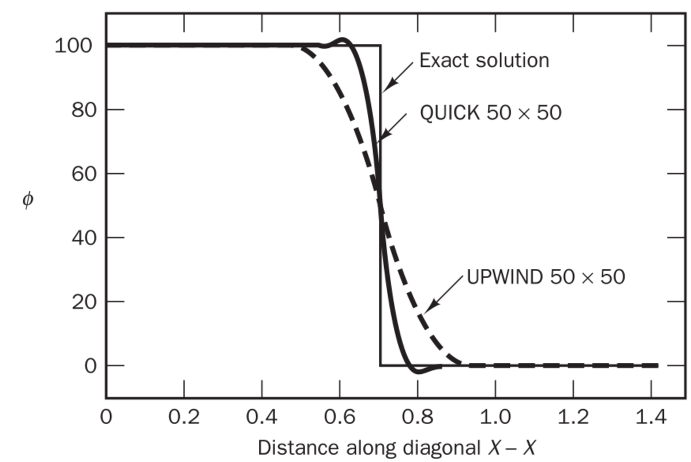
Total Variation Diminishing (TVD) methods

Wiggles (Spurious oscillations)

- High order methods such as QUICK, Central, 2nd order Upwind and so on give minor undershoots and overshoots, specially near discontinuity or sharp gradient points



- The class of TVD (total variation diminishing) schemes has been specially formulated to achieve oscillation-free solutions and has proved to be useful in CFD calculations

TVD Schemes

- Upwind differencing (UD) scheme is the most stable and unconditionally bounded scheme, but it introduces a high level of false diffusion due to its low order of accuracy (first-order).
- Higher-order schemes can give spurious oscillations when the Peclet number is high
- When higher-order schemes are used to solve for turbulent quantities, wiggles can give physically unrealistic negative values and instability.
- TVD schemes are designed to address this undesirable oscillatory behaviour of higher-order schemes.

Upwind-biased discretisation schemes

- Let's look at different schemes for the east face value of ϕ_e:

Upwind differencing (UD):

$$\phi_e = \phi_P$$

Linear upwind differencing (LUD):

$$\phi_e = \phi_P + \frac{(\phi_P - \phi_W)}{\delta x}\frac{\delta x}{2}$$
$$= \phi_P + \frac{1}{2}(\phi_P - \phi_W)$$

Central differencing (CD):

$$\phi_e = \frac{(\phi_P + \phi_E)}{2}$$
$$= \phi_P + \frac{1}{2}(\phi_E - \phi_P)$$

QUICK:

$$\phi_e = \phi_P + \frac{1}{8}[3\phi_E - 2\phi_P - \phi_W]$$

General form:

$$\boxed{\phi_e = \phi_P + \frac{1}{2}\psi(\phi_E - \phi_P)}$$

Upwind-biased discretisation schemes

- In choosing this form we express the convective flux at the east face as the sum of the flux $F_e\phi_P$ that is obtained when we use UD and an additional convective flux $F_e\psi(\phi_E - \phi_P)/2$.
- The extra contribution is connected in some way to the gradient of the transported quantity φ at the east face, as indicated by its central difference approximation $(\phi_E - \phi_P)$.

$$\boxed{\phi_e = \phi_P + \frac{1}{2}\psi(\phi_E - \phi_P)}$$

CD: $\psi = 1$

UD: $\psi = 0$

LUD: $\psi = (\phi_P - \phi_W)/(\phi_E - \phi_P)$

$$\phi_e = \phi_P + \frac{1}{2}\left(\frac{\phi_P - \phi_W}{\phi_E - \phi_P}\right)(\phi_E - \phi_P)$$

QUICK: $\psi = \left(3 + \frac{\phi_P - \phi_W}{\phi_E - \phi_P}\right)\frac{1}{4}$

$$\phi_e = \phi_P + \frac{1}{2}\left[\left(3 + \frac{\phi_P - \phi_W}{\phi_E - \phi_P}\right)\frac{1}{4}\right](\phi_E - \phi_P)$$

Upwind-biased discretisation schemes

- It can be seen the ratio of upwind-side gradient to downwind-side gradient (r) determines the value of function ψ and the nature of the scheme:

$$\boxed{\psi = \psi(r)}$$

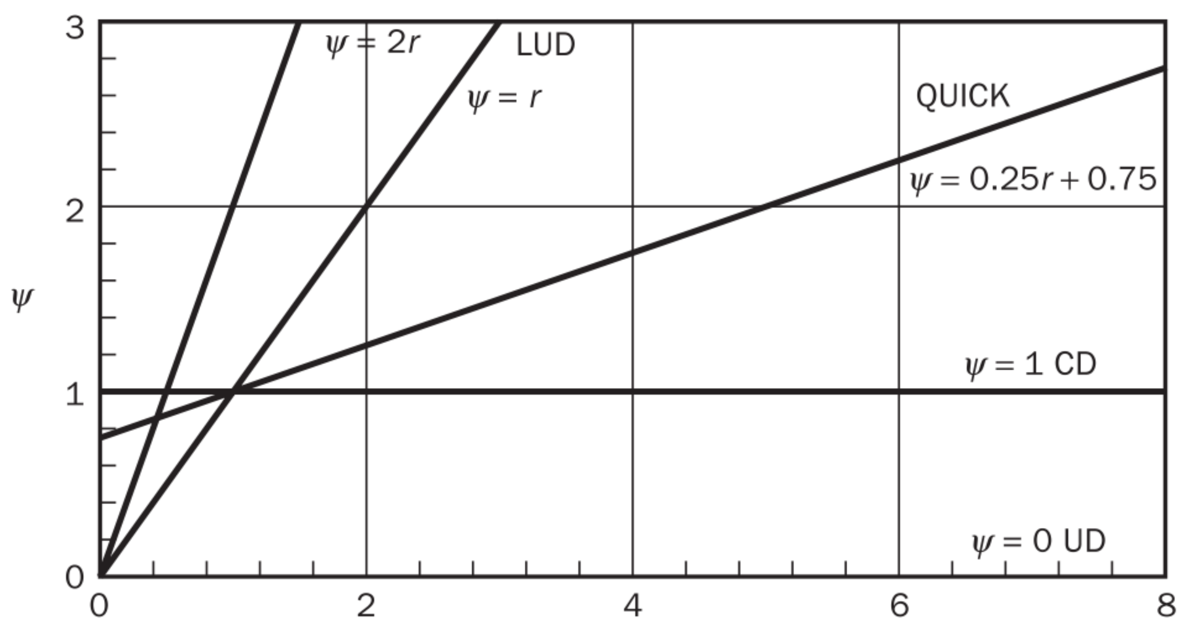
$$\boxed{r = \left(\frac{\phi_P - \phi_W}{\phi_E - \phi_P} \right)}$$

$$\boxed{\phi_e = \phi_P + \frac{1}{2}\psi(r)(\phi_E - \phi_P)}$$

For the UD scheme $\psi(r) = 0$
For the CD scheme $\psi(r) = 1$
For the LUD scheme $\psi(r) = r$
For the QUICK scheme $\psi(r) = (3 + r)/4$

Upwind-biased discretisation schemes

- The r–ψ diagram is shown.
- It is assumed that the flow direction is positive (i.e. from west to east). Similar expressions exist for negative flow direction and r will still be the ratio of upwind-side gradient to downwind-side gradient.

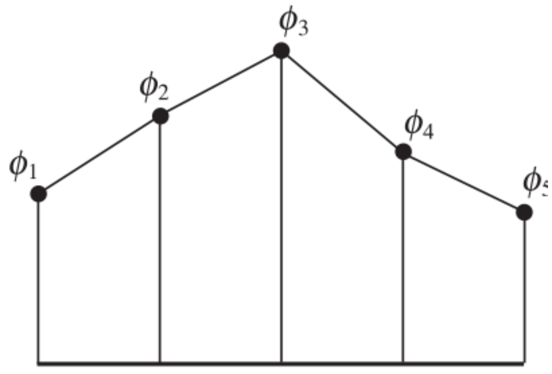


TVD schemes

- The desirable property for a stable, non-oscillatory, higher-order scheme is monotonicity preserving.
- In simple terms, monotonicity-preserving schemes do not create new undershoots and overshoots in the solution or accentuate existing extremes.
- Monotonicity-preserving schemes have implications for the so-called **total variation** of discretised solutions.
- For monotonicity to be satisfied,

  TV must not increase

$$\boxed{\begin{aligned} TV(\phi) &= |\phi_2 - \phi_1| + |\phi_3 - \phi_2| + |\phi_4 - \phi_3| + |\phi_5 - \phi_4| \\ &= |\phi_3 - \phi_1| + |\phi_5 - \phi_3| \end{aligned}}$$



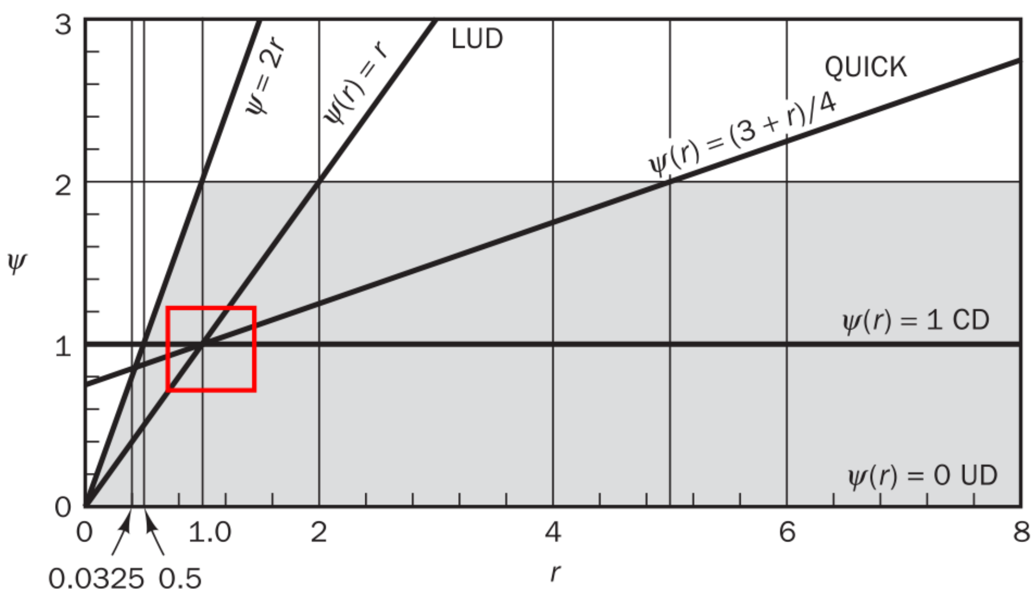
TVD schemes

- Monotonicity-preserving schemes have the property that the total variation of the discrete solution should diminish with time.
- Hence they are called total variation diminishing or TVD schemes.
- Total variation is therefore considered at every time step and a solution is said to be total variation diminishing (or TVD) if

$$TV(\phi^{n+1}) \leq TV(\phi^{n})$$

Criteria for TVD schemes

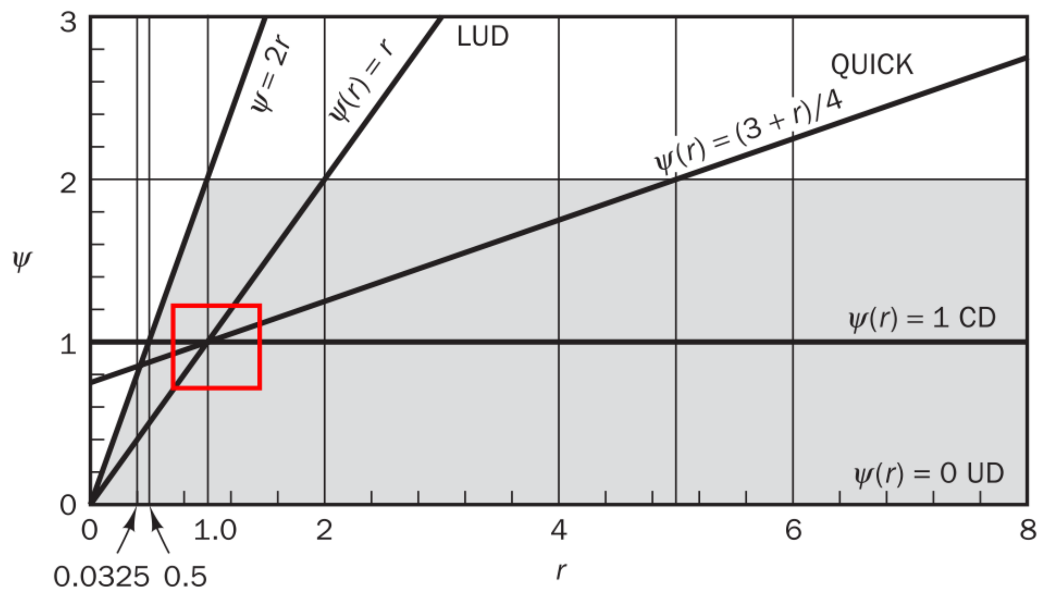
- Sweby (1984) has given necessary and sufficient conditions for a scheme to be TVD in terms of the r – ψ relationship



- If $0 < r < 1$ the upper limit is $\psi(r) = 2r$, so for TVD schemes $\psi(r) \leq 2r$
- If $r \geq 1$ the upper limit is $\psi(r) = 2$, so for TVD schemes $\psi(r) \leq 2$

Criteria for TVD schemes

- the UD scheme is TVD
- the LUD scheme is not TVD for $r > 2.0$
- the CD scheme is not TVD for $r < 0.5$
- the QUICK scheme is not TVD for $r < 3/7$ and $r > 5$



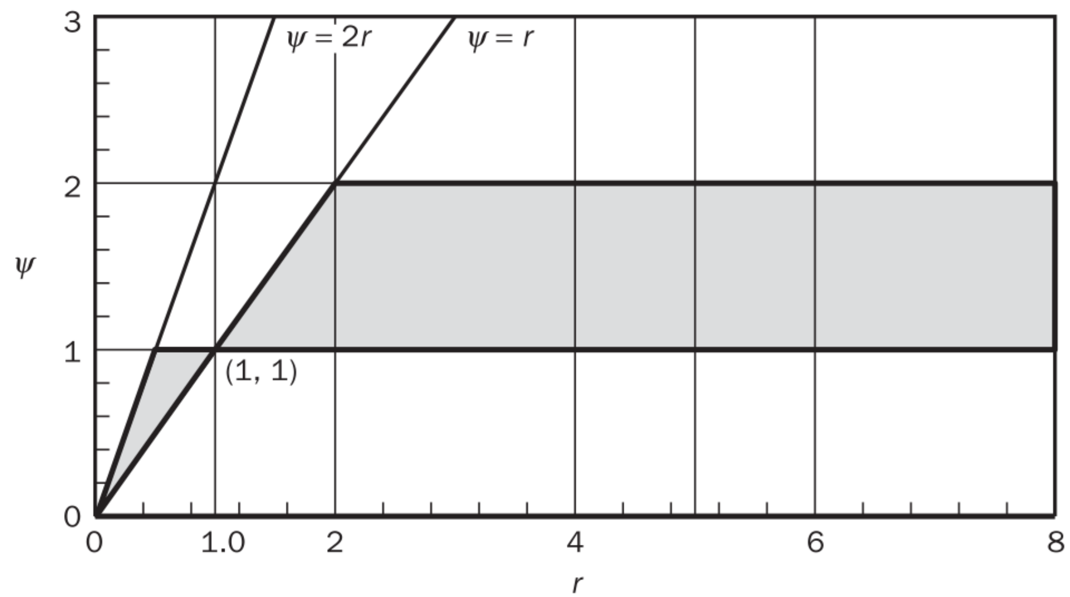
Criteria for TVD schemes

- The idea of designing a TVD scheme is to introduce a modification to the above schemes so as to force the r–ψ relationship to remain within the shaded region for all possible values of r.
- Since ψ(r) is limiting the estimated flux that is called a ***flux limiter*** function
- Sweby (1984) also introduced the following requirement for second-order accuracy in terms of the relationship ψ = ψ(r):

The flux limiter function of a second-order accurate scheme should pass through the point (1, 1) in the r–ψ diagram

Criteria for TVD schemes

- Sweby also showed that the range of possible second-order schemes is bounded by CD and LUD:



- If $0 < r < 1$ the lower limit is $\psi(r) = r$, the upper limit is $\psi(r) = 1$, so for TVD schemes $r \leq \psi(r) \leq 1$
- If $r \geq 1$ the lower limit is $\psi(r) = 1$, the upper limit is $\psi(r) = r$, so for TVD schemes $1 \leq \psi(r) \leq r$

Criteria for TVD schemes

- Sweby finally introduced the symmetry property for limiter functions:

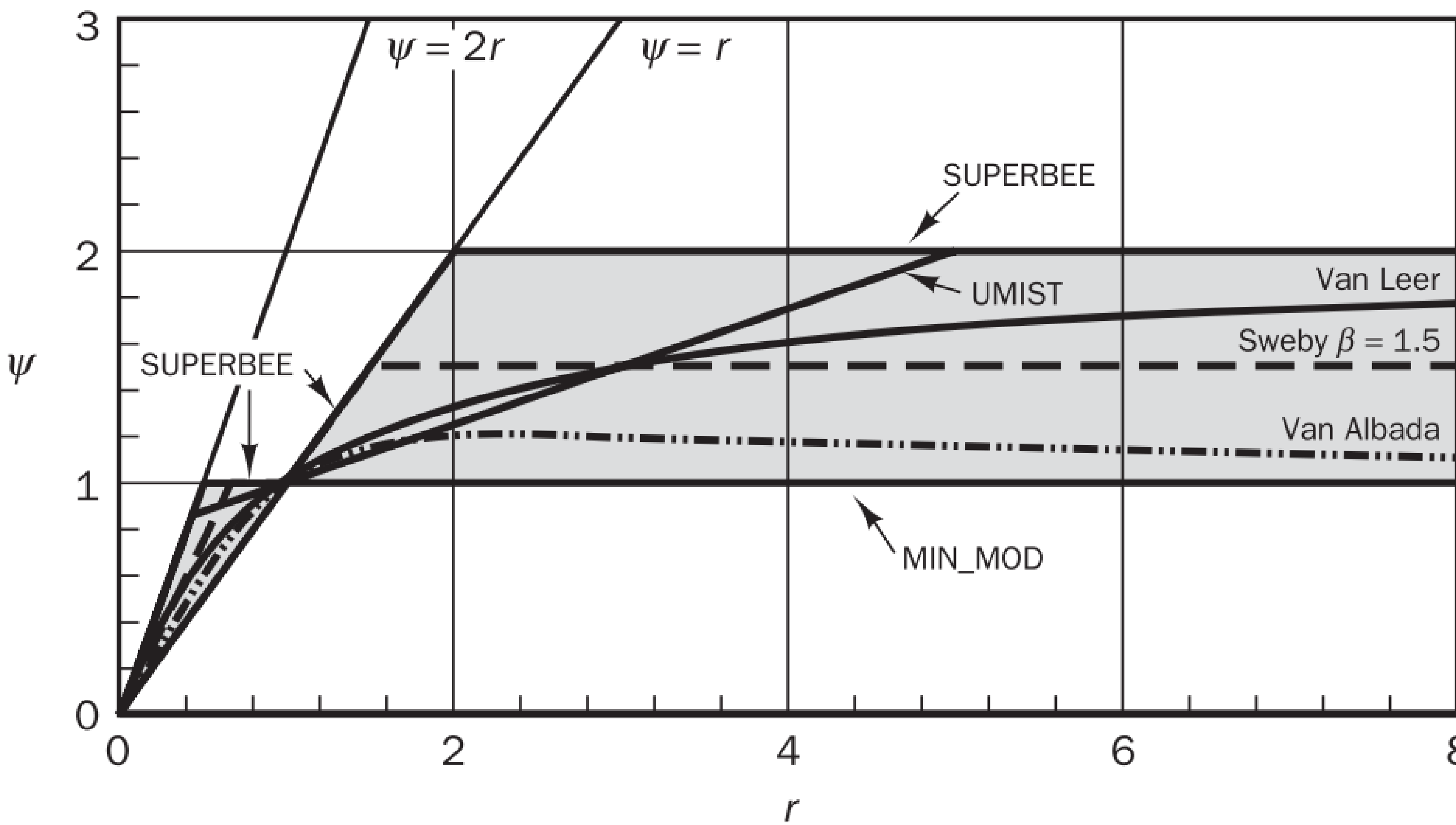
$$\frac{\psi(r)}{r} = \psi(1/r)$$

- A limiter function that satisfies the symmetry property ensures that backward- and forward-facing gradients are treated in the same fashion without the need for special coding.

Flux limiter functions

Name	*Limiter function* $\psi(r)$	*Source*
Van Leer	$\dfrac{r+\lvert r\rvert}{1+r}$	Van Leer (1974)
Van Albada	$\dfrac{r+r^2}{1+r^2}$	Van Albada *et al*. (1982)
Min-Mod	$\psi(r)=\begin{cases}\min(r,1) & \text{if } r>0\\ 0 & \text{if } r\le 0\end{cases}$	Roe (1985)
SUPERBEE	$\max[0, \min(2r, 1), \min(r, 2)]$	Roe (1985)
Sweby	$\max[0, \min(\beta r, 1), \min(r, \beta)]$	Sweby (1984)
QUICK	$\max[0, \min(2r, (3+r)/4, 2)]$	Leonard (1988)
UMIST	$\max[0, \min(2r, (1+3r)/4, (3+r)/4, 2)]$	Lien and Leschziner (1993)

Flux limiter functions



Flux limiter functions

- All the limiter functions stay inside the TVD region and pass through the point (1, 1), so they all represent second-order accurate TVD schemes.
- Van Leer and Van Albada's limiters are smooth functions, whereas all the others are piecewise linear expressions.
- The Min-Mod limiter function exactly traces the lower limit of the TVD region
- Roe's SUPERBEE scheme follows the upper limit.
- Sweby's expression is a generalisation of the Min-Mod and SUPERBEE limiters by means of a single parameter β. The limiter becomes the Min-Mod limiter when $\beta = 1$ and the SUPERBEE limiter of Roe when $\beta = 2$.
- To stay within the TVD region we only consider the range of values $1 \leq \beta \leq 2$.
- Leonard's QUICK limiter function is the only one that is non-symmetric
- Lien and Leschziner's UMIST limiter function was designed as a symmetrical version of the QUICK limiter.

Implementation of TVD schemes

- The diffusion term is discretised using central differencing as before, but the convective flux is now evaluated using a TVD scheme.

$$F_e\phi_e - F_w\phi_w = D_e(\phi_E - \phi_P) - D_w(\phi_P - \phi_W)$$

- For flow in the positive x-direction u > 0

$$\phi_e = \phi_P + \frac{1}{2}\psi(r_e)(\phi_E - \phi_P)$$

$$\phi_w = \phi_W + \frac{1}{2}\psi(r_w)(\phi_P - \phi_W)$$

$$\text{where } r_e = \left(\frac{\phi_P - \phi_W}{\phi_E - \phi_P}\right) \text{ and } r_w = \left(\frac{\phi_W - \phi_{WW}}{\phi_P - \phi_W}\right)$$

$$\longrightarrow \quad F_e\left[\phi_P + \frac{1}{2}\psi(r_e)(\phi_E - \phi_P)\right] - F_w\left[\phi_W + \frac{1}{2}\psi(r_w)(\phi_P - \phi_W)\right]$$

$$= D_e(\phi_E - \phi_P) - D_w(\phi_P - \phi_W)$$

$$\longrightarrow$$

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + \boxed{S_u^{DC}}$$

$$\text{where } a_W = D_w + F_w$$

$$a_E = D_e$$

$$a_P = a_W + a_E + (F_e - F_w)$$

$$S_u^{DC} = -F_e\left[\frac{1}{2}\psi(r_e)(\phi_E - \phi_P)\right] + F_w\left[\frac{1}{2}\psi(r_w)(\phi_P - \phi_W)\right]$$

Implementation of TVD schemes

- As mentioned earlier, the above derivation is for the positive flow direction. To note the flow direction we use a superscript '+'. Therefore both r_e and r_w are replaced with r_w^+ and r_e^+.

$$S_u^{DC} = -F_e\left[\frac{1}{2}\psi(r_e^+)(\phi_E - \phi_P)\right] + F_w\left[\frac{1}{2}\psi(r_w^+)(\phi_P - \phi_W)\right]$$

- For u < 0, it can be shown in the same manner:

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + S_u^{DC}$$

$$\text{where } a_W = D_w$$

$$a_E = D_e - F_e$$

$$a_P = a_W + a_E + (F_e - F_w)$$

$$S_u^{DC} = F_e\left[\frac{1}{2}\psi(r_e^-)(\phi_E - \phi_P)\right] - F_w\left[\frac{1}{2}\psi(r_w^-)(\phi_P - \phi_W)\right]$$

$$\text{where } r_e^- = \left(\frac{\phi_{EE} - \phi_E}{\phi_E - \phi_P}\right) \text{ and } r_w^- = \left(\frac{\phi_E - \phi_P}{\phi_P - \phi_W}\right)$$

Implementation of TVD schemes

- Thus the TVD scheme for one-dimensional convection–diffusion problems may be written as

$$\boxed{a_P\phi_P = a_W\phi_W + a_E\phi_E + S_u^{DC}}$$

$$\boxed{a_P = a_W + a_E + (F_e - F_w)}$$

	TVD neighbour coefficients
a_W	$D_w + \max(F_w, 0)$
a_E	$D_e + \max(-F_e, 0)$
	TVD deferred correction source term
S_u^{DC}	$\frac{1}{2}F_e[(1-\alpha_e)\psi(r_e^-) - \alpha_e \,.\, \psi(r_e^+)](\phi_E - \phi_P) + \frac{1}{2}F_w[\alpha_w \,.\, \psi(r_w^+) - (1-\alpha_w)\psi(r_w^-)](\phi_P - \phi_W)$

where

$$\alpha_w = 1 \text{ for } F_w > 0 \text{ and } \alpha_e = 1 \text{ for } F_e > 0$$
$$\alpha_w = 0 \text{ for } F_w < 0 \text{ and } \alpha_e = 0 \text{ for } F_e < 0$$

Treatment at the boundaries

- At inlet/outlet boundaries it is necessary to generate upstream/downstream values to evaluate the values of r.
- These can be obtained using the extra-polated mirror node practice that was demonstrated for the QUICK scheme

$$F_e\left[\phi_P + \frac{1}{2}\psi(\underline{r_e})(\phi_E - \phi_P)\right] \boxed{- F_A\phi_A} = D_e(\phi_E - \phi_P) - D_A^*(\phi_P - \phi_A)$$

$$(D_e + F_e + D_A^*)\phi_P = D_e\phi_E + (D_A^* + F_A)\phi_A - F_e\frac{1}{2}\psi(r_e)(\phi_E - \phi_P)$$

with $D_A^* = \Gamma/\delta x$

The problem is to find

$$r_e = \left(\frac{\phi_P - \boxed{\phi_W}}{\phi_E - \phi_P}\right) \longrightarrow \qquad \phi_o = 2\phi_A - \phi_P \quad \text{so} \quad r_e = \left(\frac{\phi_P - \phi_o}{\phi_E - \phi_P}\right) = \frac{2(\phi_P - \phi_A)}{\phi_E - \phi_P}$$