Note on Mathematical Logic: from Zero to Hero
数理逻辑笔记: 从入门到入土

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Contents

I Propositional Logic **3**

1 Proposition Logic **4**

1.1 Lindenbaum Lemma via Zorn's Lemma 4

II First-order Logic **5**

2 First Order Logic **6**

2.1 Syntax 6

2.1.1 terms 6

2.1.2 formulas 6

2.1.3 free variables, bound variables 6

2.2 Semantics 6

2.2.1 Interpreting terms 7

2.2.2 Substitution and the satisfaction relation 7

2.3 Isomorphisms between structures 7

2.4 Substructures and elementary substructures 8

2.5 Definable sets and Tarski's Criterion 8

2.6 Countable sets 9

2.7 The Downward Löwenheim-Skolen Theorem 9

3 The Gödel Completeness Theorem **10**

3.1 Proof 10

3.2 Soundness 11

3.3 The Henkin property 11

3.4 Extensions of consistent sets of formulas 11

3.5 The Gödel Completeness Theorem 11

3.6 The Craig Interpolation Theorem 11

Notes on $\mathfrak{Mathematical\ Logic}$

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Textbook:

Figure 0. books

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Table 1. some **adequate** and *not* adequate sets of connectives

adequate set of connectives	comments and notes
$\{\neg, \vee\}$, $\{\neg, \wedge\}$, $\{\neg, \rightarrow\}$	a common and unremarkable combination
$\{\bot, \rightarrow\}$	$\neg\varphi \coloneqq \varphi \rightarrow \bot$
$\{\downarrow\}$	***nor*** ("neither $\cdots$ nor $\cdots$")
	$v(\varphi \downarrow \psi) = \begin{cases} 1 & v(\varphi) = 1 \textit{ and } v(\psi) = 1 \\ 0 & \textit{otherwise} \end{cases}$
$\{\mid\}$	***nand*** ("not both $\cdots$ and $\cdots$"), also called **Sheffer stroke**
	$v(\varphi \mid \psi) = \begin{cases} 1 & v(\varphi) = 1 \textit{ or } v(\psi) = 1 \\ 0 & \textit{otherwise} \end{cases}$
not adequate set of connectives	
$\{\wedge, \vee, \rightarrow, \leftrightarrow\}$	
$\{\neg, \leftrightarrow\}$	

Table 2. some properties of PL

property	comments or notes
Criag Interpolation	
Lyndon Interpolation	
Compactness	

Table 3. PL v.s. FOL

	PL	FOL
Soundness and Completeness	$\surd$	$\surd$
Decidability	$\surd$	$\times$

PART I

Propositional Logic

CHAPTER

1

Proposition Logic

SECTION 1.1

Lindenbaum Lemma via Zorn's Lemma

Zorn's Lemma

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PART

II

First-order Logic

CHAPTER 2

First Order Logic

Section 2.1

Syntax

- The *logical symbols*:
  ( ) $\neg$ $\to$ $\forall$
- The *variable symbols*
- The *constant symbols*
- The *function symbols*
- The *predicate symbols*

Subsection 2.1.1

terms

$$Term \ni t ::= x \mid c \mid f(t_1 \cdots t_n).$$

Subsection 2.1.2

formulas

Subsection 2.1.3

free variables, bound variables

Definition 2.1 (Scope of quantifier) 内容...

Definition 2.2 (Occurrences)

Definition 2.3 (Sentences) A first-order formulas ϕ is a **sentence** iff it has no free variables.

Section 2.2

Semantics

Definition 2.4 (Structure) (M, I)

Subsection 2.2.1
Interpreting terms

$t[a_1, \ldots, a_n]$

$\mathfrak{M} \models \varphi[a_1, \ldots, a_n]$ indicates the proposition '$(\mathfrak{M}, v) \models \varphi$' holds, where v is any $\mathfrak{M}$-assignment such that for all $i \leq n$, $v(x_i) = a_i$.

If φ is a sentence (with on free variable), then we weite $\mathfrak{M} \models \varphi$ or $\mathfrak{M} \not\models \varphi$.

Subsection 2.2.2
Substitution and the satisfaction relation

Theorem 2.5 (Substitution) Let $\mathfrak{M} = (M, I)$ be an $\mathcal{L}_{\mathcal{A}}$-structure and v be an $\mathfrak{M}$-assignment.

Section 2.3
Isomorphisms between structures

Definition 2.6 (Isomorphisms) Suppose that $\mathfrak{M} = (M, I)$ and $\mathfrak{N} = (N, J)$ are $\mathcal{L}_{\mathcal{A}}$-structures. A *bijection* $e \colon M \to N$ is an **isomorphism** (同构) between $\mathfrak{M}$ and $\mathfrak{N}$ iff the following conditions hold:

1. $e(c^{\mathfrak{M}}) = c^{\mathfrak{N}}$
2. $(fa_1 \ldots f_n)^{\mathfrak{M}} = a_{n+1} \quad \Leftrightarrow \quad (fe(a_1) \ldots e(a_n))^{\mathfrak{N}} = e(a_{n+1})$
3. $(a_1, \ldots, a_n) \in P^{\mathfrak{M}} \quad \Leftrightarrow \quad (e(a_1), \ldots, e(a_n)) \in P^{\mathfrak{N}}$

When $\mathfrak{M} = \mathfrak{N}$, we say that e is an **automorphism** (自同构).
$\mathfrak{M} \cong \mathfrak{N}$

The *inverse* of an isomorphism is also an isomorphism.
The *composition* of two isomorphisms is also an isomorphism.
Isomorphism preserves the interpretation of terms.
$e(\bar{v}(t)) = \overline{e \circ v}(t)$

Definition 2.7 (Elementarily equivalent) $\mathcal{L}_{\mathcal{A}}$-structures $\mathfrak{M}$ and $\mathfrak{N}$ are **elementarily equivalent** iff for each $\mathcal{L}_{\mathcal{A}}$-sentence φ:

$$\mathfrak{M} \models \varphi \quad \Leftrightarrow \quad \mathfrak{N} \models \varphi.$$

We write $\mathfrak{M} \equiv \mathfrak{N}$ that these structures are elementarily equivalent.

Theorem 2.8 (类似于模态中的 H-M property) Suppose $\mathfrak{M}$ and $\mathfrak{N}$ are *finite* $\mathcal{L}_{\mathcal{A}}$-structures, the following are equivalent:

1. $\mathfrak{M} \cong \mathfrak{N}$.
2. $\mathfrak{M} \equiv \mathfrak{N}$.

Section 2.4
Substructures and elementary substructures

Definition 2.9 (Substructures) $\mathfrak{M} = (M, I)$ is a **substructure** of $\mathfrak{N} = (N, J)$ iff $M \subseteq N$ and the following conditions hold.

1. $c^{\mathfrak{M}} = c^{\mathfrak{N}}$
2. $f^{\mathfrak{M}} = f^{\mathfrak{N}} \restriction M^n$, where n is the arity of f. That is, f^M is the restriction of $f^{\mathfrak{N}}$ to M^n.
3. $P^{\mathfrak{M}} = P^{\mathfrak{N}} \cap M^n$

We will write $\mathfrak{M} \subseteq \mathfrak{N}$ to indicate that $\mathfrak{M}$ is a substructure of $\mathfrak{N}$.

Theorem 2.10 Suppose $\mathfrak{M} = (M, I)$ and $\mathfrak{N} = (N, J)$ are $\mathcal{L}_{\mathcal{A}}$-structures with $M \subseteq N$, then the following are equivalent.

1. $\mathfrak{M} \subseteq \mathfrak{N}$
2. For all atomic formulas α, and for all $\mathfrak{M}$-assignment v,

$$\mathfrak{M}, v \models \alpha \iff \mathfrak{N}, v \models \alpha.$$

Definition 2.11 (Elementary substructure) Suppose $\mathfrak{M} = (M, I)$ and $\mathfrak{N} = (N, J)$ are $\mathcal{L}_{\mathcal{A}}$-structures with $\mathfrak{M} \subseteq \mathfrak{N}$, $\mathfrak{M}$ is an **elementary substructure** of $\mathfrak{N}$ iff for all formulas φ and for all $\mathfrak{M}$-assignments v,

$$\mathfrak{M}, v \models \varphi \iff \mathfrak{N}, v \models \varphi.$$

We write $\mathfrak{M} \preceq \mathfrak{N}$ to indicate that $\mathfrak{M}$ is an elementary substructure of $\mathfrak{N}$.

Section 2.5

Definable sets and Tarski's Criterion

Tarski's Theorem below gives an elegant characterization of when a substructure of $\mathfrak{N}$ is an elementary substructure. Tarski's criterion is given in terms of definable sets.

Definition 2.12 (Definability) Suppose $\mathfrak{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure.

1. Assume $X \subseteq M$. A set $Y \subseteq M^n$ is **definable in $\mathfrak{M}$ with parameters from X** iff there are elements $b_1, \dots, b_m \in X$ and an $\mathcal{L}_{\mathcal{A}}$-formulas φ such that the following conditions hold.
   (a) φ has $n + m$ free variable $x_{j_1}, \dots, x_{j_n}$ and $x_{k_1}, \dots, x_{k_m}$.
   (b) For each $(a_1, \dots, a_n) \in M^n$, $(a_1, \dots, a_n) \in Y$ iff there is an $\mathfrak{M}$-assignment v such that
      i. for each $i \leq n$, $v(x_{j_i}) = a_i$
      ii. for each $i \leq m$, $v(x_{k_i}) = b_i$
      iii. $\mathfrak{M}, v \models \varphi$
2. A set $Y \subseteq M^n$ is **definable in $\mathfrak{M}$ without parameters** iff it is definable with parameters from $\emptyset$.
3. When $n = 1$, we identify M^1 with M and speak of **definable subsets** of M.

上面定义的等价定义：
$Y \subseteq M^n$ is definable in $\mathfrak{M}$ with parameters from X, if

$$Y \coloneqq \{(a_1, \ldots, a_n) \in M^n \mid \mathfrak{M} \models \varphi[a_1, \ldots, a_n, b_1, \ldots, b_m]\}$$

Definability within a structure is one of the central concepts in Mathematical Logic.

Theorem 2.13 (Tarski) Suppose that $\mathfrak{M} = (M, I)$ and $\mathfrak{N} = (N, J)$ are $\mathcal{L}_{\mathcal{A}}$-structures, and $\mathfrak{M}$ is a substructure of $\mathfrak{N}$. The following are equivalent.

1. $\mathfrak{M} \preceq \mathfrak{N}$
2. $\mathfrak{M} \subseteq \mathfrak{N}$ and for each nonempty set $A \subseteq N$, if A is definable in $\mathfrak{N}$ with parameters from M, then $A \cap M \neq \emptyset$.

Proof 内容... ■

Section 2.6

Countable sets

Definition 2.14 (Countable set) A set A is **countable** if either it is empty or there is a surjective map from $\mathbb{N}$ to A.

Theorem 2.15 (Cantor) The set of real number is not countable.

Proof It suffices to show that the real interval $(0, 1)$ is uncountable. Using **diagonal method**. ■

Section 2.7

The Downward Löwenheim-Skolen Theorem

The Downward Löwenheim-Skolem Theorem is an important application of Tarski's Theorem.

$\mathfrak{M} = (M, I)$ is countable if M is a countable set.

Theorem 2.16 (**Downward Löwenheim-Skolem**) Suppose that $\mathfrak{N} = (N, J)$ is an $\mathcal{L}_{\mathcal{A}}$-structure, then there exists an elementary substructure $\mathfrak{M} = (M, I) \preceq \mathfrak{N}$ such that M is countable.
[note that: $\mathfrak{N}$ maybe uncountable here]

Proof If $\mathfrak{N}$ is a countable structure, let $\mathfrak{M} = \mathfrak{N}$, then we done.
If $\mathfrak{N}$ is uncountable. ■

CHAPTER 3

The Gödel Completeness Theorem

Section 3.1

Proof

Definition 3.1 (Validity) A **validity** is an formulas φ which is satisfied in any and every interpretation. That is, for all $\mathfrak{M}$ and all v: $\mathfrak{M}, v \models \varphi$.

Definition 3.2 (Substitutable) Suppose that φ is an $\mathcal{L}_{\mathcal{A}}$formulas and that t is a term.

1. Suppose that x is a free variable of φ.
   (a) The term t is **substitutable** for x iff every variable of t is free for x in φ.
   (b) If t is **substitutable** for x, then $\varphi(x;t)$ denotes the formulas obtained by substituting t for each free occurrence of x in φ. Similar meaning for $\varphi(x_1, \ldots, x_n; t_1, \ldots, t_n)$.
2. Suppose that c is a constant symbol.
   (a) The term t is **substitutable** for c iff for every variable x_j of t, no occurrence of c in φ is within the scope of an occurrence of $\forall x_j$.
   (b) If t is **substitutable** for c, then $\varphi(c;t)$ denotes the formulas obtained by substituting t for each occurrence of c in φ. Similar meaning for $\varphi(c_1, \ldots, c_n; t_1, \ldots, t_n)$.

Definition 3.3 (Axioms) The **set of axioms**, denoted Δ, is the smallest set of formulas which satisfies the following closure properties.

1. Instance of Propositional Tautologies.
2. Suppose φ is a formula, t is a term, and t is *substitutable* for x in φ, then

$$\forall x \varphi \rightarrow \varphi(x;t) \in \Delta.$$

3. Suppose φ and ψ are formulas, then

$$\forall x(\varphi \rightarrow \psi) \rightarrow (\forall x \varphi \rightarrow \forall x \psi) \in \Delta.$$

4. Suppose φ is a formulas, and x is not a free variable of φ, then
$$\varphi \to \forall x \varphi \in \Delta.$$

5. For every variable x,
$$x \hat{=} x \in \Delta.$$

6. Suppose that φ and ψ are formulas, and x_j is substitutable for x_i in φ, ψ, then
$$\psi(x_i; x_j) = \varphi(x_i, x_j) \;\Rightarrow\; x_i \hat{=} x_j \to (\varphi \to \psi) \in \Delta.$$

7. Suppose $\varphi \in \Delta$, then $\forall x \varphi \in \Delta$.

Definition 3.4 (Deduction)

Section 3.2
Soundness

Section 3.3
The Henkin property

Section 3.4
Extensions of consistent sets of formulas

Section 3.5
The Gödel Completeness Theorem

Section 3.6
The Craig Interpolation Theorem

Bibliography