Notes on Modal Logic

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Textbook: the Blue Book
Recommended reading: Davey and Priestley, *Introduction to Lattices and Order*, CUP 2nd edition, 2002.

陈老师教授的方法论:

Definition....
⋮
Example....
⋮
Proposition...
⋮
Lemma...
⋮
Theorem...
⋮
Corollary...
⋮

Table 1: 文章的一般结构

中间的内容一般是说明性的，或者是过渡段。但有时候这些内容也会影响对概念的理解。

Contents

1 Basic Concepts

1.1 Relational structures

定义 1.1 (relational structures)**.** A **relational structure** is a tuple $\mathfrak{F} = (W, R_i)_{i \in I}$, where $W \neq \emptyset$ and $R_i \subseteq W^n$ is a n-ary relation on W for each $i \in I \neq \emptyset$ and $n \in \mathbb{N}$. ⊣

Note:

1. R_i can with arbitrary arity.

2. $\mathfrak{F}$ contains at least one relation since $I \neq \emptyset$.

There are many examples for relational structure (W, R):

- *strict partial order*: irreflexive + transitive
- *linear order* (*total order*): irreflexive + transitive + trichotomy
- *partial order*: transitive + reflexive + antisymmetric
-

定义 1.2 (reflexive closure and transitive closure)**.** For any *binary* relation R on a non-empty set W,

- R^+, the **reflexive closure** of R is the smallest transitive relation on W that contains R.
- R^*, the **reflexive transitive closure** of R is the smallest reflexive and transitive relation on W containing R.

⊣

命题 1.3. For any binary relation R on W:

1. $R^+ = \bigcap\{R' \subseteq W \mid R' \text{ is transitive } \& \ R \subseteq R'\}$

2. $R^* = \bigcap\{R' \subseteq W \mid R' \text{ is transitive and reflexive } \& \ R \subseteq R'\}$

3. $R^+uv \ \Leftrightarrow$ there is a sequence $u = w_0, w_1, \ldots, w_n = v$ $(n > 0)$ such that Rw_iw_{i+1} for each $i < n$. (R^+uv means that v is reachable from u in a finite number of R-steps)

4. $R^*uv \ \Leftrightarrow \ u = v$ or there is a sequence $u = w_0, w_1, \ldots, w_n = v$ $(n > 0)$ such that Rw_iw_{i+1} for each $i < n$. (R^+uv means that $u = v$ or v is reachable from u in a finite number of R-steps)

⊣

Proof. 内容... ■

Selected exercises:

1.2 Modal languages

定义 1.4 (Basic modal language)**.** Given a set of countable number of propositional variables Prop and an unary modal operator $\Diamond$. The **basic modal language** $\mathcal{L}_\Diamond$ is given by following BNF rule:

$$\mathcal{L}_\Diamond \ni \varphi ::= p \mid \bot \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \Diamond\varphi$$

where $p \in \mathsf{Prop}$. $\dashv$

NB: Because the bottom $\bot \notin \mathsf{Prop}$, then $\mathcal{L}_\Diamond \neq \emptyset$ if $\mathsf{Prop} = \emptyset$.

定义 1.5 (Modal similarity type)**.** A **modal similarity type** is a pair $\tau = (O, \rho)$ where O is a non-empty set of modal operators and $\rho\colon O \to \mathbb{N}$ assigns to each modal operator a finite *arity*. $\dashv$

定义 1.6 (Modal language under τ)**.** Given a modal similarity type τ and Prop, the **model language** $\mathcal{L}_{(\tau,\mathsf{Prop})}$ is defined by following BNF rule:

$$\mathcal{L}_{(\tau,\mathsf{Prop})} \ni \varphi ::= p \mid \bot \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \triangle(\varphi_1, \dots, \varphi_{\rho(\triangle)})$$

where $p \in \mathsf{Prop}$ and $\triangle \in \tau$. $\dashv$

Dual operators (*nabla*):

$$\nabla(\varphi_1, \dots, \varphi_n) \coloneqq \neg\triangle(\neg\varphi_1, \dots, \neg\varphi_n)$$

注记 1.7.

1. the name of *similarity type* is from *universal algebra*.
2. τ 说明了一个语言的模态词有哪些以及这些模态词的元数.

$\dashv$

定义 1.8 (Substitution)**.** Given a modal language $\mathcal{L}_{(\tau,\mathsf{Prop})}$, a **substitution** is a function $\sigma\colon \mathsf{Prop} \to \mathcal{L}_{(\tau,\mathsf{Prop})}$. We can extend a substitution by $(\cdot)^\sigma\colon \mathcal{L}_{(\tau,\mathsf{Prop})} \to \mathcal{L}_{(\tau,\mathsf{Prop})}$ which recursively define as follows:

$$\begin{array}{rcl} p^\sigma & = & \sigma(p) \\ \bot^\sigma & = & \bot \\ (\neg\varphi)^\sigma & = & \neg\varphi^\sigma \\ (\varphi \vee \psi) & = & \varphi^\sigma \vee \psi^\sigma \\ (\triangle(\varphi_1, \dots, \varphi_n))^\sigma & = & \triangle(\varphi_1^\sigma, \dots, \varphi_n^\sigma) \end{array}$$

Saying that χ is a **substitution instance** of φ if there is some substitution σ such that $\chi = \varphi^\sigma$. $\dashv$

1.3 Models and Frames

When talking about model/frame we often say that, a model/frame for *some language*.

For basic language

定义 1.9 (Modal and frame for basic modal language $\mathcal{L}_\Diamond$)**.** A **frame** for $\mathcal{L}_\Diamond$ is a pair $\mathfrak{F} = (W, R)$ where $W \neq \emptyset$ and $R \subseteq W \times W$.

A **model** for $\mathcal{L}_\Diamond$ is structure $\mathfrak{M} = (W, R, V)$, where (W, R) is a frame and V , called a **valuation**, is a map: $\mathsf{Prop} \to \wp(W)$.

Given a model $\mathfrak{M} = (\mathfrak{F}, V)$, we say that $\mathfrak{M}$ is *based on* $\mathfrak{F}$, and $\mathfrak{F}$ is the frame *underlying* $\mathfrak{M}$. $\dashv$

注记 1.10. A benefit of the definition of V is that, a model can be viewed as a *first-order structure* (or a relational structure) in a natural way, namely

$$\mathfrak{M} = (W, R, V(p), V(q), V(r), \dots)$$

where $V(p)$ is an unary relation on W, i.e., a *predicate*, also for $V(q), V(r), \dots$

But there are many other ways to define valuation, maybe not equivalent. $\dashv$

定义 1.11 (Satisfiability)**.** For any model $\mathfrak{M} = (W, R, V)$ and $w \in W$, a formula φ **satisfied** in $(\mathfrak{M}, w)$, notation $\mathfrak{M}, w \Vdash \varphi$, recursively define as follows:

$$\begin{array}{lll} \mathfrak{M}, w \Vdash p & :\Leftrightarrow & w \in V(p) \qquad p \in \mathsf{Prop} \\ \mathfrak{M}, w \Vdash \bot & & never \\ \mathfrak{M}, w \Vdash \neg\varphi & :\Leftrightarrow & \mathfrak{M}, w \nVdash \varphi \\ \mathfrak{M}, w \Vdash \varphi \vee \psi & :\Leftrightarrow & \mathfrak{M}, w \Vdash \varphi \ or\ \mathfrak{M}, w \Vdash \psi \\ \mathfrak{M}, w \Vdash \Diamond\varphi & :\Leftrightarrow & \exists v \in W, Rwv, \mathfrak{M}, v \Vdash \varphi \end{array}$$

A formula φ is **satisfiable** if there is a model $\mathfrak{M}$ and some state w in $\mathfrak{M}$ such that $\mathfrak{M}, w \Vdash \varphi$. ⊣

定义 1.12 (Truth set)**.** Given a model $\mathfrak{M} = (W, R, V)$, the **truth set** of φ in $\mathfrak{M}$ is given by:

$$[\![\varphi]\!]_{\mathfrak{M}} \coloneqq \{w \in W \mid \mathfrak{M}, w \Vdash \varphi\}$$

⊣

命题 1.13. Given a model $\mathfrak{M} = (W, R, V)$, then

$$[\![p]\!]_{\mathfrak{M}} = V(p) \qquad [\![\bot]\!]_{\mathfrak{M}} = \emptyset \qquad [\![\neg\varphi]\!]_{\mathfrak{M}} = W \setminus [\![\varphi]\!]_{\mathfrak{M}} \qquad [\![\varphi \vee \psi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathfrak{M}} \cup [\![\psi]\!]_{\mathfrak{M}}$$

$$[\![\Diamond\varphi]\!]_{\mathfrak{M}} = \{w \in W \mid \exists v, Rwv, v \in [\![\varphi]\!]_{\mathfrak{M}}\}$$

$$[\![\Box\varphi]\!]_{\mathfrak{M}} = \{w \in W \mid \forall v, Rwv \Rightarrow v \in [\![\varphi]\!]_{\mathfrak{M}}\}$$

⊣

For more general language

$$\begin{array}{lll} \mathfrak{M}, w \Vdash \triangle(\varphi_1, \ldots, \varphi_n) & :\Leftrightarrow & \exists v_1, \ldots, v_n \in W, (w, v_1, \ldots, v_n) \in R_\triangle, \forall i \in \{1, 2, \ldots, n\}, \mathfrak{M}, v_i \Vdash \varphi_i \\ \mathfrak{M}, w \Vdash \nabla(\varphi_1, \ldots, \varphi_n) & :\Leftrightarrow & \forall v_1, \ldots, v_n \in W, (w, v_1, \ldots, v_n) \in R_\triangle \Rightarrow \exists i \in \{1, 2, \ldots, n\}, \mathfrak{M}, v_i \Vdash \varphi_i \\ \mathfrak{M}, w \Vdash \bigcirc & :\Leftrightarrow & w \in R_\bigcirc \end{array}$$

where $\bigcirc$ is a *nullary modality*.

注记 1.14. Graded modality $\Diamond^{\geq n}$ is a good example to understood this general definition.

Validity

定义 1.15 (Validity and Logic)**.** There are different validity on different levels.

1. $\mathfrak{F}, w \Vdash \varphi$: $\quad \forall V \in \wp(W)^{\mathsf{Prop}}$[1], $(\mathfrak{F}, V), w \Vdash \varphi$.
2. $\mathfrak{F} \Vdash \varphi$: $\quad \forall w \in W, (\mathfrak{F}, w) \Vdash \varphi$.
3. $\mathsf{F} \Vdash \varphi$: $\quad \forall \mathfrak{F} \in \mathsf{F}, \mathfrak{F} \Vdash \varphi$.
4. $\Vdash \varphi$: $\quad \forall \mathfrak{F}, \mathfrak{F} \Vdash \varphi$.

The set of all valid formulae in a class of frame F is called the **logic of** F, notation Λ_{F}, that is $\Lambda_{\mathsf{F}} \coloneqq \{\varphi \mid \mathsf{F} \Vdash \varphi\}$. ⊣

1.4 General Frames (skip)

1.5 Modal Consequence Relation

定义 1.16 (Local semantic consequence)**.** Let S be a class of models or frames, for any formula φ and set of formulae Σ. We say φ is a **local semantic consequence** of Σ over S, notation $\Sigma \Vdash_{\mathsf{S}} \varphi$, if for all models $\mathfrak{M}$ in S and all states w in $\mathfrak{M}$: $\mathfrak{M}, w \Vdash \Sigma \Rightarrow \mathfrak{M}, w \Vdash \varphi$. ⊣

定义 1.17 (Global semantic consequence)**.** Let S be a class of models or frames, for any formula φ and set of formulae Σ. We say φ is a **gocal semantic consequence** of Σ over S, notation $\Sigma \Vdash^g_{\mathsf{S}} \varphi$, if for all structure $\mathfrak{G}$ in S ($\mathfrak{G}$ could be a model or a frame): $\mathfrak{G} \Vdash \Sigma \Rightarrow \mathfrak{G} \Vdash \varphi$. ⊣

[1] For any set A, B, $B^A \coloneqq \{f \mid f\colon A \to B\}$.

1.6 Normal Modal Logics

定义 1.18 (Axiom system **K**)**.** The axiom system **K** is containing following axioms and rules:

- Axioms
  1. **PC**: all propositional tautologies;
  2. K: $\Box(p \to q) \to (\Box p \to \Box q)$ (also known as *distribution axiom*)
  3. Dual: $\Diamond p \leftrightarrow \neg\Box\neg p$
- Rules
  1. MP: $\dfrac{\varphi \to \psi, \varphi}{\psi}$
  2. Sub: $\dfrac{\varphi}{\varphi^\sigma}$ where σ is a substitution
  3. $\mathrm{Gen}_\Box$: $\dfrac{\varphi}{\Box\varphi}$

A **K-proof** is a finite sequence of formulae $\varphi_1, \ldots, \varphi_n$, for each φ_i ($1 \leq i \leq n$), either φ_i is an axiom of **K**, or φ_i is obtained by one or more earlier formulae in the sequence by applying a rule of **K**.

If $\varphi_1, \ldots, \varphi_n$ is a **K**-proof and $\varphi = \varphi_n$, then we say that φ is **K**-provable, notation $\vdash_{\mathbf{K}} \varphi$, and say φ is a **theorem** of **K**. $\dashv$

注记 1.19. There are some comments on the three rules:

- MP:
  1. MP preserves *validity*: $\Vdash \varphi \to \psi, \Vdash \varphi \Rightarrow \Vdash \psi$
  2. MP preserves *satisfiability*: $\mathfrak{M}, w \Vdash \varphi \to \psi, \mathfrak{M}, w \Vdash \varphi \Rightarrow \mathfrak{M}, w \Vdash \psi$
  3. MP preserves *global truth*: $\mathfrak{M} \Vdash \varphi \to \psi, \mathfrak{M} \Vdash \varphi \Rightarrow \mathfrak{M} \Vdash \psi$
- Sub:
  1. Sub preserves *validity*: $\Vdash \varphi \Rightarrow \Vdash \varphi^\sigma$
  2. Sub not preserve *satisfiability*
  3. Sub not preserve *global truth*
- $\mathrm{Gen}_\Box$
  1. $\mathrm{Gen}_\Box$ preserves *validity*: $\Vdash \varphi \Rightarrow \Vdash \Box\varphi$
  2. $\mathrm{Gen}_\Box$ not preserve *satisfiability*
  3. $\mathrm{Gen}_\Box$ preserves *global truth*: $\mathfrak{M} \Vdash \varphi \Rightarrow \mathfrak{M} \Vdash \Box\varphi$

$\dashv$

定义 1.20 (Normal modal logics)**.** A **normal modal logic** Λ is a set of formulae that contains all tautologies, K-axiom, Dual-axiom and is closed under MP, Sub and $\mathrm{Gen}_\Box$.

The smallest normal modal logic is **K**. $\dashv$

命题 1.21. Let F be a class of frames, then $\Lambda_{\mathsf{F}} := \{\varphi \mid \mathsf{F} \Vdash \varphi\}$ is a normal modal logic. $\dashv$

Proof. See here . ■

1.7 Selected exercises for Ch.1

1.1.1

1.1.2

1.1.3

1.3.1

1.3.4

1.3.5

1.6.7 Let F be a class of frames. Show that Λ_{F} is a normal modal logic.

Proof. Because all tautologies is valid on any frame, so is for the axioms K and Dual, then we only need to show that Λ_{F} is closed under MP, Sub and Nec.

(1) MP: if $\phi, \phi \to \psi \in \Lambda_{\mathsf{F}}$, then take any model $\mathfrak{M}$ from F and any state w in $\mathfrak{M}$ we have $\mathfrak{M}, w \models \phi$ and $\mathfrak{M}, w \models \phi \to \psi$, hence $\mathfrak{M}, w \models \psi$, because $\mathfrak{M}$ and w are arbitrary from F, then ψ is valid on F, that is $\psi \in \Lambda_{\mathsf{F}}$.

★ (2) Sub: we need a lemma here:

> ***lemma***: Suppose $M = (W, R, V)$ is a model, and $\phi^\sigma = \phi[\psi_1/p_1, \cdots, \psi_n/p_n]$ is the substitution instance of ϕ under substitution σ. Define $M' = (W, R, V')$ by $V'(p_i) = \{w \in W \mid M, w \Vdash \psi_i\}$. Then for any $w \in W$:
> $$M, w \Vdash \phi^\sigma \;\Leftrightarrow\; M', w \Vdash \phi.$$

Assume $\phi \in \Lambda_{\mathsf{F}}$, that is, $\mathsf{F} \Vdash \phi$, but $\phi^\theta \notin \Lambda_{\mathsf{F}}$ for some substitution θ, i.e $\mathsf{F} \nVdash \phi^\theta$. Then for some model $M = (W, R, V)$ from F and some $w \in W$ we have $M, w \nVdash \phi^\theta$, hence $M', w \nVdash \phi$ by above lemma, but this is contradicts to ϕ is valid in F. Therefore, if $\phi \in \Lambda_{\mathsf{F}}$ then $\phi^\theta \in \Lambda_{\mathsf{F}}$ for any substitution θ.

(3) Nec: suppose $\phi \in \Lambda_{\mathsf{F}}$ but $\Box\phi \notin \Lambda_{\mathsf{F}}$, then there are a frame $F = (W, R)$ from F, a valuation V and a state $w \in W$ such that $(F, V), w \nVdash \Box\phi$. Hence there must be a state $u \in W$ for which Rwu and $(F, V), u \Vdash \neg\phi$, but this contradicts with ϕ is valid on F. Therefore $\Box\phi \in \Lambda_{\mathsf{F}}$ ■

homework No.02 (2023.03,01)

1.3.1 Show that when evaluating a formula ϕ in a model, the only relevant information in the valuation is the assignments it makes to the propositional letters actually occurring in ϕ. More precisely, let $\mathfrak{F}$ be a frame, and V and V' be two valuations on $\mathfrak{F}$ such that$V(p) = V'(p)$ for all proposition letters p in ϕ. Show that $(\mathfrak{F}, V) \Vdash \phi$ iff $(\mathfrak{F}, V') \Vdash \phi$. Work in the basic modal language.

Proof. Let $\mathfrak{F} = (W, R)$, V and V' are two valuations as mentioned above, we firstly prove the following lemma by induction on ϕ:

$$(*) \quad \forall w \in W : (\mathfrak{F}, V), w \Vdash \phi \;\Leftrightarrow\; (\mathfrak{F}, V'), w \Vdash \phi.$$

Basic cases:

- If ϕ is a propositional letter p, then for all $w \in W$

$$\begin{array}{rcll} (\mathfrak{F}, V), w \Vdash p & \Leftrightarrow & w \in V(p), & \text{(by definition)} \\ & \Leftrightarrow & w \in V'(p), & \text{(by assumption)} \\ & \Leftrightarrow & (\mathfrak{F}, V'), w \Vdash p. & \text{(by definition)} \end{array}$$

- If $\phi = \bot$, then for all $w \in W$, $(\mathfrak{F}, V), w \Vdash \phi \;\Leftrightarrow\; (\mathfrak{F}, V'), w \Vdash \phi$ trivially.

Inductive steps:
If ϕ is of the form $\neg\psi$ or $\psi \vee \chi$, this is easily done. The crucial case is the form $\Diamond\psi$.

$$\begin{array}{rcll} (\mathfrak{F}, V), w \Vdash \Diamond\psi & \Leftrightarrow & \exists v, Rwv, (\mathfrak{F}, V), v \Vdash \psi, & \text{(by definition)} \\ & \Leftrightarrow & \exists v, Rwv, (\mathfrak{F}, V'), v \Vdash \psi, & \text{(by induction hypothesis)} \\ & \Leftrightarrow & (\mathfrak{F}, V'), w \Vdash \Diamond\psi. & \text{(by definition)} \end{array}$$

Then the proposition

$$(\mathfrak{F}, V) \Vdash \phi \;\Leftrightarrow\; (\mathfrak{F}, V') \Vdash \phi$$

is just a corollary of $(*)$. ∎

1.3.4 Show that every formula that has the form of a propositional tautology is valid. Further, show that $\Box(p \to q) \to (\Box p \to \Box q)$ is valid.

Proof.
(1) (we only work in the basic modal language here)
Firstly, we give a formal definition for what is a formula has the form of a propositional tautology.
Definition (tautology) : A formula ϕ is called a *tautology* (shouldn't be confused with *proposition tautology*), if $\phi = \alpha^\sigma$ where σ is a substitution, α is a formula of propositional logic and α is a propositional tautology.
Therefore we have to show that:

$$(\circledast) \qquad \text{Every tautology is valid.}$$

To do that, we need following lemma in the first place.

> **Lemma 1** Suppose θ is a modal-free formula whose propositional variables are $p_1, \ldots, p_n$, let $\phi_1, \ldots, \phi_n$ be modal formulas, and σ is a substitution such that $\sigma(p_i) = \phi_i$ for each $1 \leq i \leq n$, that is $[\phi_i/p_i, \cdots, \phi_n/p_n]$ in another notation. If for any propositional assignment v, any model $M = (W, R, V)$, and any $w \in W$ such that $v(p_i) = 1$ iff $M, w \Vdash \phi_i$, then $v \models \theta$ iff $M, w \Vdash \theta^\sigma$.

We will prove lemma 1 by induction on θ (propositional logic formula).
Basic cases:

- if $\theta = \bot$, then $\bot^\sigma = \bot$, both $v \not\models \bot$ and $M, w \nVdash \bot$.

- if $\theta = p_i$, then

$$\begin{aligned} v \models p_i \quad &\Leftrightarrow \quad v(p_i) = 1 \\ &\Leftrightarrow \quad M, w \Vdash \phi_i \quad \text{(by assumption)} \\ &\Leftrightarrow \quad M, w \Vdash p_i^\tau \quad (\text{since} \quad p_i^\sigma = \sigma(p_i) = \phi_i, \text{by the definition of } \sigma). \end{aligned}$$

Inductive steps

- if $\theta = \neg\chi$, then

$$\begin{aligned} v \models \neg\chi \quad &\Leftrightarrow \quad v \not\models \chi \\ &\Leftrightarrow \quad M, w \not\Vdash \chi^\sigma \qquad \text{(by induction hypothesis)} \\ &\Leftrightarrow \quad M, w \Vdash \neg\chi^\sigma \\ &\Leftrightarrow \quad M, w \Vdash (\neg\chi)^\sigma \quad \text{(by the definition of substitution)} \end{aligned}$$

- if $\theta = \psi \vee \chi$, then

$$\begin{aligned} v \models (\psi \vee \chi) \quad &\Leftrightarrow \quad v \models \psi \text{ or } v \models \chi \\ &\Leftrightarrow \quad M, w \Vdash \psi^\sigma \text{ or } M, w \Vdash \chi^\sigma \quad \text{(by induction hypothesis)} \\ &\Leftrightarrow \quad M, w \Vdash \psi^\sigma \vee \chi^\sigma \\ &\Leftrightarrow \quad M, w \Vdash (\psi \vee \chi)^\sigma \qquad \text{(by the definition of substitution)} \end{aligned}$$

Hence we complete the induction proof for **Lemma 1**.

Then we prove $(\circledast)$ by contraposition.
Suppose φ is a tautology but not valid,
then by the definition of tautology above,
there is a proposition tautology θ and a substitution σ such that $\varphi = \theta^\sigma$ is invalid.
Namely $M, w \not\Vdash \theta^\sigma$ for some model M and some state w in M.
Moreover, we assume only $p_i, \ldots, p_n$ are occurring in θ,
and V satisfies $v(p_i) = \phi_i$ for each $1 \leq i \leq n$.
Now we define a propositional assignment v such that

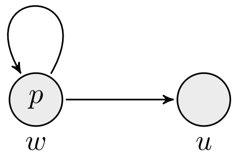
$$v(p_i) = 1 \;\Leftrightarrow\; M, w \Vdash \phi_i$$

Then, by **lemma 1**, we have that : $v \models \theta \;\Leftrightarrow\; M, w \Vdash \theta^\sigma$.
Since $M, w \not\Vdash \theta^\sigma$, therefore $v \not\models \theta$.
But this contradicts with θ is a proposition tautology.
Consequently, $(\circledast)$ is holds, that is, every tautology is valid.

(2)
Following we show that $\Box(p \to q) \to (\Box p \to \Box q)$ is valid.
Take any frame $\mathfrak{F}$ and any state w in $\mathfrak{F}$, and let V be a valuation on $\mathfrak{F}$.
We have to show that if $(\mathfrak{F}, V), w \Vdash \Box(p \to q)$ and $(\mathfrak{F}, V), w \Vdash \Box p$, then $(\mathfrak{F}, V), w \Vdash \Box q$.
So assume that $(\mathfrak{F}, V), w \Vdash \Box(p \to q)$ and $(\mathfrak{F}, V), w \Vdash \Box p$.
Then, by definition for any state v such that Rwv we have $(\mathfrak{F}, V), v \Vdash p \to q$ and $(\mathfrak{F}, V), v \Vdash p$,
hence $(\mathfrak{F}, V), v \Vdash q$, but since Rwv and v is an arbitrary state,
then by definition we have $(\mathfrak{F}, V), w \Vdash \Box q$. ∎

1.3.5 Show that every formula of the following formulas is not valid by constructing a frame $\mathfrak{F} = (W, R)$ that refutes it.
(a) $\Box\bot$ (b) $\Diamond p \to \Box p$ (c) $p \to \Box\Diamond p$ (d) $\Diamond\Box p \to \Box\Diamond p$.

Proof. Let's consider following frame $\mathfrak{F}$, then we show that this frame refutes all above formulas.
Let $\mathfrak{F} = (W, R)$ where $W = \{w, u\}$ and $R = \{(w, w), (w, u)\}$,
we visualize $\mathfrak{F}$ (with a valuation) as follows:



Now we define a valuation V on $\mathfrak{F}$ by

$$V(q) = \begin{cases} \{w\} & q = p \\ \emptyset & q \neq p \end{cases}$$

We use $w \Vdash \varphi$ instead of $(\mathfrak{F}, V), w \Vdash \varphi$ for convenience. Then we know:
$w \Vdash \Diamond p$ since Rww and $w \Vdash p$;
$w \nVdash \Box p$ since Rwv but $u \nVdash p$;
$w \nVdash \Box\Diamond p$ since Rwu but u has no successors, which means $u \nVdash \Diamond p$;
$w \Vdash \Diamond\Box p$ since Rwu and v is a '*dead end*', that is $u \Vdash \Box p$.
Then for those four formulas:
(a) $w \nVdash \Box\bot$ since Rwu but $u \nVdash \bot$;
(b) $w \nVdash \Diamond p \to \Box p$ since $w \Vdash \Diamond p$ but $w \nVdash \Box p$
(c) $w \nVdash p \to \Box\Diamond p$ since $w \Vdash p$ but $w \nVdash \Box\Diamond p$
(d) $w \nVdash \Diamond\Box p \to \Box\Diamond p$ since $w \Vdash \Diamond\Box p$ but $w \nVdash \Box\Diamond p$ ■

Show that **K** *is sound with respect to the class of all frames*.

Proof. We already known that:
(1) All axioms of **K** are valid.
(all tautologies are valid and the K-axiom is valid (see exercise 1.3.4, p27), moreover the Dual-axiom is valid (see the discussion in paragraph 5 of p34))
(2) Furthermore, we assume that all rules of **K** are preserve validity, we will give a proof in the last.

Then to show **K** is *sound*, it is sufficient to show that all **K**-provable formulas are valid.
For any formula φ, suppose φ is **K**-provable,
then there is finite a sequence of formulas $\psi_1, \ldots, \psi_n$ such that $\varphi = \psi_n$.
By induction on n.
Basic case:

- If $n = 1$, then by the definition of **K**-proof, that means φ is an axiom of **K**,
  but all axioms of **K** are valid, hence φ is valid.

Inductive step: Suppose φ has a proof of length $n > 1$.

- If φ is an axiom of **K**, then φ is valid as same as basic case.

- If φ is obtained by MP from previous formulas $\chi \to \varphi$ and χ,
  by induction hypothesis, $\chi \to \varphi$ and χ are valid,
  and MP preserves validity, hence φ is valid.

- If φ is obtained by Sub or Gen$_\Box$ from χ,
  by inductive hypothesis, χ is valid,
  and Sub or Gen$_\Box$ both preserve validity, therefore φ is valid.

From basic case and inductive step, we complete the induction proof.

In the end, we show that *modus ponens* (MP), *uniform substitution* (Sub) and *Generalization* (Gen$_\Box$) are preserve validity.

- For MP.
  That is to show: if $\varphi \to \psi$ and ψ are valid, then so is ψ.

  Suppose $\Vdash \phi, \Vdash \phi \to \psi$,
  Then $M, w \models \phi$ and $M, w \models \phi \to \psi$ for some model M and some w in M since $\varphi \to \psi, \varphi$ are valid.
  Hence $M, w \models \psi$ by the definition.
  Therefore $\Vdash \psi$ because M and w are arbitrary.

- For Gen$_\Box$.
  That is to show: if φ is valid, then so is $\Box\varphi$.

  Assume $\Vdash \varphi$. To show $\Vdash \Box\varphi$, let $M = (W, R, V)$ be any model and $w \in W$.
  For any $u \in W$, if Rwu then $M, u \Vdash \varphi$ since φ is valid, and hence $M, u \Vdash \Box\varphi$ by the definition.
  Since M and w are arbitrary, then $\Vdash \Box\varphi$.

- For Sub.
  That is to show: if ϕ is valid, then so is ϕ^σ for any substitution σ.

  First we need a lemma:

  > **lemma**: Suppose ϕ only contains $p_1, \ldots, p_n$ as its propositional letters, and ϕ^σ is the substitution instance of ϕ under substitution σ, where $\sigma(p_i) = \psi_i$ for each $1 \leq i \leq n$.
  > For any models $M = (W, R, V)$, define $M' = (W, R, V')$ by $V'(p_i) = \{w \in W \mid M, w \Vdash \psi_i\}$. Then for any $w \in W$:
  > $$M, w \Vdash \phi^\sigma \;\Leftrightarrow\; M', w \Vdash \phi.$$
  >
  > -----
  >
  > Proving this lemma by induction on ϕ.
  >
  > *Basic case*:
  >
  > – if $\psi = p$, then $p_i^\sigma = \psi_i$.
  > Hence $M, w \Vdash \psi_i \;\Leftrightarrow\; M', w \Vdash p_i$ by the definition of V'.
  >
  > – if $\phi = \bot$, then $\bot^\sigma = \bot$.
  > Both $M, w \nVdash \bot$ and $M', w \nVdash \bot$.
  >
  > *Inductive step*:
  >
  > If ϕ is of the form $\neg\psi$ or $\psi \vee \chi$, this is easily done. The more crucial case is the form $\Diamond\psi$.
  >
  > – if $\phi = \Diamond\psi$,
  >
  > $$\begin{array}{rcll} M, w \Vdash (\Diamond\psi)^\sigma & \Leftrightarrow & M, w \Vdash \Diamond\psi^\sigma & \\ & \Leftrightarrow & M, u \Vdash \psi^\sigma & \text{for some } u \text{ such that } Rwu \\ & \Leftrightarrow & M', u \Vdash \psi & \text{by inductive hypothesis} \\ & \Leftrightarrow & M', w \Vdash \Diamond\psi & \text{since } Rwu \end{array}$$
  >
  > Hence we complete the induction proof of above lemma.

  Assume ϕ is valid, but ϕ^σ is invalid for some substitution σ, and $\sigma(p_i) = \psi_i$.
  Then $M, w \nVdash \phi^\sigma$ for some model $M = (W, R, V)$ and some $w \in W$,
  hence we have $M', w \nVdash \phi$ by above **lemma**,
  but this contradicts with that ϕ is valid.
  Therefore, if ϕ is valid, then so is ϕ^σ for any substitution σ.

$\blacksquare$

2 Models