Observational Cosmology: Homework No.3

*Student ID:*__________ *Name:*__________

Note:
1. All the calculations should be formulated step by step. Solution with only a number will get **0** score.
2. Write all the solutions in the answer sheets.
3. Be free to search for the references on the internet, but cite them carefully.
4. The "Optional" questions are not necessary to be done. Finish them if possible, to earn additional scores.

A. Short questions

1. A beam of light from a laser has a wavelength of 600nm, what is the frequency of this light? What is the energy of a photon whose wavelength is 600 nm?

2. Consider an electron in an atom which can jump (transit) between different energy levels. Transition A produces light with a wavelength of 600 nm. Transition B involves half as much energy as A. What wavelength of light does it produce?

3. In the laboratory you measure that a spectral line has a wavelength of 700 nm. You observe a distant galaxy and find the same spectral line, but it appears at a wavelength of 710 nm. Use the Doppler equation to find the velocity of this galaxy toward or away from us. (A "+" sign should appear in your answer if the galaxy is moving away from us. A "-" sign means it is moving toward us.)

4. (Einstein's mass-energy relation) Einstein's mass-energy formula $E = mc^2$ tells us that mass can be converted into other forms of energy.

   (1)Does this violate the conservation of energy?

   (2)What amount of mass can be converted to energy needed by a 10 hours walk for a man ($\approx 10^7$ Joules)?

   (3)What amount of energy can the annihilation of a electron-positron pair produce? (The masses of electron and positron are identical)

   (4) The nuclear fusion in the Sun converts 4 hydrogen nuclei to a helium nucleus, and about 0.7% of the mass is "lost" in this process. Based on the luminosity of the Sun, estimate what amount of hydrogen nuclei is fused in the core of the Sun in a second.

5. (Our Sun) The Sun is the nearest star to us so that it is the best example for us to study the world of stars.

   (1) Based on the mass ($M_\odot$) and the size ($R_\odot$) of the Sun, estimate the average density of the Sun.

   (2) The core accounts for half of the total mass of the Sun, while the radius of the core is only 20% of the Sun's radius. Estimate the density of the core in the Sun, and explain why the density in the core can be so high.

   (2) Give a brief summary of the structures of the Sun (give some sketches if possible). Which part emits the most of the visible light received in the Earth? Which parts are the main sources of the Sun's X-ray and ultraviolet radiation?

   (3) Give a brief summary of the types of solar activity and answer what causes the solar activity.

6. (Hertzsprung-Russell diagram) Look at the H-R diagram in page 321 of the textbook *The Essential Cosmic Perspective* and answer the following questions.

   (1) What are the four major groups of stars in the H-R diagram? Which typically appears brighter and which typically appears dimmer? What life stages of stars are these groups on?

(2) For the main-sequence stars, the luminosity(L)-temperature(T) relation can be approximated by a power-law function $L = AT^\alpha$ in a wide range of temperature, where A, α are constants. Based on the H-R diagram, estimate the value of A and α.

(3) The electromagnetic radiation of a star can be well described by the black body radiation. Based on the Stefan-Boltzmann Law $f(T) = \sigma T^4$, and the relation obtained in (2), derive the relation between the radius (r) and the temperature (T) of a star.

(4) The luminosity(L) of a main-sequence star is approximately proportional to the fourth power of its mass(M). Based on what you have known about the Sun (the main-sequence life time of the Sun is about 10 billion years) and the energy source of stars, derive a relation between the main-sequence life time(τ_{life}) and the mass(M) of a star.

(5) For the star Betelgeuse, whose surface temperature is about 3400K, use what you have obtained in (2)(3)(4) to estimate its luminosity, radius, mass, and main-sequence life time. Compare its life time with the age of our universe.

7. The most famous open cluster - the Pleiades (also called Subaru) consists of several thousand stars. The hottest, most luminous main-sequence star in Pleiades has the spectral type B6.

(1) From the spectral type, what can you tell about this star?

(2) Based on the lifetimes of different stars, what can you tell about Pleiades?

B. Questions that need more thinking

1. (Black body radiation) In 1900, Max Planck successfully derived the famous law for black body radiation - the Planck's law. The spectral energy density $I_\lambda(\lambda, T)$ of the electromagnetic radiation emitted by a black body (energy emitted in unit time in unit surface area of the black body, over unit wavelength) is given by

$$I_\lambda(\lambda, T) = \frac{2\pi hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda kT}} - 1},$$

where T is the temperature of the black body in Kelvin (which can be converted from Celsius degree with $T[\text{K}] = T[°\text{C}] + 273.15$), h is the Planck constant, c is the speed of light, and k is the Boltzmann constant.

(1)Use the Planck's law and approximation, derive the Wien's law, $I_\lambda(\lambda, T) = \frac{A}{\lambda^5} e^{-\frac{B}{\lambda T}}$, for short-wavelength radiation with low temperature, and derive the Rayleigh-Jeans law, $I_\lambda(\lambda, T) = \frac{CT}{\lambda^4}$, for long-wavelength radiation with high temperature. Here A, B, C are constants which can be expressed by physical constants h, c and k. (hint: $e^x \approx 1 + x$ for $|x| \ll 1$)

(2)Use the Planck's law, derive the Wien's displacement law $\lambda_{\text{max}} = \frac{2.9 \times 10^6}{T}$nm, where λ_{max} is the wavelength at which the $I_\lambda(\lambda, T)$ reaches its maximum.

(3) (Optional) Use the Planck's law, derive the Stefan-Boltzmann Law $f(T) = \sigma T^4$. That is, the total energy emission by a black body per unit time in unit surface area. Here $\sigma = 5.7 \times 10^{-8}$J/(s m^2 K^4). (refer to any resource in the internet if needed, but cite your references carefully)

2. (Hydrogen atom) A hydrogen atom can be viewed a proton-electron pair. The proton is massive enough so that it can be viewed static without any motion, while the electron move around the proton at a circular orbit(just like the planetary motion around the Sun). The attractive force between the electron and the proton is similar to the inverse square law of gravity

$$f = k\frac{e^2}{r^2},$$

where $e = 1.602 \times 10^{-19}$C is the charge of the electron, $k = 8.988 \times 10^9$Nm^2/C^2 is a constant, and r is the distance between the proton and the electron. The total energy E of the hydrogen atom is the addition of the potential energy V due to electromagnetic interaction and the kinetic energy T of the electron, where

$$V = -k\frac{e^2}{r}, T = \frac{1}{2}mv^2.$$

Here m, v is the mass and velocity of the electron, respectively.

(1) Similar to the planetary motion, derive the velocity of the electron (v), the energy of the hydrogen atom (E), and the angular momentum of the electron motion with respect to the proton ($L = rmv$), as a function of radius of the circular orbit (r).

(2) The microscopic world is somehow dominated by the quantum law. That is, some physical quantities in the microscopic world can only be discrete values. Here for the electron motion in hydrogen atom, the angular momentum L can only take discrete values: $L = n\hbar$, where n can be any positive integer, and $\hbar = h/(2\pi)$ is also called Planck constant. Use what you have obtained in question (1), prove that the orbital radius r and the energy E of each orbit, should also be discrete. That is, $r_n = 0.052n^2$ nm and $E_n = -13.6$ eV$/n^2$, where each E_n is called the energy level, n is the index of each energy level, and eV $= 1.602 \times 10^{-19}$Joule.

(3) The transition in hydrogen atom (the jump of the electron between different energy level) will cause the electromagnetic radiation/absorption. Use what you have obtained above and the law of energy conservation, prove that the jump between energy levels m and n will cause the emission/absorption of a photon with wavelength

$$\lambda = \frac{91.0 \text{ nm}}{|\frac{1}{n^2} - \frac{1}{m^2}|}.$$

(4) The Balmer series is a set of spectral lines produced by the emission/absorption of photons of hydrogen atom when the transitions between energy level $m = 2$ and n ($n > 2$) happen. Calculate the corresponding wavelengths of photons for $n = 3, 4, 5, 6$ and for $n = \infty$.

3. (Magnitude system) Astronomers usually use apparent magnitude(m) to describe the apparent brightness of a astronomical light source they observe, which can be formulated as

$$m = -2.5 \log_{10}(F/F_{\text{ref}}),$$

where F is the flux density (energy received by the observer per unit time, per unit area, over unit wavelength) of the radiation, F_{ref} is a reference flux that is fixed in given magnitude system. One of the famous magnitude system is called AB magnitude system, which chooses $F_{\text{ref}} \approx 3.63 \times 10^{-9}$erg s^{-1} cm^{-2} Å^{-1}. (1 erg=10^{-7} Joule)

To describe the intrinsic radiation property of a source, we define the absolute magnitude(M), which is just the apparent magnitude assuming the source is 10 pc away and related to apparent magnitude with

$$m - M = 5 \ \log_{10}(d/10)$$

where d is the distance of the object from the observer, in the unit of pc.

(1) We know that the V-band luminosity of the Sun is 4.64×10^{32} erg s^{-1} and the effective width for V-band is about $\Delta\lambda = 880$Å. What is its V-band apparent magnitude? What is its V-band absolute magnitude?(All in AB magnitude system)

(2) A star cluster is just a group of star which are bound together though gravity. If there is a star cluster containing 200 stars with V-band magnitude $M = 3.3$ and 20 stars with V-band magnitude $M = 0.7$, what is the magnitude of this star cluster?

(3) Assuming in a Universe, there are only two kinds of stars, one with magnitude $M = 4$ and the other kind with $M = 3$, while each kind of stars are uniformly distributed in this Universe with number density $n_1 = n_2 = 1$ pc^{-3}. You, as a poor astronomer, only have a small telescope which can detect source with apparent magnitude brighter than 13. So how many stars can you see with this telescope?

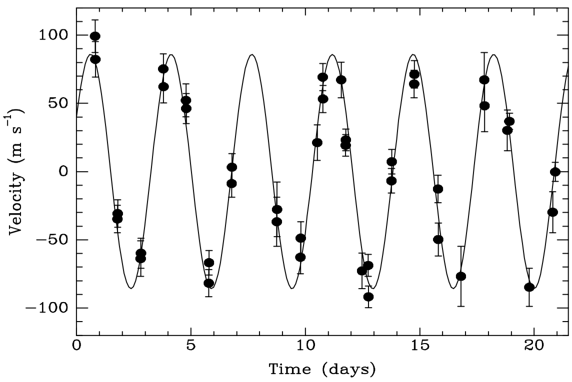

Figure 1: Variation of radial velocity.

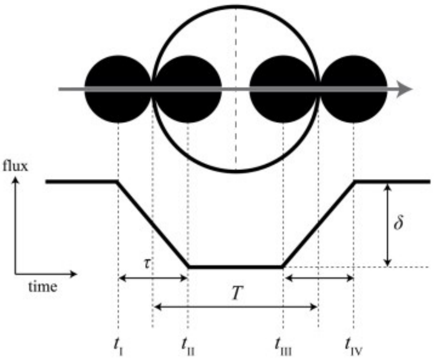

Figure 2: Shadowing of the star by the exoplanet.

4. (Measure the radius of a star) Stars are normally too small to measure their sizes directly, thus we usually call them point sources. However, there is a way to calculate the radius of a star using the property of black body radiation.

   (1) We have learned Stefan-Boltzmann Law and Wien's displacement law in class. Could you use these two laws to derive the relation among luminosity(L, energy emitted per unit time), radius of a star(r) and λ_{max}, i.e. $L = f(r, \lambda_{\text{max}})$.

   (2) When we observe a star on earth, we can only measure its flux, F, which is related to its luminosity by $F = L/(4\pi d^2)$ and d is the distance from that star to us. Now, Proxima Centauri is the nearest star from Sun and we have measured its $\lambda_{\text{max}} = 944\text{nm}$ and $F = 3.2 \times 10^{-11}\text{W/m}^2$, what is the radius of this star?

5. (Optional)(Measure the orbital parameters of exoplanets) Exoplanet searching is now a very hot topic in astronomy. Here we present a method to detect exoplanet with just Newtonian mechanics. Figure 1 shows a radial velocity measurement of a star and the periodic signal means that star is in a circular motion which indicates a possible existence of an exoplanet. We have measured that the stars maximum velocity is $V = 86\text{km/s}$, the period is $T_{\text{orbit}} = 3.5$ days and we can calculate the star's mass from M its spectra type, where $M = 1.1\text{M}_\odot$.

   (1) Under the assumption that the orbit of that star is parallel to our line of sight and $M \gg m$ where m is the mass of the exoplanet, calculate the mass m and orbital radius r_2 of this exoplanet.

   (2) If the orbital plane is inclined to the line of sight by an angle θ, what is the mass of this exoplanet inferred from this observation?

   (3) Actually, there is a way to measure the inclined angle. If the angle is not too large, the exoplanet will go across the line-of-sight between the star and the observer, thus shadow the flux of the star, which is illustrated in Figure 2. There are several reasonable assumptions that $\tau \ll T$, $T \ll T_{\text{orbit}}$ and $r_2 \ll D$, where T_{orbit} is the orbital period of the exoplanet's circular motion and D is the distance of this system to Earth. Besides, we can also infer the radius of the star, R, from the spectra type. Using these knowledge, derive the inclined angle θ with (T, R, V, M and T_{orbit}).