Introduction to Turbulence Modelling

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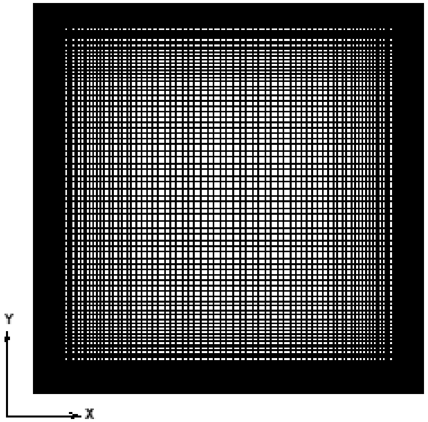
Navier-Stokes equation

- Incompressible Navier-Stokes equations

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[v(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) \right]$$

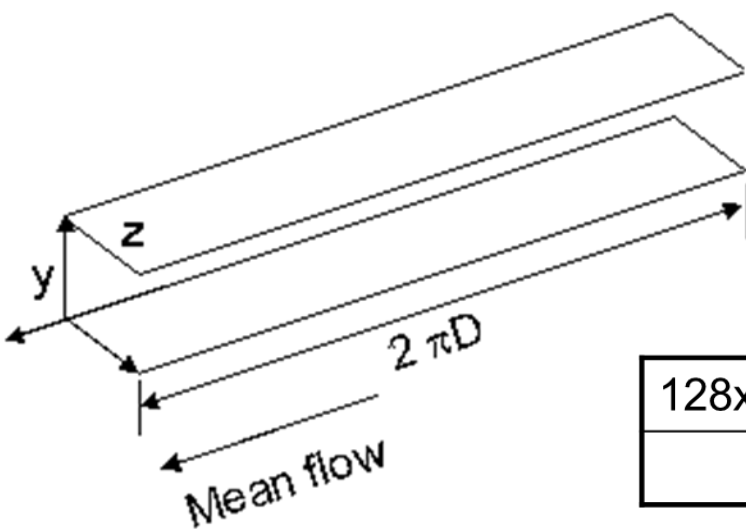
- To resolve the near wall structure
  - Grid is clustered in wall normal direction

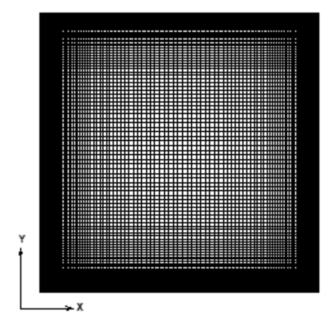


$$(x_i = \xi[1 - \frac{\tanh\gamma(\xi - \hat{x}_i)}{\tanh\gamma\xi}], \xi = \frac{D}{2}, i = 1,2)$$

Direct Numerical simulation

- Direct numerical simulations applied to the turbulent Poiseuille flow in a plane channel.
- The frictional Reynolds numbers is 360 based on frictional velocity and channel height.
- Simulation results are compared with DNS results of Abe et al. (2001) and Moser et al. (1999)at the same Ret.

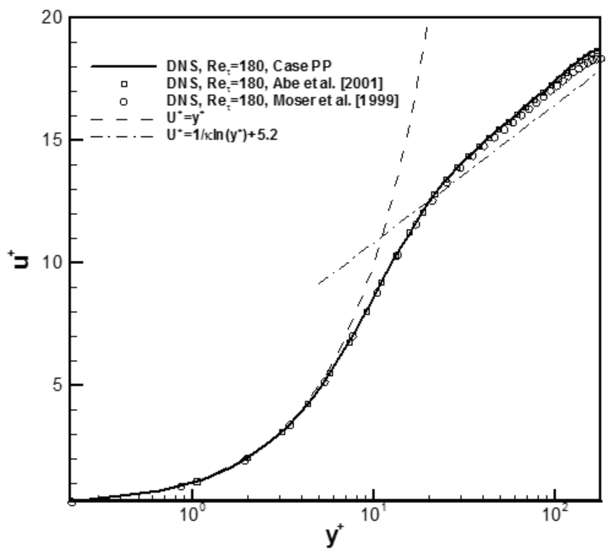




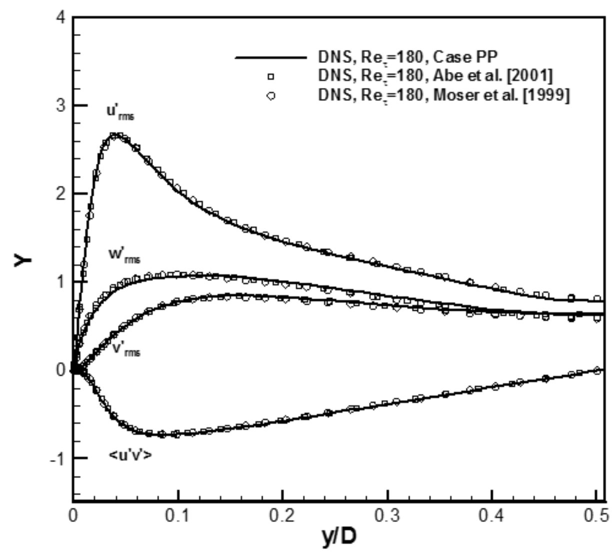
128x128x128	Δx^+	Δy^+	Δz^+
DNS	8.8357	0.2126~5.927	4.4879

Direct Numerical simulation

- Mean streamwise velocity profile



- Turbulence intensities and Reynolds stress



Reynolds averaged Navier-Stokes equations

- Reynolds averaged NS equation

$$\frac{\partial}{\partial x_j}(\rho U_i U_j) = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}[\mu(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}) - \rho\overline{u_i u_j}]$$

- How to model the Reynolds stress ??

Eddy viscosity models

- Navier Stokes equation

$$\frac{\partial \rho U_j U_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \rho\overline{u_i u_j}\right]$$

- Boussinesq Approximation

$$-\rho\overline{u_i u_j} = \mu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\rho k$$

$$\frac{\partial \rho U_j U_i}{\partial x_j} = -\frac{\partial (P + \frac{2}{3}\rho k)}{\partial x_i} + \frac{\partial}{\partial x_j}\left[(\mu + \mu_t)\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\right]$$

Eddy viscosity models

- k-εmodel (Launder and Jones 1972)

$$-\rho\overline{u_i u_j} = \mu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\rho k,$$

$$\mu_t = C_\mu \rho k^a \varepsilon^b = C_\mu \rho k^2/\varepsilon$$

- K equation

$$\frac{\partial \rho U_j k}{\partial x_j} = \frac{\partial}{\partial x_j}\left[-\frac{1}{2}\rho\overline{u_j u_i u_i} - \rho\overline{u_j p}\right] - \rho\overline{u_i u_j}\frac{\partial U_i}{\partial x_j} - \underbrace{\mu\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}}_{\rho\varepsilon}$$

BE Launder, DB Spalding, 1974, The numerical computation of turbulent flows, Computer methods in applied mechanics and engineering 3 (2), 269-289

k- εmodels-k equation

$$\underbrace{-\rho\overline{u_i u_j}\frac{\partial U_i}{\partial x_j}}_{P_k} = \left[\mu_t\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)-\frac{2}{3}\delta_{ij}\rho k\right]\frac{\partial U_i}{\partial x_j}$$

$$= \mu_t\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j}$$

$$-\frac{1}{2}\rho\overline{u_j u_i u_i}+\rho\overline{u_j p} = \frac{\mu_t}{\sigma_k}\frac{\partial k}{\partial x_j}$$

$$\frac{\partial \rho U_j k}{\partial x_j} = \frac{\partial}{\partial x_j}\left[(\mu+\frac{\mu_t}{\sigma_k})\frac{\partial k}{\partial x_j}\right]+\mu_t\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j}-\rho\varepsilon$$

k- εmodels-ε equation

$$\frac{\partial \rho U_j k}{\partial x_j} = \frac{\partial}{\partial x_j}\left[(\mu + \frac{\mu_t}{\sigma_k})\frac{\partial k}{\partial x_j}\right] + P_k - \rho\varepsilon$$

$$\frac{\partial \rho U_j \varepsilon}{\partial x_j} = \frac{\partial}{\partial x_j}\left[(\mu + \frac{\mu_t}{\sigma_\varepsilon})\frac{\partial \varepsilon}{\partial x_j}\right] + \frac{\varepsilon}{k}(C_{\varepsilon 1}P_k - C_{\varepsilon 2}\rho\varepsilon)$$

$$-If \ \mathrm{k} \sim \mathrm{x}^{-\mathrm{n}} \quad \text{decaying grid turbulence } \mathrm{P_k} = 0$$

$$\frac{dk}{dx} = -\varepsilon \qquad \Rightarrow -nx^{-n-1} = -\varepsilon \Rightarrow \varepsilon = nx^{-n-1}$$

$$\frac{d\varepsilon}{dx} = -C_{\varepsilon 2}\frac{\varepsilon^2}{k} \Rightarrow -(n+1)nx^{-n-2} = -C_{\varepsilon 2}\frac{n^2 x^{-2n-2}}{x^{-n}}$$

$$C_{\varepsilon 2} = \frac{n+1}{n} \approx 1.92 \leftarrow n \approx 1.1$$

$$\underbrace{U\frac{\partial k}{\partial x}+V\frac{\partial k}{\partial y}}_{A-convection} = \underbrace{-\overline{uv}\frac{\partial U}{\partial y}}_{B-production} \underbrace{-\frac{\partial}{\partial y}(\overline{vk'}+\frac{\overline{vp}}{\rho})}_{C-diffusion} \underbrace{-\nu\overline{(\frac{\partial u_i}{\partial x_j})}}_{D-dissipation}$$

Kinetic energy of turbulence

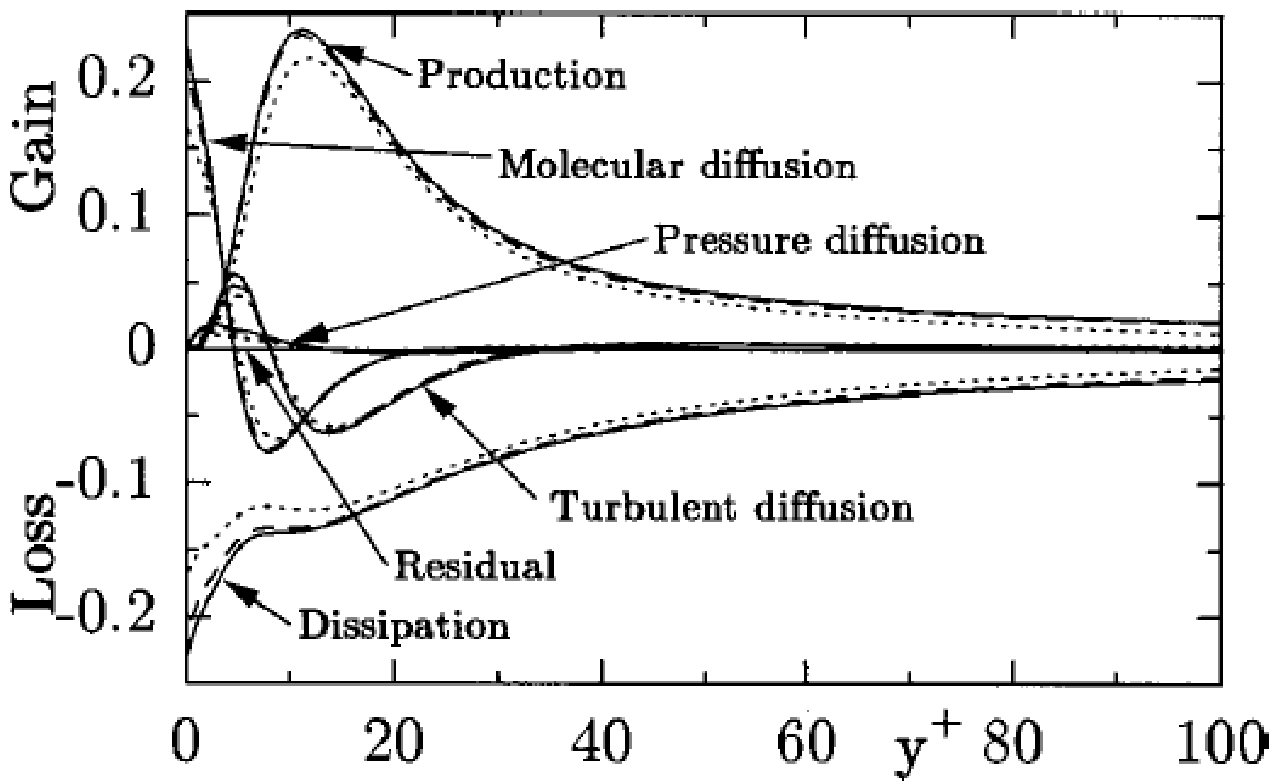


Fig. 13 Budget of turbulent kinetic energy: ——, Re_τ=640; — — —, Re_τ=395; - - -, Re_τ=180

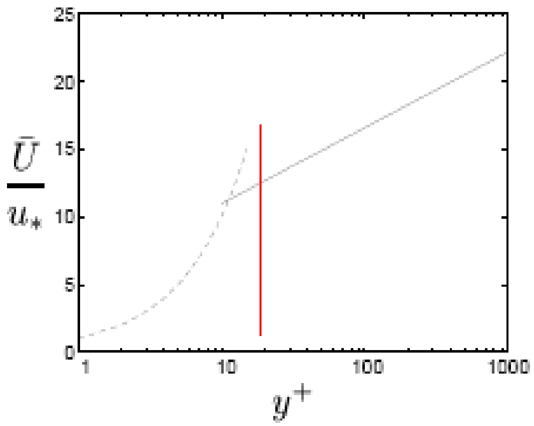
k- εmodels-local equlibrium

$-$For homogeneous shear flow $S = \frac{\partial U}{\partial y} \neq 0$

$$P_k = \mu_t \frac{\partial U}{\partial y}\frac{\partial U}{\partial y} = \rho\varepsilon \Leftarrow \tau = \mu_t \frac{\partial U}{\partial y} = \rho|\overline{uv}|$$

$$\rho^2\overline{uv}^2 = \tau^2 = \mu_t{}^2\left(\frac{\partial U}{\partial y}\right)^2 = \mu_t\rho\varepsilon = \rho C_\mu \frac{k^2}{\varepsilon}\rho\varepsilon$$

$$C_\mu = \frac{\overline{uv}^2}{k^2} \approx 0.3^2 = 0.09 \Leftrightarrow \left|\frac{\overline{uv}}{k}\right| \approx 0.3 \leftarrow \text{inner wall}$$



k- εmodels-local equlibrium P=ε

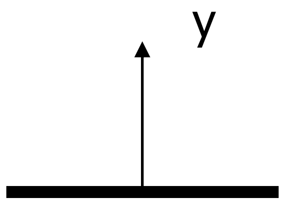
Near wall turbulence kinetic energy level

$$-The\ \text{log layer}-\text{constant stress layer}$$

$$|\overline{uv}| = u_*^2, and\ \log\ \ law\ \frac{\partial U}{\partial y} = \frac{u_*}{\kappa y}$$

$$u_*^2 = -\overline{uv} = \nu_t \frac{\partial U}{\partial y} = C_\mu \frac{k^2}{\varepsilon}\frac{u_*}{\kappa y} \Rightarrow \varepsilon = C_\mu \frac{k^2}{u_* \kappa y}$$

$$P = -\overline{uv}\frac{\partial U}{\partial y} = u_*^2 \frac{u_*}{\kappa y} = \varepsilon = C_\mu \frac{k^2}{u_* \kappa y} \Rightarrow k = \frac{u_*^2}{\sqrt{C_\mu}}$$



k- ε models-local equilibrium

$$0 = \frac{\partial}{\partial y}\left[\frac{\mu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial y}\right] + \frac{\varepsilon}{k}\left(C_{\varepsilon 1}P_k - C_{\varepsilon 2}\rho\varepsilon\right), \mu_t \simeq \rho u \ell$$

$$= \rho u_* \kappa y$$

$$\frac{\partial}{\partial y}\left[\frac{\mu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial y}\right] = \frac{\partial}{\partial y}\left[\frac{\rho u_* \kappa y}{\sigma_\varepsilon}\frac{\partial}{\partial y}\rho\frac{u_*^3}{\kappa y}\right] = \rho^2\frac{u_*^4}{\sigma_\varepsilon y^2}$$

$$0 = \left(C_{\varepsilon 1} - C_{\varepsilon 2}\right)\frac{u_*^3}{\kappa y}\frac{1}{C_\mu^{-1/2}u_*^2}\frac{u_*^3}{\kappa y} + \frac{u_*^4}{\sigma_\varepsilon y^2}$$

$$C_{\varepsilon 2} - C_{\varepsilon 1} = \frac{\kappa^2}{C_\mu^{1/2}\sigma_\varepsilon} = \frac{0.41^2}{0.3x1.3} = 0.43$$

k- εmodels

- Jet spreading rate
- Plane jet in the self-similar region

$$\frac{d\delta^{1/2}}{dx} \approx 0.11 \Leftarrow C_{\varepsilon 1} = 1.44$$

$$C_{\varepsilon 2} - C_{\varepsilon 1} = 1.92 - 1.44 = 0.48$$

The difference controls the spreading rate of jets.

Final form of k- ε models

$$\frac{\partial \rho U_j k}{\partial x_j} = \frac{\partial}{\partial x_j}\left[(\mu + \frac{\mu_t}{\sigma_k})\frac{\partial k}{\partial x_j}\right] + \underbrace{\mu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j}}_{P_k} - \rho\varepsilon$$

$$\frac{\partial \rho U_j \varepsilon}{\partial x_j} = \frac{\partial}{\partial x_j}\left[(\mu + \frac{\mu_t}{\sigma_\varepsilon})\frac{\partial \varepsilon}{\partial x_j}\right] + \frac{\varepsilon}{k}(C_{\varepsilon 1}P_k - C_{\varepsilon 2}\rho\varepsilon)$$

$$-\rho\overline{u_i u_j} = \mu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\rho k, \mu_t = C_\mu \rho k^2/\varepsilon$$

C_μ	$C_{\varepsilon 1}$	$C_{\varepsilon 2}$	σ_k	σ_ε
0.09	1.44	1.92	1.	1.3

K-ω model- Wilcox 1993

$$\frac{\partial \rho U_j k}{\partial x_j} = \frac{\partial}{\partial x_j}\left[(\mu + \frac{\mu_t}{\sigma_k})\frac{\partial k}{\partial x_j}\right] + \underbrace{\mu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j}}_{P_k} - \rho k \omega$$

$$\frac{\partial \rho U_j \omega}{\partial x_j} = \frac{\partial}{\partial x_j}\left[(\mu + \frac{\mu_t}{\sigma_\omega})\frac{\partial \omega}{\partial x_j}\right] + C_\mu C_{\omega 1}\rho\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} - C_{\omega 2}\rho\omega^2$$

$$\mu_t = C_\mu \rho \frac{k}{\omega}, \qquad \varepsilon = k\omega$$

C_μ	C_{w1}	C_{w2}	σ_k	σ_w
0.09	5/9	5/6	2	2

Low Reynolds number k-ϵ modeling

In the near wall region, the model has to be modified to account for the diminishing length scale

$$\frac{\partial U_j k}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] - \frac{1}{2}\frac{\partial}{\partial x_j}\left(\nu \frac{k}{\varepsilon}\frac{\partial \hat{\varepsilon}}{\partial x_j}\right)$$

$$-\overline{u_i u_j}\frac{\partial U_i}{\partial x_j} - (\tilde{\varepsilon} + \hat{\varepsilon})$$

$$\frac{\partial U_j \tilde{\varepsilon}}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \tilde{\varepsilon}}{\partial x_j}\right] - \frac{\partial}{\partial x_j}\left(\nu \frac{\tilde{\varepsilon}}{k}\frac{\partial k}{\partial x_j}\right)$$

$$-C_{\varepsilon 1} f_1 \frac{\tilde{\varepsilon}}{k}\overline{u_i u_j}\frac{\partial U_i}{\partial x_j} - C_{\varepsilon 2} f_2 \frac{\tilde{\varepsilon}^2}{k}$$

Low Reynolds number modeling

$$\nu_t = (k^2/\tilde{\varepsilon}) C_\mu f_\mu (y_\lambda)$$

The coefficients and the damping functions are

$$C_\mu = 0.09, \qquad C_{\varepsilon 1} = 1.44, \qquad C_{\varepsilon 2} = 1.92$$

$$f_\mu = 1 - \exp\left(-0.01 y_\lambda - 0.008 y_\lambda^3\right)$$

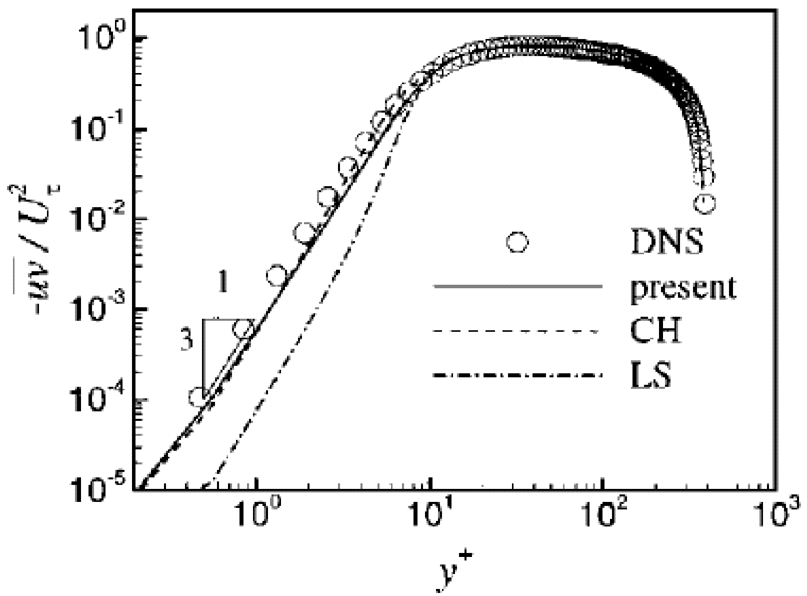
$$\sigma_k = 1.4 - 1.1 \exp[-(y_\lambda / 10)]$$

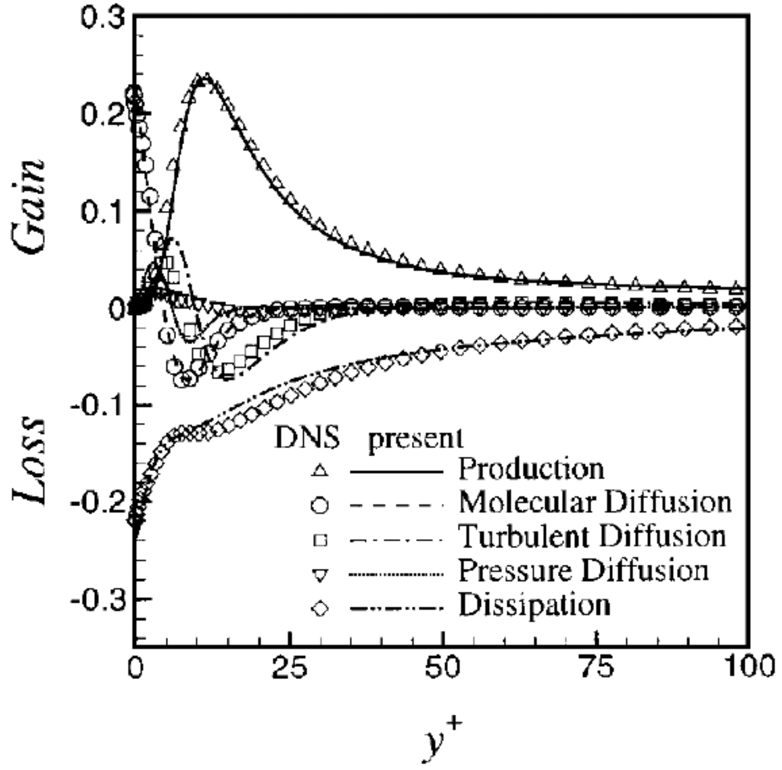
$$\sigma_\varepsilon = 1.3 - 1.0 \exp[-(y_\lambda / 10)]$$

$$f_1 = 1, \qquad f_2 = 1$$

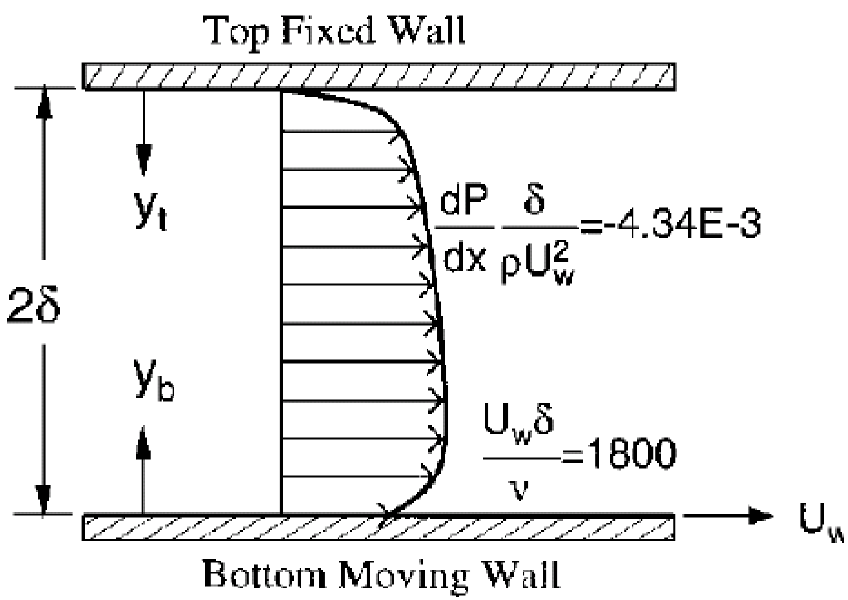
Hwang, C. B. and Lin, C. A., 1998, "An improved Low-Reynolds-Number k-e model based on Direct Numerical Simulation Data", AIAA J., Vol. 36, No. 1, pp. 38-43.

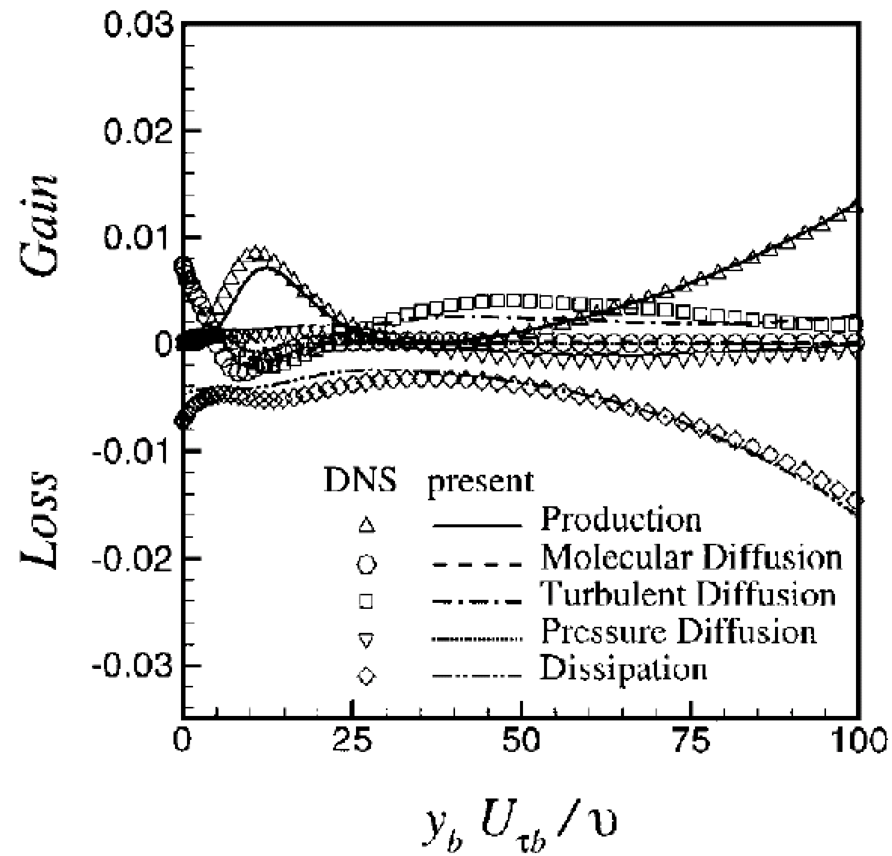
Low Reynolds number modeling-Channel flow





Low Reynolds number modeling-Couette Poiseuille flow





Reynolds stress model

$$\frac{\partial}{\partial x_j}(\rho U_i U_j) = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}[\mu(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}) - \rho\overline{u_i u_j}]$$

- Transport equation for Reynolds stress

$$\underbrace{\frac{\partial \rho U_k \overline{u_i u_j}}{\partial x_k}}_{C_{ij}} = \underbrace{\frac{\partial}{\partial x_k}\left[-\rho\overline{u_k u_i u_j} - \rho\delta_{jk}\overline{u_i p} - \rho\delta_{ik}\overline{u_j p} + \mu\frac{\partial\overline{u_i u_j}}{\partial x_k}\right]}_{d_{ij}}$$

$$\underbrace{-\rho\overline{u_i u_k}\frac{\partial U_j}{\partial x_k} - \rho\overline{u_j u_k}\frac{\partial U_i}{\partial x_k}}_{P_{ij}} + \underbrace{\overline{p\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}}_{\Phi_{ij}} - \underbrace{\mu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}}_{\varepsilon_{ij}}$$

Reynolds stress model

– Normal stresses clearly not describable by eddy-viscosity scheme.

$$
\begin{aligned}
-\rho\overline{uu} &= 2\mu_t\frac{\partial U}{\partial x} - \frac{2}{3}\rho k \\
-\rho\overline{vv} &= 2\mu_t\frac{\partial V}{\partial y} - \frac{2}{3}\rho k \\
-\rho\overline{ww} &= 2\mu_t\frac{\partial W}{\partial z} - \frac{2}{3}\rho k
\end{aligned}
$$

– Shear with streamline curvature

Wake curvature $\partial V/\partial x \sim 0.01\ \partial U/\partial y$

Yet such small extra strains produce a significant effect on turbulence. Now

$$P_{\overline{uv}} = -\rho[\overline{uu}\frac{\partial V}{\partial x} + \overline{vv}\frac{\partial U}{\partial y}]$$

If eddy viscosity model is adopted, then

$$-\rho\overline{uv} = \mu_t[\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}]$$

Reynolds stress model

$$d_{ij} = \frac{\partial}{\partial x_k}\left(C_s \rho \frac{k}{\varepsilon}\overline{u_k u_\ell}\frac{\partial \overline{u_i u_j}}{\partial x_\ell}\right) \text{ or } \frac{\partial}{\partial x_k}\left(\frac{\mu_t}{\sigma}\frac{\partial \overline{u_i u_j}}{\partial x_k}\right)$$

$-$slow part

$$\Phi_{ij1} = -C_1 \frac{\varepsilon}{k}\left(\overline{u_i u_j} - \frac{2}{3}\delta_{ij} k\right)$$

$-$ rapid part

$$\Phi_{ij2} = -C_2\left(P_{ij} - \frac{2}{3}\delta_{ij} P_k\right)$$

$$\varepsilon_{ij} = \frac{2}{3}\delta_{ij}\rho\varepsilon$$

BE Launder, GJ Reece, W Rodi, 1975, Progress in the development of a Reynolds-stress turbulence closure, Journal of fluid mechanics 68 (03), 537-566

Reynolds stress model

- The WET model of turbulence – Launder
  - Latter lectures focus on term-by-term approximations to the unknown fluctuating products in the second-moment equations.
  - The WET hypothesis avoids these complexities,

$$Wealth \propto Earning \;\; x \;\; Time$$

  - Translated to turbulence. This means,

$$Value\,of\;2nd\;moment \propto generation\,of\;2nd\;moment \;\; x \;\; turbulent\,time\,scale$$

  - Direct application of WET hypothesis to Reynolds stress gives

$$\overline{u_i u_j} \propto P_{ij} T$$

Reynolds stress model

- Reinterpretation of WET model
  - Now in isotropic turbulence
  - It seems reasonable to suppose that WET hypothesis should apply to departure of the 2nd moment from their isotropic state.

$$(\overline{u_i u_j} - \frac{1}{3}\delta_{ij}\overline{u_k u_k}) = C_s T (P_{ij} - \frac{1}{3}\delta_{ij}P_{kk})$$

$$T = \frac{k}{\epsilon}$$

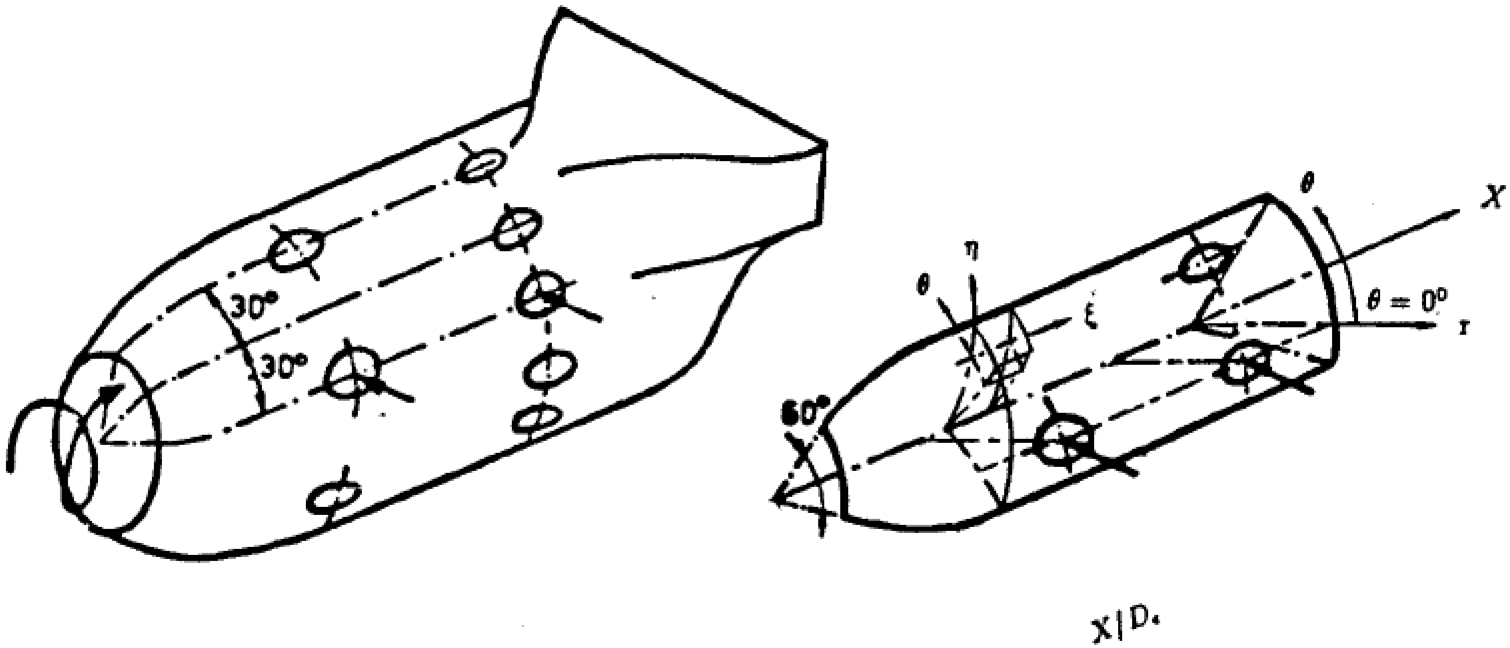
$$\phi_{ij} = \overline{p(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})}$$

$$a_{ij} = \frac{\overline{u_i u_j} - \frac{1}{3}\delta_{ij}\overline{u_k u_k}}{k} \propto \frac{P_{ij} - \frac{1}{3}\delta_{ij}P_{kk}}{\epsilon}$$

$$\phi_{ij1} = -1.8\rho\frac{\epsilon}{k}(\overline{u_i u_j} - \frac{2}{3}\delta_{ij}k)$$

$$\phi_{ij2} = -\frac{C_2+8}{11}(P_{ij} - \frac{1}{3}\delta_{ij}P_{kk}) - \frac{30C_2-2}{55}\rho k(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}) - \frac{8C_2-2}{11}(D_{ij} - \frac{1}{3}\delta_{ij}D_{kk})$$

Combustion Chamber



Combustion Chamber

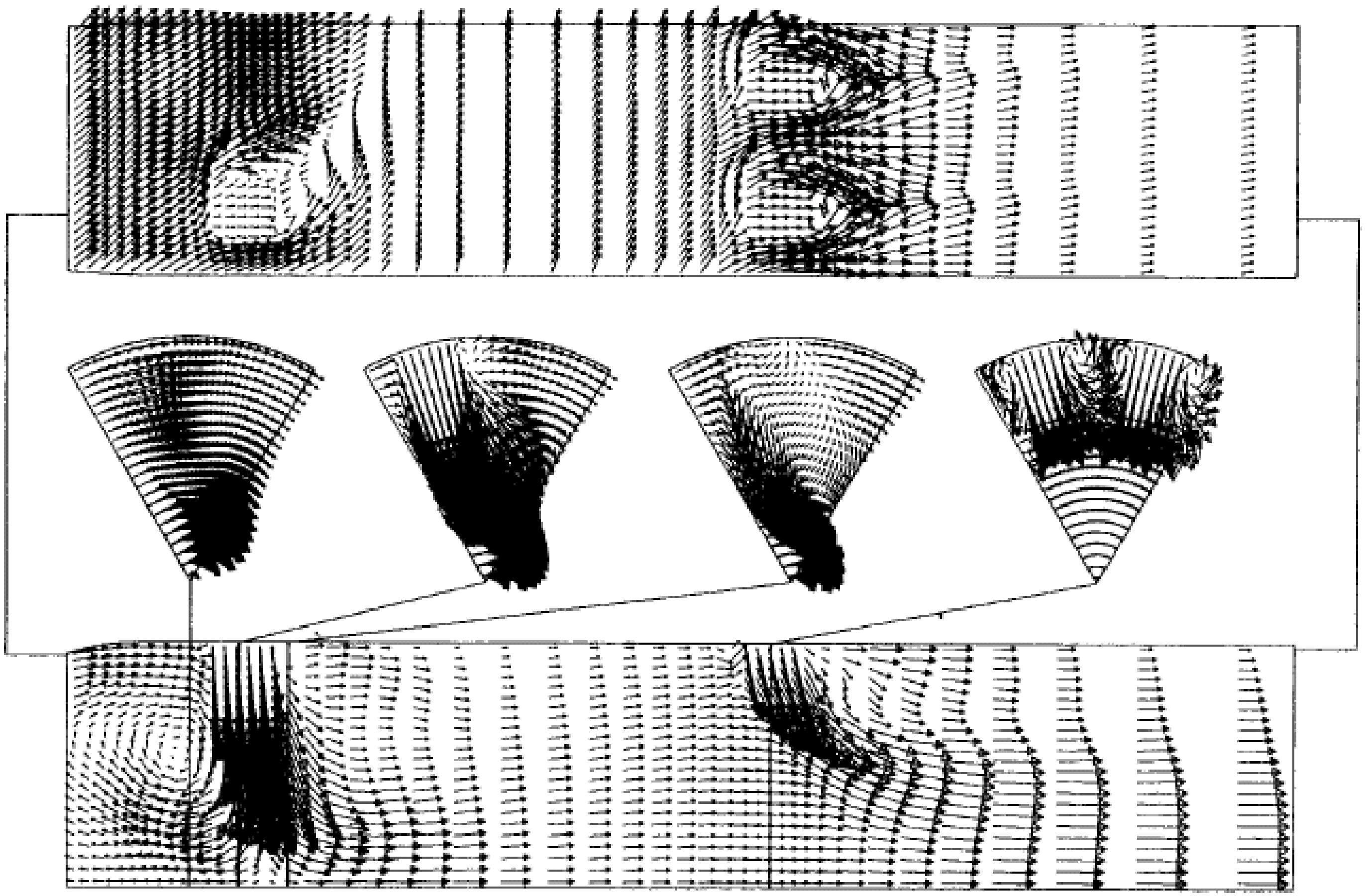
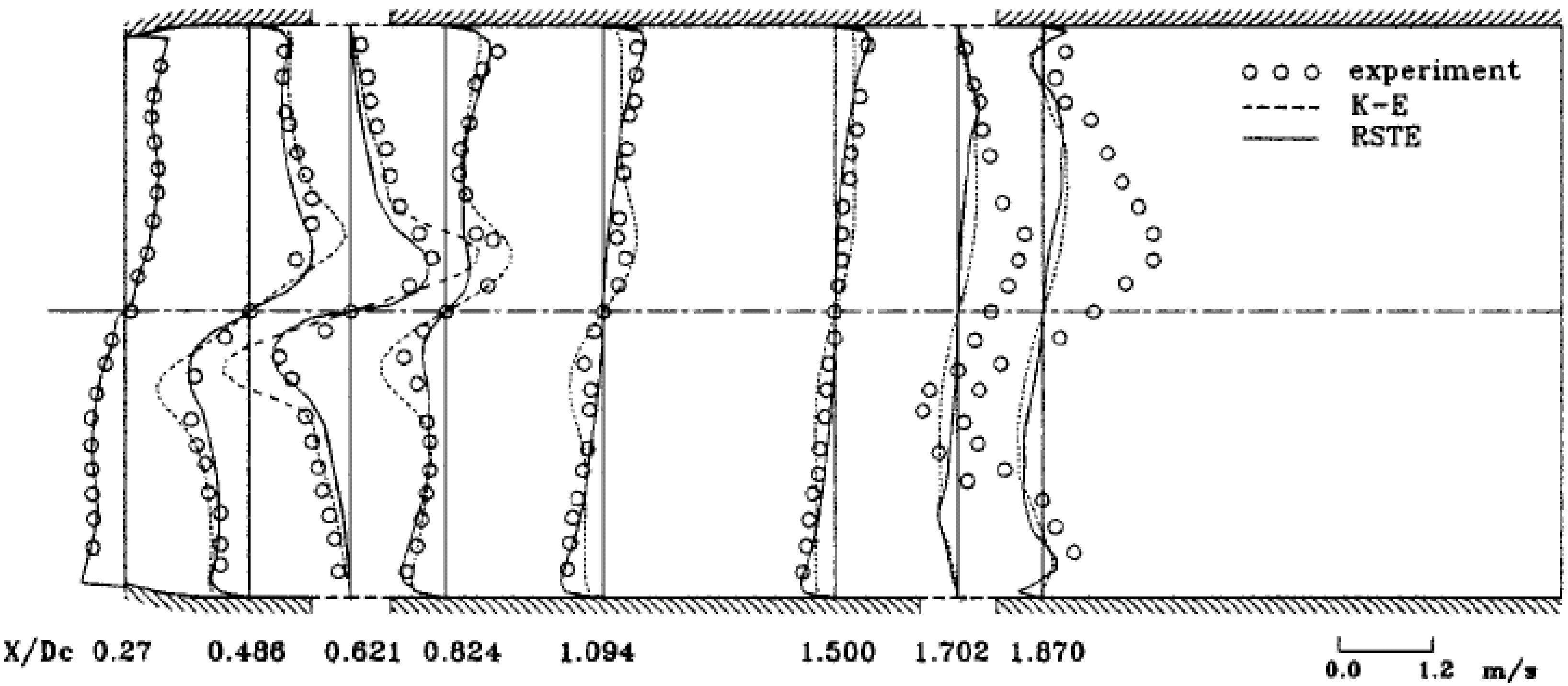


Figure 3. Overall view of the combustor flow—swirler 1.

Combustion Chamber



Lin, C. A. and Lu, C. M., 1994, "Modelling Three-Dimensional Gas-Turbine-Combustor-Model Flow using Second-Moment Closure", AIAA Journal, Vol. 32 no 7, pp. 1416-1422.

Swirling flow

INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS
Int. J. Numer. Meth. Fluids **30**: 493–508 (1999)

COMPUTATIONS OF STRONGLY SWIRLING FLOWS WITH SECOND-MOMENT CLOSURES

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$$\frac{\partial(\rho U_i)}{\partial x_i} = 0,$$

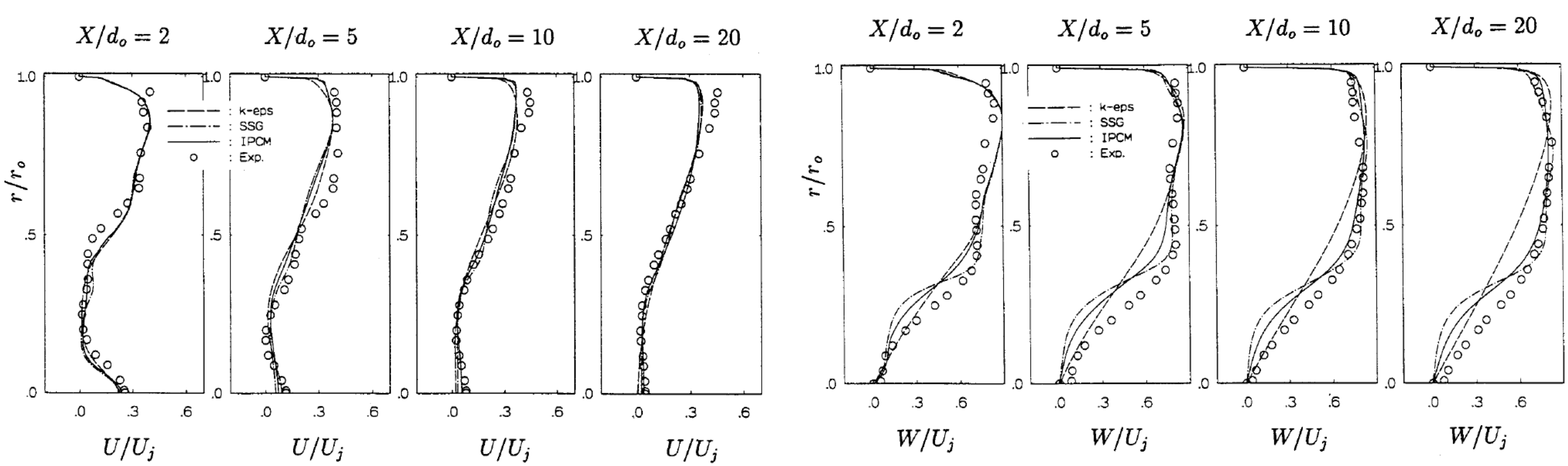
$$\frac{\partial(\rho U_i U_j)}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu_l\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \rho\overline{u_i u_j}\right], \qquad \frac{\partial}{\partial x_k}(\rho U_k \overline{u_i u_j}) = d_{ij} + P_{ij} + \epsilon_{ij} + \phi_{ij},$$

Chen, J. C. and Lin, C. A., 1999, ``Computations of Strongly Swirling Flows with Second-Moment Closure", I. J. of Numerical Methods in Fluids, Vol. 30, pp. 493-508.

Swirling flow

$$S = \frac{\int_0^{r_o} U W r^2 \, \mathrm{d}r}{r_o \int_0^{r_o} U^2 r \, \mathrm{d}r}$$

So et al.'s strongly swirling flow $S = 2.25$



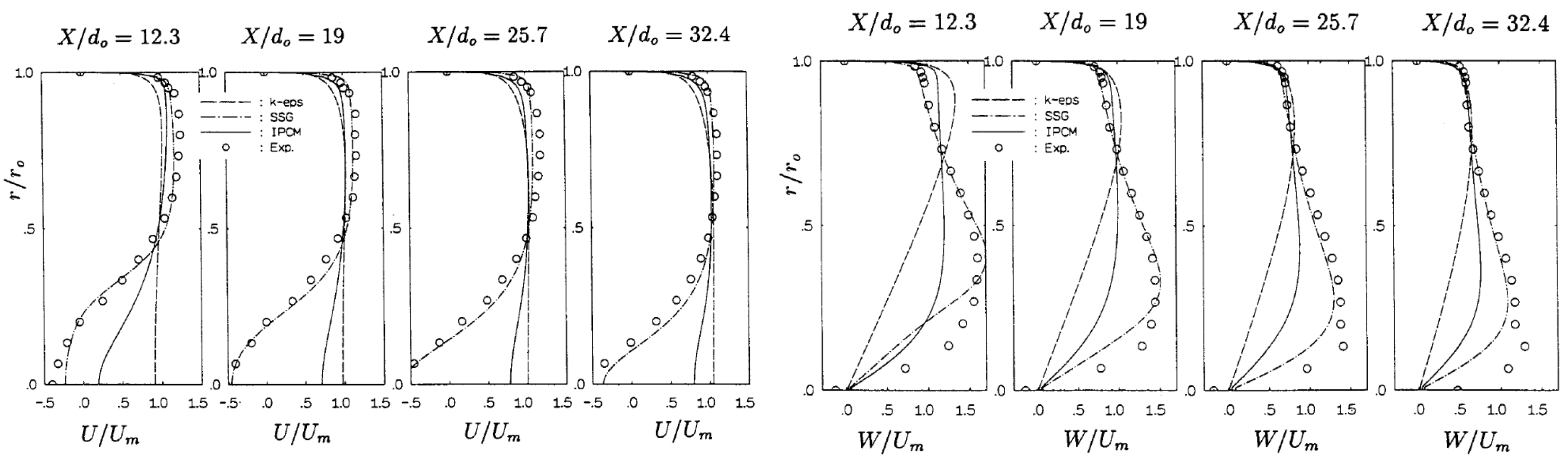
Chen, J. C. and Lin, C. A., 1999, ``Computations of Strongly Swirling Flows with Second-Moment Closure", I. J. of Numerical Methods in Fluids, Vol. 30, pp. 493-508.

$$\phi_{ij1} = -(3.4\epsilon + 1.8P_k)b_{ij} + 4.2\epsilon\left(b_{ik}b_{kj} - \frac{1}{3}b_{kl}b_{kl}\delta_{ij}\right), \tag{12}$$

$$\phi_{ij2} = (0.8 - 1.3\sqrt{b_{kl}b_{kl}})kS_{ij} + 1.25k\left(b_{ik}S_{jk} + b_{jk}S_{ik} - \frac{2}{3}b_{kl}S_{kl}\delta_{ij}\right) + 0.4k(b_{ik}W_{jk} + b_{jk}W_{ik}), \tag{13}$$

Swirling flow

Kitoh's strongly swirling flow $S = 0.85$



Chen, J. C. and Lin, C. A., 1999, ``Computations of Strongly Swirling Flows with Second-Moment Closure", I. J. of Numerical Methods in Fluids, Vol. 30, pp. 493-508.

Algebraic stress modeling

$$\underbrace{\frac{\partial}{\partial x_k}(U_k\overline{u_iu_j})}_{C_{ij}} - \underbrace{\frac{\partial}{\partial x_k}\left[\nu\frac{\partial\overline{u_iu_j}}{\partial x_k} - \overline{u_iu_ju_k} - \frac{1}{\rho}(\overline{pu_i}\delta_{jk} + \overline{pu_j}\delta_{ik})\right]}_{d_{ij}}$$

$$= \underbrace{-\overline{u_iu_k}\frac{\partial U_j}{\partial x_k} - \overline{u_ju_k}\frac{\partial U_i}{\partial x_k}}_{P_{ij}} + \underbrace{\overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}}_{\phi_{ij}} - \underbrace{\left(\frac{2}{3}\delta_{ij}\varepsilon + \varepsilon_{ij,D}\right)}_{\varepsilon_{ij}}$$

ASM-linearization

$$\underbrace{\frac{\partial}{\partial x_j}(U_j k)}_{C_k} - \underbrace{\frac{\partial}{\partial x_j}\left(\nu \frac{\partial k}{\partial x_j} - \frac{1}{2}\overline{u_i u_i u_j} - \frac{1}{\rho}\overline{p u_j}\right)}_{d_k} = \underbrace{-\overline{u_i u_j}\frac{\partial U_i}{\partial x_j}}_{P} - \varepsilon$$

$$C_{ij} - d_{ij} = \frac{\overline{u_i u_j}}{k}(C_k - d_k) = \frac{\overline{u_i u_j}}{k}(P - \varepsilon)$$

assumption

$$\frac{\overline{u_i u_j}}{k}(P - \varepsilon) = P_{ij} - \frac{2}{3}\delta_{ij}\varepsilon + \underbrace{\phi_{ij} - \varepsilon_{ij,D}}_{\pi_{ij}}$$

Rodi W. The prediction of free turbulent boundary layers by use of a two-equation model of turbulence. Ph.D. Thesis, University of London, 1972.

ASM-linearization

$$b_{ij} = \left[\frac{\overline{u_i u_j}}{2k} - \frac{1}{3}\delta_{ij}\right]$$

$$\frac{\overline{u_i u_j}}{k}(P - \varepsilon) = P_{ij} - \frac{2}{3}\delta_{ij}\varepsilon + \underbrace{\phi_{ij} - \varepsilon_{ij,D}}_{\pi_{ij}}$$

$$\pi_{ij} = -\left(C_1^0 + C_1^1\frac{P}{\varepsilon}\right)\varepsilon b_{ij} + C_2\varepsilon S_{ij} + C_3\varepsilon\left(b_{ik}S_{jk} + b_{jk}S_{ik} - \frac{2}{3}b_{mn}S_{mn}\delta_{ij}\right)$$

$$+C_4\varepsilon(b_{ik}W_{jk} + b_{jk}W_{ik})$$

$$S_{ij} = \frac{1}{2}\frac{k}{\varepsilon}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right), \quad W_{ij} = \frac{1}{2}\frac{k}{\varepsilon}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right)$$

$$S = \sqrt{2S_{ij}S_{ij}}, \quad \Omega = \sqrt{2W_{ij}W_{ij}}$$

$$b_{ij}\left[(C_1^0 - 2) + (C_1^1 + 2)\frac{P}{\varepsilon}\right] = \left(C_2 - \frac{4}{3}\right)S_{ij} + (C_3 - 2)\left(b_{ik}S_{jk} + b_{jk}S_{ik} - \frac{2}{3}b_{mn}S_{mn}\delta_{ij}\right)$$

$$+(C_4 - 2)(b_{ik}W_{jk} + b_{jk}W_{ik})$$

ASM-linearization

$$b_{ij}\left[(C_1^0-2)+(C_1^1+2)\frac{P}{\varepsilon}\right]=\left(C_2-\frac{4}{3}\right)S_{ij}+(C_3-2)\left(b_{ik}S_{jk}+b_{jk}S_{ik}-\frac{2}{3}b_{mn}S_{mn}\delta_{ij}\right)$$

$$+(C_4-2)(b_{ik}W_{jk}+b_{jk}W_{ik})$$

Explicit solution: Gatski and Speziale (1993)

$$b_{ij}=-\frac{3}{3-2\eta_1-6\eta_2}\left[\beta_1 S_{ij}+\beta_2(S_{ik}W_{kj}+S_{jk}W_{ki})-2\beta_3\left(S_{ik}S_{kj}-\frac{1}{3}S_{kl}S_{kl}\delta_{ij}\right)\right]$$

where $\beta_1=(4/3-C_2)/\beta_4$, $\beta_2=\beta_1(2-C_4)/\beta_4$, $\beta_3=\beta_1(2-C_3)/\beta_4$, $\beta_4=(C_1^0-2)+(C_1^1+2)P/\varepsilon$, $\eta_1=\xi_1(2-C_3)^2/\beta_4^2$ and $\eta_2=\xi_2(2-C_4)^2/\beta_4^2$. $\xi_1=S_{ij}S_{ij}$ and $\xi_2=W_{ij}W_{ij}$.

Gatski TB, Speziale CG. On explicit algebraic models for complex turbulent flows. Journal of Fluid Mechanics, 1993; 254:59 –78.

Non-linear eddy viscosity model

$$b_{ij} = \left[\frac{\overline{u_i u_j}}{2k} - \frac{1}{3}\delta_{ij}\right]$$

Craft TJ, Launder BE, Suga K. Prediction of turbulent transitional phenomena with a nonlinear eddy-viscosity model. International Journal of Heat and Fluid Flow 1997; 18:15–28.

$$= -C_\mu S_{ij} + C_1 C_\mu \left[S_{ik}S_{kj} - \frac{1}{3}\delta_{ij}S_{kl}S_{kl}\right] + C_2 C_\mu [W_{ik}S_{kj} + W_{jk}S_{ki}]$$

$$+C_3 C_\mu \left[W_{ik}W_{kj} - \frac{1}{3}\delta_{ij}W_{kl}W_{kl}\right] + C_4 C_\mu^2 \left(S_{ki}W_{lj} + S_{kj}W_{li} - \frac{2}{3}S_{km}W_{lm}\delta_{ij}\right)S_{kl}$$

$$+C_6 C_\mu^2 S_{ij}S_{kl}S_{kl} + C_7 C_\mu^2 S_{ij}W_{kl}W_{kl}$$

$$S_{ij} = \frac{1}{2}\frac{k}{\varepsilon}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right), \quad W_{ij} = \frac{1}{2}\frac{k}{\varepsilon}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right)$$

$$S = \sqrt{2S_{ij}S_{ij}}, \quad \Omega = \sqrt{2W_{ij}W_{ij}}$$

Large eddy simulation

– Filtering operation

$$\widetilde{f(\vec{x})} = \int f(\vec{x})G(\vec{x},\overrightarrow{x'})d\overrightarrow{x'}$$

$$\frac{\partial \rho \widetilde{U_i}}{\partial x_i} = 0$$

$$\frac{\partial \rho \widetilde{U}_i}{\partial t} + \frac{\partial \rho \widetilde{U}_j \widetilde{U}_i}{\partial x_j} = -\frac{\partial \tilde{P}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial \widetilde{U}_i}{\partial x_j} + \frac{\partial \widetilde{U}_j}{\partial x_i}\right) - \tau_{ij}\right]$$

$$\tau_{ij} = \rho\widetilde{U_i U_j} - \rho\widetilde{U}_j\widetilde{U}_i$$

Table 13.2. *Filter functions and transfer functions for one-dimensional filters: the box filter function has the same second moment as the Gaussian ($\frac{1}{12}\Delta^2$); the other filters have the same value of the transfer function at the characteristic wavenumber $\kappa_c \equiv \pi/\Delta$, i.e.,* $\hat{G}(\kappa_c) = \exp(-\pi^2/24)$

Name	Filter function	Transfer function
General	$G(r)$	$\hat{G}(\kappa) \equiv \int_{-\infty}^{\infty} e^{i\kappa r} G(r)\, dr$
Box	$\frac{1}{\Delta} H(\frac{1}{2}\Delta - \|r\|)$	$\frac{\sin(\frac{1}{2}\kappa\Delta)}{\frac{1}{2}\kappa\Delta}$
Gaussian	$\left(\frac{6}{\pi\Delta^2}\right)^{1/2} \exp\left(-\frac{6r^2}{\Delta^2}\right)$	$\exp\left(-\frac{\kappa^2\Delta^2}{24}\right)$
Sharp spectral	$\frac{\sin(\pi r/\Delta)}{\pi r}$	$H(\kappa_c - \|\kappa\|)$, $\kappa_c \equiv \pi/\Delta$
Cauchy	$\frac{a}{\pi\Delta[(r/\Delta)^2 + a^2]}$, $a = \frac{\pi}{24}$	$\exp(-a\Delta\|\kappa\|)$
Pao		$\exp\left(-\frac{\pi^{2/3}}{24}(\Delta\|\kappa\|)^{4/3}\right)$

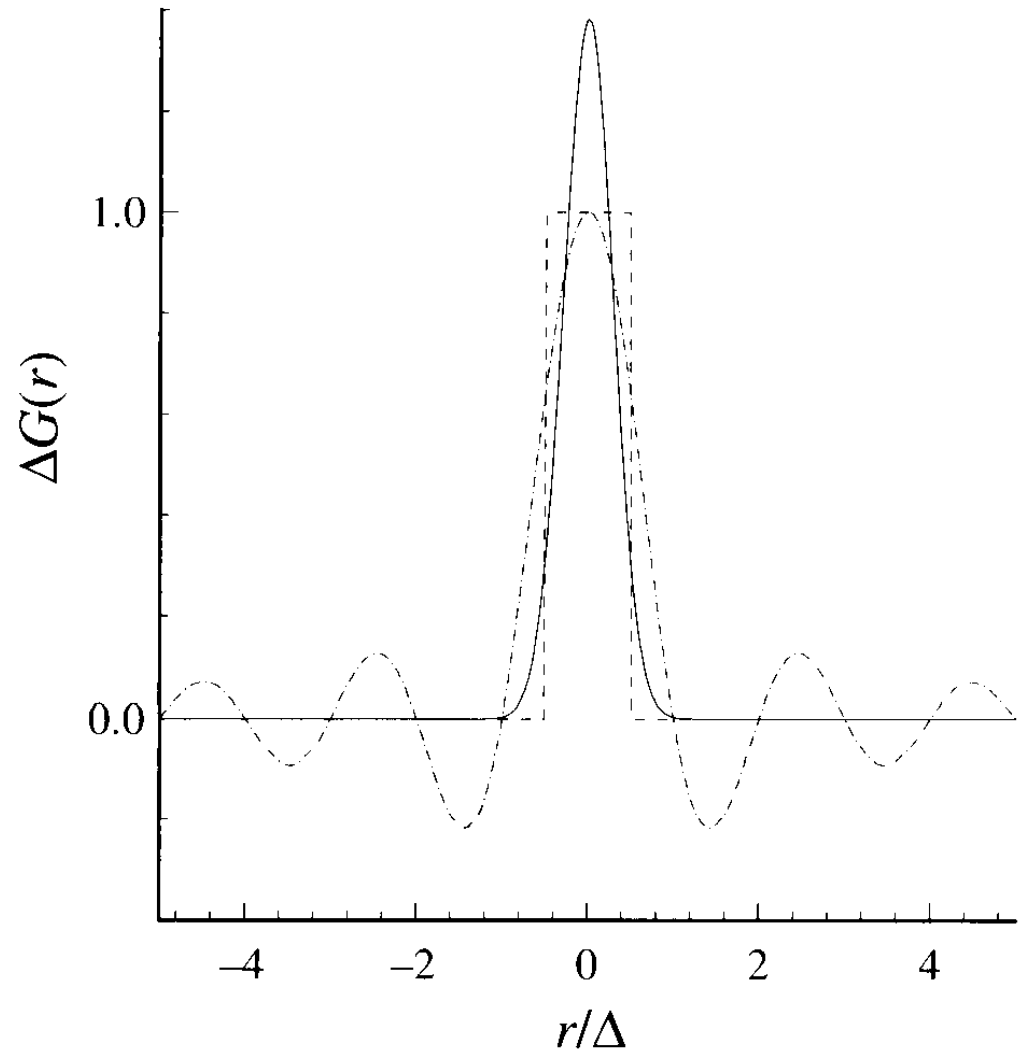


Fig. 13.1. Filters $G(r)$: box filter, dashed line; Gaussian filter, solid line; sharp spectr
filter, dot dashed line.

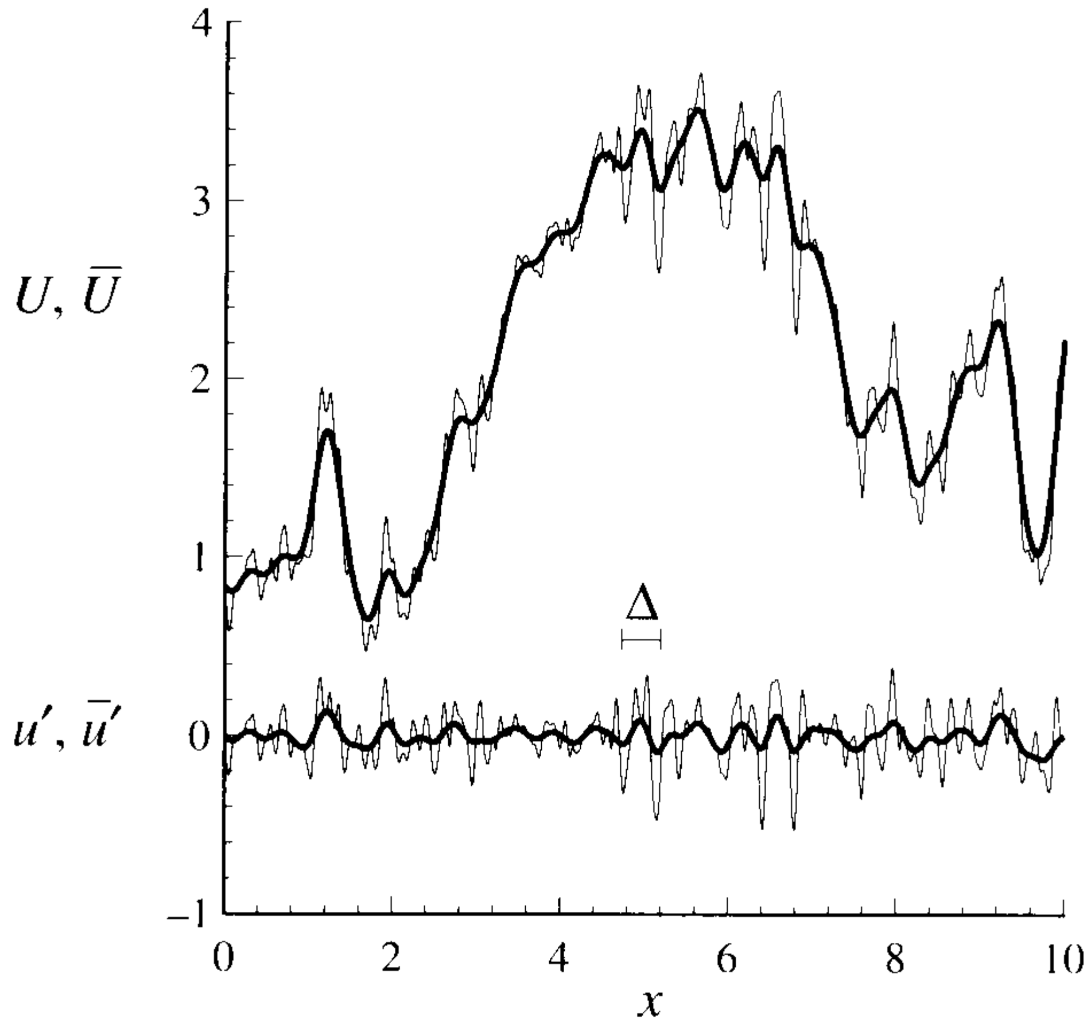


Fig. 13.2. Upper curves: a sample of the velocity field $U(x)$ and the corresponding filtered field $\overline{U}(x)$ (bold line), using the Gaussian filter with $\Delta \approx 0.35$. Lower curves: the residual field $u'(x)$ and the filtered residual field $\overline{u'(x)}$ (bold line).

Large eddy simulation

$$\frac{\partial\rho\widetilde{U}_i}{\partial t}+\frac{\partial\rho\widetilde{U}_j\widetilde{U}_i}{\partial x_j}=-\frac{\partial\tilde{P}}{\partial x_i}+\frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial\widetilde{U}_i}{\partial x_j}+\frac{\partial\widetilde{U}_j}{\partial x_i}\right)-\tau_{ij}\right]$$

$-$Smargorinsky model (eddy viscosity model)

$$\tau_{ij}-\frac{\delta_{ij}}{3}\tau_{kk}=-\mu_t\widetilde{S_{ij}}=-\mu_t\left(\frac{\partial\widetilde{U_i}}{\partial x_j}+\frac{\partial\widetilde{U_j}}{\partial x_i}\right)$$

$$\mu_t=(C_S\Delta)^2\left|\tilde{S}\right|,\qquad \tilde{S}=(\widetilde{S_{ij}}/2)^2$$

$$C_S\approx 0.1,\qquad \Delta$$

$$=(\Delta_x\Delta_y\Delta_z)^{1/3}\text{ or }\sqrt{\Delta_x\Delta_y+\Delta_x\Delta_z+\Delta_y\Delta_z}$$

Massimo Germano; Ugo Piomelli; Parviz Moin; William H. Cabot, 1991, A dynamic subgrid-scale eddy viscosity model, Phys. Fluids 3, 1760–1765 (citation:10232)

LES-dynamic model

$$\widehat{\Delta} = 2\widetilde{\Delta}$$

$$\tau_{ij} = \widetilde{u_i u_j} - \widetilde{u}_i \widetilde{u}_j$$

$$T_{ij} = \widehat{\widetilde{u_i u_j}} - \widehat{\widetilde{u}}_i \widehat{\widetilde{u}}_j$$

$$\tau_{ij} - \frac{\delta_{ij}}{3}\tau_{kk} = -2C\widetilde{\Delta}^2|\widetilde{S}|\widetilde{S}_{ij} = -2C\beta_{ij}$$

$$T_{ij} - \frac{\delta_{ij}}{3}T_{kk} = -2C\widehat{\Delta}^2|\widehat{S}|\widehat{S}_{ij} = -2C\alpha_{ij}$$

Six equations, one unknown

$$\begin{aligned} L_{ij} &= T_{ij} - \widehat{\tau_{ij}} \\ &= \widehat{\widetilde{u_i u_j}} - \widehat{\widetilde{u}}_i \widehat{\widetilde{u}}_j - \widehat{\widetilde{u_i u_j}} + \widehat{\widetilde{u}_i \widetilde{u}_j} \\ &= \widehat{\widetilde{u}_i \widetilde{u}_j} - \widehat{\widetilde{u}}_i \widehat{\widetilde{u}}_j \end{aligned}$$

$$\begin{aligned} L_{kk} &= T_{kk} - \widehat{\tau_{kk}} \\ &= \widehat{\widetilde{u}_k \widetilde{u}_k} - \widehat{\widetilde{u}}_k \widehat{\widetilde{u}}_k \end{aligned}$$

$$\begin{aligned} L^a_{ij} &= L_{ij} - \frac{\delta_{ij}}{3}L_{kk} \\ &= -2C\alpha_{ij} + 2\widehat{C\beta_{ij}} \\ &= -2C\widehat{\Delta}^2|\widehat{S}|\widehat{S}_{ij} + 2\widehat{C\widetilde{\Delta}^2|\widetilde{S}|\widetilde{S}_{ij}} \\ &= 2CM_{ij} \end{aligned}$$

Lilly, DK, 1992, A proposed modification of the Germano subgrid-scale closure method, Phys. Fluids 4, 633–635 (Citation: 5797)

$$\begin{aligned} L^a_{ij} &= L_{ij} - \frac{\delta_{ij}}{3} L_{kk} \\ &= -2C\alpha_{ij} + 2\widehat{C\beta_{ij}} \\ &= -2C\widehat{\Delta}^2|\widehat{S}|\widehat{S}_{ij} + 2C\widehat{\widetilde{\Delta}^2|\widetilde{S}|\widetilde{S}_{ij}} \\ &= 2CM_{ij} \end{aligned}$$

LES-dynamic model

Lilly (1992)-least square error to determine C

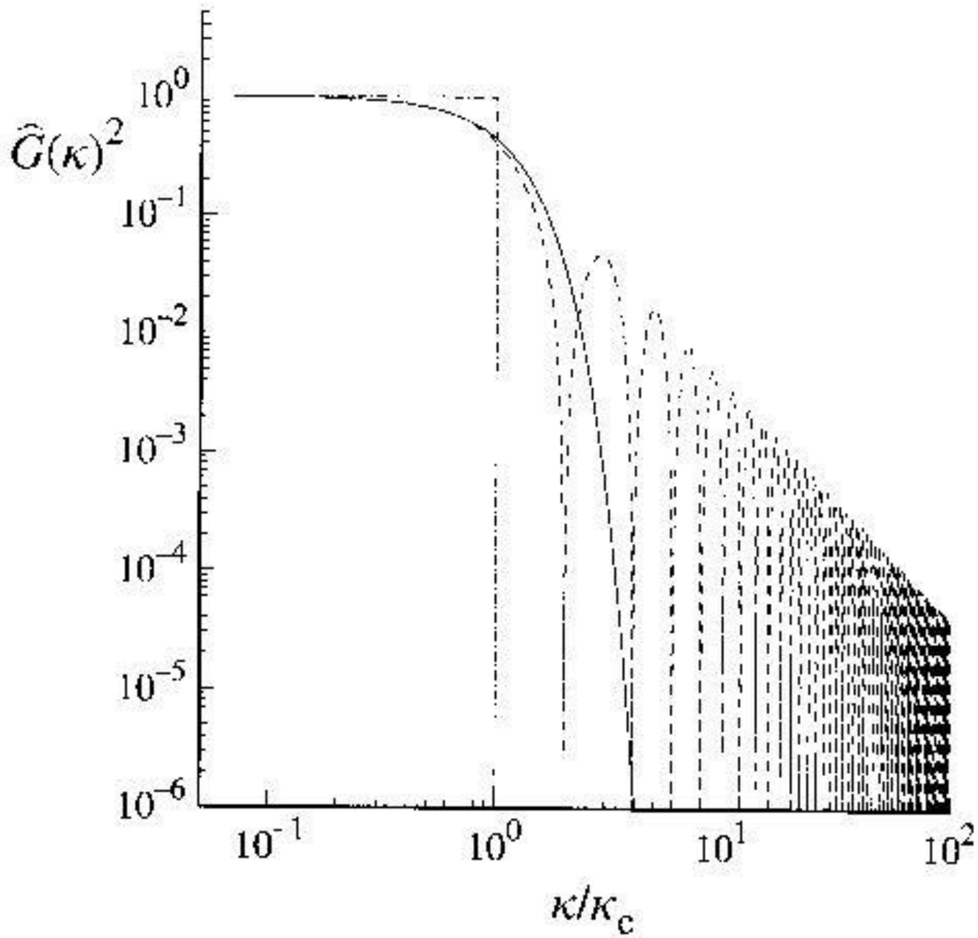
$$\epsilon = (L^a_{ij} - 2CM_{ij})^2$$

$$\frac{1}{2}\frac{\partial \epsilon}{\partial C} = -M_{ij}(L^a_{ij} - 2CM_{ij}) = 0$$

$$C = \frac{L^a_{ij}M_{ij}}{M_{kl}M_{kl}} = \frac{L_{ij}M_{ij}}{M_{kl}M_{kl}}$$

Average in the periodic direction to prevent instability

$$C(x,t) = \frac{< L_{ij}M_{ij} >}{< M_{kl}M_{kl} >}$$



g. 13.4. Attenuation factors $\widehat{G}(\kappa)^2$: box filter, dashed line; Gaussian filter, solid line; arp spectral filter, dot–dashed line.