Report on VAN BENTHEM CHARACTERIZATION THEOREM but in mathematical logic class June 7, 2023, XIN CHEN

Target

- Which first-order formulas are equivalent to the modal formulas on model level?
- the proof strategy of van Benthem's Theorem
- some tools in (first-order or modal) Model Theory

Keywords invariant under bisimilarity/modal-equivalence, compactness, detour strategy, ω-saturation $\rightsquigarrow$ m-saturation, ultrafilters $\rightsquigarrow$ ultraproducts $\rightsquigarrow$ ultrapowers, Łoś's Theorem....

Recap

1. *Languages*

$$\mathcal{L}_\Diamond \ni \varphi ::= p_i \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \Diamond\varphi.$$
$$\mathcal{L}_1 \ni \alpha ::= x \triangleq y \mid P_i x \mid Rxy \mid \neg\alpha \mid (\alpha \vee \alpha) \mid \exists x\alpha.$$

2. *Model*: $\mathfrak{M} = (W, R, V)$ (also a *first-order* structure)
3. *Standard translation* $ST_x : \mathcal{L}_\Diamond \to \mathcal{L}_1 \quad \Rightarrow \quad \mathcal{L}_\Diamond$ is a fragment of $\mathcal{L}_1$
4. *m-saturation*: $\,\underline{\leftrightarrow}\, = \leftrightsquigarrow$, (but in general $\,\underline{\leftrightarrow}\, \subsetneqq \leftrightsquigarrow$)
5. *ultrafilter*, *principal ultrafilter* (generated filter).
   (finite intersection property (FIP) $\rightsquigarrow$ ultrafilter)
6. *ultrafilter extension* $\mathfrak{M}^{\mathfrak{ue}} \quad \Rightarrow \quad$ *m-saturated*
7. ultrafilter extension *not* preserve the truth value of first-order formulas

Section 1

proof 1

Subsection 1.1

A simple characterization

Definition 1.1 (**Invariant under modal equivalence/bisimilarity**) A first-order formula $\alpha(x) \in \mathcal{L}_1$ is **invariant under modal equivalence**, if for any $\mathfrak{M}, w$ and $\mathfrak{N}, v$:

$$\mathfrak{M}, w \leftrightsquigarrow \mathfrak{N}, v \text{ implies } \mathfrak{M} \models \alpha(x)[w] \ \Leftrightarrow \ \mathfrak{N} \models \alpha(x)[v].$$

And $\alpha(x)$ is **invariant under bisimilarity**, if

$$\mathfrak{M}, w \,\underline{\leftrightarrow}\, \mathfrak{N}, v \text{ implies } \mathfrak{M} \models \alpha(x)[w] \ \Leftrightarrow \ \mathfrak{N} \models \alpha(x)[v].$$

Theorem 1.2 (**A characterization via modal equivalence**) Let $\alpha(x)$ be a first-order formulas in $\mathcal{L}_1$ with one free variable.

$\alpha(x)$ is invariant under modal equivalence $\Leftrightarrow$ it is equivalent[1] to the standard translation of a modal formula in $\mathcal{L}_\Diamond$.

[1] *semantic equivalence*, 如果这里想得到的是语形等价，非常复杂或者几乎是不可能。

PROOF (Proof of this simple characterization)

$\Leftarrow$ This direction is trivial. If for some modal formula $\varphi \in \mathcal{L}_\Diamond$ such that $\alpha(x) = ST_x(\varphi)$, and further suppose that $\mathfrak{M}, w \leftrightsquigarrow \mathfrak{N}, v$.

Then $\mathfrak{M}, w \Vdash \varphi \iff \mathfrak{N}, v \Vdash \varphi$, by *local correspondence*, $\mathfrak{M} \models \alpha(x)[w] \iff \mathfrak{N} \models \alpha(x)[v]$. That is, $\alpha(x)$ is invariant under model equivalence.

$\Rightarrow$ (Depends on Compactness[2] of FOL)

Suppose $\alpha(x)$ is invariant under model equivalence. Let the *modal consequence* of $\alpha(x)$ be

$$MOC(\alpha(x)) \coloneqq \{ST_x(\varphi) \mid \alpha(x) \models ST_x(\varphi) \ \& \ \varphi \in \mathcal{L}_\Diamond\}.$$

We have two claims:

Claim 1: If $MOC(\alpha(x)) \models \alpha(x)$ then there is a modal formula φ such that $ST_x(\varphi)$ is (semantic) equivalent to $\alpha(x)$.

Claim 2: $MOC(\alpha(x)) \models \alpha(x)$ is indeed holds.

If these two claims is true, then we have done.

.... Proof of Claim 1 ....

The first claim can be proved by an argument based on the Compactness of FOL (Σ is finitely satisfiable $\Rightarrow$ Σ is satisfiable).

Suppose $MOC(\alpha(x)) \models \alpha(x)$, then $MOC(\alpha(x)) \cup \{\neg\alpha(x)\}$ is not satisfiable, by (contraposition of) Compactness, there is a *finite* subset Z of $MOC(\alpha(x)) \cup \{\neg\alpha(x)\}$ which is unsatisfiable. There are two cases:

1. $\alpha(x) \notin Z$, let $Z = X$, then X is *finite*, $X \subseteq MOC(\alpha(x))$ and $X \cup \{\neg\alpha(x)\}$ is unsatisfiable.

2. $\alpha(x) \in Z$, then there is a *finite* set $X \subseteq MOC(\alpha(x))$ and $\alpha(x) \notin X$ such that $Z = X \cup \{\neg\alpha(x)\}$

Therefore, there exists a *finite* $X \subseteq MOC(\alpha(x))$ such that $X \cup \{\neg\alpha(x)\}$ is unsatisfiable, that is, $X \models \alpha(x)$.[3]

Since X is finite, thus $\models \bigwedge X \to \alpha(x)$, moreover, $\models \alpha(x) \to \bigwedge X$ (by the definition of $MOC(\alpha(x))$), then $\models \bigwedge X \leftrightarrow \alpha(x)$.

Assume $X = \{ST_x(\psi_1), \ldots, ST_x(\psi_n)\}$, let $\varphi = \psi_1 \wedge \cdots \wedge \psi_n$, then $ST_x(\varphi) = \bigwedge X$. Therefore, there is a modal formula φ such that $ST_x(\varphi)$ is (semantic) equivalent to $\alpha(x)$.

.... Proof of Claim 2 ....

Suppose for any model $\mathfrak{M}$ we have $\mathfrak{M} \models MOC(\alpha(x))[w]$, then we only need to show that $\mathfrak{M} \models \alpha(x)[w]$.

Let

$$\Gamma = Th(\mathfrak{M}, w) \coloneqq \{\varphi \in \mathcal{L}_\Diamond \mid \mathfrak{M}, w \Vdash \varphi\}$$

and

$$ST_x(\Gamma) = \{ST_x(\varphi) \mid \varphi \in \Gamma\}$$

If $ST_x(\Gamma) \cup \{\alpha(x)\}$ is satisfiable (in first-order sense) in some pointed model $\mathfrak{N}, v$, then $\mathfrak{M}, w \leftrightsquigarrow \mathfrak{N}, v$ since they satisfy same *modal* formulas.[4] While $\alpha(x)$ is invariant under modal equivalence, hence $\mathfrak{M} \models \alpha(x)[w]$.

Therefore it suffices to show that $ST_x(\Gamma) \cup \{\alpha(x)\}$ is satisfiable (in first-order sense) in some pointed model $\mathfrak{N}, v$. (again by a compactness argument)

[2] 想说一个东西存在，就先划一个范围，然后用如同**紧致性**这样的性质把该对象逼出来。

如果 $\alpha(x)$ 的模态对应存在的话，$MOC(\alpha(x))$ 相当于划定了 $\alpha(x)$ 模态对应的范围，然后再从这个范围里「逼出」$\alpha(x)$ 的模态对应。先划范围，然后再逼近，这是一种常见且有用的证明思路

[3] 如果熟悉一阶逻辑的紧致性，从 $MOC(\alpha(x)) \models \alpha(x)$ 直接可得 $X \models \alpha(x)$。这里也可以使用**可靠性**。

[4] $\mathfrak{N}, v$ 满足的模态公式会比 Γ 中的多吗？— 不可能。从另一个角度看，Γ 是一个 *MCS*（*w.r.t.* $\mathbf{K}$）。

Suppose $ST_x(\Gamma)\cup\{\alpha(x)\}$ is unsatisfiable, then $ST_x(\Gamma)\models\neg\alpha(x)$, by Compactness, there exists a *finite* subset Y of $ST_x(\Gamma)$ such that $Y\models\neg\alpha(x)$. Hence $\alpha(x)\models\neg\bigwedge Y$.

By the definition of $MOC(\alpha(x))$, we have $\neg\bigwedge Y\in MOC(\alpha(x))$, by assumption $\mathfrak{M}\models MOC(\alpha(x))[w]$, thus $\mathfrak{M}\models\neg\bigwedge Y[w]$. But $\bigwedge Y\in ST_x(\Gamma)$, it follows that $\mathfrak{M}\models\bigwedge Y[w]$. Contradiction! ■

SUBSECTION 1.2

van Benthem Characterization Theorem

Theorem 1 (**van Benthem Characterization Theorem**) Let $\alpha(x)$ be a first-order formula in $\mathcal{L}_1$. [5]

$\alpha(x)$ is invariant under bisimilarity $\Leftrightarrow$ it is equivalent to the standard translation of a modal formula.

[5] $\Leftarrow$ *this direction is trivial.*

To prove this theorem based on the previous *simple characterization result*, we only need to show that:

Lemma 1.3 $\alpha(x)$ is invariant under bisimilarity $\Leftrightarrow$ $\alpha(x)$ is invariant for modal equivalence.

Right-to-Left is trivial, since bisimilarity implies modal equivalence[6]:

[6] *that is* $\leftrightarrows\subseteq\leftrightsquigarrow$.

$$\begin{array}{llcl}
\text{Left} & \mathfrak{M},w\leftrightarrows\mathfrak{N},v & \Rightarrow & (\mathfrak{M}\models\alpha(x)[w]\ \Leftrightarrow\ \mathfrak{N}\models\alpha(x)[v]) \\
 & & \Uparrow & (\text{ trivial direction}) \\
\text{Right} & \mathfrak{M},w\leftrightsquigarrow\mathfrak{N},v & \Rightarrow & (\mathfrak{M}\models\alpha(x)[w]\ \Leftrightarrow\ \mathfrak{N}\models\alpha(x)[v])
\end{array}$$

Left-to-Right is hard. It is not trivial since $\leftrightsquigarrow\neq\leftrightarrows$ in general.[7]

[7] 某种意义上，*bisimilarity* 比 *modal equivalence* 更细致.

SUBSECTION 1.3

A detour strategy

Since $\leftrightarrows\subseteq\leftrightsquigarrow$, we only need to prove that if $\leftrightarrows/\alpha(x)$ then $\leftrightsquigarrow/\alpha(x)$. [8]

Now assume that $\leftrightarrows/\alpha(x)$ and $\mathfrak{M},w\leftrightsquigarrow\mathfrak{N},v$. We need to show that $\mathfrak{M}\models\alpha(x)[w]$ iff $\mathfrak{N}\models\alpha(x)[v]$. The strategy that:

[8] $\leftrightarrows/\alpha(x)$: *$\alpha(x)$ is invariant under bisimilarity.* $\leftrightsquigarrow/\alpha(x)$: *$\alpha(x)$ is invariant under modal equivalence*

$$\begin{array}{lcccl}
1.\ \alpha(x) & \mathfrak{M},w & \leftrightsquigarrow & \mathfrak{N},v & 4.\ \alpha(x) \\
 & \equiv_{FOL} & & \equiv_{FOL} & \\
2.\ \alpha(x) & \mathfrak{M}^*,w^* & \leftrightsquigarrow\ =\ \leftrightarrows & \mathfrak{N}^*,v^* & 3.\ \alpha(x)
\end{array}$$

Figure 1. A detour strategy(曲线救国): 1-2-3-4 and 4-3-2-1

How to construct $\mathfrak{M}^*,w^*$ and $\mathfrak{N}^*,v^*$? They at least need be **m-saturated**, since for m-saturated models: $\leftrightarrows$ coincides with $\leftrightsquigarrow$.

Remark *If two (pointed) models such that FOL formulas are preserved, thus modal formulas are preserved too!*

First candidate: **Ultrafilter extension**[9]. Though Ultrafilter extension preserve truth value of modal formulas, but *does not preserve the truth value of first-order formulas* . Pass! To see that, considering the ultrafilter extension of $(\mathbb{N},<)$: [10]

[9] *note that* $\mathfrak{M},w\leftrightsquigarrow\mathfrak{M}^{ue},\pi_w$, *Prop 2.59 in Blue Book .*

[10] *p.95 in Blue Book without transitive arrows.*

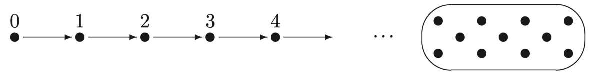

Figure 2. the ultrafilter extension of $(\mathbb{N}, <)$

There is a "cluster" of *reflexive non-principal ultrafilters* at the "end" of the chain of natural numbers. Every non-principal ultrafilters is reachable from π_0[11]. Thus the first-order formula $\exists y(Rxy \wedge Ryy)$ is satisfiable at $((\mathbb{N}, <)^{\mathfrak{ue}}, \pi_0)$ but not at $((\mathbb{N}, <), 0)$.

[11] *the principal ultrafilter generated by 0.*

Hence we need a model construction method which can:

1. make the models m-saturated, and
2. preserve truth values of first-order formulas.

SUBSECTION 1.4

Ultraproducts

1.4.1 Ultrafilters again

Intuition An intuition[12] behind (ultra)filters: "small" subsets are out, only "large" subsets stay (imagine a *filter* in the basin that we use everyday, or a coffee/tea filter).

[12] *another intuition is that, an ultrafilter often seen as the extension of a MCS.*

Figure 3. a coffee filter

每个 filter 可以被视为一些「很大」的子集的集合

Ultrafilters were originally used to define a collection of subsets of a nonempty set W which can be regarded as "large" subsets of W in a consistent mathematical sense.

Therefore given an *index set* I of a family of models $\{\mathfrak{M}_i\}_{i \in I}$, if φ holds on some $\mathfrak{M}_i, w_i$, and $\{i \mid \mathfrak{M}_i, w_i \Vdash \varphi\}$ is in a (non-principal) ultrafilter over I, then we can say that φ holds on "almost every" in the family of models. We use this idea to define ultraproducts of models.

1.4.2 Ultraproducts

Definition 1.4 (**Ultraproducts over sets**) Given a family of sets $\{W_i\}_{i \in I}$ and an ultrafilter U over the nonempty index set I. Define the *equivalence relation* $\sim_U$[13] as

$$\sim_U = \left\{ (f, g) \mid f, g \in \prod_{i \in I} W_i \text{ and } \{i \in I \mid f(i) = g(i)\} \in U \right\}.$$

The equivalence class of f w.r.t. $\sim_U$ is

$$f_U = \{g \in \prod_{i \in I} W_i \mid g \sim_U f\}.$$

The **ultraproduct of W_i modulo U**, denoted as $\prod_U W_i$, is the set of all equivalence classes of $\sim_U$:

$$\prod_U W_i = \{f_U \mid f \in \prod_{i \in I} W_i\}.$$

If for all i have $W_i = W$ then the ultraproduct is called the **ultrapower of W modulo U**, denoted by $\prod_U W$.

[13] $\sim_U$ 有自反性和对称性很显然；因为 U 是超滤且超滤对交封闭，易知 $\sim_U$ 是传递的。

Intuition Two sequences (or functions) f, g are considered the same if they coincide "*almost everywhere*" $f(i) = g(i)$ for all the i belongs to some *large set* in the ultrafilter U.

The elements in $\prod_{i\in I} W_i$ are (may infinite) sequences $\langle w_1, w_2, w_3, \dots, w_i, \dots\rangle$, but from another perspective, a sequence is a function $f\colon I \to \bigcup_{i\in I} W_i$ such that for a given index $i \in I$, f chooses an element $f(i)$ from W_i, hence $f(i)$ is just the i-th parameter in the sequence $\langle w_1, w_2, w_3, \dots\rangle$. See the following diagram[14]:

$$
\begin{array}{ccccccc}
W_1 & W_2 & W_3 & \cdots & W_i & \cdots \\
{\scriptstyle f(1)}\big\downarrow & {\scriptstyle f(2)}\big\downarrow & {\scriptstyle f(3)}\big\downarrow & & {\scriptstyle f(i)}\big\downarrow & \\
w_1 & w_2 & w_3 & \cdots & w_i & \cdots
\end{array}
$$

$$f \quad = \quad \langle w_1, w_2, w_3, \dots, w_i, \dots\rangle$$

[14] 虽然这里集合的下标是 $1, 2, 3, \dots$，但一般来说指标集不必是自然数集，此处的写法只是为了方便起见

And $f \sim_U g$ means that those elements selected respectively by f and g are same "almost everywhere"[15].

[15] I 上的超滤 U 暗含了"几乎所有"的意思，因为超滤是那些很大的子集的集合。

Definition 1.5 (**Ultraproduct over models** with a binary relation) Let $\{\mathfrak{M}_i\}_{i\in I}$ be a family of models. Given an ultrafilter U over I, the **ultraproduct of $\{\mathfrak{M}_i\}_{i\in I}$ modulo U** is a triple $\prod_U \mathfrak{M}_i = (W, \to, V)$ where:

- $W = \prod_U W_i$, where W_i is the universe of $\mathfrak{M}_i$.
- $f_U \to g_U \quad\Leftrightarrow\quad \{i \mid f(i) \xrightarrow{\mathfrak{M}_i} g(i)\} \in U$, where $\xrightarrow{\mathfrak{M}_i}$ is the binary relation of $\mathfrak{M}_i$.
- $f_U \in V(p) \quad\Leftrightarrow\quad \{i \mid f(i) \in V_i(p)\} \in U$

If for all i have $\mathfrak{M}_i = \mathfrak{M}$, then $\prod_U \mathfrak{M}_i = \prod_U \mathfrak{M}$ is called the **ultrapower of $\mathfrak{M}$ modulo U.**

Intuition Massage many models into *one* such that if *most* models satisfy something then this merged one also satisfies something. [16]

[16] 把一堆模型揉成一个模型，并且最终的成品保留"大多数"模型都满足的性质。指标集上的超滤 $\approx$ 大多数，因此考虑"大多数"对象要满足某种性质的时候，超滤是一个强有力的工具。

Remark The above is *well-defined*. Considering the valuation V, for example, suppose $f \sim_U g$, we need check that $\{i \mid f(i) \in V_i(p)\} \in U \quad\Leftrightarrow\quad \{i \mid g(i) \in V_i(p)\} \in U$.

Theorem 1.6 (**Łoś's Theorem** one free variable case) [17] Let U be an ultrafilter over an nonempty index set I, given any first-order formula $\alpha(x)$:

$$\prod_U \mathfrak{M}_i \models \alpha(x)[f_U] \quad\Leftrightarrow\quad \{i \mid \mathfrak{M}_i \models \alpha(x)[f(i)]\} \in U.$$

[17] *also called the* ***fundamental theorem of ultraproducts****. This theorem due to Jerzy Łoś, the surname is pronounced approximately "wash"* - 沃希定理.
Chaff: 螺蛳（没有粉）定理

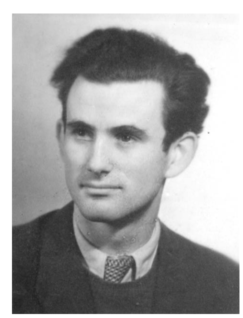

Figure 4. Jerzy Łoś

Intuition The **Right** of above theorem means that: in the family $\{\mathfrak{M}_i\}_{i\in I}$ of models, $\alpha(x)$ is satisfiable in "almost every" model.

PROOF By induction on $\alpha(x)$. Cf. Theorem A.19 in [*Blue Book* p.493]. ■

Definition 1.7 (**Elementary embedding**) Given any two models $\mathfrak{A}$ and $\mathfrak{B}$ for $\mathcal{L}_1$ with universe A and B respectively. A function $f\colon A \to B$ is an **elementary embedding** from $\mathfrak{A}$ to $\mathfrak{B}$, notation $f\colon \mathfrak{A} \preceq \mathfrak{B}$, if for any first-order formulas $\alpha(x_1, \dots, x_n)$ and $a_1, \dots, a_n \in A$,

$$\mathfrak{A} \models \alpha(x_1, \dots, x_n)[a_1, \dots, a_n] \quad\Leftrightarrow\quad \mathfrak{B} \models \alpha(x_1, \dots, x_n)[f(a_1), \dots, f(a_n)].$$

Define the **diagonal mapping** $d : \mathfrak{M} \to \prod_U \mathfrak{M}$ such that

$$w \mapsto (f_w)_U$$

where f_w is a *constant function* such that $f(i) = w$ for all $i \in I$. (in other words, f_w is the sequence $\langle w, w, w, \dots, w, \dots \rangle$)

Prop. 1.8 The diagonal mapping $d\colon \mathfrak{M} \to \prod_U \mathfrak{M}$ such that $d(w) = (f_w)_U$ is an *elementary embedding* from $\mathfrak{M}$ to $\prod_U \mathfrak{M}$, that is, $d\colon \mathfrak{M} \preceq \prod_U \mathfrak{M}$.

PROOF Let $\alpha(x)$ be a first-order formula and a an element of $\mathfrak{M}$,

$$\begin{aligned} \textstyle\prod_U \mathfrak{M} \models \alpha(x)[d(a)] &\Leftrightarrow \textstyle\prod_U \mathfrak{M} \models \alpha(x)[(f_a)_U] && (\text{since } d(a) = (f_a)_U) \\ &\Leftrightarrow \{i \in I \mid \mathfrak{M} \models \alpha(x)[a]\} \in U && (\text{by Łoś's theorem }) \\ &\Leftrightarrow \mathfrak{M} \models \alpha(x)[a] && \blacksquare \end{aligned}$$

Corollary 1.9 (**Ultrapower**) Let $\prod_U \mathfrak{M}$ be an ultrapower of $\mathfrak{M}$, then for all first-order *formula* $\alpha(x)$ given any first-order formula $\alpha(x)$:

$$\prod_U \mathfrak{M} \models \alpha(x)[(f_w)_U] \Leftrightarrow \mathfrak{M} \models \alpha(x)[w]$$

PROOF

$$\begin{aligned} \textstyle\prod_U \mathfrak{M} \models \alpha(x)[(f_w)_U] &\Leftrightarrow \{i \mid \mathfrak{M}_i \models \alpha(x)[f_w(i)]\} \in U && (\text{by Łoś's theorem }) \\ &\Leftrightarrow \{i \mid \mathfrak{M} \models \alpha(x)[w]\} \in U && (\mathfrak{M} = \mathfrak{M}_i, f_w(i) = w) \\ &\Leftrightarrow \mathfrak{M} \models \alpha(x)[w] && \blacksquare \end{aligned}$$

Theorem 1.10 (**Łoś's Theorem for modal logic**) Fixing a U, given any modal formula φ:

$$\prod_U \mathfrak{M}_i, f_U \Vdash \varphi \Leftrightarrow \{i \mid \mathfrak{M}_i, f(i) \Vdash \varphi\} \in U.$$

PROOF For any modal formula $\varphi \in \mathcal{L}_\Diamond$,

$$\begin{aligned} \textstyle\prod_U \mathfrak{M}_i, f_U \Vdash \varphi &\Leftrightarrow \textstyle\prod_U \mathfrak{M}_i, \models ST_x(\varphi)[f_U] && \text{local correspondence} \\ &\Leftrightarrow \{i \mid \mathfrak{M}_i \models ST_x(\varphi)[f(i)]\} \in U && \text{Łoś's theorem for FOL} \\ &\Leftrightarrow \{i \mid \mathfrak{M}_i, f(i) \Vdash \varphi\} \in U && \text{local correspondence} \quad \blacksquare \end{aligned}$$

Corollary 1.11 (**Ultrapower in modal logic**) Let $\prod_U \mathfrak{M}$ be an ultrapower of $\mathfrak{M}$. Then for all modal formula φ we have :

$$\prod_U \mathfrak{M}, (f_w)_U \Vdash \varphi \Leftrightarrow \mathfrak{M}, w \Vdash \varphi.$$

SUBSECTION 1.5

Saturation

Definition 1.12 (**Type** and **Realization**) A **type** is a set $\Gamma(x)$ of first-order formulas such that for any $\alpha \in \Gamma$, x is the unique variable may occur free in α.

A first-order model $\mathfrak{M}$ **realizes** type $\Gamma(x)$ if there is an element w in $\mathfrak{M}$ such that for all $\alpha \in \Gamma(x)$, $\mathfrak{M} \models \alpha[w]$.

Definition 1.13 (**Expansions** of language and model) Let $\mathfrak{M}$ (with domain W) be a model for the first-order language $\mathcal{L}_1$. For any subset $A \subseteq W$, $\mathcal{L}_1[A]$, given by

$$\mathcal{L}_1[A] := \mathcal{L}_1 \cup \{\underline{a} \mid a \in A\},$$

is the language obtained by extending $\mathcal{L}_1$ with new constants $\underline{a}$ for all $a \in A$.

$\mathfrak{M}_A$ is the **expansion** of $\mathfrak{M}$ to a structure for $\mathcal{L}_1[A]$ in which each $\underline{a}$ is interpreted as a.

Definition 1.14 (**ω-Saturated models**) [18] Suppose $\mathfrak{M}$ with domain W is a model for first-order language $\mathcal{L}$.

$\mathfrak{M}$ is **ω-saturated** iff for any *finite* subset $A \subseteq W$ and any type $\Gamma(x)$ of $\mathcal{L}[A]$, if the expansion $\mathfrak{M}_A$ realizes every *finite* subset of $\Gamma(x)$ then $\mathfrak{M}_A$ realizes $\Gamma(x)$.

[18] 此处定义参考 [文学锋，定义 *10.3.14*]

Theorem 1.15 Any ω-saturated model for language $\mathcal{L}_1$ is m-saturated. It follows that the class of ω-saturated models has the Hennessy-Miliner property. [19]

[19] *Theorem 2.65 in Blue Book*

PROOF Suppose $\mathfrak{M} = (W, R, V)$ (viewed as a first-order structure) is ω-saturated.

Let a be a state in $\mathfrak{M}$ and Σ is a set of modal formulas which is finitely satisfiable in $R(a)$ [20]. It suffices to show that Σ is satisfiable in $R(a)$.

[20] *the successor set of a.*

Let Σ' be

$$\Sigma' = \{R\underline{a}x\} \cup ST_x(\Sigma)^{21}$$

[21] 尽管这里的关系符号和模型中的关系都是用 R 表示，但根据上下文容易区分每处 R 指的是语言中的符号还是模型中的解释。

then Σ' is a type of $\mathcal{L}_a$. For any *finite* subset X of Σ', there are two cases:

1. $R\underline{a}x \notin X$. Then $X \subseteq ST_x(\Sigma)$, since Σ is finitely satisfiable in $R(a)$, by local correspondence, hence X is realized in some state b such that Rab, thus $\mathfrak{M}_a$ realizes X.

2. $R\underline{a}x \in X$. Then $X = Y \cup \{R\underline{a}x \in X\}$ and Y is a finite subset of $ST_x(\Sigma)$. Similarly, Y is realized in some state b such that Rab, clearly $\mathfrak{M}_a$ realizes $Y \cup \{R\underline{a}x\}$.

Therefore, $\mathfrak{M}_a$ realizes every *finite* subset of Σ'. By ω-saturation, $\mathfrak{M}_a$ realizes Σ', that is, $\mathfrak{M}_a \models \{R\underline{a}x\} \cup ST_x(\Sigma)[b]$ for some b. By $\mathfrak{M}_a \models R\underline{a}x[b]$ it follows that b is a successor of a. Since $\mathfrak{M}_a \models ST_x(\Sigma)[b]$, by local correspondence, $\mathfrak{M}, b \Vdash \Sigma$.

Thus Σ is satisfiable in $R(a)$. Then we complete the proof of that all ω-saturated models are m-saturated. ■

Recap

$$\begin{array}{llcll}
1.\ \alpha(x) & \mathfrak{M}, w & \rightsquigarrow & \mathfrak{N}, v & 4.\ \alpha(x) \\
 & \equiv_{FOL} & & \equiv_{FOL} & \\
2.\ \alpha(x) & \prod_U \mathfrak{M}, (f_w)_U & \leftrightsquigarrow = \Leftrightarrow & \prod_U \mathfrak{N}, (f_v)_U & 3.\ \alpha(x)
\end{array}$$

the detour strategy — use ultrapower

The ultrapower works! We need to show that ultrapowers over certain ultrafilters are m-saturated.

1.5.1 Construct saturated models

Definition 1.16 (**Countably incomplete ultrafilter**) An ultrafilter U over I is **countably incomplete** if it is not closed under countable intersections. (i.e. there exists $E \subseteq U$, E is countable but $\bigcap E \notin U$) [22]

[22] *note that, ultrafilter is closed under (finite) intersections*

A *principle ultrafilter* is not countably incomplete, since any intersection must contains the element which generated this principle ultrafilter.

Thus a countably incomplete ultrafilter must be *non-principal*. Such ultrafilter exists (recall that the non-principal ultrafilters over $\mathbb{N}$)!

Note 1.17 However, the existence of *non-principal countably complete* (closed under countable intersections) ultrafilter is not provable in ZFC.

Theorem 1.18 If U is a *countably incomplete ultrafilter* over a nonempty set I, then the ultrapower $\prod_U \mathfrak{M}$ is ω-saturated, thus it is m-saturated. [23]

[23] *Lemma 2.73 in Blue Book .*

Proof. This theorem is dependent on the language $\mathcal{L}_1$ and $\mathcal{L}_\diamond$ is countable. The detail can cf. p.384 of [Chang & Keisler 1990]. ■

Subsection 1.6

Back to plotline

Recap

$$\begin{array}{llcll}
1.\ \alpha(x) & \mathfrak{M}, w & \leftrightsquigarrow & \mathfrak{N}, v & 4.\ \alpha(x) \\
 & \equiv_{FOL} & & \equiv_{FOL} & \\
2.\ \alpha(x) & \prod_U \mathfrak{M}, (f_w)_U & \leftrightsquigarrow\ =\ \Leftrightarrow & \prod_U \mathfrak{N}, (f_v)_U & 3.\ \alpha(x)
\end{array}$$

Again: the detour strategy

Let above U be a ***countably incomplete ultrafilter***, that is ensure the ultrapower is ω-saturated, hence m-saturated.

Now we complete the proof.

Section 2

van Benthem Characterization Theorem: proof-2

Lemma 2.1 (**Detour Lemma**) [24] Let $\mathfrak{M}$ and $\mathfrak{N}$ be two models with state w and v respectively. Then the following are equivalent:

(i) $\mathfrak{M}, w \leftrightsquigarrow \mathfrak{N}, v$.

(ii) $\mathfrak{M}^{\mathfrak{ue}}, \pi_w \leftrightarroweq \mathfrak{N}^{\mathfrak{ue}}, \pi_v$.

(iii) There exist ω-saturated models $\mathfrak{M}^*, w^*$ and $\mathfrak{N}^*, v^*$ and *elementary embeddings* $f\colon \mathfrak{M} \preceq \mathfrak{M}^*$ and $g\colon \mathfrak{N} \preceq \mathfrak{N}^*$ such that

(a) $f(w) = w^*$ and $g(v) = v^*$,

(b) $\mathfrak{M}^*, w^* \leftrightarroweq \mathfrak{N}^*, v^*$.

[24] *Lemma 2.66 in Blue Book .*

Proof (i) $\Leftrightarrow$ (ii) It is just Theorem 2.62 in *Blue Book* .

(i) $\Rightarrow$ (iii)
Let

$$\mathfrak{M}^*, w^* \text{ be } \prod_U \mathfrak{M}, (f_w)_U$$

and

$$\mathfrak{N}^*, v^* \text{ be } \prod_U \mathfrak{N}, (f_v)_U$$

where U is a *countably incomplete ultrafilter*,by the argument in the previous section, then we have done.

(iii) $\Rightarrow$ (i) Trivially, since first-order satisfaction in invariant under elementary embeddings, so is for modal satisfaction. ■

Theorem 2.2 (**van Benthem Characterization Theorem**) For any $\alpha(x) \in \mathcal{L}_1$. Then $\alpha(x)$ is invariant for bisimulations iff it is equivalent to $ST_x(\varphi)$ for a modal formula in $\varphi \in \mathcal{L}_\Diamond$.

Th 2.68 in *Blue Book*

Proof $\Leftarrow$ [25] Suppose $\alpha(x)$ is equivalent to $ST_x(\varphi)$ for some $\varphi \in \mathcal{L}_\Diamond$, $\mathfrak{M}, w$ and $\mathfrak{N}, v$ are two arbitrary pointed models with $\mathfrak{M}, w \leftrightarroweq \mathfrak{N}, v$.

Clearly $\mathfrak{M}, w \Vdash \varphi \Leftrightarrow \mathfrak{N}, v \Vdash \varphi$. By *Local Correspondence on Models*, $\mathfrak{M} \models ST_x(\varphi)[w] \Leftrightarrow \mathfrak{N} \models ST_x(\varphi)[v]$. Therefore $\alpha(x)$ is invariant for bisimulations.

[25] *this direction is so easy since* $\leftrightarroweq \subseteq \leftrightsquigarrow$.

$$\begin{array}{ccccc}
\mathfrak{M}, w \leftrightarroweq \mathfrak{N}, v & \Rightarrow & \mathfrak{M}, w \Vdash \varphi & \Leftrightarrow & \mathfrak{N}, v \Vdash \varphi \\
 & & \Updownarrow & & \Updownarrow \\
 & & \mathfrak{M} \models ST_x(\varphi)[w] & \Leftrightarrow & \mathfrak{N} \models ST_x(\varphi)[v]
\end{array}$$

(proof sketch of Right-to-Left)

$\Rightarrow$ Assume that $\alpha(x)$ is invariant for bisimulations and consider the set of *modal consequences* of $\alpha(x)$:

$$MOC(\alpha(x)) = \{ST_x(\varphi) \in \mathcal{L}_1 \mid \varphi \in \mathcal{L}_\Diamond \text{ and } \alpha(x) \models ST_x(\varphi)\}.$$

Again, we have two claims:

Claim 1: if $MOC(\alpha(x)) \models \alpha(x)$, then $\alpha(x)$ is equivalent to the standard translation of a modal formula.

Claim 2: $MOC(\alpha(x)) \models \alpha(x)$ indeed.

.... proof of Claim 1 ....

Suppose $MOC(\alpha(x)) \models \alpha(x)$, by Compactness of FOL, there exists a *finite* subset X of $MOC(\alpha(x))$ such that $X \models \alpha(x)$. Hence $\models \bigwedge X \to \alpha(x)$, moreover, $\models \alpha(x) \to \bigwedge X$ (be the definition of $MOC(\alpha(x))$), thus $\models \alpha(x) \leftrightarrow \bigwedge X$. But $\bigwedge X$ is the standard translation of some modal formula, then Claim 1 is deserved.

.... proof of Claim 2 ....

Assume $\mathfrak{M} \models MOC(\alpha(x))[w]$, it suffices to show that $\mathfrak{M} \models \alpha(x)[w]$.

Considering the modal theory Γ in $\mathfrak{M}, w$, that is:

$$\Gamma = Th(\mathfrak{M}, w) \coloneqq \{\varphi \in \mathcal{L}_\Diamond \mid \mathfrak{M}, w \Vdash \varphi\},$$

let

$$ST_x(\Gamma) = \{ST_x(\varphi) \mid \varphi \in \Gamma\}.$$

It easy to check that, by compactness argument (in a similar way in the previous section, page 2), $ST_x(\Gamma) \cup \{\alpha(x)\}$ is satisfiable.

Suppose $\mathfrak{N} \models ST_x(\Gamma) \cup \{\alpha(x)\}[v]$ for some $\mathfrak{N}, v$. By local correspondence, $\mathfrak{N}, v \Vdash \Gamma$, thus $\mathfrak{M}, w \leftrightsquigarrow \mathfrak{N}, v$.

By Detour Lemma, for the ultrapowers $\prod_U \mathfrak{M}$ and $\prod_U \mathfrak{N}$ (U is a *countably incomplete* ultrafilter) of $\mathfrak{M}$ and $\mathfrak{N}$ respectively, we have

$$\begin{array}{llcll}
4.\ \alpha(x) & \mathfrak{M}, w & \leftrightsquigarrow & \mathfrak{N}, v & 1.\ \alpha(x) \\
 & \equiv_{FOL} & & \equiv_{FOL} & \\
3.\ \alpha(x) & \prod_U \mathfrak{M}, (f_w)_U & \leftrightsquigarrow \ = \ \underline{\leftrightarrow} & \prod_U \mathfrak{N}, (f_v)_U & 2.\ \alpha(x)
\end{array}$$

The reason for $\leftrightsquigarrow \ = \ \underline{\leftrightarrow}$ in above is that those two ultrapowers are ω-saturated, hence m-saturated. And $\equiv_{FOL}$ since there are elementary embeddings (i.e. the *diagonal mapping*).

Since $\mathfrak{N} \models \alpha(x)[v]$, then by the assumption (that is $\alpha(x)$ is invariant under bisimulation) and along the path 1-2-3-4, we have $\mathfrak{M} \models \alpha(x)[w]$.

....

This proves the theorem. ■

Section 3

Summary

A summary of previous proofs:

Summary

- highly non-trivial and non-constructive.
- using heavy constructions w.r.t. FOL, not “modal”enough.
- using compactness of FOL.