisfies $\frac{r+1}{6} \in \mathbb{Z}$. Therefore, $\frac{r+1}{6} \notin \mathbb{Z}$, then $\frac{r+1}{6} + q + m \notin \mathbb{Z}$ (since $m, q \in \mathbb{Z}$). And also from our previous induction we know that $\frac{y-2}{6} = \frac{r+1}{6} + q + m$, so $6 \nmid y - 2$.

<u>Part 1: Prove $6 \nmid y - 5$</u>
From b), we know $6 \mid 3^k - 3$. since $k \in \mathbb{Z}^+$ So, $\exists m \in \mathbb{Z}, 3^k - 3 = 6m$.

Then,
$$\begin{aligned}y-5 &= t + 3^k - 5\\ y-5 &= t + 6m - 2\\ \frac{y-5}{6} &= \frac{t-2}{6} + m \qquad (2)\end{aligned}$$

We will show that $\frac{y-5}{6} \notin \mathbb{Z}$:
From Quotient Remainder Theorem (QRT), we know since $t \in \mathbb{Z}$ and $6 \in \mathbb{Z}^+$, there exist $q, r \in \mathbb{Z}$ such that $t = 6q + r$ and $0 \leq r < 6$ where r is unique. We will show $r \neq 5$ by contradiction.

Assume $r = 2$, then $t = 6 \cdot q + 2$. We will show that $6 \mid t - 5$, i.e, $i \in \mathbb{Z}, t - 5 = 6 \times i$.
Let i = q. Then, we have:
$t - 2 = 6q + 2 - 2 = 6q = 6i$
So, $6 \mid t - 2$ and this is contradiction to our previous conclusion $6 \nmid t - 2$..
Therefore, $r \neq 2$.

Now, back to (2),
$$\begin{aligned}\frac{y-5}{6} &= \frac{t-2}{6} + m\\ &= \frac{6q+r-2}{6} + m\\ &= \frac{r-2}{6} + q + m\end{aligned}$$

Since $r \in \mathbb{Z}, 0 \leq r < 6$ and $r \neq 2$, r can only be 0, 1,, 3, 4, 5. But we know that:

$$\frac{0-2}{6} = -\frac{1}{3} \notin \mathbb{Z}$$
$$\frac{1-2}{6} = -\frac{1}{6} \notin \mathbb{Z}$$
$$\frac{3-2}{6} = \frac{1}{6} \notin \mathbb{Z}$$
$$\frac{4-2}{6} = \frac{1}{3} \notin \mathbb{Z}$$
$$\frac{5-2}{6} = \frac{1}{2} \notin \mathbb{Z}$$

So, there is no integer r that satisfies $\frac{r-2}{6} \in \mathbb{Z}$. Therefore, $\frac{r-2}{6} \notin \mathbb{Z}$, then $\frac{r-2}{6}+q+m \notin \mathbb{Z}$ (since $m, q \in \mathbb{Z}$). And also from our previous induction we know that $\frac{y-5}{6} = \frac{r-2}{6}+q+m$, so $6 \nmid y-5$.

Hence, we have proven that $\forall k \in \mathbb{Z}^+, \forall y \in \mathbb{N}, (y$ is $(k+1)$-digit positively balanced$) \Rightarrow 6 \nmid y-2 \wedge 6 \nmid y-5$.

Combined with our basic case, we have completed the proof that
$\forall n \in \mathbb{Z}^+, \forall x \in \mathbb{N}, (x$ is n-digit positively balanced$) \Rightarrow 6 \nmid x-2 \wedge 6 \nmid x-5$.

□

3. (a) We are going to disprove it by proving its negation: $\forall k \in \mathbb{N}, n^n \notin \mathcal{O}(n^k)$, i.e, $\forall k \in \mathbb{N}, \forall c, n_0 \in R^+, \exists n \in \mathbb{N}, n \geq n_0 \wedge n^n > c \cdot n^k$.

Let $k \in \mathbb{N}$. Also let $c, n_0 \in R^+$. We want to show that $\exists n \in \mathbb{N}, n \geq n_0 \wedge (n^n > c \cdot n^k)$.

Let n $= \lceil n_0 + c + k + 1 \rceil$.

Since we know that $n_0, c \in R^+$, so, $n_0 > 0, c > 0$.

And since $k \in \mathbb{N}$, so $k \geq 0$, then $k+1 \geq 1$.

Therefore, $n_0 + c + k + 1 > n_0$, $n_0 + c + k + 1 > c$, and $n_0 + c + k + 1 > k + 1$.

So, by the definition of ceiling, we know that:
$\lceil n_0 + c + k + 1 \rceil \geq n_0 + c + k + 1 > n_0$
$\lceil n_0 + c + k + 1 \rceil \geq n_0 + c + k + 1 > c$
$\lceil n_0 + c + k + 1 \rceil \geq n_0 + c + k + 1 > k + 1$

Thus, $n > n_0$, $n > c$, and $n > k+1$. First, $n \geq n_0$ is satisfied.
And since $n > k+1$ and $k \geq 0$, so $n > 1$. Note that we define $0^0 = 0$ for this question.
Then we can calculate:

$$n^n = n \cdot n^{n-1}$$
$$n^n > c \cdot n^{n-1} \ (Because\, n > c\, from\, our\, previous\, induction)$$
$$n^n > c \cdot n^k \ (Because\, n > k+1,\ so\, n-1 > k)$$

Therefore we have proved the negation as required, which disproves the original statement.

(b) We are going to prove that $165n^5 + n^2 \in \mathcal{O}(n^5 - n^3)$, i.e, $\exists n_0, c \in R^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow 165n^5 + n^2 \leq c \cdot (n^5 - n^3)$.

Let $n_0 = 3$, $c = 200$. And Let $n \in \mathbb{N}$, assume $n \geq n_0$.
So, we have:

$$\begin{aligned}
c \cdot (n^5 - n^3) &= 200 \cdot (n^5 - n^3) \\
&= 200n^5 - 200n^3 \\
&= 165n^5 + 35n^5 - 200n^3 \; (Because\ 200n^5 = 165n^5 + 35n^5) \\
&= 165n^5 + 35n^2 \cdot n^3 - 200n^3 \; (Because\ n^5 = n^2 \cdot n^3) \\
&\geq 165n^5 + 35 \times 9 \cdot n^3 - 200n^3 \; (Since\ n \geq n_0\ and\ n_0 = 3,\ so\ n \geq 3. Thus,\ n^2 \geq 3^2,\ so\ n^2 \geq 9) \\
&\geq 165n^5 + 315n^3 - 200n^3 \\
&\geq 165n^5 + 115n^3 \\
&\geq 165n^5 + 115 \cdot n \cdot n^2 \\
&\geq 165n^5 + n^2 \; (Because\ n \geq 3\ so\ 115 \cdot n \geq 1)
\end{aligned}$$

Hence, we have proved that $\exists n_0, c \in R^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow 165n^5 + n^2 \leq c \cdot (n^5 - n^3)$. That is, $165n^5 + n^2 \in \mathcal{O}(n^5 - n^3)$.

(c) We will be disproving the statement: $4^{n^2} \in \Theta(4^{n^2+n})$ by proving its negation: $4^{n^2} \notin \Theta(4^{n^2+n})$, i.e, $4^{n^2} \notin \mathcal{O}(4^{n^2+n}) \vee 4^{n^2} \notin \Omega(4^{n^2+n})$.

For the negation to be true we just need to prove $4^{n^2} \notin \mathcal{O}(4^{n^2+n})$ or $4^{n^2} \notin \Omega(4^{n^2+n})$. We will prove $4^{n^2} \notin \Omega(4^{n^2+n})$, i.e. $\forall c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, n \geq n_0 \wedge 4^{n^2} < c \cdot 4^{n^2+n}$.

Let $c, n_0 \in \mathbb{R}^+$. Let $n = max(\lceil n_0 \rceil, \lceil -log_4 c \rceil + 1)$. By the way we defined n, $n \geq \lceil n_0 \rceil \wedge n \geq (\lceil -log_4 c \rceil + 1)$
So, $n \geq n_0 \wedge n \geq -log_4 c + 1$, by the definition of ceiling functions.

And since $n \geq -log_4 c + 1$, we know that $n > -log_4 c$.

Then we can calculate:

$$\begin{aligned}
n &> -log_4 c \\
n + log_4 c &> 0 \\
n^2 + n + log_4 c &> n^2
\end{aligned}$$

Rewrite n^2+n as $log_4(4^{n^2+n})$ and rewrite n^2 as $log_4 4^{n^2}$ which we can do by properties of logarithmic functions. We have:

$$\begin{aligned}
log_4 4^{n^2+n} + log_4 c &> log_4 4^{n^2} \\
log_4 c \cdot 4^{n^2+n} &> log_4 4^{n^2} \; (from\ the\ properties\ of\ logarithms).
\end{aligned}$$

And we then notice that because the base $4 > 1$, so, $log_4 n$ is a strictly increasing function (non-decreasing). So, $\forall x, y \in R^{\geq 0}, x \leq y \Rightarrow log_4 x \leq log_4 y$. And the contrapositive is $\forall x, y \in R^{\geq 0}, log_4 x > log_4 y \Rightarrow x > y$
And we know since $c \in \mathbb{R}^+, n \in \mathbb{N}$, so $c \cdot 4^{n^2+n}, 4^{n^2} \in R^+$. And because $log_4 c \cdot 4^{n^2+n} > log_4 4^{n^2}$, so $c \cdot 4^{n^2+n} > 4^{n^2}$. That is, $4^{n^2} < c \cdot 4^{n^2+n}$.

Therefore we have proved the negation as required, which disproves the original statement.

(d) We want to prove that for every function $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$, if f is non-decreasing and $f(n) = n^2$ for every $n \in \mathbb{N}$ that is a power of two, then $f \in \Theta(n^2)$, i.e we want to prove:

$\forall f : \mathbb{N} \to \mathbb{R}^{\geq 0}, (\forall x, y \in \mathbb{N}, x \leq y \Rightarrow f(x) \leq f(y)) \land (\forall n \in \mathbb{N}, (\exists k \in \mathbb{N}, n = 2^k) \Rightarrow f(n) = n^2) \Rightarrow f \in \mathcal{O}(n^2) \land f \in \Omega(n^2)$.

<u>$proof:$</u>
Let $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$. Assume $(\forall x, y \in \mathbb{N}, x \leq y \Rightarrow f(x) \leq f(y))$ and $(\forall n \in \mathbb{N}, (\exists k \in \mathbb{N}, n = 2^k) \Rightarrow f(n) = n^2)$. We want to show $f \in \mathcal{O}(n^2) \land f \in \Omega(n^2)$.

<u>**Part 1:** $(f \in \mathcal{O}(n^2))$:</u> We are going to prove $f \in \mathcal{O}(n^2)$, i.e, $\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow f(n) \leq c \cdot n^2$.

Let $c = 4, n_0 = 1$. Let $n \in \mathbb{N}$, and assume that $n \geq n_0$. We are going to show that $f(n) \leq c \cdot n^2$

First, Since we know that $n = 2^{log_2 2^n}$ by properties of logarithmic functions and by the definition of ceiling we know that $log_2 2^n \leq \lceil log_2 2^n \rceil$, then $n = 2^{log_2 2^n} \leq 2^{\lceil log_2 2^n \rceil}$

And since from our assumption, we know that $\forall x, y \in \mathbb{N}, x \leq y \Rightarrow f(x) \leq f(y)$, so we get $f(n) = f(2^{log_2 2^n}) \leq f(2^{\lceil log_2 2^n \rceil})$.

Since base $2 > 0$, $\lceil log_2 2^n \rceil \geq \lceil log_2 2^0 \rceil$ (because $n \in \mathbb{N}$, so $n \geq 0$), so $\lceil log_2 2^n \rceil \geq 0$, Therefore $\lceil log_2 2^n \rceil \in \mathbb{N}$.
And we are going to show that $2^{\lceil log_2 2^n \rceil}$ is a power of 2, i.e, $\exists k \in \mathbb{N}, 2^{\lceil log_2 2^n \rceil} = 2^k$.

Let $k = \lceil log_2 2^n \rceil$.

Then, $2^{\lceil log_2 2^n \rceil} = 2^k$. So, $2^{\lceil log_2 2^n \rceil}$ is a power of two.

So by assumption, since we have proven that $2^{\lceil log_2 2^n \rceil}$ is a power of two, then we can say that $f(2^{\lceil log_2 2^n \rceil}) = (2^{\lceil log_2 2^n \rceil})^2$.

And since from our previous induction, we know that $f(n) \leq f(2^{\lceil log_2 2^n \rceil})$, we have $f(n) \leq (2^{\lceil log_2 2^n \rceil})^2$.

And since from the definition of ceiling, we know $\exists \epsilon \in \mathbb{R}, \lceil log_2 2^n \rceil = log_2 2^n + \epsilon \land 0 \leq \epsilon < 1$. So we will rewrite $\lceil log_2 2^n \rceil$ as $log_2 2^n + \epsilon$.

Then we can calculate:

$$
\begin{aligned}
f(n) &\leq f(2^{\lceil log_2 2^n \rceil}) \\
&\leq (\lceil log_2 2^n \rceil)^2 \ (Because\ from\ our\ previous\ assumption\ we\ know\ 2^{\lceil log_2 2^n \rceil}\ is\ a\ power\ of\ 2) \\
&\leq (2^{log_2 2^n + \epsilon})^2 \ (Since\ \lceil log_2 2^n \rceil = log_2 2^n + \epsilon) \\
&\leq (2^{log_2 2^n + 1})^2 \ (Because\ \epsilon < 1,\ so\ \epsilon \leq 1) \\
&\leq (2 \times 2^{log_2 2^n})^2 \\
&\leq (2 \cdot n)^2 \\
&\leq 4 \cdot n^2 \\
&\leq c \cdot n^2 \ (Because\ c = 4)
\end{aligned}
$$

So, we have prove that $\forall n \in \mathbb{N}, n \geq n_0 \Rightarrow f(n) \leq c \cdot n^2$, so $f(n) \in \mathcal{O}(n^2)$

Part 2: ($f \in \Omega(n^2)$): We are going to prove $f \in \Omega(n^2)$, i.e, $\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow f(n) \geq c \cdot n^2$.

Let $c = \frac{1}{2}, n_0 = 1$. Let $n \in \mathbb{N}$, and assume that $n \geq n_0$. We are going to show that $f(n) \geq c \cdot n^2$

First, Since we know that $n = 2^{log_2 2^n}$. And by the definition of floor, we know that $log_2 2^n \geq \lfloor log_2 2^n \rfloor$. Hence, $n = 2^{log_2 2^n} \geq 2^{\lfloor log_2 2^n \rfloor}$

From our assumption, we know that $\forall x, y \in \mathbb{N}, x \leq y \Rightarrow f(x) \leq f(y)$, so we get $f(n) = f(2^{log_2 2^n}) \geq f(2^{\lfloor log_2 2^n \rfloor})$ because $n \geq 0$.

Notice also that $\lfloor log_2 2^n \rfloor \in \mathbb{N}$ by the definition of the floor function.We now are going to show that $2^{\lfloor log_2 2^n \rfloor}$ is a power of 2, i.e, $\exists k \in \mathbb{N}, 2^{\lfloor log_2 2^n \rfloor} = 2^k$ using this fact.

Let $k = \lfloor log_2 2^n \rfloor$.

Then, $2^{\lfloor log_2 2^n \rfloor} = 2^k$. So, $2^{\lfloor log_2 2^n \rfloor}$ is a power of two.

So by assumption, since $2^{\lfloor log_2 2^n \rfloor}$ is a power of two, then we can say that $f(2^{\lfloor log_2 2^n \rfloor}) = (2^{\lfloor log_2 2^n \rfloor})^2$.

And since from our previous induction, we know that $f(n) \geq f(2^{\lfloor log_2 2^n \rfloor})$, we have $f(n) \geq (2^{\lfloor log_2 2^n \rfloor})^2$.

And since from the definition of flooring, we know $\exists \epsilon \in \mathbb{R}, \lfloor log_2 2^n \rfloor = log_2 2^n - \epsilon \wedge 0 \leq \epsilon < 1$. So we will rewrite $\lfloor log_2 2^n \rfloor$ as $log_2 2^n - \epsilon$.

Then we can calculate:

$$
\begin{aligned}
f(n) &\geq f(2^{\lfloor log_2 2^n \rfloor}) \\
&\geq (\lfloor log_2 2^n \rfloor)^2 \ (Because\ from\ our\ previous\ assumption\ we\ know\ 2^{\lfloor log_2 2^n \rfloor}\ is\ a\ power\ of\ 2) \\
&\geq (2^{log_2 2^n - \epsilon})^2 \ (Since\ \lfloor log_2 2^n \rfloor = log_2 2^n - \epsilon) \\
&\geq (2^{log_2 2^n - 1})^2 \ (Because\ \epsilon < 1,\ so\ \epsilon \leq 1) \\
&\geq (2^{-1} \times 2^{log_2 2^n})^2 \\
&\geq (\frac{1}{2} \cdot n)^2 \\
&\geq \frac{1}{4} \cdot n^2 \\
&\geq c \cdot n^2 \ (Because\ c = \frac{1}{4})
\end{aligned}
$$

So, we have prove that $\forall n \in \mathbb{N}, n \geq n_0 \Rightarrow f(n) \geq c \cdot n^2$, so $f(n) \in \Omega(n^2)$
Therefore, we have proved that $f(n) \in \mathcal{O}(n^2)$ and $f(n) \in \Omega(n^2)$. So, $f(n) \in \Theta(n^2)$ as required.