Lab00-Proof

CS214-Algorithm and Complexity, Xiaofeng Gao, Spring 2021.

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1. Prove that for any integer $n > 2$, there is a prime p satisfying $n < p < n!$. (Hint: consider a prime factor p of $n! - 1$ and prove by contradiction)

   Proof. Assume that $\forall t$ satisfying $n < t < n!$, t is a composite number.

   Then $\forall t$ satisfying $n < t < n!$, t is not a prime factor of $n! - 1$.

   Since $n! - 1$ is a composite number, there exists a prime factor p of $n! - 1$ satisfying $1 < p \leqslant n$. By definition of $n!$, p is a prime factor of $n!$.

   Then p is a common divisor of $n! - 1$ and $n!$, which contradicts the conclusion that the greatest common divisor of $n! - 1$ and $n!$ is 1. □

2. Use the minimal counterexample principle to prove that for any integer $n \geq 7$, there exists integers $i_n \geq 0$ and $j_n \geq 0$, such that $n = i_n \times 2 + j_n \times 3$.

   Proof. Define $P(n)$ be the statement that "there exists integers $i_n \geq 0$ and $j_n \geq 0$, such that $n = i_n \times 2 + j_n \times 3$". We will try to prove that $P(n)$ is true for every $n \geq 7$.

   If $P(n)$ is not true for every $n \geq 7$, then there are values of n for which $P(n)$ is false, and there must be a smallest such value, say $n = k$.

   Since $P(7) = 2 \times 2 + 1 \times 3$, we have $k \geq 8$, and $k - 1 \geq 7$.

   Since k is the smallest value for which $p(k)$ is false, $P(k-1)$ is true. Thus there exists integers $i_{k-1} \geq 0$ and $j_{k-1} \geq 0$, such that $k - 1 = i_{k-1} \times 2 + j_{k-1} \times 3$, and i_{k-1} and j_{k-1} cannot be 0 at the same time.

   If $i_{k-1} \geq 1$, we have

   $$\begin{aligned} k &= k - 1 + 1 \\ &= i_{k-1} \times 2 + j_{k-1} \times 3 + 1 \\ &= i_{k-1} \times 2 + j_{k-1} \times 3 + 3 - 2 \\ &= (i_{k-1} - 1) \times 2 + (j_{k-1} + 1) \times 3 \end{aligned}$$

   Let i_k be $i_{k-1} - 1$ and j_k be $j_{k-1} + 1$, then $k = i_k \times 2 + j_k \times 3$.

   If $j_{k-1} \geq 1$, we have

   $$\begin{aligned} k &= k - 1 + 1 \\ &= i_{k-1} \times 2 + j_{k-1} \times 3 + 1 \\ &= i_{k-1} \times 2 + j_{k-1} \times 3 + 4 - 3 \\ &= (i_{k-1} + 2) \times 2 + (j_{k-1} - 1) \times 3 \end{aligned}$$

   Let i_k be $i_{k-1} + 2$ and j_k be $j_{k-1} - 1$, then $k = i_k \times 2 + j_k \times 3$.

   We have derived a contradiction, which allows us to conclude that our original assumption is false. □

3. Suppose the function f be defined on the natural numbers recursively as follows: $f(0) = 0$, $f(1) = 1$, and $f(n) = 5f(n-1) - 6f(n-2)$, for $n \geq 2$. Use the strong principle of mathematical induction to prove that for all $n \in N$, $f(n) = 3^n - 2^n$.

   Proof. Let $P(n)$ be the statement $f(n) = 3^n - 2^n$. We will try to prove that $P(n)$ is true for every $n \in N$.

   $f(2)$ is $5 \times 1 - 6 \times 0 = 5 = 3^2 - 2^2$, which satisfies $f(n) = 3^n - 2^n$. Also, $f(0)$ is $0 = 3^0 - 2^0$ and $f(1)$ is $1 = 3^1 - 2^1$, which satisfies $f(n) = 3^n - 2^n$. Obviously, $P(n)$ is true for $n = 0, 1, 2$.

   Assume for $k \geq 2$ and $2 \leq n \leq k$,$P(n)$ is true. Now let us prove that $P(k+1)$ is true.

   By definition, we have

$$\begin{aligned} f(k+1) &= 5f(k) - 6f(k-1) \\ &= 5 \times (3^k - 2^k) - 6 \times (3^{k-1} - 2^{k-1}) \\ &= 5 \times 3^k - 5 \times 2^k - 2 \times 3^k + 3 \times 2^k \\ &= 3^{k+1} - 2^{k+1} \end{aligned}$$

   Therefore, $P(k+1)$ is true.

   According to the strong principle of mathematical induction, for every $n \in N$, $P(n)$ is true, and $f(n) = 3^n - 2^n$. □

4. An n-team basketball tournament consists of some set of $n \geq 2$ teams. Team p beats team q iff q does not beat p, for all teams $p \neq q$. A sequence of distinct teams p_1, p_2,..., p_k, such that team p_i beats team p_{i+1} for $1 \leq i < k$ is called a ranking of these teams. If also team p_k beats team p_1, the ranking is called a *k-cycle*.

   Prove by mathematical induction that in every tournament, either there is a "champion" team that beats every other team, or there is a 3-cycle.

   Proof. Let $P(n)$ be the statement "for a directed complete graph G_n with n vertices, either there is a vertex with an out-degree of $n-1$, or there is a 3-vertex loop". We will try to prove that $P(n)$ is true for every $n \geq 2$. This is equivalent to the original problem.

   For a $n = 2$, by the definition, there must be a vertex whose out-degree is 1. For $n = 3$, we can enumerate all the possible graphs and find that either there is a vertex with an out-degree of 2, or there is a 3-vertex loop. Obviously, $P(n)$ is true for $n = 2, 3$.

   Assume $P(k)$ is true for some $k \geq 3$. Now let us prove that $P(k+1)$ is true.

   The construction of a directed complete graph G_{k+1} with $k+1$ vertices can be viewed as adding a new vertex V_{k+1} to a directed complete graph G_k with k vertices and connecting this vertex with the others respectively.

   If there is a 3-vertex loop in G_k, that loop is also in G_{k+1}. Then $P(k+1)$ is true.

   If there is not a 3-vertex loop in G_k, there must be a vertex with an out-degree of $k-1$ in G_k which is denoted by V_p. If the edge between V_{k+1} and V_p is directed to V_{k+1}. Then V_p is the vertex with an out-degree of k in G_{k+1}. If that edge is directed to V_p, there is another vertex V_q satisfying V_p is directed to V_q, V_q is directed to V_{k+1} and V_{k+1} is directed to V_p, which compose a 3-vertex loop (if V_q doesn't exist, then V_{k+1} is the vertex with an out-degree of k). Therefore, $P(k+1)$ is true.

   According to the mathematical induction, $P(n)$ is true for every $n \geq 2$, and the original proposition is true. □

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