frontiers in Applied Mathematics and Statistics
published: 16 June 2021

ORIGINAL RESEARCH
doi: 10.3389/fams.2021.691582

Volumetric Lattice Boltzmann Models in General Curvilinear Coordinates: Theoretical Formulation

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Front. Appl. Math. Stat. 7:691582 doi: 10.3389/fams.2021.691582
Published: 16 June 2021

A volumetric formulation for lattice Boltzmann models on general curvilinear coordinates is presented, preserving one-to-one advection and recovering the Navier–Stokes equations in the hydrodynamic limit.

Presenter: Chen Peng Chung

Outline

Core Problem

In **general curvilinear coordinates**, macroscopic values (ρ, $\rho\vec{u}$) experience **momentum loss** due to:

- Curvature effects
- Nonuniform grid properties

Key Question

How to maintain **momentum conservation** in the streaming step of LBM when using curvilinear coordinates?

Solution Approach

Define a **velocity-space discretized force field** to compensate for the momentum loss (interpreted as an **inertial force**).

Lattice Boltzmann BGK Equation

The standard LBGK equation in curvilinear coordinates:

$$\boxed{N_\alpha(\vec{q} + \vec{e}_\alpha \delta t, t+1) = N_\alpha(\vec{q}, t) + \Omega_\alpha + \delta N_\alpha(\vec{q}, t)}$$

where the correction term is defined as:

$$\delta N_\alpha(\vec{q}, t) \equiv N_\alpha(\vec{q}, t) - N^\star_\alpha(\vec{q} - \vec{e}_\alpha \delta t, t-1)$$

- $\vec{q}$: nondimensional position (q_1, q_2, q_3)
- $N^\star$: **post-collision** distribution function
- Ω_α : collision operator
- $\vec{e}_\alpha$: discrete velocity directions

Mass Conservation vs Momentum Loss

Mass Conservation (✓)

$$\sum_{\alpha=0}^{q-1} \delta N_\alpha(\vec{q}, t) = 0$$

Reason: Sum of distributions over velocity space is conserved.

Momentum Loss (×)

$$\sum_{\alpha=1}^{q} \vec{e}_\alpha \, \delta N_\alpha \neq 0$$

Reason: Discrete velocity vectors $\vec{e}_\alpha$ vary with position in curvilinear coords!

Key Insight: $N_\alpha(\vec{q}, t) \;\neq\; N_\alpha^\star(\vec{q} - \vec{e}_\alpha \delta t, t - 1)$

Pre-streaming $\neq$ Post-collision at different positions

Definition: Nondimensional Curvature Change

Definition

The **nondimensional change of curvature tangent vector**:

$$\boxed{\vec{\Theta}_i^j(\vec{q}+\vec{e}_\alpha\delta t,\vec{q}) = [\vec{g}_i(\vec{q}+\vec{e}_\alpha\delta t) - \vec{g}_i(\vec{q})]\cdot\vec{g}^{\,j}(\vec{q})}$$

where the **contravariant basis vector** $\vec{g}^j$ is:

$$\vec{g}^1 = \frac{\vec{g}_2\times\vec{g}_3}{(\vec{g}_2\times\vec{g}_3)\cdot\vec{g}_1}$$

- $\vec{g}_i$: covariant basis vectors (tangent to coordinate curves)
- $\vec{g}^j$: contravariant basis vectors (normal to coordinate surfaces)
- Θ_i^j measures how the tangent vector changes along $\vec{e}_\alpha$

Definition: Momentum Loss

Definition

Total Momentum Loss at point $(\vec{q}, t)$:

$$\vec{M}_{\text{Loss}}(\vec{q}, t) \equiv \sum_{\alpha=1}^{q} \left[N_\alpha(\vec{q}, t)\, \vec{e}_\alpha(\vec{q}) - N^\star_\alpha(\vec{q} - \vec{e}_\alpha \delta t, t-1)\, \vec{e}_\alpha(\vec{q} - \vec{e}_\alpha \delta t) \right]$$

Physical Meaning:

- Momentum carried by particles **arriving** at $\vec{q}$
- Minus momentum carried by particles **leaving** from $\vec{q} - \vec{e}_\alpha \delta t$
- Difference arises because $\vec{e}_\alpha(\vec{q}) \neq \vec{e}_\alpha(\vec{q} - \vec{e}_\alpha \delta t)$

Inner & Outer Momentum Loss

The paper separates momentum loss into **two contributions**:

Definition

Inner Momentum Loss (particles entering):

$$\mathcal{J}(\vec{q})\,\vec{\chi}^{\mathrm{in}}(\vec{q},t) \equiv -\sum_{\alpha=1}^{q} N_\alpha(\vec{q},t)\cdot\left(\vec{e}_\alpha(\vec{q}) - \vec{e}_\alpha(\vec{q}-\vec{e}_\alpha\delta t)\right)$$

Definition

Outer Momentum Loss (particles leaving):

$$\mathcal{J}(\vec{q})\,\vec{\chi}^{\mathrm{out}}(\vec{q},t) \equiv -\sum_{\alpha=1}^{q} \left(\vec{e}_\alpha(\vec{q}+\vec{e}_\alpha\delta t) - \vec{e}_\alpha(\vec{q})\right)\cdot N_\alpha^{\star}(\vec{q},t)$$

where $\mathcal{J}$ is the Jacobian of the coordinate transformation.

Discrete Velocity in Curvilinear Coordinates

The **discrete particle velocity set** in curvilinear coordinates:

$$\boxed{\vec{e}_\alpha(\vec{q}) = \sum_{i=1}^{3} (c_\alpha^i)\, \vec{g}_i \, \frac{\Delta x}{\Delta t}}$$

- c_α^i : lattice velocity components (integers like $0, \pm 1$)
- $\vec{g}_i$: local covariant basis vectors
- $\Delta x / \Delta t$: lattice speed ratio

Important

Unlike Cartesian coordinates, $\vec{e}_\alpha$ depends on position $\vec{q}$ through the basis vectors $\vec{g}_i(\vec{q})$.

Momentum Conservation Constraint

The author imposes the **momentum constraint** for the streaming step:

$$\sum_{\alpha=1}^{q} \vec{e}_\alpha(\vec{q})\,\delta N_\alpha(\vec{q}) = \mathcal{J} \cdot \frac{\vec{\chi}^{\text{in}}(\vec{q},t) + \vec{\chi}^{\text{out}}(\vec{q},t)}{2}$$

Expanding:

$$= \mathcal{J} \cdot \frac{-\sum_\alpha N_\alpha(\vec{q})\,(\vec{e}_\alpha(\vec{q}) - \vec{e}_\alpha(\vec{q} - \vec{e}_\alpha\delta t)) - \sum_\alpha N^\star_\alpha(\vec{q})\,(\vec{e}_\alpha(\vec{q} + \vec{e}_\alpha\delta t) - \vec{e}_\alpha(\vec{q}))}{2}$$

- Blue: contribution from pre-collision distribution
- Red: contribution from post-collision distribution

Velocity-Space Discretized Force Field

Definition

The **velocity-space discretized force field** in curvilinear coordinates:

$$\boxed{F^i(\vec{q},t) = \frac{\vec{\chi}^{\text{in}}(\vec{q},t) + \vec{\chi}^{\text{out}}(\vec{q},t)}{2} \cdot \vec{g}^{\,i}(\vec{q})}$$

Physical Interpretation:

- This force represents the **inertial force** arising from curvilinear coordinates
- Analogous to centrifugal/Coriolis forces in rotating frames
- Compensates for the "loss" in momentum during streaming

General Force Definition

$$\mathcal{J}(\vec{q})F^i(\vec{q},t) \equiv \frac{\text{Momentum Loss}}{\Delta t} \cdot \vec{e}_j \frac{1}{|\vec{g}_j|}$$

Final Force Formula

After derivation, the force field becomes:

$$\boxed{\mathcal{J}(\vec{q})F^j(\vec{q},t) = \frac{-1}{2}\left(\sum_\alpha N_\alpha\, c^i_\alpha \Theta^j_i(\vec{q}-\vec{e}_\alpha \delta t, \vec{q}) + \sum_\alpha N^\star_\alpha\, c^i_\alpha \Theta^j_i(\vec{q}+\vec{e}_\alpha \delta t, \vec{q})\right)}$$

Key Components:

Symbol	Meaning
$\mathcal{J}$	Jacobian of transformation
c^i_α	Lattice velocity component
Θ^j_i	Curvature change tensor
N_α	Pre-collision distribution
$N^\star_\alpha$	Post-collision distribution

Derivation Outline

Three-Step Derivation:

1. **Define Momentum Loss:**

$$\sum_{\alpha=1}^{q} \vec{e}_\alpha(\vec{q})\,\delta N_\alpha(\vec{q},t) \equiv \text{Momentum Loss}$$

2. **Express via Inner/Outer contributions:**

$$\sum_{\alpha=1}^{q} \vec{e}_\alpha(\vec{q})\,\delta N_\alpha \equiv \mathcal{J}(\vec{q}) \cdot \frac{\vec{\chi}^{\text{in}} + \vec{\chi}^{\text{out}}}{2}$$

3. **Project onto coordinate directions:**

$$\mathcal{J}(\vec{q})F^i \equiv \frac{\text{Momentum Loss}}{\Delta t} \cdot \vec{e}_j \frac{1}{|\vec{g}_j|}$$

Basic Equations for Curvilinear LBM

Correction term:

$$\delta N_\alpha(\vec{q}, t) = w_\alpha \left(\frac{c_\alpha^i F^i}{c_s^2} + \frac{c_\alpha^i c_\alpha^j - c_s^2 \delta^{ij}}{2c_s^4} \right)$$

Moment constraints (for Chapman-Enskog $\rightarrow$ Navier-Stokes):

$$\sum_\alpha f^{eq} = \rho$$

$$\sum_\alpha f^{eq} c_\alpha^i = \rho u^i$$

$$\sum_\alpha f^{eq} c_\alpha^i c_\alpha^j = \rho c_s^2 g^{ij} + \rho \left(u^i + \frac{F^i}{2\rho} \right) \left(u^j + \frac{F^j}{2\rho} \right)$$

Note: The force F^i enters the velocity shift: $u^i \rightarrow u^i + \frac{F^i}{2\rho}$

Summary & Key Takeaways

Main Contributions

1. Identified **momentum loss** in streaming step for curvilinear LBM
2. Introduced **curvature change tensor** Θ_i^j to quantify geometric effects
3. Derived **velocity-space discretized force field** to restore conservation

Physical Insight

The force field F^i acts as an **inertial force** (like centrifugal force), arising purely from the curvilinear coordinate system.

Practical Use

Add correction term δN_α to LBGK equation:

$$N_\alpha(\vec{q} + \vec{e}_\alpha \delta t, t + 1) = N_\alpha(\vec{q}, t) + \Omega_\alpha + \delta N_\alpha$$