CSC165H1: Problem Set 3

Due Thursday March 7 2019, before 4pm

General instructions

Please read the following instructions carefully before starting the problem set. They contain important information about general problem set expectations, problem set submission instructions, and reminders of course policies.

- Your problem sets are graded on both correctness and clarity of communication. Solutions that are technically correct but poorly written will not receive full marks. Please read over your solutions carefully before submitting them.
- Each problem set may be completed in groups of up to three. If you are working in a group for this problem set, please consult `https://github.com/MarkUsProject/Markus/wiki/Student_Groups` for a brief explanation of how to create a group on MarkUs.

  Exception: Problem Set 0 must be completed individually.
- Solutions must be typeset electronically, and submitted as a PDF with the correct filename. **Hand-written submissions will receive a grade of ZERO.**

  The required filename for this problem set is **problem_set3.pdf**.
- Problem sets must be submitted online through MarkUs. If you haven't used MarkUs before, give yourself plenty of time to figure it out, and ask for help if you need it! If you are working with a partner, you must form a group on MarkUs, and make one submission per group. "I didn't know how to use MarkUs" is not a valid excuse for submitting late work.
- Your submitted file(s) should not be larger than 9MB. You might exceed this limit if you use a word processor like Microsoft Word to create a PDF; if it does, you should look into PDF compression tools to make your PDF smaller, although please make sure that your PDF is still legible before submitting!
- Submissions must be made *before* the due date on MarkUs. You may use *grace tokens* to extend the deadline; please see the Homework page for details on using grace tokens.
- The work you submit must be that of your group; you may not use or copy from the work of other groups, or external sources like websites or textbooks.

Additional instructions

- When doing a proof by induction, always label the step(s) that use the induction hypothesis.
- You may not use forms of induction we have not covered in lecture.
- Please follow the same guidelines as Problem Set 2 for all proofs (although of course you may use induction on this problem set!).

1. **[7 marks] Induction and sequences.** Consider the following definition of a sequence of numbers:

$$d_n = \begin{cases} 1 & \text{if } n = 0 \\ \dfrac{n}{d_{n-1}} & \text{otherwise} \end{cases}$$

For example, $d_0 = 1$, $d_1 = 1$, $d_2 = 2$, $d_3 = \dfrac{3}{2}$, and $d_4 = \dfrac{8}{3}$. Our goal is to prove the following statement: "For all natural numbers n, $d_n > \sqrt{n}$ if and only if n is even."

Complete the following two proofs, which together will prove the above statement.

(a) Prove the following using induction: $\forall n \in \mathbb{Z}^+,\ d_{2n-1} \leq \sqrt{2n-1}$.
HINTS: in the induction step, write d_{2k+1} in terms of d_{2k-1} using the given definition of the sequence. Later as an intermediate step, use difference of squares: $(2k-1)(2k+1) = 4k^2 - 1$.

(b) Prove the following (with or without using induction): $\forall n \in \mathbb{N},\ d_{2n} > \sqrt{2n}$.
You may use the statement you proved in part (a) in this part.

2. **[9 marks] Number Representations.** As you might suspect, it is possible to represent numbers in other ways besides decimal (base-10) and binary (base-2). One intriguing representation is *balanced ternary.*

   In balanced ternary, numbers are represented as sequences of digits $(d_{k-1}d_{k-2}\cdots d_1d_0)_{bt}$, where each digit d_i is T, 0, or 1, with "T" used to represent the value -1. The value of sequence $(d_{k-1}d_{k-2}\cdots d_1d_0)_{bt}$ is then simply $\sum_{i=0}^{k-1} d_i \times 3^i$.

   For example, $(T01)_{bt}$ represents the number $(-1) \times 3^2 + 0 \times 3^1 + 1 \times 3^0 = -9 + 1 = -8$.

   (a)
      (i) Write the decimal value of the balanced ternary number $(T011T)_{bt}$.
      (ii) Write the balanced ternary representation of the decimal number 210 that doesn't have any leading zeroes.

   (b) Prove using induction that $\forall n \in \mathbb{Z}^+,\ 6 \mid 3^n - 3$.

   (c) Let $x \in \mathbb{N}$ and $n \in \mathbb{Z}^+$. We say that x is n**-digit positively balanced** if and only if it has a balanced ternary representation $(d_{n-1}d_{n-2}\cdots d_1d_0)_{bt}$ in which none of the digits are equal to T. For example, the decimal number 31 is 4-digit positively balanced, with the representation $(1011)_{bt}$, and it also is 6-digit positively balanced, with the representation $(001011)_{bt}$.

      Prove, by induction, the following statement:

      $$\forall n \in \mathbb{Z}^+,\ \forall x \in \mathbb{N},\ (x \text{ is } n\text{-digit positively balanced}) \Rightarrow 6 \nmid x - 2 \land 6 \nmid x - 5$$

      You may use part (b) and the Quotient-Remainder Theorem (QRT) in this question.

      HINT: part of what the Quotient-Remainder Theorem says is that remainders are unique. So, for example, if you prove that $6 \mid x - 4$, then x has remainder 4 when divided by 6, and so the QRT allows you to conclude that $6 \nmid x - 0$, $6 \nmid x - 1$, $6 \nmid x - 2$, $6 \nmid x - 3$, and $6 \nmid x - 5$.

3. **[14 marks] Properties of Asymptotic Notation.** Prove or disprove each of the following statements.

You *may* use the "max", ceiling, and floor functions in your solutions. However, you may not use any external facts of Big-Oh/Omega/Theta, and instead should only be using their *definitions* in your proofs.

The following definitions apply to part (d).

Definition 1 (non-decreasing)**.** Let $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$. We say that f is **non-decreasing** if and only if for all $x, y \in \mathbb{N}$, if $x \leq y$ then $f(x) \leq f(y)$ (note that $f(x)$ and $f(y)$ can be equal).

Definition 2 (power of two)**.** Let $n \in \mathbb{N}$. We say that n is a **power of two** if and only if there exists a $k \in \mathbb{N}$ such that $n = 2^k$.

(a) $\exists k \in \mathbb{N},\ n^n \in \mathcal{O}(n^k)$. (Note: we define $0^0 = 0$ for the purpose of this question.)

(b) $165n^5 + n^2 \in \mathcal{O}(n^5 - n^3)$.

(c) $4^{n^2} \in \Theta(4^{n^2+n})$.

(d) For every function $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$, if f is non-decreasing and $f(n) = n^2$ for every $n \in \mathbb{N}$ that is a power of two, then $f \in \Theta(n^2)$.

You may use these facts about ceiling and floor:

$$\forall x \in \mathbb{R},\ \exists \varepsilon \in \mathbb{R},\ \lceil x \rceil = x + \varepsilon \wedge 0 \leq \varepsilon < 1 \qquad \text{(Fact 1)}$$

$$\forall x \in \mathbb{R},\ \exists \varepsilon \in \mathbb{R},\ \lfloor x \rfloor = x - \varepsilon \wedge 0 \leq \varepsilon < 1 \qquad \text{(Fact 2)}$$