4.Volumetric Lattice Boltzmann Models in General Curvature

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一 Momentum Loss of Distribution Function in Propagation

In general curvilinear coordinates, for macroscopic values i.e., ρ, $\rho\vec{u}$ exist the loss due to the curvature and uniform properties. And this paper shows the value below:

$$\delta N_\alpha(\vec{q},t) \equiv N_\alpha(q,t) - N^\star_\alpha(q-1,t-1) \tag{一.1}$$

where q is the non-dimension position, and the origin form of it is $\vec{q} = (q_1, q_2, q_3)$. The value $N^\star$ represents post-collision distribution function. We can review the Lattice Boltzman Equation-BGK (LBGK) and see the position of the value in the equation.

$$N_\alpha(q+1,t+1) = N_\alpha(q,t) + \Omega_\alpha + \delta N_\alpha(q,t) \tag{一.2}$$

For general orthogonal curvilinear coordinate in three dimensions, we have the loss of the momentum and loss of the density is 0, so we have the relation:

$$\begin{aligned}\sum_{\alpha_0}^{q-1}\delta N_\alpha(q,t) &= \sum_{\alpha_0}^{q-1} N_\alpha(q,t) - N^\star_\alpha(q-1,t-1)\\ &= \sum_{\alpha_0}^{q-1} N_\alpha(q,t) - \sum_{\alpha_0}^{q-1} N^\star_\alpha(q-1,t-1) = 0\end{aligned} \tag{一.3}$$

But for momentum loss, we have to take the physical meaning as "inertial force" at first. The value in equation (一.1) is modified term in the process of the streaming step. The reason why we have to consider the loss is to achieve momentum conservation through adding the term.

Definition 一、1：

$$\vec{\Theta}^j_i(\vec{q},t) = \Big[\vec{g}_i(\vec{q},t) - \vec{g}_i(\vec{q} - \vec{c_i}\Delta t, t)\Big]\cdot \vec{g}^j \tag{一.4}$$

we call the value is "non-dimension change of curvature tangent vector". where $\vec{g}^j$ can be defined as below:

$$\vec{g}^1 = \frac{\vec{g_2}\times\vec{g_3}}{(\vec{g_2}\times\vec{g_3})\cdot\vec{g_1}} \tag{一.5}$$

Substitute the vector in equation (一.6)

$$\vec{\Theta}^j_i(\vec{q},t) = \Big[\vec{g}_i(\vec{q},t) - \vec{g}_i(\vec{q} - \vec{c_i}\Delta t, t)\Big]\cdot \frac{\vec{g_2}\times\vec{g_3}}{(\vec{g_2}\times\vec{g_3})\cdot\vec{g_1}} \tag{一.6}$$

Definition 一、2：

$$\begin{aligned}\vec{M_{Loss}}(q,t) &\equiv \sum_{\alpha=1}^{q} N_\alpha(q,t)\vec{e_\alpha}(q,t) - N^\star_\alpha(q-1,t-1)\vec{e_\alpha}(q-1,t-1)\\ \vec{M_{Loss}}(q+1,t+1) &\equiv \sum_{\alpha=1}^{q} N_\alpha(q+1,t+1)\vec{e_\alpha}(q+1,t+1) - N^\star_\alpha(q,t)\vec{e_\alpha}(q,t)\end{aligned} \tag{一.7}$$

For curvilinear coordinate, the loss of momentum always exists in streaming step. The basic reason of the loss of macroscopic value is:

$$N_\alpha(\vec{q}, t) \neq N_\alpha^\star(\vec{q} - \vec{\tilde{e}}_\alpha \Delta t, t - 1) \tag{一.8}$$

But this paper defines two values about inner and outer at the point (q, t) as shown:

Definition 一、3：

$$\begin{aligned}
\mathcal{J}(q)\vec{\chi}^{in}(q,t) &\equiv \sum_{\alpha=1}^{q} N_\alpha(q,t)\left(\vec{e_\alpha}(q,t) - \vec{e_\alpha}(q-1,t-1)\right) \\
\mathcal{J}(q)\vec{\chi}^{out}(q,t) &\equiv \sum_{\alpha=1}^{q} \left(\vec{e_\alpha}(q+1,t+1) - \vec{e_\alpha}(q,t)\right) N_\alpha^\star(q,t)
\end{aligned} \tag{一.9}$$

In this paper, the momentum loss and its definition can be separated into two directions about the point (q, t) i.e., inner and outer. The difference from the true "momentum loss of the propagation" is: take post-collision distribution function at $(q-1, t-1)$: $(N_\alpha)^\star(q-1, t-1)$; take the pre-collision function at (q, t): $N_\alpha(q, t)$ for curvilinear normalized discrete particle velocity set: $\vec{\tilde{e}}_\alpha(q-1, t-1)$ and $\vec{\tilde{e}}_\alpha(q, t)$