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Finite-difference Solution of Poisson's Equation in Rectangles of Arbitrary Proportions[1])

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1. Introduction

We consider the problem of getting an approximation of reasonably good accuracy by finite-difference methods for the function $u(x, y)$ which satisfies Poisson's equation

$$\nabla^2 u(x, y) = f(x, y) \tag{1.1}$$

inside a rectangle R, and satisfies various boundary conditions on the boundary of R. When $f(x, y) \equiv 0$, (1.1) reduces to Laplace's equation, and the problem is appreciably simpler.

This problem has been much studied. A common approach is to cover R exactly with a mesh or grid of small rectangles, after which one can replace (1.1) by a finite-difference approximation involving values of $u(x, y)$ at the grid points. One then tries to solve this finite-difference analogue of (1.1) to a suitable degree of accuracy. In order to employ this approach when high accuracy is required, it has been necessary to require that the ratio of the sides of R must be rational since use of high order methods usually requires that one covers R exactly with a grid of squares. However, the conformal transformation method of Papamichael and Whiteman [2] will lead more often than not to a rectangle in which the ratio is not rational, and covering with a grid of squares is not possible. Even when the ratio is rational, there may be difficulties. Suppose, from some engineering problem, one is confronted with a rectangle R of base $6\frac{5}{8}$ and height $5\frac{7}{8}$. If this is to be covered exactly with squares, there must be $53N$ squares along the base and $47N$ squares along a vertical side, where N is a positive integer. With such a covering, many popular methods would operate at less than maximum efficiency.

Accordingly, we will propose a method of getting good accuracy with moderate labor for rectangles of arbitrary proportions.

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2. Formulation of the Problem

By rotation, translation, and scaling, as needed, we can take the rectangle R to be that shown in Figure 1. By rotating through another 90° and translating and scaling again, if need be, we can assure that $a \geq \pi$. If $a = \pi$, we have a square, and familiar approaches suffice. So we assume $a > \pi$.

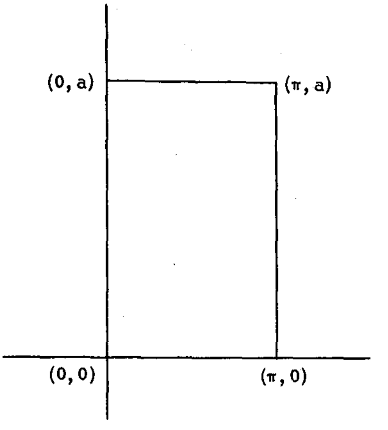

Figure 1
The rectangle R.

We consider first the case of Dirichlet boundary conditions. That is, we wish to approximate the function $u(x, y)$ which is continuous on and inside R, satisfies

$$\nabla^2 u(x, y) = f(x, y) \tag{2.1}$$

inside R, and on the sides of R satisfies the Dirichlet boundary conditions

$$u(0, y) = g_0(y) \quad 0 < y < a \tag{2.2}$$
$$u(\pi, y) = g_\pi(y) \quad 0 < y < a \tag{2.3}$$
$$u(x, 0) = h_0(x) \quad 0 < x < \pi \tag{2.4}$$
$$u(x, a) = h_a(x) \quad 0 < x < \pi. \tag{2.5}$$

Because we seek a $u(x, y)$ which is continuous on R, as well as inside, we are thereby assuming that $g_0(y)$ and $g_\pi(y)$ are continuous for $0 \leq y \leq a$, that $h_0(x)$ and $h_a(x)$ are continuous for $0 \leq x \leq \pi$, and that

$$g_0(0) = h_0(0), \tag{2.6}$$
$$g_0(a) = h_a(0), \tag{2.7}$$
$$g_\pi(0) = h_0(\pi), \tag{2.8}$$
$$g_\pi(a) = h_a(\pi). \tag{2.9}$$

If there should be discontinuities in the boundary conditions, or their derivatives, this would induce still another source of errors in the solutions, besides those due to truncation and round off. See Rosser [3]. 'Jump' discontinuities can be 'removed' by the methods on pp. 221–222 of Milne [4]. More complicated discontinuities can some-times be 'removed', but one cannot count on doing this. For the present treatment, we assume that the boundary conditions and their low order derivatives are continuous. This includes continuity at the corners, as exemplified by (2.6) through (2.9). Or, if we replace (2.2) by

$$u_x(0, y) = j_0(y) \quad 0 < y < a,$$

then continuity of the first derivatives at the corners would require

$$j_0(0) = h_0'(0)$$
$$j_0(a) = h_a'(0).$$

3. Finite-difference Approximations

There are finite-difference approximations of various orders. The higher order methods of solution, involving the higher order approximations, can be used effectively only when the function $f(x, y)$ which appears in (2.1) has suitable high order smoothness; that is, when it is continuous and has continuous derivatives of suitable orders. Thus the reader must exercise discrimination in choosing which order method to use. When they can be used, the high order methods permit the use of coarse meshes. This can greatly reduce the labor of computation.

For difference approximations of order 2, one can use mesh elements which are rectangles, rather than squares. See Hockney [1]. In this case, there would be no trouble if the ratio of the sides of R were irrational. For difference approximations of order 4, one can also use mesh elements which are rectangles. See Rosser [5]. For difference approximations of order 6, it appears that the mesh elements have to be squares. Details are presented in Rosser [5]. If $f(x, y)$ in (2.1) is sufficiently smooth, this permits one to use quite a coarse mesh, greatly reducing the computational labor. However, this raises the question how to proceed if the ratio of the sides of R is irrational.

4. Ill-proportioned Rectangles

We take h to be the side of the square mesh element. We arrange that the squares can be fitted along the base of R. That is, we take M to be a positive integer, and define

$$h = \frac{\pi}{M}. \tag{4.1}$$

We take N to be the integer part of aM/π; in symbols

$$N = \left[\frac{aM}{\pi}\right]. \tag{4.2}$$

Then

$$Nh \leq a, \tag{4.3}$$
$$(N + 1)h > a. \tag{4.4}$$

If

$$Nh = a, \tag{4.5}$$

then we can fill up the rectangle R exactly with MN squares of side h, and the methods of Rosser [5] are applicable. So we are interested only in the case $Nh < a$. We could assume this, but it is not required for the analysis which follows. If we should have (4.5) holding, then some of the steps of the subsequent analysis would be quite trivial but not incorrect in any way.

We begin by defining

$$b = Nh \tag{4.6}$$

$$c = a - b = a - Nh. \tag{4.7}$$

We take R_b to be the rectangle with corners $(0, 0)$, $(0, b)$, $(\pi, 0)$, and (π, b), and take R_c to be the rectangle with corners $(0, c)$, $(0, a)$, (π, c) and (π, a).

We choose $h_b(x)$ to be a smooth function such that

$$h_b(0) = g_0(b)$$
$$h_b(\pi) = g_\pi(b).$$

The better we can choose $h_b(x)$ to approximate $u(x, b)$, the more we can curtail certain computations later. With the limited information available at this stage, we content ourselves with taking

$$h_b(x) = h_a(x) + \left(1 - \frac{x}{\pi}\right)(g_0(b) - h_a(0)) + \frac{x}{\pi}(g_\pi(b) - h_a(\pi)).$$

We take $u_b(x, y)$ to be the function which is continuous on and inside R_b, satisfies (2.1) inside R_b, and on the sides of R_b satisfies the boundary conditions

$$u_b(0, y) = g_0(y) \quad 0 \leq y \leq b \tag{4.8}$$
$$u_b(\pi, y) = g_\pi(y) \quad 0 \leq y \leq b \tag{4.9}$$
$$u_b(x, 0) = h_0(x) \quad 0 \leq x \leq \pi \tag{4.10}$$
$$u_b(x, b) = h_b(x) \quad 0 \leq x \leq \pi. \tag{4.11}$$

We take $u_c(x, y)$ to be the function which is continuous on and inside R_c, satisfies (2.1) inside R_c, and on the sides of R_c satisfies the boundary conditions

$$u_c(0, y) = g_0(y) \quad c \leq y \leq a \tag{4.12}$$
$$u_c(\pi, y) = g_\pi(y) \quad c \leq y \leq a \tag{4.13}$$
$$u_c(x, c) = u_b(x, c) \quad 0 \leq x \leq \pi \tag{4.14}$$
$$u_c(x, a) = h_a(x) \quad 0 \leq x \leq \pi. \tag{4.15}$$

By our definition of $h_b(x)$, we see that $u_b(x, y)$ has continuous boundary conditions around the rectangle R_b. Then it follows by (4.14) that the same holds for $u_c(x, y)$ relative to the rectangle R_c. This is why in (4.8) through (4.15) we can use $\leq$ rather than $<$.

By (4.1) and (4.6) we can fill up the rectangle R_b exactly with MN squares of side h. Thus we can use the 9-point difference approximation of Rosser [5] to get accurate approximations for $u_b(x, y)$ inside R_b at the grid points (mh, nh). From these, we can get accurate approximations for $u_b(mh, c)$. By (4.14) these are part of the boundary values for $u_c(x, y)$. Thus it is necessary to determine them to order h^6. By the principle of the maximum, it is also sufficient. For a given m, the point (mh, c) is on a vertical grid line. Thus one can determine $u_b(mh, c)$ to order h^6 by using a high order interpolation formula in one dimension on the values at the six grid points $(mh, 0)$, (mh, h), $(mh, 2h)$, $(mh, 3h)$, $(mh, 4h)$, and $(mh, 5h)$.

By (4.14), this gives us good approximations to $u_c(x, c)$ at $x = h, 2h, \ldots, (M - 1)h$. By (4.1) and (4.7) we can fill up the rectangle R_c exactly with MN squares of side h. Thus we can use the 9-point difference approximation of Rosser [5] to get accurate approximations for $u_c(x, y)$ inside R_c at the grid points $(mh, c + nh)$. Then we can get accurate approximations for $u_c(mh, b)$ by the method mentioned earlier.

We define R_{bc} to be the rectangle which is the intersection of the rectangles R_b and R_c. In R_{bc}, the function $u_c(x, y) - u_b(x, y)$ is harmonic. Also, it is zero along the bottom and along the two vertical sides. So on and inside R_{bc} we have

$$u_c(x, y) - u_b(x, y) = \sum_{r=1}^{\infty} a_r \frac{\sinh r(y - c)}{\sinh r(b - c)} \sin rx \tag{4.16}$$

where

$$a_r = \frac{2}{\pi} \int_0^{\pi} \{u_c(x, b) - u_b(x, b)\} \sin rx \, dx. \tag{4.17}$$

Clearly the $|a_r|$ are bounded by

$$2 \max_{0 \le x \le \pi} |u_c(x, b) - u_b(x, b)|. \tag{4.18}$$

We recall (see (4.11)) that

$$u_b(x, b) = h_b(x).$$

Presumably $u_c(x, b)$ is fairly close to $u(x, b)$. If also we were lucky enough to choose $h_b(x)$ fairly close to $u(x, b)$, then by (4.18) the a_r will be fairly small. This will save computational effort later.

On and inside R define

$$v(x, y) = \sum_{r=1}^{\infty} a_r b_r \frac{\sinh r(a - y)}{\sinh ra} \sin rx, \tag{4.19}$$

where

$$b_r = \frac{\sinh rc}{\sinh r(b - c)}. \tag{4.20}$$

On and inside R_b define

$$u(x, y) = u_b(x, y) + v(x, y) + \sum_{r=1}^{\infty} a_r \frac{\sinh r(y - c)}{\sinh r(b - c)} \sin rx. \tag{4.21}$$

We see that $u(x, y)$ is continuous on and inside the rectangle R_b, satisfies (2.1) inside R_b, and on three sides satisfies the boundary conditions (4.8), (4.9), and (4.10). By (4.16), we see that on and inside R_{bc} we have

$$u(x, y) = u_c(x, y) + v(x, y). \tag{4.22}$$

We use (4.22) to define $u(x, y)$ for the rest of the rectangle R_c. Then $u(x, y)$ is continuous on and inside the rectangle R_c, satisfies (2.1) inside R_c, and on three sides satisfies the boundary conditions (4.12), (4.13), and (4.15).

Thus we see that $u(x, y)$ is exactly the function $u(x, y)$ that we were seeking to obtain.

We have obtained accurate approximations for $u_b(x, y)$ and $u_c(x, y)$ at various grid points. If M is of reasonable size, then c is small, since $0 \leq c \leq h$ by (4.7), (4.3), and (4.4). As a is greater than π, and $b = a - c$ by (4.7), we see that the series on the right of (4.19) is rapidly convergent for $0 \leq y \leq a$. Also, the series appearing on the right of (4.21) is rapidly convergent for small y, certainly for $0 \leq y \leq h$. If in addition the a_r are all quite small (see (4.18)), then very few terms of the series are needed to get high accuracy. So, using the known approximations for $u_b(mh, nh)$, we can get approximate values for $u(x, y)$ for small y by (4.21). For all other values of y, we can use the known approximations for $u_c(mh, c + nh)$ to get approximate values for $u(x, y)$ by (4.22).

The calculation of the a_r presents no problem. Not more than four or five will be required; fewer if the a_r are all small. Observe that the values of $u_b(x, b)$ are given by (4.11). Also, we had got accurate approximations for $u_c(mh, b)$. So we can use a numerical quadrature formula to calculate the a_r by (4.17).

CAUTION. If r is not fairly small compared to N, then there will be fairly few abscissa points in each cycle of $\sin rx$ in (4.17); in such case the usual quadrature formulas are not trustworthy. One can get twice, or four times, or eight times, as many abscissa points by interpolating to get approximations for $u_c(x, b)$ at the additional abscissa points (recall that $u_b(x, b)$ is given by (4.11)). For this interpolation one can use a high order one dimensional interpolation formula on the values $u_c(0, b)$, $u_c(h, b)$, $u_c(2h, b), \ldots$.

We need high accuracy for only the first one or two of the a_r, because of the very rapid convergence of the series appearing on the right of (4.19) and (4.21). In any case, one should increase the number of abscissa points, as needed, to the point where one can use a quadrature formula with assurance. Also, by a little foresight in the choice of M, one can arrange that, after increasing the number of abscissa points if needed, one can use a high order quadrature formula, like Bode's Rule, for example.

5. Tests for Accuracy

One advantage of using the 9-point difference approximation when one can exactly fill up the rectangle with squares is that one can make a first calculation, for less than a quarter of the calculating effort, with squares twice as large on a side, and then repeat with the smaller squares. Because the error is of the order of h^6, one can get an estimate of the error.

This can be done with the present procedure by choosing M divisible by 2. If N is not divisible by 2, the values of b and c which are used with the squares of side $2h$ will not be the same as those which are used with the squares of side h. However, this does not matter.

One dividend that will accrue from making an initial calculation with squares of side $2h$ is that from this calculation one can derive a very good approximation to take for $h_b(x)$. Then, for the calculation with squares of side h, the a_r will be very small, so that not more than two or three of them will be needed.

6. Neumann Boundary Conditions

Suppose we have the same rectangle R, and impose on $u(x, y)$ the same conditions as before, except that on top of the rectangle R we specify values to be taken by $u_y(x, a)$. That is we replace (2.5) by the Neumann condition

$$u_y(x, a) = k_a(x) \quad 0 < x < \pi. \tag{6.1}$$

We postpone to the latter part of the section a discussion of how one would handle this in the case in which a/π is rational, so that one can fill up R exactly with squares of side h. For the moment, let us assume that this can be done, and explain how to generalize to the case in which a/π is irrational.

We proceed very nearly as in Section 4. Instead of the definition given there of $h_b(x)$, we use

$$h_b(x) = \left(1 - \frac{x}{\pi}\right) g_0(b) + \frac{x}{\pi} g_\pi(b). \tag{6.2}$$

We take $u_b(x, y)$ as before, but for $u_c(x, y)$ we replace (4.15) by the analogue of (6.1), namely

$$\frac{\partial}{\partial y} u_c(x, a) = k_a(x) \quad 0 < x < \pi. \tag{6.3}$$

Everything now goes the same, down to the definition of $v(x, y)$. Let us pause a moment, and think what we require of $v(x, y)$. Clearly it should be harmonic, so that $u(x, y)$, as defined in part by (4.21) and in part by (4.22), will satisfy (2.1) inside R. Also, we wish $v(x, y)$ to be zero on the vertical sides of R, so that there $u(x, y)$ will satisfy the proper boundary conditions. Also, on the bottom of R, we must have

$$v(x, y) = \sum_{r=1}^{\infty} a_r \frac{\sinh rc}{\sinh r(b - c)} \sin rx \quad 0 < x < \pi \tag{6.4}$$

so that by (4.21) $u(x, y)$ will satisfy the right boundary conditions on the bottom of R. Finally, looking at (4.22), we see that if $u(x, y)$ is to satisfy the right boundary conditions on the top of R, we must have

$$v_y(x, a) = 0 \quad 0 < x < \pi. \tag{6.5}$$

All these conditions can be met by simply replacing the factor

$$\frac{\sinh r(a - y)}{\sinh ra}$$

in the definition of $v(x, y)$ by

$$\frac{\cosh r(a - y)}{\cosh ra}.$$

In this case, since it is unlikely that (6.2) makes $h_b(x)$ come out very close to $u(x, b)$, we cannot count on the a_r being particularly small, so that two or three more of them might have to be calculated. It might be better to turn the rectangle R upside down and proceed as follows.

Consider next the case in which the Neumann condition is at the bottom of R. That is, $u(x, y)$ satisfies (2.2), (2.3), and (2.5), but (2.4) is replaced by

$$u_y(x, 0) = k_0(x) \quad 0 < x < \pi. \tag{6.6}$$

Again, we proceed nearly as in Section 4. We can now take $h_b(x)$ the same as in Section 4, which should lead to smaller values of the a_r, so that we can get by with calculating fewer of them. For the definition of $u_b(x, y)$, we replace (4.10) by the analogue of (6.6), namely

$$\frac{\partial}{\partial y} u_b(x, 0) = k_0(x) \quad 0 < x < \pi. \tag{6.7}$$

We take $u_c(x, y)$ as in Section 4, and continue the same down to the definition of $v(x, y)$. A key requirement is that $u(x, y)$, as defined by (4.21), shall satisfy the proper boundary conditions at the bottom of R. In Section 4, this required that

$$v(x, y) + \sum_{r=1}^{\infty} a_r \frac{\sinh r(y - c)}{\sinh r(b - c)} \sin rx \tag{6.8}$$

should be zero when $y = 0$. This was accomplished by the proper choice of the b_r. Now we must assure that the partial derivative of (6.8) with respect to y shall be zero when $y = 0$. Again, this is accomplished by the proper choice of the b_r; specifically we now take

$$b_r = \frac{-\sinh ra}{\sinh r(b - c)} \frac{\cosh rc}{\cosh ra}. \tag{6.9}$$

All else remains the same.

Next consider the case in which there are Neumann conditions both at the top and the bottom of R. That is, $u(x, y)$ satisfies (2.2) and (2.3), but (2.4) is replaced by (6.6) and (2.5) is replaced by (6.1). We proceed much as in Section 4. In the definition of $u_b(x, y)$ we replace (4.10) by (6.7), and in the definition of $u(x, y)$ we replace (4.15) by (6.3). We define $h_b(x)$ by (6.2). It is then easily verified that we should replace

$$\frac{\sinh r(a-y)}{\sinh ra}$$

in the definition of $v(x, y)$ by

$$\frac{\cosh r(a-y)}{\cosh ra}$$

and define

$$b_r = \frac{\cosh ra}{\sinh r(b-c)} \frac{\cosh rc}{\sinh ra}. \tag{6.10}$$

One can of course have Neumann conditions on one or both of the vertical sides. Let us consider first the case in which there are Neumann conditions on both vertical sides, but Dirichlet conditions at the top and bottom. Rotation by 90° would reduce this to the case just considered. However, this is not desirable, since we would then lose the qualification that the height is greater than the base. It was this that assured the rapid convergence of the Fourier series in (4.19) and (4.21).

So we assume that (2.4) and (2.5) hold, but that (2.2) and (2.3) are replaced by

$$u_x(0, y) = j_0(y) \quad 0 < y < a \tag{6.11}$$
$$u_x(\pi, y) = j_\pi(y) \quad 0 < y < a. \tag{6.12}$$

We proceed analogously to Section 4, except that we use cosines instead of sines throughout. Because it is desirable to have $u_x(x, y)$ continuous around the boundary we define

$$h_b(x) = h_a(x) + \frac{1}{2\pi}(x-\pi)^2(h_a'(0) - j_0(b)) + \frac{x^2}{2\pi}(j_\pi(b) - h_a'(\pi)). \tag{6.13}$$

We define $u_b(x, y)$ and $u_c(x, y)$ as in Section 4, except that they now have Neumann conditions on their vertical sides. We replace (4.16) and (4.17) by

$$u_c(x, y) - u_b(x, y) = \sum_{r=0}^{\infty} a_r \frac{\sinh r(y-c)}{\sinh r(b-c)} \cos rx \tag{6.14}$$

where

$$a_0 = \frac{1}{\pi}\int_0^\pi \{u_c(x, b) - u_b(x, b)\}\, dx \tag{6.15}$$

$$a_r = \frac{2}{\pi}\int_0^\pi \{u_c(x, b) - u_b(x, b)\} \cos rx\, dx. \tag{6.16}$$

When $r = 0$, we define

$$\frac{\sinh r(y-c)}{\sinh r(b-c)} = \frac{y-c}{b-c}.$$

Exactly analogous changes are made in (4.19) and (4.21).

If, in addition to the Neumann conditions on the vertical sides, we replace one or both of the Dirichlet conditions on the top or bottom by Neumann conditions, we can modify the procedure just outlined quite analogously to the way in which we modified the procedure of Section 4 earlier in this section.

It will be noted that we are allowing the possibility of Neumann conditions on all four sides. For this, there will be a solution only if the boundary conditions satisfy a certain criterion. If they do, the solution is not unique, but any two solutions differ by a constant. The procedure outlined will produce one of this infinity of solutions if and only if there is a solution.

To handle the case of a Dirichlet condition on the left side and a Neumann condition on the right side, we replace $\sin rx$ by

$$\sin\left(r - \tfrac{1}{2}\right) x,$$

with suitable related changes. To handle the case of a Dirichlet condition on the right side and a Neumann condition on the left side, we replace $\sin rx$ by

$$\cos\left(r - \tfrac{1}{2}\right)x.$$

We consider finally how to handle the case in which the rectangle has a rational ratio of the sides, and we have filled it exactly with squares of side h, and wish to approximate $u(x, y)$ at the grid points. At interior grid points, we can use one of the formulas of Rosser [5]. On boundaries where there are Dirichlet boundary conditions, we assign $\bar{u}_{m,n}$ the specified value. This leaves only the boundary points where there is a Neumann condition to be dealt with. Suppose, for example, that the condition (6.11) holds on the left side of R. We note that

$$\begin{aligned} hf_x(x, y) \cong & -\frac{137}{60} f(x, y) + 5f(x+h, y) - 5f(x+2h, y) \\ & + \frac{10}{3} f(x+3h, y) - \frac{5}{4} f(x+4h, y) + \frac{1}{5} f(x+5h, y) \end{aligned} \tag{6.17}$$

holds to within terms of order h^6. If we take $x = 0$ and $y = nh$, we get by (6.11)

$$\begin{aligned} hj_0(nh) \cong & -\frac{137}{60} \bar{u}_{0,n} + 5\bar{u}_{1,n} \\ & -5\bar{u}_{2,n} + \frac{10}{3} \bar{u}_{3,n} - \frac{5}{4} \bar{u}_{4,n} + \frac{1}{5} \bar{u}_{5,n}. \end{aligned} \tag{6.18}$$

One could use a higher order formula than (6.17), but it probably suffices. A heuristic argument for this is as follows. By the principle of the maximum, if we wish to determine interior points to order h^6, it is sufficient to determine the boundary points to order h^6. However, if the interior points are given to order h^6, (6.18) will determine $\bar{u}_{0,n}$ to order h^6.

Use of (6.18) with the formulas of Rosser [5] results in a rather messy matrix of coefficients of the $\bar{u}_{m,n}$. However, one is probably using such a coarse mesh that this matrix would be less than 100×100, perhaps even less than 50×50. If so, probably the quickest method of solution is to use the standard computer routine for solving simultaneous linear equations. If this is done, it does not much matter if the matrix is messy or not.

If it happens that one is solving the Laplace equation, with $f(x, y) \equiv 0$, and has a zero normal derivative along one side, say $j_0(y) \equiv 0$, one can use the reflection principle to replace (6.18) by something which seems conceptually simpler. However, it involves three boundary grid points and three interior points, and so is probably about as much bother on a computer as (6.18), which also involves six grid points.

If one has Neumann conditions on one or more sides, and so is using (6.18), one might consider the following procedure, which would bypass the treatment in Section 4 altogether. Almost always, there is at least one side with Dirichlet conditions. By rotating and relinquishing the qualification $a > \pi$, if need be, we can arrange to have Dirichlet conditions on top. If, in the notation of Section 4, we have $0 < c < h$, the difficulty is that we have no good way to write down an equivalent of (3.7) of Rosser [5] for the values of $u(x, y)$ at the row of grid points (mh, Nh), $1 \leq m \leq M - 1$. As a substitute, write down (3.7) of Rosser [5] for the 9-point formula centered at $(mh, a - h)$. It involves values of $u(x, y)$ at $((m - 1)h, a - h)$, $((m - 1)h, a - 2h)$, $(mh, a - h)$, $(mh, a - 2h)$, $((m + 1)h, a - h)$, $((m + 1)h, a - 2h)$, as well as at the boundary points $((m - 1)h, a)$ (mh, a), and $((m + 1)h, a)$, at which latter points $u(x, y)$ is known. Now, by a high order one dimensional interpolation formula, we can write each of $u(rh, a - h)$ and $u(rh, a - 2h)$, approximately as a linear combination of $u(rh, nh)$ for $n \leq N$; we do this for $r = m - 1$, $r = m$, and $r = m + 1$. So we get a formula involving $u(rh, Nh)$, $u(rh, (N - 1)h)$, etc., for $r = m - 1, m, m + 1$, which we can use in place of (3.7) of Rosser [5]. Probably interpolation of order eight should be used. This makes the matrix still messier, but if we are having to deal with a messy matrix anyhow, because of the Neumann conditions, the idea might be worth considering.

References

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Abstract

The recently devised sixth order method for solving difference analogues of Poisson's equation affords much economy of computing effort if squares can be used for the grid elements. However, some developments result in rectangles with one side an irrational multiple of the other, for which the grid elements cannot be squares. A method is presented for handling this situation.

Zusammenfassung

Die vor kurzem hergeleitete Methode sechster Ordnung für die Lösung der Poissonschen Differentialgleichung analogen Differenzengleichungen führt zu grosser Ökonomie im Rechenaufwand, falls man Quadrate für die Netzelemente benutzen kann. In gewissen Situationen hat man es jedoch mit Rechtecken zu tun, deren eine Seite ein irrationales Vielfaches der anderen Seite ist, für die die Netzelemente also keine Quadrate sein können. Eine Methode zur Handhabung solcher Situationen wird angegeben.

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