

# Discrete lattice effects on the forcing term in the lattice Boltzmann method

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We show that discrete lattice effects must be considered in the introduction of a force into the lattice Boltzmann equation. A representation of the forcing term is then proposed. With the representation, the Navier-Stokes equation is derived from the lattice Boltzmann equation through the Chapman-Enskog expansion. Several other existing force treatments are also examined.

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The rapidly developing lattice Boltzmann method (LBM), a technique for modeling complex fluid systems, has attracted much attention in a variety of fields [1,2]. There is a wide range of fluid problems in which an external or internal force is involved, such as water waves and multiphase or multicomponent fluids. To design lattice Boltzmann models for these systems, the force must be treated appropriately in order to obtain the correct hydrodynamics. In this paper a representation of the forcing term is proposed in which discrete lattice effects are considered. Several other existing methods are also examined.

The lattice Boltzmann equation (LBE) without a force can be expressed as

$$f_i(\mathbf{x} + \mathbf{e}_i \Delta t, t + \Delta t) - f_i(\mathbf{x}, t) = -\frac{1}{\tau} [f_i(\mathbf{x}, t) - f_i^{(eq)}(\mathbf{x}, t)], \quad (1)$$

where  $f_i(\mathbf{x}, t)$  is the distribution function (DF) for particles with velocity  $\mathbf{e}_i$  at position  $\mathbf{x}$  and time  $t$ , and  $\Delta t$  is the time increment.  $f_i^{(eq)}$  is the equilibrium distribution function (EDF) and  $\tau$  is the nondimensional relaxation time. The fluid density  $\rho$  and velocity  $\mathbf{u}$  are determined by the DF,

$$\rho = \sum_i f_i, \quad \rho \mathbf{u} = \sum_i \mathbf{e}_i f_i. \quad (2)$$

The EDF  $f_i^{(eq)}$  in Eq. (1) must be chosen such that the mass and momentum are conserved and some symmetry requirements are satisfied in order that the resulting macroscopic equations describe the correct hydrodynamics of the fluid being simulated. For example, in the D2Q9 [3] model, the particle velocities are defined by  $\mathbf{e}_0 = (0, 0)$ ,  $\mathbf{e}_i = (\cos[\pi(i-1)/2], \sin[\pi(i-1)/2])c$  for  $i = 1-4$ , and  $\mathbf{e}_i = \sqrt{2}(\cos[\pi(i-9/2)/2], \sin[\pi(i-9/2)/2])c$  for  $i = 5-8$ . Here  $c = \Delta x / \Delta t$ , and  $\Delta x$  is the lattice spacing. The EDFs of D2Q9 are chosen to be  $f_i^{(eq)} = E_i(\rho, \mathbf{u})$ , where

$$E_i(\rho, \mathbf{u}) = \omega_i \rho \left[ 1 + \frac{\mathbf{e}_i \cdot \mathbf{u}}{c_s^2} + \frac{\mathbf{u} \mathbf{u} : (\mathbf{e}_i \mathbf{e}_i - c_s^2 \mathbf{I})}{2c_s^4} \right], \quad (3)$$

with  $\omega_0 = 4/9$ ,  $\omega_i = 1/9$  for  $i = 1-4$ , and  $\omega_i = 1/36$  for  $i = 5-8$ .  $c_s = c/\sqrt{3}$  is the sound speed of the model. A two-scale analysis in time will lead to the Navier-Stokes equation.

In the presence of a body force density  $\mathbf{F} = \rho \mathbf{g}$ , where  $\mathbf{g}$  is the acceleration due to  $\mathbf{F}$ , the LBE must be modified to account for the force. We do this by adding an additional term to the LBE:

$$f_i(\mathbf{x} + \mathbf{e}_i \Delta t, t + \Delta t) - f_i(\mathbf{x}, t) = -\frac{1}{\tau} [f_i(\mathbf{x}, t) - f_i^{(eq)}(\mathbf{x}, t)] + \Delta t F_i, \quad (4)$$

where the EDF  $f_i^{(eq)}$  is defined by

$$f_i^{(eq)} = E_i(\rho, \mathbf{u}^*) \quad \text{with} \quad \rho \mathbf{u}^* \equiv \sum_i \mathbf{e}_i f_i + m \mathbf{F} \Delta t. \quad (5)$$

Here  $m$  is a constant to be determined.

The forcing term  $F_i$  can be written in a power series in the particle velocity [4],

$$F_i = \omega_i \left[ A + \frac{\mathbf{B} \cdot \mathbf{e}_i}{c_s^2} + \frac{\mathbf{C} : (\mathbf{e}_i \mathbf{e}_i - c_s^2 \mathbf{I})}{2c_s^4} \right], \quad (6)$$

where  $A$ ,  $\mathbf{B}$ , and  $\mathbf{C}$  are functions of  $\mathbf{F}$  to be determined by requiring that the moments of  $F_i$  are consistent with the hydrodynamic equations. After some calculation, we can obtain the zeroth to second moments of  $F_i$ ,

$$\sum_i F_i = A, \quad \sum_i \mathbf{e}_i F_i = \mathbf{B}, \quad \sum_i \mathbf{e}_i \mathbf{e}_i F_i = c_s^2 A \mathbf{I} + \frac{1}{2} [\mathbf{C} + \mathbf{C}^T]. \quad (7)$$

The macrodynamic behavior arising from the LBE (5) can be found from a multiscale analysis using an expansion parameter  $\epsilon$ , which is proportional to the ratio of the lattice spacing to a characteristic macroscopic length. To do this, the following expansions are introduced [5]:

$$f_i = f_i^{(0)} + \epsilon f_i^{(1)} + \epsilon^2 f_i^{(2)} + \dots, \quad (8a)$$

$$\frac{\partial}{\partial t} = \epsilon \frac{\partial}{\partial t_1} + \epsilon^2 \frac{\partial}{\partial t_2}, \quad \nabla = \epsilon \nabla_1, \quad (8b)$$

$$\mathbf{F} = \epsilon \mathbf{F}_1, \quad A = \epsilon A_1, \quad \mathbf{B} = \epsilon \mathbf{B}_1, \quad \mathbf{C} = \epsilon \mathbf{C}_1. \quad (8c)$$

Expanding  $f_i(\mathbf{x} + \mathbf{e}_i \Delta t, t + \Delta t)$  in Eq. (4) about  $\mathbf{x}$  and  $t$ , and applying the above multiscale expansions to the resulting continuous equation, we can obtain the following equations in consecutive order of the parameter  $\epsilon$ :

$$O(\epsilon^0): \quad f_i^{(0)} = f_i^{(eq)}, \quad (9a)$$

$$O(\epsilon^1): \quad D_{1i} f_i^{(0)} = -\frac{1}{\tau \Delta t} f_i^{(1)} + F_{1i}, \quad (9b)$$

$O(\epsilon^2):$

$$\frac{\partial f_i^{(0)}}{\partial t_2} + \left(1 - \frac{1}{2\tau}\right) D_{1i} f_i^{(1)} = -\frac{1}{\tau \Delta t} f_i^{(2)} - \frac{\Delta t}{2} D_{1i} F_{1i}, \quad (9c)$$

where  $D_{1i} = \partial/\partial t_1 + \mathbf{e}_i \cdot \nabla_1$ .

Taking moments of Eq. (9b), we can obtain the following macroscopic equations on the  $t_1 = \epsilon t$  time scale and  $\mathbf{x}_1 = \epsilon \mathbf{x}$  space scale:

$$\frac{\partial \rho}{\partial t_1} + \nabla_1 \cdot (\rho \mathbf{u}^*) = A_1, \quad (10a)$$

$$\frac{\partial(\rho \mathbf{u}^*)}{\partial t_1} + \nabla_1 \cdot \Pi^{(0)} = \left(n + \frac{m}{\tau}\right) \mathbf{F}_1, \quad (10b)$$

where we assume that  $\mathbf{B}_1 = n \mathbf{F}_1$ , and  $n$  is a constant to be determined.  $\Pi^{(0)}$  is the zeroth-order momentum flux tensor given by  $\Pi_{\alpha\beta}^{(0)} = \sum_i \mathbf{e}_{i\alpha} \mathbf{e}_{i\beta} f_i^{(0)} = c_s^2 \rho \delta_{\alpha\beta} + \rho \mathbf{u}_\alpha^* \mathbf{u}_\beta^*$ . To recover the Euler equations from Eqs. (10), we must choose

$$A = 0, \quad n + \frac{m}{\tau} = 1. \quad (11)$$

The first-order momentum flux  $\Pi^{(1)} \equiv \sum_i \mathbf{e}_i \mathbf{e}_i f_i^{(1)}$  can be simplified using Eq. (10) with the constraint Eq. (11). After some standard algebra, we obtain that

$$\begin{aligned} \Pi_{\alpha\beta}^{(1)} = & -\tau \Delta t \left[ (\mathbf{u}_\alpha^* \mathbf{F}_{1\beta} + \mathbf{u}_\beta^* \mathbf{F}_{1\alpha}) + c_s^2 \rho (\nabla_{1\alpha} \mathbf{u}_\beta^* + \nabla_{1\beta} \mathbf{u}_\alpha^*) \right. \\ & \left. - \frac{1}{2} (\mathbf{C}_{1\alpha\beta} + \mathbf{C}_{1\beta\alpha}) \right] \end{aligned} \quad (12)$$

where the terms of order  $O(u^3)$  or higher have been neglected. If we take  $\mathbf{C} = 2\mathbf{u}^* \mathbf{F}$  or  $\mathbf{C} = \mathbf{u}^* \mathbf{F}_1 + \mathbf{F}_1 \mathbf{u}^*$ , then the momentum flux reduces to the Navier-Stokes expression for the viscous stresses, i.e.,  $\Pi_{\alpha\beta}^{(1)} = \sigma_{1\alpha\beta} \equiv \nu (\nabla_{1\alpha} \mathbf{u}_\beta^* + \nabla_{1\beta} \mathbf{u}_\alpha^*)$ , where the kinematic viscosity  $\nu$  is given by  $\nu = c_s^2 \tau \Delta t$ . This is the expression for  $\mathbf{C}$  given in Refs. [6–8], and the kinematic viscosity is the same as in the solution of the continuous Boltzmann equation. For the LBE, however, the viscos-

ity and the force are modified due to discrete lattice effects. These corrections are from the  $t_2 = \epsilon^2 t$  time scale.

The macroscopic equations on the  $t_2 = \epsilon^2 t$  time scale are derived by taking moments of Eq. (9c). With the aid of Eqs. (10) and (11), the final equations can be written as

$$\frac{\partial \rho}{\partial t_2} = \Delta t \left( m - \frac{1}{2} \right) \nabla_1 \cdot \mathbf{F}_1, \quad (13a)$$

$$\frac{\partial(\rho \mathbf{u}^*)}{\partial t_2} = \Delta t \left( m - \frac{1}{2} \right) \frac{\partial \mathbf{F}_1}{\partial t_1} + \nabla_1 \cdot \boldsymbol{\sigma}_1 \quad (13b)$$

where the stress tensor  $\boldsymbol{\sigma}_1$  is now given by

$$\begin{aligned} \sigma_{1\alpha\beta} = & \left(1 - \frac{1}{2\tau}\right) \nabla_1 \cdot \Pi_{\alpha\beta}^{(1)} - \frac{\Delta t}{4} (\mathbf{C}_{1\alpha\beta} + \mathbf{C}_{1\beta\alpha}) \\ = & \left(\tau - \frac{1}{2}\right) c_s^2 \Delta t \rho (\nabla_{1\alpha} \mathbf{u}_\beta^* + \nabla_{1\beta} \mathbf{u}_\alpha^*) \\ & + \Delta t \left[ \left(\tau - \frac{1}{2}\right) (\mathbf{u}_\alpha^* \mathbf{F}_{1\beta} + \mathbf{u}_\beta^* \mathbf{F}_{1\alpha}) - \frac{\tau}{2} (\mathbf{C}_{1\alpha\beta} + \mathbf{C}_{1\beta\alpha}) \right]. \end{aligned} \quad (14)$$

Clearly, there are additional contributions to the viscous stress due to the discrete lattice effects and the presence of the body force. It is well known that the artifact due to the lattice effect can be absorbed into a redefined viscosity,

$$\nu = \left(\tau - \frac{1}{2}\right) c_s^2 \Delta t. \quad (15)$$

The contribution to the stress due to the force can also be canceled by choosing a proper definition of  $\mathbf{C}$ . One suitable choice is taking

$$\mathbf{C} = \left(1 - \frac{1}{2\tau}\right) 2\mathbf{u}^* \mathbf{F} \quad \text{or} \quad \mathbf{C} = \left(1 - \frac{1}{2\tau}\right) (\mathbf{u}^* \mathbf{F} + \mathbf{F} \mathbf{u}^*). \quad (16)$$

Equations (13) also indicate that the spatial and temporal derivatives influence the density and momentum changes, respectively, on the  $t_2$  time scale. To eliminate these unexpected effects, one must take

$$m = \frac{1}{2} \quad \text{or} \quad \rho \mathbf{u}^* = \sum_i \mathbf{e}_i f_i + \frac{\Delta t}{2} \mathbf{F}. \quad (17)$$

Combining the results on the  $t_1$  and  $t_2$  time scales, Eqs. (10) and (11) together with Eqs. (11), (16), and (17), we now obtain the final macroscopic equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0 \quad (18a)$$

and

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = -\nabla p + \nu \nabla \cdot [\rho(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)] + \mathbf{F}, \quad (18b)$$

where  $p = c_s^2 \rho$  is the pressure, the shear viscosity  $\nu$  is given by Eq. (15), and  $\mathbf{v}$  is the fluid velocity defined by

$$\rho \mathbf{v} = \sum_i \mathbf{e}_i f_i + \frac{\Delta t}{2} \mathbf{F}. \quad (19)$$

As seen, Eqs. (18) are just the Navier-Stokes equations with a body force.

From the above discussions, we can conclude that in order to match the correct Navier-Stokes equations, the forcing term should satisfy the constraints Eqs. (11) and (16) together with Eq. (17), which gives

$$F_i = \left(1 - \frac{1}{2\tau}\right) \omega_i \left[ \frac{\mathbf{e}_i \cdot \mathbf{v}}{c_s^2} + \frac{(\mathbf{e}_i \cdot \mathbf{v})}{c_s^4} \mathbf{e}_i \right] \cdot \mathbf{F}, \quad (20)$$

and the equilibrium velocity  $\mathbf{u}^*$  and the fluid velocity  $\mathbf{v}$  should be given by Eqs. (17) and (19), respectively.

Now we examine some other existing treatments for the body force in the LBM. The usually used method (referred to as method 1) [9] takes  $m=0$ ,  $A=0$ ,  $\mathbf{B}=\mathbf{F}$ , and  $\mathbf{C}=\mathbf{0}$ , i.e.,  $F_i = \omega_i \mathbf{e}_i \cdot \mathbf{F} / c_s^2$ , and the equilibrium velocity  $\mathbf{u}^*$  and fluid velocity  $\mathbf{v}$  are defined by  $\rho \mathbf{v} = \rho \mathbf{u}^* = \sum_i \mathbf{e}_i f_i$ . This method satisfies the constraint Eq. (11), and thus obeys the Euler equations on the  $t_1$  time scale. However, neither the contributions to the momentum due to the body force, nor the influences on the density and momentum due to the spatial and temporal variations of the force, are considered in this treatment. The final macroscopic equations corresponding to this method are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = -\frac{\Delta t}{2} \nabla \cdot \mathbf{F}, \quad (21a)$$

$$\begin{aligned} \frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = & -\nabla p + \nu \nabla \cdot [\rho(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)] + \mathbf{F} \\ & - \frac{\Delta t}{2} \epsilon \frac{\partial \mathbf{F}}{\partial t_1} + \left(\tau - \frac{1}{2}\right) \Delta t \nabla \cdot (\mathbf{v} \mathbf{F} + \mathbf{F} \mathbf{v}). \end{aligned} \quad (21b)$$

As can be seen, to match the Navier-Stokes equations, the spatial and temporal changes of the body force  $\mathbf{F}$  should vary slightly, and the last term in Eq. (18b) must be negligible. In practical applications, this method is mainly used for flows exposed to a constant body force. However, this last term may not be small as  $\mathbf{F}$  is a nonzero constant due to the velocity gradient.

A recent representation of the forcing term was proposed by two groups independently starting from kinetic equations (referred to as method 2) [6–8], which uses  $m=0$ ,  $A=0$ ,  $\mathbf{B}=\mathbf{F}$ , and  $\mathbf{C}=2\mathbf{F}\mathbf{u}^*$ , namely,  $F_i = \omega_i [(\mathbf{e}_i \cdot \mathbf{u}^*) / c_s^2 + (\mathbf{e}_i \cdot \mathbf{u}^*) \mathbf{e}_i / c_s^4] \cdot \mathbf{F}$ , and the equilibrium velocity  $\mathbf{u}^*$  and

fluid velocity  $\mathbf{v}$  are defined by  $\rho \mathbf{v} = \rho \mathbf{u}^* = \sum_i \mathbf{e}_i f_i$ . In this treatment, the contribution of the external force to the momentum flux is considered. Unfortunately, the discrete lattice effect is not taken into account. The corresponding macroscopic equations are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = -\frac{\Delta t}{2} \nabla \cdot \mathbf{F}, \quad (22a)$$

$$\begin{aligned} \frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = & -\nabla p + \nu \nabla \cdot [\rho(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)] + \mathbf{F} \\ & - \frac{\Delta t}{2} \epsilon \frac{\partial \mathbf{F}}{\partial t_1} + \frac{\Delta t}{2} \nabla \cdot (\mathbf{v} \mathbf{F} + \mathbf{F} \mathbf{v}). \end{aligned} \quad (22b)$$

The additional terms in Eqs. (22) are similar to those in Eqs. (21), and the arguments for method 1 also apply to method 2.

Two improved versions of method 2 were proposed by Ladd and Verberg [4]. The first improvement (method 2a) uses a redefined  $\mathbf{C}$ ,  $\mathbf{C} = (1 - 1/2\tau)(\mathbf{F}\mathbf{u}^* + \mathbf{u}^*\mathbf{F})$ , and the other parameters and the definitions of  $\mathbf{u}^*$  and  $\mathbf{v}$  are the same as in method 2. With this redefined  $\mathbf{C}$ , the contribution to the momentum flux due to the body force, i.e., the term relative to  $\mathbf{u}^*\mathbf{F} + \mathbf{F}\mathbf{u}^*$ , is canceled. However, the influences of spatial and temporal variations of the force are still not considered. In fact, Ladd and Verberg assumed that the force  $F$  is time independent, and that  $F$  is spatial uniform or the acceleration  $\mathbf{g}$  is uniform [4]. The macroscopic equations of this method are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = -\frac{\Delta t}{2} \nabla \cdot \mathbf{F}, \quad (23a)$$

$$\begin{aligned} \frac{\partial(\rho \mathbf{u}^*)}{\partial t} + \nabla \cdot (\rho \mathbf{u}^* \mathbf{u}^*) = & -\nabla p + \nu \nabla \cdot [\rho(\nabla \mathbf{u}^* + (\nabla \mathbf{u}^*)^T)] \\ & + \mathbf{F} - \frac{\Delta t}{2} \epsilon \frac{\partial \mathbf{F}}{\partial t_1}. \end{aligned} \quad (23b)$$

Clearly, errors in the momentum equation due to the presence of an external force are efficiently reduced. In fact, if  $\mathbf{F}$  is a constant, Eqs. (23) will match the correct hydrodynamic equations.

Another improved version proposed by Ladd and Verberg (method 2b) uses the same representation of  $F_i$  and the definition of  $\mathbf{u}^*$  as used in method 2, but the fluid velocity is redefined as  $\rho \mathbf{v} = \sum_i \mathbf{e}_i f_i + (\Delta t/2)\mathbf{F}$ . In this treatment the influence on the density due to the spatial variation of the force is considered, but the discrete lattice effects on the momentum flux are ignored. As a result, the macroscopic equations become

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (24a)$$

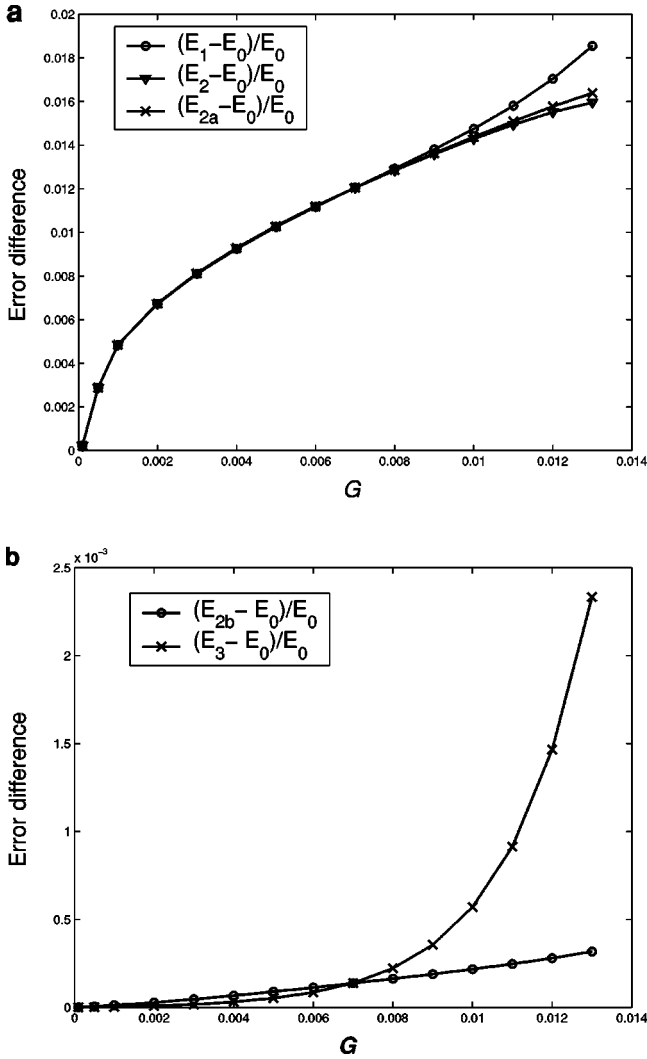


FIG. 1. Relative error differences between other methods and the present method for the Poiseuille flow.  $E_0$ , present;  $E_x$ , others.

$$\begin{aligned} \frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = & -\nabla p + \nu \nabla \cdot [\rho(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)] + \mathbf{F} \\ & + \frac{\Delta t}{2} \epsilon^2 \frac{\partial \mathbf{F}}{\partial t_2} - \frac{3 \Delta t^2}{4} \nabla \cdot (\rho \mathbf{g} \mathbf{g}) \\ & - \frac{\Delta t}{2} \nu \nabla \cdot [\rho(\nabla \mathbf{g} + (\nabla \mathbf{g})^T)]. \end{aligned} \quad (24b)$$

As seen, the continuity equation is already correct. But some additional terms still appear in the momentum equation. The term  $(\Delta t/2)\epsilon^2 \partial \mathbf{F}/\partial t_2$  is negligible since  $\epsilon$  and  $\Delta t$  are small parameters. But the terms of order  $\Delta t^2 \nabla \cdot (\rho \mathbf{g} \mathbf{g})$  and  $\Delta t \nabla \cdot (\rho \nabla \mathbf{g})$  may be large for spatially varying forces.

The last method (method 3) considered in this paper was proposed by Buick and Greated [5]; it uses the same parameters and definitions of  $\nu$  and  $\mathbf{u}^*$  as used in the method proposed in this work except for taking  $\mathbf{C} = \mathbf{0}$ , i.e.,  $F_i = \omega_i(1 - 1/2\tau)(\mathbf{e}_i \cdot \mathbf{F})/c_s^2$ . The resulting macroscopic equations derived from this method are

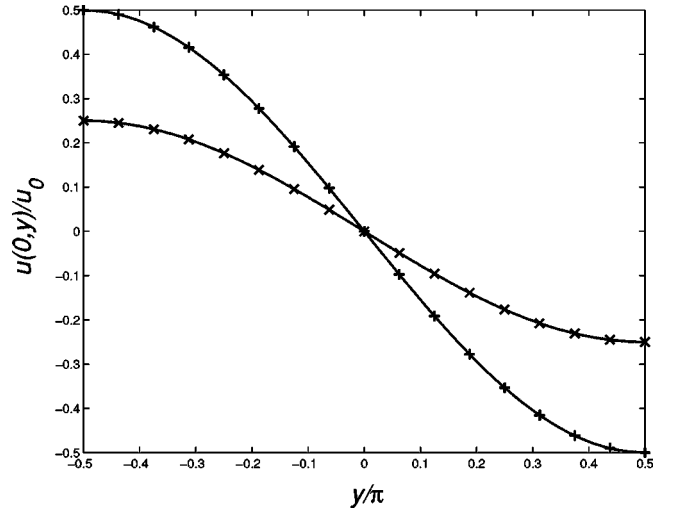


FIG. 2. Numerical velocity  $u(0,y)$  at time  $t = t_c$  and  $2t_c$  for Taylor vortex flow. Solid lines are the analytical solutions.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (25a)$$

$$\begin{aligned} \frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = & -\nabla p + \nu \nabla \cdot [\rho(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)] + \mathbf{F} \\ & + \left( \tau - \frac{1}{2} \right) \Delta t \nabla \cdot (\mathbf{v} \mathbf{F} + \mathbf{F} \mathbf{v}). \end{aligned} \quad (25b)$$

Obviously, this work takes account of the fact that in order to obtain the correct hydrodynamic equations a redefinition of the fluid velocity is needed to take account of the spatial and temporal variations of the body force. The discrete lattice effects are also considered in this treatment. Unfortunately, the contributions of the force to the momentum flux are not considered, and the momentum equation differs from the true Navier-Stokes equation by an additional term of order  $\Delta t \nabla \cdot (\mathbf{u}^* \mathbf{F} + \mathbf{F} \mathbf{u}^*)$ . This result is different from what has been obtained in [5], and it should be pointed out that the momentum equation derived in Ref. [5] is incorrect.

From the discussion above, we can see that none of the five related methods considered can model the general Navier-Stokes equations correctly. Method 1, method 2, and method 2a lead to a continuity equation with an additional term of order  $\Delta t \nabla \cdot \mathbf{F}$ , and although method 2b and method 3 both give the correct continuity equation, neither produces the true momentum equation. It is demonstrated that in order to obtain the correct continuity equation, the fluid velocity must be defined such that the effect of the external force is included, and to obtain the correct momentum equation, the contributions of the force to the momentum flux must be canceled. The method proposed in this paper matches both conditions, and gives the correct equations of hydrodynamics.

It is noted that there exist some other methods in which the body force is not included into the LBM by adding a forcing term into the LBE. The recent work [5] reviewed several such methods. It is also noticed that in the method

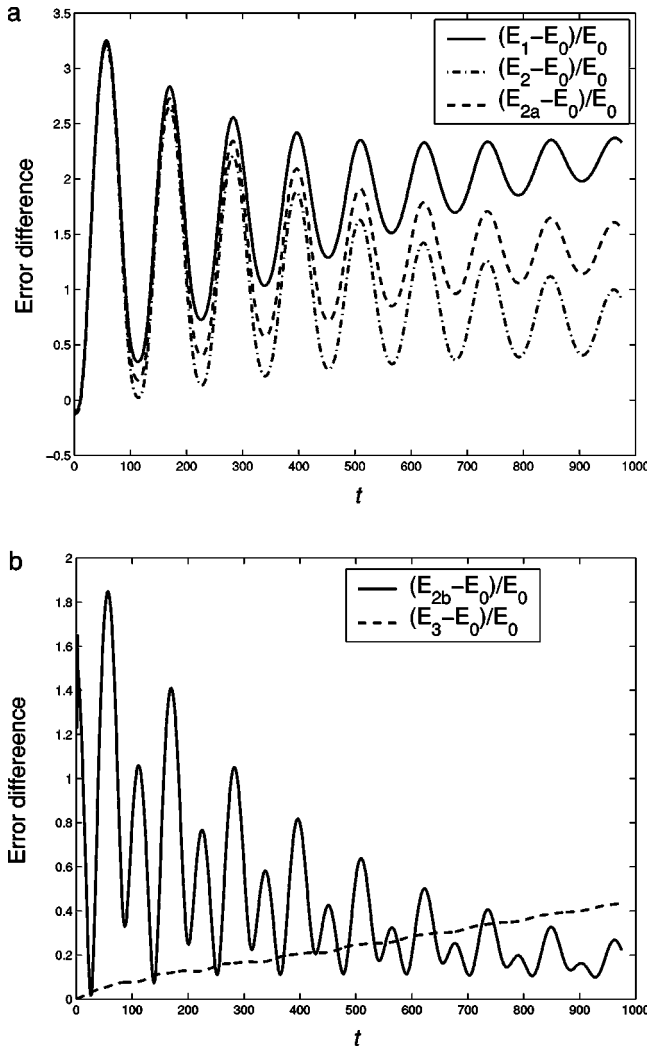


FIG. 3. Relative error differences between other methods and the present method for the Taylor vortex flow.  $E_0$ , present;  $E_x$ , others.

proposed by He *et al.* [10] the force is explicitly included in the EDF, and a forcing term is added into the LBE without rigorous proof. However, both the redefined EDF and the forcing term contain terms of order  $u^3$ , which is inconsistent with the whole system.

To verify the arguments mentioned above, we first applied the present method and the other five methods to steady Poiseuille flow driven by a pressure gradient  $\partial p/\partial x = -\rho G_1$  together with a body force  $\rho G_2$ , where  $G_1$  and  $G_2$  are two constants. The Poiseuille flow in a channel of width  $2L$  has the following steady analytical solution:

$$u_a = u_0 \left( 1 - \frac{y^2}{L^2} \right), \quad v = 0, \quad (26)$$

where  $u_0 = GL^2/2\nu$  is the peak velocity, and  $G = G_1 + G_2$  is the total acceleration.

In simulations, the extrapolation scheme [11] is applied to the upper and bottom walls of the channel for no-slip boundary conditions, and to the inlet and exit for pressure bound-

ary conditions. The pressure gradient and the body force are set to be equal, or  $G_1 = G_2 = G/2$ . The lattice size is fixed at  $N_x \times N_y = 34 \times 18$ . A set of runs is carried out with different values of  $G$ . In each case, the flow reaches its steady state after a number of iterations. The relative global errors are measured at the steady state between the LBM solution and the analytical solution given by Eq. (26); the error is defined by

$$E = \frac{\sqrt{\sum (u - u_a)^2}}{\sqrt{\sum u_a^2}} \quad (27)$$

where  $u$  is the numerical solution, and the summation is taken over the entire system. It is found that the numerical results of the present forcing term are the most accurate in all cases considered. In Fig. 1, the relative differences between the errors of the other five methods and that of the present method,  $(E_x - E_0)/E_0$ , are plotted against the total acceleration  $G$ . Here  $E_0$  represents the error of the present method, and  $E_x$  represents the error of any one of the other five methods. One can observe that the errors produced by methods 1–3 are more or less larger than that of the present method. More specifically, method 1, method 2, and method 2a, which do not satisfy the continuity equation, demonstrate similar behaviors and always produce larger errors than the other two methods, in which the continuity equation is satisfied. It is also noted that the differences increase as  $G$  increases. This is because when  $G$  is small the macroscopic equations of these six methods are nearly identical. But as  $G$  becomes larger, the discrete lattice effects cannot be neglected any more.

A simulation for unsteady flow where the force depends on both space and time was also carried out. The test problem is the two-dimensional Taylor vortex flow in a square box, which has the following analytical solution:

$$u_a = -u_0 \cos(k_1 x) \sin(k_2 y) \exp[-\nu(k_1^2 + k_2^2)t],$$

$$v_a = u_0 \frac{k_1}{k_2} \sin(k_1 x) \cos(k_2 y) \exp[-\nu(k_1^2 + k_2^2)t], \quad (28)$$

and the body force  $\mathbf{F} = (F_x, F_y)$  is given by  $F_x = -(\rho k_1 G/2) \sin(2k_1 x) \exp[-2\nu(k_1^2 + k_2^2)t]$ ,  $F_y = -(\rho k_1^2 G/2k_2) \sin(2k_2 y) \exp[-2\nu(k_1^2 + k_2^2)t]$ , where  $G = u_0^2$  is the amplitude of the force. In simulation, the flow is confined in the domain  $-\pi/2 \leq x, y \leq \pi/2$ , which is covered by a lattice of size  $N_x \times N_y = 65 \times 65$ . The wave numbers are set to be  $k_1 = k_2 = 1.0$ , and the amplitude of the force is chosen to be  $G = 0.001$  so that the compressibility of the fluid is negligible. The shear viscosity  $\nu$  is set to be 0.005. The flow is initialized by evaluating the analytical solution at  $t = 0$ , and the extrapolation scheme [11] is again applied to the four boundaries for velocity boundary conditions. Numerical solutions at  $t = t_c$  and  $t = 2t_c$  are plotted in Fig. 2 together with the analytical solutions, where  $t_c = \ln 2 / [\nu(k_1^2 + k_2^2)]$  is the time

when the amplitude of vortex is halved. One can see that the agreement between the numerical and analytical solutions is excellent.

We also applied the other five methods to Taylor vortex flow under the same conditions. The relative global errors in velocity field produced by each method were measured and compared with that by the present method. Here the relative global error is defined by

$$E(t) = \frac{\sqrt{\sum [u(t) - u_a(t)]^2 + [v(t) - v_a(t)]^2}}{\sqrt{\sum [u(t) - u_a(t)]^2}} \quad (29)$$

where the summation is taken over the whole system. In Fig. 3, the relative differences between the error obtained by each of the five methods and that by the present method are plotted as time proceeds. It is seen that the present method is the most accurate for this unsteady flow where the force changes in both space and time. It is again observed that method 1,

method 2, and method 2a demonstrate similar behaviors for this flow, and produce larger errors than the other two methods (method 2b and method 3) which satisfy the continuity equation. From these observations, we can see that discrete lattice effects do have influences on the behavior of the LBM, and should be considered in modeling fluids involving external or internal forces.

In summary, we have presented a method to include the body force into the LBM, in which the discrete lattice effect and the contributions of the body force to the momentum flux are both considered. The LBE with the proposed forcing term can lead to the exact Navier-Stokes equations. Some related methods were also examined. It is found that none of these methods match the Navier-Stokes equations in the general case. Therefore, the present work should be of benefit in designing lattice Boltzmann models for fluids exposed to external and/or internal forces.

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