Acceleration of steady-state lattice Boltzmann simulations on non-uniform mesh using local time step method

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Outline

GILBM and ISLBM

Extend **Interpolation-Supplemented LBM (ISLBM)** to general curvilinear coordinates through conformal mapping, without changing the lattice system. But the point of mapping is How can I deal with discrete particle velocity set.

Key Features:

- Cartesian → Curvilinear via coordinate mapping
- Curved particle paths in computational domain
- Lattice structure remains unchanged
- Pull-back interpolation for streaming

$$f_\alpha(\vec{\xi}, t + \Delta t) = f^\star_\alpha(\vec{\xi} - \Delta\vec{\xi}_\alpha, t) \qquad (1)$$

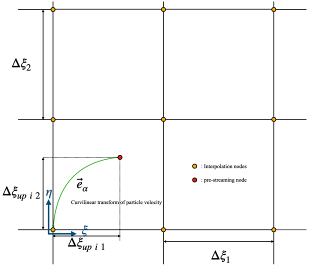


Figure: GILBM: curved paths in ξ-space

Jacobian Transformation Matrix:

$$\begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \frac{1}{J} \begin{bmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{bmatrix}, \quad J = x_\xi y_\eta - x_\eta y_\xi$$

Lamé Coefficient & Unit Vector:

$$h_1 \equiv \left| \frac{\partial \vec{r}}{\partial q_1} \right|, \quad \vec{e}_1 \equiv \frac{1}{h_1} \frac{\partial \vec{r}}{\partial q_1}$$

Position vector differentials:

$$\frac{\partial \vec{r}}{\partial \xi} = h_\xi \vec{e}_\xi = x_\xi \vec{e}_x + y_\xi \vec{e}_y$$

$$\frac{\partial \vec{r}}{\partial \eta} = h_\eta \vec{e}_\eta = x_\eta \vec{e}_x + y_\eta \vec{e}_y$$

Physical Meaning:

- h_i: Scale factor (metric coefficient)
- $\vec{e}_i$: Local tangent direction
- J: Jacobian determinant (area scaling)
- Transforms derivatives between coordinates

Standard LBE (BGK Collision Operator):

$$f_\alpha(\vec{x} + \vec{c}_\alpha \Delta t, t + \Delta t) = f_\alpha(\vec{x}, t) + \omega(f_\alpha - f_\alpha^{eq})$$

LBE with MRT Collision Operator:

$$\mathrm{M}\vec{f}(\vec{x} + \vec{c}_\alpha \Delta t, t + \Delta t) = \mathrm{M}\vec{f}(\vec{x}, t) + \mathrm{SM}\,(\vec{f} - \vec{f}^{eq})$$

Collision Step:
Position appears only at grid nodes; collision has no spatial derivatives.
$\Rightarrow$ Direct substitution: $\vec{x} \to \vec{\xi}$

$$\boxed{f_i^\star(\vec{\xi}, t) = f_i + \omega(f_i^{eq} - f_i)}$$

Streaming Step:
Requires path integration of particle velocity in curvilinear space.

$$\boxed{f_\alpha(\vec{\xi}, t) = f_\alpha^\star(\vec{\xi} - \Delta\vec{\xi}_\alpha, t)}$$

(Pull-back from pre-streaming position via interpolation)

Definition

Transform Cartesian lattice velocity to curvilinear coordinates:

$$\boxed{e_\alpha^j = c_\alpha^i \frac{\partial \xi_j}{\partial x_i}} \ldots \text{(assume laatice speed = 1)}$$

- c_α^i: Dimensionless discrete velocity (e.g., D2Q9: $0, \pm 1$)
- $\frac{\partial \xi_j}{\partial x_i}$: Coordinate transformation (from Jacobian)
- $\vec{e}_\alpha^{\,j}$: Shows velocity distortion due to curvature

Volumetric Lattice Boltzmann in Curvilinear Coordinates:

$$\vec{e}_\alpha = \underbrace{c_\alpha^i}_{\text{an integer}} \cdot \underbrace{\vec{g}_i(\vec{\xi})}_{\text{tangent vector}} \cdot \underbrace{\frac{\Delta x}{\Delta t}}_{\text{lattice speed}\approx 1}$$

Redefine the particle's discrete velocity: Enforce the direction of velocity to be the same as the tangent vector.

From straming step :

$$f_\alpha(\vec{\xi}, t + \Delta t) = f_\alpha^\star(\vec{\xi} - \delta\vec{\xi}_\alpha, t)$$

In curvilinear coordinate, we have to calculate the position of the pre-streaming particles.

Streaming Length:

$$\delta\vec{\xi}_\alpha = \int_0^{\Delta t} \vec{e}_\alpha(\vec{\xi}(t))\, dt$$

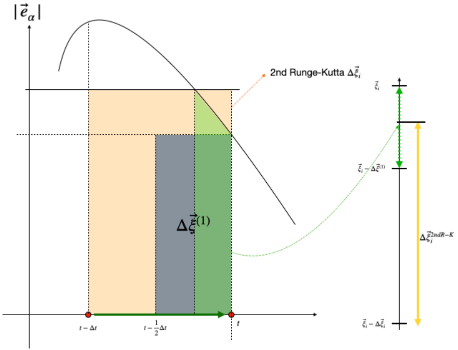
2nd Order Runge-Kutta:

Step 1: $\Delta\vec{\xi}_\alpha^{(1)} = \frac{1}{2}\Delta t\, \vec{e}_\alpha(\vec{\xi})$

Step 2: $\Delta\vec{\xi}_\alpha = \Delta t\, \vec{e}_\alpha(\vec{\xi} - \Delta\vec{\xi}_\alpha^{(1)})$

Why not 1st order Euler?

- Euler: $O(\Delta t)$ error $\to$ insufficient
- Need $O(\Delta t^2)$ to recover Navier-Stokes
- Stress term requires 2nd order accuracy



material derivative for the discrete-velocity Boltzmann equation:

$$\mathcal{D}_t = \partial_t + \frac{\delta \vec{\xi}_\alpha}{\delta t} \cdot \vec{\nabla}_\xi$$

and using Taylor expansion on lattice Bolrtzmann equation :

$$\left(\partial_t + \frac{\delta \vec{\xi}_\alpha}{\delta t} \cdot \vec{\nabla}_\xi\right) f_\alpha + \frac{1}{2}\left(\partial_t + \frac{\delta \vec{\xi}_\alpha}{\delta t} \cdot \vec{\nabla}_\xi\right)^2 f_\alpha \Delta t \Bigg|_{(\vec{\xi} - \Delta\vec{\xi}_\alpha, t)} + O(\Delta t^2) = \omega(f_\alpha(\vec{\xi} - \Delta\vec{\xi}_\alpha, t) - f_\alpha^{eq}(\rho, \vec{u}, t))$$

Therefore, we must use a method that provides higher-order temporal accuracy for the path integration, ensuring that the differential term of the velocity-discrete Boltzmann equation has at least second-order temporal accuracy.

Three Time Scales:

- $K^{(0)}$: Collision & streaming (mesoscopic)
- $K^{(1)}$: Convection / advection (macroscopic)
- $K^{(2)}$: Diffusion (macroscopic)

For three types of time scales, we can define three time variables and two types of position vector :

Scale Operators:

$$\partial_t = K\partial_t^{(1)} + K^2\partial_t^{(2)}$$
$$\vec{\nabla} = K\vec{\nabla}^{(1)}$$

Distribution Expansion:

$$f_\alpha = f_\alpha^{eq} + Kf_\alpha^{(1)} + K^2 f_\alpha^{(2)} + \cdots$$

Order K^1 Equation:

$$\left(\partial_t^{(1)} + \vec{e}_\alpha \cdot \vec{\nabla}^{(1)}\right) f_\alpha^{eq} = -\frac{1}{\tau} f_\alpha^{(1)}$$

Order K^2 Equation:

$$\left(\partial_t^{(2)} + \vec{e}_\alpha \cdot \vec{\nabla}^{(2)}\right) f_\alpha^{eq} + \left(1 - \frac{\delta t}{2\tau}\right) \times \left(\partial_t^{(1)} + \vec{e}_\alpha \cdot \vec{\nabla}^{(1)}\right) f_\alpha^{(1)} = -\frac{1}{\tau} f_\alpha^{(2)}$$

Maxwell-Boltzmann Equilibrium (Hermite Expansion):

$$f_\alpha^{eq} = w_\alpha \rho \left(1 + \frac{e_\alpha^i u_i}{c_s^2} + \frac{(e_\alpha^i e_\alpha^j - c_s^2 \delta^{ij}) u_i u_j}{2 c_s^4} \right)$$

Derivatives of f^{eq}:

$$\frac{\partial f_\alpha^{eq}}{\partial u_i} \approx \frac{e_\alpha^i - u_i}{c_s^2} f_\alpha^{eq}, \quad \frac{\partial f_\alpha^{eq}}{\partial \rho} = \frac{f_\alpha^{eq}}{\rho}$$

Conservation Laws (1st order scale):

$$\partial_t^{(1)} \rho = -\vec{\nabla}^{(1)} \cdot (\rho \vec{u})$$

$$\partial_t^{(1)} \vec{u} = -\vec{u} \cdot \vec{\nabla}^{(1)} \vec{u} - \frac{1}{\rho} \vec{\nabla}^{(1)} p$$

Assumptions at Boundary:

1. Constant pressure: $\nabla p = 0$
2. Incompressible: $\nabla \rho = 0$

Distribution Function at Boundary:

$$\boxed{f_\alpha\Big|_{bc} = f_\alpha^{eq} + \omega \delta t \left(\frac{(e_\alpha^i - u_i)(e_\alpha^\beta - u_\beta)}{c_s^2} \frac{\partial u_i}{\partial x_\beta} - \frac{\partial u_\beta}{\partial x_\beta} \right) f_\alpha^{eq}}$$

Velocity Gradient in Curvilinear Coordinates:

$$\frac{\partial u_i}{\partial x_\beta} = \frac{\partial u_i}{\partial \xi_j} \cdot \frac{\partial \xi_j}{\partial x_\beta}$$

2nd Order One-Side Finite Difference:

$$\left.\frac{\partial u}{\partial \xi}\right|_{i+\frac{1}{2}} = \frac{3u_i - 4u_{i-1} + u_{i-2}}{2\Delta\xi} + O(\Delta\xi^2)$$

GILBM Initialization:

1. Set Re, relaxation factor ω
2. Contravariant velocity:
$$e_\alpha^i = c_\alpha^\beta \frac{\partial \xi_i}{\partial x_\beta}$$
3. Global time step (CFL):
$$\Delta t = \text{CFL} \cdot \min \left| \frac{1}{e_\alpha^i} \right|$$
4. Relaxation time: $\tau = \Delta t / \omega$
5. Viscosity: $\nu = (\tau - 0.5\Delta t) c_s^2$

Runtime Loop:

1. **Collision:** $f_i^* = f_i + \omega(f_i^{eq} - f_i)$
2. **Streaming:** $f_i(\vec{\xi}) = f_i^*(\vec{\xi} - \Delta\vec{\xi}_i)$ (with interpolation)
3. **Boundary:** Apply $f_\alpha|_{bc}$
4. **Macroscopic:** $\rho = \sum_i f_i$, $\vec{u} = \frac{1}{\rho} \sum_i f_i \vec{c}_i$

Pre-calculation reduces runtime by $\sim$50%

GILBM: Key Points

1. **Coordinate Transform:** Jacobian relation maps Cartesian $\leftrightarrow$ curvilinear
2. **Collision:** Direct substitution $\vec{x} \rightarrow \vec{\xi}$ (no spatial derivatives)
3. **Streaming:** Path integration with contravariant velocity e_α^j
4. **Accuracy:** 2nd order RK required for N-S recovery
5. **Boundary:** Distribution from Chapman-Enskog with $\nabla p = 0$, $\nabla \rho = 0$
6. **Efficiency:** Pre-calculated contravariant velocity saves $\sim$50% runtime

Limitation: MRT operator extension to curvilinear coordinates not yet developed.