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Zero Forcing in Claw-Free Cubic Graphs

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1 Introduction

The zero forcing is introduced in [1] for controlling the maximum nullity. It was independently introduced in [3] for quantum control. It is also closely related the notion of power domination [3].

The zero forcing process considers the *color change rule* defined on a simple graph whose vertices are either filled or unfilled: If u is filled, and all neighbors of u are filled except for excatly one vertex v, then changed v to filled.

Let Z be a subset of $V(G)$. By make all vertices in Z filled and other vertices unfilled initially, we may apply the color change rule repeatedly until no more action can be done. At this stage, the set of filled vertices is called the *closure* of Z. We say Z is a *zero forcing set* if the closure of Z is $V(G)$. The *zero forcing number* $Z(G)$ is defined as the minimum cardinality of a zero forcing set.

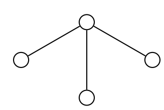

Figure 1: A claw graph.

A claw, as shown in Figure 1, is a common structure in a graph. A claw-free graph is a graph without claw as an induced subgraph, and a cubic graph is a graph whose vertices all have degree 3. Davila and Henning [3] consider the cubic claw-free graphs and shows that $Z(G) \leq \alpha(G) + 1$, where $\alpha(G)$ is the independence number.

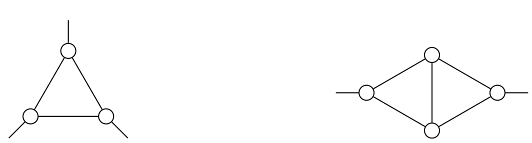

Figure 2: A triangle and a diamond.

It is known that every cubic claw-free graph has a triangle-diamond decomposition; see Figure 2. Also, a cubic claw-free graph on n vertices has $\frac{3n}{2}$ edges. Since $\frac{3n}{2}$ is a positive integer, n has to be an even number. Moreover, if a cubic claw-free graph has t triangles and d diamonds, then $n = 3t + 4d$ and t is also an even number. In this report, we studies the zero forcing numbers of a cubic claw-free graph when $t = 0, 2, 4$.

2 Main results

Since every claw-free cubic graph has the triangle-diamond decomposition, we analyze the zero forcing number of a claw-free cubic graph based on its number of triangles and diamonds. The results are summarized in Table 1.

triangles (t)	diamonds (d)	case	$Z(G)$	illustration
0	d	1	$d+2$	Figure 5
2	d	1	$d+3$	Figure 6
2	d	2	$d+3$	Figure 7
4	d	1	$d+4$	Figure 8
4	d	2	$d+5$	Figure 9
4	d	3	$d+4$	Figure 10
4	d	4	$d+4$	Figure 11
4	d	5	$d+4$	Figure 12

Table 1: Claw-free cubic graphs and their zero forcing numbers

Before we do the case analysis, we introduce two lemmas that will be used oftenly.

Lemma 2.1. *In a claw-free cubic graph, each diamond has at least one vertices in any zero forcing set.*

Proof. We consider a diamond in a claw-free cubic graph; see Figure 3. If the vertices b, c are unfilled at the beginning, then b, c will not be changed no matter other vertices are filled or unfilled. This is because a and d are the only possible vertices to make b or c filled, but each of a and d have at least two unfilled neighbors, namely, b and c. Thus, at least one of the b, c vertices has to be in the zero forcing set. ☐

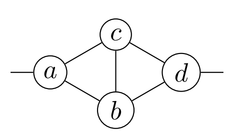

Figure 3: A diamond.

Lemma 2.2. *Let G be a claw-free cubic graph that contains a triangle on $\{a, b, c\}$. If a is a cut vertex and b, c are connected by a sequence of d diamonds (see Figure 4), then $\{b, c\}$ and the vertices on the d diamonds contain at least $d + 1$ colored vertices in any zero forcing set. Moreover, if the zero forcing set has exactly $d + 1$ vertices on this part, then the center of each diamond contains at least one filled vertices and at least one of $\{b, c\}$ is filled.*

Proof. Let Z be a zero forcing set and W the set of vertices on G that is the union of $\{b, c\}$ and the vertices on the d diamonds. According to Lemma 2.1, we know that Z contains at least one vertex in each of the n diamonds. If $|Z \cap W| = d$, then Z is not a zero forcing set since none of the filled vertices in $Z \cap W$ can perform a force and a has at least two unfilled neighbors b, c. Thus, we have $|Z \cap W| \geq d + 1$. Note that if $|Z \cap W| = d + 1$ and b, c are unfilled, then the same argument implies that Z is not a zero forcing set. Therefore, Z contains a vertex in each diamond and one vertex in $\{b, c\}$ if $|Z \cap W| = d + 1$. $\square$

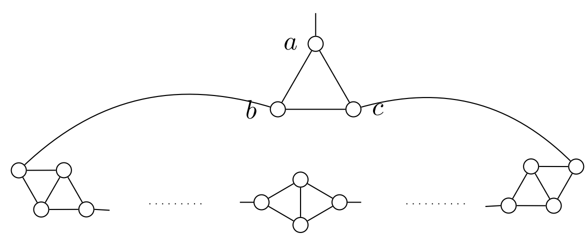

Figure 4: A triangle surrounded by n diamonds.

For the following, we will assume G is a connected claw-free cubic graph with t triangles and d diamonds. A connected claw-free cubic graph without any triangle is called a *necklace*. It is necessarily composed of d diamonds arranged in a circle; see Figure 5. A necklace on d diamonds is denoted as N_d.

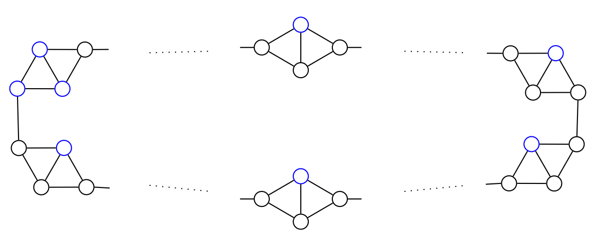

Figure 5: A necklace of n diamonds.

Lemma 2.3. *The zero forcing number of N_d is $d + 2$.*

Proof. Let Z be a zero forcing set of N_d. By Lemma 2.1, we know Z contains at least one vertex in each of the d diamonds. Since G is a cubic graph, so we must have one filled vertices and two filled neighbors to initiate the first force. If $|Z| = d$, then these d vertices are necessarily in the center of each diamond and are independent, so Z is not a zero forcing set. Similarly, $|Z| = d + 1$ does not provide enough filled vertices to initiate a force. Therefore, $|Z| \geq d + 2$. Since taking a central vertex from each of the d diamonds along with any two adjacent remaining vertices forms a zero forcing set, $Z(N_d) = d + 2$. □

Now consider the claw-free cubic graph with two triangles $t = 2$ and d diamonds. There are two cases in this situation, as depicted in Figures 6 and 7.

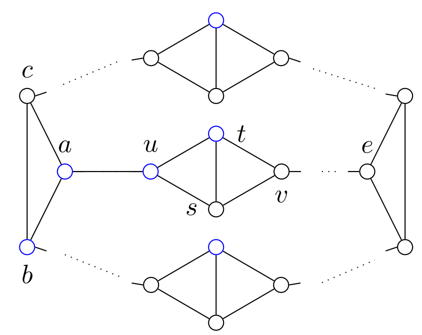

Figure 6: Two triangles and n diamonds: Case 1.

Lemma 2.4. *If G is a claw-free cubic graph with $t = 2$ triangles and d diamonds with the structure shown in Figure 6, then $Z(G) = d + 3$.*

Proof. Let Z be a zero forcing set of G. Since every diamond contains at least an element in Z and the zero forcing process requires at least one vertex with all-but-one neighbors filled to start, so we know $Z(G) \geq d + 2$. After the the first two steps, we may assume one of the diamonds has all its vertices filled. However, this is not enough to be a zero forcing set, so $Z(G) > d + 2$. Instead, if Z contains one element in each of diamond and additional three vertices a, b, and u, then it can work, so $Z(G) = n + 3$. □

Lemma 2.5. *If G is a claw-free cubic graph with $t = 2$ triangles and d diamonds with the structure shown in Figure 7, then $Z(G) = d + 3$.*

Proof. By Lemma 2.2, we know that $Z(G) \geq d + 2$. If Z is a zero forcing set with $|Z| = d+2$, then Z is an independent set, according to Lemma 2.2, so it is not a zero forcing set, a contradiction. In contrast, if Z contains d vertices from each of the d diamonds and additional vertices $\{j, g, e\}$, as shown in Figure 7, then Z is a zero forcing set. Therefore, $Z(G) = d + 3$. □

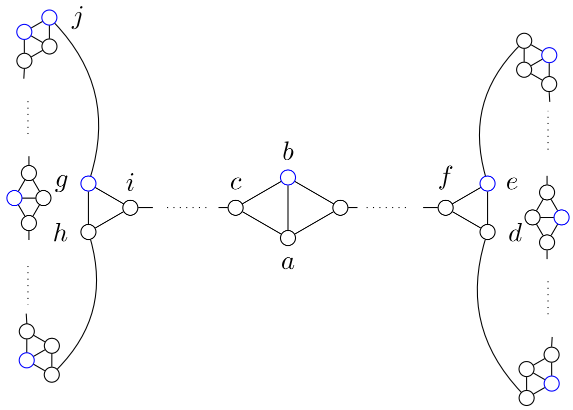

Figure 7: Two triangles and n diamonds: Case 2.

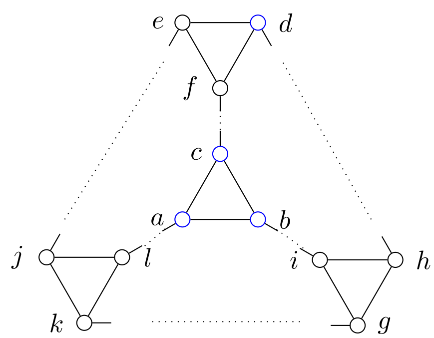

Figure 8: Four triangles and n diamonds: Case 1.

Now consider four triangles in this graph.

Lemma 2.6. *If G is a claw-free cubic graph with $t = 4$ triangles and d diamonds with the structure shown in Figure 8, then $Z(G) = d + 4$.*

Proof. Let Z be a zero forcing set of G. Since every diamond contains at least an element in Z and the zero forcing process requires at least one vertex with all-but-one neighbors filled to start, so we know $Z(G) \geq d+2$. Since adding 3 extra filled vertices, e.g. , a, b, c, can make at most three diamond paths to be fully filled, $d + 3$ vertices are not enough to form a zero forcing set. If Z contains d vertices from each of the d diamonds and additional vertices $\{a, b, c, d\}$, as shown in Figure 8, then Z is a zero forcing set. Therefore, $Z(G) = d + 4$. □

Lemma 2.7. *If G is a claw-free cubic graph with $t = 4$ triangles and d diamonds with the structure shown in Figure 9, then $Z(G) = d + 5$.*

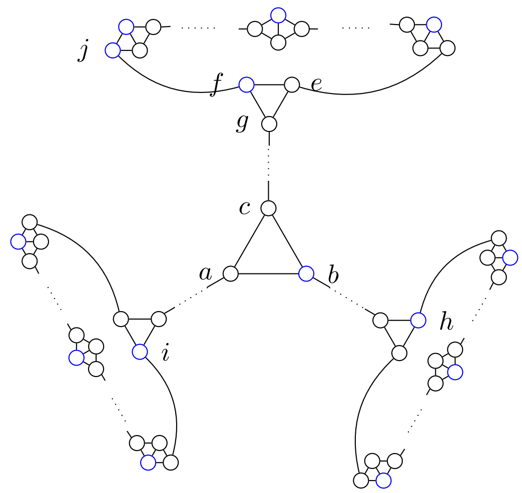

Figure 9: Four triangles and n diamonds: Case 2.

Proof. By Lemma 2.2, we know that $Z(G) \geq d+3$. If Z is a zero forcing set with $|Z| = d+3$, then Z is an independent set, according to Lemma 2.2, so it is not a zero forcing set, a contradiction. If we add one more filled vertex to one of the three branches, then there are two branches whose number of filled vertices is number of diamonds plus 1, and there is no way to propagate new filled vertices into these two branches. Therefore, we consider the case that we add one more filled vertex in $\{a, b, c\}$, but then it does not work, either. Thus, every zero forcing set contains at least $d+5$ vertices. In contrast, if Z contains d vertices from each of the d diamonds and additional vertices $\{j, h, i, f, b\}$, as shown in Figure 9, then Z is a zero forcing set. Therefore, $Z(G) = d+5$. $\square$

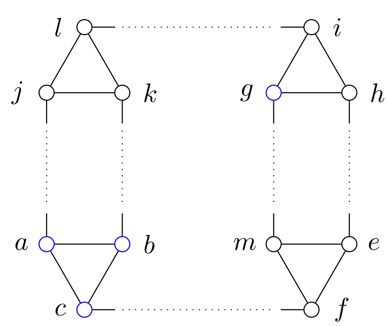

Figure 10: Four triangles and n diamonds: Case 3.

Lemma 2.8. *If G is a claw-free cubic graph with $t = 4$ triangles and d diamonds with the structure shown in Figure 10, then $Z(G) = d + 4$.*

Proof. Let Z be a zero forcing set of G. Since every diamond contains at least an element in Z and the zero forcing process requires at least one vertex with all-but-one neighbors filled to start, so we know $Z(G) \geq d + 2$. Since adding 3 extra filled vertices, e.g. , a, b, c, can make at most four diamond paths (two from the left, and the bottom and the top) to be fully filled, $d + 3$ vertices are not enough to form a zero forcing set. If Z contains d vertices from each of the d diamonds and additional vertices $\{a, b, c, g\}$, as shown in Figure 10, then Z is a zero forcing set. Therefore, $Z(G) = d + 4$. $\square$

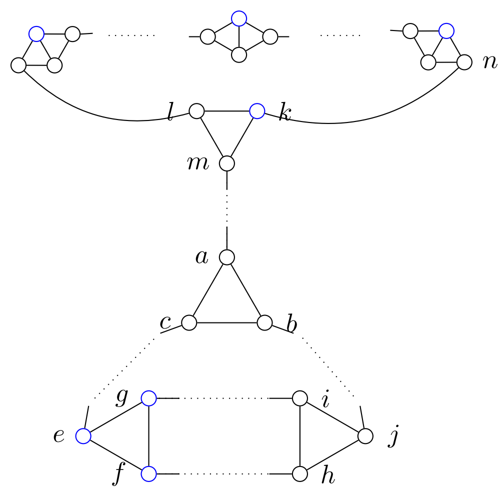

Figure 11: Four triangles and n diamonds: Case 4.

Lemma 2.9. *If G is a claw-free cubic graph with $t = 4$ triangles and d diamonds with the structure shown in Figure 11, then $Z(G) = d + 4$.*

Proof. As illustrated in Figure 11, let U be all vertices above l and k (inclusive) and let d_1 the number of diamonds in U. Let Z be a zero forcing set. By Lemma 2.2, $|Z \cap U| \geq d_1 + 1$. Note that $d_1 + 2$ inital filled vertices are enough to fill every vertices in U, so we consider two cases $|Z \cap U| = d_1 + 1$ and $|Z \cap U| = d_1 + 2$.

By Lemma 3, we know every diamond contains an element in Z. Based on these d vertices, we argue that $d + 4$ vertices are required. Note that $d + 4$ is sufficient by adding $\{k, g, e, f\}$ to the existing d vertices.

Case 1: $|Z \cap U| = d_1 + 1$. In this case, the vertices in U cannot make any force until new filled vertices come in. By adding 2 extra vertices in $V(G) \setminus U$, at most one diamond path can be filled so there is no way to propagate new filled vertices to U. Therefore, we know $d + 3$ is not enough and $d + 4$ is required.

Case 2: $|Z \cap U| = d_1 + 2$**.** In this case, all vertices will be filled and all diamond path between m and a will be filled. However, by adding one extra vertex, at most two diamond paths can be filled, so it is not enough to make the whole graph filled. Therefore, $d + 4$ is required. □

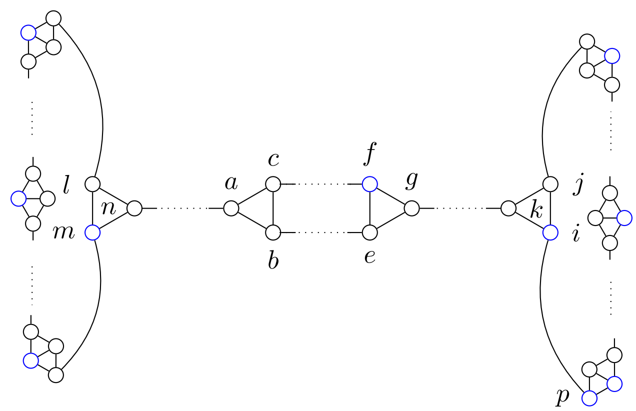

Figure 12: Four triangles and n diamonds: Case 5.

Lemma 2.10. *If G is a claw-free cubic graph with $t = 4$ triangles and d diamonds with the structure shown in Figure 12, then $Z(G) = d + 4$.*

Proof. As illustrated in Figure 12, let L be all vertices on the left of l and m (inclusive) and let d_1 the number of diamonds in L. Let R be all vertices on the right of i and j (inclusive) and let d_2 the number of diamonds in R. Let Z be a zero forcing set. By Lemma 2.2, $|Z \cap L| \geq d_1 + 1$ and $|Z \cap R| \geq d_2 + 1$. Note that $d_1 + 2$ initial filled vertices are enough to fill every vertices in L, so $|Z \cap L| \leq d_1 + 2$. Similarly, we have $|Z \cap R| \leq d_2 + 2$. Therefore, we consider three cases.

1. $|Z \cap L| = d_1 + 1$ and $|Z \cap R| = d_2 + 1$,
2. $|Z \cap L| = d_1 + 1$ and $|Z \cap R| = d_2 + 2$, and
3. $|Z \cap L| = d_1 + 2$ $|Z \cap R| = d_2 + 2$.

By Lemma 3, we know every diamond contains an element in Z. Based on these d vertices, we argue that $d + 4$ vertices are required. Note that $d + 4$ is sufficient by adding $\{i, p, f, m\}$ to the existing d vertices.
Case 1: $|Z \cap L| = d_1 + 1$ **and** $|Z \cap R| = d_2 + 1$ **.** In this case, the vertices in $L \cup R$ cannot make any force until new filled vertices come into them. By adding 1 extra vertices in $V(G) \setminus (L \cup R)$, we are not able to activate any force. Therefore, we know $d + 3$ is not enough and $d + 4$ is required.

Case 2: $|Z \cap L| = d_1 + 1$ **and** $|Z \cap R| = d_2 + 2$ **.** In this case, all vertices, where in R will be filled and all diamond path between g and k will be filled. But other vertices remain unfilled. Therefore, we know $d + 3$ is not enough and $d + 4$ is required.
Case 3: $|Z \cap L| = d_1 + 2$ **and** $|Z \cap R| = d_2 + 2$ **.** In this case, we already have $d + 4$ vertices in the set, but it is not a zero forcing set.

□

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