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A Simple Model for Spatial-Temporal Processes

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Suppose a spatial-temporal process is observed at a fixed network of sites at a number of observation times. The process is modeled as the sum of a random field fixed in time plus a second independent random field that varies both spatially and temporally. We (essentially) assume that values of this second field taken at two different observation times are uncorrelated. Using this model, we obtain a simple expression for the kriging predictor of this process. This model also suggests a method of prediction of the average change in time in the process over a region.

1. INTRODUCTION

Consider a process $z(x, t)$ that takes on values in space and time, where x indicates the location and t the time. Suppose we observe the process at some fixed set of sites at specified times or intervals of time. That is, we observe $z(x, t)$ at (x_i, t_j) for $i = 1, \cdots, n$ and $j = 1, \cdots, k$. On the basis of these observations we will consider two problems. The first is the prediction of $z(x, t_j)$ at a location x other than where we have observed $z(\ , t_j)$. The second is the prediction of the average change in the $z(\ ,\ )$ field from one time to another over some specified region. For example, we may want to decide whether or not some quantity, such as rainfall or acid deposition, increased in some region from one year to the next.

We will use the model

$$z(x, t) = m(x) + \mu(t) + e(x, t) \tag{1}$$

We will assume that $\mu(t_1), \cdots, \mu(t_k)$ is an unknown and arbitrary deterministic time series. The $m(\ )$ term represents spatial variations in the $z(\ ,\ )$ field that do not change over the time range being considered. The $e(\ ,\ )$ field represents spatial-temporal variations in the $z(\ ,\ )$ field not accounted for by the other terms. We will model the $e(\ ,\ )$ field as a stochastic process satisfying

$$Ee(x, t) = 0 \tag{2}$$

$$\tfrac{1}{2}E(e(x, t) - e(x', t))^2 = \gamma(x - x') \tag{3}$$

$$E(e(x, t_i) - e(x', t_i))(e(x'', t_j) - e(x''', t_j)) = 0 \qquad \text{if} \quad i \neq j \tag{4}$$

The function $\gamma(\ )$ is the semivariogram of the $e(\ ,\ )$ field at a given time; we assume that it is independent of the time t. Equation (4) says that differences in the $e(\ ,\ )$ field at one time at which observations are taken are uncorrelated with differences from another time. These assumptions do not allow for any temporal patterns or spatial-temporal interactions, such as a trend in the averages in time or time-lagged spatial correlations. Thus this model may be inappropriate with observations in time more frequent than annually in which we may expect to observe more complicated structure in the $e(\ ,\ )$ field or at least seasonal trends in $\mu(t_1), \cdots, \mu(t_k)$.

These assumptions are sufficient for allowing us to predict the average change in the $z(\ ,\ )$ field over some region from one time to another (see section 3). However, in order to obtain predictors of the $z(\ ,\ )$ field at a given time t_j, we must make some assumptions about the $m(\ )$ field. We will assume $m(\ )$ is a random field independent of the $e(\ ,\ )$ field satisfying

$$Em(x) = 0 \tag{5}$$

$$\tfrac{1}{2}E(m(x) - m(x'))^2 = \eta(x - x') \tag{6}$$

so that $\eta(\ )$ is the semivariogram of the $m(\ )$ field. The model specified in (1)–(6) is sufficient to determine the variance of the error of a linear unbiased predictor of $z(x, t_j)$. Assuming $\gamma(\ )$ and $\eta(\ )$ are specified, we give an expression for the best linear unbiased (kriging) predictor of $z(x, t_j)$ that only involves inverting matrices of size $n \times n$ in section 2. We also give a simple interpretation of this predictor appropriate when k is large. In Appendix A, these results are generalized to allow the $m(\ )$ and $e(\ ,\ )$ fields to have nonstationary means.

By considering observations from other than time t_j as covariates, the prediction of $z(x, t_j)$ can be viewed as a special case of the cokriging problem [*Myers*, 1982]. More specifically, we are assuming that $\gamma(x) + \eta(x)$ is the semivariogram for observations from the same time, and $\eta(x)$ is the cross-variogram for observations from different times. Thus the kriging predictor of $z(x, t_j)$ could also be obtained using the general cokriging formulation; however, we would have to invert a $kn \times kn$ matrix to compute it using the general expression.

In section 3 we will consider two approaches to predicting a difference between two times of the average of a spatial-temporal process in some region. One is take differences of the observations at the two times and predict the average difference of the field. In Appendix B we show that this predictor is the same as the kriging predictor based on all of the observations. The second is to predict the average value of the field for each of the two times and take differences of these predictions, computing the estimated standard deviation of the error by assuming the errors of the two predictions are uncorrelated. The second procedure assumes that $m(x) \equiv 0$, and if it is not, the first procedure may produce considerably smaller estimates of the standard deviation. The first procedure may be compared to a paired t test and the second to an unpaired t test.

2. PREDICTION AT A GIVEN TIME

The model for $z(x, t)$ given in (1)–(6) is sufficient to specify the variance of the error of all linear unbiased predictors of $z(x_0, t_\beta)$, where x_0 is some location, and t_β is one of the times $t_1, \cdots, t_k$ at which we observe the process [*Stein*, 1984, p. 134].

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We can thus obtain the kriging predictor for $z(x_0, t_\beta)$ if the semivariograms $\gamma(\ )$ and $\eta(\ )$ are specified. Define $N = (\eta(x_i - x_j))$, the $n \times n$ matrix whose ijth element is $\eta(x_i - x_j)$, $\mathbf{n} = (\eta(x_i - x_0))$, the vector of length n whose ith element is $\eta(x_i - x_0)$, $\Gamma = (\gamma(x_i - x_j))$, and $\mathbf{g} = (\gamma(x_i - x_0))$. Also define vectors

$$\mathbf{z}_j = (z(x_1, t_j) \cdots z(x_n, t_j))'$$

$$\bar{\mathbf{z}} = k^{-1} \sum_{j=1}^{k} \mathbf{z}_j$$

and let $\mathbf{1}$ be a vector of ones of length n. Then the kriging predictor of $z(x_0, t_\beta)$ is given by [*Stein*, 1984, p. 136]

$$\hat{z}(x_0, t_\beta) = \mathbf{a}'\bar{\mathbf{z}} + \mathbf{b}'(\mathbf{z}_\beta - \bar{\mathbf{z}}) \tag{7}$$

where

$$\mathbf{b} = \Gamma^{-1}\mathbf{g} + \Gamma^{-1}\mathbf{1}(1 - \mathbf{1}'\Gamma^{-1}\mathbf{g})/\mathbf{1}'\Gamma^{-1}\mathbf{1} \tag{8}$$

$$\mathbf{a} = (kN + \Gamma)^{-1}(k\mathbf{n} + \mathbf{g}) + (kN + \Gamma)^{-1} \cdot \mathbf{1}\{1 - \mathbf{1}'(kN + \Gamma)^{-1}(k\mathbf{n} + \mathbf{g})\}/\mathbf{1}'(kN + \Gamma)^{-1}\mathbf{1} \tag{9}$$

This predictor is a special case of the results in Appendix A in which we allow for nonstationary mean functions. We note that even though we have kn observations, because of the simple structure of the model and the fact that the observation sites are fixed in time, we only have to invert matrices of size $n \times n$, not $kn \times kn$, to obtain the kriging predictor. If $m(x) = 0$, then $\hat{z}(x_0, t_\beta) = \mathbf{b}'\mathbf{z}_\beta$; that is, observations from other times do not help us to predict $z(x_0, t_\beta)$ assuming $\gamma(\ )$ is specified.

If we let k, the number of replications in time, tend to infinity in (9), we get

$$\mathbf{a} \to N^{-1}\mathbf{n} + N^{-1}\mathbf{1}(1 - \mathbf{1}'N^{-1}\mathbf{n})/\mathbf{1}'N^{-1}\mathbf{1}$$

Thus for k large we have in a rough sense,

$$\hat{z}(x_0, t_\beta) \approx \left\{\Gamma^{-1}\mathbf{g} + \Gamma^{-1}\mathbf{1}\left(\frac{1 - \mathbf{1}'\Gamma^{-1}\mathbf{g}}{\mathbf{1}'\Gamma^{-1}\mathbf{1}}\right)\right\}'(\mathbf{z}_\beta - \bar{\mathbf{z}}) + \left\{N^{-1}\mathbf{n} + N^{-1}\mathbf{1}\left(\frac{1 - \mathbf{1}'N^{-1}\mathbf{n}}{\mathbf{1}'N^{-1}\mathbf{1}}\right)\right\}'\bar{\mathbf{z}} \tag{10}$$

This result is easily interpretable. The first term essentially kriges the $e(\ ,\ )$ field at time t_β, and the second term kriges the $m(\ )$ field. More specifically, if we define $\mathbf{e}_\beta = (e(x_1, t_\beta), \cdots, e(x_n, t_\beta))'$ and $\mathbf{m} = (m(x_1), \cdots, m(x_n))'$, then the ordinary kriging predictor of $e(x_0, t_\beta)$ based on $\mathbf{e}_\beta$ is

$$\left\{\Gamma^{-1}\mathbf{g} + \Gamma^{-1}\mathbf{1}\left(\frac{1 - \mathbf{1}'\Gamma^{-1}\mathbf{g}}{\mathbf{1}'\Gamma^{-1}\mathbf{1}}\right)\right\}'\mathbf{e}_\beta$$

and the ordinary kriging predictor of $m(x_0)$ based on $\mathbf{m}$ is

$$\left\{N^{-1}\mathbf{n} + N^{-1}\mathbf{1}\left(\frac{1 - \mathbf{1}'N^{-1}n}{\mathbf{1}'N^{-1}\mathbf{1}}\right)\right\}'\mathbf{m}$$

If only observations from time t_β are used to predict $z(x_0, t_\beta)$, then the kriging predictor based on observations from that time is

$$\{(N + \Gamma)^{-1}(\mathbf{n} + \mathbf{g}) + (N + \Gamma)^{-1}\mathbf{1} \cdot (1 - \mathbf{1}'(N + \Gamma)^{-1}(\mathbf{n} + \mathbf{g}))/\mathbf{1}'(N + \Gamma)^{-1}\mathbf{1}\}'\mathbf{z}_\beta \tag{11}$$

By restricting ourselves to observations from time t_β, we lose information about the part of the spatial structure of the data that does not vary in time. If a substantial amount of the spatial structure is independent of time, then the predictor given by (7)–(9) may be considerably better than the one in (11).

It is instructive to consider how these results are changed if we use the following modified model for $z(\ ,\ )$:

$$z(x, t) = \mu + m(x) + e(x, t) \tag{12}$$

That is, we make μ independent of time; however, we assume (2)–(6) still hold. The assumptions in (2)–(6) are no longer sufficient to obtain the variance of the error all linear unbiased predictors. Therefore we make the additional assumption

$$\tfrac{1}{2}E(e(x, t_i) - e(x', t_j))^2 = \gamma^* \qquad i \neq j \tag{13}$$

Suppose $\gamma(r)$ tends to a limit, denoted by $\gamma(\infty)$, when $|r| \to \infty$. Then *Stein* [1984, appendix F] shows that (3), (4), and (13) define a valid spatial-temporal semivariogram for the $e(\ ,\ )$ field if and only if $\gamma^* \geq \gamma(\infty)$.

If $\gamma^* = \infty$, then the kriging predictor of $z(x_0, t_\beta)$ is still given by (7)–(9) [*Stein*, 1984, p. 139]. If $\gamma^* < \infty$, then the kriging predictor of $z(x_0, t_\beta)$ is given by

$$\hat{z}(x_0, t_\beta) = \mathbf{a}'\bar{\mathbf{z}} + \mathbf{c}'(\mathbf{z}_\beta - \bar{\mathbf{z}})$$

where

$$\mathbf{c} = \Gamma^{-1}\mathbf{g} + \{\gamma^*(1 - \mathbf{1}'\Gamma^{-1}\mathbf{g})/(\gamma^*\mathbf{1}'\Gamma^{-1}\mathbf{1} - 1)\}\Gamma^{-1}\mathbf{1}$$

and $\mathbf{a}$ is defined as in (9). If we also assume that $Ee(x, t)^2 = \gamma^*$ for all x and t, then we have

$$\sigma(x - x') = Ee(x, t)e(x', t) = \gamma^* - \gamma(x - x')$$

Define $\Sigma = (\sigma(x_i - x_j))$ and $\mathbf{s} = (\sigma(x_i - x_0))$. Then $\mathbf{c} = \Sigma^{-1}\mathbf{s}$. Thus for k large we have roughly

$$\hat{z}(x_0, t_\beta) \approx \mathbf{s}'\Sigma^{-1}(\mathbf{z}_\beta - \bar{\mathbf{z}}) + \{N^{-1}\mathbf{n} + N^{-1}\mathbf{1}(1 - \mathbf{1}'N^{-1}\mathbf{n})/\mathbf{1}'N^{-1}\mathbf{1}\}'\bar{\mathbf{z}} \tag{14}$$

which we can compare to (10), the analogous result with μ varying in time. When μ is fixed, for k large, $\mathbf{z}_\beta - \bar{\mathbf{z}}$ is approximately $\mathbf{e}_\beta$. Thus we can essentially krige a field with known mean of zero; the first term in (14) is the best linear unbiased predictor of a field with known mean zero. When μ varies arbitrarily in time, $\mathbf{z}_\beta - \bar{\mathbf{z}}$ will be approximately equal to $\mathbf{e}_\beta + c\mathbf{1}$, where c is unknown. Thus we must krige a field with an unknown constant mean; the first term in (10) is the best linear unbiased predictor in this situation. In each case, the kriging of the $m(\ )$ field (the second term in equations (10) and (14)) is the same.

We can obtain kriging predictors under this model even when the observation sites are not identical at each time by using the general cokriging formula [*Myers*, 1982]. However, we will have to invert a matrix of the size of the total number of observations to obtain the kriging predictors, whereas when the observation sites are fixed, we only have to invert two matrices of size n, the number of observations in a single time.

3. Prediction of Changes Over Time

In this section we consider the problem of predicting the average change in $z(\ ,\ )$ in a region between two different times. That is, given a region R with area $A(R)$, we wish to predict

$$A(R)^{-1} \int_R (z(x, t_\alpha) - z(x, t_\beta))\, dx \tag{15}$$

We note that this quantity is independent of the $m(\ )$ field, and thus it is natural to predict it using a procedure that is independent of the $m(\ )$ field. Suppose we observe $z(\ ,\ )$ at $x_1, \cdots, x_n$ at times t_α and t_β. Define $\Delta z(x) = z(x, t_\alpha) - z(x, t_\beta)$ and $\Delta \mathbf{z} = (\Delta z(x_1) \cdots \Delta z(x_n))'$. We can predict the quantity in (15) based on $\Delta \mathbf{z}$, which does not depend on $m(\ )$. Using assumptions (1)–(4), but making no assumptions about $m(\ )$, we have that

$$E\Delta z(x) = \mu(t_\alpha) - \mu(t_\beta)$$

$$\tfrac{1}{2}E(\Delta z(x) - \Delta z(x'))^2 = 2\gamma(x - x')$$

where $\gamma(\ )$ is the semivariogram of the $e(\ , t)$ field (see equation (3)). Thus using the methods for prediction of an areal average described in the work by *Journel and Huijbregts* [1978, p. 412] we can obtain the kriging predictor based on $\Delta \mathbf{z}$ of (15) if $\gamma(\ )$ is specified. The problem of modeling the $m(\ )$ field is avoided. Also, if the assumptions (5) and (6) about the $m(\ )$ field are valid (we can even allow the variogram of $m(\ )$ to be nonhomogeneous), then this predictor will be best among all linear unbiased predictors, not just those that are a function of $\Delta \mathbf{z}$ (see Appendix B).

An alternative way of predicting (15) is to predict $A(R)^{-1} \int_R z(x, t)\, dx$ separately for time t_α and t_β, take the difference of these predictions, and estimate the standard deviation of the error of this difference assuming the two prediction errors are uncorrelated. While these two procedures will usually give approximately the same predictions, the estimated standard deviations may be radically different. If a substantial part of the spatial variation in $z(\ ,\ )$ persists from t_α to t_β, then the second procedure will produce much higher estimated standard deviations. The problem is analogous to using a paired t test rather than a two-sample t test. That is, if the observations within a pair (from the same site) tend to be similar, then we can lower the standard deviation of our estimator of the average pairwise difference (equation (15)) by using a paired t test (first procedure).

4. CONCLUSIONS

We have considered the use of a simple stochastic model, defined in (1)–(6), for spatial-temporal processes. This model allows for the persistence in time of spatial structure by including a term (the $m(\ )$ field) that is fixed in time. We can apply this model to the problem of prediction at places other than where the process was observed; assuming that the observation sites are fixed in time, a simple expression for the kriging predictor under this model is given in section 2. This model can be especially useful for predicting average changes between two times of the process over some region, as is described in section 3. When the process is observed at the same sites for these two times, we are led to a procedure that predicts the average change based only on the differences between observations at the same locations for the two times. This procedure makes it unnecessary to model the component of the process that is fixed in time and thus is easy to use.

The simplicity of the model used here is sometimes a virtue, but it makes the model clearly inapplicable in many situations. For example, the lack of spatial-temporal interactions makes the model inappropriate for many daily and weekly data sets. More complex models, such as those considered in the work by *Rodriquez-Iturbe and Mejia* [1974], are needed in these situations.

APPENDIX A

We consider a generalization of the model presented in (1)–(6) in which we allow the mean functions of $m(x)$ and $e(x, t)$ not to be constant. Specifically, assume

$$Em(x) = \boldsymbol{\alpha}_m'\mathbf{f}(x) \tag{16}$$

$$Ee(x, t) = \boldsymbol{\alpha}_e(t)'\mathbf{h}(x) \tag{17}$$

where $\mathbf{f}(x)$ and $\mathbf{h}(x)$ are known vector-valued functions, and $\boldsymbol{\alpha}_m$ and $\boldsymbol{\alpha}_e(t_1), \cdots, \boldsymbol{\alpha}_e(t_k)$ are vectors of unknown coefficients. By taking the first component of $\mathbf{h}(\ )$ to be identically one, we can absorb the $\mu(\ )$ term in (1) into the mean of the $e(\ ,\ )$ field, so there is no need to have a separate term for $\mu(t)$. If the mean functions are polynomials, we could allow $-\eta(\ )$ and $-\gamma(\ )$ to be generalized covariance functions [*Delfiner*, 1976]. Linear unbiased predictors of $z(x_0, t_\beta)$ will be of the form

$$\hat{z}(x_0, t_\beta) = \sum_{ij} \lambda_{ij} z(x_i, t_j)$$

By symmetry considerations, the best linear unbiased predictor will have the same weight for each observation at x_i other than $z(x_i, t_\beta)$. That is, the optimal λ_{ij}'s will satisfy

$$\lambda_{ij} = w_i + b_i\delta_{j\beta} \tag{18}$$

where $\delta_{j\beta}$ equals one if $j = \beta$ and zero otherwise. To obtain unbiasedness, we must have

$$\begin{aligned}\boldsymbol{\alpha}_m'\mathbf{f}(x) + \boldsymbol{\alpha}_e(t_\beta)'\mathbf{h}(x) &= E\left[\sum_{ij} z(x_i, t_j)\right] \\ &= \sum_{ij} [w_i + b_i\delta_{j\beta}][\boldsymbol{\alpha}_m'\mathbf{f}(x_i) + \boldsymbol{\alpha}_e(t_j)'\mathbf{h}(x_i)] \\ &= \sum_i (kw_i + b_i)\boldsymbol{\alpha}_m'\mathbf{f}(x_i) + \sum_i w_i\left[\sum_j \boldsymbol{\alpha}_e(t_j)'\right]\mathbf{h}(x_i) \\ &\quad + \sum_i b_i\boldsymbol{\alpha}_e(t_\beta)'\mathbf{h}(x_i)\end{aligned}$$

for all values of $\boldsymbol{\alpha}_m$ and $\boldsymbol{\alpha}_e(t_1), \cdots, \boldsymbol{\alpha}_e(t_k)$. That is,

$$\sum_i (kw_i + b_i)\mathbf{f}(x_i) = \mathbf{f}(x) \tag{19a}$$

$$\sum_i w_i\mathbf{h}(x_i) = 0 \tag{19b}$$

$$\sum_i b_i\mathbf{h}(x_i) = \mathbf{h}(x) \tag{19c}$$

For estimators satisfying (18) and (19) we can show that

$$\begin{aligned}\mathrm{Var}\,(\hat{z}(x_0, t_\beta) - z(x_0, t_\beta)) &= 2\sum_i (kw_i + b_i)\eta(x_i - x_0) \\ &\quad + 2\sum_i (w_i + b_i)\gamma(x_i - x_0) \\ &\quad - \sum_{ij} (k^2w_iw_j + 2kw_ib_j + b_ib_j)\eta(x_i - x_j) \\ &\quad - \sum_{ij} (kw_iw_j + 2w_ib_j + b_ib_j)\gamma(x_i - x_j)\end{aligned} \tag{20}$$

Minimizing (20) subject to (19) we obtain that the kriging predictor satisfies

$$\begin{bmatrix} k^2N + k\Gamma & kN + \Gamma & kF' & H' & O \\ kN + \Gamma & N + \Gamma & F' & O & H' \\ kF & F & O & O & O \\ H & O & O & O & O \\ O & H & O & O & O \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{b} \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} k\mathbf{n} + \mathbf{g} \\ \mathbf{n} + \mathbf{g} \\ \mathbf{f}(x) \\ \mathbf{0} \\ \mathbf{h}(x) \end{bmatrix} \tag{21}$$

where $F = (\mathbf{f}(x_1), \cdots, \mathbf{f}(x_n))$, $H = (\mathbf{h}(x_1), \cdots, \mathbf{h}(x_n))$, $\mathbf{w} = (w_1, \cdots, w_n)'$, and $\mathbf{b} = (b_1, \cdots, b_n)'$, O is a matrix of zeroes', and $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are vectors of Lagrange multipliers. We note that the model in (1)–(6) is just the special case $\mathbf{h}(x) \equiv 1$ and $\mathbf{f}(x) \equiv 0$ in (16) and (17). The unbiasedness constraints in (19) may contain some redundancies. For example, if a component of $\mathbf{f}(\ )$ is identical to a component of $\mathbf{h}(\ )$, which will occur if we are modelling the means of $m(\ )$ and $e(\ ,\ )$ as polynomials, then we will have a redundant constraint that will cause the matrix on the left-hand side of (21) to be singular. We will assume that any such redundancies have been eliminated by removing the appropriate components of $\mathbf{f}(\ )$ throughout (21). Then lengthy but straightforward calculations yield

$$\mathbf{b} = \Gamma^{-1}\mathbf{g} + \Gamma^{-1}H'(H\Gamma^{-1}H')^{-1}(\mathbf{h}(x) - H\Gamma^{-1}\mathbf{g})$$

$$\begin{aligned}\mathbf{a} \equiv \mathbf{b} + k\mathbf{w} = [&R + kRF'WFR - kRF'WB'AHR \\ &- kRH'ABWFR - RH'AHR + kRH'ABWB'AHR](k\mathbf{n} + \mathbf{g}) \\ &- kRF'W\mathbf{f}(x) + kRH'ABW\mathbf{f}(x) \\ &+ [RH'WB'A + k^{-1}RH'A - RH'ABWB'A]\mathbf{h}(x)\end{aligned}$$

where

$$R = (kN + \Gamma)^{-1}$$
$$A = (HR^{-1}H')^{-1}$$
$$B = HR^{-1}F'$$
$$W = (-kFRF' + kB'AB)^{-1}$$

We have, similar to (7),

$$\hat{z}(x_0, t_\beta) = \mathbf{a}'\bar{\mathbf{z}} + \mathbf{b}'(\mathbf{z}_\beta - \bar{\mathbf{z}}) \tag{22}$$

Thus we can compare $\hat{z}(x_0, t_\beta)$ only computing inverses of matrices of size $n \times n$. We see that $\mathbf{b}$ does not depend on k and that as $k \to \infty$,

$$\begin{aligned}\mathbf{a} \to [N^{-1} + N^{-1}UFN^{-1} - N^{-1}F'UD'CHN^{-1} \\ - N^{-1}H'CDUFN^{-1} - N^{-1}H'CHN^{-1} \\ + N^{-1}H'CDUD'CHN^{-1}]\mathbf{n} \\ - N^{-1}F'U\mathbf{f}(x) + N^{-1}H'CDU\mathbf{f}(x)\end{aligned}$$

where

$$C = (HN^{-1}H')^{-1}$$
$$D = HN^{-1}F'$$
$$U = (-FN^{-1}F' + D'CD)^{-1}$$

While the second term in (22) can be interpreted as kriging the $e(\ , t_\beta)$ field for large k, the first term no longer has an apparent simple interpretation.

We note that while we have allowed $\boldsymbol{\alpha}_e(t)$ to depend on t, we have chosen $\mathbf{h}(x)$ to be independent of time. While this restriction is natural when the mean function is a polynomial in x, it does not allow us to use an observed quantity that varies in time as a component of the mean function of $e(\ ,\ )$. If we allowed $\mathbf{h}(\ )$ to depend on time, we could still compute the kriging predictor, but it would no longer necessarily be of the form given in (18). Hence we would be forced to invert a $kn \times kn$ matrix to compute the kriging predictor.

APPENDIX B

In this appendix we show that the kriging predictor of

$$A(R)^{-1} \int_R (z(x, t_\alpha) - z(x, t_\beta))\, dx \tag{15}$$

is a linear function of

$$\Delta\mathbf{z} = (z(x_1, t_\alpha) - z(x_1, t_\beta) \cdots z(x_n, t_\alpha) - (x_n, t_\beta))'$$

We assume that (1)–(5) hold and that the $m(\ )$ field is independent of the $e(\ ,\ )$ field. However, it is unnecessary to make any assumptions about second moments of the $m(\ )$ field other than that $E(m(x) - m(x'))^2$ is finite for all x and x'. We also assume that $z(\ ,\ )$ is observed at the same set of sites, $x_1, \cdots, x_n$, at times t_α and t_β. At the other times, $z(\ ,\ )$ may be observed at any set of sites. That is, we can observe $z(x, t_j)$ at $x_{i1}, \cdots, x_{in_j}$, with only the restrictions that $n_\alpha = n_\beta = n$, and $x_{i\alpha} = x_{i\beta} = x_i$ for $i = 1, \cdots, n$. A linear predictor of (15) will be of the form

$$\sum_{j=1}^{k} \sum_{i=1}^{n_j} \lambda_{ij} z(x_{ij}, t_j)$$

By symmetry considerations, the kriging predictor must satisfy

$$\lambda_{i\alpha} = -\lambda_{i\beta} \qquad i = 1, \cdots, n$$

By unbiasedness considerations,

$$\sum_{j \neq \alpha,\beta} \sum_{i=1}^{n_j} \lambda_{ij} = 0$$

The kriging predictor can be expressed as

$$\mathbf{v}'\Delta\mathbf{z} + \sum_{j \neq \alpha,\beta} \sum_{i=1}^{n_j} \lambda_{ij} z(x_{ij}, t_j) \tag{23}$$

where $\mathbf{v} = (\lambda_{1\alpha} \ldots \lambda_{n\alpha})'$. But $\Delta\mathbf{z}$ is independent of the $m(\ )$ field, and together with assumption (4) about the $e(\ ,\ )$ field we can conclude that the two terms in (23) are uncorrelated. It follows that for the kriging predictor of (15), $\lambda_{ij} = 0$ if $j \neq \alpha, \beta$. Equivalently, the kriging predictor of (15) is a function of $\Delta\mathbf{z}$. This result also holds if (1)–(4) hold and $m(\ )$ is considered to be an unknown, arbitrary deterministic field.

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