

$$\begin{aligned} \mathbf{12.} \quad & \frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2} + u f\left(\frac{w}{u}\right) - w g\left(\frac{w}{u}\right) + \frac{u}{\sqrt{u^2 + w^2}} h\left(\frac{w}{u}\right), \\ & \frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + w f\left(\frac{w}{u}\right) + u g\left(\frac{w}{u}\right) + \frac{w}{\sqrt{u^2 + w^2}} h\left(\frac{w}{u}\right). \end{aligned}$$

Solution:

$$u = r(x, t)\cos\varphi(t), \quad w = r(x, t)\sin\varphi(t),$$

where the function $\varphi = \varphi(t)$ is determined by the separable first-order ordinary differential equation

$$\varphi_t' = g(\tan \varphi),$$

and the function r = r(x, t) satisfies the linear equation

$$\frac{\partial r}{\partial t} = a \frac{\partial^2 r}{\partial x^2} + r f(\tan \varphi) + h(\tan \varphi). \tag{1}$$

The change of variable

$$r = F(t) \Big[Z(x,t) + \int \frac{h(\tan\varphi) \, dt}{F(t)} \Big], \qquad F(t) = \exp\Big[\int f(\tan\varphi) \, dt \Big]$$

brings (1) to the linear heat equation

$$\frac{\partial Z}{\partial t} = a \frac{\partial^2 Z}{\partial x^2}.$$

Copyright © 2004 Andrei D. Polyanin

http://eqworld.ipmnet.ru/en/solutions/syspde/spde2112.pdf