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Functional Separation of Variables in Nonlinear PDEs: General Approach, New Solutions of Diffusion-Type Equations

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Abstract: The study gives a brief overview of existing modifications of the method of functional separation of variables for nonlinear PDEs. It proposes a more general approach to the construction of exact solutions to nonlinear equations of applied mathematics and mathematical physics, based on a special transformation with an integral term and the generalized splitting principle. The effectiveness of this approach is illustrated by nonlinear diffusion-type equations that contain reaction and convective terms with variable coefficients. The focus is on equations of a fairly general form that depend on one, two or three arbitrary functions (such nonlinear PDEs are most difficult to analyze and find exact solutions). A lot of new functional separable solutions and generalized traveling wave solutions are described (more than 30 exact solutions have been presented in total). It is shown that the method of functional separation of variables can, in certain cases, be more effective than (i) the nonclassical method of symmetry reductions based on an invariant surface condition, and (ii) the method of differential constraints based on a single differential constraint. The exact solutions obtained can be used to test various numerical and approximate analytical methods of mathematical physics and mechanics.

Keywords: functional separation of variables; generalized separation of variables; exact solutions; nonlinear reaction-diffusion equations; nonlinear partial differential equations; equations of mathematical physics; splitting principle; nonclassical method of symmetry reductions; invariant surface condition; differential constraints

Contents

1. Introduction

1.1. A Brief Overview of Modifications of the Method of Functional Separation of Variables

The methods of generalized and functional separation of variables (and their various modifications) are among the most effective methods for constructing exact solutions to various classes of nonlinear equations of mathematical physics and mechanics (including partial differential equations of fairly general forms that involve arbitrary functions). In [1–36], many exact solutions to equations of heat and mass transfer theory, wave theory, hydrodynamics, gas dynamics, nonlinear optics, and mathematical biology were obtained using these methods.

To be specific, we will consider nonlinear PDEs of mathematical physics with two independent variables

$$F(x, t, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0, \tag{1}$$

where $u = u(x, t)$ is the unknown function.

The methods of generalized and functional separation of variables are based on setting a priori a structural form of u that depends on several free functions (the specific form of these functions is determined subsequently by analyzing the arising functional-differential equations).

Exact solutions in the form of the sum or product of two functions that depend on different arguments,

$$u = \xi(x) + \eta(t) \quad \text{or} \quad u = \xi(x)\eta(t),$$

are called ordinary separable solutions. Examples of nonlinear PDEs with such solutions can be found in [10,18,23].

Often (in a narrow sense) the term 'solution with functional separation of variables' (or 'functional separable solution') is used for exact solutions of the form (e.g., see [1–4,7,8,18,22,23])

$$u = \varphi(z), \quad z = \xi(x) + \eta(t), \tag{2}$$

where the functions $\varphi(z)$, $\xi(x)$, and $\eta(t)$ are determined in a subsequent analysis. Sometimes the external function $\varphi(z)$ is specified from a priori considerations, while the internal functions $\xi(x)$ and $\eta(t)$ are to be found [19,23].

Importantly, in functional separation of variables, the search for exact solutions of the form $u = \varphi(\xi(x)\eta(t))$ and $u = \varphi(\xi(x) + \eta(t))$ leads to the same results, since $\varphi(\xi(x)\eta(t)) = \varphi_1(\xi_1(x) + \eta_1(t))$, where $\varphi_1(z) = \varphi(e^z)$, $\varphi_1(x) = \ln \varphi(x)$, and $\psi_1(t) = \ln \psi(t)$ (here, without loss of generality, it is assumed that $\varphi > 0$ and $\psi > 0$).

Generalized traveling wave solutions of the form $u = \varphi(z)$, where $z = \xi(t)x + \eta(t)$ are treated as solutions with functional separation of variables [18,23].

In [10,17,20,22], the representation of functional separable solutions in implicit form

$$\psi(u) = \xi(x) + \eta(t), \tag{3}$$

was used. All three functions $\psi(u)$, $\xi(x)$, and $\eta(t)$ were to be found. More complex functionally separable solutions of the form $u = \varphi(z)$, where $z = \eta_1(t)\xi(x) + \eta_2(t)$, were considered in [9,23,37].

The studies [33–35] described a new direct method for constructing exact solutions with functional separation of variables. It is based on an implicit integral representation of solutions in the form

$$\int \zeta(u)\,du = \xi_1(x)\eta(t) + \xi_2(x),\qquad(4)$$

where the functions $\zeta(u)$, $\xi_1(x)$, $\xi_2(x)$, and $\eta(t)$ are determined by the splitting method in the subsequent analysis. This method allowed to find more than 40 exact solutions of nonlinear reaction-diffusion equations and wave type equations with variable coefficients involving one or more arbitrary functions. In [36], it was shown that some of the solutions given in [34,35] cannot be obtained using the nonclassical method of symmetry reductions [38–45] (see also [18,23]) based on the use of the invariant surface condition (a first-order differential constraint equivalent to the relation (4)).

Please note that constructing solutions in implicit form with the integral term (4) often allows us to reduce the order of the resulting functional-differential equations [33,34].

In the general case, the term 'functional separable solution' regarding nonlinear PDEs (1) will be used for exact solutions that can be represented as

$$u = \varphi(z),\quad z = Q(x,t),\qquad(5)$$

where the desired functions $\varphi(z)$ and $Q(x,t)$ are described respectively by overdetermined systems of ODEs and PDEs. In the simplest cases, each of these functions can be described by a single equation. Representation (5) was used in [30–32] to construct exact solutions with functional separation of variables to some classes of nonlinear reaction-diffusion, convective-diffusion, and wave type equations.

It is necessary to distinguish between direct and indirect functional separation of variables based on one of the representations of solutions (2), (3), (4), or (5). At the first stage of direct functional separation of variables, the representation of solution is substituted into the original PDE, after which the resulting equation is analyzed (e.g., see [23,30,31,33–35]). At the first stage of indirect functional separation of variables, the representation of solution is replaced by one or more equivalent differential constraints, and then the overdetermined system of PDEs obtained in this way is analyzed for compatibility (e.g., see [9,20,22,36]).

To construct exact solutions of nonlinear partial differential equations, this paper proposes to use a direct method based on a special transformation with an integral term as well as the generalized splitting principle. This approach is technically simpler and more convenient than finding a solution in the form (5); it generalizes the dependence (4) and allows one to find various solutions in a uniform manner without specifying their structure a priori.

1.2. The Concept of 'Exact Solution' for Nonlinear PDEs

In what follows, the term 'exact solution' regarding nonlinear partial differential equations is used in the following cases:

(i)   the solution is expressible in terms of elementary functions;
(ii) the solution is expressible in terms of elementary functions, functions included in the equation in question, and indefinite or/and definite integrals;
(iii) the solution is expressible in terms of solutions to ordinary differential equations or systems of such equations.

Combinations of cases (i) and (iii) as well as (ii) and (iii) are also allowed.

Case (i) is especially isolated from the more general case (ii) as its simplest variant. In cases (i) and (ii), an exact solution can be represented in explicit, implicit or parametric form.

2. Direct Functional Separation of Variables. General Approach

2.1. Method Description. The Generalized Splitting Principle

To construct exact solutions of Equation (1), we first introduce a new dependent variable ϑ using the nonlinear transformation

$$\vartheta = \int \zeta(u)\, du. \tag{6}$$

Both functions $\vartheta = \vartheta(x, t)$ and $\zeta = \zeta(u)$ will be found simultaneously in the subsequent analysis. Once these functions are determined, the integral relation (6) will specify an exact solution of the equation in question in implicit form (in some cases, the solution may be represented explicitly).

Differentiating (6) with respect to the independent variables, we find the partial derivatives

$$u_x = \frac{\vartheta_x}{\zeta}, \quad u_t = \frac{\vartheta_t}{\zeta}, \quad u_{xx} = \frac{\vartheta_{xx}}{\zeta} - \frac{\vartheta_x^2 \zeta_u'}{\zeta^3}, \quad u_{xt} = \frac{\vartheta_{xt}}{\zeta} - \frac{\vartheta_x \vartheta_t \zeta_u'}{\zeta^3}, \quad \ldots \tag{7}$$

We assume that after substituting expressions (7) into (1), the resulting equation can be converted to the following form:

$$\sum_{n=1}^{N} \Phi_n \Psi_n = 0, \tag{8}$$

where

$$\begin{aligned} \Phi_n &= \Phi_n(x, t, \vartheta_x, \vartheta_t, \vartheta_{xx}, \dots), \\ \Psi_n &= \Psi_n(u, \zeta, \zeta_u', \zeta_{uu}'', \dots). \end{aligned} \tag{9}$$

To construct exact solutions of Equations (8) and (9), we use the splitting principle described below.

The generalized splitting principle. We consider linear combinations of two sets of elements $\{\Phi_j\}$ and $\{\Psi_j\}$ included in (8), which are connected by relations

$$\begin{aligned} \sum_{n=1}^{N} \alpha_{ni} \Phi_n &= 0, \quad i = 1, \dots, l; \\ \sum_{n=1}^{N} \beta_{nj} \Psi_n &= 0, \quad j = 1, \dots, m, \end{aligned} \tag{10}$$

where $1 \le l \le N - 1$ and $1 \le m \le N - 1$. The constants α_{ni} and β_{nj} in (10) are chosen so that the bilinear equality (8) is satisfied identically (this can always be done as shown below). Importantly, relations (10) are purely algebraic in nature and are independent of any particular expressions of the differential forms (9).

Once relations (10) are obtained, we substitute the differential forms (9) into them to arrive at systems of differential equations (often overdetermined) for the unknown functions $\vartheta = \vartheta(x, t)$ and $\zeta = \zeta(u)$ that appear in (6).

Remark 1. *Degenerate cases where one or more of the differential forms Φ_n and/or Ψ_n vanish in addition to the linear relations (10) must be treated separately.*

Remark 2. *The main ideas of the direct method of functional separation of variables based on transformation (6) were expressed in the brief note [46], where four exact solutions of a generalized porous medium equation with a nonlinear source were obtained. The present paper demonstrates the effectiveness of this method by constructing exact solutions (more than 30 solutions have been obtained in total) to a nonlinear diffusion-type equation involving several arbitrary functions. In addition, it will be shown that the direct method is more efficient than indirect methods.*

Remark 3. *Bilinear functional-differential equations that are similar in appearance to* (8) *and* (9) *arise when one searches for exact solutions to nonlinear equations of mathematical physics using the methods of generalized and functional separation of variables with a priori given solution structure. However, there is a fundamental difference in this case: the differential forms Φ_n and Ψ_n in* (9) *depend, in view of transformation* (6), *on the same independent variables x and t, whereas when the methods of generalized and functional separation of variables* [18,23,34–36] *(see also* [47]*) are used, the differential forms depend on different independent variables. This circumstance significantly expands the possibilities of constructing exact solutions by switching to equivalent equations (see Section* 2.3 *for details).*

Remark 4. *Instead of transformation* (6), *we can use the transformation $\vartheta = Z(u)$, which leads to slightly more complex equations. The method for constructing functional separable solutions described above is more convenient and is based on a substantial generalization of traveling wave type solutions of various classes of nonlinear PDEs. To illustrate this, consider the nonlinear heat equation*

$$u_t = [f(u)u_x]_x. \tag{11}$$

For arbitrary $f(u)$, Equation (11) *admits the traveling wave solution*

$$u = u(z), \quad z = \kappa x + \lambda t, \tag{12}$$

where κ and λ are arbitrary constants. Substituting (12) *in* (11) *yields the ODE $\lambda u'_z = \kappa^2[f(u)u'_z]'_z$, the integration of which gives its solution in implicit form*

$$\kappa x + \lambda t + C_1 = \kappa^2 \int \frac{f(u)\, du}{\lambda u + C_2}, \tag{13}$$

where C_1 and C_2 are arbitrary constants. On the left-hand side of (13), *z has been replaced with the original variables using* (12).

The representation of the solution in the form (6) *is an essential generalization of the traveling wave solution* (13), *which is carried out as follows:*

$$\kappa x + \lambda t + C_1 \implies \vartheta(x,t), \quad \frac{\kappa^2 f(u)}{\lambda u + C_2} \implies \zeta(u).$$

2.2. Some Formulas Allowing the Satisfaction of Relation (8) *Identically*

1. For any N, equality (8) can be satisfied if all Φ_i but one are put proportional to a selected element Φ_j ($j \neq i$). As a result, we get

$$\begin{aligned} \Phi_i &= -A_i\Phi_j, \quad i = 1, \ldots, j-1, j+1, \ldots, N; \\ \Psi_j &= A_1\Psi_1 + \cdots + A_{j-1}\Psi_j + A_{j+1}\Psi_{j+1} + \cdots + A_N\Psi_N, \end{aligned} \tag{14}$$

where A_i are arbitrary constants. In formula (14), the symbols can be swapped, $\Phi_n \rightleftarrows \Psi_n$.

2. For even N, equality (8) is satisfied if $N/2$ individual pairwise sums $\Phi_i\Psi_i + \Phi_j\Psi_j$ vanish. In this case, we have the relations

$$\Phi_i - A_{ij}\Phi_j = 0, \quad A_{ij}\Psi_i + \Psi_j = 0 \quad (i \neq j),$$

where A_{ij} are arbitrary constants and the indices i and j together take all values from 1 to N.

3. For $N \geq 3$, equality (8) is also satisfied identically if we choose the linear relations

$$\Phi_m - A_m \Phi_{N-1} - B_m \Phi_N = 0, \quad m = 1, 2, \ldots, N-2;$$

$$\Psi_{N-1} + A_1 \Psi_1 + \cdots + A_{N-2} \Psi_{N-2} = 0, \tag{15}$$

$$\Psi_N + B_1 \Psi_1 + \cdots + B_{N-2} \Psi_{N-2} = 0,$$

where A_i and B_i are arbitrary constants. In formulas (15), the symbols can be swapped, $\Phi_n \rightleftarrows \Psi_n$, or simultaneous pairwise substitutions $\Phi_i \rightleftarrows \Phi_j$ and $\Psi_i \rightleftarrows \Psi_j$ can be made.

To construct more complex linear combinations of the form (10) that would identically satisfy the bilinear relation (8) for any N, one can use the formulas for the coefficients α_{ni} and β_{nj} given in the books [18,23] (in sections devoted to generalized separation of variables).

2.3. Possible Generalizations Based on the Use of Equivalent Equations

Other exact solutions of Equation (1) can be obtained if, instead of (8) and (9), we consider equivalent differential equations that reduce to (8) and (9) on the set of functions satisfying relation (6). Indicated below are two classes of such equations, which will be used later in Section 3.3.

1. One can use equations of the form

$$\sum_{n=1}^{N} \widetilde{\Phi}_n \widetilde{\Psi}_n = 0, \quad \widetilde{\Phi}_n = \Phi_n \eta_n(\vartheta), \quad \widetilde{\Psi}_n = \Psi_n / \eta_n(Z), \quad Z = \int \zeta(u)\, du, \tag{16}$$

which preserve the bilinear structure and, by virtue of (6) (i.e., $\vartheta = Z$), are equivalent to Equations (8) and (9) for any functions $\eta_n(\vartheta)$.

2. One can use equations of the form

$$G(x, t, u, \vartheta) - G(x, t, u, Z) + \sum_{n=1}^{N} \Phi_n \Psi_n = 0, \tag{17}$$

which for any functions $G(x, t, u, \vartheta)$ are equivalent to Equations (8) and (9). Furthermore, in Section 3.3, specific examples of using equations of the form (17) for

$$G(x, t, u, \vartheta) = \lambda(x, t, u)\vartheta$$

will be given. The functions G and λ can explicitly depend on ϑ and ζ (and their derivatives) and the functional coefficients of the original PDE (which suggests implicit dependence on the original variables x, t, and u).

In the generic case, applying the splitting principle to Equations (16) and (17) will lead to other exact solutions of the original Equation (1) than applying this principle to Equation (8).

Remark 5. *Further generalizations are also possible. In particular, the sum $\sum_{n=1}^{N} \Phi_n \Psi_n$ in (17) can be replaced with $\sum_{n=1}^{N} \widetilde{\Phi}_n \widetilde{\Psi}_n$, where the tilde quantities are defined in (16). The functions $G(x, t, u, \vartheta)$ and $G(x, t, u, Z)$ can be multiplied by $\eta_{N+1}(\vartheta)/\eta_{N+1}(Z)$ and $\eta_{N+2}(\vartheta)/\eta_{N+2}(Z)$ respectively.*

3. Exact Solutions of Nonlinear Equations of Reaction-Diffusion Type

3.1. The Class of Equations under Consideration. Reduction to the Bilinear Form

We consider a wide class of nonlinear diffusion equations,

$$u_t = [a(x)f(u)u_x]_x + b(x)g(u)u_x + c(x)h(u), \tag{18}$$

which contain reaction and convective terms with variable coefficients.

Please note that the exact solutions of some simpler equations belonging to class (18) can be found, for example, in [8,12,18,19,22,23,30,31,34,40,45,48–61].

Using the method described in Section 2, we further obtain many new exact solutions to equations of the form (18), in which at least two functional coefficients $a(x)$ and $f(u)$ are given arbitrarily (and the others are expressed through them). Below, for brevity, the arguments of the functions included in transformation (6) and Equation (18) will often be omitted.

Having made the transformation (6), we substitute the derivatives (7) in (18). After simple rearrangements we get

$$-\vartheta_t + (a\vartheta_x)_x f + a\vartheta_x^2 \left(\frac{f}{\zeta}\right)'_u + b\vartheta_x g + ch\zeta = 0. \tag{19}$$

For $\zeta = 1$, Equation (19) coincides with the original Equation (18), where $u = \vartheta$. Therefore, at this stage, no solutions are lost.

We introduce the following notation:

$$\begin{aligned}
\Phi_1 &= -\vartheta_t, & \Phi_2 &= (a\vartheta_x)_x, & \Phi_3 &= a\vartheta_x^2, & \Phi_4 &= b\vartheta_x, & \Phi_5 &= c; \\
\Psi_1 &= 1, & \Psi_2 &= f, & \Psi_3 &= (f/\zeta)'_u, & \Psi_4 &= g, & \Psi_5 &= h\zeta.
\end{aligned} \tag{20}$$

As a result, Equation (19) can be represented in the bilinear form (8) with $N = 5$:

$$\sum_{n=1}^{5} \Phi_n \Psi_n = 0. \tag{21}$$

We now turn to the construction of exact solutions of nonlinear equations of the form (18) based on relations (20) and (21) using the approach described in Section 2.1.

3.2. Exact Solutions Obtained by Analyzing Equation (19)

Solution 1. Equation (21) can be satisfied identically if we use the linear relations

$$\begin{aligned}
\Phi_1 &= -\Phi_5, & \Phi_2 &= 0, & k\Phi_3 &= -\Phi_4; \\
\Psi_1 &= \Psi_5, & \Psi_3 &= k\Psi_4,
\end{aligned} \tag{22}$$

where k is an arbitrary constant. Substituting (20) into (22), we arrive at the equations

$$\begin{aligned}
\vartheta_t &= c, & (a\vartheta_x)_x &= 0, & ka\vartheta_x^2 &= -b\vartheta_x; \\
h\zeta &= 1, & (f/\zeta)'_u &= kg.
\end{aligned} \tag{23}$$

The solution of the overdetermined system consisting of the first three Equation (23) has the form

$$\vartheta(x,t) = c_0 t - \frac{b_0}{k} \int \frac{dx}{a(x)} + C_1, \quad b(x) = b_0, \quad c(x) = c_0, \tag{24}$$

where $a(x)$ is an arbitrary function and b_0, c_0, and C_1 are arbitrary constants. The solution of the system consisting of the last two equations in (23) can be written as follows:

$$h = \frac{kG(u) + C_2}{f}, \quad \zeta = \frac{f}{kG(u) + C_2}, \quad G(u) = \int g(u)\, du, \tag{25}$$

where $f(u)$ and $g(u)$ are arbitrary functions. From formulas (24) and (25) for $b_0 = c_0 = 1$ we obtain the equation

$$u_t = [a(x)f(u)u_x]_x + g(u)u_x + \frac{kG(u) + C_2}{f(u)}, \tag{26}$$

which admits a generalized traveling wave solution in the implicit form

$$\int \frac{f(u)\,du}{kG(u) + C_2} = t - \frac{1}{k}\int \frac{dx}{a(x)} + C_1. \tag{27}$$

Please note that Equation (26) contains three arbitrary functions $a(x)$, $f(u)$, and $g(u)$ and two arbitrary constants C_2 and k.

Solution 2. Equation (21) can be satisfied if we take

$$\Phi_1 = -\Phi_4, \quad \Phi_2 = 0, \quad k\Phi_3 = -\Phi_5;$$
$$\Psi_1 = \Psi_4, \quad \Psi_3 = k\Psi_5, \tag{28}$$

where k is an arbitrary constant. Substituting (20) into (28), we arrive at the equations

$$\vartheta_t = b\vartheta_x, \quad (a\vartheta_x)_x = 0, \quad ka\vartheta_x^2 = -c;$$
$$g = 1, \quad (f/\zeta)'_u = kh\zeta. \tag{29}$$

The solutions of the first three equations (29) are

$$\vartheta(x,t) = \lambda t + C_1 \int \frac{dx}{a(x)} + C_2, \quad b(x) = \frac{\lambda}{C_1}a(x), \quad c(x) = -\frac{kC_1^2}{a(x)}, \tag{30}$$

where $a(x)$ is an arbitrary function and C_1, C_2, and λ are arbitrary constants. The last two equations (29) give two functions

$$g(u) = 1, \quad \zeta(u) = \pm f(u)\left(2k\int f(u)h(u)\,du + C_3\right)^{-1/2}, \tag{31}$$

where $f = f(u)$ and $h = h(u)$ are arbitrary functions and C_3 is an arbitrary constant.

Setting $C_1 = 1$ in (30) and (31), we obtain the equation

$$u_t = [a(x)f(u)u_x]_x + \lambda a(x)u_x - \frac{k}{a(x)}h(u), \tag{32}$$

where $a(x)$, $f(u)$, and $h(u)$ are arbitrary functions, while k and λ are arbitrary constants. This equation admits two exact solutions

$$\pm \int f(u)\left(2k\int f(u)h(u)\,du + C_3\right)^{-1/2} du = \lambda t + \int \frac{dx}{a(x)} + C_2, \tag{33}$$

where C_2 and C_3 are arbitrary constants.

Solution 3. Equation (21) can be satisfied by setting

$$\Phi_1 = -k_1\Phi_5, \quad \Phi_2 = -k_2\Phi_5, \quad \Phi_4 = -k_3\Phi_5;$$
$$\Psi_3 = 0, \quad \Psi_5 = k_1\Psi_1 + k_2\Psi_2 + k_3\Psi_4, \tag{34}$$

where k_1, k_2, and k_3 are arbitrary constants. Substituting (20) in (34), we get

$$\vartheta_t = k_1 c, \quad (a\vartheta_x)_x = -k_2 c, \quad b\vartheta_x = -k_3 c;$$
$$(f/\zeta)'_u = 0, \quad h\zeta = k_1 + k_2 f + k_3 g. \tag{35}$$

The solution of the overdetermined system consisting of the first three equations (35) can be represented as

$$\vartheta(x, t) = c_0 k_1 t - c_0 k_2 \int \frac{x \, dx}{a(x)} - C_1 \int \frac{dx}{a(x)} + C_2,$$
$$b(x) = \frac{c_0 k_3 a(x)}{c_0 k_2 x + C_1}, \quad c(x) = c_0, \tag{36}$$

where $a(x)$ is an arbitrary function, while c_0, C_1, and C_2 are arbitrary constants. From the last two equations (35) we obtain

$$h = \frac{k_1}{f} + k_2 + k_3 \frac{g}{f}, \quad \zeta = f, \tag{37}$$

where $f = f(u)$ and $g = g(u)$ are arbitrary functions.

For $c_0 = k_3 = 1$, formulas (36) and (37) lead to the equation

$$u_t = [a(x)f(u)u_x]_x + \frac{a(x)}{k_2 x + C_1} g(u)u_x + \frac{k_1 + k_2 f(u) + g(u)}{f(u)},$$

which has the generalized traveling wave solution

$$\int f(u) \, du = k_1 t - k_2 \int \frac{x \, dx}{a(x)} - C_1 \int \frac{dx}{a(x)} + C_2.$$

Solution 4. Equation (21) holds if we set

$$\Phi_1 = -\Phi_5, \quad \Phi_2 = -k\Phi_4;$$
$$\Psi_1 = \Psi_5, \quad k\Psi_2 = \Psi_4, \quad \Psi_3 = 0, \tag{38}$$

where k is an arbitrary constant. Substituting (20) into (38) yields

$$\vartheta_t = c, \quad (a\vartheta_x)_x = -kb\vartheta_x;$$
$$h\zeta = 1, \quad kf = g, \quad (f/\zeta)'_u = 0. \tag{39}$$

The general solution of the overdetermined system consisting of the first two equations (39) has the form

$$\vartheta(x, t) = c(x)t + s(x),$$
$$c(x) = C_1 \int \exp\left(-k \int \frac{b}{a} \, dx\right) \frac{dx}{a} + C_2,$$
$$s(x) = C_3 \int \exp\left(-k \int \frac{b}{a} \, dx\right) \frac{dx}{a} + C_4, \tag{40}$$

where $a = a(x)$ and $b = b(x)$ are arbitrary functions, while C_1, C_2, C_3, and C_4 are arbitrary constants. The solution of the system consisting of the last three equations (39) is given by

$$g = kf, \quad h = \frac{1}{f}, \quad \zeta = f. \tag{41}$$

Given relations (40) and (41), we obtain the equation

$$u_t = [a(x)f(u)u_x]_x + kb(x)f(u)u_x + \frac{c(x)}{f(u)}, \tag{42}$$

which admits an exact solution in the implicit form

$$\int f(u)\,du = c(x)t + s(x). \tag{43}$$

Here $a(x)$, $b(x)$, and $f(u)$ are arbitrary functions, and the functions $c(x)$ and $s(x)$ are defined in (40). In particular, for $C_2 = \lambda$, $C_1 = 0$, and $k = 1$, we get the equation

$$u_t = [a(x)f(u)u_x]_x + b(x)f(u)u_x + \frac{\lambda}{f(u)}, \tag{44}$$

which has the solution

$$\int f(u)\,du = \lambda t + C_3 \int \exp\left(-\int \frac{b(x)}{a(x)}\,dx\right) \frac{dx}{a(x)} + C_4. \tag{45}$$

Solution 5. Equation (21) can be satisfied by setting

$$\begin{aligned}
\Phi_1 + \Phi_2 + \Phi_4 &= 0, \quad \Phi_3 = -k\Phi_5; \\
\Psi_2 = \Psi_1, \quad \Psi_4 &= \Psi_1, \quad k\Psi_3 = \Psi_5,
\end{aligned} \tag{46}$$

where k is an arbitrary constant. Substituting (20) in (46), we get

$$\begin{aligned}
-\vartheta_t + (a\vartheta_x)_x + b\vartheta_x &= 0, \quad a\vartheta_x^2 = -kc; \\
f = g = 1, \quad k(f/\zeta)_u' &= h\zeta.
\end{aligned} \tag{47}$$

The first two equations (47) admit the solution

$$\vartheta(x,t) = \lambda t + \int r(x)\,dx + C_1, \quad b = \frac{\lambda}{r} - \frac{(ar)_x'}{r}, \quad c = -\frac{ar^2}{k}, \tag{48}$$

where $a = a(x)$ and $r = r(x)$ are arbitrary functions, while λ and C_1 are arbitrary constants. From the last Equation (47) we get $k\zeta^{-3}\zeta_u' = -h$, which gives two solutions

$$\zeta = \pm\left(\frac{2}{k}\int h\,du + C_2\right)^{-1/2}, \tag{49}$$

where $h = h(u)$ is an arbitrary function and C_2 is an arbitrary constant.

Solution 6. Equation (21) holds if we set

$$\begin{aligned}
\Phi_1 = \lambda\Phi_5, \quad \Phi_2 = k_1\Phi_5, \quad \Phi_4 &= k_2\Phi_3; \\
\lambda\Psi_1 + k_1\Psi_2 + \Psi_5 = 0, \quad \Psi_3 &= -k_2\Psi_4,
\end{aligned} \tag{50}$$

where k_1, k_2, and λ are arbitrary constants. Substituting (20) into (50), we obtain

$$\begin{aligned}
\vartheta_t = -\lambda c, \quad (a\vartheta_x)_x = k_1 c, \quad b\vartheta_x &= k_2 a\vartheta_x^2; \\
\lambda + k_1 f + h\zeta = 0, \quad (f/\zeta)_u' &= -k_2 g.
\end{aligned} \tag{51}$$

The solution of the first three equations (51) is expressed as

$$\vartheta(x,t) = -\lambda t + k_1 \int \frac{x\,dx}{a(x)} + C_1 \int \frac{dx}{a(x)} + C_2,$$

$$b(x) = k_2(k_1 x + C_1), \quad c(x) = 1, \tag{52}$$

where $a(x)$ is an arbitrary function, while C_1 and C_2 are arbitrary constants. The solution of the last two equations (51) is given by

$$h = \frac{k_1 f + \lambda}{f}\left(k_2 \int g\,du + C_3\right), \quad \zeta = -f\left(k_2 \int g\,du + C_3\right)^{-1}, \tag{53}$$

where $f = f(u)$ and $g = g(u)$ are arbitrary functions, while C_3 is an arbitrary constant.

Setting $k_1 = k$ and $k_2 = 1$ in (52) and (53), we arrive at the equation

$$u_t = [a(x)f(u)u_x]_x + (kx + C_1)g(u)u_x + \frac{kf(u)+\lambda}{f(u)}G(u), \quad G(u) = \int g(u)\,du + C_3,$$

where $a(x)$, $f(u)$, and $g(u)$ are arbitrary functions, while C_1, C_3, k, and λ are arbitrary constants. This equation admits the exact solution in implicit form

$$\int \frac{f(u)}{G(u)}\,du = \lambda t - k \int \frac{x\,dx}{a(x)} - C_1 \int \frac{dx}{a(x)} - C_2.$$

Solution 7. Equation (21) can be satisfied by setting

$$\Phi_2 = k_1 \Phi_5, \quad \Phi_3 = -k_2^2 \Phi_1, \quad \Phi_4 = -k_3 \Phi_1;$$

$$\Psi_5 = -k_1 \Psi_2, \quad \Psi_1 - k_2^2 \Psi_3 - k_3 \Psi_4 = 0, \tag{54}$$

where k_1, k_2, and k_3 are arbitrary constants. Substituting (20) in (54) yields

$$(a\vartheta_x)_x = k_1 c, \quad a\vartheta_x^2 = k_2^2 \vartheta_t, \quad b\vartheta_x = k_3 \vartheta_t;$$

$$h\zeta = -k_1 f, \quad 1 - k_2^2 (f/\zeta)'_u - k_3 g = 0. \tag{55}$$

The solutions of the first three equations (55) can be represented as

$$\vartheta(x,t) = \lambda t + k_2\sqrt{\lambda}\int \frac{dx}{\sqrt{a}} + C_1, \quad b(x) = \frac{k_3}{k_2}\sqrt{\lambda a}, \quad c(x) = \frac{k_2\sqrt{\lambda}}{2k_1}\frac{a'_x}{\sqrt{a}}, \tag{56}$$

where $a = a(x)$ is an arbitrary constant, while C_1 and λ are arbitrary constants. The solutions of the last two equations (55) are given by

$$g = \frac{1}{k_3}\left(1 + \frac{k_2^2}{k_1}h'_u\right), \quad \zeta = -k_1\frac{f}{h}, \tag{57}$$

where $f = f(u)$ and $h = h(u)$ are arbitrary functions.

Setting $k_1 = k_3 = 1$, $k_2 = 1/\sqrt{\lambda}$, and $C_2 = -C$ in (56) and (57), we arrive at the equation

$$u_t = [a(x)f(u)u_x]_x + \sqrt{a(x)}[\lambda + h'_u(u)]u_x + \frac{1}{2}\frac{a'_x(x)}{\sqrt{a(x)}}h(u), \tag{58}$$

which contains three arbitrary functions $a(x)$, $f(u)$, and $h(u)$ and has the exact solution

$$\int \frac{f(u)}{h(u)} \, du = -\lambda t - \int \frac{dx}{\sqrt{a(x)}} + C. \tag{59}$$

Solution 8. Equation (21) can be satisfied if we take

$$\Phi_1 = -k_1 \Phi_4, \quad \Phi_2 = -k_2 \Phi_4, \quad \Phi_3 = -\Phi_5;$$
$$\Psi_4 = k_1 \Psi_1 + k_2 \Psi_2, \quad \Psi_3 = \Psi_5, \tag{60}$$

where k_1 and k_2 are arbitrary constants. Substituting (20) into (60), we arrive at the equations

$$\vartheta_t = k_1 b \vartheta_x, \quad (a\vartheta_x)_x = -k_2 b \vartheta_x, \quad a\vartheta_x^2 = -c;$$
$$g = k_1 + k_2 f, \quad (f/\zeta)_u' = h\zeta. \tag{61}$$

The solutions of the first three equations (61) are

$$\vartheta(x,t) = \lambda t - \frac{k_2 \lambda}{k_1} \int \frac{x + C_1}{a(x)} \, dx + C_2,$$
$$b(x) = -\frac{a(x)}{k_2(x + C_1)}, \quad c(x) = -\frac{k_2^2 \lambda^2 (x + C_1)^2}{k_1^2 a(x)}, \tag{62}$$

where $a(x)$ is an arbitrary function, while C_1, C_2, and λ are arbitrary constants. The last two equations (61) give two solutions

$$g(u) = k_1 + k_2 f(u), \quad \zeta(u) = \pm f(u) \left(2 \int f(u)h(u) \, du + C_3 \right)^{-1/2}, \tag{63}$$

where $f = f(u)$ and $h = h(u)$ are arbitrary functions and C_3 is an arbitrary constant.

Setting $C_1 = s$, $k_1 = -1$, $k_2 = k$, and $\lambda = k$ in (62) and (63), we obtain the equation

$$u_t = [a(x)f(u)u_x]_x - \frac{a(x)}{x + s}[k + f(u)]u_x - \frac{(x + s)^2}{a(x)}h(u), \tag{64}$$

where $a(x)$, $f(u)$, and $h(u)$ are arbitrary functions, while k and s are arbitrary constants. This equation admits the exact solutions

$$\pm \int f(u) \left(2 \int f(u)h(u) \, du + C_3 \right)^{-1/2} du = kt - \int \frac{x + s}{a(x)} \, dx + C_2, \tag{65}$$

where C_2 and C_3 are arbitrary constants.

In the special case $k = -1$, $f(u) = 1$, and $s = 0$, Equation (64) is reduced to a simpler equation,

$$u_t = [a(x)u_x]_x - \frac{x^2}{a(x)}h(u),$$

which was considered in [34]. Setting $h(u) = 0$, $C_3 = 0$, and $s = 0$ in (64), and renaming $a(x)$ to $xa(x)$, we obtain the equation

$$u_t = [xa(x)f(u)u_x]_x - a(x)[k + f(u)]u_x,$$

whose solutions are

$$\pm \int f(u) \, du = kt - \int \frac{dx}{a(x)} + C_2.$$

Solution 9. Equation (21) holds if we set

$$\Phi_1 + \Phi_3 + k_1\Phi_4 + \Phi_5 = 0, \quad \Phi_2 + k_2\Phi_4 = 0;$$
$$\Psi_3 = \Psi_1, \quad \Psi_4 = k_1\Psi_1 + k_2\Psi_2, \quad \Psi_5 = \Psi_1, \tag{66}$$

where k_1 and k_2 are arbitrary constants. Substituting (20) into (66), we obtain the equations

$$-\vartheta_t + a\vartheta_x^2 + k_1 b\vartheta_x + c = 0, \quad (a\vartheta_x)_x + k_2 b\vartheta_x = 0;$$
$$(f/\zeta)'_u = 1, \quad g = k_1 + k_2 f, \quad h\zeta = 1. \tag{67}$$

In the special case $k_1 = k_2 = 0$, the solution of system (67) leads to the equation

$$u_t = [a(x)f(u)u_x]_x + \left[\lambda - \frac{\beta^2}{a(x)}\right]\frac{u}{f(u)}, \tag{68}$$

where $a(x)$ and $f(u)$ are arbitrary functions, while β and λ are arbitrary constants. This equation admits two exact solutions

$$\int \frac{f(u)}{u}\,du = \lambda t \pm \beta \int \frac{dx}{a(x)} + C_1. \tag{69}$$

Solution 10. Equation (21) can be satisfied if we take

$$\Phi_1 = -k_1\Phi_5, \quad \Phi_2 = -k_2\Phi_3, \quad \Phi_4 = -k_3\Phi_5;$$
$$\Psi_5 = k_1\Psi_1 + k_3\Psi_4, \quad \Psi_3 = k_2\Psi_2, \tag{70}$$

where k_1, k_2, and k_3 are arbitrary constants. Substituting (20) into (70), we arrive at the equations

$$\vartheta_t = k_1 c, \quad (a\vartheta_x)_x = -k_2 a\vartheta_x^2, \quad b\vartheta_x = -k_3 c;$$
$$h\zeta = k_1 + k_3 g, \quad (f/\zeta)'_u = k_2 f. \tag{71}$$

The solutions of the first three equations (71) are

$$\vartheta(x,t) = k_1 t + \frac{1}{k_2}\ln\left(k_2 \int \frac{dx}{a(x)} + C_1\right) + C_2,$$
$$b(x) = -k_3 a(x)\left(k_2 \int \frac{dx}{a(x)} + C_1\right), \quad c(x) = 1, \tag{72}$$

where $a(x)$ is an arbitrary function, while C_1 and C_2 are arbitrary constants. The solutions of the last two equations (71) are given by

$$h(u) = \frac{k_1 + k_3 g(u)}{f(u)}\left[k_2 \int f(u)\,du + C_3\right],$$
$$\zeta(u) = f(u)\left[k_2 \int f(u)\,du + C_3\right]^{-1}, \tag{73}$$

where $f(u)$ and $g(u)$ are arbitrary functions and C_3 is an arbitrary constant.

In particular, setting $a(x) = x^n$, $C_1 = C_2 = 0$, $C_3 = m$, $k_1 = k$, $k_2 = 1 - n$, and $k_3 = 1$ in (72) and (73), we obtain the equation

$$u_t = [x^n f(u)u_x]_x - xg(u)u_x + \frac{k + g(u)}{f(u)}\left[(1-n)\int f(u)\,du + m\right].$$

Solution 11. Equation (21) can be satisfied if we use the relations

$$\Phi_3 = \Phi_1, \quad \Phi_4 = k_1\Phi_1 + k_2\Phi_2, \quad \Phi_5 = \Phi_1;$$
$$\Psi_1 + \Psi_3 + k_1\Psi_4 + \Psi_5 = 0, \quad \Psi_2 + k_2\Psi_4 = 0,$$

(74)

where k_1 and k_2 are arbitrary constants. Substituting (20) in (74) yields

$$a\vartheta_x^2 = -\vartheta_t, \quad b\vartheta_x = -k_1\vartheta_t + k_2(a\vartheta_x)_x, \quad c = -\vartheta_t;$$
$$1 + (f/\zeta)'_u + k_1 g + h\zeta = 0, \quad f + k_2 g = 0.$$

(75)

The first three equations (75) admit two solutions, which are given by

$$\vartheta(x,t) = -t \pm \int \frac{dx}{\sqrt{a}} + C_1, \quad b(x) = \pm k_1\sqrt{a} + \frac{1}{2}k_2 a'_x, \quad c(x) = 1,$$

(76)

where $a = a(x)$ is an arbitrary function and C_1 is an arbitrary constant (in both formulas, the upper or lower signs are taken simultaneously). From the last Equation (75) we get $g = -f/k_2$; then the penultimate equation, which serves to determine the function ζ, is converted to the Abel equation of the second kind

$$\xi\xi'_u + \left(1 - \frac{k_1}{k_2}f\right)\xi + fh = 0, \quad \zeta = f/\xi.$$

(77)

Setting $k_1 = \pm k$ and $k_2 = 1$ in (76) and (77), we obtain the equation

$$u_t = [a(x)f(u)u_x]_x - [k\sqrt{a(x)} + \tfrac{1}{2}a'_x(x)]f(u)u_x + h(u),$$

which has two exact solutions that can be represented in implicit form

$$\int \frac{f(u)}{\xi(u)} \, du = -t \pm \int \frac{dx}{\sqrt{a(x)}} + C_1,$$

(78)

where the function $\xi = \xi(u)$ is described by the Abel equation

$$\xi\xi'_u + [1 \mp kf(u)]\xi + f(u)h(u) = 0.$$

Exact solutions of the Abel equations for various functions $f(u)$ and $h(u)$ can be found in [62].

Solution 12. We set $a = b = c = 1$ in (19) and then make the substitution

$$\vartheta = \bar{\vartheta} + \alpha x + \beta t,$$

(79)

where α and β are free parameters, to obtain

$$-\bar{\vartheta}_t + \bar{\vartheta}_{xx}f + (\bar{\vartheta}_x + \alpha)^2\left(\frac{f}{\zeta}\right)'_u + \bar{\vartheta}_x g - \beta + \alpha g + h\zeta = 0.$$

(80)

Below we give three solutions of Equation (80), which lead to different solutions of the original PDE (18).

1. A particular solution to Equation (80) is sought in the form

$$\bar{\vartheta} = C_1 e^{\lambda t + \gamma x} + C_2, \quad \zeta = f,$$

(81)

where C_1 and C_2 are arbitrary constants. We get

$$C_1(-\lambda + \gamma^2 f + \gamma g)e^{\lambda t + \gamma x} - \beta + \alpha g + h\zeta = 0,$$

which leads to the defining system of equations

$$-\lambda + \gamma^2 f + \gamma g = 0, \quad -\beta + \alpha g + h\zeta = 0. \tag{82}$$

By virtue of the second equality (81), the solutions of these equations are

$$g = \frac{\lambda}{\gamma} - \gamma f, \quad h = \alpha\gamma + \left(\beta - \frac{\alpha\lambda}{\gamma}\right)\frac{1}{f}, \quad \zeta = f. \tag{83}$$

Thus, we arrive at the equation

$$u_t = [f(u)u_x]_x + \left[\frac{\lambda}{\gamma} - \gamma f(u)\right]u_x + \alpha\gamma + \left(\beta - \frac{\alpha\lambda}{\gamma}\right)\frac{1}{f(u)}, \tag{84}$$

which depends on an arbitrary function $f = f(u)$ and admits the exact solution in implicit form

$$\int f(u)\,du = \alpha x + \beta t + C_1 e^{\lambda t + \gamma x} + C_2. \tag{85}$$

Setting $\lambda/\gamma = \sigma$, $\beta - (\alpha\lambda/\gamma) = \mu$, and $\alpha\gamma = \varepsilon$ in (84) and (85), we obtain the more compact equation

$$u_t = [f(u)u_x]_x + [\sigma - \gamma f(u)]u_x + \varepsilon + \frac{\mu}{f(u)}, \tag{86}$$

which has the exact solution

$$\int f(u)\,du = \frac{\varepsilon}{\gamma}x + \left(\mu + \frac{\varepsilon\sigma}{\gamma}\right)t + C_1 e^{\gamma\sigma t + \gamma x} + C_2. \tag{87}$$

2. For $g \equiv 0$, Equation (80) has the steady-state particular solution

$$\bar{\vartheta} = -kx^2 + C, \quad h = \frac{\beta}{f} + 2k, \quad \zeta = f, \tag{88}$$

where $f = f(u)$ is an arbitrary function, while C and k are arbitrary constants. This leads to the PDE [23]

$$u_t = [f(u)u_x]_x + 2k + \frac{\beta}{f(u)}. \tag{89}$$

This equation admits a solution in the implicit form $\int f(u)\,du = -kx^2 + \alpha x + \beta t + C$.

3. For $g \equiv 0$ and $\alpha = 0$, Equation (80) has another steady-state particular solution

$$\bar{\vartheta} = \ln(C_1 x + C_2), \quad h = \beta\frac{F}{f}, \quad \zeta = \frac{f}{F}, \quad F = \int f(u)\,du, \tag{90}$$

which also leads to the equation considered in [23].

Solution 13. In (19) we set $\zeta = f$ and then make the transformation

$$\vartheta = \bar{\vartheta} + \beta t + k \int \frac{dx}{a(x)}, \tag{91}$$

where β and k are free parameters, to obtain

$$-\bar{\vartheta}_t + (a\bar{\vartheta}_x)_x f + b\bar{\vartheta}_x g - \beta + k\frac{b}{a}g + cfh = 0. \tag{92}$$

We are looking for a steady-state solution $\bar{\vartheta} = \bar{\vartheta}(x)$ of Equation (92). After the splitting procedure, we get the equations

$$g = -\mu f + \lambda, \quad h = \gamma + (\sigma/f),$$
$$(a\bar{\vartheta}'_x)'_x - \mu b\bar{\vartheta}'_x + c\gamma - k\mu(b/a) = 0,$$
$$b\lambda\bar{\vartheta}'_x g + \sigma - \beta + k\lambda(b/a) = 0,$$

where μ, λ, γ, and σ are arbitrary constants. These equations admit the solution

$$\lambda = 0, \quad \gamma = k\mu, \quad \sigma = \beta, \quad g(u) = -\mu f(u), \quad h(u) = k\mu + \frac{\beta}{f(u)},$$

$$\bar{\vartheta}(x) = C_1 \int \frac{e^{\mu x}}{a(x)} \, dx + C_2, \quad b(x) = a(x), \quad c(x) = 1,$$

(93)

where C_1, C_2, and μ are arbitrary constants. Taking into account relation (91), we obtain the equation

$$u_t = [a(x)f(u)u_x]_x - \mu a(x)f(u)u_x + \sigma + \frac{\beta}{f(u)},$$

which admits the exact solution

$$\int f(u) \, du = \beta t + \frac{\sigma}{\mu} \int \frac{dx}{a(x)} + C_1 \int \frac{e^{\mu x}}{a(x)} \, dx + C_2.$$

Solution 14. We seek a particular solution to Equation (92) as the product of functions with different arguments

$$\bar{\vartheta} = e^{\lambda t}\xi(x).$$

(94)

As a result, we arrive at the equations

$$-\lambda\xi + (a\xi'_x)'_x f + b\xi'_x g = 0,$$
$$-\beta + k\frac{b}{a}g + cfh = 0.$$

(95)

For $g = \text{const}$, we obtain $f = \text{const}$ and $h = \text{const}$, which corresponds to a linear equation. Therefore, we further assume that $g \neq \text{const}$.

The first Equation (95) is satisfied if we put

$$(a\xi'_x)'_x - Ab\xi'_x = 0, \quad Bb\xi'_x - \lambda\xi = 0, \quad g = B - Af,$$

(96)

where A and B are arbitrary constants ($A \neq 0$). The first two equations (96) involve three functions $a = a(x)$, $b = b(x)$, and $\xi = \xi(x)$, one of which can be considered arbitrary.

Assuming that the function $\xi = \xi(x)$ in (96) is given, we find that

$$a = \frac{1}{\xi'_x}\left(\frac{A\lambda}{B}\int \xi \, dx + C_1\right), \quad b = \frac{\lambda\xi}{B\xi'_x},$$

(97)

If we assume that the function $b = b(x)$ is given, then the solutions of the first two equations (96) can be written as

$$a(x) = b(x)\exp\left(-\frac{\lambda}{B}\int \frac{dx}{b(x)}\right)\left[A\int \exp\left(\frac{\lambda}{B}\int \frac{dx}{b(x)}\right)dx + C_1\right],$$

$$\xi(x) = C_2\exp\left(\frac{\lambda}{B}\int \frac{dx}{b(x)}\right),$$

(98)

where C_1 and C_2 are arbitrary constants ($C_2 \neq 0$).

In particular, for $B = 1$ and $b(x) = 1$, from (98) we find that

$$a(x) = \frac{A}{\lambda} + C_1 e^{-\lambda x}, \quad \xi(x) = C_2 e^{\lambda x},$$

and for $B = 1$ and $b(x) = x$ we get

$$a(x) = \frac{A}{\lambda + 1} x^2 + C_1 x^{1-\lambda}, \quad \xi(x) = C_2 x^\lambda.$$

The last Equation (95) can be satisfied in two cases, which are considered below.

1. For $\beta = 0$, the solution of the last Equation (95) is given by

$$c(x) = k \frac{b(x)}{a(x)}, \quad h(u) = A - \frac{B}{f(u)}, \tag{99}$$

in the derivation of which the last relation in (96) was taken into account. Thus, the equation

$$u_t = [a(x)f(u)u_x]_x + b(x)[B - Af(u)]u_x + k\frac{b(x)}{a(x)}\left[A - \frac{B}{f(u)}\right],$$

where $b(x)$ and $f(u)$ are arbitrary functions, and $a = a(x)$ is expressed via $b = b(x)$ by (98), admits the solution

$$\int f(u)\, du = k \int \frac{dx}{a(x)} + C_2 e^{\lambda t} \exp\left(\frac{\lambda}{B} \int \frac{dx}{b(x)}\right).$$

2. For $k = 0$, the solution of the last Equation (95) is

$$c(x) = 1, \quad h(u) = \frac{\beta}{f(u)}. \tag{100}$$

As a result, we obtain the equation

$$u_t = [a(x)f(u)u_x]_x + b(x)[B - Af(u)]u_x + \frac{\beta}{f(u)},$$

where $b(x)$ and $f(u)$ are arbitrary functions, and $a = a(x)$ is expressed via $b = b(x)$ by (98), which has the solution

$$\int f(u)\, du = \beta t + C_2 e^{\lambda t} \exp\left(\frac{\lambda}{B} \int \frac{dx}{b(x)}\right).$$

Solution 15. Equation (21) can be satisfied if we take Φ_i ($i = 1, 2, 3, 4$) proportional to Φ_5. As a result, we get

$$\Phi_1 = k_1\Phi_5, \quad \Phi_2 = k_2\Phi_5, \quad \Phi_3 = k_3\Phi_5, \quad \Phi_4 = k_4\Phi_5,$$
$$k_1\Psi_1 + k_2\Psi_2 + k_3\Psi_3 + k_4\Psi_4 + \Psi_5 = 0. \tag{101}$$

Substituting (20) in (101) yields

$$\vartheta_t = -k_1 c, \quad (a\vartheta_x)_x = k_2 c, \quad a\vartheta_x^2 = k_3 c, \quad b\vartheta_x = k_4 c;$$
$$k_1 + k_2 f + k_3 (f/\zeta)'_u + k_4 g + h\zeta = 0. \tag{102}$$

Consider two cases.

1. The simplest solution of the first four equations (102),

$$a(x) = b(x) = c(x) = 1, \quad \theta(x,t) = -k_1 t + k_4 x + C_1, \quad k_2 = 0, \quad k_3 = k_4^2,$$

leads to a traveling wave solution of the original reaction-diffusion Equation (18) (this solution will not be discussed here).

2. The first four equations (102) also admit a different solution

$$
\begin{aligned}
a(x) &= x^2, \quad b(x) = x, \quad c(x) = 1, \\
\vartheta(x,t) &= -k_1 t + k_2 \ln x + C_1, \quad k_3 = k_2^2, \quad k_4 = k_2.
\end{aligned}
\tag{103}
$$

Setting $k = k_1$ and $k_2 = 1$ in (103) and using the last equation in (102), we arrive at the reaction-diffusion-type equation

$$
u_t = [x^2 f(u) u_x]_x + x g(u) u_x + h(u),
\tag{104}
$$

where

$$
h(u) = -\frac{\xi(u)}{f(u)} \left[k + f(u) + g(u) + \xi'_u(u) \right], \quad \xi(u) = \frac{f(u)}{\zeta(u)},
\tag{105}
$$

and $f = f(u)$, $g = g(u)$, and $\xi = \xi(u)$ are arbitrary functions. This equation admits the exact invariant solution

$$
\int \frac{f(u)}{\xi(u)} \, du = -kt + \ln x + C_1.
\tag{106}
$$

Please note that the invariant solution (106) of Equation (104) can be obtained in the standard way in the form $u = U(z)$ with $z = -kt + \ln x$ (in this case, relation (105) between f, g, h, and ξ is not used). The function $U(z)$ is described by the ordinary differential equation

$$
[f(U) U'_z]'_z + [f(U) + g(U) + k] U'_z + h(U) = 0.
$$

Solution 16. Equation (21) holds if we set

$$
\begin{aligned}
\Phi_1 &= k_1 \Phi_4 + k_2 \Phi_5, \quad \Phi_2 = -k_3 \Phi_3; \\
\Psi_3 &= k_3 \Psi_2, \quad \Psi_4 = -k_1 \Psi_1, \quad \Psi_5 = -k_2 \Psi_1,
\end{aligned}
\tag{107}
$$

where k_1, k_2, and k_3 are arbitrary constants. Thus, we obtain the equations

$$
\begin{aligned}
\vartheta_t &= -k_1 b \vartheta_x - k_2 c, \quad (a \vartheta_x)_x = -k_3 a \vartheta_x^2; \\
(f/\zeta)'_u &= k_3 f, \quad g = -k_1, \quad h\zeta = -k_2.
\end{aligned}
\tag{108}
$$

The solutions of the first two equations (108) are given by

$$
\begin{aligned}
\vartheta(x,t) &= \lambda t + \frac{1}{k_3} \ln\left(k_3 \int \frac{dx}{a(x)} + C_1 \right) + C_2, \\
b(x) &= -\frac{k_2 c(x) + \lambda}{k_1} a(x) \left(k_3 \int \frac{dx}{a(x)} + C_1 \right),
\end{aligned}
\tag{109}
$$

where $c(x)$ and $c(x)$ are arbitrary functions, while C_1 and C_2 are arbitrary constants. From the last three equations (108) we get

$$
h(u) = -\frac{k_2}{f(u)} \left(k_3 \int f(u) \, du + C_3 \right), \quad \zeta(u) = f(u) \left(k_3 \int f(u) \, du + C_3 \right)^{-1}.
\tag{110}
$$

Substituting $C_1 = C_3 = 0$ and $k_3 = 1$ in (109) and (110), we obtain the equation

$$u_t = [a(x)f(u)u_x]_x + a(x)[c(x) + \lambda]\left(\int \frac{dx}{a(x)}\right)u_x - \frac{c(x)}{f(u)}\int f(u), \tag{111}$$

which has the solution

$$\int f(u)\,du = C_4 e^{\lambda t}\int \frac{dx}{a(x)}, \tag{112}$$

where C_4 is an arbitrary constant. When deriving formula (112), the equality

$$\int f\left(\int f(u)\,du + C_3\right)^{-1}du = \ln\left(\int f(u)\,du + C_3\right) + \text{const}$$

was taken into account. Please note that the diffusion term of Equation (111) vanishes on solution (112), $[a(x)f(u)u_x]_x = 0$.

3.3. Exact Solutions Obtained by Analyzing Equivalent Equations

Now, using the considerations outlined in Section 2.3, we will obtain some other exact solutions to Equation (1). To this end, instead of (8) and (9), we consider equivalent differential equations that reduce to (8) and (9) on the set of functions satisfying relation (6).

Solution 17. Let us return to the class of reaction-diffusion equations of the form (18). Having made substitution (6), instead of Equation (19), we consider the more complex equation

$$-e^{\lambda\vartheta}e^{-\lambda Z}\vartheta_t + (a\vartheta_x)_x f + a\vartheta_x^2\left(\frac{f}{\zeta}\right)_u' + b\vartheta_x g + ch\zeta = 0, \tag{113}$$

where $Z = \int \zeta\,du$ and λ is an arbitrary constant. Equations (19) and (113) are equivalent since, by virtue of transformation (6), the relation $\vartheta = Z$ holds.

Equation (113) can be represented in the bilinear form (21) where

$$\begin{aligned}
\Phi_1 &= -e^{\lambda\vartheta}\vartheta_t, & \Phi_2 &= (a\vartheta_x)_x, & \Phi_3 &= a\vartheta_x^2, & \Phi_4 &= b\vartheta_x, & \Phi_5 &= c; \\
\Psi_1 &= e^{-\lambda Z}, & \Psi_2 &= f, & \Psi_3 &= (f/\zeta)_u', & \Psi_4 &= g, & \Psi_5 &= h\zeta.
\end{aligned} \tag{114}$$

As previously, Equation (21) can be satisfied using relations (22). Substituting (114) into (22), we arrive at the equations

$$\begin{aligned}
e^{\lambda\vartheta}\vartheta_t &= c, & (a\vartheta_x)_x &= 0, & ka\vartheta_x^2 &= -b\vartheta_x; \\
h\zeta &= e^{-\lambda Z}, & (f/\zeta)_u' &= kg,
\end{aligned} \tag{115}$$

which for $\lambda = 0$ coincide with (23). The solution of the overdetermined system consisting of the first three equations (115) has the form

$$\begin{aligned}
\vartheta(x,t) &= \frac{1}{\lambda}\ln(t + C_2) - \frac{b_0}{k}\int \frac{dx}{a(x)} + C_1, \\
b(x) &= b_0, \quad c(x) = \frac{1}{\lambda}\exp\left(-\frac{b_0\lambda}{k}\int \frac{dx}{a(x)} + C_1\lambda\right),
\end{aligned} \tag{116}$$

where $a(x)$ is an arbitrary function and b_0, C_1, C_2, k, and λ are arbitrary constants. The solution of the system consisting of the last two equations (115) is written as

$$\zeta(u) = \frac{f(u)}{kG(u) + C_2}, \quad h(u) = \frac{1}{\zeta(u)}\exp\left(-\lambda\int \zeta(u)\,du\right), \quad G(u) = \int g(u)\,du, \tag{117}$$

where $f(u)$ and $g(u)$ are arbitrary functions.

Solution 18. Equation (21) can also be satisfied using relations (34). Substituting (114) into (34) yields

$$e^{\lambda\vartheta}\vartheta_t = k_1 c, \quad (a\vartheta_x)_x = -k_2 c, \quad b\vartheta_x = -k_3 c;$$
$$(f/\zeta)'_u = 0, \quad h\zeta = k_1 e^{-\lambda Z} + k_2 f + k_3 g. \tag{118}$$

The solution of the overdetermined system consisting of the first three equations (118) is

$$\vartheta(x,t) = \frac{1}{\lambda}\ln[k_1(\lambda t + C_1)c(x)],$$
$$a(x) = \frac{c(x)}{c'_x(x)}\left(C_2 - k_2\lambda\int c(x)\,dx\right), \quad b(x) = -\frac{k_3\lambda c^2(x)}{c'_x(x)}, \tag{119}$$

where $c(x)$ is an arbitrary function (other than a constant), while C_1, C_2, and λ are arbitrary constants. The solutions of the last two equations (118) are expressed as

$$h(u) = \frac{1}{f(u)}\left[k_1\exp\left(-\lambda\int f(u)\,du\right) + k_2 f(u) + k_3 g(u)\right], \quad \zeta(u) = f(u), \tag{120}$$

where $f = f(u)$ and $g = g(u)$ are arbitrary functions.

Solution 19. As before, Equation (21) can be satisfied using relations (38). Substituting (114) into (38), we get the equations

$$e^{\lambda\vartheta}\vartheta_t = c, \quad (a\vartheta_x)_x = -kb\vartheta_x;$$
$$h\zeta = e^{-\lambda Z}, \quad kf = g, \quad (f/\zeta)'_u = 0. \tag{121}$$

The solution of the overdetermined system consisting of the first two equations (121) can be written as

$$\vartheta(x,t) = \frac{1}{\lambda}\ln(t + C_1) + C_2\int\exp\left(-k\int\frac{b}{a}\,dx\right)\frac{dx}{a} + C_3,$$
$$c(x) = \frac{1}{\lambda}\exp\left[C_2\lambda\int\exp\left(-k\int\frac{b}{a}\,dx\right)\frac{dx}{a} + C_3\lambda\right], \tag{122}$$

where $a = a(x)$ and $b = b(x)$ are arbitrary functions, while C_1, C_2, C_3, k, and λ are arbitrary constants. The solution to the system consisting of the last three equations (121) is given by

$$g = kf, \quad h = \frac{1}{mf}\exp\left(-m\lambda\int f\,du\right), \quad \zeta = mf, \tag{123}$$

where $m \neq 0$ is an arbitrary constant.

Solution 20. Substituting (114) into (101), we arrive at the equations

$$e^{\lambda\vartheta}\vartheta_t = -k_1 c, \quad (a\vartheta_x)_x = k_2 c, \quad a\vartheta_x^2 = k_3 c, \quad b\vartheta_x = k_4 c;$$
$$k_1 e^{-\lambda Z} + k_2 f + k_3(f/\zeta)'_u + k_4 g + h\zeta = 0. \tag{124}$$

The first four equations of system (124) admit a solution for the functional coefficients in exponential form:

$$a(x) = b(x) = c(x) = e^{\lambda x}, \quad \vartheta(x,t) = \frac{1}{\lambda}\ln t + x,$$
$$k_1 = -\frac{1}{\lambda}, \quad k_2 = \lambda, \quad k_3 = k_4 = 1. \tag{125}$$

Using the last equation in (124), we obtain the reaction-diffusion-type equation

$$u_t = [e^{\lambda x}f(u)u_x]_x + e^{\lambda x}g(u)u_x + e^{\lambda x}h(u), \tag{126}$$

where

$$h(u) = -\frac{1}{\zeta}\left[-\frac{1}{\lambda}e^{-\lambda Z} + \lambda f + \left(\frac{f}{\zeta}\right)'_u + g\right], \quad Z = \int \zeta\,du, \tag{127}$$

and $f = f(u)$, $g = g(u)$, and $\zeta = \zeta(u)$ are arbitrary functions. Equation (126) admits the exact invariant solution

$$\int \zeta(u)\,du = \frac{1}{\lambda}\ln t + x. \tag{128}$$

Please note that the invariant solution (128) of Equation (126) can be represented in the standard form $u = U(z)$ with $z = \frac{1}{\lambda}\ln t + x$ (in this case, relation (127) linking f, g, h and ζ is not used). The function $U(z)$ is described by the ordinary differential equation

$$\frac{1}{\lambda}U'_z = [e^{\lambda z}f(U)U'_z]'_z + e^{\lambda z}g(U)U'_z + e^{\lambda z}h(U).$$

Solution 21. The first four equations of system (124) also admit a solution for power-law functional coefficients:

$$a(x) = x^n, \quad b(x) = x^{n-1}, \quad c(x) = x^{n-2}, \quad \vartheta(x,t) = \frac{1}{n-2}\ln t + \ln x,$$

$$\lambda = n-2, \quad k_1 = -\frac{1}{n-2}, \quad k_2 = n-1, \quad k_3 = k_4 = 1. \tag{129}$$

Using the last equation in (124), we arrive at the reaction-diffusion-type equation

$$u_t = [x^n f(u)u_x]_x + x^{n-1}g(u)u_x + x^{n-2}h(u), \tag{130}$$

where

$$h(u) = -\frac{1}{\zeta}\left[-\frac{1}{n-2}e^{-(n-2)Z} + (n-1)f + \left(\frac{f}{\zeta}\right)'_u + g\right], \quad Z = \int \zeta\,du, \tag{131}$$

and $f = f(u)$, $g = g(u)$, and $\zeta = \zeta(u)$ are arbitrary functions. Equation (130) admits the exact invariant solution

$$\int \zeta(u)\,du = \frac{1}{n-2}\ln t + \ln x. \tag{132}$$

The self-similar solution (132) of Equation (130) can be sought in the standard form $u = U(z)$ with $z = xt^{1/(n-2)}$ (in this case, relation (131) linking f, g, h, and ζ is not used). The function $U(z)$ is described by the ODE

$$\frac{1}{n-2}zU'_z = [z^n f(U)U'_z]'_z + z^{n-1}g(U)U'_z + z^{n-2}g(U).$$

Solution 22. Equation (21) can be satisfied if we take Ψ_i ($i = 1, 3, 4, 5$) proportional to Ψ_2. As a result, we get

$$\Psi_1 = k_1\Psi_2, \quad \Psi_3 = k_2\Psi_2, \quad \Psi_4 = k_3\Psi_2, \quad \Psi_5 = k_4\Psi_2,$$

$$k_1\Phi_1 + \Phi_2 + k_2\Phi_3 + k_3\Phi_4 + k_4\Phi_5 = 0. \tag{133}$$

Substituting (114) into (133), we obtain the equations

$$e^{-\lambda Z} = k_1 f, \quad (f/\zeta)'_u = k_2 f, \quad g = k_3 f, \quad h\zeta = k_4 f,$$

$$-k_1 e^{\lambda \vartheta}\vartheta_t + (a\vartheta_x)_x + k_2 a\vartheta_x^2 + k_3 b\vartheta_x + k_4 c = 0. \tag{134}$$

The first four equations of system (134) admit a solution for exponential functional coefficients:

$$f(u) = g(u) = h(u) = e^{-\lambda u}, \quad \zeta = 1, \quad Z = u;$$
$$k_1 = k_3 = k_4 = 1, \quad k_2 = -\lambda. \tag{135}$$

In this case, we obtain the reaction-diffusion-type equation

$$u_t = [a(x)e^{\beta u}u_x]_x + b(x)e^{\beta u}u_x + c(x)e^{\beta u}, \quad \lambda = -\beta, \tag{136}$$

which has the exact solution with additive separation of variables

$$u = -\frac{1}{\beta}\ln t + \eta(x), \tag{137}$$

with the function $\eta = \eta(x)$ described by the ordinary differential equation

$$-\frac{1}{\beta} = [a(x)e^{\beta\eta}\eta_x']_x + b(x)e^{\beta\eta}\eta_x' + c(x)e^{\beta\eta}. \tag{138}$$

Equations (136) and (138) contain three arbitrary functions $a(x)$, $b(x)$, and $c(x)$.

Please note that Equation (138) reduces with the substitution $\zeta = e^{\beta\eta}$ to the linear second-order ODE

$$[a(x)\zeta_x']_x + b(x)\zeta_x' + \beta c(x)\zeta + 1 = 0.$$

Solution 23. The first four equations of system (134) also admit a solution for the power-law functional coefficients

$$f(u) = u^n, \quad g(u) = u^n, \quad h(u) = u^{n+1}, \quad \zeta(u) = 1/u, \quad Z = \ln u,$$
$$\lambda = -n, \quad k_1 = k_3 = k_4 = 1, \quad k_2 = n + 1. \tag{139}$$

In this case, the solution of the last equation in (134) is determined by the formula $\vartheta = -(1/n)\ln t + \eta(x)$, with the function $\eta = \eta(x)$ satisfying the ODE

$$\frac{1}{n}e^{-n\eta} + (a\eta_x')_x' + (n+1)a(\eta_x')^2 + b\eta_x' + c = 0. \tag{140}$$

As a result, we get the reaction-diffusion-type equation

$$u_t = [a(x)u^n u_x]_x + b(x)u^n u_x + c(x)u^{n+1}, \tag{141}$$

the exact solution of which can be represented as the product of functions with different arguments $u = t^{-1/n}\zeta(x)$, with the function $\zeta(x) = e^{\eta}$ described by ODE

$$[a(x)\zeta^n\zeta_x']_x + b(x)\zeta^n\zeta_x' + c(x)\zeta^{n+1} + \frac{1}{n}\zeta = 0.$$

Solution 24. Let us return to the class of reaction-diffusion equations of the form (18). Having made the substitution (6), instead of Equation (19), we consider the more complex equation

$$-\vartheta_t + (a\vartheta_x)_x f + a\vartheta_x^2 \left(\frac{f}{\zeta}\right)_u' + b\vartheta_x g + ch\zeta\frac{\vartheta}{Z} = 0, \tag{142}$$

where $Z = \int \zeta\, du$. Equations (19) and (142) are equivalent, because, by virtue of transformation (6), the relation $\vartheta = Z$ holds.

Equation (142) can be represented in bilinear form (21), where

$$
\begin{aligned}
\Phi_1 &= -\vartheta_t, \quad \Phi_2 = (a\vartheta_x)_x, \quad \Phi_3 = a\vartheta_x^2, \quad \Phi_4 = b\vartheta_x, \quad \Phi_5 = c\vartheta; \\
\Psi_1 &= 1, \quad\; \Psi_2 = f, \quad\;\;\; \Psi_3 = (f/\zeta)'_u, \quad \Psi_4 = g, \quad\;\; \Psi_5 = h\zeta/Z.
\end{aligned}
\tag{143}
$$

Equation (21) can be satisfied by using the relations (34). Substituting (143) in (34), we get

$$
\begin{aligned}
\vartheta_t &= k_1 c\vartheta, \quad (a\vartheta_x)_x = -k_2 c\vartheta, \quad b\vartheta_x = -k_3 c\vartheta; \\
(f/\zeta)'_u &= 0, \quad h\zeta/Z = k_1 + k_2 f + k_3 g,
\end{aligned}
\tag{144}
$$

where k_1, k_2, and k_3 are arbitrary constants. Let $a = a(x)$, $f = f(u)$, and $g = g(u)$ be arbitrary functions. Then the solutions of equations (144) are given by

$$
\begin{aligned}
b(x) &= -\frac{k_3\lambda}{k_1}\frac{\omega}{\omega'_x}, \quad c(x) = \frac{\lambda}{k_1} = \text{const}, \quad \vartheta(x,t) = e^{\lambda t}\omega(x), \\
h &= \frac{1}{f}(k_1 + k_2 f + k_3 g)F, \quad \zeta = f, \quad F = \int f\, du,
\end{aligned}
\tag{145}
$$

where λ is an arbitrary constant, and the function $\omega = \omega(x)$ solves the linear second-order ODE $(a\omega'_x)'_x = -(k_2\lambda/k_1)\omega$. In the special case $a(x) = \text{const}$ and $k_3 = 0$, formulas (145) lead to the nonlinear reaction-diffusion equation and its solution, which were considered in [23].

Solution 25. Consider the special case

$$
a = b = c = 1, \quad \zeta = f.
\tag{146}
$$

We look for a solution of Equation (142) under conditions (146) in the form

$$
\vartheta = (\gamma x + \delta)e^{\alpha x + \beta t}.
\tag{147}
$$

Substituting (147) into (142) and taking into account (146), we obtain

$$
\begin{aligned}
&\gamma x e^{\alpha x + \beta t}\big[-\beta + \alpha^2 f + \alpha g + (f/F)h\big] \\
&+ e^{\alpha x + \beta t}\big[-\beta\delta + (\alpha^2\delta + 2\alpha\gamma)f + (\alpha\delta + \gamma)g + \delta(f/F)h\big] = 0,
\end{aligned}
\tag{148}
$$

where $F = \int f\, du$. Equating the expressions in square brackets in (189) with zero, we arrive at the equations

$$
\begin{aligned}
-\beta + \alpha^2 f + \alpha g + (f/F)h &= 0, \\
-\beta\delta + (\alpha^2\delta + 2\alpha\gamma)f + (\alpha\delta + \gamma)g + \delta(f/F)h &= 0.
\end{aligned}
$$

Solving these equations for g and h, we get

$$
g = -2\alpha f, \quad h = \left(\alpha^2 + \frac{\beta}{f}\right)F.
$$

As a result, we obtain the equation

$$
u_t = [f(u)u_x]_x - 2\alpha f(u)u_x + \left[\alpha^2 + \frac{\beta}{f(u)}\right]\int f(u)\,du,
\tag{149}
$$

which has the exact solution

$$
\int f(u)\,du = (\gamma x + \delta)e^{\alpha x + \beta t},
\tag{150}
$$

where γ and δ are arbitrary constants.

Solution 26. We look for a solution to Equation (142) under conditions (146) in the form

$$\vartheta = Ae^{\alpha x + \beta t} + Be^{\gamma x + \delta t}.$$

Omitting the intermediate calculations, we arrive at the equation

$$u_t = [f(u)u_x]_x + \left[\frac{\delta - \beta}{\gamma - \alpha} - (\alpha + \gamma)f(u) \right] u_x$$

$$+ \left[\alpha\gamma + \frac{\beta\gamma - \alpha\delta}{\gamma - \alpha} \frac{1}{f(u)} \right] \int f(u)\, du, \tag{151}$$

which has the solution

$$\int f(u)\, du = Ae^{\alpha x + \beta t} + Be^{\gamma x + \delta t}, \tag{152}$$

where A and B are arbitrary constants.

Example 1. *In the special case $\gamma = -\alpha$ and $\delta = \beta$, Equation (151) simplifies and takes the form*

$$u_t = [f(u)u_x]_x + \left[-\alpha^2 + \frac{\beta}{f(u)} \right] \int f(u)\, du$$

and its solution is written as

$$\int f(u)\, du = e^{\beta t}(Ae^{\alpha x} + Be^{-\alpha x}).$$

Solution 27. Assuming that conditions (146) hold, we look for a solution to Equation (142) in the form

$$\vartheta = Ae^{\alpha t} \sin(\beta x + \sigma t + \delta),$$

where A and δ are arbitrary constants. After simple rearrangements, we obtain the equation

$$u_t = [f(u)u_x]_x + \gamma u_x + \left[\beta^2 + \frac{\alpha}{f(u)} \right] \int f(u)\, du \tag{153}$$

with $\gamma = \sigma/\beta$, which admits the exact solution

$$\int f(u)\, du = Ae^{\alpha t} \sin(\beta x + \beta\gamma t + \delta). \tag{154}$$

Remark 6. *In the case (146), Equation (142) also admits a more complex solution of the form*

$$\vartheta = Ae^{\mu x + \alpha t} \sin(\beta x + \sigma t + \delta),$$

which we do not consider here.

Solution 28. Instead of Equation (142), we can look at the more complex equation

$$-\frac{\vartheta^n}{Z^n}\vartheta_t + (a\vartheta_x)_x f + a\vartheta_x^2 \left(\frac{f}{\zeta} \right)'_u + b\vartheta_x g + ch\zeta\frac{\vartheta}{Z} = 0, \tag{155}$$

where $Z = \int \zeta\, du$ and n is an arbitrary constant. Equations (19) and (155) are equivalent, since, by virtue of transformation (6), the relation $\vartheta = Z$ holds.

Equation (155) can be represented in the bilinear form (21) where

$$
\begin{aligned}
&\Phi_1 = -\vartheta^n \vartheta_t, \quad \Phi_2 = (a\vartheta_x)_x, \quad \Phi_3 = a\vartheta_x^2, \quad \Phi_4 = b\vartheta_x, \quad \Phi_5 = c\vartheta; \\
&\Psi_1 = Z^{-n}, \quad \Psi_2 = f, \quad \Psi_3 = (f/\zeta)_u', \quad \Psi_4 = g, \quad \Psi_5 = h\zeta/Z.
\end{aligned}
\tag{156}
$$

Equation (21) can be satisfied by using the relations (34). Substituting (156) in (34), we get

$$
\begin{aligned}
&\vartheta^n \vartheta_t = k_1 c\vartheta, \quad (a\vartheta_x)_x = -k_2 c\vartheta, \quad b\vartheta_x = -k_3 c\vartheta; \\
&(f/\zeta)_u' = 0, \quad h\zeta/Z = k_1 Z^{-n} + k_2 f + k_3 g,
\end{aligned}
\tag{157}
$$

where k_1, k_2, and k_3 are arbitrary constants. Let $a = a(x)$, $f = f(u)$, and $g = g(u)$ be arbitrary functions. Then the solutions of equations (157) are expressed as

$$
\begin{aligned}
&b(x) = -\frac{k_3}{k_1 n} \frac{\omega^{n+1}}{\omega_x'}, \quad c(x) = \frac{\omega^n}{k_1 n}, \quad \vartheta(x,t) = t^{1/n}\omega(x), \\
&h = \frac{1}{f}(k_1 F^{-n} + k_2 f + k_3 g)F, \quad \zeta = f, \quad F = \int f\, du,
\end{aligned}
\tag{158}
$$

where the function $\omega = \omega(x)$ is a solution of a second-order nonlinear ODE of the Emden–Fowler type:

$$
(a\omega_x')_x' = -\frac{k_2}{k_1 n}\omega^{n+1}.
\tag{159}
$$

We set $k_3 = k_1 n$ and $k = k_2/(k_1 n)$. From relations (158) it follows that the nonlinear reaction-diffusion-type equation

$$
u_t = [a(x)f(u)u_x]_x - \frac{\omega^{n+1}}{\omega_x'}g(u)u_x + \omega^n \frac{F(u)}{f(u)}\left[kf(u) + g(u) + \frac{1}{n}F^{-n}(u)\right],
\tag{160}
$$

where $f(u)$, $g(u)$, and $a(x)$ are arbitrary functions, k and n are arbitrary constants, and $F(u) = \int f(u)\, du$, admits the functional separable solution in implicit form

$$
\int f(u)\, du = \omega(x)t^{1/n}.
\tag{161}
$$

The function $\omega = \omega(x)$ in (160) and (161) is described by the nonlinear ordinary differential equation

$$
[a(x)\omega_x']_x' + k\omega^{n+1} = 0.
\tag{162}
$$

Please note that for $n = -1$, the general solution of Equation (162) is

$$
\zeta = -k\int \frac{x\, dx}{a(x)} + C_1 \int \frac{dx}{a(x)} + C_2,
$$

where C_1 and C_2 are arbitrary constants.

Example 2. *Substituting $a(x) = 1$ and $k = 0$ into* (160)–(162), *we get the equation*

$$
u_t = [f(u)u_x]_x - x^{n+1}g(u)u_x + x^n \frac{1}{f(u)}\left[g(u)F(u) + \frac{1}{n}F^{1-n}(u)\right],
\tag{163}
$$

which admits the exact solution in implicit form (161). *This solution is non-invariant, and it is of a self-similar type; when substituted into Equation* (163), *it causes the term* $[f(u)u_x]_x$ *to vanish.*

Solution 29. We now consider the equation

$$-\vartheta_t + \lambda\vartheta - \lambda Z + (a\vartheta_x)_x f + a\vartheta_x^2 \left(\frac{f}{\zeta}\right)'_u + b\vartheta_x g + ch\zeta = 0, \tag{164}$$

where $\lambda = $ const, which, by virtue of (6) ($\vartheta = Z$), is equivalent to Equation (19).

Equation (164) is invariant under the transformation

$$\vartheta = \bar{\vartheta} + C_1 e^{\lambda t}, \tag{165}$$

where C_1 is an arbitrary constant.

It is easy to verify that for constant a, b, and c, which without loss of generality can be set equal to 1, Equation (164) has the particular solution

$$\bar{\vartheta} = C_2 e^{\beta t - \mu x}, \quad g = \mu f + \frac{\lambda - \beta}{\mu}, \quad h = \frac{\lambda}{f}\int f\, du, \quad \zeta = f, \tag{166}$$

where $f = f(u)$ is an arbitrary function, while C_2, β, and μ are arbitrary constants. Given (165), we obtain the equation

$$u_t = [f(u)u_x]_x + \left[\mu f(u) + \frac{\lambda - \beta}{\mu}\right]u_x + \frac{\lambda}{f(u)}\int f(u)\, du, \tag{167}$$

which has the exact solution in the implicit form

$$\int f(u)\, du = C_1 e^{\lambda t} + C_2 e^{\beta t - \mu x}. \tag{168}$$

Setting $\beta = \lambda - \sigma\mu$, Equation (167) can be rewritten in the more compact form

$$u_t = [f(u)u_x]_x + [\mu f(u) + \sigma]u_x + \frac{\lambda}{f(u)}\int f(u)\, du.$$

In this case, its solution is $\int f(u)\, du = C_1 e^{\lambda t} + C_2 e^{(\lambda - \sigma\mu)t - \mu x}$.

Solution 30. We look for a steady-state particular solution $\vartheta = \vartheta(x)$ of Equation (164). In this case, we have

$$a\vartheta_x^2 = k_1\vartheta, \quad (a\vartheta_x)_x = k_2, \quad b\vartheta_x = k_3, \quad c = 1; \tag{169}$$
$$\lambda + k_1(f/\zeta)'_u = 0, \quad -\lambda Z + k_2 f + k_3 g + h\zeta = 0,$$

where k_1, k_2, and k_3 are arbitrary constants. The solution of the first three equations (169) with $k_1 k_2 \neq 0$ can be represented as

$$a(x) = \frac{1}{C_2 k_1}(k_2 x + C_3)^{2 - (k_1/k_2)}, \quad b(x) = \frac{k_3}{C_2 k_1}(k_2 x + C_3)^{1 - (k_1/k_2)}, \tag{170}$$
$$\vartheta(x) = C_2(k_2 x + C_3)^{k_1/k_2},$$

where C_2 and C_3 are arbitrary constants. The solution to the system consisting of the last two equations (169) is written as follows:

$$\zeta = -\frac{k_1}{\lambda}\frac{f}{u + C_4}, \quad h = \frac{\lambda(u + C_4)}{k_1 f}\left(k_2 f + k_3 g + k_1\int\frac{f\, du}{u + C_4}\right), \tag{171}$$

where $f = f(u)$ and $g = g(u)$ are arbitrary functions, C_4 is an arbitrary constant.

Substituting $C_2 = 1/k$, $C_3 = C_4 = 0$, $k_1 = k$, $k_2 = k_3 = 1$, and $\lambda = k\sigma$ in (170) and (171), we arrive at the equation

$$u_t = [x^{2-k}f(u)u_x]_x + x^{1-k}g(u)u_x + \frac{\sigma u}{f(u)}\left[f(u) + g(u) + k \int \frac{f(u)}{u}\,du\right]. \tag{172}$$

For $k \neq 0$, this equation admits the exact solution

$$\int \frac{f(u)}{u}\,du = Ce^{k\sigma t} - \frac{\sigma}{k}x^k, \qquad C = -C_1\sigma, \tag{173}$$

in the construction of which the invariance of Equation (164) with respect to transformation (165) was taken into account.

Solution 31. In Equation (164), we set $\zeta = f$ and $\lambda = p(x)f(u)$ (recall that λ can be any function dependent on x, t, and u; see Item 2 in Section 2.3). On dividing by f, we get

$$-\vartheta_t\frac{1}{f} + (a\vartheta_x)_x + p\vartheta + b\vartheta_x\frac{g}{f} + ch - pF = 0, \tag{174}$$

where $F = \int f(u)\,du$.

Assuming the function f to be given arbitrarily, we look for the functions g and h in the form

$$g = f\left(k_1 + k_2\frac{1}{f} + k_3F\right), \qquad h = m_1 + m_2\frac{1}{f} + m_3F, \tag{175}$$

where k_i and m_i are some constants ($i = 1, 2, 3$). Substituting (175) into (174), we arrive at the equations

$$\begin{aligned}
(a\vartheta_x)_x + p\vartheta + k_1 b\vartheta_x + m_1 c &= 0, \\
-\vartheta_t + k_2 b\vartheta_x + m_2 c &= 0, \\
-p + k_3 b\vartheta_x + m_3 c &= 0,
\end{aligned} \tag{176}$$

Equations (176) admit the following exact solution

$$k_2 = k_3 = 0, \qquad \vartheta = m_2 c(x)t + \eta(x), \qquad p = m_3 c(x), \tag{177}$$

where the three functions $a = a(x)$, $b = b(x)$, and $c = c(x)$ are connected by one equation

$$(ac'_x)'_x + k_1 bc'_x + m_3 c^2 = 0, \tag{178}$$

and the function η are described by the linear ODE

$$(a\eta'_x)'_x + k_1 b\eta'_x + m_3 c\eta + m_1 c = 0. \tag{179}$$

Please note that for given functions a and c, Equation (178) is algebraic with respect to b, for given b and c it is a first-order linear ODE with respect to a (which is readily integrated), and for given a and b it is a second-order ODE with a quadratic nonlinearity with respect to c.

To sum up, we have obtained the nonlinear reaction-diffusion-type equation

$$u_t = [a(x)f(u)u_x]_x + b(x)f(u)u_x + c(x)\left[m_1 + \frac{m_2}{f(u)} + m_3 \int f(u)\,du\right], \tag{180}$$

where $f(u)$ is an arbitrary function, and any two of the three functions $a = a(x)$, $b = b(x)$, and $c = c(x)$ can be given arbitrarily, while the remaining function satisfies Equation (178) with $k_1 = 1$. Equation (180) has the exact solution in implicit form

$$\int f(u)\, du = m_2 c(x)t + \eta(x),\tag{181}$$

where the function $\eta(x)$ is determined by ODE (179) with $k_1 = 1$.

Remark 7. *The more general equation*

$$u_t = [a(x)f(u)u_x]_x + b(x)f(u)u_x + m(x) + \frac{c(x)}{f(u)} + n(x)\int f(u)\, du,$$

where $f = f(u)$ and $m = m(x)$ are arbitrary functions, and the four functions $a = a(x)$, $b = b(x)$, $c = c(x)$, and $n = n(x)$ are connected by one equation (algebraic in b and n, and differential in a and c)

$$(ac'_x)'_x + bc'_x + cn = 0,$$

admits the exact solution

$$\int f(u)\, du = c(x)t + \eta(x),$$

with the function $\eta(x)$ determined by the ODE

$$(a\eta'_x)'_x + b\eta'_x + n\eta + m = 0.$$

Solution 32. Solutions of Equation (174) can be sought in the form

$$g = f\left(k_1 f^{-1} + k_2\right), \quad h = k_3 f^{-1} + k_4, \quad F = k_5 f^{-1} + k_6,\tag{182}$$

where k_n are some constants; the last relation (182) is used to determine the function f. By setting $k_1 = 0$, $k_2 = 1$, $k_5 = 2$, and $k_6 = 0$ in (182), we obtain $f = g = u^{-1/2}$, $h = k_3 u^{1/2} + k_4$, and $F = 2u^{1/2}$. The corresponding nonlinear reaction-diffusion-type equation

$$u_t = [a(x)u^{-1/2}u_x]_x + b(x)u^{-1/2}u_x + c(x)(k_3 u^{1/2} + k_4),\tag{183}$$

where $a(x)$, $b(x)$, and $c(x)$ are arbitrary functions, while and k_3 and k_4 are arbitrary constants, has an exact solution in implicit form $F = \xi(x)t + \eta(x)$, which can be expressed in explicit form as

$$u = \tfrac{1}{4}[\xi(x)t + \eta(x)]^2.\tag{184}$$

The functions $\xi = \xi(x)$ and $\eta = \eta(x)$ are determined by solving the ordinary differential equations

$$\begin{aligned}(a\xi'_x)'_x + b\xi'_x + \tfrac{1}{2}k_3 c\xi - \tfrac{1}{2}\xi^2 &= 0,\\(a\eta'_x)'_x + b\eta'_x + \tfrac{1}{2}k_3 c\eta - \tfrac{1}{2}\xi\eta + k_4 c &= 0.\end{aligned}\tag{185}$$

For $c(x) = 1$, the first Equation (185) can be satisfied if we take $\xi(x) = k_3$.

Remark 8. *The equation*

$$u_t = [a(x)u^{-1/2}u_x]_x + b(x)u^{-1/2}u_x + c(x)u^{1/2} + d(x),\tag{186}$$

which is more general than (183), *has an exact solution of the form* (184). *In the case* $d(x)/c(x) = \text{const}$, *Equation* (186) *belongs to the class of equations* (18) *in question.*

Remark 9. *The nonlinear delay PDE*

$$u_t = [a_1(x)u^{-1/2}u_x]_x + [a_2(x)w^{-1/2}w_x]_x + b_1(x)u^{-1/2}u_x + b_2(x)w^{-1/2}w_x$$
$$+ c_1(x)u^{1/2} + c_2(x)w^{1/2} + d(x), \quad w = u(x, t - \tau),$$

where τ *is the delay time and* $a_1(x)$, $a_2(x)$, $b_1(x)$, $b_2(x)$, $c_1(x)$, $c_2(x)$, *and* $d(x)$ *are arbitrary functions, also admits an exact solution of the form* (184).

    Solution 33. Now we consider the equation

$$-\vartheta_t + \vartheta_{xx}f + \vartheta_x^2\left(\frac{f}{\zeta}\right)'_u - k\left(\frac{f}{\zeta}\right)'_u\vartheta + k\left(\frac{f}{\zeta}\right)'_u Z + h\zeta = 0. \tag{187}$$

which, by virtue of (6), is equivalent to Equation (19) for $a = c = 1$ and $b = 0$.
    An exact solution of Equation (188) is sought in the form

$$\vartheta = Ax^2 + Bx + Ce^{-\lambda t},$$

where A, B, C, and λ are constants to be found. Omitting the intermediate calculations, we ultimately arrive at the equation

$$u_t = [f(u)u_x]_x - \frac{1}{2}\lambda u - \gamma^2 \frac{u}{f(u)} - \lambda \frac{u}{f(u)} \int \frac{f(u)}{u}\, du, \tag{188}$$

which has two exact solutions

$$\int \frac{f(u)}{u}\, du = \frac{1}{4}\lambda x^2 \pm \gamma x + \beta e^{-\lambda t}, \tag{189}$$

where β, γ, and λ are arbitrary constants.

Remark 10. *The described approach also makes it possible to obtain other exact solutions of Equation* (18), *which are not considered here (recall that in this article we only look at nonlinear equations of a fairly general form that depend on arbitrary functions).*

Remark 11. *This approach can also be used to construct exact solutions of nonlinear ordinary differential equations with variable coefficients.*

3.4. Using Transformation (6) to Simplify Equations

    Transformation (6) can also be used to simplify nonlinear PDEs. To illustrate this, we consider the equation

$$u_t = au_{xx} + f(u)u_x^2 + b(x)g(u)u_x + c(x)h(u), \tag{190}$$

where a is a constant.
    Transformation (6) reduces Equation (190) to the form

$$\vartheta_t = a\vartheta_{xx} + \vartheta_x^2\frac{1}{\zeta}\left[f(u) - a\frac{\zeta'_u}{\zeta}\right] + b(x)g(u)\vartheta_x + c(x)h(u)\zeta. \tag{191}$$

In (191), we set

$$f(u) - a\frac{\zeta'_u}{\zeta} = 0, \quad g(u) = 1, \quad h(u)\zeta = 1.$$

Whence

$$\zeta = \exp\left[\frac{1}{a}\int f(u)\,du\right], \quad h(u) = \exp\left[-\frac{1}{a}\int f(u)\,du\right].$$

As a result, we obtain the nonlinear equation

$$u_t = au_{xx} + f(u)u_x^2 + b(x)u_x + c(x)\exp\left[-\frac{1}{a}\int f(u)\,du\right], \tag{192}$$

where $b(x)$, $c(x)$, and $f(u)$ are arbitrary functions, which can be reduced with the transformation

$$\vartheta = \int \exp\left[\frac{1}{a}\int f(u)\,du\right]du \tag{193}$$

to the linear equation

$$\vartheta_t = a\vartheta_{xx} + b(x)\vartheta_x + c(x). \tag{194}$$

Some exact solutions of this equation can be found in [63].

Remark 12. *Please note that in Equations (192) and (194), the functional coefficients $a(x)$ and $b(x)$ can be replaced with $a(x,t)$ and $b(x,t)$.*

Example 3. *In the special case $a = 1$ and $f(u) = 1$, Equation (192) becomes*

$$u_t = u_{xx} + u_x^2 + b(x)u_x + c(x)e^{-u},$$

and transformation (193) can be written in explicit form as $u = \ln\vartheta$. As a result, we obtain Equation (194) with $a = 1$.

4. Indirect Functional Separation of Variables

4.1. Functional Separation of Variables Based on the Nonclassical Method of Symmetry Reductions

The method of functional separation of variables based on transformation (6) is closely related to the nonclassical method of symmetry reductions which is based on an invariant surface condition [38]. To show this, we differentiate formula (6) with respect to t to obtain

$$u_t = Q(x,t)\phi(u) \tag{195}$$

where $Q(x,t) = \vartheta_t$ and $\phi(u) = 1/\zeta(u)$.

Relation (195) can be treated as a first-order differential constraint or an invariant surface condition of a special form (in general, an invariant surface condition is a quasilinear first-order PDE of general form), which can be used to find exact solutions of Equation (18) through a compatibility analysis of the overdetermined pair of differential Equations (18) and (195) with the single unknown u. The invariant surface condition (195) is equivalent to relation (6); at the initial stage, both functions $Q(x,t)$ and $\phi(u)$ included on the right-hand side of (195) are considered arbitrary, and the specific form of these functions is determined in the subsequent analysis.

A description of the nonclassical method of symmetry reductions and examples of its application to construct exact solutions of nonlinear PDEs can be found, for example, in [9,18,23,38–44]. Although the invariant surface condition (195) is equivalent to the functional relation (6), the subsequent procedure for finding exact solutions by the nonclassical method of symmetry reductions (or by the method of differential constraints) and that by the direct method for seeking functional separable solutions differ significantly. Let us compare the effectiveness of these methods by the example of the reaction-diffusion-type Equation (18) (since its functional separable solutions have already been obtained in Sections 3.2 and 3.3). To construct exact solutions by the nonclassical method of symmetry reductions, we will use relation (195) as an invariant surface condition.

Remark 13. *The nonclassical method of symmetry reductions, based on the invariant surface condition (195), and the method of differential constraints [64], based on the single differential constraint (195), end up in the same result. A description of the method of differential constraints and examples of its application to construct exact solutions of nonlinear PDEs can be found, for example, in [7,18,23,45,50,65–69].*

Taking into account the last remark, below we use the method of differential constraints [23]. We solve Equation (18) for the highest derivative u_{xx} and eliminate u_t with the help of (195) to obtain

$$u_{xx} = -\frac{f'_u}{f}u_x^2 - \left(\frac{a'_x}{a} + \frac{b}{a}\frac{g}{f}\right)u_x + \frac{Q\phi - ch}{af}. \tag{196}$$

Differentiating (195) twice with respect to x and taking into account relation (196), we get

$$
\begin{aligned}
u_{tx} &= Q\phi'_u u_x + Q_x\phi, \\
u_{txx} &= Q\phi'_u u_{xx} + Q\phi''_{uu}u_x^2 + 2Q_x\phi'_u u_x + Q_{xx}\phi \\
&= Q\left(\phi''_u - \frac{f'_u}{f}\phi'_u\right)u_x^2 + A_1(x,t,u)u_x + A_0(x,t,u), \\
A_1(x,t,u) &= \left[2Q_x - Q\left(\frac{a'_x}{a} + \frac{b}{a}\frac{g}{f}\right)\right]\phi'_u, \\
A_0(x,t,u) &= Q_{xx}\phi - \frac{cQ}{a}\frac{h}{f}\phi'_u + \frac{Q^2}{a}\frac{\phi}{f}\phi'_u.
\end{aligned}
\tag{197}
$$

where $A_1(x,t,u)$ and $A_0(x,t,u)$ are independent of the derivative u_x and are expressed in terms of the functions appearing in the original PDE (18) and the invariant surface condition (195).

Differentiating (196) with respect to t and taking into account relation (195) and the first formula of (197), we find the mixed derivative in a different way,

$$
\begin{aligned}
u_{xxt} &= -Q\left[\phi\left(\frac{f'_u}{f}\right)'_u + 2\frac{f'_u}{f}\phi'_u\right]u_x^2 + B_1(x,t,u)u_x + B_0(x,t,u), \\
B_1(x,t,u) &= -2Q_x\phi\frac{f'_u}{f} - \frac{a_xQ}{a}\phi'_u - \frac{b}{a}Q\left(\frac{g\phi}{f}\right)'_u, \\
B_0(x,t,u) &= -\frac{a_xQ_x}{a}\phi - \frac{bQ_x}{a}\frac{g}{f}\phi - \frac{cQ}{a}\phi\left(\frac{h}{f}\right)'_u + \frac{Q_t}{a}\frac{\phi}{f} + \frac{Q^2}{a}\phi\left(\frac{\phi}{f}\right)'_u,
\end{aligned}
\tag{198}
$$

where $B_1(x,t,u)$ and $B_0(x,t,u)$ are independent of u_x.

Equating the third-order mixed derivatives (197) and (198), we get the following relation, quadratic in u_x:

$$Ku_x^2 + Mu_x + N = 0, \tag{199}$$

where

$$K = Q\left[\phi_u'' + \phi_u'\frac{f_u'}{f} + \phi\left(\frac{f_u'}{f}\right)_u'\right],$$

$$M = 2Q_x\left(\phi_u' + \frac{f_u'}{f}\phi\right) + \frac{bQ}{a}\left(\frac{g}{f}\right)_u'\phi, \tag{200}$$

$$N = Q_{xx}\phi + Q_x\phi\left(\frac{a_x'}{a} + \frac{b}{a}\frac{g}{f}\right) - \frac{Q_t}{a}\frac{\phi}{f} + \frac{cQ}{a}\left[\phi\left(\frac{h}{f}\right)_u' - \frac{h}{f}\phi_u'\right] + \frac{Q^2}{a}\frac{\phi^2 f_u'}{f^2}.$$

The functional coefficients K, M, and N depend on $a, b, c, f, g, h, Q, \phi$ and their derivatives (and are independent of u_x). By equating in (199) the functional coefficients K, M, and N with zero (the procedure of splitting by the derivative u_x), one obtains the determining system of equations $K = 0$, $M = 0$, $N = 0$. Next, we only need the first equation of this system (corresponding to $K = 0$), which after dividing by Q, takes the form

$$\phi_u'' + \phi_u'\frac{f_u'}{f} + \phi\left(\frac{f_u'}{f}\right)_u' = 0. \tag{201}$$

This equation admits the first integral

$$\phi_u' + \phi\frac{f_u'}{f} = C_1. \tag{202}$$

Considering f to be an arbitrary function and ϕ to be an unknown function and integrating (202), we find the general solution of Equation (201):

$$\phi = \frac{1}{f}\left(C_1\int f\,du + C_2\right), \tag{203}$$

where C_1 and C_2 are arbitrary constants. Thus, the nonclassical method of symmetry reductions with the invariant surface condition (195) leads to exact solutions in which the functions f and ϕ (involved in the original equation and the invariant surface condition) are related by (203).

Using the invariant surface condition (195) is equivalent to representing the solution in the form (6). Since $\phi = 1/\zeta$, solution (203) can be rewritten in terms of f and ζ as

$$\zeta = f\left(C_1\int f\,du + C_2\right)^{-1}. \tag{204}$$

We now consider some solutions obtained in Sections 3.2 and 3.3 by the method of functional separation of variables. Solution (69) of Equation (68) and solution (173) of Equation (172) are special cases of solutions (6) with $\zeta = f/u$. These solutions are different from (204); consequently, they cannot be obtained by the nonclassical method of symmetry reductions with the invariant surface condition (195). Also, more complex solutions (25), (33), (53), (59), (63), (78), and (117), in which the function ζ depends not only on $f(u)$ but also on other functional coefficients $g(u)$ or/and $h(u)$ of the considered class of equations (18), cannot be obtained by the nonclassical method of symmetry reductions with condition (195).

Remark 14. *It can be shown that exact solutions listed above cannot be obtained by the nonclassical method of symmetry reductions using an invariant surface condition of the form $u_t = U(x, t, u)$, which is more general than (195).*

Remark 15. *Importantly, the vast majority of solutions constructed in this paper are non-invariant (that is, they cannot be obtained using the classical group analysis of differential equations [70–72]).*

4.2. Some Remarks on Weak Symmetries

In applying the nonclassical method of symmetry reductions to Equation (18), the loss of some exact solutions occurred when the splitting procedure in powers of u_x was applied to relation (199)–(200). Theoretically, in order to avoid such losses, we can further search for weak symmetries [73–75]. Consider two possible algorithms for finding weak solutions by looking at the example of the nonlinear Equation (18).

The first algorithm. This algorithm consists of two stages.

1. The first (composite) stage suggests obtaining relation (199) and so leads to the same results as applying the nonclassical method until the splitting procedure in powers of u_x.

2. The second stage suggests analyzing three PDEs (195), (196), and (199)–(200) for consistency (in order to obtain the determining equation, which must then be integrated).

The compatibility analysis of these PDEs is carried out as follows. Equation (199) is differentiated with respect to t, after which the derivatives u_t and u_{xt} are eliminated from the resulting expression using relation (195) and the first formula of (197). As a result, we obtain

$$Pu_x^2 + Qu_x + R = 0, \tag{205}$$

where

$$\begin{aligned}
P &= K_t + UK_u + 2U_uK, \\
Q &= M_t + UM_u + U_uM + 2U_xK, \\
R &= N_t + UN_u + U_xM.
\end{aligned} \tag{206}$$

For brevity, short notations are used here:

$$K = K(x,t,u), \quad M = M(x,t,u), \quad N = N(x,t,u), \quad U = Q(x,t)\phi(u).$$

Having further eliminated the derivative u_x from Equations (199) and (205), we obtain the determining equation, which in the nondegenerate case ($MP - KQ \not\equiv 0$) has the form

$$K(NP - KR)^2 - M(MP - KQ)(NP - KR) + N(MP - KQ)^2 = 0. \tag{207}$$

Equation (207) is a very complex nonlinear PDE, which includes third-order derivatives Q_{xxt} and ϕ'''_{uuu} (recall that Q and ϕ are both unknown functions), whose length in expanded form (taking into account the relations (199) and (206)) occupies almost an entire page. In addition, Equation (207), which includes one or more arbitrary functions $f(u), g(u)$, etc., must be solved together with Equations (195) and (196) (or the original equation). As a result, instead of one Equation (18) (or Equation (19) together with (6)), it is necessary in this case to deal with a much more complex system of coupled nonlinear PDEs.

Example 4. *For greater clarity, let us look at the linear heat equation $u_t = u_{xx}$, which is obtained from (18) by setting*

$$a(x) = 1, \quad b(x) = 0, \quad c(x) = 0, \quad f(u) = 1.$$

In this case, one has to substitute into Equation (207) the following functions:

$$\begin{aligned}
K &= U_{uu}, \quad M = 2U_{xu}, \quad N = U_{xx} - U_t, \quad U = Q\phi; \\
P &= U_{tuu} + UU_{uuu} + 2U_uU_{uu}, \\
Q &= 2(U_{xtu} + UU_{xuu} + U_uU_{xu} + U_xU_{uu}, \\
R &= U_{xxt} - U_{tt} + U(U_{xxu} - U_{tu}) + 2U_xU_{xu}.
\end{aligned} \tag{208}$$

One can see that the nonlinear third-order Equations (207) and (208) becomes isolated (can be solved independently of the original equation); it is far more complicated than the linear heat equation under consideration.

The degenerate case of $MP - KQ \equiv 0$ can be treated likewise.

It is apparent from the above examples that the method in question, based on the analysis of three PDEs (195), (196), and (199), is extremely difficult for practical use.

The second algorithm. In this case, we differentiate formula (6) with respect to t and x. As a result, we obtain two relations

$$u_t = \vartheta_t \phi(u), \quad u_x = \vartheta_x \phi(u), \tag{209}$$

which can be interpreted as two compatible differential constraints, where the functions $\vartheta = \vartheta(x, t)$ and $\phi(u) = 1/\zeta(u)$ are to be determined. Differentiating the second relation (209) with respect to x, we find the second derivative

$$u_{xx} = \vartheta_{xx}\phi + \vartheta_x \phi'_u u_x = \vartheta_{xx}\phi + \vartheta_x^2 \phi \phi'_u, \quad \phi = \phi(u). \tag{210}$$

Next, we substitute the derivatives (209) and (210) in (18). As a result, we arrive at an equation that is equivalent to Equation (19). Using further the generalized splitting principle described in Section 2.1, one can find the exact solutions obtained in Section 3.2. However, it will not be possible to find the solutions obtained in Section 3.3 in this way. To find these solutions, one must first integrate the differential relations (209) and return to the original relation (6), and then consider the equivalent equations described in Section 2.3.

Thus, it seems that the use of transformation (6) is more effective for constructing exact solutions than the use of one or two equivalent differential constraints.

5. Brief Conclusions

A general method for constructing exact solutions of nonlinear PDEs has been described, which is based on nonlinear transformations with an integral term in combination with the generalized splitting principle. The high productivity of the method has been illustrated by nonlinear equations of the reaction-diffusion type with variable coefficients that depend on one, two or three arbitrary functions. Many new exact functional separable solutions and generalized traveling wave solutions have been obtained. The effectiveness of various methods for constructing exact solutions of nonlinear differential equations has been compared.

The direct method of functional separation of variables based on transformation (6), in addition to diffusion-type equations, is also applicable to other classes of PDEs. In particular, these include nonlinear wave equations, nonlinear Klein–Gordon type equations, nonlinear telegraph-type equations, and others; these also include some third- and higher-order PDEs. This method is easy to generalize to equations with three or more independent variables.

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