4.Volumetric Lattice Boltzmann Models in General Curvature

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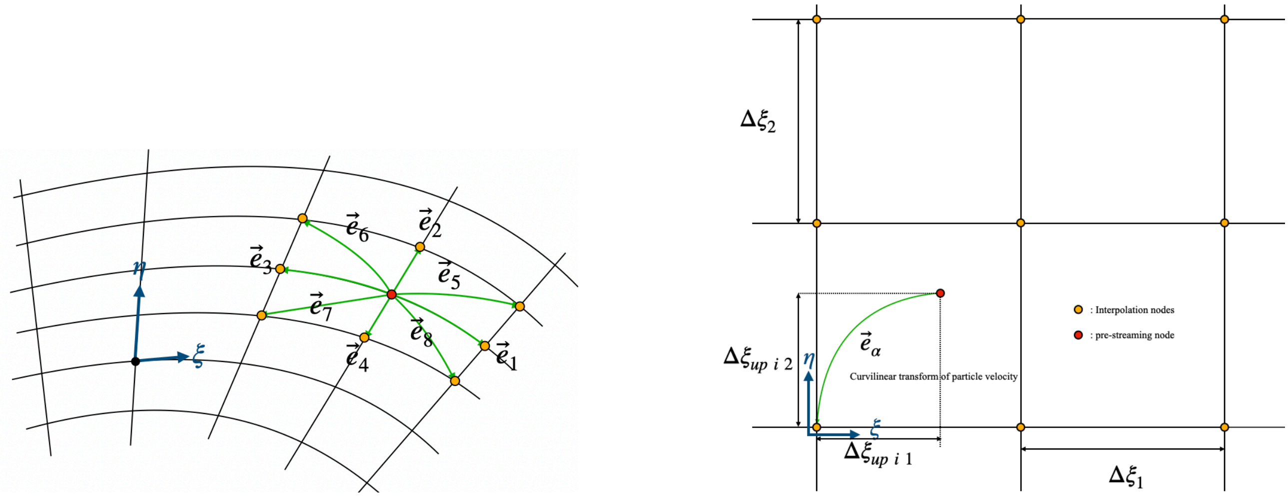
Contents

1 General Interpolation LBM

This note will promote the Interpolation-supplement Lattice Boltzmann Method for using in curvilinear coordinate. However, The first stratgy is transform Cartesian coordinate to general curvilinear coordinate through comformal mapping. Different from the preview paper, this method is extend **ISLBM** ,and do not change the lattice system to curve motion. Notice, because exist the transform based on transformation, so we still compute the curve path for pasrticle in computational domain. like the picture below :



(a) curvilinear coordinate system for SLBM **(b)** General Interpolation LBM

Figure 1.1: 2 types of curvilinear coordinate system

2 Transformation

2.1 Jacobian relation

This section woul give the proven of the Jacobian relation :

$$\begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \frac{1}{\mathrm{J}} \begin{bmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{bmatrix} \tag{2.1}$$

For general curvilinear coordinate in two dimensions, we have to define Lamé coefficient for analyzing:

$$\vec{dr}_{+1}|_{q_2,q_3}(\vec{r}) \equiv \vec{r}(q_1+\Delta q_1, q_2, q_3) - \vec{r}(q_1, q_2, q_3) = \left.\frac{\partial \vec{r}}{\partial q_1}\right|_{q_2,q_3} dq_1 = \frac{\left.\frac{\partial \vec{r}}{\partial q_1}\right|_{q_2,q_3}}{\left|\left.\frac{\partial \vec{r}}{\partial q_1}\right|_{q_2,q_3}\right|} \left|\left.\frac{\partial \vec{r}}{\partial q_1}\right|_{q_2,q_3}\right| dq_1 \tag{2.2}$$

From the expression above, we can define unit vector and coefficient for differential geometry.

$$\begin{aligned} h_1 &\equiv \left|\frac{\partial \vec{r}}{\partial q_1}\right|_{q_2,q_3} \\ \vec{e}_1 &\equiv \frac{1}{h_1} \left.\frac{\partial \vec{r}}{\partial q_1}\right|_{q_2,q_3} \end{aligned} \tag{2.3}$$

where symbol h_1 is a Lamé coefficient for the coordinate component. Let's take the definition to defferential of position vector about variable (ξ, η).

$$\begin{aligned}
\frac{\partial \vec{r}}{\partial \xi} &= h_\xi \vec{e}_\xi = \vec{e}_x x_\xi + \vec{e}_y y_\xi \\
\frac{\partial \vec{r}}{\partial \eta} &= h_\eta \vec{e}_\eta = \vec{e}_x x_\eta + \vec{e}_y y_\eta \\
\frac{\partial \vec{r}}{\partial x} &= \vec{e}_x = h_\xi \vec{e}_\xi \xi_x + h_\eta \vec{e}_\eta \eta_x \\
\frac{\partial \vec{r}}{\partial y} &= \vec{e}_y = h_\xi \vec{e}_\xi \xi_y + h_\eta \vec{e}_\eta \eta_y
\end{aligned} \tag{2.4}$$

we can do some simple computation, and get the result below :

$$\begin{aligned}
\frac{h_\xi}{x_\xi}\vec{e}_\xi - \frac{h_\eta}{x_\eta}\vec{e}_\eta = \vec{e}_y\left(\frac{y_\xi}{x_\xi} - \frac{y_\eta}{x_\eta}\right) = \vec{e}_y \frac{y_\xi x_\eta - y_\eta x_\xi}{x_\eta x_\xi} \\
\frac{h_\xi}{y_\xi}\vec{e}_\xi - \frac{h_\eta}{y_\eta}\vec{e}_\eta = \vec{e}_x\left(\frac{x_\xi}{y_\xi} - \frac{x_\eta}{y_\eta}\right) = \vec{e}_x \frac{x_\xi y_\eta - x_\eta y_\xi}{y_\eta y_\xi}
\end{aligned} \tag{2.5}$$

Furthermore, we can rewrite this as:

$$\begin{aligned}
-x_\eta h_\xi \vec{e}_\xi + x_\xi h_\eta \vec{e}_\eta = \vec{e}_y \left(x_\xi y_\eta - x_\eta y_\xi\right) \\
y_\eta h_\xi \vec{e}_\xi - y_\xi h_\eta \vec{e}_\eta = \vec{e}_x \left(x_\xi y_\eta - x_\eta y_\xi\right)
\end{aligned} \tag{2.6}$$

Substitution the equation (2.6) into (2.4)

$$\begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \frac{1}{(x_\xi y_\eta - x_\eta y_\xi)} \begin{bmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{bmatrix} \tag{2.7}$$

The equation above is Jacobian relation.

2.2 Transform for Lattice Boltzmann Equation

2.3 Boundary Condition

2.4 Algorithm