Systems of Ordinary Differential Equations > Nonlinear Systems of Two Equations

12. $x''_{tt} + a(t)x = x^{-3}f(y/x), \quad y''_{tt} + a(t)y = y^{-3}g(y/x).$

Generalized Ermakov (Yermakov) system.

1°. First integral:

$$\frac{1}{2}(xy'_t - yx'_t)^2 + \int^{y/x} \left[uf(u) - u^{-3}g(u) \right] du = C,$$

where C is an arbitrary constant.

2°. Let $\varphi = \varphi(t)$ is a nontrivial solution of the second-order linear differential equation

$$\varphi''_{tt} + a(t)\varphi = 0. \tag{1}$$

Then, the transformation

$$\tau = \int \frac{dt}{\varphi^2(t)}, \quad u = \frac{x}{\varphi(t)}, \quad v = \frac{y}{\varphi(t)} \tag{2}$$

leads to the autonomous system of equations

$$u''_{\tau\tau} = u^{-3}f(v/u), \quad v''_{\tau\tau} = v^{-3}g(v/u). \tag{3}$$

3°. Particular solution of system (3) is

$$u = A\sqrt{C_2\tau^2 + C_1\tau + C_0}, \quad v = Ak\sqrt{C_2\tau^2 + C_1\tau + C_0}, \quad A = \left[\frac{f(k)}{C_0C_2 - \frac{1}{4}C_1^2} \right]^{1/4},$$

where C_0, C_1, and C_2 are arbitrary constants, and k is a root of the algebraic (transcendental) equation

$$k^4 f(k) = g(k).$$

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