EqWorld http://eqworld.ipmnet.ru

Exact Solutions > Basic Handbooks > A. D. Polyanin, *Handbook of Linear Partial Differential Equations for Engineers and Scientists*, Chapman & Hall/CRC Press, Boca Raton, 2002

CONTENTS

Foreword . . . . . **iii**

Basic Notation and Remarks . . . . . **v**

Author . . . . . **vii**

Introduction Some Definitions, Formulas, Methods, and Solutions **1**

0.1. Classification of Second Order Partial Differential Equations 1
0.1.1. Equations with Two Independent Variables 1
0.1.2. Equations with Many Independent Variables 3

0.2. Basic Problems of Mathematical Physics 4
0.2.1. Initial and Boundary Conditions. Cauchy Problem. Boundary Value Problems 4
0.2.2. First, Second, Third, and Mixed Boundary Value Problems 6

0.3. Properties and Particular Solutions of Linear Equations 7
0.3.1. Homogeneous Linear Equations 7
0.3.2. Nonhomogeneous Linear Equations 10

0.4. Method of Separation of Variables 11
0.4.1. General Description of the Method of Separation of Variables 11
0.4.2. Solution of Boundary Value Problems for Parabolic and Hyperbolic Equations 15

0.5. Method of Integral Transforms 17
0.5.1. Main Integral Transforms 17
0.5.2. Laplace Transform and Its Application in Mathematical Physics 18
0.5.3. Fourier Transform and Its Application in Mathematical Physics 21

0.6. Representation of the Solution of the Cauchy Problem via the Fundamental Solution . . 23
0.6.1. Cauchy Problem for Equations of Parabolic Type 23
0.6.2. The Cauchy Problem for Equations of Hyperbolic Type 24

0.7. Nonhomogeneous Boundary Value Problems with One Space Variable. Representation of Solutions via the Green's Function 25
0.7.1. Problems for Equations of Parabolic Type 25
0.7.2. Problems for Equations of Hyperbolic Type 27

0.8. Nonhomogeneous Boundary Value Problems with Many Space Variables. Representation of Solutions via the Green's function 28
0.8.1. Problems for Equations of Parabolic Type 28
0.8.2. Problems for Equations of Hyperbolic Type 29
0.8.3. Problems for Equations of Elliptic Type 30
0.8.4. Comparison of the Solution Structure for Boundary Value Problems for Equations of Various Types 31

0.9. Construction of the Green's Functions. General Formulas and Relations 32
0.9.1. Green's Functions of Boundary Value Problems for Equations of Various Types in Bounded Domains 32
0.9.2. Green's Functions Admitting Incomplete Separation of Variables 33
0.9.3. Construction of Green's Functions via Fundamental Solutions 35

0.10. Duhamel's Principles in Nonstationary Problems . . . 38
0.10.1. Problems for Homogeneous Linear Equations . . . 38
0.10.2. Problems for Nonhomogeneous Linear Equations . . . 39
0.11. Transformations Simplifying Initial and Boundary Conditions . . . 40
0.11.1. Transformations That Lead to Homogeneous Boundary Conditions . . . 40
0.11.2. Transformations That Lead to Homogeneous Initial and Boundary Conditions . 41

1. Equations of Parabolic Type with One Space Variable . . . 43

1.1. Constant Coefficient Equations . . . 43
1.1.1. Heat Equation $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2}$ . . . 43
1.1.2. Equation of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + \Phi(x,t)$ . . . 51
1.1.3. Equation of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + bw + \Phi(x,t)$ . . . 54
1.1.4. Equation of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b\frac{\partial w}{\partial x} + \Phi(x,t)$ . . . 58
1.1.5. Equation of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b\frac{\partial w}{\partial x} + cw + \Phi(x,t)$ . . . 62
1.2. Heat Equation with Axial or Central Symmetry and Related Equations . . . 65
1.2.1. Equation of the Form $\frac{\partial w}{\partial t} = a\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)$ . . . 65
1.2.2. Equation of the Form $\frac{\partial w}{\partial t} = a\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)+\Phi(r,t)$ . . . 70
1.2.3. Equation of the Form $\frac{\partial w}{\partial t} = a\left(\frac{\partial^2 w}{\partial r^2} + \frac{2}{r}\frac{\partial w}{\partial r}\right)$ . . . 73
1.2.4. Equation of the Form $\frac{\partial w}{\partial t} = a\left(\frac{\partial^2 w}{\partial r^2} + \frac{2}{r}\frac{\partial w}{\partial r}\right)+\Phi(r,t)$ . . . 79
1.2.5. Equations of the Form $\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \frac{1-2\beta}{x}\frac{\partial w}{\partial x}$ . . . 81
1.2.6. Equations of the Form $\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \frac{1-2\beta}{x}\frac{\partial w}{\partial x} + \Phi(x,t)$ . . . 84
1.3. Equations Containing Power Functions and Arbitrary Parameters . . . 85
1.3.1. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(x,t)w$ . . . 85
1.3.2. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(x,t)\frac{\partial w}{\partial x}$ . . . 90
1.3.3. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(x,t)\frac{\partial w}{\partial x} + g(x,t)w + h(x,t)$ . . . 93
1.3.4. Equations of the Form $\frac{\partial w}{\partial t} = (ax+b)\frac{\partial^2 w}{\partial x^2} + f(x,t)\frac{\partial w}{\partial x} + g(x,t)w$ . . . 95
1.3.5. Equations of the Form $\frac{\partial w}{\partial t} = (ax^2+bx+c)\frac{\partial^2 w}{\partial x^2} + f(x,t)\frac{\partial w}{\partial x} + g(x,t)w$ . . . 98
1.3.6. Equations of the Form $\frac{\partial w}{\partial t} = f(x)\frac{\partial^2 w}{\partial x^2} + g(x,t)\frac{\partial w}{\partial x} + h(x,t)w$ . . . 100
1.3.7. Equations of the Form $\frac{\partial w}{\partial t} = f(x,t)\frac{\partial^2 w}{\partial x^2} + g(x,t)\frac{\partial w}{\partial x} + h(x,t)w$ . . . 104
1.3.8. Liquid-Film Mass Transfer Equation $(1-y^2)\frac{\partial w}{\partial x} = a\frac{\partial^2 w}{\partial y^2}$ . . . 105
1.3.9. Equations of the Form $f(x,y)\frac{\partial w}{\partial x} + g(x,y)\frac{\partial w}{\partial y} = \frac{\partial^2 w}{\partial y^2} + h(x,y)$ . . . 107
1.4. Equations Containing Exponential Functions and Arbitrary Parameters . . . 108
1.4.1. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(x,t)w$ . . . 108
1.4.2. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(x,t)\frac{\partial w}{\partial x}$ . . . 110
1.4.3. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(x,t)\frac{\partial w}{\partial x} + g(x,t)w$ . . . 113
1.4.4. Equations of the Form $\frac{\partial w}{\partial t} = ax^n\frac{\partial^2 w}{\partial x^2} + f(x,t)\frac{\partial w}{\partial x} + g(x,t)w$ . . . 114
1.4.5. Equations of the Form $\frac{\partial w}{\partial t} = ae^{\beta x}\frac{\partial^2 w}{\partial x^2} + f(x,t)\frac{\partial w}{\partial x} + g(x,t)w$ . . . 114
1.4.6. Other Equations . . . 117
1.5. Equations Containing Hyperbolic Functions and Arbitrary Parameters . . . 117
1.5.1. Equations Containing Hyperbolic Cosine . . . 117
1.5.2. Equations Containing Hyperbolic Sine . . . 118
1.5.3. Equations Containing Hyperbolic Tangent . . . 119
1.5.4. Equations Containing Hyperbolic Cotangent . . . 120

1.6. Equations Containing Logarithmic Functions and Arbitrary Parameters 121
1.6.1. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(x,t)\frac{\partial w}{\partial x} + g(x,t)w$ 121
1.6.2. Equations of the Form $\frac{\partial w}{\partial t} = ax^k\frac{\partial^2 w}{\partial x^2} + f(x,t)\frac{\partial w}{\partial x} + g(x,t)w$ 122
1.7. Equations Containing Trigonometric Functions and Arbitrary Parameters 123
1.7.1. Equations Containing Cosine 123
1.7.2. Equations Containing Sine 124
1.7.3. Equations Containing Tangent 125
1.7.4. Equations Containing Cotangent 126
1.8. Equations Containing Arbitrary Functions 127
1.8.1. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(x,t)w$ 127
1.8.2. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(x,t)\frac{\partial w}{\partial x}$ 130
1.8.3. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(x,t)\frac{\partial w}{\partial x} + g(x,t)w$ 133
1.8.4. Equations of the Form $\frac{\partial w}{\partial t} = ax^n\frac{\partial^2 w}{\partial x^2} + f(x,t)\frac{\partial w}{\partial x} + g(x,t)w$ 136
1.8.5. Equations of the Form $\frac{\partial w}{\partial t} = ae^{\beta x}\frac{\partial^2 w}{\partial x^2} + f(x,t)\frac{\partial w}{\partial x} + g(x,t)w$ 137
1.8.6. Equations of the Form $\frac{\partial w}{\partial t} = f(x)\frac{\partial^2 w}{\partial x^2} + g(x,t)\frac{\partial w}{\partial x} + h(x,t)w$ 138
1.8.7. Equations of the Form $\frac{\partial w}{\partial t} = f(t)\frac{\partial^2 w}{\partial x^2} + g(x,t)\frac{\partial w}{\partial x} + h(x,t)w$ 146
1.8.8. Equations of the Form $\frac{\partial w}{\partial t} = f(x,t)\frac{\partial^2 w}{\partial x^2} + g(x,t)\frac{\partial w}{\partial x} + h(x,t)w$ 148
1.8.9. Equations of the Form $s(x)\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[p(x)\frac{\partial w}{\partial x}\right] - q(x)w + \Phi(x,t)$ 151
1.9. Equations of Special Form 155
1.9.1. Equations of the Diffusion (Thermal) Boundary Layer 155
1.9.2. One-Dimensional Schrödinger Equation $i\hbar\frac{\partial w}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 w}{\partial x^2} + U(x)w$ 157

2. Equations of Parabolic Type with Two Space Variables 161

2.1. Heat Equation $\frac{\partial w}{\partial t} = a\Delta_2 w$ 161
2.1.1. Boundary Value Problems in Cartesian Coordinates 161
2.1.2. Problems in Polar Coordinates 173
2.1.3. Axisymmetric Problems 179
2.2. Heat Equation with a Source $\frac{\partial w}{\partial t} = a\Delta_2 w + \Phi(x,y,t)$ 187
2.2.1. Problems in Cartesian Coordinates 187
2.2.2. Problems in Polar Coordinates 194
2.2.3. Axisymmetric Problems 195
2.3. Other Equations 198
2.3.1. Equations Containing Arbitrary Parameters 198
2.3.2. Equations Containing Arbitrary Functions 199

3. Parabolic Equations with Three or More Space Variables 205

3.1. Heat Equation $\frac{\partial w}{\partial t} = a\Delta_3 w$ 205
3.1.1. Problems in Cartesian Coordinates 205
3.1.2. Problems in Cylindrical Coordinates 225
3.1.3. Problems in Spherical Coordinates 250
3.2. Heat Equation with Source $\frac{\partial w}{\partial t} = a\Delta_3 w + \Phi(x,y,z,t)$ 254
3.2.1. Problems in Cartesian Coordinates 254
3.2.2. Problems in Cylindrical Coordinates 257
3.2.3. Problems in Spherical Coordinates 260
3.3. Other Equations with Three Space Variables 261
3.3.1. Equations Containing Arbitrary Parameters 261
3.3.2. Equations Containing Arbitrary Functions 263
3.3.3. Equations of the Form $\rho(x,y,z)\frac{\partial w}{\partial t} = \operatorname{div}[a(x,y,z)\nabla w] - q(x,y,z)w + \Phi(x,y,z,t)$ 266

3.4. Equations with n Space Variables 268
3.4.1. Equations of the Form $\frac{\partial w}{\partial t} = a\Delta_n w + \Phi(x_1, \ldots, x_n, t)$ 268
3.4.2. Other Equations Containing Arbitrary Parameters 270
3.4.3. Equations Containing Arbitrary Functions 271

4. Hyperbolic Equations with One Space Variable 279

4.1. Constant Coefficient Equations 279
4.1.1. Wave Equation $\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2}$ 279
4.1.2. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2} + \Phi(x, t)$ 284
4.1.3. Equation of the Form $\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2} - bw + \Phi(x, t)$ 287
4.1.4. Equation of the Form $\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2} - b\frac{\partial w}{\partial x} + \Phi(x, t)$ 291
4.1.5. Equation of the Form $\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2} + b\frac{\partial w}{\partial x} + cw + \Phi(x, t)$ 293
4.2. Wave Equation with Axial or Central Symmetry 295
4.2.1. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a^2\big(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\big)$ 295
4.2.2. Equation of the Form $\frac{\partial^2 w}{\partial t^2} = a^2\big(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\big) + \Phi(r, t)$ 298
4.2.3. Equation of the Form $\frac{\partial^2 w}{\partial t^2} = a^2\big(\frac{\partial^2 w}{\partial r^2} + \frac{2}{r}\frac{\partial w}{\partial r}\big)$ 298
4.2.4. Equation of the Form $\frac{\partial^2 w}{\partial t^2} = a^2\big(\frac{\partial^2 w}{\partial r^2} + \frac{2}{r}\frac{\partial w}{\partial r}\big) + \Phi(r, t)$ 301
4.2.5. Equation of the Form $\frac{\partial^2 w}{\partial t^2} = a^2\big(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\big) - bw + \Phi(r, t)$ 302
4.2.6. Equation of the Form $\frac{\partial^2 w}{\partial t^2} = a^2\big(\frac{\partial^2 w}{\partial r^2} + \frac{2}{r}\frac{\partial w}{\partial r}\big) - bw + \Phi(r, t)$ 305
4.3. Equations Containing Power Functions and Arbitrary Parameters 308
4.3.1. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = (ax + b)\frac{\partial^2 w}{\partial x^2} + c\frac{\partial w}{\partial x} + kw + \Phi(x, t)$ 308
4.3.2. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = (ax^2 + b)\frac{\partial^2 w}{\partial x^2} + cx\frac{\partial w}{\partial x} + kw + \Phi(x, t)$ 312
4.3.3. Other Equations 314
4.4. Equations Containing the First Time Derivative 320
4.4.1. Equations of the Form $\frac{\partial^2 w}{\partial t^2} + k\frac{\partial w}{\partial t} = a^2 \frac{\partial^2 w}{\partial x^2} + b\frac{\partial w}{\partial x} + cw + \Phi(x, t)$ 320
4.4.2. Equations of the Form $\frac{\partial^2 w}{\partial t^2} + k\frac{\partial w}{\partial t} = f(x)\frac{\partial^2 w}{\partial x^2} + g(x)\frac{\partial w}{\partial x} + h(x)w + \Phi(x, t)$ . . . 326
4.4.3. Other Equations 331
4.5. Equations Containing Arbitrary Functions 333
4.5.1. Equations of the Form $s(x)\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\big[p(x)\frac{\partial w}{\partial x}\big] - q(x)w + \Phi(x, t)$ 333
4.5.2. Equations of the Form $\frac{\partial^2 w}{\partial t^2} + a(t)\frac{\partial w}{\partial t} = b(t)\big\{\frac{\partial}{\partial x}\big[p(x)\frac{\partial w}{\partial x}\big] - q(x)w\big\} + \Phi(x, t)$. 335
4.5.3. Other Equations 337

5. Hyperbolic Equations with Two Space Variables 341

5.1. Wave Equation $\frac{\partial^2 w}{\partial t^2} = a^2\Delta_2 w$ 341
5.1.1. Problems in Cartesian Coordinates 341
5.1.2. Problems in Polar Coordinates 346
5.1.3. Axisymmetric Problems 351
5.2. Nonhomogeneous Wave Equation $\frac{\partial^2 w}{\partial t^2} = a^2\Delta_2 w + \Phi(x, y, t)$ 355
5.2.1. Problems in Cartesian Coordinates 355
5.2.2. Problems in Polar Coordinates 357
5.2.3. Axisymmetric Problems 360
5.3. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a^2\Delta_2 w - bw + \Phi(x, y, t)$ 362
5.3.1. Problems in Cartesian Coordinates 362
5.3.2. Problems in Polar Coordinates 366
5.3.3. Axisymmetric Problems 371

5.4. Telegraph Equation $\frac{\partial^2 w}{\partial t^2} + k\frac{\partial w}{\partial t} = a^2\Delta_2 w - bw + \Phi(x, y, t)$ 376
5.4.1. Problems in Cartesian Coordinates 376
5.4.2. Problems in Polar Coordinates 381
5.4.3. Axisymmetric Problems 386
5.5. Other Equations with Two Space Variables 391

6. Hyperbolic Equations with Three or More Space Variables 393

6.1. Wave Equation $\frac{\partial^2 w}{\partial t^2} = a^2\Delta_3 w$ 393
6.1.1. Problems in Cartesian Coordinates 393
6.1.2. Problems in Cylindrical Coordinates 399
6.1.3. Problems in Spherical Coordinates 408
6.2. Nonhomogeneous Wave Equation $\frac{\partial^2 w}{\partial t^2} = a^2\Delta_3 w + \Phi(x, y, z, t)$ 412
6.2.1. Problems in Cartesian Coordinates 412
6.2.2. Problems in Cylindrical Coordinates 412
6.2.3. Problems in Spherical Coordinates 413
6.3. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a^2\Delta_3 w - bw + \Phi(x, y, z, t)$ 414
6.3.1. Problems in Cartesian Coordinates 414
6.3.2. Problems in Cylindrical Coordinates 420
6.3.3. Problems in Spherical Coordinates 429
6.4. Telegraph Equation $\frac{\partial^2 w}{\partial t^2} + k\frac{\partial w}{\partial t} = a^2\Delta_3 w - bw + \Phi(x, y, z, t)$ 434
6.4.1. Problems in Cartesian Coordinates 434
6.4.2. Problems in Cylindrical Coordinates 438
6.4.3. Problems in Spherical Coordinates 448
6.5. Other Equations with Three Space Variables 453
6.5.1. Equations Containing Arbitrary Parameters 453
6.5.2. Equation of the Form $\rho(x, y, z)\frac{\partial^2 w}{\partial t^2} = \operatorname{div}\big[a(x, y, z)\nabla w\big] - q(x, y, z)w + \Phi(x, y, z, t)$ 453
6.6. Equations with n Space Variables 455
6.6.1. Wave Equation $\frac{\partial^2 w}{\partial t^2} = a^2\Delta_n w$ 456
6.6.2. Nonhomogeneous Wave Equation $\frac{\partial^2 w}{\partial t^2} = a^2\Delta_n w + \Phi(x_1, \ldots, x_n, t)$ 457
6.6.3. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a^2\Delta_n w - bw + \Phi(x_1, \ldots, x_n, t)$ 460
6.6.4. Equations Containing the First Time Derivative 463

7. Elliptic Equations with Two Space Variables 467

7.1. Laplace Equation $\Delta_2 w = 0$ 467
7.1.1. Problems in Cartesian Coordinate System 467
7.1.2. Problems in Polar Coordinate System 472
7.1.3. Other Coordinate Systems. Method of Conformal Mappings 476
7.2. Poisson Equation $\Delta_2 w = -\Phi(\mathrm{x})$ 478
7.2.1. Preliminary Remarks. Structure of Solution 478
7.2.2. Problems in Cartesian Coordinate System 480
7.2.3. Problems in Polar Coordinate System 485
7.2.4. Domain of Arbitrary Shape. Method of Conformal Mappings 489
7.3. Helmholtz Equation $\Delta_2 w + \lambda w = -\Phi(\mathrm{x})$ 490
7.3.1. General Remarks, Results, and Formulas 490
7.3.2. Problems in Cartesian Coordinate System 494
7.3.3. Problems in Polar Coordinate System 503
7.3.4. Other Orthogonal Coordinate Systems. Elliptic Domain 508

7.4. Other Equations 510
7.4.1. Stationary Schrödinger Equation $\Delta_2 w = f(x,y)w$ 510
7.4.2. Equation of Convective Heat and Mass Transfer 512
7.4.3. Equations of Heat and Mass Transfer in Anisotropic Media 518
7.4.4. Other Equations Arising in Applications 526
7.4.5. Equations of the Form $a(x)\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + b(x)\frac{\partial w}{\partial x} + c(x)w = -\Phi(x,y)$ 529

8. Elliptic Equations with Three or More Space Variables 533

8.1. Laplace Equation $\Delta_3 w = 0$ 533
8.1.1. Problems in Cartesian Coordinates 533
8.1.2. Problems in Cylindrical Coordinates 535
8.1.3. Problems in Spherical Coordinates 537
8.1.4. Other Orthogonal Curvilinear Systems of Coordinates 539
8.2. Poisson Equation $\Delta_3 w + \Phi(x) = 0$ 539
8.2.1. Preliminary Remarks. Solution Structure 539
8.2.2. Problems in Cartesian Coordinates 544
8.2.3. Problems in Cylindrical Coordinates 554
8.2.4. Problems in Spherical Coordinates 558
8.3. Helmholtz Equation $\Delta_3 w + \lambda w = -\Phi(x)$ 561
8.3.1. General Remarks, Results, and Formulas 561
8.3.2. Problems in Cartesian Coordinates 567
8.3.3. Problems in Cylindrical Coordinates 580
8.3.4. Problems in Spherical Coordinates 588
8.3.5. Other Orthogonal Curvilinear Coordinates 591
8.4. Other Equations with Three Space Variables 593
8.4.1. Equations Containing Arbitrary Functions 593
8.4.2. Equations of the Form $\operatorname{div}[a(x,y,z)\nabla w] - q(x,y,z)w = -\Phi(x,y,z)$ 595
8.5. Equations with n Space Variables 597
8.5.1. Laplace Equation $\Delta_n w = 0$ 597
8.5.2. Other Equations 598

9. Higher Order Partial Differential Equations 601

9.1. Third Order Partial Differential Equations 601
9.2. Fourth Order One-Dimensional Nonstationary Equations 602
9.2.1. Equations of the Form $\frac{\partial w}{\partial t} + a^2\frac{\partial^4 w}{\partial x^4} = \Phi(x,t)$ 602
9.2.2. Equations of the Form $\frac{\partial^2 w}{\partial t^2} + a^2\frac{\partial^4 w}{\partial x^4} = 0$ 605
9.2.3. Equations of the Form $\frac{\partial^2 w}{\partial t^2} + a^2\frac{\partial^4 w}{\partial x^4} = \Phi(x,t)$ 606
9.2.4. Equations of the Form $\frac{\partial^2 w}{\partial t^2} + a^2\frac{\partial^4 w}{\partial x^4} + kw = \Phi(x,t)$ 608
9.2.5. Other Equations 611
9.3. Two-Dimensional Nonstationary Fourth Order Equations 613
9.3.1. Equations of the Form $\frac{\partial w}{\partial t} + a^2\left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4}\right) = \Phi(x,y,t)$ 613
9.3.2. Two-Dimensional Equations of the Form $\frac{\partial^2 w}{\partial t^2} + a^2\Delta\Delta w = 0$ 615
9.3.3. Three- and n-Dimensional Equations of the Form $\frac{\partial^2 w}{\partial t^2} + a^2\Delta\Delta w = 0$ 617
9.3.4. Equations of the Form $\frac{\partial^2 w}{\partial t^2} + a^2\Delta\Delta w + kw = \Phi(x,y,t)$ 619
9.3.5. Equations of the Form $\frac{\partial^2 w}{\partial t^2} + a^2\left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4}\right) + kw = \Phi(x,y,t)$ 620
9.4. Fourth Order Stationary Equations 621
9.4.1. Biharmonic Equation $\Delta\Delta w = 0$ 621
9.4.2. Equations of the Form $\Delta\Delta w = \Phi(x,y)$ 625

9.4.3. Equations of the Form $\Delta\Delta w - \lambda w = \Phi(x, y)$ 626
9.4.4. Equations of the Form $\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} = \Phi(x, y)$ 628
9.4.5. Equations of the Form $\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + kw = \Phi(x, y)$ 629
9.4.6. Stokes Equation (Axisymmetric Flows of Viscous Fluids) 630
9.5. Higher Order Linear Equations with Constant Coefficients 633
9.5.1. Fundamental Solutions. Cauchy Problem 633
9.5.2. Differential Equations of Elliptic Type 635
9.5.3. Differential Equations of Hyperbolic Type 637
9.5.4. Regular Equations. Number of Initial Conditions in the Cauchy Problem 637
9.5.5. Some Equations of Special Type 640
9.6. Higher Order Linear Equations with Variable Coefficients 644
9.6.1. Equations Containing the First Time Derivative 644
9.6.2. Equations Containing the Second Time Derivative 648
9.6.3. Nonstationary Problems with Many Space Variables 649
9.6.4. Some Equations of Special Type 651

Supplement A. Special Functions and Their Properties 655

A.1. Some Symbols and Coefficients 655
A.1.1. Factorials 655
A.1.2. Binomial Coefficients 655
A.1.3. Pochhammer Symbol 656
A.1.4. Bernoulli Numbers 656
A.2. Error Functions and Exponential Integral 656
A.2.1. Error Function and Complementary Error Function 656
A.2.2. Exponential Integral 656
A.2.3. Logarithmic Integral 657
A.3. Sine Integral and Cosine Integral. Fresnel Integrals 657
A.3.1. Sine Integral 657
A.3.2. Cosine Integral 658
A.3.3. Fresnel Integrals 658
A.4. Gamma and Beta Functions 659
A.4.1. Gamma Function 659
A.4.2. Beta Function 660
A.5. Incomplete Gamma and Beta Functions 660
A.5.1. Incomplete Gamma Function 660
A.5.2. Incomplete Beta function 661
A.6. Bessel Functions 661
A.6.1. Definitions and Basic Formulas 661
A.6.2. Integral Representations and Asymptotic Expansions 663
A.6.3. Zeros and Orthogonality Properties of Bessel Functions 664
A.6.4. Hankel Functions (Bessel Functions of the Third Kind) 665
A.7. Modified Bessel Functions 666
A.7.1. Definitions. Basic Formulas 666
A.7.2. Integral Representations and Asymptotic Expansions 667
A.8. Airy Functions 668
A.8.1. Definition and Basic Formulas 668
A.8.2. Power Series and Asymptotic Expansions 668

A.9. Degenerate Hypergeometric Functions 669
A.9.1. Definitions and Basic Formulas 669
A.9.2. Integral Representations and Asymptotic Expansions 671
A.10. Hypergeometric Functions 672
A.10.1. Definition and Some Formulas 672
A.10.2. Basic Properties and Integral Representations 672
A.11. Whittaker Functions 672
A.12. Legendre Polynomials and Legendre Functions 674
A.12.1. Definitions. Basic formulas 674
A.12.2. Zeros of Legendre Polynomials and the Generating Function 674
A.12.3. Associated Legendre Functions 674
A.13. Parabolic Cylinder Functions 675
A.13.1. Definitions. Basic Formulas 675
A.13.2. Integral Representations and Asymptotic Expansions 675
A.14. Mathieu Functions 675
A.14.1. Definitions and Basic Formulas 675
A.15. Modified Mathieu Functions 677
A.16. Orthogonal Polynomials 677
A.16.1. Laguerre Polynomials and Generalized Laguerre Polynomials 678
A.16.2. Chebyshev Polynomials and Functions 679
A.16.3. Hermite Polynomial 679
A.16.4. Jacobi Polynomials 680

Supplement B. Methods of Generalized and Functional Separation of Variables in Nonlinear Equations of Mathematical Physics 681

B.1. Introduction 681
B.1.1. Preliminary Remarks 681
B.1.2. Simple Cases of Variable Separation in Nonlinear Equations 682
B.1.3. Examples of Nontrivial Variable Separation in Nonlinear Equations 683
B.2. Methods of Generalized Separation of Variables 684
B.2.1. Structure of Generalized Separable Solutions 684
B.2.2. Solution of Functional Differential Equations by Differentiation 685
B.2.3. Solution of Functional Differential Equations by Splitting 688
B.2.4. Simplified Scheme for Constructing Exact Solutions of Equations with Quadratic Nonlinearities 690
B.3. Methods of Functional Separation of Variables 691
B.3.1. Structure of Functional Separable Solutions 691
B.3.2. Special Functional Separable Solutions 692
B.3.3. Method of Differentiation 694
B.3.4. Splitting Method. Reduction to a Functional Equation with Two Variables 698
B.3.5. Some Functional Equations and Their Solutions. Exact Solutions of Heat and Wave Equations 699
B.4. First Order Nonlinear Equations 704
B.4.1. Preliminary Remarks 704
B.4.2. Individual Equations 705
B.5. Second Order Nonlinear Equations 707
B.5.1. Parabolic Equations 707
B.5.2. Hyperbolic Equations 719
B.5.3. Elliptic Equations 726
B.5.4. Equations Containing Mixed Derivatives 733

B.5.5. Equations of General Form 736
B.6. Third Order Nonlinear Equations 739
B.6.1. Equations of Stationary Hydrodynamic Boundary Layer 739
B.6.2. Equations of Nonstationary Hydrodynamic Boundary Layer 741
B.7. Fourth Order Nonlinear Equations 749
B.7.1. Stationary Hydrodynamic Equations (Navier–Stokes Equations) 749
B.7.2. Nonstationary Hydrodynamic Equations 752
B.8. Higher Order Nonlinear Equations 757
B.8.1. Equations of the Form $\frac{\partial w}{\partial t} = F\big(x, t, w, \frac{\partial w}{\partial x}, \ldots, \frac{\partial^n w}{\partial x^n}\big)$ 757
B.8.2. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = F\big(x, t, w, \frac{\partial w}{\partial x}, \ldots, \frac{\partial^n w}{\partial x^n}\big)$ 761
B.8.3. Other Equations 765

References **769**

Index **777**