Supplementary material to "Additive Model Building for Spatial Regression" by Siddhartha

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1 Spatial covariance functions

Table 1 gives the expression of spatial covariance functions that we have considered in the paper.

Name	Parameters	$\delta(m{h})$
Exponential, $\text{Exp}(\rho)$	ρ	$\sigma^2 \exp(-\rho \mathbf{h})$
Matérn, $\mathrm{Mat}_{ u}(ho)$	u, ho	$\sigma^2 \frac{2^{1- u}}{\Gamma(u)} \left(\frac{\sqrt{2 u} \boldsymbol{h} }{ ho} \right)^{ u} K_{ u} \left(\frac{\sqrt{2 u} \boldsymbol{h} }{ ho} \right)$
Inverse Multiqudratics, InvMQ(ρ)	ho	$\sigma^2 (1 + (\mathbf{h})^2)^{-\rho}$
Gaussian, Gauss(ρ)	ho	$\sigma^2 \exp(- ho^2 m{h} ^2)$

Table 1: Spatial covariance functions, $\delta(h)$, considered in the simulation study for a covariance model and a spatial weight matrix.

2 Derivation of the assumption (K 2)'

We restate the assumption (K 2)'.

(K 2)'
$$\frac{\lambda_{n1}\sqrt{m_n}}{\lambda_{n2}} + \frac{\lambda_{n2}m_n}{n} = o(1).$$

Using the given choice of r_n ,

$$\frac{n^2}{\lambda_{n2}^2 r_n^2 m_n} \geq \frac{n^2}{\lambda_{n2}^2 m_n} \frac{\rho_{\max}^2(\boldsymbol{L}) m_n^3 \log(J m_n)}{n}$$

$$= \frac{n \rho_{\max}^2(\boldsymbol{L}) m_n^2 \log(J m_n)}{\lambda_{n2}^2}$$

$$= \frac{n \rho_{\max}^2(\boldsymbol{L}) m_n \log(s_n m_n)}{\lambda_{n2}^2 r_n^2} \left\{ \frac{\log(J m_n)}{\log(s_n m_n)} r_n^2 m_n \right\}.$$
(1)

Since $\left\{\frac{\log(Jm_n)}{\log(s_nm_n)}r_n^2m_n\right\}\longrightarrow\infty$, if L.H.S of (1) goes to 0, then $\frac{\sqrt{\rho_{\max}^2(\boldsymbol{L})nm_n\log(s_nm_n)}}{\lambda_{n2}r_n}\longrightarrow0$. Also by same choice of r_n and using the fact that $m_n\asymp n^{1/2\tau+1}$, we have

$$\frac{n^2}{\lambda_{n2}^2 r_n^2 m_n} = \frac{n^2 \rho_{\max}^2(\mathbf{L}) m_n^3 \log(J m_n)}{\lambda_{n2}^2 n m_n}$$

$$= \frac{m_n^2 n \rho_{\max}^2(\mathbf{L}) \log(J m_n)}{\lambda_{n2}^2} = \frac{m_n \lambda_{n1}^2}{\lambda_{n2}^2}.$$

Therefore under (H 1) to (H 7), (K 2) can be replaced by (K 2)'.

3 Proofs of Lemmas

To be self-contained, we restate Lemmas together with proofs.

Lemma 1 (Lemma 1 in Huang et al. (2010b)). Suppose that $f \in \mathcal{F}$ and $\mathbb{E}f(X_j) = 0$. Then under (**H 4**) and (**H 5**) in Assumption 1, there exists an $f_n \in \mathcal{S}_{nj}^0$ such that

$$||f_n - f||_2 = O_p\left(m_n^{-\tau} + \sqrt{\frac{m_n}{n}}\right).$$
 (2)

Particularly, under the choice of $m_n = O(n^{\frac{1}{2\tau+1}})$, we have

$$||f_n - f||_2 = O_p(m_n^{-\tau}) = O_p(n^{-\frac{\tau}{2\tau+1}}).$$
 (3)

Lemma 2. Suppose that |A| is bounded by a fixed constant independent of n and J. Let $h_n
subseteq m_n^{-1}$. Then under (**H 4**) and (**H 5**) in Assumption 1, with probability converging to one,

$$\rho_{\min}(\Sigma_W^{-1})d_1h_n \le \rho_{\min}(\Omega_A) \le \rho_{\max}(\Omega_A) \le \rho_{\max}(\Sigma_W^{-1})d_2h_n \tag{4}$$

Additionally under (H 7), (4) becomes,

$$c_1 h_n \le \rho_{\min}(\Omega_A) \le \rho_{\max}(\Omega_A) \le c_2 h_n$$
 (5)

where d_1 , d_2 , c_1 and c_2 are some positive constants.

Proof. One can follow the proof of the Lemma 3 in Huang *et al.* (2010b) but after observing that,

$$\rho_{\min}(\boldsymbol{\Sigma}_{W}^{-1}) \left(\frac{\mathbb{B}_{A}^{c} / \mathbb{B}_{A}^{c}}{n} \right) \leq \Omega_{A} = \frac{\mathbb{D}_{A}^{c} / \mathbb{D}_{A}^{c}}{n} \leq \rho_{\max}(\boldsymbol{\Sigma}_{W}^{-1}) \left(\frac{\mathbb{B}_{A}^{c} / \mathbb{B}_{A}^{c}}{n} \right)$$

which gives (4). By (**H 7**), the well-conditioned property of Σ_W^{-1} , we have (5).

Lemma 3. Define M_n be a non-negative definite matrix of order n and

$$T_{jl} = \left(\frac{m_n}{n}\right)^{\frac{1}{2}} \mathbf{a}'_{jl} \mathbf{M}_n \boldsymbol{\epsilon} \qquad \forall 1 \le j \le J, 1 \le l \le m_n$$
 (6)

where $a_{jl} = (\mathbb{B}_l^c(X_j(s)), s \in S)'$ and $T_n = \max_{\substack{1 \le j \le J \\ 1 \le l \le m_n}} |T_{jl}|$. Then, under assumptions (H 2) to (H 5) in Assumption 1,

$$\mathbb{E}(T_n) \le C_1 \rho_{\max}(\boldsymbol{M}_n) \sqrt{(m_n \log(Jm_n))},\tag{7}$$

for some $C_1 > 0$.

Proof. Since $\epsilon \sim \text{Gaussian}(\mathbf{0}, \Sigma_T)$, $T_{jl} \sim \text{Gaussian}(\mathbf{0}, \frac{m_n}{n} \mathbf{a}'_{jl} \mathbf{M}_n \Sigma_T \mathbf{M}'_n \mathbf{a}_{jl})$. Therefore we can use maximal inequalities of sub-Gaussian random variables [van der Vaart and Wellner (1996), Lemmas 2.2.1 and 2.2.2]. Let $\|\cdot\|_{\phi}$ be the Orlicz norm, defined by $\|X\|_{\phi} = \inf\{k \in (0,\infty) \mid \mathbb{E}(\phi(|X|/k)) \leq 1\}$. Then, conditional on $\{X_j(s), s \in S, 1 \leq j \leq J\}$, we have the following

$$\| \max_{1 \leq j \leq J, 1 \leq l \leq m_n} |T_{jl}| | X_{j}(\boldsymbol{s}), \boldsymbol{s} \in \boldsymbol{S}, 1 \leq j \leq J \|_{\phi_2}$$

$$\leq K \sqrt{\frac{m_n}{n} \log(1 + Jm_n)} \max_{1 \leq j \leq J, 1 \leq l \leq m_n} \| |\boldsymbol{a}'_{jl} \boldsymbol{M}_n \boldsymbol{\epsilon}| | X_{j}(\boldsymbol{s}), \boldsymbol{s} \in \boldsymbol{S}, 1 \leq j \leq J \|_{\phi_2}$$

$$\leq K \sqrt{\frac{m_n}{n} \log(Jm_n)} \max_{1 \leq j \leq J, 1 \leq l \leq m_n} \sqrt{\boldsymbol{a}'_{jl} \boldsymbol{M}_n \boldsymbol{\Sigma}_T \boldsymbol{M}'_n \boldsymbol{a}_{jl}},$$

where K > 0 is a generic constant and $\phi_p(x) = e^{x^p} - 1$. Now taking expectation with respect to $\{X_j(s), s \in S, 1 \le j \le J\}$ on both sides of the above inequality,

$$\begin{aligned}
& \max_{1 \leq j \leq J, 1 \leq l \leq m_n} |T_{jl}| \|_{\phi_2} \\
& \leq K \sqrt{\frac{m_n}{n}} \log(Jm_n) \mathbb{E} \left(\max_{1 \leq j \leq J, 1 \leq l \leq m_n} \sqrt{\boldsymbol{a}'_{jl} \boldsymbol{M}_n \boldsymbol{\Sigma}_T \boldsymbol{M}'_n \boldsymbol{a}_{jl}} \right) \\
& = K \sqrt{\frac{m_n}{n}} \log(Jm_n) \mathbb{E} \left(\sqrt{\max_{1 \leq j \leq J, 1 \leq l \leq m_n} \boldsymbol{a}'_{jl} \boldsymbol{M}_n \boldsymbol{\Sigma}_T \boldsymbol{M}'_n \boldsymbol{a}_{jl}} \right) \\
& \leq K \rho_{\max}(\boldsymbol{M}_n) \sqrt{\frac{m_n}{n}} \log(Jm_n) \sqrt{\mathbb{E} \left(\max_{1 \leq j \leq J, 1 \leq l \leq m_n} \boldsymbol{a}'_{jl} \boldsymbol{\Sigma}_T \boldsymbol{a}_{jl} \right)}.
\end{aligned} \tag{8}$$

Since $\mathbb{B}_{l}^{c}(x)$ are normalized B-splines, we have

$$\mathbb{E}\left(\max_{1\leq j\leq J, 1\leq l\leq m_n} \sum_{\boldsymbol{s}\in \boldsymbol{S}} \sum_{\boldsymbol{s}'\in \boldsymbol{S}} \mathbb{B}_l^c(X_j(\boldsymbol{s}))\delta(\boldsymbol{s}-\boldsymbol{s}')\mathbb{B}_l^c(X_j(\boldsymbol{s}'))\right)$$

$$\leq 4\sum_{\boldsymbol{s}\in \boldsymbol{S}} \sum_{\boldsymbol{s}'\in \boldsymbol{S}} \delta(\boldsymbol{s}-\boldsymbol{s}')$$

$$\leq K\sum_{\boldsymbol{s}\in \boldsymbol{S}} \int_{\boldsymbol{h}\in C\boldsymbol{D}_n} \delta(\boldsymbol{h})d\boldsymbol{h}$$

$$\leq Kn\int_{\boldsymbol{h}\in C\boldsymbol{D}_n} \delta(\boldsymbol{h})d\boldsymbol{h}.$$
(9)

for some K, C > 0. From (8) and (9),

$$\| \max_{1 \le j \le J, 1 \le l \le m_n} |T_{jl}| \|_{\phi} \le K \rho_{\max}(\boldsymbol{M}_n) \sqrt{m_n \log(Jm_n)} \sqrt{\int_{\boldsymbol{h} \in C} \boldsymbol{D}_n} \delta(\boldsymbol{h}) d\boldsymbol{h}$$
$$\le K \rho_{\max}(\boldsymbol{M}_n) \sqrt{m_n \log(Jm_n)}.$$

Finally, (35) follows from $||X||_{L^1} \le C||X||_{L^2} \le ||X||_{\phi_2}$, where $||X||_{L^p} = (\mathbb{E}(|X|^p))^{1/p}$.

Lemma 4. Under the Assumption 1 with $\lambda_{n1} > C\rho_{\max}(L)\sqrt{nm_n\log(Jm_n)}$ for a sufficiently large constant C. we have $|\tilde{A}_{\beta}| \leq M_1|A_*|$ for a finite constant $M_1 > 1$ with w.p. converging to 1.

Proof. Along with considering the approximation error for spline regression, we also have to take care of the dependence structure of a Gaussian random vector $\boldsymbol{\epsilon}$ according to $(\mathbf{H} \ 3)$. To emphasize the dependence on n, we denote write $\boldsymbol{\epsilon}_n$ instead of $\boldsymbol{\epsilon}$ and similar notation for others as well. Recall $\boldsymbol{\pi}_n = \boldsymbol{\epsilon}_n + \boldsymbol{\theta}_n$, where $\boldsymbol{\theta}_n = (\theta_n(\boldsymbol{s}); \boldsymbol{s} \in \boldsymbol{S})'$ with $\theta_n(\boldsymbol{s}) = \sum_{j=1}^J (f_j(X_j(\boldsymbol{s})) - f_{nj}(X_j(\boldsymbol{s})))$. Note that $\|\boldsymbol{\theta}_n\|_2 = \boldsymbol{O}(n^{1/2}q^{1/2}m_n^{-\tau}) = \boldsymbol{O}(q^{1/2}n^{1/(4\tau+2)})$ by Lemma 1 since $m_n = \boldsymbol{O}(n^{1/(2\tau+1)})$. Define $\lambda_{n,J} = 2\sqrt{K\rho_{\max}^2(\boldsymbol{L})m_nn\log(Jm_n)}$ for some K>0 and $\lambda_{n1} \geq \max\{\lambda_0,\lambda_{n,J}\}$, where $\lambda_0 = \inf\{\lambda:M_1(\lambda)q^*+1\leq q_0\}$ for some finite $q_0>0$ and $M_1(\lambda)>1$, which will be specified later in the proof. Without loss of generality, we will assume the infimum of an empty set to be ∞ . That is, if $\{\lambda:M_1(\lambda)q^*+1\leq q_0\}$ is an empty set, it implies that $\lambda_{n1}=\lambda_0=\infty$ and which in turn implies that we drop all the components in our additive model, i.e. $|\tilde{A}_{\boldsymbol{\beta}}|=0$. So part (i) is trivial in this case and hence for the rest of the proof we will assume $\{\lambda:M_1(\lambda)q^*+1\leq q_0\}$ is a non-empty set.

First, define a new vector U_k such that $U_k = \mathbb{D}_k^{c'}(\mathbf{Z}^c - \mathbb{D}^c \hat{\boldsymbol{\beta}}_{gL})/\lambda_{n1}$ for $k = 1, \dots, J$. By Karsuh-Kuhn-Tucker (KKT) conditions of the optimization problem for $\mathbf{Q}_n(\boldsymbol{\beta}, \lambda_n)$ with the solution $\hat{\boldsymbol{\beta}}_{gL}$, we have

$$U_{k} \begin{cases}
= \frac{\hat{\boldsymbol{\beta}}_{gL,k}}{\|\hat{\boldsymbol{\beta}}_{gL,k}\|_{2}} & \text{if } \|\hat{\boldsymbol{\beta}}_{gL,k}\|_{2} > 0, \\
\leq 1 & \text{if } \|\hat{\boldsymbol{\beta}}_{gL,k}\|_{2} = 0.
\end{cases}$$
(10)

Then, the norm of U_k is

$$\|\boldsymbol{U}_{k}\|_{2} \begin{cases} = 1 & \text{if } \|\hat{\boldsymbol{\beta}}_{gL,k}\|_{2} > 0, \\ \leq 1 & \text{if } \|\hat{\boldsymbol{\beta}}_{gL,k}\|_{2} = 0. \end{cases}$$
(11)

Now we introduce the following quantities.

$$x_r = \max_{|A|=r} \max_{\|U_k\|_2=1, k \in A, B \subset A} \left| oldsymbol{\pi}_n' oldsymbol{w}_{A|B} \right|, \quad ext{and}$$
 $x_r^* = \max_{|A|=r} \max_{\|U_k\|_2=1, k \in A, B \subset A} \left| oldsymbol{\epsilon}_n' oldsymbol{w}_{A|B} \right|,$

where $\boldsymbol{w}_{A|B} = \boldsymbol{W}_{A|B} / \|\boldsymbol{W}_{A|B}\|_2$ with $\boldsymbol{W}_{A|B} = (\mathbb{D}_A^c (\mathbb{D}_A^c / \mathbb{D}_A^c)^{-1} \lambda_{n1} Q_{BA}' Q_{BA} \boldsymbol{U}_A - (\mathbb{I} - P_A) \mathbb{D}^c \boldsymbol{\beta})$. For $B \subset A$, Q_{BA} is the matrix corresponding to the selection of variables in B from A, i.e. $Q_{BA}\boldsymbol{\beta}_A = \boldsymbol{\beta}_B$. $P_A = \mathbb{D}_A^c \boldsymbol{\Omega}_A^{-1} \mathbb{D}_A^c / n$, $\boldsymbol{U}_A = (\boldsymbol{U}_k; k \in A)'$. By the triangle inequality and Cauchy-Schwarz inequality, for some set A with |A| = r > 0, we have

$$\begin{aligned} \left| \boldsymbol{\pi}_{n}' \boldsymbol{w}_{A|B} \right| &\leq \left| \boldsymbol{\epsilon}_{n}' \boldsymbol{w}_{A|B} \right| + \|\boldsymbol{\theta}_{n}\|_{2} \\ &\leq \left| \boldsymbol{\epsilon}_{n}' \boldsymbol{w}_{A|B} \right| + K_{2} q n^{1/(4\tau+2)} \\ &\leq \left| \boldsymbol{\epsilon}_{n}' \boldsymbol{w}_{A|B} \right| + K_{1} \sqrt{\frac{(r m_{n} \vee m_{n}) \rho_{\max}^{2}(\boldsymbol{L}) m_{n} \log(J m_{n})}{\rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_{1}})}} \end{aligned}$$
(12)

where the last inequality holds for a sufficiently large n for some $K_1 > 0$. By introducing the following sets,

$$\Omega_{r_0} = \left\{ (\mathbb{D}^c, \boldsymbol{\pi}_n); x_r \leq 2K_1 \sqrt{\frac{(rm_n \vee m_n)\rho_{\max}^2(\boldsymbol{L})m_n \log(Jm_n)}{\rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_1})}}, \forall r \geq r_0 \right\}, \quad \text{and}$$

$$\Omega_{r_0}^* = \left\{ (\mathbb{D}^c, \boldsymbol{\epsilon}_n); x_r^* \leq K_1 \sqrt{\frac{(rm_n \vee m_n)\rho_{\max}^2(\boldsymbol{L})m_n \log(Jm_n)}{\rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_1})}}, \forall r \geq r_0 \right\},$$

we can show $\mathbb{P}\left\{(\mathbb{D}^c, \boldsymbol{\pi}_n) \in \Omega_{r_0}\right\} \geq \mathbb{P}\left\{(\mathbb{D}^c, \boldsymbol{\epsilon}_n) \in \Omega_{r_0}^*\right\}$ for any $r_0 > 0$ since we have

$$x_r \leq x_r^* + \|\boldsymbol{\theta}_n\|_2 \leq x_r^* + K_1 \sqrt{\frac{(rm_n \vee m_n)\rho_{\max}^2(\boldsymbol{L})m_n\log(Jm_n)}{\rho_{\max}(\Omega_{\tilde{A}_1})}}$$
(13)

by recalling the definitions of x_r and x_r^* and (12).

Now, we want to show that $\mathbb{P}\left\{(\mathbb{D}^c, \boldsymbol{\pi}_n) \in \Omega_{q_1}\right\} \to 1$ implies $|\tilde{A}_{\boldsymbol{\beta}}| \leq M_1 |\tilde{A}_*| = M_1 q^*$ for some finite $M_1 > 1$, which completes the proof since $q^* \leq q$. Before proving this claim, we first show $\mathbb{P}\left\{(\mathbb{D}^c, \boldsymbol{\epsilon}_n) \in \Omega_{q_1}^*\right\} \to 1$, which implies $\mathbb{P}\left\{(\mathbb{D}^c, \boldsymbol{\pi}_n) \in \Omega_{q_1}\right\} \to 1$. We start with the following:

$$1 - \mathbb{P}\left\{(\mathbb{D}^{c}, \boldsymbol{\epsilon}_{n}) \in \Omega_{q_{1}}^{*}\right\}$$

$$\leq \sum_{r=0}^{\infty} \mathbb{P}\left(x_{r}^{*} > K_{1}\sqrt{\frac{(rm_{n} \vee m_{n})\rho_{\max}^{2}(\boldsymbol{L})m_{n}\log(Jm_{n})}{\rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_{1}})}}\right)$$

$$\leq \sum_{r=0}^{\infty} \binom{J}{r} \mathbb{P}\left(|\boldsymbol{w}_{A|B}^{\prime}\boldsymbol{\epsilon}_{n}| > K_{1}\sqrt{\frac{(rm_{n} \vee m_{n})\rho_{\max}^{2}(\boldsymbol{L})m_{n}\log(Jm_{n})}{\rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_{1}})}}\right),$$

$$(15)$$

where |A| = r. Since $w'_{A|B} \epsilon_n \sim \text{Gaussian}(0, w'_{A|B} \Sigma_T w_{A|B})$, (15) becomes

$$\leq 2 \sum_{r=0}^{\infty} {J \choose r} \exp \left(-0.5 K_1^2 \frac{(rm_n \vee m_n) \rho_{\max}^2(\boldsymbol{L}) m_n \log(Jm_n)}{(\boldsymbol{w}_{A|B}' \boldsymbol{\Sigma}_T \boldsymbol{w}_{A|B}) \rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_1})}\right)
\leq 2 \sum_{r=0}^{\infty} {J \choose r} \exp \left(-0.5 K_1^2 \frac{(rm_n \vee m_n) \rho_{\max}^2(\boldsymbol{L}) m_n \log(Jm_n)}{\rho_{\max}(\boldsymbol{\Sigma}_T) \rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_1})}\right)
= 2 \sum_{r=0}^{\infty} {J \choose r} (Jm_n)^{-0.5 K_1^2 (rm_n \vee m_n) \rho_{\max}^2(\boldsymbol{L}) m_n / \rho_{\max}(\boldsymbol{\Sigma}_T) \rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_1})}.$$
(16)

Let $K_n = 0.5K_1^2 m_n^2 \rho_{\text{max}}^2(\boldsymbol{L}) / (\rho_{\text{max}}(\boldsymbol{\Sigma}_T) \rho_{\text{max}}(\boldsymbol{\Omega}_{\tilde{A}_1}))$. Then (16) becomes,

$$= 2(Jm_n)^{-K_n} + 2\sum_{r=1}^{\infty} {J \choose r} (Jm_n)^{-rK_n}$$

$$\leq 2(Jm_n)^{-K_n} + 2\sum_{r=1}^{\infty} \frac{1}{r!} \left(\frac{J}{(Jm_n)^{K_n}}\right)^r$$

$$= 2(Jm_n)^{-K_n} + 2\exp\left(\frac{J}{(Jm_n)^{K_n}}\right) - 2.$$
(17)

Define $\|\mathbf{\Sigma}_T\|_1 = \max_{\mathbf{S} \in \mathbf{S}} \sum_{\mathbf{S}' \in \mathbf{S}} \boldsymbol{\sigma}_{\mathbf{S},\mathbf{S}'}$ and note that $\|\mathbf{\Sigma}_T\|_1 \asymp \int_{\mathbf{h} \in \mathbf{D}_n} \boldsymbol{\delta}(\mathbf{h}) d\mathbf{h} = \mathbf{O}(1)$. Therefore by using the fact, $\frac{1}{\sqrt{n}} \|\mathbf{\Sigma}_T\|_1 \le \rho_{\max}(\mathbf{\Sigma}_T) \le \sqrt{n} \|\mathbf{\Sigma}_T\|_1$ and $\rho_{\max}(\mathbf{\Sigma}_W^{-1}) \le \rho_{\max}(\mathbf{L}) \rho_{\max}(\mathbf{L}') = \rho_{\max}^2(\mathbf{L})$, we have

$$K_n \ge c_1 \frac{0.5K_1^2 m_n^3}{\sqrt{n} \|\mathbf{\Sigma}_T\|_1} \approx 0.5c_1 K_1^2 \sqrt{n^{6\gamma - 1}},$$

and $K_n \longrightarrow \infty$ by (**H 6**). This shows (17) goes to zero as $n \to \infty$.

To show $\mathbb{P}\left\{(\mathbb{D}^c, \boldsymbol{\pi}_n) \in \Omega_{q_1}\right\} \to 1$ implies $|\tilde{A}_{\boldsymbol{\beta}}| \leq M_1 |\tilde{A}_*| = M_1 q^*$, let

$$m{V}_{1j} = rac{m{\Omega}_{ ilde{A}_1}^{-rac{1}{2}} Q_{j1}' m{U}_{ ilde{A}_j} \lambda_{n1}}{\sqrt{n}}$$
 , for $j=1,3,4,$

and

$$m{u} = rac{\mathbb{D}_{ ilde{A}_1}^c m{\Omega}_{ ilde{A}_1}^{-1/2} m{V}_{14} / \sqrt{n} - m{\omega}_2}{\|\mathbb{D}_{ ilde{A}_1}^c m{\Omega}_{ ilde{A}_1}^{-1/2} m{V}_{14} / \sqrt{n} - m{\omega}_2\|_2},$$

where, for simplicity in notations, $Q_{kj}=Q_{\tilde{A}_k\tilde{A}_j}$ is the matrix corresponding to the selection of variables in \tilde{A}_k from \tilde{A}_j and $\boldsymbol{\omega}_2=(\mathbb{I}-P_{\tilde{A}_1})\mathbb{D}^c_{\tilde{A}_2}\boldsymbol{\beta}_{\tilde{A}_2}=(\mathbb{I}-P_{\tilde{A}_1})\mathbb{D}^c\boldsymbol{\beta}$. We can show that, $\boldsymbol{V}_{11}=\boldsymbol{V}_{14}+\boldsymbol{V}_{13}$ and $Q'_{31}Q_{31}+Q'_{41}Q_{41}=\mathbb{I}_{m_n|\tilde{A}_1|}$ due to the fact that $\tilde{A}_3\cup\tilde{A}_4=\tilde{A}_1,\,\tilde{A}_3\cap\tilde{A}_4=\phi$ and hence $\boldsymbol{\beta}'_{\tilde{A}_3}Q_{31}+\boldsymbol{\beta}'_{\tilde{A}_4}Q_{41}=\boldsymbol{\beta}_{\tilde{A}_1}$. Since $q_1=|\tilde{A}_1|=|\tilde{A}_3|+|\tilde{A}_4|$ and $|\tilde{A}_3|\leq q^*,\,|\tilde{A}_4|\geq (q_1-q^*)$. Then, we have the following lower bound for L_2 -norm of \boldsymbol{V}_{14} ,

$$\|\boldsymbol{V}_{14}\|_{2}^{2} \ge \frac{\lambda_{n1}^{2} \|Q_{41}^{\prime} U_{\tilde{A}_{4}}\|_{2}^{2}}{n\rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_{1}})} = \frac{\lambda_{n1}^{2} \|Q_{41}^{\prime} Q_{41} U_{\tilde{A}_{1}}\|_{2}^{2}}{n\rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_{1}})} = \frac{\lambda_{n1}^{2} m_{n} |\tilde{A}_{4}|}{n\rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_{1}})} \ge B_{1} \frac{(q_{1} - q^{*})^{+}}{q^{*}}, \quad (18)$$

with $B_1 = \frac{\lambda_{n_1}^2 m_n q^*}{n \rho_{\max}(\Omega_{\tilde{A}_1})}$. From (18), we have

$$|\tilde{A}_{\beta}| \le |\tilde{A}_1| = q_1 \le (q_1 - q^*)^+ + q^* \le q^* \frac{\|V_{14}\|_2^2}{B_1} + q^* = \left(\frac{\|V_{14}\|_2^2}{B_1} + 1\right) q^*.$$
 (19)

Thus, to show $|\tilde{A}_{\beta}| \leq M_1 q^*$ for some finite $M_1 > 1$, we need to show $\frac{\|\boldsymbol{V}_{14}\|_2^2}{B_1} + 1 \leq M_1$ for some finite $M_1 > 1$.

We start with an upper bound of $\|\boldsymbol{V}_{14}\|_2^2 + \|\boldsymbol{\omega}_2\|_2^2$. Since

$$||V_{14}||_{2}^{2} + ||\omega_{2}||_{2}^{2} = V'_{14}V_{14} + ||\omega_{2}||_{2}^{2} = V'_{14}(V_{11} - V_{13}) + ||\omega_{2}||_{2}^{2}$$

$$\leq V'_{14}V_{11} + ||V_{14}||_{2}||V_{13}||_{2} + ||\omega_{2}||_{2}^{2}, \tag{20}$$

we find upper bounds for $m{V}_{14}'m{V}_{11}$, $\|m{V}_{14}\|_2\|m{V}_{13}\|_2$ and $\|m{\omega}_2\|_2^2$, respectively. First,

$$V'_{14}V_{11} = \frac{\lambda_{n1}^{2}}{n}U'_{\tilde{A}_{4}}Q_{41}\Omega_{\tilde{A}_{1}}^{-1}U_{\tilde{A}_{1}}$$

$$= \frac{\lambda_{n1}^{2}}{n}U'_{\tilde{A}_{4}}Q_{41}\Omega_{\tilde{A}_{1}}^{-1}\left(\mathbb{D}_{\tilde{A}_{1}}^{c'}\left(\mathbf{Z}^{c} - \mathbb{D}^{c}\hat{\boldsymbol{\beta}}_{gL}\right)/\lambda_{n1}\right)$$

$$= \frac{\lambda_{n1}}{n}U'_{\tilde{A}_{4}}Q_{41}\Omega_{\tilde{A}_{1}}^{-1}\mathbb{D}_{\tilde{A}_{1}}^{c'}\left(\mathbb{D}_{\tilde{A}_{1}}^{c}\boldsymbol{\beta}_{\tilde{A}_{1}} + \mathbb{D}_{\tilde{A}_{2}}^{c}\boldsymbol{\beta}_{\tilde{A}_{2}} + \boldsymbol{\pi}_{n} - \mathbb{D}_{\tilde{A}_{1}}^{c}\hat{\boldsymbol{\beta}}_{gL,\tilde{A}_{1}}\right)$$

$$= \lambda_{n1}U'_{\tilde{A}_{4}}Q_{41}\left((\boldsymbol{\beta}_{\tilde{A}_{1}} - \hat{\boldsymbol{\beta}}_{gL,\tilde{A}_{1}}) + \frac{\Omega_{\tilde{A}_{1}}^{-1}}{n}(\mathbb{D}_{\tilde{A}_{1}}^{c'}\mathbb{D}_{\tilde{A}_{2}}^{c})\boldsymbol{\beta}_{\tilde{A}_{2}} + \frac{\Omega_{\tilde{A}_{1}}^{-1}}{n}\mathbb{D}_{\tilde{A}_{1}}^{c'}\boldsymbol{\pi}_{n}\right)$$

$$= \lambda_{n1}U'_{\tilde{A}_{4}}(\boldsymbol{\beta}_{\tilde{A}_{4}} - \hat{\boldsymbol{\beta}}_{gL,\tilde{A}_{4}}) + \lambda_{n1}U'_{\tilde{A}_{4}}Q_{41}\left(\frac{\Omega_{\tilde{A}_{1}}^{-1}}{n}(\mathbb{D}_{\tilde{A}_{1}}^{c'}\mathbb{D}_{\tilde{A}_{2}}^{c})\boldsymbol{\beta}_{\tilde{A}_{2}} + \frac{\Omega_{\tilde{A}_{1}}^{-1}}{n}\mathbb{D}_{\tilde{A}_{1}}^{c'}\boldsymbol{\pi}_{n}\right)$$

$$\leq \lambda_{n1}\sum_{k\in\tilde{A}_{4}}\|\boldsymbol{\beta}_{k}\|_{2} + \frac{\lambda_{n1}U'_{\tilde{A}_{4}}Q_{41}\Omega_{\tilde{A}_{1}}^{-1}(\mathbb{D}_{\tilde{A}_{1}}^{c'}\mathbb{D}_{\tilde{A}_{2}}^{c})\boldsymbol{\beta}_{\tilde{A}_{2}}}{n} + \frac{\lambda_{n1}U'_{\tilde{A}_{4}}Q_{41}\Omega_{\tilde{A}_{1}}^{-1}\mathbb{D}_{\tilde{A}_{1}}^{c'}\boldsymbol{\pi}_{n}}{n},$$

$$(21)$$

where the last inequality is based on $\left| \boldsymbol{U}_{\tilde{A}_{4}}' \boldsymbol{\beta}_{\tilde{A}_{4}} \right| \leq \sum_{k \in \tilde{A}_{4}} |\boldsymbol{U}_{k}' \boldsymbol{\beta}_{k}| \leq \sum_{k \in \tilde{A}_{4}} \|\boldsymbol{\beta}_{k}\|_{2}$ and $\boldsymbol{U}_{\tilde{A}_{4}}' \hat{\boldsymbol{\beta}}_{gL,\tilde{A}_{4}} = \sum_{k \in \tilde{A}_{4} \cap \tilde{A}_{\boldsymbol{\beta}}} \boldsymbol{U}_{k}' \hat{\boldsymbol{\beta}}_{gL,k} \geq 0$. For $\|\boldsymbol{V}_{14}\|_{2} \|\boldsymbol{V}_{13}\|_{2}$, we have

$$\|\boldsymbol{V}_{14}\|_{2}\|\boldsymbol{V}_{13}\|_{2} \leq \|\boldsymbol{V}_{14}\|_{2}\lambda_{n1}\sqrt{\frac{m_{n}|\tilde{A}_{3}|}{n\rho_{\min}(\Omega_{\tilde{A}_{1}})}}$$
(22)

from the definition of V_{13} . For $\|\boldsymbol{\omega}_2\|_2^2$,

$$\begin{split} \|\boldsymbol{\omega}_{2}\|_{2}^{2} &= \|(I - P_{\tilde{A}_{1}})\mathbb{D}_{\tilde{A}_{2}}^{c}\boldsymbol{\beta}_{\tilde{A}_{2}}\|_{2}^{2} \\ &= \boldsymbol{\beta}_{\tilde{A}_{2}}^{\prime}\mathbb{D}_{\tilde{A}_{2}}^{c\prime}(I - P_{\tilde{A}_{1}})\mathbb{D}_{\tilde{A}_{2}}^{c}\boldsymbol{\beta}_{\tilde{A}_{2}} \\ &= \boldsymbol{\beta}_{\tilde{A}_{2}}^{\prime}\left(n\boldsymbol{\Omega}_{\tilde{A}_{2}}\boldsymbol{\beta}_{\tilde{A}_{2}} - \frac{1}{n}\mathbb{D}_{\tilde{A}_{2}}^{c\prime}\boldsymbol{\beta}_{\tilde{A}_{2}}^{c}\boldsymbol{\Omega}_{\tilde{A}_{1}}^{-1}\mathbb{D}_{\tilde{A}_{1}}^{c\prime}\mathbb{D}_{\tilde{A}_{2}}^{c}\boldsymbol{\beta}_{\tilde{A}_{2}}\right) \\ &\leq \boldsymbol{\beta}_{\tilde{A}_{2}}^{\prime}\left(\lambda_{n1}\boldsymbol{D}_{\tilde{A}_{2}} - \mathbb{D}_{\tilde{A}_{2}}^{c\prime}\boldsymbol{\pi}_{n} - \mathbb{D}_{\tilde{A}_{2}}^{c\prime}\mathbb{D}_{\tilde{A}_{1}}^{c}\left(\boldsymbol{\beta}_{\tilde{A}_{1}} - \hat{\boldsymbol{\beta}}_{gl,\tilde{A}_{1}}\right) - \frac{1}{n}\mathbb{D}_{\tilde{A}_{2}}^{c\prime}\mathbb{D}_{\tilde{A}_{1}}^{c}\boldsymbol{\Omega}_{\tilde{A}_{1}}^{-1}\mathbb{D}_{\tilde{A}_{2}}^{c\prime}\boldsymbol{\beta}_{\tilde{A}_{2}}\right) \\ &= \boldsymbol{\beta}_{\tilde{A}_{2}}^{\prime}\left(\lambda_{n1}\boldsymbol{D}_{\tilde{A}_{2}} - \mathbb{D}_{\tilde{A}_{2}}^{c\prime}\boldsymbol{\pi}_{n} - \frac{\lambda_{n1}}{n}\mathbb{D}_{\tilde{A}_{2}}^{c\prime}\mathbb{D}_{\tilde{A}_{1}}^{c}\boldsymbol{\Omega}_{\tilde{A}_{1}}^{-1}\boldsymbol{U}_{\tilde{A}_{1}} + \frac{1}{n}\mathbb{D}_{\tilde{A}_{2}}^{c\prime}\mathbb{D}_{\tilde{A}_{1}}^{c}\boldsymbol{\Omega}_{\tilde{A}_{1}}^{-1}\mathbb{D}_{\tilde{A}_{1}}^{c\prime}\boldsymbol{\pi}_{n}\right) \\ &= \lambda_{n1}\boldsymbol{\beta}_{\tilde{A}_{2}}^{\prime}\boldsymbol{D}_{\tilde{A}_{2}} - \boldsymbol{\omega}_{2}^{\prime}\boldsymbol{\pi}_{n} - \frac{\lambda_{n1}}{n}\boldsymbol{U}_{\tilde{A}_{1}}^{\prime}\boldsymbol{\Omega}_{\tilde{A}_{1}}^{-1}\mathbb{D}_{\tilde{A}_{2}}^{c\prime}\boldsymbol{\beta}_{\tilde{A}_{2}}, \end{split}$$

where the inequality is from

$$\mathbb{D}_{\tilde{A}_2}^{c\prime} \mathbb{D}_{\tilde{A}_1}^{c} (\boldsymbol{\beta}_{\tilde{A}_1} - \hat{\boldsymbol{\beta}}_{gL,\tilde{A}_1}) + n\Omega_{\tilde{A}_2} \boldsymbol{\beta}_{\tilde{A}_2} + \mathbb{D}_{\tilde{A}_2}^{c\prime} \boldsymbol{\pi}_n \le \lambda_{n1} \boldsymbol{D}_{\tilde{A}_2}. \tag{23}$$

In (23), \mathbf{D}_A is a 0-1 vector whose k^{th} entry is $I(\|\hat{\boldsymbol{\beta}}_{k,gL}\|_2=0)$, where I(A) is the indicator function for a set A. The inequality between vectors is defined entry-wise. Note that (23) holds due to (11).

Since $V_{14} \perp \omega_2$, we have

$$\|\mathbb{D}_{\tilde{A}_1}^{c}\boldsymbol{\Omega}_{\tilde{A}_1}^{-1}Q_{41}^{\prime}\boldsymbol{U}_{\tilde{A}_4}\lambda_{n1}/n-\boldsymbol{\omega}_2\|_2^2=\|\mathbb{D}_{\tilde{A}_1}^{c}\boldsymbol{\Omega}_{\tilde{A}_1}^{-1/2}\boldsymbol{V}_{14}/\sqrt{n}-\boldsymbol{\omega}_2\|_2^2=\|\boldsymbol{V}_{14}\|_2^2+\|\boldsymbol{\omega}_2\|_2^2$$

so that

$$\left(\frac{\lambda_{n1}}{n} \bm{U}_{\tilde{A}_4}' Q_{41} \bm{\Omega}_{\tilde{A}_1}^{-1} \mathbb{D}_{\tilde{A}_1}^{c\prime} - \bm{\omega}_2'\right) \bm{\pi}_n = (\|\bm{V}_{14}\|_2^2 + \|\bm{\omega}_2\|_2^2)^{1/2} (\bm{u}' \bm{\pi}_n)$$

using the definition of u. Then, this implies

$$\|\boldsymbol{\omega}_{2}\|_{2}^{2} \leq \lambda_{n1}\boldsymbol{\beta}_{\tilde{A}_{2}}^{\prime}\boldsymbol{D}_{\tilde{A}_{2}} - \frac{\lambda_{n1}}{n}\boldsymbol{U}_{\tilde{A}_{1}}^{\prime}\boldsymbol{\Omega}_{\tilde{A}_{1}}^{-1}\mathbb{D}_{\tilde{A}_{1}}^{c\prime}\mathbb{D}_{\tilde{A}_{2}}^{c}\boldsymbol{\beta}_{\tilde{A}_{2}} + (\|\boldsymbol{V}_{14}\|_{2}^{2} + \|\boldsymbol{\omega}_{2}\|_{2}^{2})^{1/2}(\boldsymbol{u}^{\prime}\boldsymbol{\pi}_{n}) - \frac{\lambda_{n1}}{n}\boldsymbol{U}_{\tilde{A}_{4}}^{\prime}Q_{41}\boldsymbol{\Omega}_{\tilde{A}_{1}}^{-1}\mathbb{D}_{\tilde{A}_{1}}^{c\prime}\boldsymbol{\pi}_{n}.$$
(24)

Combining (21) and (24), we have

$$V'_{14}V_{11} + \|\boldsymbol{\omega}_{2}\|_{2}^{2}$$

$$\leq \lambda_{n1} \sum_{k \in \tilde{A}_{4}} \|\boldsymbol{\beta}_{k}\|_{2} - \frac{\lambda_{n1}U'_{\tilde{A}_{3}}Q_{31}\Omega_{\tilde{A}_{1}}^{-1}(\mathbb{D}_{\tilde{A}_{1}}^{c'}\mathbb{D}_{\tilde{A}_{2}}^{c})\boldsymbol{\beta}_{\tilde{A}_{2}}}{n} + \lambda_{n1}\boldsymbol{\beta}_{\tilde{A}_{2}}^{\prime}\boldsymbol{D}_{\tilde{A}_{2}}$$

$$+ (\|\boldsymbol{V}_{14}\|_{2}^{2} + \|\boldsymbol{\omega}_{2}\|_{2}^{2})^{1/2}|\boldsymbol{u}'\boldsymbol{\pi}_{n}|$$

$$= \lambda_{n1} \sum_{k \in \tilde{A}_{4}} \|\boldsymbol{\beta}_{k}\|_{2} - \boldsymbol{V}_{13}^{\prime}\Omega_{\tilde{A}_{1}}^{-1/2}(\mathbb{D}_{\tilde{A}_{1}}^{c'}\mathbb{D}_{\tilde{A}_{2}}^{c})\boldsymbol{\beta}_{\tilde{A}_{2}}/\sqrt{n} + \lambda_{n1}\boldsymbol{\beta}_{\tilde{A}_{2}}^{\prime}\boldsymbol{D}_{\tilde{A}_{2}}$$

$$+ 2\left((\|\boldsymbol{V}_{14}\|_{2}^{2} + \|\boldsymbol{\omega}_{2}\|_{2}^{2})^{1/2}/2\right)|\boldsymbol{u}'\boldsymbol{\pi}_{n}|$$

$$\leq \lambda_{n1} \sum_{k \in \tilde{A}_{4}} \|\boldsymbol{\beta}_{k}\|_{2} + \|\boldsymbol{V}_{13}\|_{2}\|\Omega_{\tilde{A}_{1}}^{-1/2}(\mathbb{D}_{\tilde{A}_{1}}^{c'}\mathbb{D}_{\tilde{A}_{2}}^{c})\boldsymbol{\beta}_{\tilde{A}_{2}}\|_{2}/\sqrt{n} + \lambda_{n1} \sum_{k \in \tilde{A}_{2}} \|\boldsymbol{\beta}_{k}\|_{2}$$

$$+ \frac{(\|\boldsymbol{V}_{14}\|_{2}^{2} + \|\boldsymbol{\omega}_{2}\|_{2}^{2})}{4} + |\boldsymbol{u}'\boldsymbol{\pi}_{n}|^{2}, \tag{25}$$

where the inequality is by the Cauchy-Schwarz inequality, triangle inequality and $2ab \le a^2 + b^2$. Then, by (22) and (25),

$$\|\boldsymbol{V}_{14}\|_{2}^{2} + \|\boldsymbol{\omega}_{2}\|_{2}^{2} \leq \boldsymbol{V}_{14}^{\prime}\boldsymbol{V}_{11} + \|\boldsymbol{V}_{14}\|_{2}\|\boldsymbol{V}_{13}\|_{2} + \|\boldsymbol{\omega}_{2}\|_{2}^{2} \\ \leq \lambda_{n1} \sum_{k \in \tilde{A}_{4}} \|\boldsymbol{\beta}_{k}\|_{2} + \|\boldsymbol{V}_{13}\|_{2} \|\boldsymbol{\Omega}_{\tilde{A}_{1}}^{-1/2}(\mathbb{D}_{\tilde{A}_{1}}^{c}\mathbb{D}_{\tilde{A}_{2}}^{c})\boldsymbol{\beta}_{\tilde{A}_{2}}\|_{2}/\sqrt{n} + \lambda_{n1} \sum_{k \in \tilde{A}_{2}} \|\boldsymbol{\beta}_{k}\|_{2} \\ + \frac{(\|\boldsymbol{V}_{14}\|_{2}^{2} + \|\boldsymbol{\omega}_{2}\|_{2}^{2})}{4} + |\boldsymbol{u}^{\prime}\boldsymbol{\pi}_{n}|^{2} + \|\boldsymbol{V}_{14}\|_{2}\lambda_{n1}\sqrt{\frac{m_{n}|\tilde{A}_{3}|}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_{1}})}}.$$
(26)

Since $\{(\mathbb{D}^c, \boldsymbol{\pi}_n) \in \Omega_{q_1}\}$ implies $|\boldsymbol{u}'\boldsymbol{\pi}_n|^2 \leq (x_{q_1})^2 \leq \frac{(q_1m_n \vee m_n)\lambda_{n_1}^2}{4n\rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_1})} = \frac{q_1m_n\lambda_{n_1}^2}{4n\rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_1})} = \frac{1}{4}\frac{q_1}{q^*}B_1$, we can show $|\boldsymbol{u}'\boldsymbol{\pi}_n|^2 \leq \frac{1}{4}\frac{q_1}{q^*}B_1 \leq \frac{1}{4}(\|V_{14}\|_2^2 + B_1)$ by using (18). Along with $\|\boldsymbol{V}_{13}\|_2 \leq$

$$\lambda_{n1}\sqrt{rac{m_n| ilde{A}_3|}{n
ho_{\min}(\mathbf{\Omega}_{ ilde{A}_1})}},$$
 (26) becomes

$$\leq \lambda_{n1} \sum_{k \in \tilde{A}_{4}} \|\boldsymbol{\beta}_{k}\|_{2} + \lambda_{n1} \sqrt{\frac{m_{n} |\tilde{A}_{3}|}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_{1}})}} \|\boldsymbol{\Omega}_{\tilde{A}_{1}}^{-1/2} \mathbb{D}_{\tilde{A}_{1}}^{c'} \mathbb{D}_{\tilde{A}_{2}}^{c} \boldsymbol{\beta}_{\tilde{A}_{2}} \|_{2} / \sqrt{n} + \lambda_{n1} \sum_{k \in \tilde{A}_{2}} \|\boldsymbol{\beta}_{k}\|_{2}$$

$$+ \frac{(\|\boldsymbol{V}_{14}\|_{2}^{2} + \|\boldsymbol{\omega}_{2}\|_{2}^{2})}{4} + \frac{(\|\boldsymbol{V}_{14}\|_{2}^{2} + B_{1})}{4} + \|\boldsymbol{V}_{14}\|_{2} \lambda_{n1} \sqrt{\frac{m_{n} |\tilde{A}_{3}|}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_{1}})}}$$

$$= \lambda_{n1} \sum_{k \in \tilde{A}_{5}} \|\boldsymbol{\beta}_{k}\|_{2} + \lambda_{n1} \sqrt{\frac{m_{n} |\tilde{A}_{3}|}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_{1}})}} \|P_{\tilde{A}_{1}}^{1/2} \mathbb{D}_{\tilde{A}_{2}}^{c} \boldsymbol{\beta}_{\tilde{A}_{2}} \|_{2} + \lambda_{n1} \sum_{k \in \tilde{A}_{4} \cup \tilde{A}_{6}} \|\boldsymbol{\beta}_{k}\|_{2}$$

$$+ \frac{\|\boldsymbol{V}_{14}\|_{2}^{2}}{2} + \frac{\|\boldsymbol{\omega}_{2}\|_{2}^{2}}{4} + \frac{B_{1}}{4} + \|\boldsymbol{V}_{14}\|_{2} \lambda_{n1} \sqrt{\frac{m_{n} |\tilde{A}_{3}|}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_{1}})}}$$

$$\leq \lambda_{n1} \sum_{k \in \tilde{A}_{5}} \|\boldsymbol{\beta}_{k}\|_{2} + \lambda_{n1} \sqrt{\frac{m_{n} |\tilde{A}_{3}|}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_{1}})}} \|P_{\tilde{A}_{1}}^{1/2} \mathbb{D}_{\tilde{A}_{2}}^{c} \boldsymbol{\beta}_{\tilde{A}_{2}} \|_{2} + \lambda_{n1} \eta_{1}$$

$$+ \frac{\|\boldsymbol{V}_{14}\|_{2}^{2}}{2} + \frac{\|\boldsymbol{\omega}_{2}\|_{2}^{2}}{4} + \frac{B_{1}}{4} + \|\boldsymbol{V}_{14}\|_{2} \lambda_{n1} \sqrt{\frac{m_{n} |\tilde{A}_{3}|}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_{1}})}}},$$

where the equality comes from $\tilde{A}_2 = \tilde{A}_5 \cup \tilde{A}_6$ with $P_{\tilde{A}_1}^{1/2} = \Omega_{\tilde{A}_1}^{-1/2} \mathbb{D}_{\tilde{A}_1}^{c'} / \sqrt{n}$ and the last inequality is due to GSC. This result gives

$$\frac{1}{2} \|\boldsymbol{V}_{14}\|_{2}^{2} + \frac{3}{4} \|\boldsymbol{\omega}_{2}\|_{2}^{2} \leq \lambda_{n1} \sum_{k \in \tilde{A}_{5}} \|\boldsymbol{\beta}_{k}\|_{2} + \lambda_{n1} \sqrt{\frac{m_{n} |\tilde{A}_{3}|}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_{1}})}} \|P_{\tilde{A}_{1}}^{1/2} \mathbb{D}_{\tilde{A}_{2}}^{c} \boldsymbol{\beta}_{\tilde{A}_{2}}\|_{2} + \lambda_{n1} \eta_{1} + \frac{B_{1}}{4} + \|\boldsymbol{V}_{14}\|_{2} \lambda_{n1} \sqrt{\frac{m_{n} |\tilde{A}_{3}|}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_{1}})}} \tag{27}$$

from which we have

$$\begin{aligned} \|\boldsymbol{V}_{14}\|_{2}^{2} & \leq 2\left(\frac{1}{2}\|\boldsymbol{V}_{14}\|_{2}^{2} + \frac{3}{4}\|\boldsymbol{\omega}_{2}\|_{2}^{2}\right) \\ & \leq 2\lambda_{n1} \sum_{k \in \tilde{A}_{5}} \|\boldsymbol{\beta}_{k}\|_{2} + 2\lambda_{n1} \sqrt{\frac{m_{n}|\tilde{A}_{3}|}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_{1}})}} \|P_{\tilde{A}_{1}}^{1/2} \mathbb{D}_{\tilde{A}_{2}}^{c} \boldsymbol{\beta}_{\tilde{A}_{2}}\|_{2} + 2\lambda_{n1} \eta_{1} \\ & + \frac{B_{1}}{2} + 2\|\boldsymbol{V}_{14}\|_{2} \lambda_{n1} \sqrt{\frac{m_{n}|\tilde{A}_{3}|}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_{1}})}}. \end{aligned}$$

Note that the largest possible \tilde{A}_1 contains all the "large" $\|\boldsymbol{\beta}_j\|_2$ then $\tilde{A}_5 = \phi$ so that $\tilde{A}_2 = \tilde{A}_6$, $\sum_{k \in \tilde{A}_5} \|\boldsymbol{\beta}_k\|_2 = 0$ and $P_{\tilde{A}_1}^{1/2} \mathbb{D}_{\tilde{A}_2}^c \boldsymbol{\beta}_{\tilde{A}_2} = P_{\tilde{A}_1}^{1/2} \mathbb{D}_{\tilde{A}_6}^c \boldsymbol{\beta}_{\tilde{A}_6}$. Finally using $\|P_{\tilde{A}_1}^{1/2} \mathbb{D}_{\tilde{A}_6}^c \boldsymbol{\beta}_{\tilde{A}_6}\|_2 \leq 1$

 $\max_{A \subset \tilde{A}_0} \|\sum_{k \in A} \mathbb{D}_k^c \boldsymbol{\beta}_k\|_2$ since $\tilde{A}_6 \subset \tilde{A}_0$, we have the following inequality.

$$\|\boldsymbol{V}_{14}\|_{2}^{2} \leq 2\eta_{2}\sqrt{B_{2}} + 2\lambda_{n1}\eta_{1} + \frac{B_{1}}{2} + 2\sqrt{B_{2}}\|\boldsymbol{V}_{14}\|_{2}, \tag{28}$$

where $\eta_2 = \max_{A \subset \tilde{A}_0} \|\sum_{k \in A} \mathbb{D}_k^c \boldsymbol{\beta}_k\|_2$ and $B_2 = \frac{\lambda_{n1}^2 m_n q^*}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_1})}$. Hence after using the fact $2\sqrt{B_2} \|\boldsymbol{V}_{14}\|_2 = 2\left(\sqrt{2B_2}\right) \left(\|\boldsymbol{V}_{14}\|_2/\sqrt{2}\right) \leq 2B_2 + \|\boldsymbol{V}_{14}\|_2^2/2$, we obtain an upper bound for $\|\boldsymbol{V}_{14}\|_2^2$ from (28),

$$\|\boldsymbol{V}_{14}\|_{2}^{2} \le B_{1} + 4\lambda_{n1}\eta_{1} + 4\eta_{2}\sqrt{B_{2}} + 4B_{2}$$

which, when combined with (19), implies

$$|\tilde{A}_{\beta}| \leq M_1 q^*,$$

where $M_1 = M_1(\lambda_{n1}) = 2 + 4r_1 + 4r_2\sqrt{C_{12}} + 4C_{12}$ with

$$r_1 = r_1(\lambda_{n1}) = \left(\frac{c_2\eta_1 n}{q^*m_n\lambda_{n1}}\right), r_2 = r_2(\lambda_{n1}) = \left(\frac{c_2\eta_2^2 n}{q^*m_n\lambda_{n1}^2}\right)^{1/2} \text{ and } C_{12} = \frac{c_2}{c_1}.$$

Note that $M_1(\lambda_{n1})$ is a decreasing function in λ_{n1} .

If $\eta_1=0$ which is referred to as a narrow-sense sparsity condition, then $r_1=r_2=0$ and hence $M_1(\lambda_{n1})=2+4C_{12}<\infty$. Note that since we are assuming $\lambda_0<\infty$, we implicitly assume that $(2+4C_{12})q^*+1\leq q_0$ holds. In general, as long as η_1 and η_2 satisfy that $\eta_1\leq \left(\frac{C_1q^*}{c_2}\right)\left(\frac{m_n\lambda_{n1}}{n}\right)$ and $\eta_2^2\leq \left(\frac{C_2q^*}{c_2}\right)\left(\frac{m_n\lambda_{n1}^2}{n}\right)$ for some finite C_1 and C_2 , we will have $r_1\leq C_1$ and $r_2\leq C_2$, which gives $M_1(\lambda_{n1})\leq (2+4C_1+4C_2\sqrt{C_{12}}+4C_{12})<\infty$. Thus, we complete the proof.

4 Extended results on different levels of smoothness for additive components

In this section, we provide extended results when we allow different smoothness levels for additive components. To do this, we need to make necessary changes in Assumption 1.

(H 4)'
$$f_j \in \mathcal{F}_j$$
 and $\mathbb{E} f_j(X_j) = 0$ for $j = 1, \dots, J$, where

$$\mathcal{F}_{j} = \left\{ f \mid \left| f^{(k_{j})}(s) - f^{(k_{j})}(t) \right| \le C \left| s - t \right|^{\nu_{j}}, \ \forall s, t \in [a, b] \right\}$$

for some nonnegative integer k_j and $\nu_j \in (0,1]$. Also suppose that $\tau_j = k_j + \nu_j > 1$, and,

(H 6)' $m_{nj} = O(n^{\gamma_j})$ with $1/6 \le \gamma_j = 1/(2\tau_j + 1) < 1/3$, $\forall j = 1, 2, ..., J$, where m_{nj} is the number of B-spline bases (or knots) to approximate the j^{th} additive component.

We provide revised lemmas and theorems in the following. We first extend the definition of S_{nj}^0 as follows:

$$S_{nj}^{0} = \left\{ f_{nj} : f_{nj} = \sum_{l=1}^{m_{nj}} b_{jl} \mathbb{B}_{l}^{c}(x), (b_{j1}, \cdots, b_{jm_{nj}}) \in \mathbb{R}^{m_{nj}} \right\}, 1 \le j \le J.$$
 (29)

With new assumptions, Lemmas 1 and 3 and Theorems 2 and 4 are changed as follows. **Lemma 1'**. Suppose that $f \in \mathcal{F}_j$ and $\mathbb{E}f(X_j) = 0$. Then, under **(H 4)'** and **(H 5)**, there exists an $f_n \in \mathcal{S}_{nj}^0$ such that

$$||f_n - f||_2 = O_p \left(m_{nj}^{-\tau} + \sqrt{\frac{m_{nj}}{n}} \right).$$
 (30)

Particularly, under the choice of $m_{nj} = O(n^{\frac{1}{2\tau_j+1}})$, we have

$$||f_n - f||_2 = O_p\left(m_{nj}^{-\tau_j}\right) = O_p\left(n^{-\frac{\tau_j}{2\tau_j+1}}\right).$$
 (31)

Lemma 3'. Define M_n be a non-negative definite matrix of order n and,

$$T_{jl} = \left(\frac{m_{nj}}{n}\right)^{\frac{1}{2}} \boldsymbol{a}'_{jl} \boldsymbol{M}_n \boldsymbol{\epsilon} \qquad \forall 1 \le j \le J, 1 \le l \le m_{nj}, \tag{32}$$

where $a_{jl} = (\mathbb{B}_l^c(X_j(s)), s \in S)'$ and $T_n = \max_{\substack{1 \le j \le J \\ 1 \le l \le m_{nj}}} |T_{jl}|$. With new Assumption 1,

$$\mathbb{E}(T_n) \le C_1 \rho_{\max}(\boldsymbol{M}_n) \sqrt{(m_n \log(Jm_n))},\tag{33}$$

for some $C_1 > 0$ and $m_n = \max_{j=1,2,...,J} m_{nj}$.

Theorem 2 '. With new Assumption 1 and $\lambda_{n1} > C\rho_{\max}(\mathbf{L}) \sqrt{nm_n \log(Jm_n)}$ for a sufficiently large constant C,

(a)
$$\|\hat{f}_{gL,j} - f_j\|_2^2 = O_p \left\{ \left(\frac{\rho_{\max}^2(\boldsymbol{L}) m_n^3 \log(J m_n)}{n} + \frac{m_n}{n} + \frac{1}{m_n^{2\tau - 1}} + \frac{4m_n^2 \lambda_{n1}^2}{n^2} \right) / m_{nj} \right\}$$
 for $j \in \tilde{A}_{\boldsymbol{\beta}} \cup A_*$, where $\tilde{A}_{\boldsymbol{\beta}}$ is the index set of nonzero gL estimates for $\boldsymbol{\beta}_j$,

(b) If $\frac{m_n^2 \lambda_{n1}^2}{m_{nj} n^2} \longrightarrow 0$ as $n \longrightarrow \infty$ for $1 \le j \le q$, all the nonzero components $f_j, 1 \le j \le q$ are selected w.p. converging to 1.

Theorem 4 '. With new Assumptions 1 and 2,

(a)
$$\mathbb{P}(\|\hat{f}_{AgL,j}\|_2 > 0, j \in A_* \text{ and } \|\hat{f}_{AgL,j}\|_2 = 0, j \notin A_*) \longrightarrow 1,$$

(b)
$$\|\hat{f}_{AgL,j} - f_j\|_2^2 = O_p \left\{ \left(\frac{\rho_{\max}^2(\boldsymbol{L}) m_n^3 \log(J_0 m_n)}{n} + \frac{m_n}{n} + \frac{1}{m_n^{2\tau-1}} + \frac{4m_n^2 \lambda_{n2}^2}{n^2} \right) / m_{nj} \right\} \ \forall j \in A_*.$$

Results are similar except a few changes due to the introduction of m_{nj} . In practice, we standardize the range and variability of all additive components and use the same number of knot points for each component which we choose as m_n . Note that the largest number of knot points m_n can cover all smooth additive components. Since the order of m_n also has to be between $\left[n^{1/6}, n^{1/3}\right]$ according to the assumption (**H 6**)', we use this bound as a guide to choose m_n . The suggestion of using the same number of knot points for each component, in practice, is also suggested by Hastie and Tibshirani (1990).

5 Results with extension of (H 3)

We extend the assumption (H 3) to cover a broader class of spatial covariance functions. The assumptions (H 6) and (K 2) are adjusted accordingly as well.

(H 3)* The random vector $\epsilon = \{\epsilon(s), s \in S\} \sim \text{Gaussian}(0, \Sigma_T)$, where $\Sigma_T = ((\sigma_{S,S'}))_{S,S' \in S}$ with $\sigma_{S,S'} = \delta(s-s')$ and $\delta(h)$ is a covariance function such that $\int_{\mathbf{D}_n} \delta(h) dh = O(n^{\alpha})$ for some $\alpha \in [0,1)$. $\mathbf{D}_n \subset \mathbb{R}^d$ is the sampling region that contains the sampling locations S. Without loss of generality, we assume that the origin of \mathbb{R}^d is in the interior of \mathbf{D}_n and \mathbf{D}_n is increasing with n.

(H 6)*
$$m_n = O(n^{\gamma})$$
 with $1/6 \le \gamma = 1/(2\tau + 1) < (1 - \alpha)/3$.

 $(K 2)^*$

$$\frac{\sqrt{\rho_{\max}^2(\mathbf{L}) \, n^{1+\alpha} m_n \log(s_n m_n)}}{\lambda_{n2} r_n} + \frac{n^2}{\lambda_{n2}^2 r_n^2 m_n} + \frac{\lambda_{n2} m_n}{n} = o(1)$$

where $s_n = J - |A_{**}|$.

Lemmas and theorems are then updated in the following way.

Lemma 3*. Define M_n be a non-negative definite matrix of order n and,

$$T_{jl} = \left(\frac{m_n}{n}\right)^{\frac{1}{2}} \boldsymbol{a}'_{jl} \boldsymbol{M}_n \boldsymbol{\epsilon} \qquad \forall 1 \le j \le J, 1 \le l \le m_n$$
 (34)

where $a_{jl} = (\mathbb{B}_l^c(X_j(s)), s \in S)'$ and $T_n = \max_{\substack{1 \le j \le J \\ 1 \le l \le m_n}} |T_{jl}|$. Then, under assumptions (H 2), (H 3)*, (H 4) and (H 5),

$$\mathbb{E}(T_n) \le C_1 \rho_{\max}(\boldsymbol{M}_n) \sqrt{(m_n \log(Jm_n)) \boldsymbol{O}(n^{\alpha})},\tag{35}$$

for some $C_1 > 0$.

Lemma 4*. Under the Assumption 1 with updated (**H 3**)* and (**H 6**)* and with $\lambda_{n1} > C\rho_{\max}(\mathbf{L}) \sqrt{n^{1+\alpha}m_n \log(Jm_n)}$ for a sufficiently large constant C, we have $|\tilde{A}_{\beta}| \leq M_1 |A_*|$ for a finite constant $M_1 > 1$ with w.p. converging to 1.

Theorem 1.*

Suppose that conditions in Assumption 1 with updated (H 3)* and (H 6)* hold and if $\lambda_{n1} > C\rho_{\max}(L) \sqrt{n^{1+\alpha}m_n\log(Jm_n)}$ for a sufficiently large constant C. Then, we have

(a)
$$\sum_{j=1}^{J} \|\hat{\boldsymbol{\beta}}_{gL,j} - \boldsymbol{\beta}_j\|_2^2 = O_p \left(\frac{\rho_{\max}^2(\boldsymbol{L}) m_n^3 \log(J m_n)}{n^{1-\alpha}} + \frac{m_n}{n^{1-\alpha}} + \frac{1}{m_n^{2\tau-1}} + \frac{4m_n^2 \lambda_{n1}^2}{n^2} \right),$$

(b) If $\frac{m_n^2 \lambda_{n1}^2}{n^2} \longrightarrow 0$ as $n \longrightarrow \infty$, all the nonzero components β_j , $1 \le j \le q$ are selected with probability (w.p.) converging to 1.

Theorem 2.* Suppose that conditions in Assumption 1 with updated (**H 3**)* and (**H 6**)* hold and if $\lambda_{n1} > C\rho_{\max}(\mathbf{L}) \sqrt{n^{1+\alpha}m_n\log(Jm_n)}$ for a sufficiently large constant C. Then,

(a)
$$\|\hat{f}_{gL,j} - f_j\|_2^2 = O_p\left(\frac{\rho_{\max}^2(\boldsymbol{L})m_n^2\log(Jm_n)}{n^{1-\alpha}} + \frac{1}{n^{1-\alpha}} + \frac{1}{m_n^{2\tau}} + \frac{4m_n\lambda_{n1}^2}{n^2}\right)$$
 for $j \in \tilde{A}_{\boldsymbol{\beta}} \cup A_*$, where $\tilde{A}_{\boldsymbol{\beta}}$ is the index set of nonzero gL estimates for $\boldsymbol{\beta}_j$,

(b) If $\frac{m_n \lambda_{n1}^2}{n^2} \longrightarrow 0$ as $n \longrightarrow \infty$, all the nonzero components $f_j, 1 \le j \le q$ are selected w.p. converging to 1.

Theorem 3.* Suppose that conditions in Assumptions 1 and 2 with updated $(\mathbf{H} \ \mathbf{3})^*$, $(\mathbf{H} \ \mathbf{6})^*$ and $(\mathbf{K} \ \mathbf{2})^*$ are satisfied. Then,

(a)
$$\mathbb{P}(\hat{\boldsymbol{\beta}}_{AaL} =_0 \boldsymbol{\beta}) \longrightarrow 1$$
,

(b)
$$\sum_{j=1}^{q} \|\hat{\boldsymbol{\beta}}_{AgL,j} - \boldsymbol{\beta}_{j}\|_{2}^{2} = \boldsymbol{O}_{p} \left(\frac{\rho_{\max}^{2}(\boldsymbol{L}) m_{n}^{3} \log(J_{0}m_{n})}{n^{1-\alpha}} + \frac{m_{n}}{n^{1-\alpha}} + \frac{1}{m_{n}^{2\tau-1}} + \frac{4m_{n}^{2} \lambda_{n2}^{2}}{n^{2}} \right).$$

Theorem 4.* Suppose that conditions in Assumptions 1 and 2 with updated $(\mathbf{H} \ \mathbf{3})^*$, $(\mathbf{H} \ \mathbf{6})^*$ and $(\mathbf{K} \ \mathbf{2})^*$ are satisfied. Then,

(a)
$$\mathbb{P}(\|\hat{f}_{AgL,j}\|_2 > 0, j \in A_* \text{ and } \|\hat{f}_{AgL,j}\|_2 = 0, j \notin A_*) \longrightarrow 1,$$

(b)
$$\sum_{j=1}^{q} \|\hat{f}_{AgL,j} - f_j\|_2^2 = O_p \left(\frac{\rho_{\max}^2(\boldsymbol{L}) m_n^2 \log(J_0 m_n)}{n^{1-\alpha}} + \frac{1}{n^{1-\alpha}} + \frac{1}{m_n^{2\tau}} + \frac{4m_n \lambda_{n2}^2}{n^2} \right).$$

The updated theorems show that the lower bound of the penalty parameter as well as the convergence rate are affected by α . More specifically, introduction of α increases the order in the lower bound of the penalty parameter and the order of the convergence rate is decreased (slower convergence rate) with α . Note that α does not fully characterize a spatial dependence structure but it gives some information on the level of spatial dependence such that $0 < \alpha < 1$ implies a long-range dependence. For any integrable stationary spatial covariance model, $\alpha = 0$ and this is the case for most practical situations. If $0 < \alpha < 1$, one might consider estimating

 α for calculation of the lower bound of the penalty parameter. There are some literature which provide how to estimate long-range parameters for random fields [e.g. Anh and Lunney (1995), Boissy *et al.* (2005)], but they are limited since a specific class of random fields or a parametric model is assumed. Estimation of α has its own interest but we do not pursue it since our focus is on variable selection.

6 Additional simulation results

In this section, we provide complete simulation results including the simulated data with t=3 in generating covariates, X_j . Compared to the Table 2 in the main paper, Tables 2 - 4 include the results for m=6,12,24, more choices of correlation parameters and J=15,25,35. The results are consistent with what we have in the main paper.

As discussed in Section 4 in the main paper, the correlation between X_j and X_k is higher with t=3 compared to t=1. Tables 5-7 show results with t=3 case. The selection results are worse for both our approach as well as the independent approach compared to the cases with t=1 in Tables 2-4 but our approach still performs better than the independent approach.

-				9=w	9			H	m=12			m=24	42	
-	Cov Model	Spatial Weights	GLA	GLASSO	Adapt. GLASSO	LASSO	GLASSO			Adapt. GLASSO	GLASSO			Adapt. GLASSO
			True Positive	False Positive										
-		None(Indep)	3.74(0.48)	3.63(1.83)	3.74(0.48)	3.62(1.84)	4(0)	1.6(1.3)	4(0)	1.6(1.3)	4(0)	0.55(0.77)	4(0)	0.55(0.77)
			3.17(0.82)	2.02(1.34)	3.15(0.82)	2(1.29)	4(0)	0.38(0.65)	4(0)	0.38(0.65)	4(0)	0(0)	4(0)	0)0
	Exp(0.5)	Gauss	3.01(0.85)	1.76(1.24)	3.01(0.85)	1.74(1.23)	4(0)	0.18(0.48)	4(0)	0.18(0.48)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.94(0.93)	1.58(1.44)	2.94(0.93)	1.58(1.44)	4(0)	0.1(0.3)	4(0)	0.1(0.3)	4(0)	0(0)	4(0)	0(0)
		True	1.66(1.02)	0.29(0.56)	1.66(1.02)	0.29(0.56)	4(0)	0.07(0.29)	4(0)	0.07(0.29)	4(0)	0(0)	4(0)	0(0)
		None(Indep)	3.82(0.41)	3.77(2.09)	3.82(0.41)	3.67(1.99)	4(0)	1.51(1.24)	4(0)	1.51(1.24)	4(0)	0.53(0.69)	4(0)	0.53(0.69)
		1	3.11(0.82)	2.31(1.35)	3.11(0.82)	2.31(1.35)	4(0)	0.43(0.64)	4(0)	0.43(0.64)	4(0)	0.03(0.22)	4(0)	0.03(0.22)
	Exp(1)	Gauss	3.01(0.8)	1.93(1.27)	3.01(0.8)	1.93(1.27)	4(0)	0.24(0.51)	4(0)	0.24(0.51)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.99(0.88)	1.56(1.17)	2.99(0.88)	1.54(1.13)	4(0)	0.14(0.47)	4(0)	0.14(0.47)	4(0)	0(0)	4(0)	0(0)
		True	2.33(1.05)	1.1(0.96)	2.33(1.05)	1.1(0.96)	4(0)	0.12(0.38)	4(0)	0.12(0.38)	4(0)	0(0)	4(0)	0(0)
		None(Indep)	3.8(0.45)	3.63(1.86)	3.79(0.46)	3.57(1.81)	4(0)	1.37(1.39)	4(0)	1.37(1.39)	4(0)	0.34(0.57)	4(0)	0.34(0.57)
			3.26(0.77)	1.87(1.5)	3.26(0.77)	1.87(1.5)	4(0)	0.46(0.77)	4(0)	0.46(0.77)	4(0)	0.05(0.22)	4(0)	0.05(0.22)
	Mat _{3/2} (2.5)	Gauss	2.98(0.85)	1.64(1.34)	2.98(0.85)	1.63(1.33)	4(0)	0.25(0.58)	4(0)	0.25(0.58)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.74(0.86)	1.5(1.18)	2.74(0.86)	1.5(1.18)	4(0)	0.1(0.33)	4(0)	0.1(0.33)	4(0)	0(0)	4(0)	000
		True	0.8(0.68)	0.09(0.32)	0.8(0.68)	0.09(0.32)	3.36(0.72)	0.01(0.1)	3.36(0.72)	0.01(0.1)	4(0)	0(0)	4(0)	0(0)
		None(Indep)	3.75(0.46)	4.07(2.04)	3.75(0.46)	4.06(2.04)	4(0)	1.54(1.42)	4(0)	1.54(1.42)	4(0)	0.47(0.67)	4(0)	0.47(0.67)
		-	3.23(0.71)	2.05(1.3)	3.22(0.7)	2.01(1.26)	4(0)	0.46(0.78)	4(0)	0.46(0.78)	4(0)	0.02(0.14)	4(0)	0.02(0.14)
	Mat _{3/2} (1.5)	Gauss	3.03(0.76)	1.83(1.3)	3.03(0.76)	1.8(1.31)	4(0)	0.28(0.6)	4(0)	0.28(0.6)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.84(0.9)	1.51(1.06)	2.84(0.9)	1.48(1.01)	4(0)	0.14(0.35)	4(0)	0.14(0.35)	4(0)	0(0)	4(0)	0(0)
51		True	1.01(0.88)	0.15(0.36)	1.01(0.88)	0.15(0.36)	3.82(0.46)	0.07(0.26)	3.82(0.46)	0.07(0.26)	4(0)	0(0)	4(0)	0(0)
3		None(Indep)	3.82(0.41)	3.59(2.08)	3.81(0.42)	3.53(2.04)	4(0)	1.34(1.2)	4(0)	1.34(1.2)	4(0)	0.58(0.88)	4(0)	0.58(0.88)
			3.22(0.76)	2.06(1.55)	3.22(0.76)	2.06(1.55)	4(0)	0.43(0.86)	4(0)	0.43(0.86)	4(0)	0.04(0.2)	4(0)	0.04(0.2)
	Mat _{5/2} (2.5)	Gauss	2.91(0.89)	1.72(1.35)	2.91(0.89)	1.69(1.35)	4(0)	0.26(0.66)	4(0)	0.26(0.66)	4(0)	0(0)	4(0)	0(0)
	,	InvMQ	2.97(0.89)	1.67(1.33)	2.97(0.89)	1.67(1.33)	4(0)	0.1(0.3)	4(0)	0.1(0.3)	4(0)	0(0)	4(0)	0(0)
		True	0.37(0.56)	0.04(0.24)	0.37(0.56)	0.04(0.24)	2.2(0.9)	0.06(0.24)	2.2(0.9)	0.06(0.24)	4(0)	0(0)	4(0)	0(0)
		None(Indep)	3.69(0.51)	3.85(1.92)	3.69(0.51)	3.84(1.93)	4(0)	1.74(1.52)	4(0)	1.74(1.52)	4(0)	0.58(0.75)	4(0)	0.58(0.75)
		_	3.29(0.74)	1.9(1.43)	3.27(0.75)	1.88(1.39)	4(0)	0.47(0.77)	4(0)	0.47(0.77)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
	$Mat_{5/2}(1.5)$	Gauss	3.21(0.74)	1.63(1.23)	3.21(0.74)	1.61(1.21)	4(0)	0.25(0.5)	4(0)	0.25(0.5)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.84(0.88)	1.53(1.1)	2.84(0.88)	1.52(1.1)	4(0)	0.19(0.42)	4(0)	0.19(0.42)	4(0)	0(0)	4(0)	0(0)
		True	0.59(0.65)	0.06(0.24)	0.59(0.65)	0.06(0.24)	2.79(0.99)	0.06(0.24)	2.79(0.99)	0.06(0.24)	4(0)	0(0)	4(0)	0(0)
		None(Indep)	3.76(0.49)	3.97(2.06)	3.76(0.49)	3.92(2.02)	4(0)	1.63(1.3)	4(0)	1.63(1.3)	4(0)	0.59(0.81)	4(0)	0.59(0.81)
			3.23(0.75)	2.07(1.24)	3.23(0.75)	2.07(1.24)	4(0)	0.64(0.92)	4(0)	0.64(0.92)	4(0)	0.03(0.17)	4(0)	0.03(0.17)
	Gauss(1.5)	Gauss	3.02(0.82)	1.86(1.25)	3.02(0.82)	1.86(1.25)	4(0)	0.28(0.59)	4(0)	0.28(0.59)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.76(0.79)	1.58(1.17)	2.76(0.79)	1.58(1.17)	3.99(0.1)	0.23(0.47)	3.99(0.1)	0.23(0.47)	4(0)	0(0)	4(0)	0(0)
		True	3.3(0.73)	2.32(1.55)	3.3(0.73)	2.32(1.55)	4(0)	0.98(1.05)	4(0)	0.98(1.05)	4(0)	0.12(0.33)	4(0)	0.12(0.33)
		None(Indep)	3.67(0.57)	3.97(1.89)	3.67(0.57)	3.83(1.81)	4(0)	1.77(1.46)	4(0)	1.77(1.46)	4(0)	0.55(0.78)	4(0)	0.55(0.78)
		_	3.19(0.77)	2.1(1.19)	3.19(0.77)	2.09(1.19)	4(0)	0.49(0.69)	4(0)	0.49(0.69)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
	Gauss(2.5)	Gauss	2.97(0.83)	1.85(1.34)	2.97(0.83)	1.83(1.33)	4(0)	0.24(0.47)	4(0)	0.24(0.47)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.75(0.77)	1.45(1.12)	2.75(0.77)	1.44(1.09)	4(0)	0.24(0.47)	4(0)	0.24(0.47)	4(0)	0(0)	4(0)	0(0)
		True	3.56(0.57)	3.11(1.38)	3.56(0.57)	3.11(1.38)	4(0)	1.73(1.3)	4(0)	1.73(1.3)	4(0)	0.69(0.8)	4(0)	0.69(0.8)

Independent and Dependent setup using spatially weighted group LASSO and adaptive Group LASSO algorithms when J=15Table 2: Monte Carlo Mean (Standard dev.) for the selected number of nonzero covariates using 100 datasets under both (t = 1)

				9=W	ۅ			m=12	12			m=24	42	
r	Cov Model	Spatial Weights	∀T9	GLASSO	Adapt. GLASSO	TASSO	GLASSO			Adapt. GLASSO	VT9	GLASSO		Adapt. GLASSO
			True Positive	False Positive										
		None(Indep)	3.53(0.67)	5.01(2.31)	3.53(0.67)	4.89(2.26)	4(0)	2.54(1.73)	4(0)	2.54(1.73)	4(0)	0.88(0.99)	4(0)	0.88(0.99)
			2.84(0.95)	2.76(1.78)	2.84(0.95)	2.76(1.78)	4(0)	0.89(1.14)	4(0)	0.89(1.14)	4(0)	0.04(0.2)	4(0)	0.04(0.2)
	Exp(0.5)	Gauss	2.72(0.91)	2.49(1.56)	2.72(0.91)	2.48(1.56)	4(0)	0.57(0.89)	4(0)	0.57(0.89)	4(0)	0.02(0.14)	4(0)	0.02(0.14)
		InvMQ	2.4(0.85)	2(1.36)	2.4(0.85)	2(1.36)	4(0)	0.29(0.61)	4(0)	0.29(0.61)	4(0)	000	4(0)	000
		True	1.36(0.99)	0.44(0.61)	1.36(0.99)	0.44(0.61)	3.99(0.1)	0.07(0.26)	3.99(0.1)	0.07(0.26)	4(0)	0(0)	4(0)	0(0)
		None(Indep)	3.49(0.63)	5.25(1.9)	3.49(0.63)	5.14(1.8)	4(0)	2.98(2.07)	4(0)	2.98(2.07)	4(0)	0.83(1.04)	4(0)	0.83(1.04)
		ı	2.78(0.77)	2.77(1.62)	2.78(0.77)	2.77(1.62)	4(0)	1.06(1.08)	4(0)	1.06(1.08)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
	Exp(1)	Gauss	2.7(0.85)	2.18(1.35)	2.7(0.85)	2.16(1.33)	4(0)	0.55(0.78)	4(0)	0.55(0.78)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.57(0.9)	1.8(1.32)	2.57(0.9)	1.8(1.32)	4(0)	0.37(0.63)	4(0)	0.37(0.63)	4(0)	0(0)	4(0)	0(0)
		True	2.05(1.09)	1.02(1.08)	2.05(1.09)	1.02(1.08)	4(0)	0.31(0.54)	4(0)	0.31(0.54)	4(0)	0(0)	4(0)	0(0)
1		None(Indep)	3.57(0.59)	5.09(2.16)	3.57(0.59)	5.07(2.16)	4(0)	2.29(1.82)	4(0)	2.29(1.82)	4(0)	0.83(1.1)	4(0)	0.83(1.1)
		-	2.99(0.9)	2.52(1.51)	2.99(0.9)	2.52(1.51)	4(0)	0.64(0.93)	4(0)	0.64(0.93)	4(0)	0.08(0.27)	4(0)	0.08(0.27)
	Mat _{3/2} (2.5)	Gauss	2.85(0.77)	2.38(1.49)	2.85(0.77)	2.36(1.48)	4(0)	0.39(0.71)	4(0)	0.39(0.71)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.55(0.87)	1.91(1.44)	2.55(0.87)	1.89(1.44)	4(0)	0.25(0.56)	4(0)	0.25(0.56)	4(0)	0(0)	4(0)	0(0)
		True	0.66(0.65)	0.01(0.1)	0.66(0.65)	0.01(0.1)	3.15(0.93)	0.01(0.1)	3.15(0.93)	0.01(0.1)	4(0)	0(0)	4(0)	0(0)
_		None(Indep)	3.63(0.56)	4.91(2.17)	3.63(0.56)	4.87(2.15)	4(0)	2.59(1.5)	4(0)	2.59(1.5)	4(0)	0.91(1.06)	4(0)	0.91(1.06)
		ı	2.97(0.97)	2.78(1.85)	2.97(0.97)	2.78(1.85)	4(0)	0.9(1.01)	4(0)	0.9(1.01)	4(0)	0.02(0.14)	4(0)	0.02(0.14)
	Mat _{3/2} (1.5)	Gauss	2.68(1)	2.18(1.64)	2.68(1)	2.16(1.61)	4(0)	0.53(0.74)	4(0)	0.53(0.74)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.56(0.89)	1.88(1.4)	2.56(0.89)	1.87(1.4)	4(0)	0.18(0.48)	4(0)	0.18(0.48)	4(0)	0(0)	4(0)	0(0)
35		True	0.8(0.74)	0.06(0.24)	0.8(0.74)	0.06(0.24)	3.78(0.48)	0.06(0.24)	3.78(0.48)	0.06(0.24)	4(0)	0(0)	4(0)	0(0)
ì		None(Indep)	3.67(0.53)	4.99(2.2)	3.67(0.53)	4.97(2.2)	4(0)	1.95(1.9)	4(0)	1.95(1.9)	4(0)	0.71(0.94)	4(0)	0.71(0.94)
		ı	3.07(0.86)	2.59(1.78)	3.07(0.86)	2.57(1.77)	4(0)	0.7(1.11)	4(0)	0.7(1.11)	4(0)	0.06(0.24)	4(0)	0.06(0.24)
	Mat _{5/2} (2.5)	Gauss	2.87(0.91)	2.28(1.53)	2.87(0.91)	2.26(1.5)	4(0)	0.41(0.75)	4(0)	0.41(0.75)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.56(0.83)	1.98(1.48)	2.56(0.83)	1.97(1.47)	4(0)	0.26(0.63)	4(0)	0.26(0.63)	4(0)	0(0)	4(0)	(0)0
		True	0.22(0.44)	0.03(0.17)	0.22(0.44)	0.03(0.17)	1.75(0.89)	0.07(0.29)	1.75(0.89)	0.07(0.29)	4(0)	0(0)	4(0)	0(0)
l		None(Indep)	3.49(0.61)	5.31(2.15)	3.49(0.61)	5.17(2.11)	4(0)	2.45(1.77)	4(0)	2.45(1.77)	4(0)	0.76(0.97)	4(0)	0.76(0.97)
		_	2.82(0.9)	2.63(1.53)	2.81(0.92)	2.62(1.51)	4(0)	0.84(1)	4(0)	0.84(1)	4(0)	0.03(0.17)	4(0)	0.03(0.17)
	$Mat_{5/2}(1.5)$	Gauss	2.67(0.88)	2.26(1.38)	2.67(0.88)	2.26(1.38)	4(0)	0.44(0.76)	4(0)	0.44(0.76)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.46(0.82)	1.77(1.26)	2.46(0.82)	1.76(1.27)	4(0)	0.24(0.51)	4(0)	0.24(0.51)	4(0)	0(0)	4(0)	0(0)
		True	0.4(0.67)	0.04(0.2)	0.4(0.67)	0.04(0.2)	2.46(1)	0.07(0.26)	2.46(1)	0.07(0.26)	4(0)	0(0)	4(0)	0(0)
		None(Indep)	3.5(0.63)	5.17(1.93)	3.48(0.66)	5.02(1.79)	4(0)	2.89(1.53)	4(0)	2.89(1.53)	4(0)	0.8(0.88)	4(0)	0.8(0.88)
		-	2.84(0.9)	3.23(1.6)	2.84(0.9)	3.23(1.6)	4(0)	0.8(0.84)	4(0)	0.8(0.84)	4(0)	0.04(0.2)	4(0)	0.04(0.2)
	Gauss(1.5)	Gauss	2.74(0.87)	2.9(1.46)	2.74(0.87)	2.87(1.43)	4(0)	0.38(0.69)	4(0)	0.38(0.69)	4(0)	0(0)	4(0)	(0)0
		InvMQ	2.47(0.96)	2.11(1.52)	2.47(0.96)	2.1(1.51)	4(0)	0.27(0.51)	4(0)	0.27(0.51)	4(0)	0(0)	4(0)	0(0)
		True	2.77(0.99)	3.32(1.73)	2.76(1)	3.32(1.73)	4(0)	1.4(1.11)	4(0)	1.4(1.11)	4(0)	0.21(0.46)	4(0)	0.21(0.46)
		None(Indep)	3.42(0.74)	5.47(2.27)	3.4(0.75)	5.32(2.15)	4(0)	2.89(1.85)	4(0)	2.89(1.85)	4(0)	0.95(1.06)	4(0)	0.95(1.06)
		-	2.77(0.89)	2.82(1.56)	2.77(0.89)	2.82(1.56)	4(0)	0.96(1.08)	4(0)	0.96(1.08)	4(0)	0(0)	4(0)	(0)0
	Gauss(2.5)	Gauss	2.7(0.88)	2.5(1.33)	2.7(0.88)	2.49(1.33)	4(0)	0.4(0.68)	4(0)	0.4(0.68)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.65(0.78)	1.99(1.31)	2.65(0.78)	1.99(1.31)	4(0)	0.37(0.65)	4(0)	0.37(0.65)	4(0)	0(0)	4(0)	(0)0
		True	3.36(0.75)	3.99(1.8)	3.36(0.75)	3.97(1.79)	4(0)	2.52(1.46)	4(0)	2.52(1.46)	4(0)	1.21(1.15)	4(0)	1.21(1.15)

Table 3: Monte Carlo Mean (Standard dev.) for the selected number of nonzero covariates using 100 datasets under both Independent and Dependent setup using spatially weighted group LASSO and adaptive Group LASSO algorithms when J=25(t = 1)

Cov Model	Spatial Weights	∀T9	GLASSO	Adapt. (Adapt. GLASSO	GLASSO	OSS	Adapt. GLASSO	TASSO	P CF	GLASSO	Adapt	Adapt. GLASSO
		True Positive	False Positive										
	None(Indep)	3.38(0.74)	6.22(2.39)	3.36(0.76)	6.1(2.33)	4(0)	3.74(2.68)	4(0)	3.74(2.68)	4(0)	1.49(1.55)	4(0)	1.49(1.55)
	I	2.62(0.94)	3.15(2.14)	2.6(0.95)	3.11(2.13)	4(0)	1.5(1.45)	4(0)	1.5(1.45)	4(0)	0.04(0.2)	4(0)	0.04(0.2)
Exp(0.5)	Gauss	2.39(0.98)	2.59(1.84)	2.39(0.98)	2.56(1.81)	4(0)	0.89(1.08)	4(0)	0.89(1.08)	4(0)	0.02(0.14)	4(0)	0.02(0.14)
	InvMQ	2.22(1.04)	2.38(1.79)	2.22(1.04)	2.37(1.78)	4(0)	0.41(0.75)	4(0)	0.41(0.75)	4(0)	0(0)	4(0)	000
	True	1.21(0.95)	0.48(0.78)	1.21(0.95)	0.48(0.78)	3.99(0.1)	0.1(0.3)	3.99(0.1)	0.1(0.3)	4(0)	0(0)	4(0)	000
	None(Indep)	3.36(0.7)	5.77(2.3)	3.33(0.73)	5.54(1.98)	4(0)	3.68(1.93)	4(0)	3.68(1.93)	4(0)	1.31(1.18)	4(0)	1.31(1.18)
	I	2.59(0.98)	3.3(1.8)	2.59(0.98)	3.3(1.8)	4(0)	1.39(1.36)	4(0)	1.39(1.36)	4(0)	0.09(0.29)	4(0)	0.09(0.29)
Exp(1)	Gauss	2.36(0.97)	2.98(1.69)	2.36(0.97)	2.95(1.67)	4(0)	0.67(0.95)	4(0)	0.67(0.95)	4(0)	0(0)	4(0)	000
	InvMQ	2.07(0.99)	2.25(1.61)	2.07(0.99)	2.24(1.59)	4(0)	0.71(1.04)	4(0)	0.71(1.04)	4(0)	0(0)	4(0)	000
	True	1.56(1.05)	1.32(1.38)	1.56(1.05)	1.32(1.38)	4(0)	0.51(0.7)	4(0)	0.51(0.7)	4(0)	0(0)	4(0)	000
	None(Indep)	3.48(0.67)	5.79(2.32)	3.46(0.67)	5.59(2.25)	4(0)	2.71(2.13)	4(0)	2.71(2.13)	4(0)	1.25(1.37)	4(0)	1.25(1.37)
	-	2.71(0.9)	2.94(1.75)	2.71(0.9)	2.94(1.75)	4(0)	1.22(1.28)	4(0)	1.22(1.28)	4(0)	0.1(0.39)	4(0)	0.1(0.39)
Mat _{3/2} (2.5)	Gauss	2.57(0.92)	2.59(1.6)	2.57(0.92)	2.57(1.58)	4(0)	0.69(1)	4(0)	0.69(1)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
	InvMQ	2.25(0.9)	2.24(1.68)	2.25(0.9)	2.23(1.68)	4(0)	0.62(0.97)	4(0)	0.62(0.97)	4(0)	0(0)	4(0)	000
	True	0.52(0.58)	0.12(0.41)	0.52(0.58)	0.12(0.41)	2.81(1.01)	0.04(0.2)	2.81(1.01)	0.04(0.2)	4(0)	0(0)	4(0)	000
	None(Indep)	3.24(0.74)	6.39(2.65)	3.23(0.74)	6.21(2.5)	4(0)	3.4(2.23)	4(0)	3.4(2.23)	4(0)	1.47(1.42)	4(0)	1.47(1.42)
	1	2.63(0.91)	3.15(1.93)	2.63(0.91)	3.15(1.93)	4(0)	1.47(1.51)	4(0)	1.47(1.51)	4(0)	0.05(0.22)	4(0)	0.05(0.22)
$Mat_{3/2}(1.5)$	Gauss	2.39(0.84)	2.93(1.93)	2.39(0.84)	2.9(1.88)	4(0)	0.68(0.99)	4(0)	0.68(0.99)	4(0)	0(0)	4(0)	(0)0
,	InvMQ	2.03(1)	2.26(1.57)	2.03(1)	2.25(1.55)	4(0)	0.43(0.82)	4(0)	0.43(0.82)	4(0)	0(0)	4(0)	000
	True	0.69(0.73)	0.2(0.53)	0.69(0.73)	0.2(0.53)	3.56(0.61)	0.17(0.4)	3.56(0.61)	0.17(0.4)	4(0)	0(0)	4(0)	000
	None(Indep)	3.33(0.74)	5.59(2.19)	3.33(0.74)	5.41(2.08)	4(0)	3.2(2.56)	4(0)	3.2(2.56)	4(0)	1.34(1.51)	4(0)	1.34(1.51)
	I	2.72(1)	3.06(1.97)	2.72(1)	3.06(1.97)	4(0)	0.75(1.19)	4(0)	0.75(1.19)	4(0)	0.1(0.33)	4(0)	0.1(0.33)
$Mat_{5/2}(2.5)$	Gauss	2.53(1.04)	2.57(1.81)	2.53(1.04)	2.57(1.81)	4(0)	0.41(0.81)	4(0)	0.41(0.81)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
	InvMQ	2.25(0.97)	2.17(1.68)	2.25(0.97)	2.16(1.67)	4(0)	0.38(0.91)	4(0)	0.38(0.91)	4(0)	0(0)	4(0)	000
	True	0.34(0.57)	0.05(0.26)	0.34(0.57)	0.05(0.26)	1.82(0.83)	0.05(0.22)	1.82(0.83)	0.05(0.22)	4(0)	0(0)	4(0)	0(0)
	None(Indep)	3.2(0.71)	5.66(2.16)	3.19(0.71)	5.56(2.08)	4(0)	3.57(2.38)	4(0)	3.56(2.37)	4(0)	1.3(1.32)	4(0)	1.3(1.32)
	-	2.39(1.03)	2.86(1.99)	2.39(1.03)	2.86(1.99)	4(0)	1.39(1.41)	4(0)	1.39(1.41)	4(0)	0.07(0.26)	4(0)	0.07(0.26)
$Mat_{5/2}(1.5)$	Gauss	2.35(0.91)	2.84(1.76)	2.35(0.91)	2.82(1.75)	4(0)	0.88(1.11)	4(0)	0.88(1.11)	4(0)	0(0)	4(0)	000
	InvMQ	2.2(0.89)	2.09(1.52)	2.2(0.89)	2.09(1.52)	4(0)	0.36(0.72)	4(0)	0.36(0.72)	4(0)	0(0)	4(0)	000
	True	0.28(0.51)	0(0)	0.28(0.51)	(0)0	2.38(0.91)	0.04(0.2)	2.38(0.91)	0.04(0.2)	4(0)	0(0)	4(0)	000
	None(Indep)	3.15(0.88)	6.37(1.95)	3.14(0.88)	6.24(1.83)	4(0)	3.94(1.92)	4(0)	3.94(1.92)	4(0)	1.78(1.54)	4(0)	1.78(1.54)
	-	2.44(1.02)	3.35(1.98)	2.44(1.02)	3.34(1.96)	4(0)	1.25(1.14)	4(0)	1.25(1.14)	4(0)	0.08(0.27)	4(0)	0.08(0.27)
Gauss(1.5)	Gauss	2.24(0.95)	2.83(1.61)	2.24(0.95)	2.82(1.57)	4(0)	0.63(0.73)	4(0)	0.63(0.73)	4(0)	0.02(0.14)	4(0)	0.02(0.14)
	InvMQ	2.18(0.9)	2.2(1.41)	2.18(0.9)	2.19(1.39)	4(0)	0.59(0.83)	4(0)	0.59(0.83)	4(0)	0(0)	4(0)	0(0)
	True	2.78(0.87)	3.66(1.74)	2.75(0.87)	3.61(1.69)	4(0)	1.84(1.54)	4(0)	1.84(1.54)	4(0)	0.17(0.45)	4(0)	0.17(0.45)
	None(Indep)	3.13(0.75)	6.64(2.37)	3.12(0.76)	6.45(2.24)	4(0)	4.38(2.06)	4(0)	4.38(2.06)	4(0)	1.14(1.07)	4(0)	1.14(1.07)
_	-	2.6(0.96)	3.62(1.66)	2.6(0.96)	3.62(1.66)	4(0)	1.42(1.36)	4(0)	1.42(1.36)	4(0)	0.02(0.14)	4(0)	0.02(0.14)
Gauss(2.5)	Gauss	2.44(0.86)	2.9(1.47)	2.44(0.86)	2.86(1.44)	4(0)	0.73(0.92)	4(0)	0.73(0.92)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
	InvMQ	2.17(0.94)	2.36(1.52)	2.17(0.94)	2.36(1.52)	4(0)	0.63(0.85)	4(0)	0.63(0.85)	4(0)	0(0)	4(0)	0(0)
_									((a).			

Table 4: Monte Carlo Mean (Standard dev.) for the selected number of nonzero covariates using 100 datasets under both Independent and Dependent setup using spatially weighted group LASSO and adaptive Group LASSO algorithms when J=35(t = 1)

_			_	9=w	9			m=12	12			m=24	24	
ŗ	Cov Model	Spatial Weights	GLA	GLASSO	Adapt. GLASSO	TASSO	GLASSO	OSS	Adapt. GLASSO	TASSO	GLASSO	OSS	Adapt. (Adapt. GLASSO
			True Positive	False Positive										
		None(Indep)	3.74(0.48)	3.63(1.83)	3.74(0.48)	3.62(1.84)	4(0)	1.6(1.3)	4(0)	1.6(1.3)	4(0)	0.55(0.77)	4(0)	0.55(0.77)
			3.17(0.82)	2.02(1.34)	3.15(0.82)	2(1.29)	4(0)	0.38(0.65)	4(0)	0.38(0.65)	4(0)	(0)0	4(0)	(0)0
	Exp(0.5)	Gauss	3.01(0.85)	1.76(1.24)	3.01(0.85)	1.74(1.23)	4(0)	0.18(0.48)	4(0)	0.18(0.48)	4(0)	0(0)	4(0)	(0)0
		InvMQ	2.94(0.93)	1.58(1.44)	2.94(0.93)	1.58(1.44)	4(0)	0.1(0.3)	4(0)	0.1(0.3)	4(0)	0(0)	4(0)	000
		True	1.66(1.02)	0.29(0.56)	1.66(1.02)	0.29(0.56)	4(0)	0.07(0.29)	4(0)	0.07(0.29)	4(0)	0(0)	4(0)	(0)0
		None(Indep)	3.82(0.41)	3.77(2.09)	3.82(0.41)	3.67(1.99)	4(0)	1.51(1.24)	4(0)	1.51(1.24)	4(0)	0.53(0.69)	4(0)	0.53(0.69)
		1	3.11(0.82)	2.31(1.35)	3.11(0.82)	2.31(1.35)	4(0)	0.43(0.64)	4(0)	0.43(0.64)	4(0)	0.03(0.22)	4(0)	0.03(0.22)
	Exp(1)	Gauss	3.01(0.8)	1.93(1.27)	3.01(0.8)	1.93(1.27)	4(0)	0.24(0.51)	4(0)	0.24(0.51)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.99(0.88)	1.56(1.17)	2.99(0.88)	1.54(1.13)	4(0)	0.14(0.47)	4(0)	0.14(0.47)	4(0)	0(0)	4(0)	000
		True	2.33(1.05)	1.1(0.96)	2.33(1.05)	1.1(0.96)	4(0)	0.12(0.38)	4(0)	0.12(0.38)	4(0)	0(0)	4(0)	000
		None(Indep)	3.8(0.45)	3.63(1.86)	3.79(0.46)	3.57(1.81)	4(0)	1.37(1.39)	4(0)	1.37(1.39)	4(0)	0.34(0.57)	4(0)	0.34(0.57)
		_	3.26(0.77)	1.87(1.5)	3.26(0.77)	1.87(1.5)	4(0)	0.46(0.77)	4(0)	0.46(0.77)	4(0)	0.05(0.22)	4(0)	0.05(0.22)
	Mat _{3/2} (2.5)	Gauss	2.98(0.85)	1.64(1.34)	2.98(0.85)	1.63(1.33)	4(0)	0.25(0.58)	4(0)	0.25(0.58)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
	,	InvMQ	2.74(0.86)	1.5(1.18)	2.74(0.86)	1.5(1.18)	4(0)	0.1(0.33)	4(0)	0.1(0.33)	4(0)	(0)0	4(0)	(0)0
		True	0.8(0.68)	0.09(0.32)	0.8(0.68)	0.09(0.32)	3.36(0.72)	0.01(0.1)	3.36(0.72)	0.01(0.1)	4(0)	0(0)	4(0)	(0)0
		None(Indep)	3.75(0.46)	4.07(2.04)	3.75(0.46)	4.06(2.04)	4(0)	1.54(1.42)	4(0)	1.54(1.42)	4(0)	0.47(0.67)	4(0)	0.47(0.67)
		_	3.23(0.71)	2.05(1.3)	3.22(0.7)	2.01(1.26)	4(0)	0.46(0.78)	4(0)	0.46(0.78)	4(0)	0.02(0.14)	4(0)	0.02(0.14)
	Mat _{3/2} (1.5)	Gauss	3.03(0.76)	1.83(1.3)	3.03(0.76)	1.8(1.31)	4(0)	0.28(0.6)	4(0)	0.28(0.6)	4(0)	0(0)	4(0)	000
		InvMQ	2.84(0.9)	1.51(1.06)	2.84(0.9)	1.48(1.01)	4(0)	0.14(0.35)	4(0)	0.14(0.35)	4(0)	0(0)	4(0)	000
7		True	1.01(0.88)	0.15(0.36)	1.01(0.88)	0.15(0.36)	3.82(0.46)	0.07(0.26)	3.82(0.46)	0.07(0.26)	4(0)	0(0)	4(0)	(0)0
CI		None(Indep)	3.82(0.41)	3.59(2.08)	3.81(0.42)	3.53(2.04)	4(0)	1.34(1.2)	4(0)	1.34(1.2)	4(0)	0.58(0.88)	4(0)	0.58(0.88)
		-	3.22(0.76)	2.06(1.55)	3.22(0.76)	2.06(1.55)	4(0)	0.43(0.86)	4(0)	0.43(0.86)	4(0)	0.04(0.2)	4(0)	0.04(0.2)
	Mat _{5/2} (2.5)	Gauss	2.91(0.89)	1.72(1.35)	2.91(0.89)	1.69(1.35)	4(0)	0.26(0.66)	4(0)	0.26(0.66)	4(0)	0(0)	4(0)	000
		InvMQ	2.97(0.89)	1.67(1.33)	2.97(0.89)	1.67(1.33)	4(0)	0.1(0.3)	4(0)	0.1(0.3)	4(0)	0(0)	4(0)	000
		True	0.37(0.56)	0.04(0.24)	0.37(0.56)	0.04(0.24)	2.2(0.9)	0.06(0.24)	2.2(0.9)	0.06(0.24)	4(0)	(0)0	4(0)	(0)0
		None(Indep)	3.69(0.51)	3.85(1.92)	3.69(0.51)	3.84(1.93)	4(0)	1.74(1.52)	4(0)	1.74(1.52)	4(0)	0.58(0.75)	4(0)	0.58(0.75)
		_	3.29(0.74)	1.9(1.43)	3.27(0.75)	1.88(1.39)	4(0)	0.47(0.77)	4(0)	0.47(0.77)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
	Mat _{5/2} (1.5)	Gauss	3.21(0.74)	1.63(1.23)	3.21(0.74)	1.61(1.21)	4(0)	0.25(0.5)	4(0)	0.25(0.5)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.84(0.88)	1.53(1.1)	2.84(0.88)	1.52(1.1)	4(0)	0.19(0.42)	4(0)	0.19(0.42)	4(0)	0(0)	4(0)	000
		True	0.59(0.65)	0.06(0.24)	0.59(0.65)	0.06(0.24)	2.79(0.99)	0.06(0.24)	2.79(0.99)	0.06(0.24)	4(0)	0(0)	4(0)	0(0)
		(dapuI)auoN	3.76(0.49)	3.97(2.06)	3.76(0.49)	3.92(2.02)	4(0)	1.63(1.3)	4(0)	1.63(1.3)	4(0)	0.59(0.81)	4(0)	0.59(0.81)
			3.23(0.75)	2.07(1.24)	3.23(0.75)	2.07(1.24)	4(0)	0.64(0.92)	4(0)	0.64(0.92)	4(0)	0.03(0.17)	4(0)	0.03(0.17)
	Gauss(1.5)	Gauss	3.02(0.82)	1.86(1.25)	3.02(0.82)	1.86(1.25)	4(0)	0.28(0.59)	4(0)	0.28(0.59)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.76(0.79)	1.58(1.17)	2.76(0.79)	1.58(1.17)	3.99(0.1)	0.23(0.47)	3.99(0.1)	0.23(0.47)	4(0)	0(0)	4(0)	(0)0
		True	3.3(0.73)	2.32(1.55)	3.3(0.73)	2.32(1.55)	4(0)	0.98(1.05)	4(0)	0.98(1.05)	4(0)	0.12(0.33)	4(0)	0.12(0.33)
		None(Indep)	3.67(0.57)	3.97(1.89)	3.67(0.57)	3.83(1.81)	4(0)	1.77(1.46)	4(0)	1.77(1.46)	4(0)	0.55(0.78)	4(0)	0.55(0.78)
		_	3.19(0.77)	2.1(1.19)	3.19(0.77)	2.09(1.19)	4(0)	0.49(0.69)	4(0)	0.49(0.69)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
	Gauss(2.5)	Gauss	2.97(0.83)	1.85(1.34)	2.97(0.83)	1.83(1.33)	4(0)	0.24(0.47)	4(0)	0.24(0.47)	4(0)	0(0)	4(0)	000
		InvMQ	2.75(0.77)	1.45(1.12)	2.75(0.77)	1.44(1.09)	4(0)	0.24(0.47)	4(0)	0.24(0.47)	4(0)	0(0)	4(0)	000
		True	3.56(0.57)	3.11(1.38)	3.56(0.57)	3.11(1.38)	4(0)	1.73(1.3)	4(0)	1.73(1.3)	4(0)	0.69(0.8)	4(0)	0.69(0.8)

Independent and Dependent setup using spatially weighted group LASSO and adaptive Group LASSO algorithms when J=15Table 5: Monte Carlo Mean (Standard dev.) for the selected number of nonzero covariates using 100 datasets under both (t = 3)

4 (Application) Standard Multiple Con-Noded Special Multiple Con-Noded Application Legister Con-Noded	-				9=m					m=12			m=24		
English Temporalise Februay Revokine Time Positive February Revokine Control of Con	ŗ	Cov Model	Spatial Weights		OSS	Adapt. 6	TASSO	GLA	OSS	Adapt. (SLASSO	GLA	OSS	Adapt.	GLASSO
Name Canasa 2.2560,57 2.561,58 2.560,58 2.561,58 2.560,58 4.60 0.576,89 4.60 0.576,89 4.60 0.	_				False Positive	True Positive	False Positive								
Eryth Carrot	-		None(Indep)	3.53(0.67)	5.01(2.31)	3.53(0.67)	4.89(2.26)	4(0)	2.54(1.73)	4(0)	2.54(1.73)	4(0)	0.88(0.99)	4(0)	0.88(0.99)
Exp(15) Games 27/201/34 2.48(1.54)				2.84(0.95)	2.76(1.78)	2.84(0.95)	2.76(1.78)	4(0)	0.89(1.14)	4(0)	0.89(1.14)	4(0)	0.04(0.2)	4(0)	0.04(0.2)
Figure F		Exp(0.5)	Gauss	2.72(0.91)	2.49(1.56)	2.72(0.91)	2.48(1.56)	4(0)	0.57(0.89)	4(0)	0.57(0.89)	4(0)	0.02(0.14)	4(0)	0.02(0.14)
True			InvMQ	2.4(0.85)	2(1.36)	2.4(0.85)	2(1.36)	4(0)	0.29(0.61)	4(0)	0.29(0.61)	4(0)	0(0)	4(0)	0(0)
Equit Cause 2.380(73) 2.77(16.2) 2.78(0.73) 2.77(16.2) 4400 2.98(2.0) 440 1.08(1.08) 440 0.08(1.04)			True	1.36(0.99)	0.44(0.61)	1.36(0.99)	0.44(0.61)	3.99(0.1)	0.07(0.26)	3.99(0.1)	0.07(0.26)	4(0)	0(0)	4(0)	0(0)
The Cause 1	-		None(Indep)	3.49(0.63)	5.25(1.9)	3.49(0.63)	5.14(1.8)	4(0)	2.98(2.07)	4(0)	2.98(2.07)	4(0)	0.83(1.04)	4(0)	0.83(1.04)
Hard			_	2.78(0.77)	2.77(1.62)	2.78(0.77)	2.77(1.62)	4(0)	1.06(1.08)	4(0)	1.06(1.08)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
Novelindery 25700 181,323 2570,040 181,323 4400 0370,054 4400 0370,054 4400 0310		Exp(1)	Gauss	2.7(0.85)	2.18(1.35)	2.7(0.85)	2.16(1.33)	4(0)	0.55(0.78)	4(0)	0.55(0.78)	4(0)	0(0)	4(0)	0(0)
Number (abover) 2.205(1.05) 1.02(1.04) 2.05(1.04) 1.02(1.04)			InvMQ	2.57(0.9)	1.8(1.32)	2.57(0.9)	1.8(1.32)	4(0)	0.37(0.63)	4(0)	0.37(0.63)	4(0)	0(0)	4(0)	0(0)
Numericary Numericary Signolary Si			True	2.05(1.09)	1.02(1.08)	2.05(1.09)	1.02(1.08)	4(0)	0.31(0.54)	4(0)	0.31(0.54)	4(0)	0(0)	4(0)	0(0)
Milka _{2/2} (25) 11 2.500(9) 2.52(1.51) 2.58(0.57) 2.58(1.48) 2.58(0.57) 2.58(1.48) 4.00 0.58(0.55) 4.00 0.008(0.27) 4.00 0.008(0.			None(Indep)	3.57(0.59)	5.09(2.16)	3.57(0.59)	5.07(2.16)	4(0)	2.29(1.82)	4(0)	2.29(1.82)	4(0)	0.83(1.1)	4(0)	0.83(1.1)
Math 3/2 (2.5) Chance (Control of Control of Con			-	2.99(0.9)	2.52(1.51)	2.99(0.9)	2.52(1.51)	4(0)	0.64(0.93)	4(0)	0.64(0.93)	4(0)	0.08(0.27)	4(0)	0.08(0.27)
True		Mat _{3/2} (2.5)	Gauss	2.85(0.77)	2.38(1.49)	2.85(0.77)	2.36(1.48)	4(0)	0.39(0.71)	4(0)	0.39(0.71)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
Noverlinky Oxfords O			InvMQ	2.55(0.87)	1.91(1.44)	2.55(0.87)	1.89(1.44)	4(0)	0.25(0.56)	4(0)	0.25(0.56)	4(0)	0(0)	4(0)	0(0)
Numclindep S.A590.56 248(12.15) A.600.56 4.8(12.15) A.600.26 4.8(12.15) A.600.26 4.8(12.15) A.600.56 4.8(12.15) A.600.52 A.600.23 A.600			True	0.66(0.65)	0.01(0.1)	0.66(0.65)	0.01(0.1)	3.15(0.93)	0.01(0.1)	3.15(0.93)	0.01(0.1)	4(0)	0(0)	4(0)	0(0)
Maig/2(1.5) LinkAg 2.58(1) 2.18(1.45) 2.28(1.18) 2.18(1.45) 2.28(1.18) 2.18(1.44) 4400 0.33(0.44) 4400 0.33(0.43) 4400 0.03(0.14) 4400 0.03(0.18) 4400 0.03(0.18) 4400 0.03(0.18) 4400 0.03(0.18) 4400 0.03(0.18) 4400 0.03(0.18) 4400 0.03(0.18) 4400 0.03(0.18) 4400			None(Indep)	3.63(0.56)	4.91(2.17)	3.63(0.56)	4.87(2.15)	4(0)	2.59(1.5)	4(0)	2.59(1.5)	4(0)	0.91(1.06)	4(0)	0.91(1.06)
MMa ₃ /2(1.5) Gauss 2.56(1) 2.18(1.44) 2.68(1) 1.51(1.45) 4.00 0.53(0.74) 4(0) 0.53(0.74) 4(0) 0.40(0.74) 4(0) 0.40(0.74) 4(0) 0.40(0.74) 4(0) 0.40(0.74) 4(0) 0.70(0.44) 4(0)			-	2.97(0.97)	2.78(1.85)	2.97(0.97)	2.78(1.85)	4(0)	0.9(1.01)	4(0)	0.9(1.01)	4(0)	0.02(0.14)	4(0)	0.02(0.14)
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $		Mat _{3/2} (1.5)	Gauss	2.68(1)	2.18(1.64)	2.68(1)	2.16(1.61)	4(0)	0.53(0.74)	4(0)	0.53(0.74)	4(0)	0(0)	4(0)	0(0)
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$			InvMQ	2.56(0.89)	1.88(1.4)	2.56(0.89)	1.87(1.4)	4(0)	0.18(0.48)	4(0)	0.18(0.48)	4(0)	0(0)	4(0)	0(0)
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	30		True	0.8(0.74)	0.06(0.24)	0.8(0.74)	0.06(0.24)	3.78(0.48)	0.06(0.24)	3.78(0.48)	0.06(0.24)	4(0)	0(0)	4(0)	0(0)
True	3		None(Indep)	3.67(0.53)	4.99(2.2)	3.67(0.53)	4.97(2.2)	4(0)	1.95(1.9)	4(0)	1.95(1.9)	4(0)	0.71(0.94)	4(0)	0.71(0.94)
Gauss 2.87(0.91) 2.28(1.53) 2.87(0.91) 2.26(1.53) 400 0.44(0.75) 400 0.41(0.75) 400 0.01(0.1) 400 InvMQ 2.256(0.83) 1.98(1.48) 2.256(0.83) 1.98(1.48) 2.256(0.83) 1.98(1.48) 0.04(0.20) 1.75(0.89) 0.07(0.29) 1.75(0.89) 0.07(0.29) 1.75(0.89) 0.07(0.29) 4(0) 0.07(0.29) 4(0) 0.07(0.97) 4(0) 0.00 0.			-	3.07(0.86)	2.59(1.78)	3.07(0.86)	2.57(1.77)	4(0)	0.7(1.11)	4(0)	0.7(1.11)	4(0)	0.06(0.24)	4(0)	0.06(0.24)
InwMQ 1.556(0.83) 1.98(148) 2.56(0.83) 1.98(148) 2.56(0.83) 1.98(148) 2.56(0.83) 1.98(148) 2.56(0.83) 1.98(148) 2.56(0.83) 1.98(148) 2.56(0.83) 4(0) 0.26(0.63) 4(0) 0.26(0.63) 4(0) 0.26(0.63) 4(0) 0.06(0.23) 4(0) 0.03(0.17) 4(0) 0.03(0.17) 4(0) 0.03(0.17) 4(0) 0.04(0.27) 4(0) 0.24(0.51) 4(0) 0.04(0.77) 4(0) 0.04(0.75)		$Mat_{5/2}(2.5)$	Gauss	2.87(0.91)	2.28(1.53)	2.87(0.91)	2.26(1.5)	4(0)	0.41(0.75)	4(0)	0.41(0.75)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
True 0.0220.44) 0.03(0.17) 0.22(0.44) 0.03(0.17) 1.75(0.89) 0.07(0.29) 1.75(0.89) 0.07(0.29) 4(0) 0.04(0.29) 4(0) 0.04(0.29) 4(0) 0.04(0.24) 4(0) 0.04(InvMQ	2.56(0.83)	1.98(1.48)	2.56(0.83)	1.97(1.47)	4(0)	0.26(0.63)	4(0)	0.26(0.63)	4(0)	0(0)	4(0)	0(0)
None(Indep) 3490.61) 5.31(2.15) 3490.61) 5.17(2.11) 4(0) 2.45(1.77) 4(0) 2.45(1.77) 4(0) 0.76(0.97) 4(0) 0.76(0.98) 2.62(1.53) 2.81(0.92) 2.62(1.53) 2.81(0.92) 2.62(1.53) 2.81(0.92) 2.62(1.53) 2.81(0.92) 2.62(1.53) 2.81(0.92) 2.62(1.53) 2.81(0.92) 2.62(1.53) 2.81(0.92) 2.62(1.53) 2.81(0.92) 2.62(1.53) 2.81(0.92) 2.62(1.53) 2.62(1.63) 2.62			True	0.22(0.44)	0.03(0.17)	0.22(0.44)	0.03(0.17)	1.75(0.89)	0.07(0.29)	1.75(0.89)	0.07(0.29)	4(0)	0(0)	4(0)	0(0)
True 2.82(0.9) 2.65(1.53) 2.81(0.92) 2.62(1.51) 4(0) 0.84(1) 4(0) 0.84(1) 4(0) 0.03(0.17) 4(0) (0.04(0.25) 0.25(1.38) 2.26(1.38) 2.26(1.38) 2.26(1.38) 2.26(1.38) 2.26(1.38) 2.26(1.38) 2.26(1.38) 2.26(1.38) 2.26(1.38) 2.26(1.38) 2.26(1.38) 2.26(1.38) 2.26(1.38) 2.26(1.38) 4(0) 0.24(0.51) 4(0) 0.24(0.51) 4(0) 0.24(0.51) 4(0) 0.04(0.2) 4(0)			None(Indep)	3.49(0.61)	5.31(2.15)	3.49(0.61)	5.17(2.11)	4(0)	2.45(1.77)	4(0)	2.45(1.77)	4(0)	0.76(0.97)	4(0)	0.76(0.97)
Gauss 2.67(0.88) 2.26(1.38) 2.67(0.88) 2.67(0.88) 2.67(0.88) 2.67(0.88) 2.67(0.88) 4(0) 0.24(0.75) 4(0) 0.44(0.75) 4(0) 0.44(0.75) 4(0) 0.44(0.75) 4(0) 0.24(0.51) 4(0)			-	2.82(0.9)	2.63(1.53)	2.81(0.92)	2.62(1.51)	4(0)	0.84(1)	4(0)	0.84(1)	4(0)	0.03(0.17)	4(0)	0.03(0.17)
InwNQ 2.46(0.82) 1.77(1.26) 2.46(0.82) 1.76(1.27) 4(0) 0.24(0.51) 4(0) 0.24(0.51) 4(0)		$Mat_{5/2}(1.5)$	Gauss	2.67(0.88)	2.26(1.38)	2.67(0.88)	2.26(1.38)	4(0)	0.44(0.76)	4(0)	0.44(0.76)	4(0)	0(0)	4(0)	0(0)
True 0.4(0.67) 0.04(0.27) 0.04(0.27) 0.04(0.27) 0.04(0.27) 0.04(0.07) 0.07(0.26) 2.46(1) 0.07(0.26) 2.46(1) 0.07(0.26) 2.46(1) 0.07(0.26) 4(0) 0.40(0.27) 0.07(0.26) 2.46(1) 0.07(0.26) 4(0) 0.07(0.26) 4(0) 0.07(0.26) 4(0) 0.07(0.26) 4(0) 0.07(0.26) 4(0) 0.00(0.26) 4(0)			InvMQ	2.46(0.82)	1.77(1.26)	2.46(0.82)	1.76(1.27)	4(0)	0.24(0.51)	4(0)	0.24(0.51)	4(0)	0(0)	4(0)	0(0)
None(Indep) 3.5(0.63) 5.17(1.93) 3.48(0.66) 5.02(1.79) 4(0) 2.89(1.53) 4(0) 2.89(1.53) 4(0) 0.8(0.88) 4(0) 0.8(0.88) 4(0) 0.8(0.88) 4(0) 0.8(0.84) 4(0) 4(0.84)			True	0.4(0.67)	0.04(0.2)	0.4(0.67)	0.04(0.2)	2.46(1)	0.07(0.26)	2.46(1)	0.07(0.26)	4(0)	0(0)	4(0)	0(0)
True 2.84(0.9) 3.23(1.6) 2.84(0.9) 3.23(1.6) 4(0) 0.80(0.84) 4(0) 0.80(0.84) 4(0) 0.04(0.2) 4(0) 4(0) 0.04(0.2) 4(0) 4(0) 2.74(0.87) 2.97(1.83) 2.77(0.89) 2.32(1.73) 2.77(0.89) 2.82(1.56) 2.87(1.53) 2.87(1.54) 2.87(1.55) 2.87(ı——		(July None (Indep)	3.5(0.63)	5.17(1.93)	3.48(0.66)	5.02(1.79)	4(0)	2.89(1.53)	4(0)	2.89(1.53)	4(0)	0.8(0.88)	4(0)	0.8(0.88)
Gauss 2.74(0.87) 2.9(1.46) 2.74(0.87) 2.87(1.43) 4(0) 0.238(0.69) 4(0) 0.38(0.69) 4(0) 0.00 4(0)			-	2.84(0.9)	3.23(1.6)	2.84(0.9)	3.23(1.6)	4(0)	0.8(0.84)	4(0)	0.8(0.84)	4(0)	0.04(0.2)	4(0)	0.04(0.2)
InvMQ 2.47(0.96) 2.11(1.32) 2.47(0.96) 2.11(1.51) 4(0) 0.27(0.51) 4(0) 0.27(0.51) 4(0) 0.00) 4(0)		Gauss(1.5)	Gauss	2.74(0.87)	2.9(1.46)	2.74(0.87)	2.87(1.43)	4(0)	0.38(0.69)	4(0)	0.38(0.69)	4(0)	0(0)	4(0)	0(0)
True 2.77(0.99) 3.32(1.73) 2.76(1) 3.32(1.73) 4(0) 1.4(1.11) 4(0) 1.4(1.11) 4(0) 0.21(0.46) 4(0) 0.21(0.46) 4(0) 0.21(0.46) 4(0) 0.21(0.46) 4(0) 0.21(0.46) 4(0) 0.21(0.46) 4(0) 0.21(0.46) 4(0) 0.21(0.46) 4(0) 0.21(0.46) 4(0) 0.21(0.46) 4(0) 0.21(0.46) 4(0) 0.21(0.46) 4(0) 0.21(0.46) 4(0) 0.21(0.46) 4(0) 0.21(0.46) 4(0) 4(InvMQ	2.47(0.96)	2.11(1.52)	2.47(0.96)	2.1(1.51)	4(0)	0.27(0.51)	4(0)	0.27(0.51)	4(0)	0(0)	4(0)	0(0)
NoneUndep) 3.42(0.74) 5.47(2.27) 3.4(0.75) 5.32(2.15) 4(0) 2.89(1.85) 4(0) 2.89(1.85) 4(0) 0.95(1.06) 4(0) 0.95(1.08) 2.77(0.89) 2.82(1.56) 2.77(0.88) 2.49(1.33) 2.77(0.88) 2.49(1.33) 2.56(0.78) 1.99(1.31) 4(0) 0.37(0.65) 4(0) 0.37(0.65) 4(0) 0.37(0.65) 4(0) 0.40(0) 4(0)			True	2.77(0.99)	3.32(1.73)	2.76(1)	3.32(1.73)	4(0)	1.4(1.11)	4(0)	1.4(1.11)	4(0)	0.21(0.46)	4(0)	0.21(0.46)
1 2.77(0.89) 2.82(1.56) 2.77(0.89) 2.82(1.56) 2.77(0.89) 2.48(1.56) 2.74(0.89) 2.44(1.56) 2.74(0.89) 2.44(1.56) 2.44(0.89) 2.44(1.59) 2.44(1.59) 2.44(1.59) 2.44(1.59) 2.44(1.59) 2.54(1.59) 2.54(1.59) 2.54(1.59) 2.54(1.59) 2.54(1.59) 2.54(1.59) 2.54(1.59) 2.54(1.46) 2.52(1.46) 4(0) 2.52(1.46) 4(0) 2.52(1.46) 4(0) 1.21(1.15) 4(0) 1.24(1.15) 1.24(1.	_		None(Indep)	3.42(0.74)	5.47(2.27)	3.4(0.75)	5.32(2.15)	4(0)	2.89(1.85)	4(0)	2.89(1.85)	4(0)	0.95(1.06)	4(0)	0.95(1.06)
Gauss 2.7(0.88) 2.5(1.33) 2.7(0.88) 2.49(1.33) 4(0) 0.4(0.68) 4(0) 0.4(0.68) 4(0) 0.0(0) 4(0)			-	2.77(0.89)	2.82(1.56)	2.77(0.89)	2.82(1.56)	4(0)	0.96(1.08)	4(0)	0.96(1.08)	4(0)	0(0)	4(0)	0(0)
		Gauss(2.5)	Gauss	2.7(0.88)	2.5(1.33)	2.7(0.88)	2.49(1.33)	4(0)	0.4(0.68)	4(0)	0.4(0.68)	4(0)	0(0)	4(0)	0(0)
$ \left \begin{array}{cccccccccccccccccccccccccccccccccccc$			InvMQ	2.65(0.78)	1.99(1.31)	2.65(0.78)	1.99(1.31)	4(0)	0.37(0.65)	4(0)	0.37(0.65)	4(0)	0(0)	4(0)	0(0)
			True	3.36(0.75)	3.99(1.8)	3.36(0.75)	3.97(1.79)	4(0)	2.52(1.46)	4(0)	2.52(1.46)	4(0)	1.21(1.15)	4(0)	1.21(1.15)

Table 6: Monte Carlo Mean (Standard dev.) for the selected number of nonzero covariates using 100 datasets under both Independent and Dependent setup using spatially weighted group LASSO and adaptive Group LASSO algorithms when J=25(t = 3)

Mapple Coloresiste Following GLASSO Adapt CLASSO GLASSO Adapt CLASSO GLASSO Adapt CLASSO GLASSO Adapt CLASSO Adap					y=w	يو			m=12	12			42=m	74	
True Paris True	ī	Cov Model	Spatial Weights	T9			TASSO	GLA			GLASSO	GLA			GLASSO
Exp(5) None(ladep) 3.58.0.71 p. 2.60.28 p. 3.60.73 p. 3.60.73 p. 4.00 p. 3.74.26 p. 4.00 p. 1.60.45 p. Exp(65) Grass 2.60.00 p. 3.62.00 p. 3.62.00 p. 3.62.00 p. 4.00 p. 1.60.45 p. Exp(18) 2.60.00 p. 2.60.00 p. 2.60.00 p. 2.60.00 p. 4.00 p. 0.00.00 p. Trace 1.21.00550 2.80.00 p. 2.74.10 p. 2.74.10 p. 2.74.10 p. 2.60.00 p. 4.00 p. 0.00.00 p. Exp(11) Grass 2.34.00 p. 2.74.10 p. 2.74.10 p. 2.74.10 p. 2.74.10 p. 2.74.10 p. 0.00.00 p. Exp(11) Grass 2.34.00 p. 2.74.10 p. 2.74.10 p. 2.74.10 p. 0.00.00 p. 0.00.00 p. Mat ₂ /2(1.5) 2.24.00 p. 2.74.10 p. 2.74.10 p. 2.74.10 p. 2.74.10 p. 0.00.00 p. 0.00.00 p. Mat ₂ /2(1.5) 2.24.00 p. 2.74.10 p. 2.74.10 p. 2.74.10 p. 2.74.10 p. 0.00.00 p. 0.00.00 p. 0.00.00 p. Mat ₂				True Positive	False Positive										
Euglo3 Games 2.20(0.94) 1.34(4.24) 2.20(8.81) 440 1.54(4.8) 440 0.15(4.8) 440 0.15(4.8) 440 0.04(0.37)<			None(Indep)	3.38(0.74)	6.22(2.39)	3.36(0.76)	6.1(2.33)	4(0)	3.74(2.68)	4(0)	3.74(2.68)	4(0)	1.49(1.55)	4(0)	1.49(1.55)
Exp(λ) (Games) Captilotys) Captilotysis 2.22(1.04) <th< td=""><th></th><td></td><td>ı</td><td>2.62(0.94)</td><td>3.15(2.14)</td><td>2.6(0.95)</td><td>3.11(2.13)</td><td>4(0)</td><td>1.5(1.45)</td><td>4(0)</td><td>1.5(1.45)</td><td>4(0)</td><td>0.04(0.2)</td><td>4(0)</td><td>0.04(0.2)</td></th<>			ı	2.62(0.94)	3.15(2.14)	2.6(0.95)	3.11(2.13)	4(0)	1.5(1.45)	4(0)	1.5(1.45)	4(0)	0.04(0.2)	4(0)	0.04(0.2)
Fig. 1		Exp(0.5)	Gauss	2.39(0.98)	2.59(1.84)	2.39(0.98)	2.56(1.81)	4(0)	0.89(1.08)	4(0)	0.89(1.08)	4(0)	0.02(0.14)	4(0)	0.02(0.14)
Novelleigh Sept. According Sept. Acc			InvMQ	2.22(1.04)	2.38(1.79)	2.22(1.04)	2.37(1.78)	4(0)	0.41(0.75)	4(0)	0.41(0.75)	4(0)	0(0)	4(0)	0(0)
Harry Cause Caus			True	1.21(0.95)	0.48(0.78)	1.21(0.95)	0.48(0.78)	3.99(0.1)	0.1(0.3)	3.99(0.1)	0.1(0.3)	4(0)	0(0)	4(0)	0(0)
Exp(1) Class 2.580(697) 3.34(18) 2.490(97) 2.34(18) 4 (40) 1.39(1.36) 4 (40) 0.050(02) Exp(1) ImANQ 2.250(697) 2.284(1.89) 2.240(997) 2.24(1.8) 2.490 2.24(1.8) 4 (40) 0.71(1.44) 4 (40) 0.71(1.44) 4 (40) 0.71(1.44) 4 (40) 0.71(1.44) 4 (40) 0.71(1.44) 4 (40) 0.71(1.44) 4 (40) 0.70(1.44) 0.70(1.44) 0.70(1.44) 0.70(1.44) 0.70(1.44) 0.70(1.44) 0.70(1.44) 0.70(1.44)			None(Indep)	3.36(0.7)	5.77(2.3)	3.33(0.73)	5.54(1.98)	4(0)	3.68(1.93)	4(0)	3.68(1.93)	4(0)	1.31(1.18)	4(0)	1.31(1.18)
Exp(1) Cases 2.360.079 2.346.079 3.346.079 3.34			_	2.59(0.98)	3.3(1.8)	2.59(0.98)	3.3(1.8)	4(0)	1.39(1.36)	4(0)	1.39(1.36)	4(0)	0.09(0.29)	4(0)	0.09(0.29)
The Part		Exp(1)	Gauss	2.36(0.97)	2.98(1.69)	2.36(0.97)	2.95(1.67)	4(0)	0.67(0.95)	4(0)	0.67(0.95)	4(0)	0(0)	4(0)	0(0)
True 1.55(1.05) 1.52(1.38) 3.45(1.05) 5.39(2.35) 440 0.51(1.77) 400 1.25(1.37) 400 1.01(1.39) Mats _{2/2} (2.5) 1.50(1.50) 5.39(2.32) 3.460(6.7) 5.39(2.35) 440 1.27(1.31) 400 1.25(1.37) 400 1.01(3.39) Mats _{2/2} (2.5) Image 2.55(0.92) 2.57(0.82) 2.57(0.82) 2.57(1.82) 440 0.65(0.97) 440 0.10(3.9) Noneclinder) 2.35(0.38) 0.120(4.4) 0.25(1.82) 2.57(1.82) 2.57(1.82) 2.57(1.82) 440 0.66(0.7) 440 0.10(3.9) Mats _{2/2} (1.5) 1.05(0.38) 0.120(4.4) 0.25(1.82) 2.57(1.82) 440 0.66(0.7) 440 0.00(0.1) Mats _{2/2} (1.5) 1.05(0.92) 2.37(1.93) 440 0.66(0.7) 440 0.00(0.1) Mats _{2/2} (1.5) 1.05(0.92) 2.37(1.93) 2.37(1.93) 2.37(1.93) 2.37(1.93) 440 0.00(0.1) Mats _{2/2} (1.5) 1.05(0.92) 2.37(1.93) 2.37(1.93) <th< td=""><th></th><td></td><td>InvMQ</td><td>2.07(0.99)</td><td>2.25(1.61)</td><td>2.07(0.99)</td><td>2.24(1.59)</td><td>4(0)</td><td>0.71(1.04)</td><td>4(0)</td><td>0.71(1.04)</td><td>4(0)</td><td>0(0)</td><td>4(0)</td><td>0(0)</td></th<>			InvMQ	2.07(0.99)	2.25(1.61)	2.07(0.99)	2.24(1.59)	4(0)	0.71(1.04)	4(0)	0.71(1.04)	4(0)	0(0)	4(0)	0(0)
National Carrier National C			True	1.56(1.05)	1.32(1.38)	1.56(1.05)	1.32(1.38)	4(0)	0.51(0.7)	4(0)	0.51(0.7)	4(0)	0(0)	4(0)	0(0)
Mais ₂ (2,25) (1) (2) (1) (2) (2) (1) (2) (2) (1) (2) (2) (1) (2) (2) (1) (2) (2) (2) (1) (2) (2) (2) (2) (2) (2) (2) (2) (2) (2			None(Indep)	3.48(0.67)	5.79(2.32)	3.46(0.67)	5.59(2.25)	4(0)	2.71(2.13)	4(0)	2.71(2.13)	4(0)	1.25(1.37)	4(0)	1.25(1.37)
Mali ₃ /2(1.5) Gauss 2.57(0.92) 2.54(1.68) 2.27(0.92) 2.54(1.68) 2.57(0.92) 2.54(1.68) 2.57(0.92) 2.54(1.68) 4(0) 0.06(1.01) 0.06(1.02) 4(0) 0.06(0.02) 4			_	2.71(0.9)	2.94(1.75)	2.71(0.9)	2.94(1.75)	4(0)	1.22(1.28)	4(0)	1.22(1.28)	4(0)	0.1(0.39)	4(0)	0.1(0.39)
Image 12,26(0.9) 12,24(0.4) 12,24(0.4) 12,24(0.4) 12,24(0.4) 12,24(0.4) 12,24(0.4) 12,24(0.4) 12,24(0.2) 14,7(14.) 12,24(0.2) 14,7(14.) 12,24(0.2) 14,7(14.) 12,24(0.2) 14,7(14.		$Mat_{3/2}(2.5)$	Gauss	2.57(0.92)	2.59(1.6)	2.57(0.92)	2.57(1.58)	4(0)	0.69(1)	4(0)	0.69(1)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
True 0.5250.58 0.1204.1) 0.1204.1			InvMQ	2.25(0.9)	2.24(1.68)	2.25(0.9)	2.23(1.68)	4(0)	0.62(0.97)	4(0)	0.62(0.97)	4(0)	0(0)	4(0)	0(0)
Name			True	0.52(0.58)	0.12(0.41)	0.52(0.58)	0.12(0.41)	2.81(1.01)	0.04(0.2)	2.81(1.01)	0.04(0.2)	4(0)	0(0)	4(0)	0(0)
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$			None(Indep)	3.24(0.74)	6.39(2.65)	3.23(0.74)	6.21(2.5)	4(0)	3.4(2.23)	4(0)	3.4(2.23)	4(0)	1.47(1.42)	4(0)	1.47(1.42)
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$			п	2.63(0.91)	3.15(1.93)	2.63(0.91)	3.15(1.93)	4(0)	1.47(1.51)	4(0)	1.47(1.51)	4(0)	0.05(0.22)	4(0)	0.05(0.22)
True		$Mat_{3/2}(1.5)$	Gauss	2.39(0.84)	2.93(1.93)	2.39(0.84)	2.9(1.88)	4(0)	0.68(0.99)	4(0)	0.68(0.99)	4(0)	0(0)	4(0)	0(0)
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$			InvMQ	2.03(1)	2.26(1.57)	2.03(1)	2.25(1.55)	4(0)	0.43(0.82)	4(0)	0.43(0.82)	4(0)	0(0)	4(0)	0(0)
Mat _{5/2} (2.5) Gauss(1.5) 3.33(0.44) 5.39(0.19) 3.33(0.44) 5.39(0.19) 3.33(0.44) 5.39(0.19) 3.33(0.44) 5.33(0.44) 5.33(0.44) 5.33(0.44) 5.33(0.44) 5.33(0.44) 5.33(0.44) 4(0) 0.15(1.19) 4(0) 0.15(1.19) 4(0) 0.15(1.18) 4(0) 0.15(1.18) 4(0) 0.15(1.19) 4(0) 0.15(1.18) 4(0) 0.15(1.19) 4(0) 0.15(0.13) 4(0) 0.15(0.13) 4(0) 0.15(0.13) 4(0) 0.01(0.11) 4(0) 0.01(0.11) 4(0) 0.01(0.11) 4(0) 0.01(0.11) 4(0) 0.01(0.11) 4(0) 0.01(0.11) 4(0) 0.01(0.11)	35		True	0.69(0.73)	0.2(0.53)	0.69(0.73)	0.2(0.53)	3.56(0.61)	0.17(0.4)	3.56(0.61)	0.17(0.4)	4(0)	0(0)	4(0)	0(0)
Gauss 2.52(1) 3.06(1.97) 2.06(1.97) 4(0) 0.75(1.19) 4(0) 0.75(1.19) 4(0) 0.10(33) InvMQ 2.253(104) 2.53(104) 0.05(0.26) 0.05(0.25) 1.82(0.83) 0.05(0.22) 1.82(0.83) 0.05(0.22) 4(0) 0.01(0.1) 0.00 None(Indep) 3.30(17) 5.66(2.16) 3.90(17) 5.66(2.08) 1.90(172) 4(0) 0.36(0.22) 4(0) 0.01(0.20) InvMQ 2.25(0.91) 2.39(1.03) 2.86(1.99) 4(0) 1.36(1.21) 4(0) 0.01(0.20) InvMQ 2.25(0.89) 2.90(1.52) 4(0) 0.36(0.22) 4(0) 0.01(0.20) InvMQ 2.25(0.89) 2.90(1.52) 2.86(1.78) 4(0) 0.36(0.72) 4(0) 0.01(0.20)	ì		None(Indep)	3.33(0.74)	5.59(2.19)	3.33(0.74)	5.41(2.08)	4(0)	3.2(2.56)	4(0)	3.2(2.56)	4(0)	1.34(1.51)	4(0)	1.34(1.51)
Gauss 2.55(1.04) 2.57(1.81) 2.57(1.81) 4(0) 0.41(0.81) 4(0) 0.38(0.91) 4(0) 0.01(0.1) True 1 m/MQ 2.25(0.97) 2.16(1.68) 2.25(0.97) 2.57(1.81) 4(0) 0.38(0.91) 4(0) 0.38(0.91) 4(0) 0.03(0.02) True 0.34(0.71) 5.66(2.16) 3.19(0.71) 5.56(2.08) 4(0) 3.57(2.38) 4(0) 0.38(0.21) 4(0) 0.01(0.1) None(Indep) 3.20(7.1) 5.66(2.16) 3.19(0.71) 5.56(2.08) 4(0) 1.39(1.41) 4(0) 0.38(0.23) 4(0) 0.07(0.26) Inm/MQ 2.23(0.91) 2.36(0.91) 0.40(0.72) 2.38(0.91) 0.04(0.25) 4(0) 0.07(0.26) Inm/MQ 2.23(0.51) 2.36(0.91) 2.36(0.91) 0.04(0.22) 4(0) 0.04(0.26) 0.00(0.00) Inm/MQ 2.34(1.02) 2.34(1.02) 3.34(1.02) 3.34(1.02) 4(0) 0.34(1.02) 4(0) 0.04(0.22) Inm/MQ 2.13(0.85) 2.24(1.02) 3.34(1.02				2.72(1)	3.06(1.97)	2.72(1)	3.06(1.97)	4(0)	0.75(1.19)	4(0)	0.75(1.19)	4(0)	0.1(0.33)	4(0)	0.1(0.33)
InvMQ 2.25(0.97) 2.17(1.68) 2.25(0.97) 2.16(1.67) 4(0) 0.38(0.91) 4(0) 0.38(0.91) 4(0) 0.05(0.25) (0.05(0.22) 1.82(0.83) 0.05(0.22) 1.82(0.83) 0.05(0.22) 4(0) 0.05(0.22)		$Mat_{5/2}(2.5)$	Gauss	2.53(1.04)	2.57(1.81)	2.53(1.04)	2.57(1.81)	4(0)	0.41(0.81)	4(0)	0.41(0.81)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
True 0.340,57) 0.05(0.26) 0.440,57 0.05(0.25) 1.82(0.83) 0.05(0.22) 1.82(0.83) 0.05(0.22) 1.82(0.83) 0.05(0.22) 4(0) 0.05(0.22) 4(0) 0.05(0.22) 4(0) 0.05(0.22) 4(0) 0.05(0.22) 4(0) 1.36(1.31) 4(0) 0.05(0.22) 4(0) 1.36(1.31) 4(0) 0.07(0.26) 0.07(0.2			InvMQ	2.25(0.97)	2.17(1.68)	2.25(0.97)	2.16(1.67)	4(0)	0.38(0.91)	4(0)	0.38(0.91)	4(0)	0(0)	4(0)	0(0)
None(Indep) 3.2(0.71) 5.66(2.16) 3.19(0.71) 5.56(2.08) 4(0) 3.57(2.38) 4(0) 1.39(1.41) 4(0) 1.30(1.41) 4(0) 0.70(0.26) Causs 2.38(1.03) 2.28(1.59) 2.38(1.59) 2.38(1.59) 2.38(1.51) 4(0) 0.38(1.11) 4(0) 0.07(0.26) InvMQ 2.20(8.9) 2.24(1.62) 2.24(1.			True	0.34(0.57)	0.05(0.26)	0.34(0.57)	0.05(0.26)	1.82(0.83)	0.05(0.22)	1.82(0.83)	0.05(0.22)	4(0)	0(0)	4(0)	0(0)
True 2.39(1.03) 2.86(1.99) 2.39(1.03) 2.86(1.99) 2.39(1.03) 2.86(1.99) 2.39(1.03) 2.36(1.99) 2.39(1.03) 2.36(1.99) 2.39(1.03) 2.36(1.91) 2.36(1.11) 4(0) 0.88(1.11) 4(0) 0.88(1.11) 4(0) 0.00(0.22) 4(0) 0.00(0.22) 4(0) 0.36(0.72) 4(0) 0.00(0.22) 4(0) 0.36(0.72) 4(0) 0.00(0.22) 4(0) 0			None(Indep)	3.2(0.71)	5.66(2.16)	3.19(0.71)	5.56(2.08)	4(0)	3.57(2.38)	4(0)	3.56(2.37)	4(0)	1.3(1.32)	4(0)	1.3(1.32)
Gauss 2.35(091) 2.84(1.76) 2.35(091) 2.84(1.76) 2.35(091) 2.84(1.76) 2.35(091) 2.84(1.76) 0.38(0.11) 4(0) 0.88(1.11) 4(0) 0.04(0.22) 4(0) 0.04(0.22) 2.4(0.02) 0.36(0.72) 4(0) 0.04(0.22) 4(0) 0.036(0.72) 4(0) 0.04(0.22) 4(0) 0.04(0.22) 4(0) 0.04(0.22) 4(0) 0.04(0.22) 4(0) 0.04(0.22) 4(0) 0.04(0.22) 4(0) 0.04(0.22) 4(0) 0.04(0.22) 4(0) 0.04(0.22) 4(0) 0.04(0.22) 4(0) 0.04(0.22) 4(0) 0.04(0.22) 4(0) 0.04(0.22) 4(0) 0.04(0.02)			_	2.39(1.03)	2.86(1.99)	2.39(1.03)	2.86(1.99)	4(0)	1.39(1.41)	4(0)	1.39(1.41)	4(0)	0.07(0.26)	4(0)	0.07(0.26)
InwMQ 2.2(0.89) 2.09(1.52) 2.2(0.89) 2.09(1.52) 2.4(0.89) 2.09(1.52) 2.38(0.91) 0.046(0.22) 0.046(0.22) 2.38(0.91) 0.046(0.22) 0.046(0.22) 4(0) 0.09(0.02) 0.009(0.02) 0.028(0.21) 0.028(0.21) 0.046(0.22)		$Mat_{5/2}(1.5)$	Gauss	2.35(0.91)	2.84(1.76)	2.35(0.91)	2.82(1.75)	4(0)	0.88(1.11)	4(0)	0.88(1.11)	4(0)	0(0)	4(0)	0(0)
True 0.28(0.51) 0(0) 0.28(0.51) 0(0) 0.28(0.51) 0(0) 0.04(0.2) 2.38(0.91) 0.04(0.2) 2.38(0.91) 0.04(0.2) 4(0) 0.04(0.2) 0.04(0.2) 0(0) None(Indep) 3.15(0.88) 6.37(1.95) 3.14(0.88) 6.24(1.83) 4(0) 1.25(1.14) 4(0) 1.25(1.14) 4(0) 0.08(0.27) Gauss 2.24(0.95) 2.83(1.61) 2.24(1.95) 2.24(1.95) 4(0) 0.63(0.33) 4(0) 0.02(0.14) True 2.78(0.87) 3.66(1.74) 2.19(1.39) 4(0) 0.59(0.83) 4(0) 0.59(0.83) 4(0) 0.02(0.14) None(Indep) 3.13(0.75) 6.64(2.37) 3.12(0.76) 4(0) 1.84(1.54) 4(0) 0.17(0.45) Gauss 2.6(0.96) 3.62(1.66) 4(0) 1.42(1.36) 4(0) 0.17(0.45) 4(0) 0.17(0.45) InvMQ 2.17(0.94) 2.36(1.52) 3.62(1.66) 4(0) 0.73(0.92) 4(0) 0.01(0.1) InvMQ 2.17(0.94) 2.3			InvMQ	2.2(0.89)	2.09(1.52)	2.2(0.89)	2.09(1.52)	4(0)	0.36(0.72)	4(0)	0.36(0.72)	4(0)	0(0)	4(0)	0(0)
None(Indep) 3.15(0.88) 6.37(1.95) 3.14(0.88) 6.24(1.83) 4(0) 3.94(1.92) 4(0) 3.94(1.92) 4(0) 1.78(1.54) 1.24(1.02) 3.35(1.98) 2.244(1.02) 3.34(1.96) 3.34(1.96) 4(0) 1.25(1.14) 4(0) 1.25(1.14) 4(0) 0.08(0.27) 4(0) 0.08(0.27) 4(0) 0.24(1.02) 2.24(0.95) 2.24(1.95) 2.24(1.95) 2.24(1.95) 2.24(1.95) 2.24(1.95) 2.24(1.95) 2.24(1.95) 2.24(1.95) 2.24(1.95) 2.24(1.95) 3.24(1.25) 4(0) 0.59(0.83) 4(0) 0.59(0.83) 4(0) 0.59(0.83) 4(0) 0.29(0.83) 4(0) 0.29(0.83) 4(0) 0.17(0.45			True	0.28(0.51)	0(0)	0.28(0.51)	0(0)	2.38(0.91)	0.04(0.2)	2.38(0.91)	0.04(0.2)	4(0)	0(0)	4(0)	0(0)
True 2.24(0.95) 3.55(1.86) 2.44(1.02) 3.35(1.87) 3.35(1.87) 4(0) 1.25(1.14) 4(0) 1.25(1.14) 4(0) 0.08(0.27) InwMQ 2.18(0.9) 2.24(0.95) 2.24(1.45) 4(0) 0.59(0.83) 4(0) 0.59(0.83) 4(0) 0.59(0.83) True 2.78(0.87) 3.66(1.74) 2.75(0.87) 3.61(1.69) 4(0) 1.84(1.54) 4(0) 1.84(1.54) 4(0) 0.17(0.45) True 2.78(0.87) 3.66(1.74) 2.75(0.87) 3.61(1.69) 4(0) 1.84(1.54) 4(0) 1.84(1.54) 4(0) 0.17(0.45) True 2.46(0.86) 3.62(1.66) 3.62(1.66) 3.62(1.66) 3.62(1.66) 3.62(1.44) 4(0) 0.73(0.92) 4(0) 0.73(0.92) True 2.44(0.84) 2.36(1.25) 3.62(1.85) 3.63(1.85) 3.63(1.85) 3.63(1.85) 3.63(1.85) 4(0) 3.53(2.23) 4(0) 3.53(2.23) 4(0) 3.53(2.23) 4(0) 3.53(2.23) 4(0) 3.53(2.23) 4(0) 3.53(2.23) 4(0) 1.28(1.05) True 2.44(0.84) 2.44(0.85) 3.68(1.85) 3.6			None(Indep)	3.15(0.88)	6.37(1.95)	3.14(0.88)	6.24(1.83)	4(0)	3.94(1.92)	4(0)	3.94(1.92)	4(0)	1.78(1.54)	4(0)	1.78(1.54)
Gauss 2.24(0.95) 2.83(1.61) 2.24(0.95) 2.82(1.45) 2.82(1.45) 2.82(1.45) 2.82(1.45) 2.82(1.45) 2.82(1.44) 2.18(0.95) 2.19(1.39) 4(0) 0.636(0.73) 4(0) 0.63(0.83) 4(0) 0.63(0.83) 4(0) 0.63(0.14) 0.02(0.			ı	2.44(1.02)	3.35(1.98)	2.44(1.02)	3.34(1.96)	4(0)	1.25(1.14)	4(0)	1.25(1.14)	4(0)	0.08(0.27)	4(0)	0.08(0.27)
InvMQ 2.18(0.9) 2.2(1.41) 2.18(0.9) 2.19(1.39) 4(0) 0.59(0.83) 4(0) 0.59(0.83) 4(0) 0.00) True 2.78(0.87) 3.66(1.74) 2.75(0.87) 3.61(1.64) 4(0) 1.84(1.54) 4(0) 4.38(2.06) 4(0) 1.42(1.36) 4(0) 0.02(0.14) Gauss 2.44(0.86) 2.2(1.47) 2.44(0.86) 2.86(1.44) 4(0) 0.73(0.25) 4(0) 0.73(0.25) 4(0) 0.03(0.13) True 3.0(8.3) 4.62(1.85) 3.0(8.3) 4.62(1.85) 3.0(8.3) 4.62(1.85) 3.0(8.3) 4.0(0) 3.53(2.23) 4(0) 3.53(2.23) 4(0) 3.53(2.23) 4(0) 1.28(1.05) The True 1.86(1.87) 1.28(1.85)		Gauss(1.5)	Gauss	2.24(0.95)	2.83(1.61)	2.24(0.95)	2.82(1.57)	4(0)	0.63(0.73)	4(0)	0.63(0.73)	4(0)	0.02(0.14)	4(0)	0.02(0.14)
True 2.78(0.87) 3.6(1.74) 2.75(0.87) 3.6(1.69) 4(0) 1.84(1.54) 4(0) 1.84(1.54) 4(0) 0.17(0.45) None(Indep) 3.13(0.75) 6.64(2.37) 3.12(0.76) 6.45(2.24) 4(0) 4.38(2.06) 4(0) 4.38(2.06) 4(0) 1.14(1.07) Gauss 2.44(0.86) 2.9(1.47) 2.44(0.86) 2.9(1.47) 2.44(0.86) 2.9(1.47) 2.44(0.86) 2.9(1.47) 2.36(1.52) 4(0) 0.63(0.85) 4(0) 0.63(0.85) 4(0) 0.63(0.85) True 3(0.83) 4.62(1.85) 3(0.83) 4.62(1.85) 3(0.83) 4.62(1.85) 3.53(2.23) 4(0) 3.53(2.23) 4(0) 3.53(2.23) 4(0) 1.28(1.05) True 2.78(0.87) 3.66(1.24) 3.66(1.85) 3.68(1.85) 3.68(1.85) 4(0) 3.53(2.23) 4(0) 3.53(2.23) 4(0) 1.28(1.05) True 2.78(0.87) 2.78(0.87			InvMQ	2.18(0.9)	2.2(1.41)	2.18(0.9)	2.19(1.39)	4(0)	0.59(0.83)	4(0)	0.59(0.83)	4(0)	0(0)	4(0)	0(0)
None(Indep) 3.13(0.75) 6.64(2.37) 3.12(0.76) 6.45(2.24) 4(0) 1.38(2.06) 4(0) 4.38(2.06) 4(0) 1.14(1.07) (1.07) (1.06) (True	2.78(0.87)	3.66(1.74)	2.75(0.87)	3.61(1.69)	4(0)	1.84(1.54)	4(0)	1.84(1.54)	4(0)	0.17(0.45)	4(0)	0.17(0.45)
1 2.6(0.96) 3.62(1.66) 2.6(0.96) 3.62(1.66) 4(0) 1.42(1.36) 4(0) 1.42(1.36) 4(0) 0.02(0.14) 0.002(0.			None(Indep)	3.13(0.75)	6.64(2.37)	3.12(0.76)	6.45(2.24)	4(0)	4.38(2.06)	4(0)	4.38(2.06)	4(0)	1.14(1.07)	4(0)	1.14(1.07)
Gauss 2.44(0.86) 2.9(1.47) 2.44(0.86) 2.86(1.44) 4(0) 0.73(0.92) 4(0) 0.73(0.92) 4(0) 0.01(0.1) InvMQ 2.17(0.94) 2.36(1.52) 2.17(0.94) 2.36(1.52) 4(0) 0.63(0.85) 4(0) 0.63(0.85) 4(0) 0.00(0) True 3(0.83) 4.62(1.85) 4(0) 3.53(2.23) 4(0) 3.53(2.23) 4(0) 1.28(1.05)			П	2.6(0.96)	3.62(1.66)	2.6(0.96)	3.62(1.66)	4(0)	1.42(1.36)	4(0)	1.42(1.36)	4(0)	0.02(0.14)	4(0)	0.02(0.14)
		Gauss(2.5)	Gauss	2.44(0.86)	2.9(1.47)	2.44(0.86)	2.86(1.44)	4(0)	0.73(0.92)	4(0)	0.73(0.92)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
3(0.83) 4.62(1.85) 3(0.83) 4.62(1.85) 4(0) 3.53(2.23) 4(0) 3.53(2.23) 4(0) 1.28(1.05)			InvMQ	2.17(0.94)	2.36(1.52)	2.17(0.94)	2.36(1.52)	4(0)	0.63(0.85)	4(0)	0.63(0.85)	4(0)	0(0)	4(0)	0(0)
			True	3(0.83)	4.62(1.85)	3(0.83)	4.62(1.85)	4(0)	3.53(2.23)	4(0)	3.53(2.23)	4(0)	1.28(1.05)	4(0)	1.28(1.05)

Table 7: Monte Carlo Mean (Standard dev.) for the selected number of nonzero covariates using 100 datasets under both Independent and Dependent setup using spatially weighted group LASSO and adaptive Group LASSO algorithms when J=35(t = 3)

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