3. Nonlinear Systems of Two Elliptic Partial Differential Equations

3.1. Reaction-Diffusion and Mathematical Biology Systems of the Form $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=F(u,w)$, $\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=G(u,w)$

Preliminary comments. Similar systems of equations are encountered in stationary problems of the theory of heat and mass transfer in reacting media, the theory of chemical reactors, combustion theory, mathematical biology, and biophysics.

The functions $f(\varphi)$, $g(\varphi)$, and $g(\varphi)$ appearing below are arbitrary functions of an argument $\varphi=\varphi(u,w)$; the equations are arranged in order of complicating this argument.

1. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=uf(au-bw)+g(au-bw)$, $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=wf(au-bw)+h(au-bw)$.

2. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=e^{\lambda u}f(\lambda u-\sigma w)$, $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=e^{\sigma w}g(\lambda u-\sigma w)$.

3. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=uf\left(\dfrac{u}{w}\right)$, $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=wg\left(\dfrac{u}{w}\right)$.

4. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=uf\left(\dfrac{u}{w}\right)+\dfrac{u}{w}h\left(\dfrac{u}{w}\right)$, $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=wg\left(\dfrac{u}{w}\right)+h\left(\dfrac{u}{w}\right)$.

5. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=u^n f\left(\dfrac{u}{w}\right)$, $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=w^n g\left(\dfrac{u}{w}\right)$.

6. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=uf(u^n w^m)$, $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=wg(u^n w^m)$.

7. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=uf(u^2+w^2)-wg(u^2+w^2)$, $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=wf(u^2+w^2)+ug(u^2+w^2)$.

8. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=uf(u^2-w^2)+wg(u^2-w^2)$, $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=wf(u^2-w^2)+ug(u^2-w^2)$.

3.2. Other Systems

9. $ax\dfrac{\partial u}{\partial x}+ay\dfrac{\partial u}{\partial y}=\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}-f(u,w)$, $ax\dfrac{\partial w}{\partial x}+ay\dfrac{\partial w}{\partial y}=\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}-g(u,w)$.

10. $\dfrac{\partial u}{\partial t}=\dfrac{\partial}{\partial x}\left[f\left(t,\dfrac{u}{w}\right)\dfrac{\partial u}{\partial x}\right]+ug\left(t,\dfrac{u}{w}\right)$, $\dfrac{\partial w}{\partial t}=\dfrac{\partial}{\partial x}\left[f\left(t,\dfrac{u}{w}\right)\dfrac{\partial w}{\partial x}\right]+wh\left(t,\dfrac{u}{w}\right)$.

