UNIVERSITY OF TORONTO

Faculty of Arts and Science

MAT344H1S Introduction to Combinatorics

Final Assignment

Due Tuesday April 14 at 10:00 pm
(to be submitted on Crowdmark)

Rules:

This assessment is subject to the University of Toronto Code of Behaviour on Academic Matters, available at: `https://governingcouncil.utoronto.ca/secretariat/policies/code-behaviour-academic-matters-july-1-2019`

It is your responsibility to be familiar and follow this code (in particular, see Section B1). In addition, this assessment is subject to the following rules:

- You may use all official course material, i.e. your lecture and tutorial notes, the videos, slides and other notes posted on our Quercus page, the textbook, the previous assignments in our course and their posted solutions. You may also use your own term work.
- Any aid or resource (online or offline) other than those authorized above are not permitted.
- During the assessment period do not communicate about the questions or material on the assessment with any person other than a MAT344 course instructor.
- Do not post the questions or answers to them anywhere online or otherwise.

Every single paper will be investigated, and cases of academic misconduct will be reported. Every year a handful of students are reported to OSAI (Office of Student Academic Integrity) and receive various penalties, which may range from an F in the course to expulsion from the University.

Instructions:

- You must complete and sign the Honour Pledge on the next page before you start working on the assessment. After you finish working on the assessment, you must complete and sign the Declaration Form provided on the last page of the assessment. The Honour Pledge and Declaration Form must be submitted (together with the rest of the assessment) before the deadline. The assessment will be considered as not submitted if the Honour Pledge or Declaration Form is missing (or is not completed and signed).
- The Honour Pledge, Declaration Form, and your answer to each question are to be uploaded separately on Crowdmark before the deadline. In case of technical issues, you must submit these by email to one of the course instructors before the deadline. If you are sending your solutions to a course instructor, please only attach one file per question (JPEG or PDF).

Honour Pledge

I pledge to honour myself and my community by assuring that the work I do on this assessment fully represents my own knowledge and ideas. I pledge to fully adhere to the University of Toronto Code of Behaviour on Academic Matters and the rules listed on the cover page of this assessment.

Full Name	Student Number
Signature	Date

Problems

Note: All claims (including answers to questions starting with "determine" or "find") must be justified . Answers without or with wrong justification will not be given any credit.

1. (6 points) A graph G is defined as follows: the set of vertices of G is the set of all sequences of length 4 in $\{1, 2, 3\}$ (thus G has $3^4 = 81$ vertices). Two vertices of G are adjacent if and only if the two sequences differ in exactly one position. So for instance, 1123 and 1323 are adjacent, whereas 1123 and 2223 are not.
(a) Determine if G is Eulerian.
(b) Find the chromatic number of G.

2. (5 points) Let G be the same graph as in the previous question.
(a) Find the number of edges of G.
(b) Determine if G is planar.

3. (4 points) For any positive x, let $f(x)$ be the number of elements of the set

$$\{n \in \mathbb{Z} : 0 < n \leq x,\ n \text{ is not divisible by any of 2,3, and 7}\}.$$

Find $\lim_{x\to\infty} \frac{f(x)}{x}$.

4. (5 points) Let n be a positive integer. Find the number of solutions to the equation

$$x_1 + x_2 + x_3 = n$$

in $\mathbb{Z}$ subject to $x_1, x_2, x_3 \geq 0$, x_1 even, and x_2 odd.

5. (5 points) For $n \geq 1$, let $a_n = \prod_{r=1}^{n}(3r - 2)$. Set $a_0 = 1$. Show that for every $n \geq 0$,

$$\sum_{\substack{k,\ell,m\geq 0\\ k+\ell+m=n}} \frac{a_k a_\ell a_m}{k!\,\ell!\,m!} = 3^n.$$

6. (5 points) Find a closed form formula for the sequence a_n defined by $a_0 = -9/4$, $a_1 = -23/4$, and

$$a_n = a_{n-1} + 2a_{n-2} + 2n + 3^n \qquad (n \geq 2).$$

Inductive arguments will not be accepted.

7. (5 points) The sequence a_n is defined by $a_0 = 0$ and

$$a_n = 1 + \sum_{k=0}^{n} a_k a_{n-k} \qquad (n \geq 1).$$

Find a non-recursive formula for a_n. (Your formula may involve a sum of terms the number of which depends on n.)

Declaration Form

Congratulations - you have made it to the end of your assessment for this course! We hope that you feel proud of the work that you did here because you know that it was your own and no one else's. Please know that all suspected cases of academic dishonesty will be investigated following the procedures outlined in the Code of Behaviour on Academic Matters. If you have violated that Code or other rules of this assessment mentioned on the cover page, admitting it now will significantly reduce any penalty you incur. Admitting your mistakes is as much a matter of pride as never making them from the beginning. Thus, please check the appropriate statement below and complete the rest of the form.

☐ I confirm that the work I have done here is my own and no one else's, and is in line with the principles of scholarship and the University of Toronto Code of Behaviour on Academic Matters, and with the rules given on the cover page of this assessment.

☐ I regret that I violated the Code of Behaviour on Academic Matters or other rules of this assessment and would like to admit that now so that I can take responsibility for my mistake.

I confirm that my response here is an accurate and true representation of my behaviour, knowing that by signing this declaration untruthfully I will incur an even greater penalty if it is later discovered that I have cheated or behaved dishonestly on this assessment.

Full Name	Student Number
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Q1 (6 points)

(a) Yes. G is Eulerian.

Justification:

Let $G = \langle V, E \rangle$, with V being the set of all sequences of length 4 in $\{1,2,3\}$, $E = \{(v,w) \mid v \in V, w \in V, v \text{ and } w \text{ differ in exactly one position}\}$.

① I'll show G is connected.

Let x, y be two arbitrary vertex of V.

W.T.S: $(x, y) \in E$.

denote $x = a_1 a_2 a_3 a_4$ $(a_i \in \{1,2,3\}, 1 \le i \le 4)$
$y = b_1 b_2 b_3 b_4$ $(b_i \in \{1,2,3\}, 1 \le i \le 4)$

Claim: there is a path from x to y

proof: i) if for every $1 \le i \le 4$, $a_i \ne b_i$

$(a_1a_2a_3a_4, a_1a_2a_3b_4, a_1a_2b_3b_4, a_1b_2b_3b_4, b_1b_2b_3b_4)$ is such a path because any adjacent vertice in the sequence differ in exactly one position.

2) if for some i, $1 \le i \le 4$, $a_i = b_i$.

Since by definition, a path contains no duplicate vertice, we just need to merge the same vertice in sequence $(a_1a_2a_3a_4, a_1a_2a_3b_4, a_1a_2b_3b_4, a_1b_2b_3b_4, b_1b_2b_3b_4)$ to get a valid path from $a_1a_2a_3a_4$ to $b_1b_2b_3b_4$.

because, say V_1, V_2, V_3, V_4, $V_2 = V_3$, $(V_1, V_2) \in E$, $(V_3, V_4) \in E$, then (V_1, V_3), (V_2, V_4), (V_3, V_4) must all $\in E$, so no matter we merge V_2 with V_3 or V_3 with V_2, (V_1, V_2, V_4), (V_1, V_3, V_4) is a valid path.

Hence, there is a path from $x = a_1a_2a_3a_4$ to $y = b_1b_2b_3b_4$.

And since x, y are two arbittary vertice. We know for every $x, y \in V$, there is a path from x to y.

Thus. G is connected.

② I'll show every vertex has even degree.

Pick an arbittay vertex $v \in V$.

Let's represent $v = a_1a_2a_3a_4$

i) Let's $w = a_1a_2a_3b_4$ and $(v, E) \in E$.

Therefore, w and v differ in exactly one position: the fourth character.

And since $a_4, b_4 \in \{1,2,3\}$, $b_4 \in \{1,2,3\} - \{a_4\}$.

and therefore there're 2 possibilities since $|\{1,2,3\} - \{a_4\}| = 2$.

ii) Let's $w = a_1a_2b_3a_4$ and $(v, E) \in E$.

Therefore, w and v differ in exactly one position: the third character.

And since $a_3, b_3 \in \{1,2,3\}$, $b_3 \in \{1,2,3\} - \{a_3\}$.

and therefore there're 2 possibilities since $|\{1,2,3\} - \{a_3\}| = 2$.

iii) Let's $w = a_1b_2a_3a_4$ and $(v, E) \in E$.

Therefore, w and v differ in exactly one position: the second character.

And since $a_2, b_2 \in \{1,2,3\}$, $b_2 \in \{1,2,3\} - \{a_2\}$.

and therefore there're 2 possibilities since $|\{1,2,3\} - \{a_2\}| = 2$.

iv) Let's $w = b_1 a_2 a_3 a_4$ and $(v, E) \in E$.

Therefore, w and v differ in exactly one position: the first character.

And since $a_1, b_1 \in \{1,2,3\}$, $b_1 \in \{1,2,3\} - \{a_1\}$, and therefore there're 2 possibilities since $|\{1,2,3\} - \{a_1\}| = 2$.

Thus, there're $2+2+2+2 = 8$ vertice which differ in only one position with v. And therefore v is adjacent to 8 vertice. So $\deg(v) = 8$, which is even.

And since v is an arbitrary vertex $\in V$. We know that every vertex has an even degree.

Finally, Since G is ① connected and ② every vertex has an even degree, by Theorem 5.13 "A graph G is eulerian if and only if it is connected and every vertex has an even degree.", we know that G is Eulerian.

(b) The chromatic number of G is 3.

Justification: ∠ ① We can color with only 3 colors.

I am going to show that coloring all the vertice by showing a pretty simple algorithm.

Let's call it "April 14 Algorithm":

For any vertex $v \in V$, let's represent $v = a_1 a_2 a_3 a_4$, $a_i \in \{1,2,3\}$ for $i = 1,2,3,4$. We decide v's color by this way:

$$\begin{cases} \text{if } (a_1 + a_2 + a_3 + a_4) \% 3 = 0 \Rightarrow \text{Color } v \text{ BLUE} \\ \text{if } (a_1 + a_2 + a_3 + a_4) \% 3 = 1 \Rightarrow \text{Color } v \text{ GREEN} \\ \text{if } (a_1 + a_2 + a_3 + a_4) \% 3 = 2 \Rightarrow \text{Color } v \text{ RED} \end{cases}$$

And it is a fact that Remainder of division by is 0, 1, or 2. So we can color all the vertice. On next Page, I will show this algorithm gives a proper coloring.

To show my algorithm colors the vertice in a valid way, I need to show no adjacent vertice are colored same.

That is, "$\forall v, w \in V, (v, w) \in E \Rightarrow$ v and w have different colors"

proof. Let v, w be two arbitrary vertice of V.

Assume $(v, w) \in E$, that is, v and w are adjacent, thus v and w differ in exactly one position.

W.T.S: v and w have different colors.

Let's represent $v = a_1a_2a_3a_4$, $w = b_1b_2b_3b_4$.

Denote the remainder of $a_1 + a_2 + a_3 + a_4$ divided by 3 q. so, we have:

$a_1 + a_2 + a_3 + a_4 = 3p + q$, for some $p \in \mathbb{Z}$, $q \in \{0, 1, 2\}$.

And notice since v and w differ in exactly 1 position, we have:

$b_1 + b_2 + b_3 + b_4 = a_1 + a_2 + a_3 + a_4 + t$, $t \in \{1, 2\}$.

$t \neq 0$ because v and w differ in exactly 1 position.

$t = 1$ or 2 because the "only-differ" positions may have difference 1 or 2.

W.T.P: Remainder of $(b_1 + b_2 + b_3 + b_4)$ divided by 3 is not q.

Proof by contradiction:

Assume for contradiction that

$(b_1 + b_2 + b_3 + b_4) = 3p' + q$ for some $p' \in \mathbb{Z}$.

Since $b_1 + b_2 + b_3 + b_4 = a_1 + a_2 + a_3 + a_4 + t$

So, $a_1 + a_2 + a_3 + a_4 + t = 3p' + q$

$3p + \underline{q} + t = 3p' + \underline{q}$ (since $a_1 + a_2 + a_3 + a_4 = 3p$ by previous steps)

$t = 3(p' - p)$

since $t = 3(p' - p)$ and both p' and p are integers, hence t is divisible by 3.

However, it is against the fact that t is not divisible by 3, because it is a fact that $t = 1$ or 2.

So, the remainder of $b_1+b_2+b_3+b_4$ divided by 3 is not q.

Recall that q is the remainder of $a_1+a_2+a_3+a_4$ divided by 3.

Therefore, $(a_1+a_2+a_3+a_4)\ \%\ 3 \neq (b_1+b_2+b_3+b_4)\ \%\ 3$.

So, by my algorithm, v and w are colored differently.

And since v and w are two arbitray vertice of V.

we proved: $\forall v, w \in V.\ (v, w) \in E \Rightarrow$ v and w have different colors.

That is, no adjacent vertice have same colors.

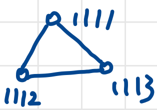
So, my algorithm gives a proper coloring.

We can color with only three colors.

② We can't color with 2 colors:

We can't color with 2 colors because triangle exists. (K_3)

For example:



So, by Theorem 5.21, we know G is not 2-colorable

K_3 is also an odd cycle.

Hence, we can't color with 2 colors.

Finally, putting everything together:

$\begin{cases} \text{We can color with 3 colors.} \\ \text{We can't color with 2 colors.} \end{cases} \Rightarrow$ 3 is the minimal number of colors to give a proper coloring.

By definition of chromatic number, we get: the chromatic number is 3.

Q2 (5 Points)

(a) 324.

Justification:

From Q1, I've showed that every vertex has degree 8.

So, by Theorem 5.9:

$\sum_{v \in V} \deg_G(v) = 2|E|$ we have,

$$2|E| = \sum_{v \in V} 8$$

$= 8 \times 81$ (since there're 81 vertices, $|V| = 81$, as given by the description of Q1).

$= 648.$

So, $|E| = 324$, that is, G has 324 edges.

Thus, the number of edges of G is 324.

(b) G is not planar.

Justification:

By theorem 5.33: A planar graph on n vertices has at most $3n-6$ edges when $n \geq 3$.

we know, let $n \geq 3$, $G = \langle V, E \rangle$ and $|V| = n$,

then G is planar $\Rightarrow$ G has at most $3n-6$ edges,

which is equivalent to (contrapositive)

$\neg$(G has at most $3n-6$ edges) $\Rightarrow$ $\neg$(G is planar),

which is, G has $> 3n-6$ edges $\Rightarrow$ G is not planar.

And in our case $\begin{cases} n = 81 \geq 3, \ 3n-6 = 237 \\ |E| = 324, \end{cases}$

so $|E| > 3n-6$, that is, G has $> 3n-6$ edges,

and therefore, G is not planar.

Q3 (4 points)

I'll use Inclusion-Exclusion formula to solve this question.

Let $K = \{n \in \mathbb{Z};\ 0 < n \le x\}$

Let $P = \{P_1, P_2, P_3\}$, $\begin{cases} P_1: \text{divisible by } 2 \\ P_2: \text{divisible by } 3 \\ P_3: \text{divisible by } 7 \end{cases}$

For each subset $S \subseteq [m]$, let $N(S)$ denote the number of elements of K which satisfy property P_i for all i in S.

since we want $\lim_{x\to\infty} \frac{f(x)}{x}$. so I'll assume that $x \geqslant 2$ as we only care about a value when $x \to \infty$)

① $N(\emptyset)$: No actual property

Since every element of K satisfies every property in S, which contains no actual property, so $N(\emptyset) = |K|$, and since K contains integers from 1 to x, $|K| = x$.

So, $N(\emptyset) = x$

② $N(\{1\})$: divisible by 2.

Among integers from 1 to x, there should be $\lfloor \frac{x}{2} \rfloor$ integers that can be divisible by 2. (since integer $= 2k$ can be divisible by 2)

And to get rid of "floor" (actually no needed because $\lim_{x\to\infty} \frac{\lfloor x \rfloor}{x} = 1$).

I'll apply proposition 7.15 because ① $x \to \infty$ ② 2 is a prime.

so the number of integers from 1 to x divisible by 2 is $\frac{x}{2}$.

So, $N(\{1\}) = \frac{x}{2}$

③ $N(\{2\})$: divisible by 3.

Similarly, since $x \to \infty$ and 3 is a prime.

By proposition 7.15, # integers from 1 to n divisible by 3 is $\frac{x}{3}$.

So, $N(\{2\}) = \frac{x}{3}$.

④ $N(\{3\})$: divisible by 7.

Similarly, since $x \to \infty$ and 7 is a prime.

By proposition 7.15, # integers from 1 to n divisible by 7 is $\frac{x}{7}$.

So, $N(\{3\}) = \frac{x}{7}$.

⑤ $N(\{1,2\})$: divisible by 2 and 3.

Since $x \to \infty$ and 2, 3 are both primes,

By proposition 7.15, # integers from 1 to n divisible by 2 and 3 is $\frac{x}{2 \times 3} = \frac{x}{6}$

So, $N(\{1,2\}) = \frac{x}{6}$.

⑥ $N(\{1,3\})$: divisible by 2 and 7

Since $x \to \infty$ and 2, 7 are both primes,

By proposition 7.15, # integers from 1 to n divisible by 2 and 7 is $\frac{x}{2 \times 7} = \frac{x}{14}$

So, $N(\{1,3\}) = \frac{x}{14}$

⑦ $N(\{2,3\})$: divisible by 3 and 7.

Since $x \to \infty$ and 3, 7 are both primes,

By proposition 7.15, # integers from 1 to n divisible by 3 and 7 is $\frac{x}{3 \times 7} = \frac{x}{21}$

So, $N(\{2,3\}) = \frac{x}{21}$.

⑧ $N(\{1,2,3\})$: divisible by 2, 3, and 7.

Since $x \to \infty$ and 2, 3, 7 are all primes.

By Proposition 7.15, # integers from 1 to n divisible by 2, 3 and 7 is $\frac{x}{2\times3\times7} = \frac{x}{42}$

So, $N(\{1,2,3\}) = \frac{x}{42}$.

So, we have
$$\begin{cases} N(\emptyset) = x \\ N(\{1\}) = \frac{x}{2} \\ N(\{2\}) = \frac{x}{3} \\ N(\{3\}) = \frac{x}{7} \\ N(\{1,2\}) = \frac{x}{6} \\ N(\{1,3\}) = \frac{x}{14} \\ N(\{2,3\}) = \frac{x}{21} \\ N(\{1,2,3\}) = \frac{x}{42} \end{cases}$$

And note that $\{n \in \mathbb{Z} : 0 < n \le x,\ n$ is not divisible by any of 2, 3, and 7$\}$ is a subset of K (since it contains some integers >0 and $\le x$), and every element in it doesn't satisfy none of the properties in P.

Hence, by Theorem 7.7 principle of Inclusion-Exclusion, we know:

$$f(x) = \sum_{S \subseteq [m]} (-1)^{|S|} N(S)$$

$$= (-1)^0 \cdot N(\emptyset) + (-1)^1 \cdot N(\{1\}) + (-1)^1 \cdot N(\{2\}) + (-1)^1 \cdot N(\{3\})$$
$$+ (-1)^2 \cdot N(\{1,2\}) + (-1)^2 \cdot N(\{1,3\}) + (-1)^2 \cdot N(\{2,3\}) + (-1)^3 \cdot N(\{1,2,3\})$$

$$= x - \frac{x}{2} - \frac{x}{3} - \frac{x}{7} + \frac{x}{6} + \frac{x}{14} + \frac{x}{21} - \frac{x}{42}$$

$$= \frac{2}{7}x$$

So, $\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\frac{2}{7}x}{x} = \lim_{x\to\infty} \frac{2}{7} = \frac{2}{7}$

Q4 (5 points)

For x_1 (even non-negative integer), we need:

$$1+x^2+x^4+x^6+\cdots+x^{2n}+\cdots$$
$$=\sum_{n=0}^{\infty} x^{2n}$$
$$=\frac{1}{1-x^2}$$

since $\sum_{n=0}^{\infty} x^{2n}=\sum_{n=0}^{\infty}(x^2)^n$, and we know $\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n$)

For x_2 (odd non-negative integer), we need:

$$x+x^3+x^5+\cdots+x^{2n+1}+\cdots$$
$$=x[1+x^2+x^4+\cdots+x^{2n}+\cdots]$$
$$=x\cdot\frac{1}{1-x^2}$$

since $1+x^2+x^4+\cdots+x^{2n}+\cdots=\frac{1}{1-x^2}$ by previous steps.

For x_3 (non-negative integer), we need:

$$1+x+x^2+x^3+\cdots+x^n+\cdots$$
$$=\sum_{n=0}^{\infty} x^n$$
$$=\frac{1}{1-x}$$

And now we can set up a generating function for $\{a_n : n \geq 0\}$, with a_n being the number of solutions to the equation $x_1+x_2+x_3=n$ with the constraints given.

$$F(x)=\sum_{n=0}^{\infty} a_n x^n$$

a_n is the coefficient of x^n

$$=\frac{1}{1-x^2}\cdot\frac{x}{1-x^2}\cdot\frac{1}{1-x}$$
$$=\frac{x}{(1-x^2)^2(1-x)}$$
$$=\frac{x}{[(1+x)(1-x)]^2(1-x)}$$
$$=\frac{x}{(1+x)^2(1-x)^3}$$

Now I am going to do some partial fraction work:

Let $\frac{x}{(1-x)^3(1+x)^2} = \frac{A}{(1-x)} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} + \frac{D}{(1+x)} + \frac{E}{(1+x)^2}$

$= \frac{A(1-x)^2(1+x)^2 + B(1-x)(1+x)^2 + C(1+x)^2 + D(1-x)^3(1+x) + E(1-x)^3}{(1-x)^3(1+x)^2}$

Note: $\begin{cases} (1-x)^2(1+x)^2 = (1-x^2)^2 = 1-2x^2+x^4 \rightarrow A \\ (1-x)(1+x)^2 = (1-x)(1+x)^2 = (1-x)(1+2x+x^2) \\ \quad = 1+2x+x^2-x-2x^2-x^3 \\ \quad = 1+x-x^2-x^3 \rightarrow B \\ (1+x)^2 = 1+2x+x^2 \rightarrow C \\ (1-x)^3(1+x) = (1-x)^2(1-x)(1+x) \\ \quad = (1-2x+x^2)(1-x^2) \\ \quad = 1-2x+x^2-x^2+2x^3-x^4 \\ \quad = 1-2x+2x^3-x^4 \rightarrow D \\ (1-x)^3 = 1-3x+3x^2-x^3 \rightarrow E \end{cases}$

So, $\frac{x}{(1-x)^3(1+x)^2} = \frac{A(1-2x^2+x^4) + B(1+x-x^2-x^3) + C(1+2x+x^2) + D(1-2x+2x^3-x^4) + E(1-3x+3x^2-x^3)}{(1-x)^3(1+x)^2}$

$= \frac{(A+B+C+D+E) + (B+2C-2D-3E)x + (-2A-B+C+3E)x^2 + (-B+2D-E)x^3 + (A-D)x^4}{(1-x)^3(1+x)^2}$

So $\begin{cases} A+B+C+D+E=0 \\ B+2C-2D-3E=1 \\ -2A-B+C+3E=0 \\ -B+2D-E=0 \\ A-D=0 \end{cases}$, solving this gives $\begin{cases} A=-\frac{1}{16} \\ B=0 \\ C=\frac{1}{4} \\ D=-\frac{1}{16} \\ E=-\frac{1}{8} \end{cases}$

So, $F(x) = \sum_{n=0}^{\infty} a_n \cdot x^n$

$$= \frac{x}{(1+x)^2(1-x)^3}$$

$$= \frac{1}{(1-x)} \times \left(-\frac{1}{16}\right) + \frac{1}{(1-x)^3} \times \frac{1}{4} + \frac{1}{1+x} \times \left(-\frac{1}{16}\right)$$

$$+ \frac{1}{(1+x)^2} \times \left(-\frac{1}{8}\right)$$

$$= -\frac{1}{16} \sum_{n=0}^{\infty} x^n + \frac{1}{4} \sum_{n=0}^{\infty} \binom{n+2}{n} x^n - \frac{1}{16} \sum_{n=0}^{\infty} (-1)^n x^n$$

$$- \frac{1}{8} \sum_{n=0}^{\infty} \binom{n+1}{n} \cdot (-1)^n \cdot x^n$$

So, the solution to our question is thus the coefficient of x^n in $F(x)$, which is:

$$-\frac{1}{16} + \frac{1}{4}\binom{n+2}{n} - \frac{1}{16}(-1)^n - \frac{1}{8}\binom{n+1}{n} \cdot (-1)^n$$

Q5 (5 points)

Claim: $a_{n+1} = a_n \cdot (3n+1)$, for $n \geq 0$

proof: ① if $n=0$.

by def $a_1 = \prod_{r=1}^{1} (3r-2) = (3\times 1 - 2) = 1$

$a_0 = 1$

so, $a_1 = a_0(3\times 0+1)$, so $a_{n+1} = a_n(3n+1)$ holds for $n=0$

② if $n \geq 1$:

by def, $a_{n+1} = \prod_{r=1}^{n+1} (3r-2) = \prod_{r=1}^{n} (3r-2) \cdot [3(r+1)-2]$

$= a_n \cdot (3r+1)$ # since $n \geq 1$, $a_n = \prod_{r=1}^{n} (3r-2)$

so, $a_{n+1} = a_n \cdot (3n+1)$ holds for $n \geq 1$

Therefore, $a_{n+1} = a_n \cdot (3n+1)$ for $n \geq 0$

I'll then define a generating function:

$F(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} \cdot x^i$ (a_i is defined by what the question gave)

Notice that:

$$F(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} \cdot x^i$$

$$= \frac{a_0}{0!} \cdot x^0 + \frac{a_1}{1!} \cdot x^1 + \frac{a_2}{2!} \cdot x^2 + \cdots + \frac{a_n}{n!} \cdot x^n + \cdots$$

So, $F'(x) = 0 + \frac{1\times a_1}{1!} \cdot x^0 + \frac{2\times a_2}{2!} \cdot x^1 + \cdots + \frac{n \cdot a_n}{n!} \cdot x^{n-1} + \cdots$

$$= \sum_{i=1}^{\infty} \frac{i \cdot a_i}{i!} \cdot x^{i-1}$$

$$= \sum_{i=1}^{\infty} \frac{a_i}{(i-1)!} \cdot x^{i-1} \quad (\text{since } i \neq 0)$$

Then, $3xF'(x) = \sum_{i=1}^{\infty} \frac{3a_i}{(i-1)!} \cdot x^i = \sum_{i=1}^{\infty} \frac{3i \cdot a_i}{i!} \cdot x^i$

Therefore, $3xF'(x) + F(x)$

$$= \sum_{i=1}^{\infty} \frac{3i \cdot a_i}{i!} \cdot x^i + \sum_{i=0}^{\infty} \frac{a_i}{i!} \cdot x^i$$

$$= \sum_{i=1}^{\infty} \frac{3i \cdot a_i}{i!} \cdot x^i + \sum_{i=1}^{\infty} \frac{a_i}{i!} \cdot x^i + \frac{a_0}{0!} \cdot x^0$$

$$= \left(\sum_{i=1}^{\infty} \frac{(3i+1)a_i}{i!} \cdot x^i\right) + \frac{a_0}{0!} \cdot x^0$$

$$= \left(\sum_{i=1}^{\infty} \frac{(3i+1)a_i}{i!} \cdot x^i\right) + \frac{(3\times 0+1)a_0}{0!} \cdot x^0 \quad \# \; 3\times 0+1 = 1$$

$$= \sum_{i=0}^{\infty} \frac{(3i+1)a_i}{i!} \cdot x^i$$

So, we have:

$$\begin{cases} F'(x) = \sum_{i=1}^{\infty} \frac{a_i}{(i-1)!} \cdot x^{i-1} \\ 3x \cdot F'(x) + F(x) = \sum_{i=0}^{\infty} \frac{(3i+1)a_i}{i!} \cdot x^i \end{cases}$$

Then, we can compute:

$$F'(x) - (3x \cdot F'(x) + F(x))$$

$$= \sum_{i=1}^{\infty} \frac{a_i}{(i-1)!} \cdot x^{i-1} - \sum_{i=0}^{\infty} \frac{(3i+1)a_i}{i!} \cdot x^i$$

$$= \frac{a_1}{0!} \cdot x^0 + \frac{a_2}{1!} \cdot x^1 + \frac{a_3}{2!} \cdot x^2 + \cdots + \frac{a_n}{(n-1)!} \cdot x^n + \cdots$$

$$- \frac{(3\times 0+1) \cdot a_0}{0!} \cdot x^0 - \frac{(3\times 1+1)}{1!} \cdot x^1 - \frac{(3\times 2+1)}{2!} \cdot x^2 - \cdots - \frac{(3n+1)}{n!} \cdot x^n - \cdots$$

$$= \sum_{i=0}^{\infty} \frac{a_{i+1} - (3i+1)a_i}{i!} \cdot x^i$$

And by Claim * proved at the beginning, we know that $a_{i+1} = (3i+1)a_i$ for every $i \geq 0$, so $a_{i+1} - (3i+1)a_i = 0$ for $i \geq 0$

Hence,

$$F'(x) - (3x F'(x) + F(x))$$

$$= \sum_{i=0}^{\infty} \frac{a_{i+1} - (3i+1)a_i}{i!} \cdot x^i$$

$$= \sum_{i=0}^{\infty} \frac{0}{i!} \cdot x^i \quad \text{\# since } i \geq 0,\ a_{i+1} - (3i+1)a_i = 0$$

$$= \sum_{i=0}^{\infty} 0$$

$$= 0.$$

That is, $F'(x) - (3x F'(x) + F(x)) = 0$

$$(1-3x) F'(x) = F(x)$$

Here, I need to use some calcuous skills to find the solution for $F(x)$.

$$(1-3x) F'(x) = F(x)$$

$$(1-3x) \frac{dF}{dx} = F$$

$\frac{1}{F} dF = \frac{1}{1-3x} dx$, integrating both sides:

$$\ln F = -\frac{1}{3} \ln(1-3x) = \ln(1-3x)^{-\frac{1}{3}}$$

Thus, $F(x) = (1-3x)^{-\frac{1}{3}}$ # I ignore the constant because we know $a_0 = 1$.

Putting everything so far together, we get:

$$F(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} \cdot x^i = (1-3x)^{-\frac{1}{3}}$$

Also, we can do a double check with Theorem 8.10 Newton's Binomial Theorem: # since $-\frac{1}{3}$ is a real number.

$$(1-3x)^{-\frac{1}{3}} = \sum_{i=0}^{\infty} \binom{-\frac{1}{3}}{i}(-3x)^i = \binom{-\frac{1}{3}}{0} \cdot (-3x)^0 + \sum_{i=1}^{\infty} \binom{-\frac{1}{3}}{i} \cdot (-3x)^i$$

$$= 1 + \sum_{i=1}^{\infty} \binom{-\frac{1}{3}}{i} \cdot (-3x)^i$$

$$= a_0 + \sum_{i=1}^{\infty} \frac{\overbrace{(-\frac{1}{3})(-\frac{1}{3}-1)(-\frac{1}{3}-2)\cdots(-\frac{1}{3}-(i-1))}^{i \text{ terms}}}{i!} \cdot (-3x)^i$$

$$= a_0 + \sum_{i=1}^{\infty} \frac{(-\frac{1}{3})(-\frac{1+3}{3})(-\frac{1+3\times 2}{3})\cdots(-\frac{1+3(i-1)}{3})}{i!} \cdot (-3x)^i$$

$$= a_0 + \sum_{i=1}^{\infty} \frac{1\times(1+3)(1+3\times 2)\cdots(1+3(i-1))}{i! \cdot (-3)!} \cdot (-3x)^i$$

$$= a_0 + \sum_{i=1}^{\infty} \frac{(1+3-3)(1+3+3-3)(1+3\times 2+3-3)\cdots(1+3(i-1)+3-3)}{i!} \cdot x^i$$

$$= a_0 + \sum_{i=1}^{\infty} \frac{(3\times 1-2)(3\times 2-2)(3\times 3-2)\cdots(3i-2)}{i!} \cdot x^i$$

$$= a_0 + \sum_{i=1}^{\infty} \frac{\prod_{r=1}^{i}(3r-2)}{i!} \cdot x^i$$

$$= a_0 + \sum_{i=1}^{\infty} a_i \cdot x^i$$ # since by def of a_n, when $n \geq 1$, $a_n = \prod_{r=1}^{\infty}(3r-2)$

$$= \sum_{i=0}^{\infty} a_i \cdot x^i$$

$$= F(x).$$

So, I am pretty sure $F(x) = \sum_{i=0}^{\infty} a_i \cdot x^i = (1-3x)^{-\frac{1}{3}}$. I'll continue my work with this in my hands.

Consider:

$$\left(\sum_{k=0}^{\infty} \frac{a_k}{k!} \cdot x^k\right)\left(\sum_{l=0}^{\infty} \frac{a_l}{l!} \cdot x^l\right)\left(\sum_{m=0}^{\infty} \frac{a_m}{m!} \cdot x^m\right)$$

The coefficient of x^n in the product is $\sum_{\substack{k,l,m \geq 0 \\ k+l+m=n}} \frac{a_k a_l a_m}{k!\,l!\,m!}$

(consider all cases when x^n appears, it's when ↗

And we know:

$$\left(\sum_{k=0}^{\infty} \frac{a_k}{k!} \cdot x^k\right)\left(\sum_{l=0}^{\infty} \frac{a_l}{l!} \cdot x^l\right)\left(\sum_{m=0}^{\infty} \frac{a_m}{m!} \cdot x^m\right)$$

$$= (1-3x)^{-\frac{1}{3}} \cdot (1-3x)^{-\frac{1}{3}} \cdot (1-3x)^{-\frac{1}{3}}$$ # since $\sum_{i=0}^{\infty} \frac{a_i}{i!} \cdot x^i = (1-3x)^{-\frac{1}{3}}$ by previous steps

$$= (1-3x)^{-1}$$

$$= \frac{1}{(1-3x)}$$

$$= 1 \cdot x^0 + 3 \cdot x^1 + 3^2 \cdot x^2 + \cdots + 3^n \cdot x^n + \cdots$$

$$= \sum_{n=0}^{\infty} 3^n \cdot x^n$$

So, the coefficient of x^n is 3^n. And we showed $\sum_{\substack{k,l,m \geq 0 \\ k+l+m=n}} \frac{a_k a_l a_m}{k!\, l!\, m!}$ is the coefficient of x^n.

Therefore, $\sum_{\substack{k,l,m \geq 0 \\ k+l+m=n}} \frac{a_k a_l a_m}{k!\, l!\, m!} = 3^n$ ∎

Q6 (5 points)

We have $a_n = a_{n-1} + 2a_{n-2} + 2n + 3^n$ $(n \geq 2)$

which is equivalent to:

$a_n - a_{n-1} - 2a_{n-2} = 2n + 3^n$ $(n \geq 2)$.

Assume $n \geq 2$.

① Find the solution for the corresponding homogeneous equation: $a_n - a_{n-1} - 2a_{n-2} = 0$.

We start by writing the given recurrence equation as an advance operator equation for a function $a(n)$:

$(A^2 - A - 2)a = 0$

Factoring $P(A) = A^2 - A - 2$ gives

$P(A) = (A+1)(A-2)$

Then, we know that $a_1(n) = C_1 \cdot (-1)^n$ and $C_2 \cdot 2^n$ are two solutions.

So, the general solution for the corresponding homogeneous equation is: $a_3(n) = C_1 \cdot (-1)^n + C_2 \cdot 2^n$

② Find a particular solution for the nonhomogeous equation:

I'll try $a_0(n) = d_1 \cdot n + d_2 + d_3 \cdot 3^n$

Notice: $a_0(n-1) = d_1(n-1) + d_2 + d_3 \cdot 3^{n-1}$

$= d_1 n - d_1 + d_2 + \frac{1}{3} d_3 \cdot 3^n$

$a_0(n-2) = d_1(n-2) + d_2 + d_3 \cdot 3^{n-2}$

$= d_1 n - 2d_1 + d_2 + \frac{d_3}{9} \cdot 3^n$

Then, we can compute:

$((A^2 - A - 2)a_0)(n-2) = A^2(a_0)(n-2) - A(a_0)(n-2) - 2a_0(n-2)$

$= a_0(n) - a_0(n-1) - 2a_0(n-2)$ # by def of Advance operator

$= d_1 \cdot n + d_2 + d_3 \cdot 3^n$

$- (d_1 \cdot n - d_1 + d_2 + \frac{1}{3} \cdot d_3 \cdot 3^n)$

$- 2(d_1 n - 2d_1 + d_2 + \frac{1}{9} d_3 \cdot 3^n)$

$$= d_1 \cdot n + d_2 + d_3 \cdot 3^n - d_1 n + d_1 - d_2 - \frac{1}{3} d_3 \cdot 3^n$$
$$-2 d_1 \cdot n + 4 d_1 - 2 d_2 - \frac{2}{9} d_3 \cdot 3^n$$
$$= -2 d_1 \cdot n + 5 d_1 - 2 d_2 + \frac{4}{9} d_3 \cdot 3^n$$

We want $2n + 3^n$, so let

$$\begin{cases} -2 d_1 = 2 \\ 5 d_1 - 2 d_2 = 0 \\ \frac{4}{9} d_3 = 1 \end{cases} \Rightarrow \begin{cases} d_1 = -1 \\ d_2 = -\frac{5}{2} \\ d_3 = \frac{9}{4} \end{cases}$$

Hence, $a_0(n) = -n - \frac{5}{2} + \frac{9}{4} \cdot 3^n$ is a particular solution to the nonhomogeous equation.

Therefore, the general solution is:

$$a(n) = -n - \frac{5}{2} + \frac{9}{4} \cdot 3^n + C_1 \cdot (-1)^n + C_2 \cdot 2^n$$

And by $a_0 = -\frac{9}{4}$, $a_1 = -\frac{23}{4}$, we get

$$a_2 = a_1 + 2 a_0 + 2 \times 2 + 3^2$$
$$= \frac{11}{4}$$
$$a_3 = a_2 + 2 a_1 + 2 \times 3 + 3^3$$
$$= \frac{97}{4}$$

And so, with $a_2 = \frac{11}{4}$, $a_3 = \frac{97}{4}$, we have:

$$\begin{cases} -2 - \frac{5}{2} + \frac{9}{4} \times 3^2 + C_1 + 4 C_2 = \frac{11}{4} \\ -3 - \frac{5}{2} + \frac{9}{4} \times 3^3 - C_1 + 8 C_2 = \frac{97}{4} \end{cases}.$$

Solving this we get $\begin{cases} C_1 = \frac{5}{3} \\ C_2 = -\frac{11}{3} \end{cases}$

Thus, we arrive at the desired solution, which is

$$a(n) = -n - \frac{5}{2} + \frac{9}{4} \cdot 3^n + \frac{5}{3} (-1)^n - \frac{11}{3} \cdot 2^n \quad , n \geq 2$$

And it is easy to check:

$$a(0) = -0 - \frac{5}{2} + \frac{9}{4} \times 3^0 + \frac{5}{3}(-1)^0 - \frac{11}{3} \cdot 2^0$$
$$= -\frac{9}{4}$$

$$a(1) = -1 - \frac{5}{2} + \frac{9}{4} \times 3^1 + \frac{5}{3}(-1)^1 - \frac{11}{3} \times 2^1$$
$$= -\frac{23}{4}$$

So, $n=0$ and $n=1$ also satisfy the equation.

Hence, $a(n) = -n - \frac{5}{2} + \frac{9}{4} \cdot 3^n + \frac{5}{3}(-1)^n - \frac{11}{3} \cdot 2^n$.

for $n \geq 0$.

That is, $a_n = -n - \frac{5}{2} + \frac{9}{4} \cdot 3^n + \frac{5}{3}(-1)^n - \frac{11}{3} \cdot 2^n$

a closed form.

Q7 (5 points)

Let $F(x)=\sum_{n=0}^{\infty} a_n x^n$ be the generating function for the sequence $\{a_n : n\geq 0\}$, where a_n is defined as $a_n=\begin{cases} 0, & n=0 \\ 1+\sum_{k=0}^{n} a_k a_{n-k}, & n\geq 1 \end{cases}$

By proposition 8.3: "Let $A(x)=\sum_{n=0}^{\infty} a_n x^n$ and $B(x)=\sum_{n=0}^{\infty} b_n x^n$ be generating functions. Then $A(x)B(x)$ is the generating function of the sequence whose n^{th} term is given by

$$a_0 b_0 + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}".$$

We have:

$$F^2(x) = \sum_{k=0}^{0} a_k a_{0-k} + \left(\sum_{k=0}^{1} a_k a_{1-k}\right)\cdot x + \left(\sum_{k=0}^{2} a_k a_{2-k}\right)x^2 + \cdots + \left(\sum_{k=n}^{0} a_k a_{n-k}\right)\cdot x^n + \cdots$$

And $F(x)=\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$

Then, we can compute:

$$F(x)-F^2(x) = \left(a_0 - \sum_{k=0}^{0} a_k a_{0-k}\right) + \left(a_1 - \sum_{k=0}^{1} a_k a_{1-k}\right)x + \left(a_2 - \sum_{k=0}^{2} a_k a_{2-k}\right)x^2 + \left(a_3 - \sum_{k=0}^{3} a_k a_{3-k}\right)x^3 + \cdots + \left(a_n - \sum_{k=0}^{n} a_k a_{n-k}\right)x^n + \cdots$$

Since we know from the definition, $a_n = 1+\sum_{k=0}^{n} a_k a_{n-k}$, $n\geq 1$, that is, $a_n - \sum_{k=0}^{n} a_k a_{n-k} = 1$ whenever $n\geq 1$.

Thus,
$$\begin{aligned} F(x)-F^2(x) &= \left(a_0 - \sum_{k=0}^{0} a_k a_{0-k}\right) + x + x^2 + x^3 + \cdots + x^n + \cdots \\ &= (a_0 - a_0 a_0) + x + x^2 + x^3 + \cdots + x^n + \cdots \\ &= x + x^2 + x^3 + \cdots + x^n + \cdots \quad \text{\# since } a_0 = 0 \\ &= x(1+x+x^2+\cdots+x^n+\cdots) = \frac{x}{1-x} \end{aligned}$$

So, $F(x) - F^2(x) = \frac{x}{1-x}$

That is, $F^2(x) - F(x) + \frac{x}{1-x} = 0$

Factoring this gives $\left(F(x) + \frac{1+\sqrt{1-\frac{4x}{1-x}}}{2}\right)\left(F(x) - \frac{1+\sqrt{1-\frac{4x}{1-x}}}{2}\right) = 0$

Hence, $\boxed{F(x) = \frac{1 \pm \sqrt{1-\frac{4x}{1-x}}}{2}}$

Notice that, we can expand $\sqrt{1-\frac{4x}{1-x}}$:

$$\sqrt{1-\frac{4x}{1-x}} = \left(1-\frac{4x}{1-x}\right)^{\frac{1}{2}}$$

$$= \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} \cdot \left(-\frac{4x}{1-x}\right)^i$$ # by Theorem 8.10 Newton's Binomial Theorem

$$= \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} \cdot (-4x)^i \cdot \left(\frac{1}{1-x}\right)^i$$

$$= \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} \cdot (-4)^i \cdot x^i \cdot \left(\frac{1}{1-x}\right)^i$$

$$= \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} \cdot (-4)^i \cdot x^i \cdot \sum_{t=0}^{\infty} \binom{t+i-1}{t} \cdot x^t$$

And also, by lemma 9.27: For each $k \geq 1$, $\binom{\frac{1}{2}}{k} = \frac{(-1)^{k-1}}{k} \cdot \frac{\binom{2k-2}{k-1}}{2^{2k-1}}$

So, $$\sqrt{1-\frac{4x}{1-x}} = \binom{\frac{1}{2}}{0} \cdot (-4)^0 \cdot x^0 \cdot \sum_{t=0}^{\infty} \binom{t-1}{t} \cdot x^t + \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \cdot \frac{\binom{2i-2}{i-1}}{2^{2i-1}} \cdot (-4)^i x^i \cdot \sum_{t=0}^{\infty} \binom{t+i-1}{t} \cdot x^t$$

$$= \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \cdot \frac{\binom{2i-2}{i-1}}{2^{2i-1}} \cdot (-4)^i \cdot x^i \cdot \sum_{t=0}^{\infty} \binom{t+i-1}{t} \cdot x^t$$

since $t-1<t$, so $\binom{t-1}{t}=0$ (we can't choose t from $t-1$)

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And therefore, the coefficient of x^n in $\sqrt{1-\frac{4x}{1-x}}$ is:

$$\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \cdot \frac{\binom{2i-2}{i-1}}{2^{2i-1}} \cdot (-4)^i \cdot \binom{n-1}{n-i}$$

by let $t = n-i$, so later the inner summation's x^{n-i} times x^i generates x^n.

So, the coefficient of x^n in

$F(x) = \frac{1 \pm \sqrt{1-\frac{4x}{1-x}}}{2}$

is $\pm \frac{1}{2}\left(\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \cdot \frac{\binom{2i-2}{i-1}}{2^{2i-1}} \cdot (-4)^i \cdot \binom{n-1}{n-i}\right)$

this is because, i starts from 1 in $\sqrt{1-\frac{4x}{1-x}}$, so $\frac{1}{2}$ doesn't affect because there is no x^0 term in $\sqrt{1-\frac{4x}{1-x}}$.

That is, $a_n = \pm \frac{1}{2}\left(\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \cdot \frac{\binom{2i-2}{i-1}}{2^{2i-1}} \cdot (-4)^i \cdot \binom{n-1}{n-i}\right)$, for $n \geq 1$. *

And notice that, we need $a_n \geq 0$ as per to the recurrence definition, and $(-1)^{i-1} \cdot (-4)^i$ always gives a negative value, since only one of $i-1$ and i can be even. So, $\left(\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \cdot \frac{\binom{2i-2}{i-1}}{2^{2i-1}} \cdot (-4)^i \cdot \binom{n-1}{n-i}\right)$ is always negative.

So, we take the minus from the "plus-or-minus" in *.

And finally, we get:

$$a_n = \begin{cases} 0, & n=0 \\ \frac{1}{2}\left[\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \cdot \frac{\binom{2i-2}{i-1}}{2^{2i-1}} \cdot (-4)^i \cdot \binom{n-1}{n-i}\right], & n \geq 1 \end{cases}$$

Also (no $\frac{1}{2}$ outside the summation)

$$a_n = \begin{cases} 0, & n = 0 \\ \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \cdot \frac{\binom{2i-2}{i-1}}{2^{2i}} \cdot (-4)^i \cdot \binom{n-1}{n-i}, & n \geq 1 \end{cases}$$

this is $\binom{n-1}{n-i}$