Observational Cosmology: Homework No.4

*Student ID:*__________ *Name:*__________

Note:

- Read the questions carefully, and write all the solutions on the answer sheets.
- All the calculations should be formulated step by step and all the explanations should be clear. Calculation with only a number as the result will get **0** score.
- Be free to search for the references on the internet or to ask others for help, but cite them carefully and make sure that you understand the answer.
- The "Optional" questions are not necessary to be done. Finish them if possible to earn additional scores.

1. (Life as a low-mass star) A low mass ($M < 2M_\odot$) star evolves slower than a massive one. Near the end of its life, it will experience the red-giant stage, and finally become an white dwarf.

   (1) Where does the formation of a proto-star happen? Why are the low temperature and high density of gas necessary for the formation of a proto-star?

   (2) For given temperature and gas density, the minimal mass of a cloud needed for gravitational collapse is given by

   $$M = 18M_\odot\sqrt{\frac{T^3}{n}}$$

   where T is the temperature of gas in K, and n is the number density of gas particles in cm^{-3}. For $T = 15$K and $n = 150\ \mathrm{cm}^{-3}$, calculate the minimal mass for gravitational collapse. What is the minimal mass for $T = 10$ K and $n = 10^5\ \mathrm{cm}^{-3}$?

   (3) What process is happening at the core of a low mass star during its main-sequence stage? Why does it become more luminous while its surface becomes cooler at the red-giant stage? Does a planetary nebula has some relation to planet?

   (4) Why is the ignition temperature of the helium fusion higher than that of the hydrogen fusion? Why is the ignition temperature for carbon fusion even higher?

   (5) Near the end of its life, the Sun's radius will extend nearly to Earth's orbit. Estimate the average matter density of the Sun at that time. How does that density compare with the density of water ($1\ \mathrm{g/cm^3}$)? How does it compare with the density of Earth's atmosphere at sea level (about $10^{-3}\ \mathrm{g/cm^3}$)?

   (6) Can the corpse - the white dwarf of a dead low mass star have chance to regain material and blaze back to life? What is the reason for a white dwarf to produce novae and supernova?

   (7) The Algol binary system consists of a 3.7 $\mathrm{M}_\odot$ star and a 0.8 $\mathrm{M}_\odot$ star with an orbital period of 2.87 days. Use Newton's version of Kepler's third law to calculate the orbital separation of the system. How does that separation compare with the typical size of a red giant star? Why is the more massive one still in the main-sequence stage while the less massive one has already been a sub-giant?

2. (White dwarf and neutron star) A neutron star is the corpse of a massive star. After the release of huge amounts of energy in the supernova at the final stage of a massive star, a dense, high-temperature and fast-rotating core will remain in the center, and that is a neutron star.

   (1) The evolution track of a star is mainly determined by the mass when it is born. What is the condition for the formation of a neutron star? Explain why this condition is necessary. Will our Sun eventually become a neutron star?

   (2) What are the differences between the stellar evolution track that will end up with a neutron star and that will end up with a white dwarf? What is the differences of the nuclear fusions in the core in the main-sequence stage between a star with solar mass and a star 3 times more massive than the sun?

(3) The gravity is strong both in the white dwarf and the neutron star. What is the source of the pressure that push the matter outward against the gravity in a white dwarf, and what is the source of pressure in a neutron star? Can this type of pressure be enough to support a neutron star that is more massive than $3M_\odot$?

(4) The typical radius R and mass M of a neutron star is $R = 10$ km, $M = 1\ M_\odot$. Calculate the average mass density ρ of the neutron star. Compare it with the average density of the Sun (radius $R_\odot \approx 0.01$ AU) and that of a white dwarf (similar mass, radius $R \approx R_\mathrm{e}$ where R_e is the radius of the Earth).

(5) The formation of a neutron star is accompanied with the supernova at the final stage of the life of a massive star, and enormous energy will be emitted in this process. Use the formula for gravitational potential energy $U = GM^2/r$, where r and M are, respectively, the radius and mass of a neutron star, to estimate the amount of potential energy released by a massive-star supernova explosion. Compare it with the energy released by the Sun during its entire main-sequence lifetime.

(6) Amazing amount of energy can be released by a massive-star supernova. What influence would you expect from such a strong energy burst on the galaxy which hosts this star?

(7) What is the difference between a white dwarf supernova and a massive-star supernova (e.g. their progenitor stars, the reasons of the explosions, their remnants, their observational phenomena, and their heavy element outputs)?

(8) What is the nature of a pulsar? Are all neutron stars pulsars? Why can a neutron star rotate fast? Why do the rotations of pulsars gradually slow down?

(9) Theoretical models of the slowing of pulsars predict that the age of a pulsar is approximately equal to $p/(2r)$, where p is the pulsar's current period and r is the rate at which the period is slowing with time. Observations of the pulsar in the Crab Nebula show that it pulses 30 times per seconds, so $p = 0.0333$ second, but the time interval between pulses is growing longer by 4.2×10^{-13} second with each passing second, so $r = 4.2 \times 10^{-13}$ second per second. Using that information, estimate the age of the Crab pulsar. How does your estimate compare with the true age of the pulsar, which was born in the supernova observed in *A.D.* 1054?

(10) What are the differences in observation between an isolated neutron star and that in a close neutron binary system?

3. (Black holes) Black hole is the most compact object in the nature, and it is also the most powerful energy source in some cases.

   (1) The Schwarzschild radius R_S gives the radius of the event horizon of a black hole. For a black hole with mass $M = 1M_\odot$, its radius R_S is about 3 km. What is the radius R_S of a black hole with mass $M = 10M_\odot$? What about $M = 10^6 M_\odot$?

   (2) (A water black hole) A clump of matter does not need to be extraordinarily dense in order to have an escape velocity greater than the speed of light, as long as its mass is large enough. You can use the formula for the Schwarzschild radius R_S to calculate the volume inside the event horizon of a black hole of mass M. What does the mass of a black hole need to be in order for its mass divided by its volume $\frac{4}{3}\pi R_S^3$ to be equal to the density of water (1 g/cm^3)?

   (3) (A visit to black hole) Suppose you and your best friend visit a black hole with mass $10M_\odot$ by a spaceship, circling on a orbit well above the event horizon. Unfortunately, when going out the spaceship for some repairing work, the rope connecting your friend and the spaceship is somehow broken. What is worse, a mysterious power pushes your friend towards the black hole. How will your friend appear to you as he falls into the black hole? Can you see him across the event horizon of the black hole? If your friend is lucky enough to survive such a disaster, what will he feel and what will he observe about you before he cross the event horizon? Can he use his iPhone X to connect you after across the event horizon?

4. Write a short paragraph to describe the main processes of star-gas-star recycle.

5. Write a short paragraph to describe the main processes of formation of our Milky Way?

6. (Stellar mass distribution) Observation of young star clusters shows that stars with different masses appear in the clusters with different probability.

   (1) Which type of stars would you expect to be more abundant, the massive ones or the low-mass ones?

   (2) Suppose $f(M)dM$ is the fraction of stars with masses between M and $M + dM$ at birth in a cluster. In observation, if you simply count the number of main-sequence stars of mass M among all you have observed, and calculate the fraction of stars with masses between M and $M + dM$, $f'(M)$, would you expect $f'(M) = f(M)$? Why?

   (3) When born from a proto-star, the mass of a star can neither be too high nor too low (typically the mass of a star is between $M_{\text{min}} = 0.08\ M_\odot$ and $M_{\text{max}} = 300\ M_\odot$). Explain the reason why stellar mass has these limitations?

   The initial mass distribution $\xi(M/M_\odot)$ of stars gives the probability that a star will be born per unit mass range (in units of $M_\odot$). That means $\xi(M/M_\odot)\mathrm{d}(M/M_\odot)$ is the probability of a star to have mass in the range $[M/M_\odot,\ (M + \mathrm{d}M)/M_\odot]$. Observations show that over a wide range of stellar mass, the initial mass distribution can be approximated by

   $$\xi(M/M_\odot) = A(M/M_\odot)^{-\alpha}$$

   where $\alpha = 2.35$ is obtained by fitting observational data. For simplicity we assume this formula is valid for the full mass range $[M_{\text{min}},\ M_{\text{max}}]$. Use what you have learned to answer the following questions.

   Questions from (4) to (7) are optional.

   (4) Calculate the normalization factor A so that the total probability in the full mass range is 1.

   (5) For each star with mass between 10 and 300 solar mass, how many stars with masses between 2 and 10 solar mass would you expect, and how many stars with masses between 0.08 and 2 solar mass would you expect?

   The mass function $F(M/M_\odot)$ of stars in a cluster is the number of main-sequence stars per unit mass range. Now consider a simple case that the stars form in the cluster at a constant rate. That means, Φ, the stellar mass forms per unit time, is a constant over the cosmic time since the beginning of the star formation in this cluster.

   (6) If all stars can live forever without death, calculate $F(M/M_\odot)$ at the time T since the beginning of star formation in this cluster.

   (7) In reality, because of the running out of hydrogen in their cores, stars will gradually leave the main sequence stage. Use what you have obtained about the relation between the life time τ_{ms} of a main sequence star and its mass M in the previous homework (Homework No.3, Question A.6), and calculate the real $F(M/M_\odot)$ at time T since the beginning of star formation in this cluster.

7. (Modified Newtonian Dynamics instead of dark matter) Surely we always talk about dark matter these days. However, since we haven't detected dark matter particles yet, there is another possibility to explain the phenomenons where we used dark matter to explain, such as the flat rotation curve of galaxies.

   (1) We know that the distance between Earth and sun is 1 AU, and the orbit velocity is 30 km/s, what is the acceleration of Earth?

   (2) Another fact is that the sun is 8 kpc away from the center of Milky Way and the orbit velocity of sun is 220 km/s, what is the acceleration of the sun?

   (3) If we modify the Newton's second law so that when the acceleration a is very small (like $10^{-10}\,\text{m/s}^2$), it approaches $F = ma^2/a_0$ where $a_0 = 1 \times 10^{-10}\,\text{m/s}^2$. Assuming most of the matter in a galaxy is concentrated in the center, what is the rotation curve now (rotation curve means the relation between circular velocity, v, of a star around the galaxy at the distance, r, to the center of the galaxy)?

   (4) (optional) Another way to produce flat rotation curve is to modify the gravitational force, $F = GMm/r^2$, when r is very large. Can you find this kind of gravitational law to reproduce the observed flat rotation curve from a point mass?