Notes on Modal Logic

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Textbook: the Blue Book
Recommended reading: Davey and Priestley, *Introduction to Lattices and Order*, CUP 2nd edition, 2002.

陈锦盛老师教授的方法论:

Definition....
⋮
Example....
⋮
Proposition...
⋮
Lemma...
⋮
Theorem...
⋮
Corollary...
⋮

Table 1: 文章的一般结构

中间的内容一般是说明性的，或者是过渡段。但有时候这些内容也会影响对概念的理解。

Contents

1 Basic Concepts

1.1 Relational structures

定义 1.1 (relational structures)**.** A **relational structure** is a tuple $\mathfrak{F} = (W, R_i)_{i\in I}$, where $W \neq \emptyset$ and $R_i \subseteq W^n$ is a n-ary relation on W for each $i \in I \neq \emptyset$ and $n \in \mathbb{N}$. ⊣

Note:

1. R_i can with arbitrary arity.

2. $\mathfrak{F}$ contains at least one relation since $I \neq \emptyset$.

There are many examples for relational structure (W, R):

- *strict partial order*: irreflexive + transitive
- *linear order* (*total order*): irreflexive + transitive + trichotomy
- *partial order*: transitive + reflexive + antisymmetric
-

定义 1.2 (reflexive closure and transitive closure)**.** For any *binary* relation R on a non-empty set W,

- R^+, the **reflexive closure** of R is the smallest transitive relation on W that contains R.
- R^*, the **reflexive transitive closure** of R is the smallest reflexive and transitive relation on W containing R.

⊣

命题 1.3. For any binary relation R on W:

1. $R^+ = \bigcap\{R' \subseteq W \mid R' \text{ is transitive } \& \ R \subseteq R'\}$

2. $R^* = \bigcap\{R' \subseteq W \mid R' \text{ is transitive and reflexive } \& \ R \subseteq R'\}$

3. $R^+uv \;\Leftrightarrow\;$ there is a sequence $u = w_0, w_1, \ldots, w_n = v$ $(n > 0)$ such that Rw_iw_{i+1} for each $i < n$. (R^+uv means that v is reachable from u in a finite number of R-steps)

4. $R^*uv \;\Leftrightarrow\; u = v$ or there is a sequence $u = w_0, w_1, \ldots, w_n = v$ $(n > 0)$ such that Rw_iw_{i+1} for each $i < n$. (R^+uv means that $u = v$ or v is reachable from u in a finite number of R-steps)

⊣

Proof. 内容... ■

Selected exercises:

1.2 Modal languages

定义 1.4 (Basic modal language)**.** Given a set of countable number of propositional variables Prop and an unary modal operator $\Diamond$. The **basic modal language** $\mathcal{L}_\Diamond$ is given by following BNF rule:

$$\mathcal{L}_\Diamond \ni \varphi ::= p \mid \bot \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \Diamond\varphi$$

where $p \in \mathsf{Prop}$. ⊣

NB: Because the bottom $\bot \notin \mathsf{Prop}$, then $\mathcal{L}_\Diamond \neq \emptyset$ if $\mathsf{Prop} = \emptyset$.

定义 1.5 (Modal similarity type)**.** A **modal similarity type** is a pair $\tau = (O, \rho)$ where O is a non-empty set of modal operators and $\rho\colon O \to \mathbb{N}$ assigns to each modal operator a finite *arity*. ⊣

定义 1.6 (Modal language under τ)**.** Given a modal similarity type τ and Prop, the **model language** $\mathcal{L}_{(\tau,\mathsf{Prop})}$ is defined by following BNF rule:

$$\mathcal{L}_{(\tau,\mathsf{Prop})} \ni \varphi ::= p \mid \bot \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \triangle(\varphi_1, \dots, \varphi_{\rho(\triangle)})$$

where $p \in \mathsf{Prop}$ and $\triangle \in \tau$. $\dashv$

Dual operators (*nabla*):

$$\nabla(\varphi_1, \dots, \varphi_n) := \neg\triangle(\neg\varphi_1, \dots, \neg\varphi_n)$$

注记 1.7.

1. the name of *similarity type* is from *universal algebra*.
2. τ 说明了一个语言的模态词有哪些以及这些模态词的元数.

$\dashv$

定义 1.8 (Substitution)**.** Given a modal language $\mathcal{L}_{(\tau,\mathsf{Prop})}$, a **substitution** is a function $\sigma\colon \mathsf{Prop} \to \mathcal{L}_{(\tau,\mathsf{Prop})}$. We can extend a substitution by $(\cdot)^\sigma\colon \mathcal{L}_{(\tau,\mathsf{Prop})} \to \mathcal{L}_{(\tau,\mathsf{Prop})}$ which recursively define as follows:

$$\begin{array}{rcl} p^\sigma & = & \sigma(p) \\ \bot^\sigma & = & \bot \\ (\neg\varphi)^\sigma & = & \neg\varphi^\sigma \\ (\varphi \vee \psi) & = & \varphi^\sigma \vee \psi^\sigma \\ (\triangle(\varphi_1, \dots, \varphi_n))^\sigma & = & \triangle(\varphi_1^\sigma, \dots, \varphi_n^\sigma) \end{array}$$

Saying that χ is a **substitution instance** of φ if there is some substitution σ such that $\chi = \varphi^\sigma$. $\dashv$

1.3 Models and Frames

When talking about model/frame we often say that, a model/frame for *some language*.

For basic language

定义 1.9 (Modal and frame for basic modal language $\mathcal{L}_\Diamond$)**.** A **frame** for $\mathcal{L}_\Diamond$ is a pair $\mathfrak{F} = (W, R)$ where $W \neq \emptyset$ and $R \subseteq W \times W$.

A **model** for $\mathcal{L}_\Diamond$ is structure $\mathfrak{M} = (W, R, V)$, where (W, R) is a frame and V , called a **valuation**, is a map: $\mathsf{Prop} \to \wp(W)$.

Given a model $\mathfrak{M} = (\mathfrak{F}, V)$, we say that $\mathfrak{M}$ is *based on* $\mathfrak{F}$, and $\mathfrak{F}$ is the frame *underlying* $\mathfrak{M}$. $\dashv$

注记 1.10. A benefit of the definition of V is that, a model can be viewed as a *first-order structure* (or a relational structure) in a natural way, namely

$$\mathfrak{M} = (W, R, V(p), V(q), V(r), \dots)$$

where $V(p)$ is an unary relation on W, i.e., a *predicate*, also for $V(q), V(r), \dots$

But there are many other ways to define valuation, maybe not equivalent. $\dashv$

定义 1.11 (Satisfiability)**.** For any model $\mathfrak{M} = (W, R, V)$ and $w \in W$, a formula φ **satisfied** in $(\mathfrak{M}, w)$, notation $\mathfrak{M}, w \Vdash \varphi$, recursively define as follows:

$$\begin{array}{llll} \mathfrak{M}, w \Vdash p & :\Leftrightarrow & w \in V(p) & p \in \mathsf{Prop} \\ \mathfrak{M}, w \Vdash \bot & & never & \\ \mathfrak{M}, w \Vdash \neg\varphi & :\Leftrightarrow & \mathfrak{M}, w \nVdash \varphi & \\ \mathfrak{M}, w \Vdash \varphi \vee \psi & :\Leftrightarrow & \mathfrak{M}, w \Vdash \varphi \ or\ \mathfrak{M}, w \Vdash \psi & \\ \mathfrak{M}, w \Vdash \Diamond\varphi & :\Leftrightarrow & \exists v \in W, Rwv, \mathfrak{M}, v \Vdash \varphi & \end{array}$$

A formula φ is **satisfiable** if there is a model $\mathfrak{M}$ and some state w in $\mathfrak{M}$ such that $\mathfrak{M}, w \Vdash \varphi$. $\dashv$

定义 1.12 (Truth set)**.** Given a model $\mathfrak{M} = (W, R, V)$, the **truth set** of φ in $\mathfrak{M}$ is given by:

$$[\![\varphi]\!]_{\mathfrak{M}} := \{w \in W \mid \mathfrak{M}, w \Vdash \varphi\}$$

$\dashv$

命题 1.13. Given a model $\mathfrak{M} = (W, R, V)$, then

$$[\![p]\!]_{\mathfrak{M}} = V(p) \qquad [\![\bot]\!]_{\mathfrak{M}} = \emptyset \qquad [\![\neg\varphi]\!]_{\mathfrak{M}} = W \setminus [\![\varphi]\!]_{\mathfrak{M}} \qquad [\![\varphi \vee \psi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathfrak{M}} \cup [\![\psi]\!]_{\mathfrak{M}}$$
$$[\![\Diamond\varphi]\!]_{\mathfrak{M}} = \{w \in W \mid \exists v, Rwv, v \in [\![\varphi]\!]_{\mathfrak{M}}\}$$
$$[\![\Box\varphi]\!]_{\mathfrak{M}} = \{w \in W \mid \forall v, Rwv \Rightarrow v \in [\![\varphi]\!]_{\mathfrak{M}}\}$$

$\dashv$

For more general language

$$\begin{array}{lll}
\mathfrak{M}, w \Vdash \triangle(\varphi_1, \ldots, \varphi_n) & :\Leftrightarrow & \exists v_1, \ldots, v_n \in W, (w, v_1, \ldots, v_n) \in R_\triangle, \forall i \in \{1, 2, \ldots, n\}, \mathfrak{M}, v_i \Vdash \varphi_i \\
\mathfrak{M}, w \Vdash \nabla(\varphi_1, \ldots, \varphi_n) & :\Leftrightarrow & \forall v_1, \ldots, v_n \in W, (w, v_1, \ldots, v_n) \in R_\triangle \Rightarrow \exists i \in \{1, 2, \ldots, n\}, \mathfrak{M}, v_i \Vdash \varphi_i \\
\mathfrak{M}, w \Vdash \bigcirc & :\Leftrightarrow & w \in R_\bigcirc
\end{array}$$

where $\bigcirc$ is a *nullary modality*.

注记 1.14. Graded modality $\Diamond^{\geq n}$ is a good example to understood this general definition.

Validity

定义 1.15 (Validity and Logic)**.** There are different validity on different levels.

1. $\mathfrak{F}, w \Vdash \varphi$: $\quad \forall V \in \wp(W)^{\mathsf{Prop}}$[1]$, (\mathfrak{F}, V), w \Vdash \varphi$.
2. $\mathfrak{F} \Vdash \varphi$: $\quad \forall w \in W, (\mathfrak{F}, w) \Vdash \varphi$.
3. $\mathsf{F} \Vdash \varphi$: $\quad \forall \mathfrak{F} \in \mathsf{F}, \mathfrak{F} \Vdash \varphi$.
4. $\Vdash \varphi$: $\quad \forall \mathfrak{F}, \mathfrak{F} \Vdash \varphi$.

The set of all valid formulae in a class of frame F is called the **logic of** F, notation Λ_{F}, that is $\Lambda_{\mathsf{F}} := \{\varphi \mid \mathsf{F} \Vdash \varphi\}$. $\dashv$

1.4 General Frames (skip)

1.5 Modal Consequence Relation

定义 1.16 (Local semantic consequence)**.** Let S be a class of models or frames, for any formula φ and set of formulae Σ. We say φ is a **local semantic consequence** of Σ over S, notation $\Sigma \Vdash_{\mathsf{S}} \varphi$, if for all models $\mathfrak{M}$ in S and all states w in $\mathfrak{M}$: $\mathfrak{M}, w \Vdash \Sigma \Rightarrow \mathfrak{M}, w \Vdash \varphi$. $\dashv$

定义 1.17 (Global semantic consequence)**.** Let S be a class of models or frames, for any formula φ and set of formulae Σ. We say φ is a **gocal semantic consequence** of Σ over S, notation $\Sigma \Vdash^g_{\mathsf{S}} \varphi$, if for all structure $\mathfrak{G}$ in S ($\mathfrak{G}$ could be a model or a frame): $\mathfrak{G} \Vdash \Sigma \Rightarrow \mathfrak{G} \Vdash \varphi$. $\dashv$

1.6 Normal Modal Logics

定义 1.18 (Axiom system **K**)**.** The axiom system **K** is containing following axioms and rules:

- Axioms
  1. **PC**: all propositional tautologies;
  2. K: $\Box(p \to q) \to (\Box p \to \Box q)$ (also known as *distribution axiom*)
  3. Dual: $\Diamond p \leftrightarrow \neg\Box\neg p$
- Rules

[1] For any set A, B, $B^A := \{f \mid f : A \to B\}$.

1. MP: $\dfrac{\varphi \to \psi, \varphi}{\psi}$
2. Sub: $\dfrac{\varphi}{\varphi^\sigma}$ where σ is a substitution
3. Gen$_\Box$: $\dfrac{\varphi}{\Box\varphi}$

A **K-proof** is a finite sequence of formulae $\varphi_1, \ldots, \varphi_n$, for each φ_i ($1 \leq i \leq n$), either φ_i is an axiom of **K**, or φ_i is obtained by one or more earlier formulae in the sequence by applying a rule of **K**.

If $\varphi_1, \ldots, \varphi_n$ is a **K**-proof and $\varphi = \varphi_n$, then we say that φ is **K**-provable, notation $\vdash_{\mathbf{K}} \varphi$, and say φ is a **theorem** of **K**. $\dashv$

注记 1.19. There are some comments on the three rules:

- MP:
  1. MP preserves *validity*: $\Vdash \varphi \to \psi, \Vdash \varphi \Rightarrow \Vdash \psi$
  2. MP preserves *satisfiability*: $\mathfrak{M}, w \Vdash \varphi \to \psi, \mathfrak{M}, w \Vdash \varphi \Rightarrow \mathfrak{M}, w \Vdash \psi$
  3. MP preserves *global truth*: $\mathfrak{M} \Vdash \varphi \to \psi, \mathfrak{M} \Vdash \varphi \Rightarrow \mathfrak{M} \Vdash \psi$
- Sub:
  1. Sub preserves *validity*: $\Vdash \varphi \Rightarrow \Vdash \varphi^\sigma$
  2. Sub not preserve *satisfiability*
  3. Sub not preserve *global truth*
- Gen$_\Box$
  1. Gen$_\Box$ preserves *validity*: $\Vdash \varphi \Rightarrow \Vdash \Box\varphi$
  2. Gen$_\Box$ not preserve *satisfiability*
  3. Gen$_\Box$ preserves *global truth*: $\mathfrak{M} \Vdash \varphi \Rightarrow \mathfrak{M} \Vdash \Box\varphi$

$\dashv$

定义 1.20 (Normal modal logics)**.** A **normal modal logic** Λ is a set of formulae that contains all tautologies, K-axiom, Dual-axiom and is closed under MP, Sub and Gen$_\Box$.

The smallest normal modal logic is **K**. $\dashv$

命题 1.21. Let F be a class of frames, then $\Lambda_{\mathsf{F}} := \{\varphi \mid \mathsf{F} \Vdash \varphi\}$ is a normal modal logic. $\dashv$

Proof. See here . ■

1.7 Selected exercises for Ch.1

1.1.1

1.1.2

1.1.3

1.3.1

1.3.4

1.3.5

1.6.7 Let F be a class of frames. Show that Λ_{F} is a normal modal logic.

Proof. Because all tautologies is valid on any frame, so is for the axioms K and Dual, then we only need to show that Λ_{F} is closed under MP, Sub and Nec.

(1) MP: if $\phi, \phi \to \psi \in \Lambda_{\mathsf{F}}$, then take any model $\mathfrak{M}$ from F and any state w in $\mathfrak{M}$ we have $\mathfrak{M}, w \models \phi$ and $\mathfrak{M}, w \models \phi \to \psi$, hence $\mathfrak{M}, w \models \psi$, because $\mathfrak{M}$ and w are arbitrary from F, then ψ is valid on F, that is $\psi \in \Lambda_{\mathsf{F}}$.

★ (2) Sub: we need a lemma here:

> ***lemma***: Suppose $M = (W, R, V)$ is a model, and $\phi^\sigma = \phi[\psi_1/p_1, \cdots, \psi_n/p_n]$ is the substitution instance of ϕ under substitution σ. Define $M' = (W, R, V')$ by $V'(p_i) = \{w \in W \mid M, w \Vdash \psi_i\}$. Then for any $w \in W$:
> $$M, w \Vdash \phi^\sigma \ \Leftrightarrow\ M', w \Vdash \phi.$$

Assume $\phi \in \Lambda_{\mathsf{F}}$, that is, $\mathsf{F} \Vdash \phi$, but $\phi^\theta \notin \Lambda_{\mathsf{F}}$ for some substitution θ, i.e $\mathsf{F} \nVdash \phi^\theta$. Then for some model $M = (W, R, V)$ from F and some $w \in W$ we have $M, w \nVdash \phi^\theta$, hence $M', w \nVdash \phi$ by above lemma, but this is contradicts to ϕ is valid in F. Therefore, if $\phi \in \Lambda_{\mathsf{F}}$ then $\phi^\theta \in \Lambda_{\mathsf{F}}$ for any substitution θ.

(3) Nec: suppose $\phi \in \Lambda_{\mathsf{F}}$ but $\Box\phi \notin \Lambda_{\mathsf{F}}$, then there are a frame $F = (W, R)$ from F, a valuation V and a state $w \in W$ such that $(F, V), w \nVdash \Box\phi$. Hence there must be a state $u \in W$ for which Rwu and $(F, V), u \Vdash \neg\phi$, but this contradicts with ϕ is valid on F. Therefore $\Box\phi \in \Lambda_{\mathsf{F}}$ ■

1.3.1 Show that when evaluating a formula ϕ in a model, the only relevant information in the valuation is the assignments it makes to the propositional letters actually occurring in ϕ. More precisely, let $\mathfrak{F}$ be a frame, and V and V' be two valuations on $\mathfrak{F}$ such that$V(p) = V'(p)$ for all proposition letters p in ϕ. Show that $(\mathfrak{F}, V) \Vdash \phi$ iff $(\mathfrak{F}, V') \Vdash \phi$. Work in the basic modal language.

Proof. Let $\mathfrak{F} = (W, R)$, V and V' are two valuations as mentioned above, we firstly prove the following lemma by induction on ϕ:

$$(*) \quad \forall w \in W : (\mathfrak{F}, V), w \Vdash \phi \;\Leftrightarrow\; (\mathfrak{F}, V'), w \Vdash \phi.$$

Base case

- If ϕ is a propositional letter p, then for all $w \in W$

$$\begin{array}{rcll} (\mathfrak{F}, V), w \Vdash p & \Leftrightarrow & w \in V(p), & \text{(by definition)} \\ & \Leftrightarrow & w \in V'(p), & \text{(by assumption)} \\ & \Leftrightarrow & (\mathfrak{F}, V'), w \Vdash p. & \text{(by definition)} \end{array}$$

- If $\phi = \bot$, then for all $w \in W$, $(\mathfrak{F}, V), w \Vdash \phi \;\Leftrightarrow\; (\mathfrak{F}, V'), w \Vdash \phi$ trivially.

Induction step:
If ϕ is of the form $\neg\psi$ or $\psi \vee \chi$, this is easily done. The crucial case is the form $\Diamond\psi$.

$$\begin{array}{rcll} (\mathfrak{F}, V), w \Vdash \Diamond\psi & \Leftrightarrow & \exists v, Rwv, (\mathfrak{F}, V), v \Vdash \psi, & \text{(by definition)} \\ & \Leftrightarrow & \exists v, Rwv, (\mathfrak{F}, V'), v \Vdash \psi, & \text{(by induction hypothesis)} \\ & \Leftrightarrow & (\mathfrak{F}, V'), w \Vdash \Diamond\psi. & \text{(by definition)} \end{array}$$

Then the desired proposition

$$(\mathfrak{F}, V) \Vdash \phi \;\Leftrightarrow\; (\mathfrak{F}, V') \Vdash \phi$$

is just a corollary of $(*)$. ∎

1.3.4 Show that every formula that has the form of a propositional tautology is valid. Further, show that $\Box(p \to q) \to (\Box p \to \Box q)$ is valid.

Proof.
(1) (we only work in the basic modal language here)
Firstly, we give a formal definition for what is a formula has the form of a propositional tautology.

Definition: tautology

A modal formula ϕ is called a *tautology* (shouldn't be confused with *proposition tautology*), if $\phi = \alpha^\sigma$ where σ is a substitution, α is a formula of propositional logic and α is a proposition tautology.

In effect, therefore, we have to show that:

$$(\circledast) \qquad \text{Every tautology is valid.}$$

To do that, we need following lemma in the first place.

Lemma 1 Suppose θ is a modal-free formula whose propositional variables are $p_1, \ldots, p_n$, let $\phi_1, \ldots, \phi_n$ be modal formulas, and σ is a substitution such that $\sigma(p_i) = \phi_i$ for each $1 \leq i \leq n$.

If for any propositional assignment v, any modal model $M = (W, R, V)$, and any $w \in W$ such that $v(p_i) = 1$ iff $M, w \Vdash \phi_i$, then $v \models \theta$ iff $M, w \Vdash \theta^\sigma$.

(where $v \models \theta$ represents the satisfiability of propositional logic)

Proof for lemma 1

By induction on θ (note θ is a proposition formula).
Base case

- if $\theta = \bot$, then $\bot^\sigma = \bot$, both $v \not\models \bot$ and $M, w \not\Vdash \bot$.

- if $\theta = p_i$, then

$$\begin{aligned} v \models p_i \quad &\Leftrightarrow \quad v(p_i) = 1 \\ &\Leftrightarrow \quad M, w \Vdash \phi_i \quad \text{(by assumption)} \\ &\Leftrightarrow \quad M, w \Vdash p_i^\sigma \quad \text{(since } \ p_i^\sigma = \sigma(p_i) = \phi_i \text{, by the definition of } \ \sigma). \end{aligned}$$

Induction step

- if $\theta = \neg\chi$, then

$$\begin{aligned} v \models \neg\chi \quad &\Leftrightarrow \quad v \not\models \chi \\ &\Leftrightarrow \quad M, w \not\Vdash \chi^\sigma \qquad \text{(by induction hypothesis)} \\ &\Leftrightarrow \quad M, w \Vdash \neg\chi^\sigma \\ &\Leftrightarrow \quad M, w \Vdash (\neg\chi)^\sigma \quad \text{(by the definition of substitution)} \end{aligned}$$

- if $\theta = \psi \vee \chi$, then

$$\begin{aligned} v \models (\psi \vee \chi) \quad &\Leftrightarrow \quad v \models \psi \text{ or } v \models \chi \\ &\Leftrightarrow \quad M, w \Vdash \psi^\sigma \text{ or } M, w \Vdash \chi^\sigma \quad \text{(by induction hypothesis)} \\ &\Leftrightarrow \quad M, w \Vdash \psi^\sigma \vee \chi^\sigma \\ &\Leftrightarrow \quad M, w \Vdash (\psi \vee \chi)^\sigma \qquad \text{(by the definition of substitution)} \end{aligned}$$

Hence we complete the induction proof for **Lemma 1**.

Then we prove ($\circledast$) by contraposition.
Suppose φ is a tautology but not valid,
then by the definition of tautology,
there is a proposition tautology θ and a substitution σ such that $\varphi = \theta^\sigma$ is invalid.
Namely $M, w \not\Vdash \theta^\sigma$ for some model M and some state w in M.
Moreover, we assume only $p_i, \ldots, p_n$ are occurring in θ,
and σ satisfies $\sigma(p_i) = \phi_i$ for each $1 \leq i \leq n$.
Now we define a propositional assignment v by

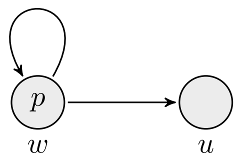
$$v(p_i) = 1 \ \Leftrightarrow \ M, w \Vdash \phi_i$$

Then, by the **lemma 1**, we have that : $v \models \theta \ \Leftrightarrow \ M, w \Vdash \theta^\sigma$.
Since $M, w \not\Vdash \theta^\sigma$, therefore $v \not\models \theta$.
But this contradicts with θ is a proposition tautology.
Consequently, ($\circledast$) is desired, that is, every tautology is valid.

(2)
Following we show that $\Box(p \to q) \to (\Box p \to \Box q)$ is valid.
Take any frame $\mathfrak{F}$ and any state w in $\mathfrak{F}$, and let V be a valuation on $\mathfrak{F}$.
We have to show that if $(\mathfrak{F}, V), w \Vdash \Box(p \to q)$ and $(\mathfrak{F}, V), w \Vdash \Box p$, then $(\mathfrak{F}, V), w \Vdash \Box q$.
So assume that $(\mathfrak{F}, V), w \Vdash \Box(p \to q)$ and $(\mathfrak{F}, V), w \Vdash \Box p$.
Then, by definition for any state v such that Rwv we have $(\mathfrak{F}, V), v \Vdash p \to q$ and $(\mathfrak{F}, V), v \Vdash p$,
hence $(\mathfrak{F}, V), v \Vdash q$.
But since Rwv and v is an arbitrary state,
then by definition we have $(\mathfrak{F}, V), w \Vdash \Box q$. ■

1.3.5 Show that every formula of the following formulas is not valid by constructing a frame $\mathfrak{F} = (W, R)$ that refutes it.
(a) $\Box\bot$ (b) $\Diamond p \to \Box p$ (c) $p \to \Box\Diamond p$ (d) $\Diamond\Box p \to \Box\Diamond p$.

Proof. Let's consider following frame $\mathfrak{F}$, then we show that this frame refutes all above formulas.
Let $\mathfrak{F} = (W, R)$ where $W = \{w, u\}$ and $R = \{(w, w), (w, u)\}$,
we visualize $\mathfrak{F}$ (with a valuation) as follows:



Now we define a valuation V on $\mathfrak{F}$ by

$$V(q) = \begin{cases} \{w\} & q = p \\ \emptyset & q \neq p \end{cases}$$

We use $w \Vdash \varphi$ instead of $(\mathfrak{F}, V), w \Vdash \varphi$ for convenience. Then we know:
$w \Vdash \Diamond p$ since Rww and $w \Vdash p$;
$w \nVdash \Box p$ since Rwv but $u \nVdash p$;
$w \nVdash \Box\Diamond p$ since Rwu but u has no successors, which means $u \nVdash \Diamond p$;
$w \Vdash \Diamond\Box p$ since Rwu and v is a '*dead end*', that is $u \Vdash \Box p$.
Then,
(a) $w \nVdash \Box\bot$ since Rwu but $u \nVdash \bot$;
(b) $w \nVdash \Diamond p \to \Box p$ since $w \Vdash \Diamond p$ but $w \nVdash \Box p$
(c) $w \nVdash p \to \Box\Diamond p$ since $w \Vdash p$ but $w \nVdash \Box\Diamond p$
(d) $w \nVdash \Diamond\Box p \to \Box\Diamond p$ since $w \Vdash \Diamond\Box p$ but $w \nVdash \Box\Diamond p$ ■

Show that **K** *is sound with respect to the class of all frames*.

Proof. We already known that:
(1) All axioms of **K** are valid.
(all tautologies are valid and the K-axiom is valid (see exercise 1.3.4, p27), moreover the Dual-axiom is valid (see the discussion in paragraph 5 of p34))
(2) Furthermore, we assume that all rules of **K** are preserve validity, we will give a proof in the last.

Then to show **K** is *sound*, it is sufficient to show that all **K**-provable formulas are valid.
Suppose φ is **K**-provable for any formula φ.
By induction on n, the length of proof for φ.
Base case:

- If $n = 1$, then by the definition of **K**-proof, that means φ is an axiom of **K**,
but all axioms of **K** are valid,
hence φ is valid.

Induction step: Suppose φ has a proof of length $n > 1$.

- If φ is an axiom of **K**, then φ is valid as same as base case.

- If φ is obtained by MP from previous formulas $\chi \to \varphi$ and χ,
by induction hypothesis, $\chi \to \varphi$ and χ are valid,
and MP preserves validity,
hence φ is valid.

- If φ is obtained by Sub or $\text{Gen}_\Box$ from χ,
by inductive hypothesis, χ is valid,
moreover Sub and $\text{Gen}_\Box$ both preserve validity,
therefore φ is valid.

In the end, we will show that *modus ponens* (MP), *uniform substitution* (Sub) and *Generalization* (Gen$_\Box$) are preserve validity.

(a) For MP.

That is to show: if $\phi \to \psi$ and ψ are valid, then so is ψ.

Suppose $\Vdash \phi, \Vdash \phi \to \psi$,

Then $M, w \models \phi$ and $M, w \models \phi \to \psi$ for some model M and some w in M.

Hence $M, w \models \psi$ by the definition.

Therefore $\Vdash \psi$ because M and w are arbitrary.

(b) For Gen$_\Box$.

That is to show: if ϕ is valid, then so is $\Box\phi$.

Assume $\Vdash \phi$. To show $\Vdash \Box\phi$, let $M = (W, R, V)$ be any model and $w \in W$.

For any $u \in W$, if Rwu then $M, u \Vdash \phi$ since ϕ is valid, and hence $M, u \Vdash \Box\phi$ by the definition.

Since M and w are arbitrary, then $\Vdash \Box\varphi$.

(c) For Sub.

That is to show: if ϕ is valid, then so is ϕ^σ for any substitution σ.

First we need a lemma:

Lemma 2: Suppose ϕ only contains $p_1, \ldots, p_n$ as its propositional letters, and ϕ^σ is the substitution instance of ϕ under substitution σ, where $\sigma(p_i) = \psi_i$ for each $1 \leq i \leq n$.

For any models $M = (W, R, V)$, define $M' = (W, R, V')$ by $V'(p_i) = \{w \in W \mid M, w \Vdash \psi_i\}$. Then for any $w \in W$: $M, w \Vdash \phi^\sigma \;\Leftrightarrow\; M', w \Vdash \phi$.

Proof for this Lemma 2

By induction on ϕ.

Base case:

· If $\phi = p$, then $p_i^\sigma = \psi_i$.
  Hence $M, w \Vdash \psi_i \;\Leftrightarrow\; M', w \Vdash p_i$ by the definition of V'.

· If $\phi = \bot$, then $\bot^\sigma = \bot$.
  Both $M, w \nVdash \bot$ and $M', w \nVdash \bot$.

Induction step

· If ϕ is of the form $\neg\psi$ or $\psi \vee \chi$, this is easily done. The more crucial case is the form $\Diamond\psi$.

· if $\phi = \Diamond\psi$, then

$$
\begin{array}{rcll}
M, w \Vdash (\Diamond\psi)^\sigma & \Leftrightarrow & M, w \Vdash \Diamond\psi^\sigma & \\
& \Leftrightarrow & M, u \Vdash \psi^\sigma & \text{for some } u \text{ such that } Rwu \\
& \Leftrightarrow & M', u \Vdash \psi & \text{by inductive hypothesis} \\
& \Leftrightarrow & M', w \Vdash \Diamond\psi & \text{since } Rwu
\end{array}
$$

Therefore we complete the induction proof of above lemma.

Then, assume ϕ is valid,

but ϕ^σ is invalid for some substitution σ, such that $\sigma(p_i) = \psi_i$.

Hence $M, w \nVdash \phi^\sigma$ for some model $M = (W, R, V)$ and some $w \in W$ since ϕ^σ is invalid,

hence we have $M', w \nVdash \phi$ by above **lemma 2**,

but this contradicts with that ϕ is valid.

Therefore, if ϕ is valid, then so is ϕ^σ for any substitution σ. ■

2 Modal model theory

2.1 Disjoint unions

2.2 Generated submodels

2.3 Bounded morphisms

2.4 Bisimulation

2.5 Finite model property (fmp)

2.6 fmp via selection

2.7 fmp via filtration

2.8 The standard translation

2.9 Modal saturation

2.10 van Benthem characterization theorem

3 Modal frame theory

4 Hybrid Logic

4.1 So many hybrid languages

4.2 Basic hybrid language $\mathcal{L}_{@}$

不同的文献对最小的正规混合逻辑有不同的公理化：

I Love Hybrid Logic: $\mathbf{K}_h$ and $\mathbf{K}_h^+$

(PC)

(K)

(Dual)

(MP)

(Gen$_\Box$)

(Gen$_@$)

(Sub$_{sorted}$)

(K$_@$) $@_i(p \to q) \to (@_i p \to @_i q)$

(Self-dual) $@_i p \leftrightarrow \neg @_i \neg p$

(Intro) $i \wedge p \to @_i p$ $\qquad (elim : i \wedge @_i p \to p)$

(Ref) $@_i i$

(Sym) $@_i j \leftrightarrow @_j i$

(Nom) $@_i j \wedge @_j p \to @_i p$

(Agree) $@_j @_i p \to @_i p$

(Back) $\Diamond @_i p \to @_i p$ $\qquad (Bridge : \Diamond i \wedge @_i p \to \Diamond p)$

$\mathbf{K}_h^+$

$$\text{(NAME)}\ \frac{j \to \theta}{\theta} \qquad \text{(PASTE)}\ \frac{@_i \Diamond j \wedge @_j \phi \to \theta}{@_i \Diamond \phi \to \theta} \quad \text{(in both } i \neq j, j \notin \theta, j \notin i)$$

Proof for Lemma 1

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Proof for Lemma 1

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