Supplementary material to "Additive Model Building for Spatial Regression" by Siddhartha Nandy, Chae Young Lim and Tapabrata Maiti

1 Spatial covariance functions

Table 1 gives the expression of spatial covariance functions that we have considered in the paper.

Name	Parameters	$\delta(\boldsymbol{h})$
Exponential, Exp(ρ)	ρ	$\sigma^2 \exp(-\rho\|\boldsymbol{h}\|)$
Matérn, Mat$_\nu(\rho)$	ν, ρ	$\sigma^2 \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\frac{\sqrt{2\nu}\|\boldsymbol{h}\|}{\rho}\right)^\nu K_\nu \left(\frac{\sqrt{2\nu}\|\boldsymbol{h}\|}{\rho}\right)$
Inverse Multiqudratics, InvMQ(ρ)	ρ	$\sigma^2(1+(\|\boldsymbol{h}\|)^2)^{-\rho}$
Gaussian, Gauss(ρ)	ρ	$\sigma^2 \exp(-\rho^2\|\boldsymbol{h}\|^2)$

Table 1: ***Spatial covariance functions, $\delta(\boldsymbol{h})$, considered in the simulation study for a covariance model and a spatial weight matrix.***

2 Derivation of the assumption (K 2)$'$

We restate the assumption **(K 2)**$'$.

(K 2)$'$

$$\frac{\lambda_{n1}\sqrt{m_n}}{\lambda_{n2}} + \frac{\lambda_{n2} m_n}{n} = o(1).$$

Using the given choice of r_n,

$$\begin{aligned} \frac{n^2}{\lambda_{n2}^2 r_n^2 m_n} &\geq \frac{n^2}{\lambda_{n2}^2 m_n} \frac{\rho_{\max}^2\left(\boldsymbol{L}\right) m_n^3 \log(J m_n)}{n} \\ &= \frac{n\rho_{\max}^2(\boldsymbol{L}) m_n^2 \log(J m_n)}{\lambda_{n2}^2} \\ &= \frac{n\rho_{\max}^2(\boldsymbol{L}) m_n \log(s_n m_n)}{\lambda_{n2}^2 r_n^2} \left\{ \frac{\log(J m_n)}{\log(s_n m_n)} r_n^2 m_n \right\}. \end{aligned} \tag{1}$$

Since $\left\{\frac{\log(Jm_n)}{\log(s_nm_n)}r_n^2m_n\right\} \longrightarrow \infty$, if L.H.S of (1) goes to 0, then $\frac{\sqrt{\rho^2_{\max}(\boldsymbol{L})nm_n\log(s_nm_n)}}{\lambda_{n2}r_n} \longrightarrow 0$. Also by same choice of r_n and using the fact that $m_n \asymp n^{1/2\tau+1}$, we have

$$\begin{aligned}\frac{n^2}{\lambda^2_{n2}r_n^2m_n} &= \frac{n^2\rho^2_{\max}(\boldsymbol{L})m_n^3\log(Jm_n)}{\lambda^2_{n2}nm_n}\\ &= \frac{m_n^2n\rho^2_{\max}(\boldsymbol{L})\log(Jm_n)}{\lambda^2_{n2}} = \frac{m_n\lambda^2_{n1}}{\lambda^2_{n2}}.\end{aligned}$$

Therefore under **(H 1)** to **(H 7)**, **(K 2)** can be replaced by **(K 2)**$'$.

3 Proofs of Lemmas

To be self-contained, we restate Lemmas together with proofs.

Lemma 1 (Lemma 1 in Huang et al. (2010b))**.** *Suppose that $f \in \mathcal{F}$ and $\mathbb{E}f(X_j) = 0$. Then under* **(H 4)** *and* **(H 5)** *in Assumption 1, there exists an $f_n \in \mathcal{S}^0_{nj}$ such that*

$$\|f_n - f\|_2 = \boldsymbol{O}_p\left(m_n^{-\tau} + \sqrt{\frac{m_n}{n}}\right). \tag{2}$$

Particularly, under the choice of $m_n = \boldsymbol{O}(n^{\frac{1}{2\tau+1}})$, we have

$$\|f_n - f\|_2 = \boldsymbol{O}_p\left(m_n^{-\tau}\right) = \boldsymbol{O}_p(n^{-\frac{\tau}{2\tau+1}}). \tag{3}$$

Lemma 2. *Suppose that $|A|$ is bounded by a fixed constant independent of n and J. Let $h_n \asymp m_n^{-1}$. Then under* **(H 4)** *and* **(H 5)** *in Assumption 1, with probability converging to one,*

$$\rho_{\min}(\boldsymbol{\Sigma}_W^{-1})d_1h_n \le \rho_{\min}(\boldsymbol{\Omega}_A) \le \rho_{\max}(\boldsymbol{\Omega}_A) \le \rho_{\max}(\boldsymbol{\Sigma}_W^{-1})d_2h_n \tag{4}$$

Additionally under **(H 7)***,* (4) *becomes,*

$$c_1h_n \le \rho_{\min}(\boldsymbol{\Omega}_A) \le \rho_{\max}(\boldsymbol{\Omega}_A) \le c_2h_n \tag{5}$$

where d_1, d_2, c_1 and c_2 are some positive constants.

Proof. One can follow the proof of the Lemma 3 in Huang *et al.* (2010b) but after observing that,

$$\rho_{\min}(\boldsymbol{\Sigma}_W^{-1})\left(\frac{\mathbb{B}_A^{c\prime}\mathbb{B}_A^c}{n}\right) \le \boldsymbol{\Omega}_A = \frac{\mathbb{D}_A^{c\prime}\mathbb{D}_A^c}{n} \le \rho_{\max}(\boldsymbol{\Sigma}_W^{-1})\left(\frac{\mathbb{B}_A^{c\prime}\mathbb{B}_A^c}{n}\right)$$

which gives (4). By **(H 7)**, the well-conditioned property of $\boldsymbol{\Sigma}_W^{-1}$, we have (5). □

Lemma 3. *Define $\boldsymbol{M}_n$ be a non-negative definite matrix of order n and,*

$$T_{jl} = \left(\frac{m_n}{n}\right)^{\frac{1}{2}}\boldsymbol{a}'_{jl}\boldsymbol{M}_n\boldsymbol{\epsilon} \qquad \forall 1 \le j \le J, 1 \le l \le m_n \tag{6}$$

where $\boldsymbol{a}_{jl} = (\mathbb{B}_l^c(X_j(\boldsymbol{s})), \boldsymbol{s} \in \boldsymbol{S})'$ *and* $T_n = \max_{\substack{1\leq j\leq J \\ 1\leq l\leq m_n}} |T_{jl}|$. *Then, under assumptions* **(H 2)** *to* **(H 5)** *in Assumption 1,*

$$\mathbb{E}(T_n) \leq C_1 \rho_{\max}(\boldsymbol{M}_n)\sqrt{(m_n \log(Jm_n))}, \tag{7}$$

for some $C_1 > 0$.

Proof. Since $\boldsymbol{\epsilon} \sim \text{Gaussian}(\boldsymbol{0}, \boldsymbol{\Sigma}_T)$, $T_{jl} \sim \text{Gaussian}(0, \frac{m_n}{n}\boldsymbol{a}_{jl}'\boldsymbol{M}_n\boldsymbol{\Sigma}_T\boldsymbol{M}_n'\boldsymbol{a}_{jl})$. Therefore we can use maximal inequalities of sub-Gaussian random variables [van der Vaart and Wellner (1996), Lemmas 2.2.1 and 2.2.2]. Let $\|\cdot\|_\phi$ be the Orlicz norm, defined by $\|X\|_\phi = \inf\{k \in (0,\infty) \,|\, \mathbb{E}(\phi(|X|/k)) \leq 1\}$. Then, conditional on $\{X_j(\boldsymbol{s}), \boldsymbol{s} \in \boldsymbol{S}, 1 \leq j \leq J\}$, we have the following

$$\begin{aligned}
&\Big\| \max_{1\leq j\leq J, 1\leq l\leq m_n} |T_{jl}| \,\Big|\, X_j(\boldsymbol{s}), \boldsymbol{s} \in \boldsymbol{S}, 1 \leq j \leq J \Big\|_{\phi_2} \\
&\quad \leq K\sqrt{\frac{m_n}{n}\log(1+Jm_n)} \max_{1\leq j\leq J, 1\leq l\leq m_n} \|\, |\boldsymbol{a}_{jl}'\boldsymbol{M}_n\boldsymbol{\epsilon}| \,\big|\, X_j(\boldsymbol{s}), \boldsymbol{s} \in \boldsymbol{S}, 1 \leq j \leq J\|_{\phi_2} \\
&\quad \leq K\sqrt{\frac{m_n}{n}\log(Jm_n)} \max_{1\leq j\leq J, 1\leq l\leq m_n} \sqrt{\boldsymbol{a}_{jl}'\boldsymbol{M}_n\boldsymbol{\Sigma}_T\boldsymbol{M}_n'\boldsymbol{a}_{jl}},
\end{aligned}$$

where $K > 0$ is a generic constant and $\phi_p(x) = e^{x^p} - 1$. Now taking expectation with respect to $\{X_j(\boldsymbol{s}), \boldsymbol{s} \in \boldsymbol{S}, 1 \leq j \leq J\}$ on both sides of the above inequality,

$$\begin{aligned}
&\| \max_{1\leq j\leq J, 1\leq l\leq m_n} |T_{jl}| \,\|_{\phi_2} \\
&\quad \leq K\sqrt{\frac{m_n}{n}\log(Jm_n)}\, \mathbb{E}\left(\max_{1\leq j\leq J, 1\leq l\leq m_n} \sqrt{\boldsymbol{a}_{jl}'\boldsymbol{M}_n\boldsymbol{\Sigma}_T\boldsymbol{M}_n'\boldsymbol{a}_{jl}}\right) \\
&\quad = K\sqrt{\frac{m_n}{n}\log(Jm_n)}\, \mathbb{E}\left(\sqrt{\max_{1\leq j\leq J, 1\leq l\leq m_n} \boldsymbol{a}_{jl}'\boldsymbol{M}_n\boldsymbol{\Sigma}_T\boldsymbol{M}_n'\boldsymbol{a}_{jl}}\right) \\
&\quad \leq K\rho_{\max}(\boldsymbol{M}_n)\sqrt{\frac{m_n}{n}\log(Jm_n)}\sqrt{\mathbb{E}\left(\max_{1\leq j\leq J, 1\leq l\leq m_n} \boldsymbol{a}_{jl}'\boldsymbol{\Sigma}_T\boldsymbol{a}_{jl}\right)}.
\end{aligned} \tag{8}$$

Since $\mathbb{B}_l^c(x)$ are normalized B-splines, we have

$$\begin{aligned}
&\mathbb{E}\left(\max_{1\leq j\leq J, 1\leq l\leq m_n} \sum_{\boldsymbol{s}\in\boldsymbol{S}}\sum_{\boldsymbol{s}'\in\boldsymbol{S}} \mathbb{B}_l^c(X_j(\boldsymbol{s}))\delta(\boldsymbol{s}-\boldsymbol{s}')\mathbb{B}_l^c(X_j(\boldsymbol{s}'))\right) \\
&\quad \leq 4\sum_{\boldsymbol{s}\in\boldsymbol{S}}\sum_{\boldsymbol{s}'\in\boldsymbol{S}} \delta(\boldsymbol{s}-\boldsymbol{s}') \\
&\quad \leq K\sum_{\boldsymbol{s}\in\boldsymbol{S}} \int_{\boldsymbol{h}\in C\boldsymbol{D}_n} \delta(\boldsymbol{h})d\boldsymbol{h} \\
&\quad \leq Kn\int_{\boldsymbol{h}\in C\boldsymbol{D}_n} \delta(\boldsymbol{h})d\boldsymbol{h}.
\end{aligned} \tag{9}$$

for some $K, C > 0$.

From (8) and (9),

$$\begin{aligned} \| \max_{1 \le j \le J, 1 \le l \le m_n} |T_{jl}| \, \|_\phi \quad & \le K \rho_{\max}(\boldsymbol{M}_n) \sqrt{m_n \log(J m_n)} \sqrt{\int_{\boldsymbol{h} \in C \boldsymbol{D}_n} \delta(\boldsymbol{h}) d\boldsymbol{h}} \\ & \le K \rho_{\max}(\boldsymbol{M}_n) \sqrt{m_n \log(J m_n)}. \end{aligned}$$

Finally, (35) follows from $\|X\|_{L^1} \le C \|X\|_{L^2} \le \|X\|_{\phi_2}$, where $\|X\|_{L^p} = (\mathbb{E}(|X|^p))^{1/p}$. □

Lemma 4. *Under the Assumption 1 with* $\lambda_{n1} > C \rho_{\max}(\boldsymbol{L}) \sqrt{n m_n \log(J m_n)}$ *for a sufficiently large constant C. we have* $|\tilde{A}_{\boldsymbol{\beta}}| \le M_1 |A_*|$ *for a finite constant* $M_1 > 1$ *with w.p. converging to* 1.

Proof. Along with considering the approximation error for spline regression, we also have to take care of the dependence structure of a Gaussian random vector $\boldsymbol{\epsilon}$ according to $(\mathbf{H\,3})$. To emphasize the dependence on n, we denote write $\boldsymbol{\epsilon}_n$ instead of $\boldsymbol{\epsilon}$ and similar notation for others as well. Recall $\boldsymbol{\pi}_n = \boldsymbol{\epsilon}_n + \boldsymbol{\theta}_n$, where $\boldsymbol{\theta}_n = (\theta_n(\boldsymbol{s}); \boldsymbol{s} \in \boldsymbol{S})'$ with $\theta_n(\boldsymbol{s}) = \sum_{j=1}^{J} (f_j(X_j(\boldsymbol{s})) - f_{nj}(X_j(\boldsymbol{s})))$. Note that $\|\boldsymbol{\theta}_n\|_2 = \boldsymbol{O}(n^{1/2} q^{1/2} m_n^{-\tau}) = \boldsymbol{O}(q^{1/2} n^{1/(4\tau+2)})$ by Lemma 1 since $m_n = \boldsymbol{O}(n^{1/(2\tau+1)})$. Define $\lambda_{n,J} = 2\sqrt{K \rho_{\max}^2(\boldsymbol{L}) m_n n \log(J m_n)}$ for some $K > 0$ and $\lambda_{n1} \ge \max\{\lambda_0, \lambda_{n,J}\}$, where $\lambda_0 = \inf\{\lambda : M_1(\lambda) q^* + 1 \le q_0\}$ for some finite $q_0 > 0$ and $M_1(\lambda) > 1$, which will be specified later in the proof. Without loss of generality, we will assume the infimum of an empty set to be ∞. That is, if $\{\lambda : M_1(\lambda) q^* + 1 \le q_0\}$ is an empty set, it implies that $\lambda_{n1} = \lambda_0 = \infty$ and which in turn implies that we drop all the components in our additive model, i.e. $|\tilde{A}_{\boldsymbol{\beta}}| = 0$. So part (i) is trivial in this case and hence for the rest of the proof we will assume $\{\lambda : M_1(\lambda) q^* + 1 \le q_0\}$ is a non-empty set.

First, define a new vector $\boldsymbol{U}_k$ such that $\boldsymbol{U}_k = \mathbb{D}_k^{c\prime} (\boldsymbol{Z}^c - \mathbb{D}^c \hat{\boldsymbol{\beta}}_{gL}) / \lambda_{n1}$ for $k = 1, \cdots, J$. By **K**arsuh-**K**uhn-**T**ucker (KKT) conditions of the optimization problem for $\boldsymbol{Q}_n(\boldsymbol{\beta}, \lambda_n)$ with the solution $\hat{\boldsymbol{\beta}}_{gL}$, we have

$$\boldsymbol{U}_k \begin{cases} = \dfrac{\hat{\boldsymbol{\beta}}_{gL,k}}{\|\hat{\boldsymbol{\beta}}_{gL,k}\|_2} & \text{if } \|\hat{\boldsymbol{\beta}}_{gL,k}\|_2 > 0, \\ \le \mathbf{1} & \text{if } \|\hat{\boldsymbol{\beta}}_{gL,k}\|_2 = 0. \end{cases} \tag{10}$$

Then, the norm of $\boldsymbol{U}_k$ is

$$\|\boldsymbol{U}_k\|_2 \begin{cases} = 1 & \text{if } \|\hat{\boldsymbol{\beta}}_{gL,k}\|_2 > 0, \\ \le 1 & \text{if } \|\hat{\boldsymbol{\beta}}_{gL,k}\|_2 = 0. \end{cases} \tag{11}$$

Now we introduce the following quantities.

$$\begin{aligned} x_r &= \max_{|A|=r} \max_{\|U_k\|_2 = 1, k \in A, B \subset A} \left| \boldsymbol{\pi}_n' \boldsymbol{w}_{A|B} \right|, \quad \text{and} \\ x_r^* &= \max_{|A|=r} \max_{\|U_k\|_2 = 1, k \in A, B \subset A} \left| \boldsymbol{\epsilon}_n' \boldsymbol{w}_{A|B} \right|, \end{aligned}$$

where $\boldsymbol{w}_{A|B} = \boldsymbol{W}_{A|B} / \left\|\boldsymbol{W}_{A|B}\right\|_2$ with $\boldsymbol{W}_{A|B} = (\mathbb{D}^c_A(\mathbb{D}^{c\prime}_A\mathbb{D}^c_A)^{-1}\lambda_{n1}Q'_{BA}Q_{BA}\boldsymbol{U}_A - (\mathbb{I} - P_A)\mathbb{D}^c\boldsymbol{\beta})$. For $B \subset A$, Q_{BA} is the matrix corresponding to the selection of variables in B from A, *i.e.* $Q_{BA}\boldsymbol{\beta}_A = \boldsymbol{\beta}_B$. $P_A = \mathbb{D}^c_A\boldsymbol{\Omega}_A^{-1}\mathbb{D}^{c\prime}_A/n$, $\boldsymbol{U}_A = (\boldsymbol{U}_k; k \in A)'$. By the triangle inequality and Cauchy-Schwarz inequality, for some set A with $|A| = r > 0$, we have

$$
\begin{aligned}
\left|\boldsymbol{\pi}_n'\boldsymbol{w}_{A|B}\right| &\leq \left|\boldsymbol{\epsilon}_n'\boldsymbol{w}_{A|B}\right| + \|\boldsymbol{\theta}_n\|_2 \\
&\leq \left|\boldsymbol{\epsilon}_n'\boldsymbol{w}_{A|B}\right| + K_2 q n^{1/(4\tau+2)} \\
&\leq \left|\boldsymbol{\epsilon}_n'\boldsymbol{w}_{A|B}\right| + K_1\sqrt{\frac{(rm_n \vee m_n)\rho^2_{\max}(\boldsymbol{L})m_n\log(Jm_n)}{\rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_1})}} \qquad (12)
\end{aligned}
$$

where the last inequality holds for a sufficiently large n for some $K_1 > 0$. By introducing the following sets,

$$
\begin{aligned}
\Omega_{r_0} &= \left\{(\mathbb{D}^c, \boldsymbol{\pi}_n); x_r \leq 2K_1\sqrt{\frac{(rm_n \vee m_n)\rho^2_{\max}(\boldsymbol{L})m_n\log(Jm_n)}{\rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_1})}}, \forall r \geq r_0\right\}, \quad \text{and} \\
\Omega^*_{r_0} &= \left\{(\mathbb{D}^c, \boldsymbol{\epsilon}_n); x^*_r \leq K_1\sqrt{\frac{(rm_n \vee m_n)\rho^2_{\max}(\boldsymbol{L})m_n\log(Jm_n)}{\rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_1})}}, \forall r \geq r_0\right\},
\end{aligned}
$$

we can show $\mathbb{P}\left\{(\mathbb{D}^c, \boldsymbol{\pi}_n) \in \Omega_{r_0}\right\} \geq \mathbb{P}\left\{(\mathbb{D}^c, \boldsymbol{\epsilon}_n) \in \Omega^*_{r_0}\right\}$ for any $r_0 > 0$ since we have

$$
x_r \leq x^*_r + \|\boldsymbol{\theta}_n\|_2 \leq x^*_r + K_1\sqrt{\frac{(rm_n \vee m_n)\rho^2_{\max}(\boldsymbol{L})m_n\log(Jm_n)}{\rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_1})}} \qquad (13)
$$

by recalling the definitions of x_r and x^*_r and (12).

Now, we want to show that $\mathbb{P}\left\{(\mathbb{D}^c, \boldsymbol{\pi}_n) \in \Omega_{q_1}\right\} \to 1$ implies $|\tilde{A}_{\boldsymbol{\beta}}| \leq M_1|\tilde{A}_*| = M_1 q^*$ for some finite $M_1 > 1$, which completes the proof since $q^* \leq q$. Before proving this claim, we first show $\mathbb{P}\left\{(\mathbb{D}^c, \boldsymbol{\epsilon}_n) \in \Omega^*_{q_1}\right\} \to 1$, which implies $\mathbb{P}\left\{(\mathbb{D}^c, \boldsymbol{\pi}_n) \in \Omega_{q_1}\right\} \to 1$. We start with the following:

$$
\begin{aligned}
& 1 - \mathbb{P}\left\{(\mathbb{D}^c, \boldsymbol{\epsilon}_n) \in \Omega^*_{q_1}\right\} \qquad (14) \\
& \leq \sum_{r=0}^{\infty} \mathbb{P}\left(x^*_r > K_1\sqrt{\frac{(rm_n \vee m_n)\rho^2_{\max}(\boldsymbol{L})m_n\log(Jm_n)}{\rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_1})}}\right) \\
& \leq \sum_{r=0}^{\infty} \binom{J}{r}\mathbb{P}\left(|\boldsymbol{w}'_{A|B}\boldsymbol{\epsilon}_n| > K_1\sqrt{\frac{(rm_n \vee m_n)\rho^2_{\max}(\boldsymbol{L})m_n\log(Jm_n)}{\rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_1})}}\right), \qquad (15)
\end{aligned}
$$

where $|A| = r$. Since $\boldsymbol{w}'_{A|B}\boldsymbol{\epsilon}_n \sim \text{Gaussian}(0, \boldsymbol{w}'_{A|B}\boldsymbol{\Sigma}_T\boldsymbol{w}_{A|B})$, (15) becomes

$$\begin{aligned}
&\leq 2\sum_{r=0}^{\infty}\binom{J}{r}\exp\left(-0.5K_1^2\frac{(rm_n \vee m_n)\rho^2_{\max}(\boldsymbol{L})m_n\log(Jm_n)}{(\boldsymbol{w}'_{A|B}\boldsymbol{\Sigma}_T\boldsymbol{w}_{A|B})\rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_1})}\right)\\
&\leq 2\sum_{r=0}^{\infty}\binom{J}{r}\exp\left(-0.5K_1^2\frac{(rm_n \vee m_n)\rho^2_{\max}(\boldsymbol{L})m_n\log(Jm_n)}{\rho_{\max}(\boldsymbol{\Sigma}_T)\rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_1})}\right)\\
&= 2\sum_{r=0}^{\infty}\binom{J}{r}(Jm_n)^{-0.5K_1^2(rm_n\vee m_n)\rho^2_{\max}(\boldsymbol{L})m_n\big/\rho_{\max}(\boldsymbol{\Sigma}_T)\rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_1})}. \qquad (16)
\end{aligned}$$

Let $K_n = 0.5K_1^2m_n^2\rho^2_{\max}(\boldsymbol{L})\big/\left(\rho_{\max}(\boldsymbol{\Sigma}_T)\rho_{\max}\left(\boldsymbol{\Omega}_{\tilde{A}_1}\right)\right)$. Then (16) becomes,

$$\begin{aligned}
&= 2(Jm_n)^{-K_n} + 2\sum_{r=1}^{\infty}\binom{J}{r}(Jm_n)^{-rK_n}\\
&\leq 2(Jm_n)^{-K_n} + 2\sum_{r=1}^{\infty}\frac{1}{r!}\left(\frac{J}{(Jm_n)^{K_n}}\right)^r\\
&= 2(Jm_n)^{-K_n} + 2\exp\left(\frac{J}{(Jm_n)^{K_n}}\right) - 2. \qquad (17)
\end{aligned}$$

Define $\|\boldsymbol{\Sigma}_T\|_1 = \max_{\boldsymbol{s}\in\boldsymbol{S}}\sum_{\boldsymbol{s}'\in\boldsymbol{S}}\boldsymbol{\sigma}_{\boldsymbol{s},\boldsymbol{s}'}$ and note that $\|\boldsymbol{\Sigma}_T\|_1 \asymp \int_{\boldsymbol{h}\in\boldsymbol{D}_n}\boldsymbol{\delta}(\boldsymbol{h})d\boldsymbol{h} = \boldsymbol{O}(1)$. Therefore by using the fact, $\frac{1}{\sqrt{n}}\|\boldsymbol{\Sigma}_T\|_1 \leq \rho_{\max}(\boldsymbol{\Sigma}_T) \leq \sqrt{n}\|\boldsymbol{\Sigma}_T\|_1$ and $\rho_{\max}(\boldsymbol{\Sigma}_W^{-1}) \leq \rho_{\max}(\boldsymbol{L})\rho_{\max}(\boldsymbol{L}') = \rho^2_{\max}(\boldsymbol{L})$, we have

$$K_n \geq c_1\frac{0.5K_1^2m_n^3}{\sqrt{n}\|\boldsymbol{\Sigma}_T\|_1} \asymp 0.5c_1K_1^2\sqrt{n^{6\gamma-1}},$$

and $K_n \longrightarrow \infty$ by **(H 6)**. This shows (17) goes to zero as $n \to \infty$.

To show $\mathbb{P}\{(\mathbb{D}^c, \boldsymbol{\pi}_n) \in \Omega_{q_1}\} \to 1$ implies $|\tilde{A}_{\boldsymbol{\beta}}| \leq M_1|\tilde{A}_*| = M_1q^*$, let

$$\boldsymbol{V}_{1j} = \frac{\boldsymbol{\Omega}_{\tilde{A}_1}^{-\frac{1}{2}}Q'_{j1}\boldsymbol{U}_{\tilde{A}_j}\lambda_{n1}}{\sqrt{n}}\text{, for } j = 1, 3, 4,$$

and

$$\boldsymbol{u} = \frac{\mathbb{D}^c_{\tilde{A}_1}\boldsymbol{\Omega}_{\tilde{A}_1}^{-1/2}\boldsymbol{V}_{14}/\sqrt{n} - \boldsymbol{\omega}_2}{\|\mathbb{D}^c_{\tilde{A}_1}\boldsymbol{\Omega}_{\tilde{A}_1}^{-1/2}\boldsymbol{V}_{14}/\sqrt{n} - \boldsymbol{\omega}_2\|_2},$$

where, for simplicity in notations, $Q_{kj} = Q_{\tilde{A}_k\tilde{A}_j}$ is the matrix corresponding to the selection of variables in $\tilde{A}_k$ from $\tilde{A}_j$ and $\boldsymbol{\omega}_2 = (\mathbb{I}-P_{\tilde{A}_1})\mathbb{D}^c_{\tilde{A}_2}\boldsymbol{\beta}_{\tilde{A}_2} = (\mathbb{I}-P_{\tilde{A}_1})\mathbb{D}^c\boldsymbol{\beta}$. We can show that, $\boldsymbol{V}_{11} = \boldsymbol{V}_{14}+\boldsymbol{V}_{13}$ and $Q'_{31}Q_{31}+Q'_{41}Q_{41} = \mathbb{I}_{m_n|\tilde{A}_1|}$ due to the fact that $\tilde{A}_3\cup\tilde{A}_4 = \tilde{A}_1$, $\tilde{A}_3\cap\tilde{A}_4 = \phi$ and hence $\boldsymbol{\beta}'_{\tilde{A}_3}Q_{31}+\boldsymbol{\beta}'_{\tilde{A}_4}Q_{41} = \boldsymbol{\beta}_{\tilde{A}_1}$. Since $q_1 = |\tilde{A}_1| = |\tilde{A}_3|+|\tilde{A}_4|$ and $|\tilde{A}_3| \leq q^*$, $|\tilde{A}_4| \geq (q_1-q^*)$. Then, we have the following lower bound for L_2-norm of $\boldsymbol{V}_{14}$,

$$\|\boldsymbol{V}_{14}\|_2^2 \geq \frac{\lambda^2_{n1}\|Q'_{41}U_{\tilde{A}_4}\|_2^2}{n\rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_1})} = \frac{\lambda^2_{n1}\|Q'_{41}Q_{41}U_{\tilde{A}_1}\|_2^2}{n\rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_1})} = \frac{\lambda^2_{n1}m_n|\tilde{A}_4|}{n\rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_1})} \geq B_1\frac{(q_1-q^*)^+}{q^*}, \qquad (18)$$

with $B_1 = \frac{\lambda_{n1}^2 m_n q^*}{n\rho_{\max}(\mathbf{\Omega}_{\tilde{A}_1})}$. From (18), we have

$$|\tilde{A}_{\boldsymbol{\beta}}| \leq |\tilde{A}_1| = q_1 \leq (q_1 - q^*)^+ + q^* \leq q^* \frac{\|\boldsymbol{V}_{14}\|_2^2}{B_1} + q^* = \left(\frac{\|\boldsymbol{V}_{14}\|_2^2}{B_1} + 1\right) q^*. \tag{19}$$

Thus, to show $|\tilde{A}_{\boldsymbol{\beta}}| \leq M_1 q^*$ for some finite $M_1 > 1$, we need to show $\frac{\|\boldsymbol{V}_{14}\|_2^2}{B_1} + 1 \leq M_1$ for some finite $M_1 > 1$.

We start with an upper bound of $\|\boldsymbol{V}_{14}\|_2^2 + \|\boldsymbol{\omega}_2\|_2^2$. Since

$$\begin{aligned}
\|\boldsymbol{V}_{14}\|_2^2 + \|\boldsymbol{\omega}_2\|_2^2 &= \boldsymbol{V}_{14}'\boldsymbol{V}_{14} + \|\boldsymbol{\omega}_2\|_2^2 = \boldsymbol{V}_{14}'(\boldsymbol{V}_{11} - \boldsymbol{V}_{13}) + \|\boldsymbol{\omega}_2\|_2^2 \\
&\leq \boldsymbol{V}_{14}'\boldsymbol{V}_{11} + \|\boldsymbol{V}_{14}\|_2\|\boldsymbol{V}_{13}\|_2 + \|\boldsymbol{\omega}_2\|_2^2,
\end{aligned} \tag{20}$$

we find upper bounds for $\boldsymbol{V}_{14}'\boldsymbol{V}_{11}$, $\|\boldsymbol{V}_{14}\|_2\|\boldsymbol{V}_{13}\|_2$ and $\|\boldsymbol{\omega}_2\|_2^2$, respectively. First,

$$\begin{aligned}
\boldsymbol{V}_{14}'\boldsymbol{V}_{11} &= \frac{\lambda_{n1}^2}{n}\boldsymbol{U}_{\tilde{A}_4}' Q_{41}\mathbf{\Omega}_{\tilde{A}_1}^{-1}\boldsymbol{U}_{\tilde{A}_1} \\
&= \frac{\lambda_{n1}^2}{n}\boldsymbol{U}_{\tilde{A}_4}' Q_{41}\mathbf{\Omega}_{\tilde{A}_1}^{-1}\left(\mathbb{D}_{\tilde{A}_1}^{c\prime}\left(\boldsymbol{Z}^c - \mathbb{D}^c\hat{\boldsymbol{\beta}}_{gL}\right)/\lambda_{n1}\right) \\
&= \frac{\lambda_{n1}}{n}\boldsymbol{U}_{\tilde{A}_4}' Q_{41}\mathbf{\Omega}_{\tilde{A}_1}^{-1}\mathbb{D}_{\tilde{A}_1}^{c\prime}\left(\mathbb{D}_{\tilde{A}_1}^{c}\boldsymbol{\beta}_{\tilde{A}_1} + \mathbb{D}_{\tilde{A}_2}^{c}\boldsymbol{\beta}_{\tilde{A}_2} + \boldsymbol{\pi}_n - \mathbb{D}_{\tilde{A}_1}^{c}\hat{\boldsymbol{\beta}}_{gL,\tilde{A}_1}\right) \\
&= \lambda_{n1}\boldsymbol{U}_{\tilde{A}_4}' Q_{41}\left((\boldsymbol{\beta}_{\tilde{A}_1} - \hat{\boldsymbol{\beta}}_{gL,\tilde{A}_1}) + \frac{\mathbf{\Omega}_{\tilde{A}_1}^{-1}}{n}(\mathbb{D}_{\tilde{A}_1}^{c\prime}\mathbb{D}_{\tilde{A}_2}^{c})\boldsymbol{\beta}_{\tilde{A}_2} + \frac{\mathbf{\Omega}_{\tilde{A}_1}^{-1}}{n}\mathbb{D}_{\tilde{A}_1}^{c\prime}\boldsymbol{\pi}_n\right) \\
&= \lambda_{n1}\boldsymbol{U}_{\tilde{A}_4}'(\boldsymbol{\beta}_{\tilde{A}_4} - \hat{\boldsymbol{\beta}}_{gL,\tilde{A}_4}) + \lambda_{n1}\boldsymbol{U}_{\tilde{A}_4}' Q_{41}\left(\frac{\mathbf{\Omega}_{\tilde{A}_1}^{-1}}{n}(\mathbb{D}_{\tilde{A}_1}^{c\prime}\mathbb{D}_{\tilde{A}_2}^{c})\boldsymbol{\beta}_{\tilde{A}_2} + \frac{\mathbf{\Omega}_{\tilde{A}_1}^{-1}}{n}\mathbb{D}_{\tilde{A}_1}^{c\prime}\boldsymbol{\pi}_n\right) \\
&\leq \lambda_{n1}\sum_{k\in\tilde{A}_4}\|\boldsymbol{\beta}_k\|_2 + \frac{\lambda_{n1}\boldsymbol{U}_{\tilde{A}_4}' Q_{41}\mathbf{\Omega}_{\tilde{A}_1}^{-1}(\mathbb{D}_{\tilde{A}_1}^{c\prime}\mathbb{D}_{\tilde{A}_2}^{c})\boldsymbol{\beta}_{\tilde{A}_2}}{n} + \frac{\lambda_{n1}\boldsymbol{U}_{\tilde{A}_4}' Q_{41}\mathbf{\Omega}_{\tilde{A}_1}^{-1}\mathbb{D}_{\tilde{A}_1}^{c\prime}\boldsymbol{\pi}_n}{n},
\end{aligned} \tag{21}$$

where the last inequality is based on $\left|\boldsymbol{U}_{\tilde{A}_4}'\boldsymbol{\beta}_{\tilde{A}_4}\right| \leq \sum_{k\in\tilde{A}_4}|\boldsymbol{U}_k'\boldsymbol{\beta}_k| \leq \sum_{k\in\tilde{A}_4}\|\boldsymbol{\beta}_k\|_2$ and $\boldsymbol{U}_{\tilde{A}_4}'\hat{\boldsymbol{\beta}}_{gL,\tilde{A}_4} = \sum_{k\in\tilde{A}_4\cap\tilde{A}_{\boldsymbol{\beta}}}\boldsymbol{U}_k'\hat{\boldsymbol{\beta}}_{gL,k} \geq 0$. For $\|\boldsymbol{V}_{14}\|_2\|\boldsymbol{V}_{13}\|_2$, we have

$$\|\boldsymbol{V}_{14}\|_2\|\boldsymbol{V}_{13}\|_2 \leq \|\boldsymbol{V}_{14}\|_2\lambda_{n1}\sqrt{\frac{m_n|\tilde{A}_3|}{n\rho_{\min}(\mathbf{\Omega}_{\tilde{A}_1})}} \tag{22}$$

from the definition of $\boldsymbol{V}_{13}$. For $\|\boldsymbol{\omega}_2\|_2^2$,

$$
\begin{aligned}
\|\boldsymbol{\omega}_2\|_2^2 &= \|(I-P_{\tilde{A}_1})\mathbb{D}^c_{\tilde{A}_2}\boldsymbol{\beta}_{\tilde{A}_2}\|_2^2 \\
&= \boldsymbol{\beta}'_{\tilde{A}_2}\mathbb{D}^{c\prime}_{\tilde{A}_2}(I-P_{\tilde{A}_1})\mathbb{D}^c_{\tilde{A}_2}\boldsymbol{\beta}_{\tilde{A}_2} \\
&= \boldsymbol{\beta}'_{\tilde{A}_2}\left(n\boldsymbol{\Omega}_{\tilde{A}_2}\boldsymbol{\beta}_{\tilde{A}_2} - \frac{1}{n}\mathbb{D}^{c\prime}_{\tilde{A}_2}\mathbb{D}^c_{\tilde{A}_1}\boldsymbol{\Omega}^{-1}_{\tilde{A}_1}\mathbb{D}^{c\prime}_{\tilde{A}_1}\mathbb{D}^c_{\tilde{A}_2}\boldsymbol{\beta}_{\tilde{A}_2}\right) \\
&\leq \boldsymbol{\beta}'_{\tilde{A}_2}\left(\lambda_{n1}\boldsymbol{D}_{\tilde{A}_2} - \mathbb{D}^{c\prime}_{\tilde{A}_2}\boldsymbol{\pi}_n - \mathbb{D}^{c\prime}_{\tilde{A}_2}\mathbb{D}^c_{\tilde{A}_1}\left(\boldsymbol{\beta}_{\tilde{A}_1} - \hat{\boldsymbol{\beta}}_{gl,\tilde{A}_1}\right) - \frac{1}{n}\mathbb{D}^{c\prime}_{\tilde{A}_2}\mathbb{D}^c_{\tilde{A}_1}\boldsymbol{\Omega}^{-1}_{\tilde{A}_1}\mathbb{D}^{c\prime}_{\tilde{A}_1}\mathbb{D}^c_{\tilde{A}_2}\boldsymbol{\beta}_{\tilde{A}_2}\right) \\
&= \boldsymbol{\beta}'_{\tilde{A}_2}\left(\lambda_{n1}\boldsymbol{D}_{\tilde{A}_2} - \mathbb{D}^{c\prime}_{\tilde{A}_2}\boldsymbol{\pi}_n - \frac{\lambda_{n1}}{n}\mathbb{D}^{c\prime}_{\tilde{A}_2}\mathbb{D}^c_{\tilde{A}_1}\boldsymbol{\Omega}^{-1}_{\tilde{A}_1}\boldsymbol{U}_{\tilde{A}_1} + \frac{1}{n}\mathbb{D}^{c\prime}_{\tilde{A}_2}\mathbb{D}^c_{\tilde{A}_1}\boldsymbol{\Omega}^{-1}_{\tilde{A}_1}\mathbb{D}^{c\prime}_{\tilde{A}_1}\boldsymbol{\pi}_n\right) \\
&= \lambda_{n1}\boldsymbol{\beta}'_{\tilde{A}_2}\boldsymbol{D}_{\tilde{A}_2} - \boldsymbol{\omega}'_2\boldsymbol{\pi}_n - \frac{\lambda_{n1}}{n}\boldsymbol{U}'_{\tilde{A}_1}\boldsymbol{\Omega}^{-1}_{\tilde{A}_1}\mathbb{D}^{c\prime}_{\tilde{A}_1}\mathbb{D}^c_{\tilde{A}_2}\boldsymbol{\beta}_{\tilde{A}_2},
\end{aligned}
$$

where the inequality is from

$$
\mathbb{D}^{c\prime}_{\tilde{A}_2}\mathbb{D}^c_{\tilde{A}_1}(\boldsymbol{\beta}_{\tilde{A}_1} - \hat{\boldsymbol{\beta}}_{gL,\tilde{A}_1}) + n\boldsymbol{\Omega}_{\tilde{A}_2}\boldsymbol{\beta}_{\tilde{A}_2} + \mathbb{D}^{c\prime}_{\tilde{A}_2}\boldsymbol{\pi}_n \leq \lambda_{n1}\boldsymbol{D}_{\tilde{A}_2}. \tag{23}
$$

In (23), $\boldsymbol{D}_A$ is a $0-1$ vector whose k^{th} entry is $I(\|\hat{\boldsymbol{\beta}}_{k,gL}\|_2 = 0)$, where $I(A)$ is the indicator function for a set A. The inequality between vectors is defined entry-wise. Note that (23) holds due to (11).

Since $\boldsymbol{V}_{14} \perp\!\!\!\perp \boldsymbol{\omega}_2$, we have

$$
\|\mathbb{D}^c_{\tilde{A}_1}\boldsymbol{\Omega}^{-1}_{\tilde{A}_1}Q'_{41}\boldsymbol{U}_{\tilde{A}_4}\lambda_{n1}/n - \boldsymbol{\omega}_2\|_2^2 = \|\mathbb{D}^c_{\tilde{A}_1}\boldsymbol{\Omega}^{-1/2}_{\tilde{A}_1}\boldsymbol{V}_{14}/\sqrt{n} - \boldsymbol{\omega}_2\|_2^2 = \|\boldsymbol{V}_{14}\|_2^2 + \|\boldsymbol{\omega}_2\|_2^2
$$

so that

$$
\left(\frac{\lambda_{n1}}{n}\boldsymbol{U}'_{\tilde{A}_4}Q_{41}\boldsymbol{\Omega}^{-1}_{\tilde{A}_1}\mathbb{D}^{c\prime}_{\tilde{A}_1} - \boldsymbol{\omega}'_2\right)\boldsymbol{\pi}_n = (\|\boldsymbol{V}_{14}\|_2^2 + \|\boldsymbol{\omega}_2\|_2^2)^{1/2}(\boldsymbol{u}'\boldsymbol{\pi}_n)
$$

using the definition of $\boldsymbol{u}$. Then, this implies

$$
\begin{aligned}
\|\boldsymbol{\omega}_2\|_2^2 \leq\; & \lambda_{n1}\boldsymbol{\beta}'_{\tilde{A}_2}\boldsymbol{D}_{\tilde{A}_2} - \frac{\lambda_{n1}}{n}\boldsymbol{U}'_{\tilde{A}_1}\boldsymbol{\Omega}^{-1}_{\tilde{A}_1}\mathbb{D}^{c\prime}_{\tilde{A}_1}\mathbb{D}^c_{\tilde{A}_2}\boldsymbol{\beta}_{\tilde{A}_2} \\
& + (\|\boldsymbol{V}_{14}\|_2^2 + \|\boldsymbol{\omega}_2\|_2^2)^{1/2}(\boldsymbol{u}'\boldsymbol{\pi}_n) - \frac{\lambda_{n1}}{n}\boldsymbol{U}'_{\tilde{A}_4}Q_{41}\boldsymbol{\Omega}^{-1}_{\tilde{A}_1}\mathbb{D}^{c\prime}_{\tilde{A}_1}\boldsymbol{\pi}_n. \qquad (24)
\end{aligned}
$$

Combining (21) and (24), we have

$$\begin{aligned}
&\boldsymbol{V}'_{14}\boldsymbol{V}_{11} + \|\boldsymbol{\omega}_2\|_2^2 \\
&\quad \leq \lambda_{n1}\sum_{k\in\tilde{A}_4}\|\boldsymbol{\beta}_k\|_2 - \frac{\lambda_{n1}\boldsymbol{U}'_{\tilde{A}_3}Q_{31}\boldsymbol{\Omega}_{\tilde{A}_1}^{-1}(\mathbb{D}^{c\prime}_{\tilde{A}_1}\mathbb{D}^{c}_{\tilde{A}_2})\boldsymbol{\beta}_{\tilde{A}_2}}{n} + \lambda_{n1}\boldsymbol{\beta}'_{\tilde{A}_2}\boldsymbol{D}_{\tilde{A}_2} \\
&\qquad + (\|\boldsymbol{V}_{14}\|_2^2 + \|\boldsymbol{\omega}_2\|_2^2)^{1/2}|\boldsymbol{u}'\boldsymbol{\pi}_n| \\
&\quad = \lambda_{n1}\sum_{k\in\tilde{A}_4}\|\boldsymbol{\beta}_k\|_2 - \boldsymbol{V}'_{13}\boldsymbol{\Omega}_{\tilde{A}_1}^{-1/2}(\mathbb{D}^{c\prime}_{\tilde{A}_1}\mathbb{D}^{c}_{\tilde{A}_2})\boldsymbol{\beta}_{\tilde{A}_2}/\sqrt{n} + \lambda_{n1}\boldsymbol{\beta}'_{\tilde{A}_2}\boldsymbol{D}_{\tilde{A}_2} \\
&\qquad + 2\left((\|\boldsymbol{V}_{14}\|_2^2 + \|\boldsymbol{\omega}_2\|_2^2)^{1/2}/2\right)|\boldsymbol{u}'\boldsymbol{\pi}_n| \\
&\quad \leq \lambda_{n1}\sum_{k\in\tilde{A}_4}\|\boldsymbol{\beta}_k\|_2 + \|\boldsymbol{V}_{13}\|_2\|\boldsymbol{\Omega}_{\tilde{A}_1}^{-1/2}(\mathbb{D}^{c\prime}_{\tilde{A}_1}\mathbb{D}^{c}_{\tilde{A}_2})\boldsymbol{\beta}_{\tilde{A}_2}\|_2/\sqrt{n} + \lambda_{n1}\sum_{k\in\tilde{A}_2}\|\boldsymbol{\beta}_k\|_2 \\
&\qquad + \frac{(\|\boldsymbol{V}_{14}\|_2^2 + \|\boldsymbol{\omega}_2\|_2^2)}{4} + |\boldsymbol{u}'\boldsymbol{\pi}_n|^2, \qquad (25)
\end{aligned}$$

where the inequality is by the Cauchy-Schwarz inequality, triangle inequality and $2ab \leq a^2+b^2$. Then, by (22) and (25),

$$\begin{aligned}
\|\boldsymbol{V}_{14}\|_2^2 + \|\boldsymbol{\omega}_2\|_2^2 &\leq \boldsymbol{V}'_{14}\boldsymbol{V}_{11} + \|\boldsymbol{V}_{14}\|_2\|\boldsymbol{V}_{13}\|_2 + \|\boldsymbol{\omega}_2\|_2^2 \\
&\leq \lambda_{n1}\sum_{k\in\tilde{A}_4}\|\boldsymbol{\beta}_k\|_2 + \|\boldsymbol{V}_{13}\|_2\|\boldsymbol{\Omega}_{\tilde{A}_1}^{-1/2}(\mathbb{D}^{c\prime}_{\tilde{A}_1}\mathbb{D}^{c}_{\tilde{A}_2})\boldsymbol{\beta}_{\tilde{A}_2}\|_2/\sqrt{n} + \lambda_{n1}\sum_{k\in\tilde{A}_2}\|\boldsymbol{\beta}_k\|_2 \\
&\quad + \frac{(\|\boldsymbol{V}_{14}\|_2^2 + \|\boldsymbol{\omega}_2\|_2^2)}{4} + |\boldsymbol{u}'\boldsymbol{\pi}_n|^2 + \|\boldsymbol{V}_{14}\|_2\lambda_{n1}\sqrt{\frac{m_n|\tilde{A}_3|}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_1})}}. \qquad (26)
\end{aligned}$$

Since $\{(\mathbb{D}^c, \boldsymbol{\pi}_n) \in \Omega_{q_1}\}$ implies $|\boldsymbol{u}'\boldsymbol{\pi}_n|^2 \leq (x_{q_1})^2 \leq \frac{(q_1 m_n \vee m_n)\lambda_{n1}^2}{4n\rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_1})} = \frac{q_1 m_n \lambda_{n1}^2}{4n\rho_{\max}(\boldsymbol{\Omega}_{\tilde{A}_1})} = \frac{1}{4}\frac{q_1}{q^*}B_1$, we can show $|\boldsymbol{u}'\boldsymbol{\pi}_n|^2 \leq \frac{1}{4}\frac{q_1}{q^*}B_1 \leq \frac{1}{4}(\|V_{14}\|_2^2 + B_1)$ by using (18). Along with $\|\boldsymbol{V}_{13}\|_2 \leq$

$\lambda_{n1}\sqrt{\frac{m_n|\tilde{A}_3|}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_1})}}$, (26) becomes

$$\begin{aligned}
&\leq \lambda_{n1}\sum_{k\in\tilde{A}_4}\|\boldsymbol{\beta}_k\|_2 + \lambda_{n1}\sqrt{\frac{m_n|\tilde{A}_3|}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_1})}}\|\boldsymbol{\Omega}_{\tilde{A}_1}^{-1/2}\mathbb{D}_{\tilde{A}_1}^{c\prime}\mathbb{D}_{\tilde{A}_2}^{c}\boldsymbol{\beta}_{\tilde{A}_2}\|_2/\sqrt{n} + \lambda_{n1}\sum_{k\in\tilde{A}_2}\|\boldsymbol{\beta}_k\|_2 \\
&+\frac{(\|\boldsymbol{V}_{14}\|_2^2 + \|\boldsymbol{\omega}_2\|_2^2)}{4} + \frac{(\|\boldsymbol{V}_{14}\|_2^2 + B_1)}{4} + \|\boldsymbol{V}_{14}\|_2\lambda_{n1}\sqrt{\frac{m_n|\tilde{A}_3|}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_1})}} \\
&= \lambda_{n1}\sum_{k\in\tilde{A}_5}\|\boldsymbol{\beta}_k\|_2 + \lambda_{n1}\sqrt{\frac{m_n|\tilde{A}_3|}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_1})}}\|P_{\tilde{A}_1}^{1/2}\mathbb{D}_{\tilde{A}_2}^{c}\boldsymbol{\beta}_{\tilde{A}_2}\|_2 + \lambda_{n1}\sum_{k\in\tilde{A}_4\cup\tilde{A}_6}\|\boldsymbol{\beta}_k\|_2 \\
&+\frac{\|\boldsymbol{V}_{14}\|_2^2}{2} + \frac{\|\boldsymbol{\omega}_2\|_2^2}{4} + \frac{B_1}{4} + \|\boldsymbol{V}_{14}\|_2\lambda_{n1}\sqrt{\frac{m_n|\tilde{A}_3|}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_1})}} \\
&\leq \lambda_{n1}\sum_{k\in\tilde{A}_5}\|\boldsymbol{\beta}_k\|_2 + \lambda_{n1}\sqrt{\frac{m_n|\tilde{A}_3|}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_1})}}\|P_{\tilde{A}_1}^{1/2}\mathbb{D}_{\tilde{A}_2}^{c}\boldsymbol{\beta}_{\tilde{A}_2}\|_2 + \lambda_{n1}\eta_1 \\
&+\frac{\|\boldsymbol{V}_{14}\|_2^2}{2} + \frac{\|\boldsymbol{\omega}_2\|_2^2}{4} + \frac{B_1}{4} + \|\boldsymbol{V}_{14}\|_2\lambda_{n1}\sqrt{\frac{m_n|\tilde{A}_3|}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_1})}},
\end{aligned}$$

where the equality comes from $\tilde{A}_2 = \tilde{A}_5\cup\tilde{A}_6$ with $P_{\tilde{A}_1}^{1/2} = \boldsymbol{\Omega}_{\tilde{A}_1}^{-1/2}\mathbb{D}_{\tilde{A}_1}^{c\prime}/\sqrt{n}$ and the last inequality is due to GSC. This result gives

$$\begin{aligned}
\frac{1}{2}\|\boldsymbol{V}_{14}\|_2^2 + \frac{3}{4}\|\boldsymbol{\omega}_2\|_2^2 \leq \lambda_{n1}\sum_{k\in\tilde{A}_5}\|\boldsymbol{\beta}_k\|_2 + \lambda_{n1}\sqrt{\frac{m_n|\tilde{A}_3|}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_1})}}\|P_{\tilde{A}_1}^{1/2}\mathbb{D}_{\tilde{A}_2}^{c}\boldsymbol{\beta}_{\tilde{A}_2}\|_2 + \lambda_{n1}\eta_1 \\
+ \frac{B_1}{4} + \|\boldsymbol{V}_{14}\|_2\lambda_{n1}\sqrt{\frac{m_n|\tilde{A}_3|}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_1})}} \qquad (27)
\end{aligned}$$

from which we have

$$\begin{aligned}
\|\boldsymbol{V}_{14}\|_2^2 \quad &\leq 2\left(\frac{1}{2}\|\boldsymbol{V}_{14}\|_2^2 + \frac{3}{4}\|\boldsymbol{\omega}_2\|_2^2\right) \\
&\leq 2\lambda_{n1}\sum_{k\in\tilde{A}_5}\|\boldsymbol{\beta}_k\|_2 + 2\lambda_{n1}\sqrt{\frac{m_n|\tilde{A}_3|}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_1})}}\|P_{\tilde{A}_1}^{1/2}\mathbb{D}_{\tilde{A}_2}^{c}\boldsymbol{\beta}_{\tilde{A}_2}\|_2 + 2\lambda_{n1}\eta_1 \\
&\qquad + \frac{B_1}{2} + 2\|\boldsymbol{V}_{14}\|_2\lambda_{n1}\sqrt{\frac{m_n|\tilde{A}_3|}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_1})}}.
\end{aligned}$$

Note that the largest possible $\tilde{A}_1$ contains all the “large" $\|\boldsymbol{\beta}_j\|_2$ then $\tilde{A}_5 = \phi$ so that $\tilde{A}_2 = \tilde{A}_6$, $\sum_{k\in\tilde{A}_5}\|\boldsymbol{\beta}_k\|_2 = 0$ and $P_{\tilde{A}_1}^{1/2}\mathbb{D}_{\tilde{A}_2}^{c}\boldsymbol{\beta}_{\tilde{A}_2} = P_{\tilde{A}_1}^{1/2}\mathbb{D}_{\tilde{A}_6}^{c}\boldsymbol{\beta}_{\tilde{A}_6}$. Finally using $\|P_{\tilde{A}_1}^{1/2}\mathbb{D}_{\tilde{A}_6}^{c}\boldsymbol{\beta}_{\tilde{A}_6}\|_2 \leq$

$\max_{A\subset\tilde{A}_0}\|\sum_{k\in A}\mathbb{D}_k^c\boldsymbol{\beta}_k\|_2$ since $\tilde{A}_6\subset\tilde{A}_0$, we have the following inequality.

$$\|\boldsymbol{V}_{14}\|_2^2\le 2\eta_2\sqrt{B_2}+2\lambda_{n1}\eta_1+\frac{B_1}{2}+2\sqrt{B_2}\|\boldsymbol{V}_{14}\|_2, \tag{28}$$

where $\eta_2=\max_{A\subset\tilde{A}_0}\|\sum_{k\in A}\mathbb{D}_k^c\boldsymbol{\beta}_k\|_2$ and $B_2=\frac{\lambda_{n1}^2 m_n q^*}{n\rho_{\min}(\boldsymbol{\Omega}_{\tilde{A}_1})}$. Hence after using the fact $2\sqrt{B_2}\|\boldsymbol{V}_{14}\|_2=2\left(\sqrt{2B_2}\right)\left(\|\boldsymbol{V}_{14}\|_2/\sqrt{2}\right)\le 2B_2+\|\boldsymbol{V}_{14}\|_2^2/2$, we obtain an upper bound for $\|\boldsymbol{V}_{14}\|_2^2$ from (28),

$$\|\boldsymbol{V}_{14}\|_2^2\le B_1+4\lambda_{n1}\eta_1+4\eta_2\sqrt{B_2}+4B_2$$

which, when combined with (19), implies

$$|\tilde{A}_{\boldsymbol{\beta}}|\le M_1q^*,$$

where $M_1=M_1(\lambda_{n1})=2+4r_1+4r_2\sqrt{C_{12}}+4C_{12}$ with

$$r_1=r_1(\lambda_{n1})=\left(\frac{c_2\eta_1 n}{q^*m_n\lambda_{n1}}\right),r_2=r_2(\lambda_{n1})=\left(\frac{c_2\eta_2^2 n}{q^*m_n\lambda_{n1}^2}\right)^{1/2}\text{ and }C_{12}=\frac{c_2}{c_1}.$$

Note that $M_1(\lambda_{n1})$ is a decreasing function in λ_{n1}.

If $\eta_1=0$ which is referred to as a narrow-sense sparsity condition, then $r_1=r_2=0$ and hence $M_1(\lambda_{n1})=2+4C_{12}<\infty$. Note that since we are assuming $\lambda_0<\infty$, we implicitly assume that $(2+4C_{12})q^*+1\le q_0$ holds. In general, as long as η_1 and η_2 satisfy that $\eta_1\le\left(\frac{C_1q^*}{c_2}\right)\left(\frac{m_n\lambda_{n1}}{n}\right)$ and $\eta_2^2\le\left(\frac{C_2q^*}{c_2}\right)\left(\frac{m_n\lambda_{n1}^2}{n}\right)$ for some finite C_1 and C_2, we will have $r_1\le C_1$ and $r_2\le C_2$, which gives $M_1(\lambda_{n1})\le(2+4C_1+4C_2\sqrt{C_{12}}+4C_{12})<\infty$. Thus, we complete the proof.

$\square$

4 Extended results on different levels of smoothness for additive components

In this section, we provide extended results when we allow different smoothness levels for additive components. To do this, we need to make necessary changes in Assumption 1.

(H 4)′ $f_j\in\mathcal{F}_j$ and $\mathbb{E}f_j(X_j)=0$ for $j=1,\ldots,J$, where

$$\mathcal{F}_j=\left\{f\mid\left|f^{(k_j)}(s)-f^{(k_j)}(t)\right|\le C\left|s-t\right|^{\nu_j},\ \forall s,t\in[a,b]\right\}$$

for some nonnegative integer k_j and $\nu_j\in(0,1]$. Also suppose that $\tau_j=k_j+\nu_j>1$, and,

(H 6)$'$ $m_{nj} = O(n^{\gamma_j})$ with $1/6 \leq \gamma_j = 1/(2\tau_j + 1) < 1/3, \ \ \forall j = 1, 2, \ldots, J$, where m_{nj} is the number of B-spline bases (or knots) to approximate the j^{th} additive component.

We provide revised lemmas and theorems in the following. We first extend the definition of $\mathcal{S}^0_{nj}$ as follows:

$$\mathcal{S}^0_{nj} \quad = \quad \left\{ f_{nj} : f_{nj} = \sum_{l=1}^{m_{nj}} b_{jl} \mathbb{B}^c_l(x), (b_{j1}, \cdots, b_{jm_{nj}}) \in \mathbb{R}^{m_{nj}} \right\}, 1 \leq j \leq J. \tag{29}$$

With new assumptions, Lemmas 1 and 3 and Theorems 2 and 4 are changed as follows.

Lemma 1$'$. Suppose that $f \in \mathcal{F}_j$ and $\mathbb{E}f(X_j) = 0$. Then, under **(H 4)**$'$ and **(H 5)**, there exists an $f_n \in \mathcal{S}^0_{nj}$ such that

$$\|f_n - f\|_2 = \boldsymbol{O}_p\left(m_{nj}^{-\tau} + \sqrt{\frac{m_{nj}}{n}}\right). \tag{30}$$

Particularly, under the choice of $m_{nj} = \boldsymbol{O}(n^{\frac{1}{2\tau_j+1}})$, we have

$$\|f_n - f\|_2 = \boldsymbol{O}_p\left(m_{nj}^{-\tau_j}\right) = \boldsymbol{O}_p\left(n^{-\frac{\tau_j}{2\tau_j+1}}\right). \tag{31}$$

Lemma 3$'$. Define $\boldsymbol{M}_n$ be a non-negative definite matrix of order n and,

$$T_{jl} = \left(\frac{m_{nj}}{n}\right)^{\frac{1}{2}} \boldsymbol{a}'_{jl} \boldsymbol{M}_n \boldsymbol{\epsilon} \qquad \forall 1 \leq j \leq J, 1 \leq l \leq m_{nj}, \tag{32}$$

where $\boldsymbol{a}_{jl} = (\mathbb{B}^c_l(X_j(\boldsymbol{s})), \boldsymbol{s} \in \boldsymbol{S})'$ and $T_n = \max_{\substack{1 \leq j \leq J \\ 1 \leq l \leq m_{nj}}} |T_{jl}|$. With new Assumption 1,

$$\mathbb{E}(T_n) \leq C_1 \rho_{\max}(\boldsymbol{M}_n) \sqrt{(m_n \log(J m_n))}, \tag{33}$$

for some $C_1 > 0$ and $m_n = \max_{j=1,2,\ldots,J} m_{nj}$.

Theorem 2 $'$. With new Assumption 1 and $\lambda_{n1} > C \rho_{\max}(\boldsymbol{L}) \sqrt{n m_n \log(J m_n)}$ for a sufficiently large constant C,

(a) $\|\hat{f}_{gL,j} - f_j\|_2^2 = \boldsymbol{O}_p \left\{ \left(\frac{\rho^2_{\max}(\boldsymbol{L}) m_n^3 \log(J m_n)}{n} + \frac{m_n}{n} + \frac{1}{m_n^{2\tau-1}} + \frac{4 m_n^2 \lambda_{n1}^2}{n^2} \right) / m_{nj} \right\}$ for $j \in \tilde{A}_{\boldsymbol{\beta}} \cup A_*$, where $\tilde{A}_{\boldsymbol{\beta}}$ is the index set of nonzero gL estimates for $\boldsymbol{\beta}_j$,

(b) If $\frac{m_n^2 \lambda_{n1}^2}{m_{nj} n^2} \longrightarrow 0$ as $n \longrightarrow \infty$ for $1 \leq j \leq q$, all the nonzero components $f_j, 1 \leq j \leq q$ are selected w.p. converging to 1.

Theorem 4 $'$. With new Assumptions 1 and 2,

(a) $\mathbb{P}(\|\hat{f}_{AgL,j}\|_2 > 0, j \in A_*$ and $\|\hat{f}_{AgL,j}\|_2 = 0, j \notin A_*) \longrightarrow 1$,

(b) $\|\hat{f}_{AgL,j} - f_j\|_2^2 = \boldsymbol{O}_p\left\{\left(\frac{\rho^2_{\max}(\boldsymbol{L})m_n^3\log(J_0m_n)}{n} + \frac{m_n}{n} + \frac{1}{m_n^{2\tau-1}} + \frac{4m_n^2\lambda_{n2}^2}{n^2}\right)/m_{nj}\right\}$ $\forall j \in A_*$.

Results are similar except a few changes due to the introduction of m_{nj}. In practice, we standardize the range and variability of all additive components and use the same number of knot points for each component which we choose as m_n. Note that the largest number of knot points m_n can cover all smooth additive components. Since the order of m_n also has to be between $\left[n^{1/6}, n^{1/3}\right)$ according to the assumption **(H 6)′**, we use this bound as a guide to choose m_n. The suggestion of using the same number of knot points for each component, in practice, is also suggested by Hastie and Tibshirani (1990).

5 Results with extension of (H 3)

We extend the assumption **(H 3)** to cover a broader class of spatial covariance functions. The assumptions **(H 6)** and **(K 2)** are adjusted accordingly as well.

(H 3)* The random vector $\boldsymbol{\epsilon} = \{\epsilon(\boldsymbol{s}), \boldsymbol{s} \in \boldsymbol{S}\} \sim \text{Gaussian}(0, \boldsymbol{\Sigma}_T)$, where $\boldsymbol{\Sigma}_T = ((\sigma_{\boldsymbol{S},\boldsymbol{S}'}))_{\boldsymbol{S},\boldsymbol{S}'\in\boldsymbol{S}}$ with $\sigma_{\boldsymbol{S},\boldsymbol{S}'} = \delta(\boldsymbol{s}-\boldsymbol{s}')$ and $\delta(\boldsymbol{h})$ is a covariance function such that $\int_{\boldsymbol{D}_n}\delta(\boldsymbol{h})d\boldsymbol{h} = \boldsymbol{O}(n^\alpha)$ for some $\alpha \in [0, 1)$. $\boldsymbol{D}_n \subset \mathbb{R}^d$ is the sampling region that contains the sampling locations $\boldsymbol{S}$. Without loss of generality, we assume that the origin of $\mathbb{R}^d$ is in the interior of $\boldsymbol{D}_n$ and $\boldsymbol{D}_n$ is increasing with n.

(H 6)* $m_n = O(n^\gamma)$ with $1/6 \leq \gamma = 1/(2\tau+1) < (1-\alpha)/3$.

(K 2)*

$$\frac{\sqrt{\rho^2_{\max}(\boldsymbol{L})\, n^{1+\alpha} m_n \log(s_n m_n)}}{\lambda_{n2} r_n} + \frac{n^2}{\lambda_{n2}^2 r_n^2 m_n} + \frac{\lambda_{n2} m_n}{n} = o(1)$$

where $s_n = J - |A_{**}|$.

Lemmas and theorems are then updated in the following way.

Lemma 3*. Define $\boldsymbol{M}_n$ be a non-negative definite matrix of order n and,

$$T_{jl} = \left(\frac{m_n}{n}\right)^{\frac{1}{2}} \boldsymbol{a}'_{jl}\boldsymbol{M}_n\boldsymbol{\epsilon} \qquad \forall 1 \leq j \leq J, 1 \leq l \leq m_n \tag{34}$$

where $\boldsymbol{a}_{jl} = (\mathbb{B}_l^c(X_j(\boldsymbol{s})), \boldsymbol{s} \in \boldsymbol{S})'$ and $T_n = \max_{\substack{1\leq j\leq J\\1\leq l\leq m_n}} |T_{jl}|$. Then, under assumptions **(H 2)**, **(H 3)***, **(H 4)** and **(H 5)**,

$$\mathbb{E}(T_n) \leq C_1\rho_{\max}(\boldsymbol{M}_n)\sqrt{(m_n\log(Jm_n))\boldsymbol{O}(n^\alpha)}, \tag{35}$$

for some $C_1 > 0$.

Lemma 4*. Under the Assumption 1 with updated **(H 3)*** and **(H 6)*** and with $\lambda_{n1} > C\rho_{\max}(\boldsymbol{L})\sqrt{n^{1+\alpha}m_n \log(Jm_n)}$ for a sufficiently large constant C, we have $|\tilde{A}_{\boldsymbol{\beta}}| \leq M_1|A_*|$ for a finite constant $M_1 > 1$ with w.p. converging to 1.

Theorem 1.*

Suppose that conditions in Assumption 1 with updated **(H 3)*** and **(H 6)*** hold and if $\lambda_{n1} > C\rho_{\max}(\boldsymbol{L})\sqrt{n^{1+\alpha}m_n \log(Jm_n)}$ for a sufficiently large constant C. Then, we have

(a) $\sum_{j=1}^{J}\|\hat{\boldsymbol{\beta}}_{gL,j} - \boldsymbol{\beta}_j\|_2^2 = \boldsymbol{O}_p\left(\frac{\rho^2_{\max}(\boldsymbol{L})m_n^3\log(Jm_n)}{n^{1-\alpha}} + \frac{m_n}{n^{1-\alpha}} + \frac{1}{m_n^{2\tau-1}} + \frac{4m_n^2\lambda_{n1}^2}{n^2}\right)$,

(b) If $\frac{m_n^2\lambda_{n1}^2}{n^2} \longrightarrow 0$ as $n \longrightarrow \infty$, all the nonzero components $\boldsymbol{\beta}_j, 1 \leq j \leq q$ are selected with probability (w.p.) converging to 1.

Theorem 2.* Suppose that conditions in Assumption 1 with updated **(H 3)*** and **(H 6)*** hold and if $\lambda_{n1} > C\rho_{\max}(\boldsymbol{L})\sqrt{n^{1+\alpha}m_n \log(Jm_n)}$ for a sufficiently large constant C. Then,

(a) $\|\hat{f}_{gL,j} - f_j\|_2^2 = \boldsymbol{O}_p\left(\frac{\rho^2_{\max}(\boldsymbol{L})m_n^2\log(Jm_n)}{n^{1-\alpha}} + \frac{1}{n^{1-\alpha}} + \frac{1}{m_n^{2\tau}} + \frac{4m_n\lambda_{n1}^2}{n^2}\right)$ for $j \in \tilde{A}_{\boldsymbol{\beta}} \cup A_*$, where $\tilde{A}_{\boldsymbol{\beta}}$ is the index set of nonzero gL estimates for $\boldsymbol{\beta}_j$,

(b) If $\frac{m_n\lambda_{n1}^2}{n^2} \longrightarrow 0$ as $n \longrightarrow \infty$, all the nonzero components $f_j, 1 \leq j \leq q$ are selected w.p. converging to 1.

Theorem 3.* Suppose that conditions in Assumptions 1 and 2 with updated **(H 3)***, **(H 6)*** and **(K 2)*** are satisfied. Then,

(a) $\mathbb{P}(\hat{\boldsymbol{\beta}}_{AgL} =_0 \boldsymbol{\beta}) \longrightarrow 1$,

(b) $\sum_{j=1}^{q}\|\hat{\boldsymbol{\beta}}_{AgL,j} - \boldsymbol{\beta}_j\|_2^2 = \boldsymbol{O}_p\left(\frac{\rho^2_{\max}(\boldsymbol{L})m_n^3\log(J_0m_n)}{n^{1-\alpha}} + \frac{m_n}{n^{1-\alpha}} + \frac{1}{m_n^{2\tau-1}} + \frac{4m_n^2\lambda_{n2}^2}{n^2}\right)$.

Theorem 4.* Suppose that conditions in Assumptions 1 and 2 with updated **(H 3)***, **(H 6)*** and **(K 2)*** are satisfied. Then,

(a) $\mathbb{P}(\|\hat{f}_{AgL,j}\|_2 > 0, j \in A_*$ and $\|\hat{f}_{AgL,j}\|_2 = 0, j \notin A_*) \longrightarrow 1$,

(b) $\sum_{j=1}^{q}\|\hat{f}_{AgL,j} - f_j\|_2^2 = \boldsymbol{O}_p\left(\frac{\rho^2_{\max}(\boldsymbol{L})m_n^2\log(J_0m_n)}{n^{1-\alpha}} + \frac{1}{n^{1-\alpha}} + \frac{1}{m_n^{2\tau}} + \frac{4m_n\lambda_{n2}^2}{n^2}\right)$.

The updated theorems show that the lower bound of the penalty parameter as well as the convergence rate are affected by α. More specifically, introduction of α increases the order in the lower bound of the penalty parameter and the order of the convergence rate is decreased (slower convergence rate) with α. Note that α does not fully characterize a spatial dependence structure but it gives some information on the level of spatial dependence such that $0 < \alpha < 1$ implies a long-range dependence. For any integrable stationary spatial covariance model, $\alpha = 0$ and this is the case for most practical situations. If $0 < \alpha < 1$, one might consider estimating

α for calculation of the lower bound of the penalty parameter. There are some literature which provide how to estimate long-range parameters for random fields [e.g. Anh and Lunney (1995), Boissy *et al.* (2005)], but they are limited since a specific class of random fields or a parametric model is assumed. Estimation of α has its own interest but we do not pursue it since our focus is on variable selection.

6 Additional simulation results

In this section, we provide complete simulation results including the simulated data with $t = 3$ in generating covariates, X_j. Compared to the Table 2 in the main paper, Tables 2 - 4 include the results for $m = 6, 12, 24$, more choices of correlation parameters and $J = 15, 25, 35$. The results are consistent with what we have in the main paper.

As discussed in Section 4 in the main paper, the correlation between X_j and X_k is higher with $t = 3$ compared to $t = 1$. Tables 5- 7 show results with $t = 3$ case. The selection results are worse for both our approach as well as the independent approach compared to the cases with $t = 1$ in Tables 2- 4 but our approach still performs better than the independent approach.

J	Cov Model	Spatial Weights	m=6				m=12				m=24			
			GLASSO		Adapt. GLASSO		GLASSO		Adapt. GLASSO		GLASSO		Adapt. GLASSO	
			True Positive	False Positive	True Positive	False Positive	True Positive	False Positive	True Positive	False Positive	True Positive	False Positive	True Positive	False Positive
15	Exp(0.5)	None(Indep)	3.74(0.48)	3.63(1.83)	3.74(0.48)	3.62(1.84)	4(0)	1.6(1.3)	4(0)	1.6(1.3)	4(0)	0.55(0.77)	4(0)	0.55(0.77)
		I	3.17(0.82)	2.02(1.34)	3.15(0.82)	2(1.29)	4(0)	0.38(0.65)	4(0)	0.38(0.65)	4(0)	0(0)	4(0)	0(0)
		Gauss	3.01(0.85)	1.76(1.24)	3.01(0.85)	1.74(1.23)	4(0)	0.18(0.48)	4(0)	0.18(0.48)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.94(0.93)	1.58(1.44)	2.94(0.93)	1.58(1.44)	4(0)	0.1(0.3)	4(0)	0.1(0.3)	4(0)	0(0)	4(0)	0(0)
		True	1.66(1.02)	0.29(0.56)	1.66(1.02)	0.29(0.56)	4(0)	0.07(0.29)	4(0)	0.07(0.29)	4(0)	0(0)	4(0)	0(0)
	Exp(1)	None(Indep)	3.82(0.41)	3.77(2.09)	3.82(0.41)	3.67(1.99)	4(0)	1.51(1.24)	4(0)	1.51(1.24)	4(0)	0.53(0.69)	4(0)	0.53(0.69)
		I	3.11(0.82)	2.31(1.35)	3.11(0.82)	2.31(1.35)	4(0)	0.43(0.64)	4(0)	0.43(0.64)	4(0)	0.03(0.22)	4(0)	0.03(0.22)
		Gauss	3.01(0.8)	1.93(1.27)	3.01(0.8)	1.93(1.27)	4(0)	0.24(0.51)	4(0)	0.24(0.51)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.99(0.88)	1.56(1.17)	2.99(0.88)	1.54(1.13)	4(0)	0.14(0.47)	4(0)	0.14(0.47)	4(0)	0(0)	4(0)	0(0)
		True	2.33(1.05)	1.1(0.96)	2.33(1.05)	1.1(0.96)	4(0)	0.12(0.38)	4(0)	0.12(0.38)	4(0)	0(0)	4(0)	0(0)
	$\mathrm{Mat}_{3/2}$(2.5)	None(Indep)	3.8(0.45)	3.63(1.86)	3.79(0.46)	3.57(1.81)	4(0)	1.37(1.39)	4(0)	1.37(1.39)	4(0)	0.34(0.57)	4(0)	0.34(0.57)
		I	3.26(0.77)	1.87(1.5)	3.26(0.77)	1.87(1.5)	4(0)	0.46(0.77)	4(0)	0.46(0.77)	4(0)	0.05(0.22)	4(0)	0.05(0.22)
		Gauss	2.98(0.85)	1.64(1.34)	2.98(0.85)	1.63(1.33)	4(0)	0.25(0.58)	4(0)	0.25(0.58)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.74(0.86)	1.5(1.18)	2.74(0.86)	1.5(1.18)	4(0)	0.1(0.33)	4(0)	0.1(0.33)	4(0)	0(0)	4(0)	0(0)
		True	0.8(0.68)	0.09(0.32)	0.8(0.68)	0.09(0.32)	3.36(0.72)	0.01(0.1)	3.36(0.72)	0.01(0.1)	4(0)	0(0)	4(0)	0(0)
	$\mathrm{Mat}_{3/2}$(1.5)	None(Indep)	3.75(0.46)	4.07(2.04)	3.75(0.46)	4.06(2.04)	4(0)	1.54(1.42)	4(0)	1.54(1.42)	4(0)	0.47(0.67)	4(0)	0.47(0.67)
		I	3.23(0.71)	2.05(1.3)	3.22(0.7)	2.01(1.26)	4(0)	0.46(0.78)	4(0)	0.46(0.78)	4(0)	0.02(0.14)	4(0)	0.02(0.14)
		Gauss	3.03(0.76)	1.83(1.3)	3.03(0.76)	1.8(1.31)	4(0)	0.28(0.6)	4(0)	0.28(0.6)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.84(0.9)	1.51(1.06)	2.84(0.9)	1.48(1.01)	4(0)	0.14(0.35)	4(0)	0.14(0.35)	4(0)	0(0)	4(0)	0(0)
		True	1.01(0.88)	0.15(0.36)	1.01(0.88)	0.15(0.36)	3.82(0.46)	0.07(0.26)	3.82(0.46)	0.07(0.26)	4(0)	0(0)	4(0)	0(0)
	$\mathrm{Mat}_{5/2}$(2.5)	None(Indep)	3.82(0.41)	3.59(2.08)	3.81(0.42)	3.53(2.04)	4(0)	1.34(1.2)	4(0)	1.34(1.2)	4(0)	0.58(0.88)	4(0)	0.58(0.88)
		I	3.22(0.76)	2.06(1.55)	3.22(0.76)	2.06(1.55)	4(0)	0.43(0.86)	4(0)	0.43(0.86)	4(0)	0.04(0.2)	4(0)	0.04(0.2)
		Gauss	2.91(0.89)	1.72(1.35)	2.91(0.89)	1.69(1.35)	4(0)	0.26(0.66)	4(0)	0.26(0.66)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.97(0.89)	1.67(1.33)	2.97(0.89)	1.67(1.33)	4(0)	0.1(0.3)	4(0)	0.1(0.3)	4(0)	0(0)	4(0)	0(0)
		True	0.37(0.56)	0.04(0.24)	0.37(0.56)	0.04(0.24)	2.2(0.9)	0.06(0.24)	2.2(0.9)	0.06(0.24)	4(0)	0(0)	4(0)	0(0)
	$\mathrm{Mat}_{5/2}$(1.5)	None(Indep)	3.69(0.51)	3.85(1.92)	3.69(0.51)	3.84(1.93)	4(0)	1.74(1.52)	4(0)	1.74(1.52)	4(0)	0.58(0.75)	4(0)	0.58(0.75)
		I	3.29(0.74)	1.9(1.43)	3.27(0.75)	1.88(1.39)	4(0)	0.47(0.77)	4(0)	0.47(0.77)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		Gauss	3.21(0.74)	1.63(1.23)	3.21(0.74)	1.61(1.21)	4(0)	0.25(0.5)	4(0)	0.25(0.5)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.84(0.88)	1.53(1.1)	2.84(0.88)	1.52(1.1)	4(0)	0.19(0.42)	4(0)	0.19(0.42)	4(0)	0(0)	4(0)	0(0)
		True	0.59(0.65)	0.06(0.24)	0.59(0.65)	0.06(0.24)	2.79(0.99)	0.06(0.24)	2.79(0.99)	0.06(0.24)	4(0)	0(0)	4(0)	0(0)
	Gauss(1.5)	None(Indep)	3.76(0.49)	3.97(2.06)	3.76(0.49)	3.92(2.02)	4(0)	1.63(1.3)	4(0)	1.63(1.3)	4(0)	0.59(0.81)	4(0)	0.59(0.81)
		I	3.23(0.75)	2.07(1.24)	3.23(0.75)	2.07(1.24)	4(0)	0.64(0.92)	4(0)	0.64(0.92)	4(0)	0.03(0.17)	4(0)	0.03(0.17)
		Gauss	3.02(0.82)	1.86(1.25)	3.02(0.82)	1.86(1.25)	4(0)	0.28(0.59)	4(0)	0.28(0.59)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.76(0.79)	1.58(1.17)	2.76(0.79)	1.58(1.17)	3.99(0.1)	0.23(0.47)	3.99(0.1)	0.23(0.47)	4(0)	0(0)	4(0)	0(0)
		True	3.3(0.73)	2.32(1.55)	3.3(0.73)	2.32(1.55)	4(0)	0.98(1.05)	4(0)	0.98(1.05)	4(0)	0.12(0.33)	4(0)	0.12(0.33)
	Gauss(2.5)	None(Indep)	3.67(0.57)	3.97(1.89)	3.67(0.57)	3.83(1.81)	4(0)	1.77(1.46)	4(0)	1.77(1.46)	4(0)	0.55(0.78)	4(0)	0.55(0.78)
		I	3.19(0.77)	2.1(1.19)	3.19(0.77)	2.09(1.19)	4(0)	0.49(0.69)	4(0)	0.49(0.69)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		Gauss	2.97(0.83)	1.85(1.34)	2.97(0.83)	1.83(1.33)	4(0)	0.24(0.47)	4(0)	0.24(0.47)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.75(0.77)	1.45(1.12)	2.75(0.77)	1.44(1.09)	4(0)	0.24(0.47)	4(0)	0.24(0.47)	4(0)	0(0)	4(0)	0(0)
		True	3.56(0.57)	3.11(1.38)	3.56(0.57)	3.11(1.38)	4(0)	1.73(1.3)	4(0)	1.73(1.3)	4(0)	0.69(0.8)	4(0)	0.69(0.8)

Table 2: ***Monte Carlo Mean (Standard dev.) for the selected number of nonzero covariates using 100 datasets under both Independent and Dependent setup using spatially weighted group LASSO and adaptive Group LASSO algorithms when*** $J = 15$ **(**$t = 1$**)**

J	Cov Model	Spatial Weights	m=6				m=12				m=24			
			GLASSO		Adapt. GLASSO		GLASSO		Adapt. GLASSO		GLASSO		Adapt. GLASSO	
			True Positive	False Positive	True Positive	False Positive	True Positive	False Positive	True Positive	False Positive	True Positive	False Positive	True Positive	False Positive
25	Exp(0.5)	None(Indep)	3.53(0.67)	5.01(2.31)	3.53(0.67)	4.89(2.26)	4(0)	2.54(1.73)	4(0)	2.54(1.73)	4(0)	0.88(0.99)	4(0)	0.88(0.99)
		I	2.84(0.95)	2.76(1.78)	2.84(0.95)	2.76(1.78)	4(0)	0.89(1.14)	4(0)	0.89(1.14)	4(0)	0.04(0.2)	4(0)	0.04(0.2)
		Gauss	2.72(0.91)	2.49(1.56)	2.72(0.91)	2.48(1.56)	4(0)	0.57(0.89)	4(0)	0.57(0.89)	4(0)	0.02(0.14)	4(0)	0.02(0.14)
		InvMQ	2.4(0.85)	2(1.36)	2.4(0.85)	2(1.36)	4(0)	0.29(0.61)	4(0)	0.29(0.61)	4(0)	0(0)	4(0)	0(0)
		True	1.36(0.99)	0.44(0.61)	1.36(0.99)	0.44(0.61)	3.99(0.1)	0.07(0.26)	3.99(0.1)	0.07(0.26)	4(0)	0(0)	4(0)	0(0)
	Exp(1)	None(Indep)	3.49(0.63)	5.25(1.9)	3.49(0.63)	5.14(1.8)	4(0)	2.98(2.07)	4(0)	2.98(2.07)	4(0)	0.83(1.04)	4(0)	0.83(1.04)
		I	2.78(0.77)	2.77(1.62)	2.78(0.77)	2.77(1.62)	4(0)	1.06(1.08)	4(0)	1.06(1.08)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		Gauss	2.7(0.85)	2.18(1.35)	2.7(0.85)	2.16(1.33)	4(0)	0.55(0.78)	4(0)	0.55(0.78)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.57(0.9)	1.8(1.32)	2.57(0.9)	1.8(1.32)	4(0)	0.37(0.63)	4(0)	0.37(0.63)	4(0)	0(0)	4(0)	0(0)
		True	2.05(1.09)	1.02(1.08)	2.05(1.09)	1.02(1.08)	4(0)	0.31(0.54)	4(0)	0.31(0.54)	4(0)	0(0)	4(0)	0(0)
	$\mathrm{Mat}_{3/2}(2.5)$	None(Indep)	3.57(0.59)	5.09(2.16)	3.57(0.59)	5.07(2.16)	4(0)	2.29(1.82)	4(0)	2.29(1.82)	4(0)	0.83(1.1)	4(0)	0.83(1.1)
		I	2.99(0.9)	2.52(1.51)	2.99(0.9)	2.52(1.51)	4(0)	0.64(0.93)	4(0)	0.64(0.93)	4(0)	0.08(0.27)	4(0)	0.08(0.27)
		Gauss	2.85(0.77)	2.38(1.49)	2.85(0.77)	2.36(1.48)	4(0)	0.39(0.71)	4(0)	0.39(0.71)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.55(0.87)	1.91(1.44)	2.55(0.87)	1.89(1.44)	4(0)	0.25(0.56)	4(0)	0.25(0.56)	4(0)	0(0)	4(0)	0(0)
		True	0.66(0.65)	0.01(0.1)	0.66(0.65)	0.01(0.1)	3.15(0.93)	0.01(0.1)	3.15(0.93)	0.01(0.1)	4(0)	0(0)	4(0)	0(0)
	$\mathrm{Mat}_{3/2}(1.5)$	None(Indep)	3.63(0.56)	4.91(2.17)	3.63(0.56)	4.87(2.15)	4(0)	2.59(1.5)	4(0)	2.59(1.5)	4(0)	0.91(1.06)	4(0)	0.91(1.06)
		I	2.97(0.97)	2.78(1.85)	2.97(0.97)	2.78(1.85)	4(0)	0.9(1.01)	4(0)	0.9(1.01)	4(0)	0.02(0.14)	4(0)	0.02(0.14)
		Gauss	2.68(1)	2.18(1.64)	2.68(1)	2.16(1.61)	4(0)	0.53(0.74)	4(0)	0.53(0.74)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.56(0.89)	1.88(1.4)	2.56(0.89)	1.87(1.4)	4(0)	0.18(0.48)	4(0)	0.18(0.48)	4(0)	0(0)	4(0)	0(0)
		True	0.8(0.74)	0.06(0.24)	0.8(0.74)	0.06(0.24)	3.78(0.48)	0.06(0.24)	3.78(0.48)	0.06(0.24)	4(0)	0(0)	4(0)	0(0)
	$\mathrm{Mat}_{5/2}(2.5)$	None(Indep)	3.67(0.53)	4.99(2.2)	3.67(0.53)	4.97(2.2)	4(0)	1.95(1.9)	4(0)	1.95(1.9)	4(0)	0.71(0.94)	4(0)	0.71(0.94)
		I	3.07(0.86)	2.59(1.78)	3.07(0.86)	2.57(1.77)	4(0)	0.7(1.11)	4(0)	0.7(1.11)	4(0)	0.06(0.24)	4(0)	0.06(0.24)
		Gauss	2.87(0.91)	2.28(1.53)	2.87(0.91)	2.26(1.5)	4(0)	0.41(0.75)	4(0)	0.41(0.75)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.56(0.83)	1.98(1.48)	2.56(0.83)	1.97(1.47)	4(0)	0.26(0.63)	4(0)	0.26(0.63)	4(0)	0(0)	4(0)	0(0)
		True	0.22(0.44)	0.03(0.17)	0.22(0.44)	0.03(0.17)	1.75(0.89)	0.07(0.29)	1.75(0.89)	0.07(0.29)	4(0)	0(0)	4(0)	0(0)
	$\mathrm{Mat}_{5/2}(1.5)$	None(Indep)	3.49(0.61)	5.31(2.15)	3.49(0.61)	5.17(2.11)	4(0)	2.45(1.77)	4(0)	2.45(1.77)	4(0)	0.76(0.97)	4(0)	0.76(0.97)
		I	2.82(0.9)	2.63(1.53)	2.81(0.92)	2.62(1.51)	4(0)	0.84(1)	4(0)	0.84(1)	4(0)	0.03(0.17)	4(0)	0.03(0.17)
		Gauss	2.67(0.88)	2.26(1.38)	2.67(0.88)	2.26(1.38)	4(0)	0.44(0.76)	4(0)	0.44(0.76)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.46(0.82)	1.77(1.26)	2.46(0.82)	1.76(1.27)	4(0)	0.24(0.51)	4(0)	0.24(0.51)	4(0)	0(0)	4(0)	0(0)
		True	0.4(0.67)	0.04(0.2)	0.4(0.67)	0.04(0.2)	2.46(1)	0.07(0.26)	2.46(1)	0.07(0.26)	4(0)	0(0)	4(0)	0(0)
	Gauss(1.5)	None(Indep)	3.5(0.63)	5.17(1.93)	3.48(0.66)	5.02(1.79)	4(0)	2.89(1.53)	4(0)	2.89(1.53)	4(0)	0.8(0.88)	4(0)	0.8(0.88)
		I	2.84(0.9)	3.23(1.6)	2.84(0.9)	3.23(1.6)	4(0)	0.8(0.84)	4(0)	0.8(0.84)	4(0)	0.04(0.2)	4(0)	0.04(0.2)
		Gauss	2.74(0.87)	2.9(1.46)	2.74(0.87)	2.87(1.43)	4(0)	0.38(0.69)	4(0)	0.38(0.69)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.47(0.96)	2.11(1.52)	2.47(0.96)	2.1(1.51)	4(0)	0.27(0.51)	4(0)	0.27(0.51)	4(0)	0(0)	4(0)	0(0)
		True	2.77(0.99)	3.32(1.73)	2.76(1)	3.32(1.73)	4(0)	1.4(1.11)	4(0)	1.4(1.11)	4(0)	0.21(0.46)	4(0)	0.21(0.46)
	Gauss(2.5)	None(Indep)	3.42(0.74)	5.47(2.27)	3.4(0.75)	5.32(2.15)	4(0)	2.89(1.85)	4(0)	2.89(1.85)	4(0)	0.95(1.06)	4(0)	0.95(1.06)
		I	2.77(0.89)	2.82(1.56)	2.77(0.89)	2.82(1.56)	4(0)	0.96(1.08)	4(0)	0.96(1.08)	4(0)	0(0)	4(0)	0(0)
		Gauss	2.7(0.88)	2.5(1.33)	2.7(0.88)	2.49(1.33)	4(0)	0.4(0.68)	4(0)	0.4(0.68)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.65(0.78)	1.99(1.31)	2.65(0.78)	1.99(1.31)	4(0)	0.37(0.65)	4(0)	0.37(0.65)	4(0)	0(0)	4(0)	0(0)
		True	3.36(0.75)	3.99(1.8)	3.36(0.75)	3.97(1.79)	4(0)	2.52(1.46)	4(0)	2.52(1.46)	4(0)	1.21(1.15)	4(0)	1.21(1.15)

Table 3: ***Monte Carlo Mean (Standard dev.) for the selected number of nonzero covariates using 100 datasets under both Independent and Dependent setup using spatially weighted group LASSO and adaptive Group LASSO algorithms when*** $J = 25$ **(**$t = 1$**)**

J	Cov Model	Spatial Weights	m=6				m=12				m=24			
			GLASSO		Adapt. GLASSO		GLASSO		Adapt. GLASSO		GLASSO		Adapt. GLASSO	
			True Positive	False Positive	True Positive	False Positive	True Positive	False Positive	True Positive	False Positive	True Positive	False Positive	True Positive	False Positive
35	Exp(0.5)	None(Indep)	3.38(0.74)	6.22(2.39)	3.36(0.76)	6.1(2.33)	4(0)	3.74(2.68)	4(0)	3.74(2.68)	4(0)	1.49(1.55)	4(0)	1.49(1.55)
		I	2.62(0.94)	3.15(2.14)	2.6(0.95)	3.11(2.13)	4(0)	1.5(1.45)	4(0)	1.5(1.45)	4(0)	0.04(0.2)	4(0)	0.04(0.2)
		Gauss	2.39(0.98)	2.59(1.84)	2.39(0.98)	2.56(1.81)	4(0)	0.89(1.08)	4(0)	0.89(1.08)	4(0)	0.02(0.14)	4(0)	0.02(0.14)
		InvMQ	2.22(1.04)	2.38(1.79)	2.22(1.04)	2.37(1.78)	4(0)	0.41(0.75)	4(0)	0.41(0.75)	4(0)	0(0)	4(0)	0(0)
		True	1.21(0.95)	0.48(0.78)	1.21(0.95)	0.48(0.78)	3.99(0.1)	0.1(0.3)	3.99(0.1)	0.1(0.3)	4(0)	0(0)	4(0)	0(0)
	Exp(1)	None(Indep)	3.36(0.7)	5.77(2.3)	3.33(0.73)	5.54(1.98)	4(0)	3.68(1.93)	4(0)	3.68(1.93)	4(0)	1.31(1.18)	4(0)	1.31(1.18)
		I	2.59(0.98)	3.3(1.8)	2.59(0.98)	3.3(1.8)	4(0)	1.39(1.36)	4(0)	1.39(1.36)	4(0)	0.09(0.29)	4(0)	0.09(0.29)
		Gauss	2.36(0.97)	2.98(1.69)	2.36(0.97)	2.95(1.67)	4(0)	0.67(0.95)	4(0)	0.67(0.95)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.07(0.99)	2.25(1.61)	2.07(0.99)	2.24(1.59)	4(0)	0.71(1.04)	4(0)	0.71(1.04)	4(0)	0(0)	4(0)	0(0)
		True	1.56(1.05)	1.32(1.38)	1.56(1.05)	1.32(1.38)	4(0)	0.51(0.7)	4(0)	0.51(0.7)	4(0)	0(0)	4(0)	0(0)
	$\text{Mat}_{3/2}(2.5)$	None(Indep)	3.48(0.67)	5.79(2.32)	3.46(0.67)	5.59(2.25)	4(0)	2.71(2.13)	4(0)	2.71(2.13)	4(0)	1.25(1.37)	4(0)	1.25(1.37)
		I	2.71(0.9)	2.94(1.75)	2.71(0.9)	2.94(1.75)	4(0)	1.22(1.28)	4(0)	1.22(1.28)	4(0)	0.1(0.39)	4(0)	0.1(0.39)
		Gauss	2.57(0.92)	2.59(1.6)	2.57(0.92)	2.57(1.58)	4(0)	0.69(1)	4(0)	0.69(1)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.25(0.9)	2.24(1.68)	2.25(0.9)	2.23(1.68)	4(0)	0.62(0.97)	4(0)	0.62(0.97)	4(0)	0(0)	4(0)	0(0)
		True	0.52(0.58)	0.12(0.41)	0.52(0.58)	0.12(0.41)	2.81(1.01)	0.04(0.2)	2.81(1.01)	0.04(0.2)	4(0)	0(0)	4(0)	0(0)
	$\text{Mat}_{3/2}(1.5)$	None(Indep)	3.24(0.74)	6.39(2.65)	3.23(0.74)	6.21(2.5)	4(0)	3.4(2.23)	4(0)	3.4(2.23)	4(0)	1.47(1.42)	4(0)	1.47(1.42)
		I	2.63(0.91)	3.15(1.93)	2.63(0.91)	3.15(1.93)	4(0)	1.47(1.51)	4(0)	1.47(1.51)	4(0)	0.05(0.22)	4(0)	0.05(0.22)
		Gauss	2.39(0.84)	2.93(1.93)	2.39(0.84)	2.9(1.88)	4(0)	0.68(0.99)	4(0)	0.68(0.99)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.03(1)	2.26(1.57)	2.03(1)	2.25(1.55)	4(0)	0.43(0.82)	4(0)	0.43(0.82)	4(0)	0(0)	4(0)	0(0)
		True	0.69(0.73)	0.2(0.53)	0.69(0.73)	0.2(0.53)	3.56(0.61)	0.17(0.4)	3.56(0.61)	0.17(0.4)	4(0)	0(0)	4(0)	0(0)
	$\text{Mat}_{5/2}(2.5)$	None(Indep)	3.33(0.74)	5.59(2.19)	3.33(0.74)	5.41(2.08)	4(0)	3.2(2.56)	4(0)	3.2(2.56)	4(0)	1.34(1.51)	4(0)	1.34(1.51)
		I	2.72(1)	3.06(1.97)	2.72(1)	3.06(1.97)	4(0)	0.75(1.19)	4(0)	0.75(1.19)	4(0)	0.1(0.33)	4(0)	0.1(0.33)
		Gauss	2.53(1.04)	2.57(1.81)	2.53(1.04)	2.57(1.81)	4(0)	0.41(0.81)	4(0)	0.41(0.81)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.25(0.97)	2.17(1.68)	2.25(0.97)	2.16(1.67)	4(0)	0.38(0.91)	4(0)	0.38(0.91)	4(0)	0(0)	4(0)	0(0)
		True	0.34(0.57)	0.05(0.26)	0.34(0.57)	0.05(0.26)	1.82(0.83)	0.05(0.22)	1.82(0.83)	0.05(0.22)	4(0)	0(0)	4(0)	0(0)
	$\text{Mat}_{5/2}(1.5)$	None(Indep)	3.2(0.71)	5.66(2.16)	3.19(0.71)	5.56(2.08)	4(0)	3.57(2.38)	4(0)	3.56(2.37)	4(0)	1.3(1.32)	4(0)	1.3(1.32)
		I	2.39(1.03)	2.86(1.99)	2.39(1.03)	2.86(1.99)	4(0)	1.39(1.41)	4(0)	1.39(1.41)	4(0)	0.07(0.26)	4(0)	0.07(0.26)
		Gauss	2.35(0.91)	2.84(1.76)	2.35(0.91)	2.82(1.75)	4(0)	0.88(1.11)	4(0)	0.88(1.11)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.2(0.89)	2.09(1.52)	2.2(0.89)	2.09(1.52)	4(0)	0.36(0.72)	4(0)	0.36(0.72)	4(0)	0(0)	4(0)	0(0)
		True	0.28(0.51)	0(0)	0.28(0.51)	0(0)	2.38(0.91)	0.04(0.2)	2.38(0.91)	0.04(0.2)	4(0)	0(0)	4(0)	0(0)
	Gauss(1.5)	None(Indep)	3.15(0.88)	6.37(1.95)	3.14(0.88)	6.24(1.83)	4(0)	3.94(1.92)	4(0)	3.94(1.92)	4(0)	1.78(1.54)	4(0)	1.78(1.54)
		I	2.44(1.02)	3.35(1.98)	2.44(1.02)	3.34(1.96)	4(0)	1.25(1.14)	4(0)	1.25(1.14)	4(0)	0.08(0.27)	4(0)	0.08(0.27)
		Gauss	2.24(0.95)	2.83(1.61)	2.24(0.95)	2.82(1.57)	4(0)	0.63(0.73)	4(0)	0.63(0.73)	4(0)	0.02(0.14)	4(0)	0.02(0.14)
		InvMQ	2.18(0.9)	2.2(1.41)	2.18(0.9)	2.19(1.39)	4(0)	0.59(0.83)	4(0)	0.59(0.83)	4(0)	0(0)	4(0)	0(0)
		True	2.78(0.87)	3.66(1.74)	2.75(0.87)	3.61(1.69)	4(0)	1.84(1.54)	4(0)	1.84(1.54)	4(0)	0.17(0.45)	4(0)	0.17(0.45)
	Gauss(2.5)	None(Indep)	3.13(0.75)	6.64(2.37)	3.12(0.76)	6.45(2.24)	4(0)	4.38(2.06)	4(0)	4.38(2.06)	4(0)	1.14(1.07)	4(0)	1.14(1.07)
		I	2.6(0.96)	3.62(1.66)	2.6(0.96)	3.62(1.66)	4(0)	1.42(1.36)	4(0)	1.42(1.36)	4(0)	0.02(0.14)	4(0)	0.02(0.14)
		Gauss	2.44(0.86)	2.9(1.47)	2.44(0.86)	2.86(1.44)	4(0)	0.73(0.92)	4(0)	0.73(0.92)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.17(0.94)	2.36(1.52)	2.17(0.94)	2.36(1.52)	4(0)	0.63(0.85)	4(0)	0.63(0.85)	4(0)	0(0)	4(0)	0(0)
		True	3(0.83)	4.62(1.85)	3(0.83)	4.62(1.85)	4(0)	3.53(2.23)	4(0)	3.53(2.23)	4(0)	1.28(1.05)	4(0)	1.28(1.05)

Table 4: ***Monte Carlo Mean (Standard dev.) for the selected number of nonzero covariates using 100 datasets under both Independent and Dependent setup using spatially weighted group LASSO and adaptive Group LASSO algorithms when*** $J = 35$ **(**$t = 1$**)**

J	Cov Model	Spatial Weights	m=6				m=12				m=24			
			GLASSO		Adapt. GLASSO		GLASSO		Adapt. GLASSO		GLASSO		Adapt. GLASSO	
			True Positive	False Positive	True Positive	False Positive	True Positive	False Positive	True Positive	False Positive	True Positive	False Positive	True Positive	False Positive
15	Exp(0.5)	None(Indep)	3.74(0.48)	3.63(1.83)	3.74(0.48)	3.62(1.84)	4(0)	1.6(1.3)	4(0)	1.6(1.3)	4(0)	0.55(0.77)	4(0)	0.55(0.77)
		I	3.17(0.82)	2.02(1.34)	3.15(0.82)	2(1.29)	4(0)	0.38(0.65)	4(0)	0.38(0.65)	4(0)	0(0)	4(0)	0(0)
		Gauss	3.01(0.85)	1.76(1.24)	3.01(0.85)	1.74(1.23)	4(0)	0.18(0.48)	4(0)	0.18(0.48)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.94(0.93)	1.58(1.44)	2.94(0.93)	1.58(1.44)	4(0)	0.1(0.3)	4(0)	0.1(0.3)	4(0)	0(0)	4(0)	0(0)
		True	1.66(1.02)	0.29(0.56)	1.66(1.02)	0.29(0.56)	4(0)	0.07(0.29)	4(0)	0.07(0.29)	4(0)	0(0)	4(0)	0(0)
	Exp(1)	None(Indep)	3.82(0.41)	3.77(2.09)	3.82(0.41)	3.67(1.99)	4(0)	1.51(1.24)	4(0)	1.51(1.24)	4(0)	0.53(0.69)	4(0)	0.53(0.69)
		I	3.11(0.82)	2.31(1.35)	3.11(0.82)	2.31(1.35)	4(0)	0.43(0.64)	4(0)	0.43(0.64)	4(0)	0.03(0.22)	4(0)	0.03(0.22)
		Gauss	3.01(0.8)	1.93(1.27)	3.01(0.8)	1.93(1.27)	4(0)	0.24(0.51)	4(0)	0.24(0.51)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.99(0.88)	1.56(1.17)	2.99(0.88)	1.54(1.13)	4(0)	0.14(0.47)	4(0)	0.14(0.47)	4(0)	0(0)	4(0)	0(0)
		True	2.33(1.05)	1.1(0.96)	2.33(1.05)	1.1(0.96)	4(0)	0.12(0.38)	4(0)	0.12(0.38)	4(0)	0(0)	4(0)	0(0)
	$\text{Mat}_{3/2}(2.5)$	None(Indep)	3.8(0.45)	3.63(1.86)	3.79(0.46)	3.57(1.81)	4(0)	1.37(1.39)	4(0)	1.37(1.39)	4(0)	0.34(0.57)	4(0)	0.34(0.57)
		I	3.26(0.77)	1.87(1.5)	3.26(0.77)	1.87(1.5)	4(0)	0.46(0.77)	4(0)	0.46(0.77)	4(0)	0.05(0.22)	4(0)	0.05(0.22)
		Gauss	2.98(0.85)	1.64(1.34)	2.98(0.85)	1.63(1.33)	4(0)	0.25(0.58)	4(0)	0.25(0.58)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.74(0.86)	1.5(1.18)	2.74(0.86)	1.5(1.18)	4(0)	0.1(0.33)	4(0)	0.1(0.33)	4(0)	0(0)	4(0)	0(0)
		True	0.8(0.68)	0.09(0.32)	0.8(0.68)	0.09(0.32)	3.36(0.72)	0.01(0.1)	3.36(0.72)	0.01(0.1)	4(0)	0(0)	4(0)	0(0)
	$\text{Mat}_{3/2}(1.5)$	None(Indep)	3.75(0.46)	4.07(2.04)	3.75(0.46)	4.06(2.04)	4(0)	1.54(1.42)	4(0)	1.54(1.42)	4(0)	0.47(0.67)	4(0)	0.47(0.67)
		I	3.23(0.71)	2.05(1.3)	3.22(0.7)	2.01(1.26)	4(0)	0.46(0.78)	4(0)	0.46(0.78)	4(0)	0.02(0.14)	4(0)	0.02(0.14)
		Gauss	3.03(0.76)	1.83(1.3)	3.03(0.76)	1.8(1.31)	4(0)	0.28(0.6)	4(0)	0.28(0.6)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.84(0.9)	1.51(1.06)	2.84(0.9)	1.48(1.01)	4(0)	0.14(0.35)	4(0)	0.14(0.35)	4(0)	0(0)	4(0)	0(0)
		True	1.01(0.88)	0.15(0.36)	1.01(0.88)	0.15(0.36)	3.82(0.46)	0.07(0.26)	3.82(0.46)	0.07(0.26)	4(0)	0(0)	4(0)	0(0)
	$\text{Mat}_{5/2}(2.5)$	None(Indep)	3.82(0.41)	3.59(2.08)	3.81(0.42)	3.53(2.04)	4(0)	1.34(1.2)	4(0)	1.34(1.2)	4(0)	0.58(0.88)	4(0)	0.58(0.88)
		I	3.22(0.76)	2.06(1.55)	3.22(0.76)	2.06(1.55)	4(0)	0.43(0.86)	4(0)	0.43(0.86)	4(0)	0.04(0.2)	4(0)	0.04(0.2)
		Gauss	2.91(0.89)	1.72(1.35)	2.91(0.89)	1.69(1.35)	4(0)	0.26(0.66)	4(0)	0.26(0.66)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.97(0.89)	1.67(1.33)	2.97(0.89)	1.67(1.33)	4(0)	0.1(0.3)	4(0)	0.1(0.3)	4(0)	0(0)	4(0)	0(0)
		True	0.37(0.56)	0.04(0.24)	0.37(0.56)	0.04(0.24)	2.2(0.9)	0.06(0.24)	2.2(0.9)	0.06(0.24)	4(0)	0(0)	4(0)	0(0)
	$\text{Mat}_{5/2}(1.5)$	None(Indep)	3.69(0.51)	3.85(1.92)	3.69(0.51)	3.84(1.93)	4(0)	1.74(1.52)	4(0)	1.74(1.52)	4(0)	0.58(0.75)	4(0)	0.58(0.75)
		I	3.29(0.74)	1.9(1.43)	3.27(0.75)	1.88(1.39)	4(0)	0.47(0.77)	4(0)	0.47(0.77)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		Gauss	3.21(0.74)	1.63(1.23)	3.21(0.74)	1.61(1.21)	4(0)	0.25(0.5)	4(0)	0.25(0.5)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.84(0.88)	1.53(1.1)	2.84(0.88)	1.52(1.1)	4(0)	0.19(0.42)	4(0)	0.19(0.42)	4(0)	0(0)	4(0)	0(0)
		True	0.59(0.65)	0.06(0.24)	0.59(0.65)	0.06(0.24)	2.79(0.99)	0.06(0.24)	2.79(0.99)	0.06(0.24)	4(0)	0(0)	4(0)	0(0)
	Gauss(1.5)	None(Indep)	3.76(0.49)	3.97(2.06)	3.76(0.49)	3.92(2.02)	4(0)	1.63(1.3)	4(0)	1.63(1.3)	4(0)	0.59(0.81)	4(0)	0.59(0.81)
		I	3.23(0.75)	2.07(1.24)	3.23(0.75)	2.07(1.24)	4(0)	0.64(0.92)	4(0)	0.64(0.92)	4(0)	0.03(0.17)	4(0)	0.03(0.17)
		Gauss	3.02(0.82)	1.86(1.25)	3.02(0.82)	1.86(1.25)	4(0)	0.28(0.59)	4(0)	0.28(0.59)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.76(0.79)	1.58(1.17)	2.76(0.79)	1.58(1.17)	3.99(0.1)	0.23(0.47)	3.99(0.1)	0.23(0.47)	4(0)	0(0)	4(0)	0(0)
		True	3.3(0.73)	2.32(1.55)	3.3(0.73)	2.32(1.55)	4(0)	0.98(1.05)	4(0)	0.98(1.05)	4(0)	0.12(0.33)	4(0)	0.12(0.33)
	Gauss(2.5)	None(Indep)	3.67(0.57)	3.97(1.89)	3.67(0.57)	3.83(1.81)	4(0)	1.77(1.46)	4(0)	1.77(1.46)	4(0)	0.55(0.78)	4(0)	0.55(0.78)
		I	3.19(0.77)	2.1(1.19)	3.19(0.77)	2.09(1.19)	4(0)	0.49(0.69)	4(0)	0.49(0.69)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		Gauss	2.97(0.83)	1.85(1.34)	2.97(0.83)	1.83(1.33)	4(0)	0.24(0.47)	4(0)	0.24(0.47)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.75(0.77)	1.45(1.12)	2.75(0.77)	1.44(1.09)	4(0)	0.24(0.47)	4(0)	0.24(0.47)	4(0)	0(0)	4(0)	0(0)
		True	3.56(0.57)	3.11(1.38)	3.56(0.57)	3.11(1.38)	4(0)	1.73(1.3)	4(0)	1.73(1.3)	4(0)	0.69(0.8)	4(0)	0.69(0.8)

Table 5: ***Monte Carlo Mean (Standard dev.) for the selected number of nonzero covariates using 100 datasets under both Independent and Dependent setup using spatially weighted group LASSO and adaptive Group LASSO algorithms when*** $J = 15$ ***(***$t = 3$***)***

J	Cov Model	Spatial Weights	m=6				m=12				m=24			
			GLASSO		Adapt. GLASSO		GLASSO		Adapt. GLASSO		GLASSO		Adapt. GLASSO	
			True Positive	False Positive	True Positive	False Positive	True Positive	False Positive	True Positive	False Positive	True Positive	False Positive	True Positive	False Positive
25	Exp(0.5)	None(Indep)	3.53(0.67)	5.01(2.31)	3.53(0.67)	4.89(2.26)	4(0)	2.54(1.73)	4(0)	2.54(1.73)	4(0)	0.88(0.99)	4(0)	0.88(0.99)
		I	2.84(0.95)	2.76(1.78)	2.84(0.95)	2.76(1.78)	4(0)	0.89(1.14)	4(0)	0.89(1.14)	4(0)	0.04(0.2)	4(0)	0.04(0.2)
		Gauss	2.72(0.91)	2.49(1.56)	2.72(0.91)	2.48(1.56)	4(0)	0.57(0.89)	4(0)	0.57(0.89)	4(0)	0.02(0.14)	4(0)	0.02(0.14)
		InvMQ	2.4(0.85)	2(1.36)	2.4(0.85)	2(1.36)	4(0)	0.29(0.61)	4(0)	0.29(0.61)	4(0)	0(0)	4(0)	0(0)
		True	1.36(0.99)	0.44(0.61)	1.36(0.99)	0.44(0.61)	3.99(0.1)	0.07(0.26)	3.99(0.1)	0.07(0.26)	4(0)	0(0)	4(0)	0(0)
	Exp(1)	None(Indep)	3.49(0.63)	5.25(1.9)	3.49(0.63)	5.14(1.8)	4(0)	2.98(2.07)	4(0)	2.98(2.07)	4(0)	0.83(1.04)	4(0)	0.83(1.04)
		I	2.78(0.77)	2.77(1.62)	2.78(0.77)	2.77(1.62)	4(0)	1.06(1.08)	4(0)	1.06(1.08)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		Gauss	2.7(0.85)	2.18(1.35)	2.7(0.85)	2.16(1.33)	4(0)	0.55(0.78)	4(0)	0.55(0.78)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.57(0.9)	1.8(1.32)	2.57(0.9)	1.8(1.32)	4(0)	0.37(0.63)	4(0)	0.37(0.63)	4(0)	0(0)	4(0)	0(0)
		True	2.05(1.09)	1.02(1.08)	2.05(1.09)	1.02(1.08)	4(0)	0.31(0.54)	4(0)	0.31(0.54)	4(0)	0(0)	4(0)	0(0)
	$\text{Mat}_{3/2}$(2.5)	None(Indep)	3.57(0.59)	5.09(2.16)	3.57(0.59)	5.07(2.16)	4(0)	2.29(1.82)	4(0)	2.29(1.82)	4(0)	0.83(1.1)	4(0)	0.83(1.1)
		I	2.99(0.9)	2.52(1.51)	2.99(0.9)	2.52(1.51)	4(0)	0.64(0.93)	4(0)	0.64(0.93)	4(0)	0.08(0.27)	4(0)	0.08(0.27)
		Gauss	2.85(0.77)	2.38(1.49)	2.85(0.77)	2.36(1.48)	4(0)	0.39(0.71)	4(0)	0.39(0.71)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.55(0.87)	1.91(1.44)	2.55(0.87)	1.89(1.44)	4(0)	0.25(0.56)	4(0)	0.25(0.56)	4(0)	0(0)	4(0)	0(0)
		True	0.66(0.65)	0.01(0.1)	0.66(0.65)	0.01(0.1)	3.15(0.93)	0.01(0.1)	3.15(0.93)	0.01(0.1)	4(0)	0(0)	4(0)	0(0)
	$\text{Mat}_{3/2}$(1.5)	None(Indep)	3.63(0.56)	4.91(2.17)	3.63(0.56)	4.87(2.15)	4(0)	2.59(1.5)	4(0)	2.59(1.5)	4(0)	0.91(1.06)	4(0)	0.91(1.06)
		I	2.97(0.97)	2.78(1.85)	2.97(0.97)	2.78(1.85)	4(0)	0.9(1.01)	4(0)	0.9(1.01)	4(0)	0.02(0.14)	4(0)	0.02(0.14)
		Gauss	2.68(1)	2.18(1.64)	2.68(1)	2.16(1.61)	4(0)	0.53(0.74)	4(0)	0.53(0.74)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.56(0.89)	1.88(1.4)	2.56(0.89)	1.87(1.4)	4(0)	0.18(0.48)	4(0)	0.18(0.48)	4(0)	0(0)	4(0)	0(0)
		True	0.8(0.74)	0.06(0.24)	0.8(0.74)	0.06(0.24)	3.78(0.48)	0.06(0.24)	3.78(0.48)	0.06(0.24)	4(0)	0(0)	4(0)	0(0)
	$\text{Mat}_{5/2}$(2.5)	None(Indep)	3.67(0.53)	4.99(2.2)	3.67(0.53)	4.97(2.2)	4(0)	1.95(1.9)	4(0)	1.95(1.9)	4(0)	0.71(0.94)	4(0)	0.71(0.94)
		I	3.07(0.86)	2.59(1.78)	3.07(0.86)	2.57(1.77)	4(0)	0.7(1.11)	4(0)	0.7(1.11)	4(0)	0.06(0.24)	4(0)	0.06(0.24)
		Gauss	2.87(0.91)	2.28(1.53)	2.87(0.91)	2.26(1.5)	4(0)	0.41(0.75)	4(0)	0.41(0.75)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.56(0.83)	1.98(1.48)	2.56(0.83)	1.97(1.47)	4(0)	0.26(0.63)	4(0)	0.26(0.63)	4(0)	0(0)	4(0)	0(0)
		True	0.22(0.44)	0.03(0.17)	0.22(0.44)	0.03(0.17)	1.75(0.89)	0.07(0.29)	1.75(0.89)	0.07(0.29)	4(0)	0(0)	4(0)	0(0)
	$\text{Mat}_{5/2}$(1.5)	None(Indep)	3.49(0.61)	5.31(2.15)	3.49(0.61)	5.17(2.11)	4(0)	2.45(1.77)	4(0)	2.45(1.77)	4(0)	0.76(0.97)	4(0)	0.76(0.97)
		I	2.82(0.9)	2.63(1.53)	2.81(0.92)	2.62(1.51)	4(0)	0.84(1)	4(0)	0.84(1)	4(0)	0.03(0.17)	4(0)	0.03(0.17)
		Gauss	2.67(0.88)	2.26(1.38)	2.67(0.88)	2.26(1.38)	4(0)	0.44(0.76)	4(0)	0.44(0.76)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.46(0.82)	1.77(1.26)	2.46(0.82)	1.76(1.27)	4(0)	0.24(0.51)	4(0)	0.24(0.51)	4(0)	0(0)	4(0)	0(0)
		True	0.4(0.67)	0.04(0.2)	0.4(0.67)	0.04(0.2)	2.46(1)	0.07(0.26)	2.46(1)	0.07(0.26)	4(0)	0(0)	4(0)	0(0)
	Gauss(1.5)	None(Indep)	3.5(0.63)	5.17(1.93)	3.48(0.66)	5.02(1.79)	4(0)	2.89(1.53)	4(0)	2.89(1.53)	4(0)	0.8(0.88)	4(0)	0.8(0.88)
		I	2.84(0.9)	3.23(1.6)	2.84(0.9)	3.23(1.6)	4(0)	0.8(0.84)	4(0)	0.8(0.84)	4(0)	0.04(0.2)	4(0)	0.04(0.2)
		Gauss	2.74(0.87)	2.9(1.46)	2.74(0.87)	2.87(1.43)	4(0)	0.38(0.69)	4(0)	0.38(0.69)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.47(0.96)	2.11(1.52)	2.47(0.96)	2.1(1.51)	4(0)	0.27(0.51)	4(0)	0.27(0.51)	4(0)	0(0)	4(0)	0(0)
		True	2.77(0.99)	3.32(1.73)	2.76(1)	3.32(1.73)	4(0)	1.4(1.11)	4(0)	1.4(1.11)	4(0)	0.21(0.46)	4(0)	0.21(0.46)
	Gauss(2.5)	None(Indep)	3.42(0.74)	5.47(2.27)	3.4(0.75)	5.32(2.15)	4(0)	2.89(1.85)	4(0)	2.89(1.85)	4(0)	0.95(1.06)	4(0)	0.95(1.06)
		I	2.77(0.89)	2.82(1.56)	2.77(0.89)	2.82(1.56)	4(0)	0.96(1.08)	4(0)	0.96(1.08)	4(0)	0(0)	4(0)	0(0)
		Gauss	2.7(0.88)	2.5(1.33)	2.7(0.88)	2.49(1.33)	4(0)	0.4(0.68)	4(0)	0.4(0.68)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.65(0.78)	1.99(1.31)	2.65(0.78)	1.99(1.31)	4(0)	0.37(0.65)	4(0)	0.37(0.65)	4(0)	0(0)	4(0)	0(0)
		True	3.36(0.75)	3.99(1.8)	3.36(0.75)	3.97(1.79)	4(0)	2.52(1.46)	4(0)	2.52(1.46)	4(0)	1.21(1.15)	4(0)	1.21(1.15)

Table 6: ***Monte Carlo Mean (Standard dev.) for the selected number of nonzero covariates using 100 datasets under both Independent and Dependent setup using spatially weighted group LASSO and adaptive Group LASSO algorithms when*** $J = 25$ **(**$t = 3$**)**

J	Cov Model	Spatial Weights	m=6				m=12				m=24			
			GLASSO		Adapt. GLASSO		GLASSO		Adapt. GLASSO		GLASSO		Adapt. GLASSO	
			True Positive	False Positive	True Positive	False Positive	True Positive	False Positive	True Positive	False Positive	True Positive	False Positive	True Positive	False Positive
35	Exp(0.5)	None(Indep)	3.38(0.74)	6.22(2.39)	3.36(0.76)	6.1(2.33)	4(0)	3.74(2.68)	4(0)	3.74(2.68)	4(0)	1.49(1.55)	4(0)	1.49(1.55)
		I	2.62(0.94)	3.15(2.14)	2.6(0.95)	3.11(2.13)	4(0)	1.5(1.45)	4(0)	1.5(1.45)	4(0)	0.04(0.2)	4(0)	0.04(0.2)
		Gauss	2.39(0.98)	2.59(1.84)	2.39(0.98)	2.56(1.81)	4(0)	0.89(1.08)	4(0)	0.89(1.08)	4(0)	0.02(0.14)	4(0)	0.02(0.14)
		InvMQ	2.22(1.04)	2.38(1.79)	2.22(1.04)	2.37(1.78)	4(0)	0.41(0.75)	4(0)	0.41(0.75)	4(0)	0(0)	4(0)	0(0)
		True	1.21(0.95)	0.48(0.78)	1.21(0.95)	0.48(0.78)	3.99(0.1)	0.1(0.3)	3.99(0.1)	0.1(0.3)	4(0)	0(0)	4(0)	0(0)
	Exp(1)	None(Indep)	3.36(0.7)	5.77(2.3)	3.33(0.73)	5.54(1.98)	4(0)	3.68(1.93)	4(0)	3.68(1.93)	4(0)	1.31(1.18)	4(0)	1.31(1.18)
		I	2.59(0.98)	3.3(1.8)	2.59(0.98)	3.3(1.8)	4(0)	1.39(1.36)	4(0)	1.39(1.36)	4(0)	0.09(0.29)	4(0)	0.09(0.29)
		Gauss	2.36(0.97)	2.98(1.69)	2.36(0.97)	2.95(1.67)	4(0)	0.67(0.95)	4(0)	0.67(0.95)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.07(0.99)	2.25(1.61)	2.07(0.99)	2.24(1.59)	4(0)	0.71(1.04)	4(0)	0.71(1.04)	4(0)	0(0)	4(0)	0(0)
		True	1.56(1.05)	1.32(1.38)	1.56(1.05)	1.32(1.38)	4(0)	0.51(0.7)	4(0)	0.51(0.7)	4(0)	0(0)	4(0)	0(0)
	$\mathrm{Mat}_{3/2}$(2.5)	None(Indep)	3.48(0.67)	5.79(2.32)	3.46(0.67)	5.59(2.25)	4(0)	2.71(2.13)	4(0)	2.71(2.13)	4(0)	1.25(1.37)	4(0)	1.25(1.37)
		I	2.71(0.9)	2.94(1.75)	2.71(0.9)	2.94(1.75)	4(0)	1.22(1.28)	4(0)	1.22(1.28)	4(0)	0.1(0.39)	4(0)	0.1(0.39)
		Gauss	2.57(0.92)	2.59(1.6)	2.57(0.92)	2.57(1.58)	4(0)	0.69(1)	4(0)	0.69(1)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.25(0.9)	2.24(1.68)	2.25(0.9)	2.23(1.68)	4(0)	0.62(0.97)	4(0)	0.62(0.97)	4(0)	0(0)	4(0)	0(0)
		True	0.52(0.58)	0.12(0.41)	0.52(0.58)	0.12(0.41)	2.81(1.01)	0.04(0.2)	2.81(1.01)	0.04(0.2)	4(0)	0(0)	4(0)	0(0)
	$\mathrm{Mat}_{3/2}$(1.5)	None(Indep)	3.24(0.74)	6.39(2.65)	3.23(0.74)	6.21(2.5)	4(0)	3.4(2.23)	4(0)	3.4(2.23)	4(0)	1.47(1.42)	4(0)	1.47(1.42)
		I	2.63(0.91)	3.15(1.93)	2.63(0.91)	3.15(1.93)	4(0)	1.47(1.51)	4(0)	1.47(1.51)	4(0)	0.05(0.22)	4(0)	0.05(0.22)
		Gauss	2.39(0.84)	2.93(1.93)	2.39(0.84)	2.9(1.88)	4(0)	0.68(0.99)	4(0)	0.68(0.99)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.03(1)	2.26(1.57)	2.03(1)	2.25(1.55)	4(0)	0.43(0.82)	4(0)	0.43(0.82)	4(0)	0(0)	4(0)	0(0)
		True	0.69(0.73)	0.2(0.53)	0.69(0.73)	0.2(0.53)	3.56(0.61)	0.17(0.4)	3.56(0.61)	0.17(0.4)	4(0)	0(0)	4(0)	0(0)
	$\mathrm{Mat}_{5/2}$(2.5)	None(Indep)	3.33(0.74)	5.59(2.19)	3.33(0.74)	5.41(2.08)	4(0)	3.2(2.56)	4(0)	3.2(2.56)	4(0)	1.34(1.51)	4(0)	1.34(1.51)
		I	2.72(1)	3.06(1.97)	2.72(1)	3.06(1.97)	4(0)	0.75(1.19)	4(0)	0.75(1.19)	4(0)	0.1(0.33)	4(0)	0.1(0.33)
		Gauss	2.53(1.04)	2.57(1.81)	2.53(1.04)	2.57(1.81)	4(0)	0.41(0.81)	4(0)	0.41(0.81)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.25(0.97)	2.17(1.68)	2.25(0.97)	2.16(1.67)	4(0)	0.38(0.91)	4(0)	0.38(0.91)	4(0)	0(0)	4(0)	0(0)
		True	0.34(0.57)	0.05(0.26)	0.34(0.57)	0.05(0.26)	1.82(0.83)	0.05(0.22)	1.82(0.83)	0.05(0.22)	4(0)	0(0)	4(0)	0(0)
	$\mathrm{Mat}_{5/2}$(1.5)	None(Indep)	3.2(0.71)	5.66(2.16)	3.19(0.71)	5.56(2.08)	4(0)	3.57(2.38)	4(0)	3.56(2.37)	4(0)	1.3(1.32)	4(0)	1.3(1.32)
		I	2.39(1.03)	2.86(1.99)	2.39(1.03)	2.86(1.99)	4(0)	1.39(1.41)	4(0)	1.39(1.41)	4(0)	0.07(0.26)	4(0)	0.07(0.26)
		Gauss	2.35(0.91)	2.84(1.76)	2.35(0.91)	2.82(1.75)	4(0)	0.88(1.11)	4(0)	0.88(1.11)	4(0)	0(0)	4(0)	0(0)
		InvMQ	2.2(0.89)	2.09(1.52)	2.2(0.89)	2.09(1.52)	4(0)	0.36(0.72)	4(0)	0.36(0.72)	4(0)	0(0)	4(0)	0(0)
		True	0.28(0.51)	0(0)	0.28(0.51)	0(0)	2.38(0.91)	0.04(0.2)	2.38(0.91)	0.04(0.2)	4(0)	0(0)	4(0)	0(0)
	Gauss(1.5)	None(Indep)	3.15(0.88)	6.37(1.95)	3.14(0.88)	6.24(1.83)	4(0)	3.94(1.92)	4(0)	3.94(1.92)	4(0)	1.78(1.54)	4(0)	1.78(1.54)
		I	2.44(1.02)	3.35(1.98)	2.44(1.02)	3.34(1.96)	4(0)	1.25(1.14)	4(0)	1.25(1.14)	4(0)	0.08(0.27)	4(0)	0.08(0.27)
		Gauss	2.24(0.95)	2.83(1.61)	2.24(0.95)	2.82(1.57)	4(0)	0.63(0.73)	4(0)	0.63(0.73)	4(0)	0.02(0.14)	4(0)	0.02(0.14)
		InvMQ	2.18(0.9)	2.2(1.41)	2.18(0.9)	2.19(1.39)	4(0)	0.59(0.83)	4(0)	0.59(0.83)	4(0)	0(0)	4(0)	0(0)
		True	2.78(0.87)	3.66(1.74)	2.75(0.87)	3.61(1.69)	4(0)	1.84(1.54)	4(0)	1.84(1.54)	4(0)	0.17(0.45)	4(0)	0.17(0.45)
	Gauss(2.5)	None(Indep)	3.13(0.75)	6.64(2.37)	3.12(0.76)	6.45(2.24)	4(0)	4.38(2.06)	4(0)	4.38(2.06)	4(0)	1.14(1.07)	4(0)	1.14(1.07)
		I	2.6(0.96)	3.62(1.66)	2.6(0.96)	3.62(1.66)	4(0)	1.42(1.36)	4(0)	1.42(1.36)	4(0)	0.02(0.14)	4(0)	0.02(0.14)
		Gauss	2.44(0.86)	2.9(1.47)	2.44(0.86)	2.86(1.44)	4(0)	0.73(0.92)	4(0)	0.73(0.92)	4(0)	0.01(0.1)	4(0)	0.01(0.1)
		InvMQ	2.17(0.94)	2.36(1.52)	2.17(0.94)	2.36(1.52)	4(0)	0.63(0.85)	4(0)	0.63(0.85)	4(0)	0(0)	4(0)	0(0)
		True	3(0.83)	4.62(1.85)	3(0.83)	4.62(1.85)	4(0)	3.53(2.23)	4(0)	3.53(2.23)	4(0)	1.28(1.05)	4(0)	1.28(1.05)

Table 7: ***Monte Carlo Mean (Standard dev.) for the selected number of nonzero covariates using 100 datasets under both Independent and Dependent setup using spatially weighted group LASSO and adaptive Group LASSO algorithms when*** $J = 35$ **(**$t = 3$**)**

References

[1] Anh, V. V. and Lunney, K. E. (1995), "Parameter estimation of random fields with long-range dependence," *Mathematical and computer modelling*, 21(9), 67-77.

[2] Boissy, Y., Bhattacharyya, B. B., Li, X. and Richardson, G.D. (2005), "Parameter estimates for fractional autoregressive spatial processes", *The Annals of Statistics*, 33, 2553-2567.

[3] Hastie, T. J. and Tibshirani, R. J. (1990)," Generalized additive models," *CRC Press.*

[4] Huang, J., Horowitz, J. L. and Wei, F. (2010b), "Variable selection in nonparametric additive models," *The Annals of Statistics*, 38, 2282-2313.

[5] Van der Vart, A. W. and Wellner, J.A. (1996), "Weak Convergence and Emperical Processes:With Application to Statistics," *Springer, New York.* MR 13851671.