4.Volumetric Lattice Boltzmann Models in General Curvature

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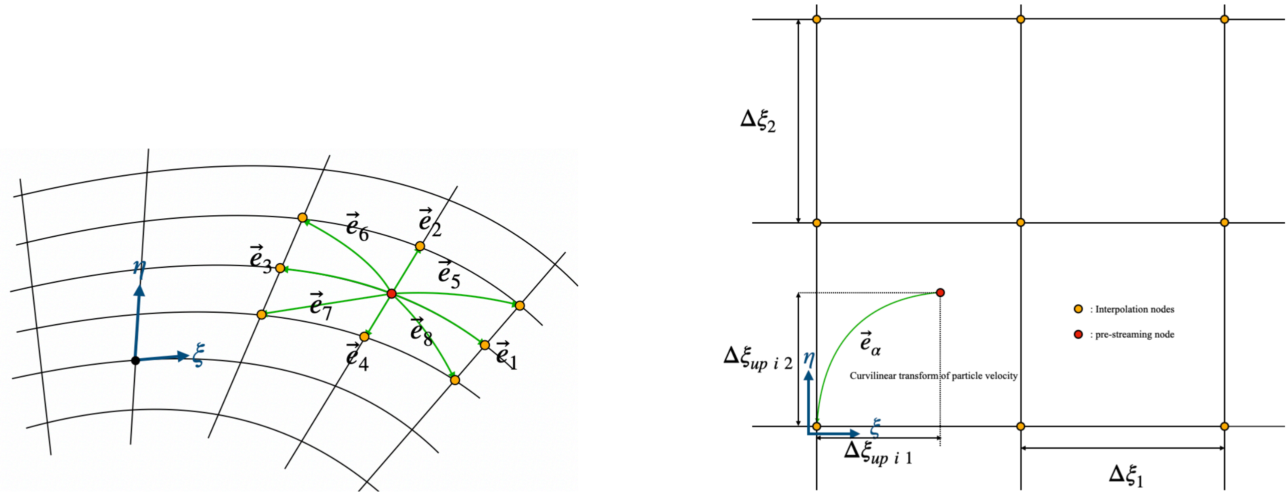
1 General Interpolation LBM

This note presents an extension of the Interpolation-Supplemented Lattice Boltzmann Method (ISLBM) for use in curvilinear coordinate systems. The first strategy is to transform the Cartesian coordinate system to a general curvilinear coordinate system through conformal mapping. Different from previous papers, this method extends **ISLBM** without changing the lattice system to accommodate curved motion. Note that because the transformation is based on coordinate mapping, we still compute the curved particle paths in the computational domain, as shown in the figure below:



(a) curvilinear coordinate system for SLBM **(b)** General Interpolation LBM

Figure 1.1: 2 types of curvilinear coordinate system

2 Transformation

2.1 Jacobian relation

This section woul give the proven of the Jacobian relation :

$$\begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \frac{1}{\mathrm{J}} \begin{bmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{bmatrix} \tag{2.1}$$

For general curvilinear coordinate in two dimensions, we have to define Lamé coefficient for analyzing:

$$\vec{dr}_{+1}|_{q_2,q_3}(\vec{r}) \equiv \vec{r}(q_1+\Delta q_1, q_2, q_3) - \vec{r}(q_1, q_2, q_3) = \left.\frac{\partial \vec{r}}{\partial q_1}\right|_{q_2,q_3} dq_1 = \frac{\left.\frac{\partial \vec{r}}{\partial q_1}\right|_{q_2,q_3}}{\left|\left.\frac{\partial \vec{r}}{\partial q_1}\right|_{q_2,q_3}\right|} \left|\left.\frac{\partial \vec{r}}{\partial q_1}\right|_{q_2,q_3}\right| dq_1 \tag{2.2}$$

From the expression above, we can define unit vector and coefficient for differential geometry.

$$\begin{aligned} h_1 &\equiv \left|\frac{\partial \vec{r}}{\partial q_1}\right|_{q_2,q_3} \\ \vec{e}_1 &\equiv \frac{1}{h_1} \left.\frac{\partial \vec{r}}{\partial q_1}\right|_{q_2,q_3} \end{aligned} \tag{2.3}$$

where symbol h_1 is a Lamé coefficient for the coordinate component. Let's take the definition to defferential of position vector about variable (ξ, η).

$$
\begin{aligned}
\frac{\partial \vec{r}}{\partial \xi} &= h_\xi \vec{e}_\xi = \vec{e}_x x_\xi + \vec{e}_y y_\xi \\
\frac{\partial \vec{r}}{\partial \eta} &= h_\eta \vec{e}_\eta = \vec{e}_x x_\eta + \vec{e}_y y_\eta \\
\frac{\partial \vec{r}}{\partial x} &= \vec{e}_x = h_\xi \vec{e}_\xi \xi_x + h_\eta \vec{e}_\eta \eta_x \\
\frac{\partial \vec{r}}{\partial y} &= \vec{e}_y = h_\xi \vec{e}_\xi \xi_y + h_\eta \vec{e}_\eta \eta_y
\end{aligned}
\tag{2.4}
$$

we can do some simple computation, and get the result below :

$$
\begin{aligned}
\frac{h_\xi}{x_\xi}\vec{e_\xi} - \frac{h_\eta}{x_\eta}\vec{e_\eta} &= \vec{e}_y\left(\frac{y_\xi}{x_\xi} - \frac{y_\eta}{x_\eta}\right) = \vec{e}_y\frac{y_\xi x_\eta - y_\eta x_\xi}{x_\eta x_\xi} \\
\frac{h_\xi}{y_\xi}\vec{e_\xi} - \frac{h_\eta}{y_\eta}\vec{e_\eta} &= \vec{e}_x\left(\frac{x_\xi}{y_\xi} - \frac{x_\eta}{y_\eta}\right) = \vec{e}_x\frac{x_\xi y_\eta - x_\eta y_\xi}{y_\eta y_\xi}
\end{aligned}
\tag{2.5}
$$

Furthermore, we can rewrite this as:

$$
\begin{aligned}
-x_\eta h_\xi \vec{e_\xi} + x_\xi h_\eta \vec{e_\eta} &= \vec{e}_y\left(x_\xi y_\eta - x_\eta y_\xi\right) \\
y_\eta h_\xi \vec{e_\xi} - y_\xi h_\eta \vec{e_\eta} &= \vec{e}_x\left(x_\xi y_\eta - x_\eta y_\xi\right)
\end{aligned}
\tag{2.6}
$$

Substitution the equation (2.6) into (2.4)

$$
\begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \frac{1}{(x_\xi y_\eta - x_\eta y_\xi)} \begin{bmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{bmatrix}
\tag{2.7}
$$

The equation above is Jacobian relation.

2.2 Transform for Lattice Boltzmann Equation

First, review the basic governing equation for standard lattice boltzman method i.e., lattice boltzmann equation in Cartesian coordinate with 3 dimension:

$$
f_i(\vec{x} + \vec{c_i}\Delta t, t+1) = f_i(\vec{x}, t) + \Omega(f_i(\vec{x}, t), f_i^{eq}(\rho, \vec{u}, t))
\tag{2.8}
$$

where the symbol Ω is a coillision operator, so it can apart two types of forms.

$$
f_i(\vec{x} + \vec{c_i}\Delta t, t+1) = f_i(\vec{x}, t) + \omega(f_i(\vec{x}, t) - f_i^{eq}(\rho, \vec{u}, t))
\tag{2.9}
$$

If the collision operator is chosen as the BGK operator, we call the equation the **LBGK** equation. On the other hand, we can allow each momentum component to have a different relaxation effect. To achieve this, we introduce a basis transformation matrix that transforms the function from velocity space to momentum space. This means that the streaming step and collision step are performed in separate spaces. Let's show the transform below :

$$\begin{aligned}\mathbf{M}\vec{f}(\vec{x}+\vec{c}_{i\Delta t,t+1}) &= \mathbf{M}\vec{f}(\vec{x},t)+\mathbf{SM}(\vec{f}(\vec{x},t,\vec{f}^{eq}(\rho,\vec{u},t)) \\ &= \mathbf{M}\vec{f}(\vec{x},t)+\mathbf{S}\left(\vec{m}(\vec{x},t)-\vec{m}^{eq}(\vec{x},t,f_i^{eq}(\rho,\vec{u},t))\right)\end{aligned} \tag{2.10}$$

In this work, we first apply the LBGK equation and use coordinate mapping to modify the streaming process, thereby extending ISLBM to curvilinear coordinate systems. The limitation of this approach is that we have not yet extended the MRT operator to general coordinate systems, as we have not fully grasped the underlying mathematical theory and physical mechanisms.

2.2.1 Transform Collision Step

Equations (2.9) and (2.10) are described in three-dimensional Cartesian coordinates. To transform these equations to general curvilinear coordinates, we note that the position vector only appears at discrete grid nodes, and the collision operator does not involve derivatives with respect to position. Therefore, we can directly substitute $\vec{x}$ with $\vec{\xi}$, where $\vec{x}$ denotes the Cartesian coordinate position and $\vec{\xi}$ denotes the curvilinear coordinate position.

we can say "position information only have position variable located at the grid node, so we can simply to transform by substituting .

$$f_i^{\star}(\vec{\xi},t) = f_i(\vec{\xi},t)+\omega(f_i(\vec{\xi},t)-f_i^{eq}(\rho,\vec{u},t)) \tag{2.11}$$

2.2.2 Transform Streaming Step

Definition of normalized discrete velocity set In general curvilinear coordinate, the variation of position vector or velocity or any physical variable related length dimension need to condsider transform factor for basis transformation. This is complexity for analiyzing the motion of particles. Let's define normalized discrete velocity set.

$$\vec{e}_\alpha = \underbrace{c_\alpha^i}_{\text{non-dimension}} \; \underbrace{\vec{g}_i(\vec{\xi})}_{\text{curvature tangent vector}} \; \underbrace{\frac{\Delta x}{\Delta t}}_{\text{(lattice speed)}\approx 1} \tag{2.12}$$

where the vector $\vec{e}_\alpha$ is the non-dimension discrete particle velocity set, and vector $\vec{g}_i$ is the tangent vector related to the curvature at the position. $\frac{\Delta x}{\Delta t}$ is the lattice speed, and its function in this definition is velocity dimension. so we rewrite the definition to another form below based on differential geometry.

$$\vec{e}_\alpha^{\,j} = c_\alpha^i \frac{\partial \xi_j}{\partial x_i} \tag{2.13}$$

where x represent the index of Cartesian coordinate and ξ is the index of general curvilinear coordinate, and the differential represent the basis transform from Cartesian coordinate to curve coordinate. Therefore, we can see the particle velocity distortion through the factor of the transform.

Path integration of the particle velocity The streaming step function

2.3 Boundary Condition

2.4 Algorithm