

Acceleration of steady-state lattice Boltzmann simulations on non-uniform mesh using local time step method

Taro Imamura^{a,*}, Kojiro Suzuki^b, Takashi Nakamura^a, Masahiro Yoshida^a

^a *Japan Aerospace Exploration Agency, Institute of Space Technology and Aeronautics,
7-44-1, Jindai-ji Higashi-mati, Chofu-si, Tokyo 182-8522, Japan*

^b *Department of Advanced Energy, The University of Tokyo,
5-1-5 Kashiwa-no-ha, Kashiwa, Chiba 277-8562, Japan*

Journal of Computational Physics 202 (2005) 645–663

Received 29 December 2003; revised 21 June 2004; accepted 2 August 2004

Presenter: Chen Peng Chung

GILBM and ISLBM

Extend **Interpolation-Supplemented LBM (ISLBM)** to general curvilinear coordinates through **conformal mapping**, without changing the lattice system. But the point of mapping is How can I deal with discrete particle velocity set.

Key Features:

- Cartesian \rightarrow Curvilinear via coordinate mapping
- Curved particle paths in computational domain
- Lattice structure remains **unchanged**
- Pull-back interpolation for streaming

$$f_{\alpha}(\vec{\xi}, t + \Delta t) = f_{\alpha}^*(\vec{\xi} - \Delta \vec{\xi}_{\alpha}, t) \quad (1)$$

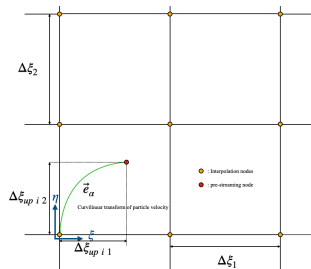


Figure: GILBM: curved paths in ξ -space

Jacobian Transformation Matrix:

$$\begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \frac{1}{J} \begin{bmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{bmatrix}, \quad J = x_\xi y_\eta - x_\eta y_\xi$$

Lamé Coefficient & Unit Vector:

$$h_1 \equiv \left| \frac{\partial \vec{r}}{\partial q_1} \right|, \quad \vec{e}_1 \equiv \frac{1}{h_1} \frac{\partial \vec{r}}{\partial q_1}$$

Position vector differentials:

$$\frac{\partial \vec{r}}{\partial \xi} = h_\xi \vec{e}_\xi = x_\xi \vec{e}_x + y_\xi \vec{e}_y$$

$$\frac{\partial \vec{r}}{\partial \eta} = h_\eta \vec{e}_\eta = x_\eta \vec{e}_x + y_\eta \vec{e}_y$$

Physical Meaning:

- h_i : Scale factor (metric coefficient)
- \vec{e}_i : Local tangent direction
- J : Jacobian determinant (area scaling)
- Transforms derivatives between coordinates

Standard LBE (BGK Collision Operator):

$$f_{\alpha}(\vec{x} + \vec{c}_{\alpha}\Delta t, t + \Delta t) = f_{\alpha}(\vec{x}, t) + \omega(f_{\alpha} - f_{\alpha}^{eq})$$

LBE with MRT Collision Operator:

$$M\vec{f}(\vec{x} + \vec{c}_{\alpha}\Delta t, t + \Delta t) = M\vec{f}(\vec{x}, t) + SM(\vec{f} - \vec{f}^{eq})$$

Collision Step:

Position appears only at grid nodes; collision has **no spatial derivatives**.

⇒ Direct substitution: $\vec{x} \rightarrow \vec{\xi}$

$$f_i^*(\vec{\xi}, t) = f_i + \omega(f_i^{eq} - f_i)$$

Streaming Step:

Requires **path integration** of particle velocity in curvilinear space.

$$f_{\alpha}(\vec{\xi}, t) = f_{\alpha}^*(\vec{\xi} - \Delta\xi_{\alpha}, t)$$

(Pull-back from pre-streaming position via interpolation)

Definition

Transform Cartesian lattice velocity to curvilinear coordinates:

$$e_{\alpha}^j = c_{\alpha}^i \frac{\partial \xi_j}{\partial x_i} \dots (\text{assume lattice speed} = 1)$$

- c_{α}^i : Dimensionless discrete velocity (e.g., D2Q9: $0, \pm 1$)
- $\frac{\partial \xi_j}{\partial x_i}$: Coordinate transformation (from Jacobian)
- \vec{e}_{α}^j : Shows **velocity distortion** due to curvature

Volumetric Lattice Boltzmann in Curvilinear Coordinates:

$$\vec{e}_{\alpha} = \underbrace{c_{\alpha}^i}_{\text{an integer}} \cdot \underbrace{\vec{g}_i(\vec{\xi})}_{\text{tangent vector}} \cdot \underbrace{\frac{\Delta x}{\Delta t}}_{\text{lattice speed} \approx 1}$$

Redefine the particle's discrete velocity: Enforce the direction of velocity to be the same as the tangent vector.

From streaming step :

$$f_{\alpha}(\vec{\xi}, t + \Delta t) = f_{\alpha}^*(\vec{\xi} - \delta \vec{\xi}_{\alpha}, t)$$

In curvilinear coordinate, we have to calculate the position of the pre-streaming particles.

Streaming Length:

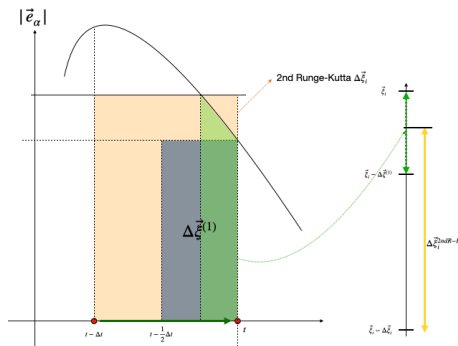
$$\delta \vec{\xi}_{\alpha} = \int_0^{\Delta t} \vec{e}_{\alpha}(\vec{\xi}(t)) dt$$

**2nd
Order
Runge-
Kutta:**

Step 1: $\Delta \vec{\xi}_{\alpha}^{(1)} = \frac{1}{2} \Delta t \vec{e}_{\alpha}(\vec{\xi})$

Step 2: $\Delta \vec{\xi}_{\alpha} = \Delta t \vec{e}_{\alpha}(\vec{\xi} - \Delta \vec{\xi}_{\alpha}^{(1)})$

**Why
not**



Three Time Scales:

- $K^{(0)}$: Collision & streaming (mesoscopic)
- $K^{(1)}$: Convection / advection (macroscopic)
- $K^{(2)}$: Diffusion (macroscopic)

Scale Operators:

$$\partial_t = K\partial_t^{(1)} + K^2\partial_t^{(2)}$$
$$\vec{\nabla} = K\vec{\nabla}^{(1)}$$

Distribution Expansion:

$$f_\alpha = f_\alpha^{eq} + Kf_\alpha^{(1)} + K^2f_\alpha^{(2)} + \dots$$

Order K^1 Equation:

$$\left(\partial_t^{(1)} + \vec{e}_\alpha \cdot \vec{\nabla}^{(1)}\right)f_\alpha^{eq} = -\frac{1}{\tau}f_\alpha^{(1)}$$

\Rightarrow First-order non-equilibrium:

$$f_\alpha^{(1)} = -\tau \left(\partial_t^{(1)} + \vec{e}_\alpha \cdot \vec{\nabla}^{(1)}\right)f_\alpha^{eq}$$

Maxwell-Boltzmann Equilibrium (Hermite Expansion):

$$f_{\alpha}^{eq} = w_{\alpha} \rho \left(1 + \frac{e_{\alpha}^i u_i}{c_s^2} + \frac{(e_{\alpha}^i e_{\alpha}^j - c_s^2 \delta^{ij}) u_i u_j}{2c_s^4} \right)$$

Derivatives of f_{α}^{eq} :

$$\frac{\partial f_{\alpha}^{eq}}{\partial u_i} \approx \frac{e_{\alpha}^i - u_i}{c_s^2} f_{\alpha}^{eq}, \quad \frac{\partial f_{\alpha}^{eq}}{\partial \rho} = \frac{f_{\alpha}^{eq}}{\rho}$$

Conservation Laws (1st order scale):

$$\begin{aligned} \partial_t^{(1)} \rho &= -\vec{\nabla}^{(1)} \cdot (\rho \vec{u}) \\ \partial_t^{(1)} \vec{u} &= -\vec{u} \cdot \vec{\nabla}^{(1)} \vec{u} - \frac{1}{\rho} \vec{\nabla}^{(1)} p \end{aligned}$$

Assumptions at Boundary:

- ① Constant pressure: $\nabla p = 0$
- ② Incompressible: $\nabla \rho = 0$

Distribution Function at Boundary:

$$f_{\alpha}|_{bc} = f_{\alpha}^q + \omega \delta t \left(\frac{(e_{\alpha}^i - u_i)(e_{\alpha}^{\beta} - u_{\beta})}{c_s^2} \frac{\partial u_i}{\partial x_{\beta}} - \frac{\partial u_{\beta}}{\partial x_{\beta}} \right) f_{\alpha}^q$$

Velocity Gradient in Curvilinear Coordinates:

$$\frac{\partial u_i}{\partial x_{\beta}} = \frac{\partial u_i}{\partial \xi_j} \cdot \frac{\partial \xi_j}{\partial x_{\beta}}$$

2nd Order One-Side Finite Difference:

$$\left. \frac{\partial u}{\partial \xi} \right|_{i+\frac{1}{2}} = \frac{3u_i - 4u_{i-1} + u_{i-2}}{2\Delta\xi} + O(\Delta\xi^2)$$

GILBM Initialization:

- ❶ Set Re , relaxation factor ω
- ❷ Contravariant velocity:

$$e_{\alpha}^i = c_{\alpha}^{\beta} \frac{\partial \xi_i}{\partial x_{\beta}}$$

- ❸ Global time step (CFL):

$$\Delta t = \text{CFL} \cdot \min \left| \frac{1}{e_{\alpha}^i} \right|$$

- ❹ Relaxation time: $\tau = \Delta t / \omega$
- ❺ Viscosity: $\nu = (\tau - 0.5\Delta t)c_s^2$

Runtime Loop:

- ❶ **Collision:** $f_i^* = f_i + \omega(f_i^{eq} - f_i)$
- ❷ **Streaming:** $f_i(\vec{\xi}) = f_i^*(\vec{\xi} - \Delta \vec{\xi}_i)$ (with interpolation)
- ❸ **Boundary:** Apply $f_{\alpha}|_{bc}$
- ❹ **Macroscopic:** $\rho = \sum_i f_i, \vec{u} = \frac{1}{\rho} \sum_i f_i \vec{c}_i$

Pre-calculation reduces runtime by $\sim 50\%$

GILBM: Key Points

- ① **Coordinate Transform:** Jacobian relation maps Cartesian \leftrightarrow curvilinear
- ② **Collision:** Direct substitution $\vec{x} \rightarrow \vec{\xi}$ (no spatial derivatives)
- ③ **Streaming:** Path integration with contravariant velocity e_{α}^j
- ④ **Accuracy:** 2nd order RK required for N-S recovery
- ⑤ **Boundary:** Distribution from Chapman-Enskog with $\nabla p = 0$, $\nabla \rho = 0$
- ⑥ **Efficiency:** Pre-calculated contravariant velocity saves $\sim 50\%$ runtime

Limitation: MRT operator extension to curvilinear coordinates not yet developed.