4.Volumetric Lattice Boltzmann Models in General Curvature

Chen Peng Chung

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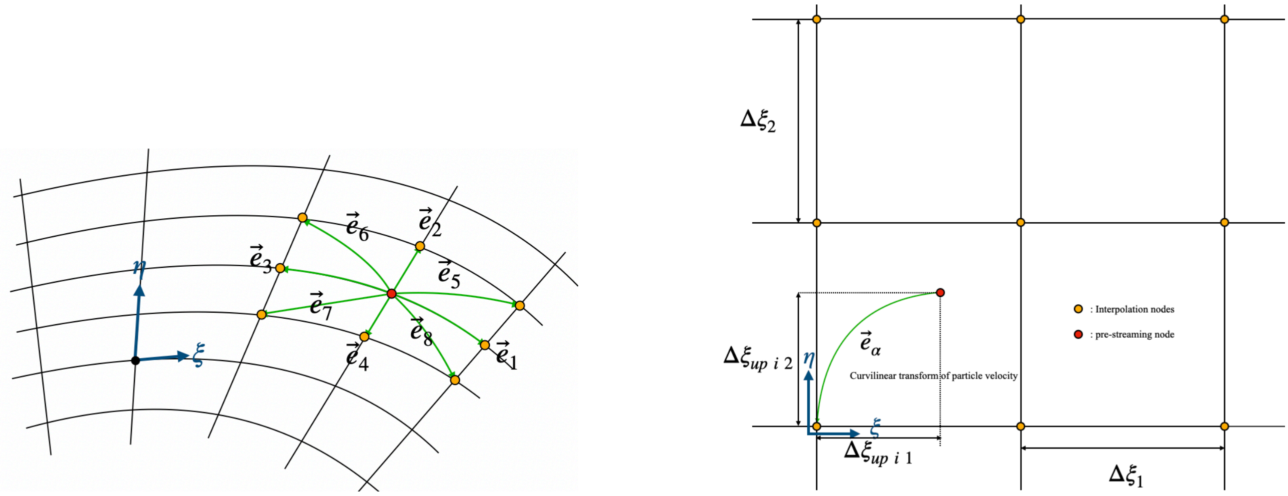
Contents

1 General Interpolation LBM

This note presents an extension of the Interpolation-Supplemented Lattice Boltzmann Method (ISLBM) for use in curvilinear coordinate systems. The first strategy is to transform the Cartesian coordinate system to a general curvilinear coordinate system through conformal mapping. Different from previous papers, this method extends **ISLBM** without changing the lattice system to accommodate curved motion. Note that because the transformation is based on coordinate mapping, we still compute the curved particle paths in the computational domain, as shown in the figure below:



(a) curvilinear coordinate system for SLBM

(b) General Interpolation LBM

Figure 1.1: 2 types of curvilinear coordinate system

2 Transformation

2.1 Jacobian relation

This section woul give the proven of the Jacobian relation :

$$\begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \frac{1}{\mathrm{J}} \begin{bmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{bmatrix} \tag{2.1}$$

For general curvilinear coordinate in two dimensions, we have to define Lamé coefficient for analyzing:

$$\vec{dr}_{+1}|_{q_2,q_3}(\vec{r}) \equiv \vec{r}(q_1+\Delta q_1, q_2, q_3) - \vec{r}(q_1, q_2, q_3) = \left.\frac{\partial \vec{r}}{\partial q_1}\right|_{q_2,q_3} dq_1 = \frac{\left.\frac{\partial \vec{r}}{\partial q_1}\right|_{q_2,q_3}}{\left|\left.\frac{\partial \vec{r}}{\partial q_1}\right|_{q_2,q_3}\right|} \left|\left.\frac{\partial \vec{r}}{\partial q_1}\right|_{q_2,q_3}\right| dq_1 \tag{2.2}$$

From the expression above, we can define unit vector and coefficient for differential geometry.

$$\begin{aligned} h_1 &\equiv \left|\frac{\partial \vec{r}}{\partial q_1}\right|_{q_2,q_3} \\ \vec{e}_1 &\equiv \frac{1}{h_1}\left.\frac{\partial \vec{r}}{\partial q_1}\right|_{q_2,q_3} \end{aligned} \tag{2.3}$$

where symbol h_1 is a Lamé coefficient for the coordinate component. Let's take the definition to defferential of position vector about variable (ξ, η).

$$\begin{aligned}
\frac{\partial \vec{r}}{\partial \xi} &= h_\xi \vec{e}_\xi = \vec{e}_x x_\xi + \vec{e}_y y_\xi \\
\frac{\partial \vec{r}}{\partial \eta} &= h_\eta \vec{e}_\eta = \vec{e}_x x_\eta + \vec{e}_y y_\eta \\
\frac{\partial \vec{r}}{\partial x} &= \vec{e}_x = h_\xi \vec{e}_\xi \xi_x + h_\eta \vec{e}_\eta \eta_x \\
\frac{\partial \vec{r}}{\partial y} &= \vec{e}_y = h_\xi \vec{e}_\xi \xi_y + h_\eta \vec{e}_\eta \eta_y
\end{aligned} \tag{2.4}$$

we can do some simple computation, and get the result below :

$$\begin{aligned}
\frac{h_\xi}{x_\xi}\vec{e}_\xi - \frac{h_\eta}{x_\eta}\vec{e}_\eta &= \vec{e}_y\left(\frac{y_\xi}{x_\xi} - \frac{y_\eta}{x_\eta}\right) = \vec{e}_y \frac{y_\xi x_\eta - y_\eta x_\xi}{x_\eta x_\xi} \\
\frac{h_\xi}{y_\xi}\vec{e}_\xi - \frac{h_\eta}{y_\eta}\vec{e}_\eta &= \vec{e}_x\left(\frac{x_\xi}{y_\xi} - \frac{x_\eta}{y_\eta}\right) = \vec{e}_x \frac{x_\xi y_\eta - x_\eta y_\xi}{y_\eta y_\xi}
\end{aligned} \tag{2.5}$$

Furthermore, we can rewrite this as:

$$\begin{aligned}
-x_\eta h_\xi \vec{e}_\xi + x_\xi h_\eta \vec{e}_\eta &= \vec{e}_y\left(x_\xi y_\eta - x_\eta y_\xi\right) \\
y_\eta h_\xi \vec{e}_\xi - y_\xi h_\eta \vec{e}_\eta &= \vec{e}_x\left(x_\xi y_\eta - x_\eta y_\xi\right)
\end{aligned} \tag{2.6}$$

Substitution the equation (2.6) into (2.4)

$$\begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \frac{1}{(x_\xi y_\eta - x_\eta y_\xi)} \begin{bmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{bmatrix} \tag{2.7}$$

The equation above is Jacobian relation.

2.2 Transform for Lattice Boltzmann Equation

First, review the basic governing equation for standard lattice boltzman method i.e., lattice boltzmann equation in Cartesian coordinate with 3 dimension:

$$f_i(\vec{x} + \vec{c}_i \Delta t, t+1) = f_i(\vec{x}, t) + \Omega(f_i(\vec{x}, t), f_i^{eq}(\rho, \vec{u}, t)) \tag{2.8}$$

where the symbol Ω is a coillision operator, so it can apart two types of forms.

$$f_i(\vec{x} + \vec{c}_i \Delta t, t+1) = f_i(\vec{x}, t) + \omega(f_i(\vec{x}, t) - f_i^{eq}(\rho, \vec{u}, t)) \tag{2.9}$$

If the collision operator is chosen as the BGK operator, we call the equation the **LBGK** equation. On the other hand, we can allow each momentum component to have a different relaxation effect. To achieve this, we introduce a basis transformation matrix that transforms the function from velocity space to momentum space. This means that the streaming step and collision step are performed in separate spaces. Let's show the transform below :

$$\begin{aligned}\mathbf{M}\vec{f}(\vec{x}+\vec{c}_{i\Delta t,t+1}) &= \mathbf{M}\vec{f}(\vec{x},t)+\mathbf{SM}(\vec{f}(\vec{x},t,\vec{f}^{eq}(\rho,\vec{u},t)) \\ &= \mathbf{M}\vec{f}(\vec{x},t)+\mathbf{S}\left(\vec{m}(\vec{x},t)-\vec{m}^{eq}(\vec{x},t,f_i^{eq}(\rho,\vec{u},t))\right)\end{aligned} \tag{2.10}$$

In this work, we first apply the LBGK equation and use coordinate mapping to modify the streaming process, thereby extending ISLBM to curvilinear coordinate systems. The limitation of this approach is that we have not yet extended the MRT operator to general coordinate systems, as we have not fully grasped the underlying mathematical theory and physical mechanisms.

2.2.1 Transform Collision Step

Equations (2.9) and (2.10) are described in three-dimensional Cartesian coordinates. To transform these equations to general curvilinear coordinates, we note that the position vector only appears at discrete grid nodes, and the collision operator does not involve derivatives with respect to position. Therefore, we can directly substitute $\vec{x}$ with $\vec{\xi}$, where $\vec{x}$ denotes the Cartesian coordinate position and $\vec{\xi}$ denotes the curvilinear coordinate position.

we can say ”position information only have position variable located at the grid node, so we can simply to transform by substituting .

$$f_i^{\star}(\vec{\xi},t) = f_i(\vec{\xi},t)+\omega(f_i(\vec{\xi},t)-f_i^{eq}(\rho,\vec{u},t)) \tag{2.11}$$

2.2.2 Transform Streaming Step

Definition of normalized discrete velocity set In general curvilinear coordinate, the variation of position vector or velocity or any physical variable related length dimension need to condsider transform factor for basis transformation. This is complexity for analiyzing the motion of particles. Let's define normalized discrete velocity set.

$$\vec{e}_\alpha = \underbrace{c_\alpha^i}_{\text{non-dimension}} \quad \underbrace{\vec{g}_i(\vec{\xi})}_{\text{curvature tangent vector}} \quad \underbrace{\frac{\Delta x}{\Delta t}}_{\text{(lattice speed)}\approx 1} \tag{2.12}$$

where the vector $\vec{e}_\alpha$ is the non-dimension discrete particle velocity set, and vector $\vec{g}_i$ is the tangent vector related to the curvature at the position. $\frac{\Delta x}{\Delta t}$ is the lattice speed, and its function in this definition is velocity dimension. so we rewrite the definition to another form below based on differential geometry.

$$\vec{e}_\alpha^{\,j} = c_\alpha^i \frac{\partial \xi_j}{\partial x_i} \tag{2.13}$$

where x represent the index of Cartesian coordinate and ξ is the index of general curvilinear coordinate, and the differential represent the basis transform from Cartesian coordinate to curve coordinate. Therefore, we can see the particle velocity distortion through the factor of the transform.

Path integration of the particle velocity The streaming step equation

$$f_\alpha(\vec{x},t) = f^\star_\alpha(\vec{x} - +\vec{c}_\alpha \delta t, t)$$

we can see the essential issue is compute the integration of the dis crete particle velocity to know the position where is post streaming position and streaming length. The length of the streaming step is defined as below :

$$\delta\vec{\xi}_\alpha = \int_0^{\Delta t} d\vec{\xi}_\alpha = \int_0^{\Delta t} \vec{e}_\alpha dt \tag{2.14}$$

1st explicit Euler method :

$$\delta\vec{\xi}_\alpha \approx \vec{e}_\alpha \Delta t \ldots O(\Delta t) \tag{2.15}$$

2nd order Runge-Kutta method :
using 2nd Runge-Kutta method to solving the integration, then it is apart into two step :

$$\begin{aligned} &\text{step 1 : } \Delta\vec{\xi}^{(1)}_\alpha = \frac{1}{2}\Delta t \vec{e}_\alpha \\ &\text{step 2 : } \Delta\vec{\xi}_\alpha = \Delta t \vec{e}_\alpha(\vec{\xi} - \vec{\xi}^{(1)}_\alpha) \end{aligned} \tag{2.16}$$

The result of discrete integration (2.16) have 2nd order accuracy in time space. The figure show that the method of 2nd Runge-Kutta to treat integration.

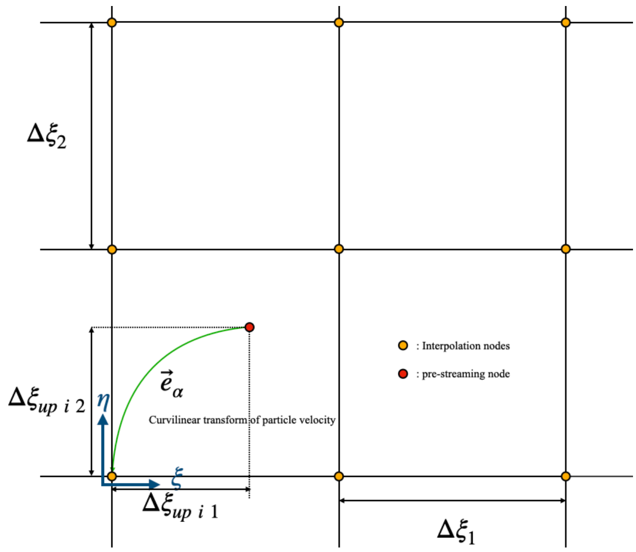


Figure 2.2: integration skill-2nd Runge-Kutta

In Interpolation-Supplemented LBM, the streaming step is implemented by pulling data backward from the pre-streaming position to the current grid node through interpolation. This approach ensures that information does not spread from grid nodes to non-physical locations. Therefore, when integrating the particle path, we integrate backward from $\vec{\xi}_\alpha$ to $\vec{\xi}_\alpha - \Delta\vec{\xi}_\alpha$, rather than forward from $\vec{\xi}_\alpha$ to $\vec{\xi}_\alpha + \Delta\vec{\xi}_\alpha$.

Streaming step after transformation When we get the result of the path integration of particle discrete velocity along the direction, we can rewrite the streaming step equation below : The discrete process is little similar as the process of distribution for time-space discretizing. However, the term discretized is different, one is time integration of distribution and the other is time integration of particle path. In curvilinear coordinate , we have

$$f_\alpha(\vec{\xi}, t) = f^\star_\alpha(\vec{\xi} - \Delta\vec{\xi}_\alpha, t) \tag{2.17}$$

where$\Delta\vec{\xi}_\alpha$ the change of position have 2nd order accuracy in time space, and R.H.S term is post-collision function.

Accuracy effect in time space for lattice bolrtzmann equation - BGK operator LBGK in curvilinear coordinate can be rewrite below :

$$f_\alpha(\vec{\xi}, t+1) - f_\alpha(\vec{\xi} - \Delta\vec{\xi}_\alpha, t) = \omega(f_\alpha(\vec{\xi} - \Delta\vec{\xi}_\alpha, t) - f^{eq}_\alpha(\rho, \vec{u}, t)) \ldots \text{(curvilinear coordinate)} \tag{2.18}$$

and we can define a material derivative for discrete-velocity Boltzmann equation

$$\mathcal{D}_t = \partial_t + \frac{\delta\vec{\xi}_\alpha}{\delta t} \cdot \vec{\nabla}_\xi \tag{2.19}$$

we call above operator as "discrete-velocity material derivation". Using Taylor expansion on equation (2.18)

$$\left(\partial_t + \frac{\delta\vec{\xi}_\alpha}{\delta t} \cdot \vec{\nabla}_\xi\right) f_\alpha \Delta t + \frac{1}{2}\left(\partial_t + \frac{\delta\vec{\xi}_\alpha}{\delta t} \cdot \vec{\nabla}_\xi\right)^2 f_\alpha \Delta t^2 \Bigg|_{(\vec{\xi} - \Delta\vec{\xi}_\alpha, t)} = \omega(f_\alpha(\vec{\xi} - \Delta\vec{\xi}_\alpha, t) - f^{eq}_\alpha(\rho, \vec{u}, t)) \tag{2.20}$$

left hand side and right hand side are both divided by Δt, can reover N-S equation in differential form.

$$\left(\partial_t + \frac{\delta\vec{\xi}_\alpha}{\delta t} \cdot \vec{\nabla}_\xi\right) f_\alpha + \frac{1}{2}\left(\partial_t + \frac{\delta\vec{\xi}_\alpha}{\delta t} \cdot \vec{\nabla}_\xi\right)^2 f_\alpha \Delta t \Bigg|_{(\vec{\xi} - \Delta\vec{\xi}_\alpha, t)} + O(\Delta t^2) = \omega(f_\alpha(\vec{\xi} - \Delta\vec{\xi}_\alpha, t) - f^{eq}_\alpha(\rho, \vec{u}, t)) \tag{2.21}$$

The second term of the velocity-discrete Boltzmann equation corresponds to the shear stress field through Chapman-Enskog expansion. If we use the explicit Euler method to discretize the particle path integration, the first term has only zeroth-order temporal accuracy, while the stress term has only first-order temporal accuracy. This is insufficient to recover the Navier-Stokes equation. Therefore, we must use a method that provides higher-order temporal accuracy for the path integration, ensuring that the differential term of the velocity-discrete Boltzmann equation has at least second-order temporal accuracy.

2.3 Multiple Scale Expansion

For collision and straming in mesoscopic space, and advevtion and diffusion in macroscpic space, we have three types of time scale :

1. $K^{(0)}$: The first time scale, related to collision and streaming in mesoscopic space.
2. $K^{(1)}$: The second time scale, related to convection in macroscopic space .

3. $K^{(2)}$: The third time scale, related to diffusion in macroscopic space.

If K = 0.01, then so we can define three types of time variables : (the value is about $K^{(0)}$, $K^{(1)}$, $K^{(2)}$)

$$
\begin{aligned}
& t \in K^{(2)} \\
& t^{(1)} \equiv K * t \in K^{(1)} \\
& t^{(2)} \equiv K^2 * t \in K^{(0)}
\end{aligned}
\tag{2.22}
$$

and define two types of space variables :

$$
\vec{r}^{(1)} \equiv K * \vec{r} \tag{2.23}
$$

Therefore, the differential about time variable is :

$$
\frac{\partial}{\partial t} = K\frac{\partial}{\partial t^{(1)}} + K^2\frac{\partial}{\partial t^{(2)}} = K\partial_t^{(1)} + K\partial_t^{(2)} \tag{2.24}
$$

In physical space, we have the relation about two variables :

$$
\begin{aligned}
& \frac{\partial}{\partial \vec{\xi}} = K\frac{\partial}{\partial \vec{\xi}^{(1)}} \\
& \Rightarrow \vec{\nabla}_{\vec{\xi}} = K\vec{\nabla}_{\vec{\xi}^{(1)}}
\end{aligned}
\tag{2.25}
$$

Finally, for three types of time variable, multi-scale expansion of dis distribution funtion :

$$
f_i = f_i^{eq} + Kf_i^{(1)} + K^2 f_i^{(2)} + \dots \tag{2.26}
$$

We have to known that time scale is a domain for time space like $K^{(0)}, K^{(1)}$, and the element of the time scale is a variable, and their scale have different property.

Multi-Scale Technique and LBGK Before analyzing the LBGK, first, review the form of the basic equation :

$$
f_\alpha(\vec{r} + \vec{e}_\alpha \delta t, t + \delta t) - f_\alpha(\vec{r}, t) = -\frac{\delta t}{\tau}\left[f^{neq}(\vec{r}, t)\right] \tag{2.27}
$$

Using particle material derivation :

$$
\mathcal{D}_t = \partial_t + \vec{e}_\alpha \cdot \vec{\nabla}
$$

and Taylor expansion to analiyzing the left hand side of the LBGK :

$$
\mathcal{D}_t \delta t + \frac{\delta t^2}{2}\mathcal{D}_t^2 f_\alpha \bigg|_{\vec{r},t} = -\frac{\delta t}{\tau}\left[f^{neq}(\vec{r}, t)\right] \tag{2.28}
$$

$$
\left(\partial_t + \vec{e}_\alpha \cdot \vec{\nabla}\right) f_\alpha + \frac{\delta t^1}{2}\left(\partial_t + \vec{e}_\alpha \cdot \vec{\nabla}\right)^2 f_\alpha = -\frac{1}{\tau}\left[f^{neq}(\vec{r}, t)\right] \tag{2.29}
$$

substitue above equation using the discrete velocity Boltzmann equation :

$$\left(\partial_t + \vec{e}_\alpha \cdot \vec{\nabla}\right) f_\alpha = -\frac{1}{\tau}\left[f^{neq}(\vec{r},t)\right]$$

we have the equation below :

$$\left(\partial_t + \vec{e}_\alpha \cdot \vec{\nabla}\right) f_\alpha - \frac{\delta t^1}{2}\left(\partial_t + \vec{e}_\alpha \cdot \vec{\nabla}\right) \frac{1}{\tau}\left[f_\alpha^{neq}(\vec{r},t)\right] = -\frac{1}{\tau}\left[f_\alpha^{neq}(\vec{r},t)\right] \tag{2.30}$$

For analiyzing the equation above, review the multiple scale relation as shown below:

$$\begin{aligned}
\partial_t &= K\partial_t^{(1)} + K^2\partial_t^{(2)} + \dots \\
\vec{\nabla} &= K\vec{\nabla}^{(1)} + K^2\vec{\nabla}^{(2)} + \dots \\
f_\alpha^{neq} &= f_\alpha - f_\alpha^{eq} = Kf_\alpha^{(1)} + K^2 f_\alpha^{(2)} + \dots
\end{aligned} \tag{2.31}$$

and go back to the equation (2.30), we have

$$\begin{aligned}
&K\left(\partial_t^{(1)} + \vec{e}_\alpha \cdot \vec{\nabla}^{(1)}\right) f_\alpha = -K\frac{1}{\tau}\left[f_\alpha^{(1)}(\vec{r},t)\right] \\
&K^2\left(\partial_t^{(2)} + \vec{e}_\alpha \cdot \vec{\nabla}^{(2)}\right) f_\alpha - \frac{\delta t K^2}{2\tau}\left(\partial_t^{(1)} + \vec{e}_\alpha \cdot \vec{\nabla}^{(1)}\right)\left[f_\alpha^{(1)}(\vec{r},t)\right] = -K^2\frac{1}{\tau}\left[f_\alpha^{(2)}(\vec{r},t)\right]
\end{aligned} \tag{2.32}$$

But $f_\alpha = f_\alpha^{eq} + f_\alpha^{neq} = f_\alpha^{eq} + Kf_\alpha^{(1)} + \dots$, the equation can be shown below :

$$\begin{aligned}
&K\left(\partial_t^{(1)} + \vec{e}_\alpha \cdot \vec{\nabla}^{(1)}\right) f_\alpha^{eq} = -K\frac{1}{\tau}\left[f_\alpha^{(1)}(\vec{r},t)\right] \\
&K^2\left(\partial_t^{(2)} + \vec{e}_\alpha \cdot \vec{\nabla}^{(2)}\right) f_\alpha^{eq} - \frac{\delta t K^2}{2\tau}\left(\partial_t^{(1)} + \vec{e}_\alpha \cdot \vec{\nabla}^{(1)}\right)\left[f_\alpha^{(1)}(\vec{r},t)\right] + K^2\left(\partial_t^{(1)} + \vec{e}_\alpha \cdot \vec{\nabla}^{(1)}\right) f_\alpha^{(1)} \\
&= -K^2\frac{1}{\tau}\left[f_\alpha^{(2)}(\vec{r},t)\right]
\end{aligned} \tag{2.33}$$

2.4 Boundary Condition

2.5 Algorithm