# Continuous symmetries of Lagrangians and exact solutions of discrete equations

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## Abstract

One of the difficulties encountered when studying physical theories in discrete space-time is that of describing the underlying continuous symmetries (like Lorentz, or Galilei invariance). One of the ways of addressing this difficulty is to consider point transformations acting simultaneously on difference equations and lattices. In a previous article we have classified ordinary difference schemes invariant under Lie groups of point transformations. The present article is devoted to an invariant Lagrangian formalism for scalar single-variable difference schemes. The formalism is used to obtain first integrals and explicit exact solutions of the schemes. Equations invariant under two- and three- dimensional groups of Lagrangian symmetries are considered.

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# 1 Introduction

A recent article was devoted to a symmetry classification of second order ordinary difference equations [1]. This was modeled on a paper by Sophus Lie, in which he provided a symmetry classification of second order differential equations (ODEs) [2]. As a matter of fact, the classification of difference schemes goes over into Lie's classification of ODEs in the continuous limit [1].

S. Lie showed that a second order ODE can be invariant under a group $G_r$ of dimension $N = 0, 1, 2, 3,$ or 8. For $N \geq 2$ the equation can be integrated in quadratures. This can be done by transforming the equation to one of the "canonical" forms, integrated by Lie himself [2]. Virtually all standard methods of integrating second order ODEs analytically can be interpreted in this manner (though this is not mentioned in most elementary textbooks).

The situation with difference equations is much less developed. This is not surprising, since applications of Lie group theory to difference equations are much more recent [3,...,27]. Several different approaches are being pursued. One possibility is to consider the difference equations on a fixed lattice [3,...,13] and consider only transformations that do not act on the lattice. In order to obtain physically interesting symmetries in this approach, it is necessary to go beyond point symmetries and to let the transformations act on more than one point of the lattice. Lie algebra contractions occur in the continuous limit and some "generalized" symmetries may "contract" to point ones [10].

The second possibility is to consider group transformations acting both on the difference equations and on the lattice [1,17,...,27]. Technically, for systems involving one dependent and one independent variable, this is achieved by considering a difference scheme, consisting of two equations, one representing the actual difference equation, the other the lattice.

This is the approach that we will follow in the present article. More specifically, we will consider the same three–point scheme as in our previous article [1]. The continuous limit of the scheme, if it exists, will be a second order ODE.

Thus, we consider two variables, $x$ and $y$, with $x$ the independent one and $y$ dependent. The variable $x$ runs through an infinite set of values $\{x = x_k, \ k \in \mathbb{Z}\}$ that are not necessarily equally spaced and are not prescribed a priori. Instead, we give two relations between any three neighboring points

$$F(x, x_-, x_+, y, y_-, y_+) = 0, \tag{1.1}$$

$$\Omega(x, x_-, x_+, y, y_-, y_+) = 0 \tag{1.2}$$

and also specify some initial conditions like $x_0$, $x_1$, $y_0 = y(x_0)$, $y_1 = y(x_1)$. In the continuous limit eq. (1.1) goes into an ODE, (1.2) into an identity (like $0 = 0$), if the continuous limit exists.

The group transformations considered in this approach are of the same type as for ODEs. They are generated by a Lie algebra of vector fields of the form

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}. \tag{1.3}$$

The corresponding transformations are purely point ones, since the coefficients $\xi$ and $\eta$ depend only on $(x, y)$, not on the shifted points $(x_+, y_+)$ or $(x_-, y_-)$.

In [1] we showed how Lie group theory can be used to classify such pairs of equations as (1.1) and (1.2). Possible dimensions of the symmetry group $G$ of eq. (1.1), (1.2) are $N = 0, 1, 2, 3, 4, 5$, and 6. The highest dimension, $N = 6$, occurs only for difference schemes equivalent to

$$\frac{y_+ - 2y + y_-}{(x_+ - x)^2} = 0, \qquad x_+ - 2x + x_- = 0.$$

The purpose of this article is to provide a Lagrange formalism and difference analog of Noether's theorem for second order difference schemes of the form (1.1) and (1.2), admitting Lie point symmetry groups. The Lagrangians will be used to obtain first integrals and exact analytic solutions of the difference schemes.

## 2  General theory

### 2.1  Definitions and notations

We study the difference system (1.1) and (1.2). In general, we assume that these equations can be solved to express $x_+$ and $y_+$ explicitly in terms of $(x, y, x_-, y_-)$ and also vice verse, i.e. $(x_-, y_-)$ in terms of the other quantities. We also make use of the following quantities

$$h_+ = x_+ - x, \qquad h_- = x - x_-, \qquad y_x = \frac{y_+ - y}{h_+}, \qquad y_{\bar{x}} = \frac{y - y_-}{h_-},$$
$$\tag{2.1}$$
$$y_{x\bar{x}} = \frac{2}{h_+ + h_-}(y_x - y_{\bar{x}}),$$

i.e. the up and down spacings in $x$, the right and left discrete first derivatives and the discrete second derivative, respectively. It is also convenient to use the following total shift and discrete differentiation operators:

$$\underset{\pm h}{S} f(x) = f(x_\pm), \quad \underset{\pm h}{D} = \frac{\underset{\pm h}{S} - 1}{\pm h_\pm}.$$

Continuous first and second derivatives are denoted $y'$ and $y''$, respectively.

When acting on differential equations, the vector fields (1.3) must be prolonged to act on derivatives. For difference schemes, the prolongation of a vector field acts on variables at other points of the lattice. It is obtained by shifting the coefficients to the corresponding points. For three point schemes we have

$$\mathbf{pr}X = X + \xi(x_-,y_-)\frac{\partial}{\partial x_-} + \xi(x_+,y_+)\frac{\partial}{\partial x_+} + \eta(x_-,y_-)\frac{\partial}{\partial y_-} + \eta(x_+,y_+)\frac{\partial}{\partial y_+}.$$
(2.2)

## 2.2 Lagrangian formulation for second order ODEs

It has been known since E. Noether's fundamental work that conservation laws for differential equations are connected with their symmetry properties [28,...,31]. For convenience we present here some well–known results adapted to the case of second order ODEs.

Let us consider the functional

$$\mathbb{L}(y) = \int_I L(x,y,y')dx, \quad I \subset \mathbb{R}^1,$$
(2.3)

where $L(x,y,y')$ is called a first order Lagrangian. The functional (2.3) achieves its extremal values when $y(x)$ satisfies the Euler-Lagrange equation

$$\frac{\delta L}{\delta y} = \frac{\partial L}{\partial y} - D\left(\frac{\partial L}{\partial y'}\right) = 0, \qquad D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots,$$
(2.4)

where $D$ is the total derivative operator. The equation (2.4) is an ODE that can be rewritten as

$$y'' = f(x,y,y').$$
(2.5)

Let us consider a Lie point transformation $G$ generated by the vector field (1.3). The group $G$ is a "variational symmetry" of the functional $\mathcal{L}$ if and only if the Lagrangian satisfies

$$\mathrm{pr}X(L) + LD(\xi) = 0,$$
(2.6)

when $\mathrm{pr}X$ is the first prolongation of the vector field $X$ for $y'$. We will actually need a weaker invariance condition than given by eq. (2.6). The vector field $X$ is an "infinitesimal divergence symmetry" of the functional $\mathcal{L}(y)$ if there exists a function $V(x,y)$ such that [28]

$$\mathrm{pr}X(L) + LD(\xi) = D(V), \qquad V = V(x,y).$$
(2.7)

The two important statements for us are:

**1.** If $X$ is an infinitesimal divergence symmetry of the functional $\mathcal{L}$, it generates a symmetry group of the corresponding Euler-Lagrange equation. The

symmetry group of eq. (2.4) can of course be larger than the one generated by symmetries of the Lagrangian.

**2.** Noether's theorem [28,...,31] can be based on the following Noether-type identity [31], which holds for any vector field and any function $L$:

$$\mathrm{pr}X(L) + LD(\xi) = (\eta - \xi y')\frac{\delta L}{\delta y} + D(\xi L + (\eta - \xi y')\frac{\partial L}{\partial y'}). \qquad (2.8)$$

It follows that if $X$ is a divergence symmetry of $L$, i.e. (2.6) or (2.7) is satisfied, then there exists a first integral

$$\xi L + (\eta - \xi y')\frac{\partial L}{\partial y'} - V = K = const \qquad (2.9)$$

of the corresponding Euler-Lagrange equation.

The above considerations tell us how to obtain invariant ODEs and conservation laws from divergence invariant Lagrangians. They do not tell us how to obtain invariant Lagrangians for invariant equations. This amounts to "variational integration", as opposed to variational differentiation.

A procedure that we shall use below to find invariant Lagrangians for differential equations can be summed up as follows.

Start from a given ODE $y'' = f(x, y, y')$ and its symmetry algebra with basis

$$X_\alpha = \xi_\alpha(x, y)\frac{\partial}{\partial x} + \eta_\alpha(x, y)\frac{\partial}{\partial y}, \qquad \alpha = 1, ..., k.$$

Find the invariants of $X_\alpha$ in the space $\{x, y, y', \Lambda\}$, where $\Lambda$ is the Lagrangian. The appropriate prolongation in this case is

$$prX = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta^1\frac{\partial}{\partial y'} - (D\xi)\Lambda\frac{\partial}{\partial \Lambda}, \qquad \zeta^1 = D(\eta) - y'D(\xi) \qquad (2.10)$$

and we require that $L(x, y, y')$ should satisfy

$$prX(\Lambda - L)|_{\Lambda=L} = 0. \qquad (2.11)$$

Each basis element $X_\alpha$ provides us with an equation of the form

$$\xi_\alpha\frac{\partial L}{\partial x} + \eta_\alpha\frac{\partial L}{\partial y} + \zeta^1_\alpha\frac{\partial L}{\partial y'} - LD(\xi_\alpha) = 0. \qquad (2.12)$$

Solve the partial differential equations (2.12). This will give us the general form of an invariant Lagrangian. It may involve arbitrary functions of the invariants of $X$.

Request that the Euler-Lagrange equation (2.4) should coincide with the equation we started from. This will further restrict the invariant Lagrangian and determine whether one exists.

If this procedure does not yield a suitable Lagrangian, then step 1 can be weakened. We can request that the Lagrangian be invariant under some subgroup of the symmetry group of the given ODE, rather then the entire group. We then go through step 2, then verify whether the obtained Lagrangian is divergence invariant under the entire group, or at least a larger subgroup. In any case, each divergence symmetry of the Lagrangian will provide a first integral of the ODE.

For ODEs the Lagrangian formalism is not the only integration method. The existence of one-parameter symmetry group provides a reduction to a first-order ODE directly. The existence of a two-parameter symmetry group makes it possible to integrate in quadratures. An invariant Lagrangian provides an alternative. Indeed, assume that we know two first integrals

$$f_1(x, y, y') = A, \qquad f_2(x, y, y') = B, \tag{2.13}$$

then we eliminate $y'$ from these two equations and obtain the general solution

$$y = F(x, A, B), \tag{2.14}$$

of the corresponding ODE (2.5) by purely *algebraic* manipulations. It is this method of invariant Lagrangians that generalizes to difference equations and is particularly useful when direct methods fail.

## 2.3 Lagrangian formalism for second order difference equations

The variational formulation of discrete equations and a discrete analog of Noether's theorem are much more recent [19,25,26,27]. Here we briefly overview the results that we shall need below.

Let us consider a finite difference functional

$$\mathbb{L}_h = \sum_{\Omega} \mathcal{L}(x, x_+, y, y_+)h_+, \tag{2.15}$$

defined on some one–dimensional lattice $\Omega$ with step $h_+$ that generally can depend on the solution

$$h_+ = \varphi(x, y, x_+, y_+). \tag{2.16}$$

The Lagrangian (2.15) must be considered together with a lattice (2.16). On different lattices it can have different continuous limits and in this limit the lattice equation itself vanishes (turns into an identity like $0 = 0$)

In the continuous case, a Lagrangian $L$ provides an equation (the Euler-Lagrange equation) that inherits all the symmetries of $L$. In the discrete case we wish the Lagrangian (2.15) to provide two equations: the entire difference system (1.1),(1.2). Moreover, the three-point difference system should inherit the symmetries of two-point Lagrangian.

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Let us again consider a Lie group of point transformations, generated by a Lie algebra of vector fields $X_\alpha$ of the form (1.3). The infinitesimal invariance condition of the functional (2.15) on the lattice (2.16) is given by two equations [19,25,27]:

$$\xi\frac{\partial\mathcal{L}}{\partial x} + \xi^+\frac{\partial\mathcal{L}}{\partial x_+} + \eta\frac{\partial\mathcal{L}}{\partial y} + \eta^+\frac{\partial\mathcal{L}}{\partial y_+} + \mathcal{L}\underset{+h}{D}(\xi) = 0,$$

$$(2.17)$$

$$\underset{+h}{S}(\xi) - \xi = X(\varphi),$$

where

$$\xi^+ = \xi(x_+, y_+), \quad \eta^+ = \eta(x_+, y_+). \tag{2.18}$$

Let us consider a variation of the difference functional (2.15) along some curve $y = \phi(x)$ at some point $(x, y)$. The variation will effect only two terms in the sum (2.15):

$$\mathbb{L}_h = ... + \mathcal{L}(x, x_-, y, y_-)h_- + \mathcal{L}(x, x_+, y, y_+)h_+ + ..., \tag{2.19}$$

so we get the following expression for the variation of the difference functional

$$\delta L = \frac{\delta\mathcal{L}}{\delta x}\delta x + \frac{\delta\mathcal{L}}{\delta y}\delta y, \tag{2.20}$$

where $\delta y = \phi'\delta x$ and

$$\frac{\delta\mathcal{L}}{\delta x} = h_+\frac{\partial\mathcal{L}}{\partial x} + h_-\frac{\partial\mathcal{L}^-}{\partial x} + \mathcal{L}^- - \mathcal{L}, \qquad \frac{\delta\mathcal{L}}{\delta y} = h_+\frac{\partial\mathcal{L}}{\partial y} + h_-\frac{\partial\mathcal{L}^-}{\partial y},$$

where $\mathcal{L}^- = \underset{-h}{S}(\mathcal{L})$.

Thus, for an arbitrary curve the stationary value of difference functional is given by any solution of the *two* equations, called *quasiextremal equations*

$$\frac{\delta\mathcal{L}}{\delta x} = 0, \qquad \frac{\delta\mathcal{L}}{\delta y} = 0. \tag{2.21}$$

Both of them tend to the differential Euler-Lagrange equation in the continuous limit. Together they represent the entire difference scheme and could be called "the discrete Euler-Lagrange system". The difference between these two equations, or some other function of them that vanishes in the continuous limit will represent the lattice.

Now let us consider a vector field (1.3) with given coefficients $\xi(x, y)$ and $\eta(x, y)$. Variations along the integral curves of this vector field are given by $\delta x = \xi da$ and $\delta y = \eta da$, where $da$ is a variation of a group parameter. A stationary value of the difference functional (2.15) along the flow generated by this vector field is given by the equation:

$$\xi\frac{\delta\mathcal{L}}{\delta x} + \eta\frac{\delta\mathcal{L}}{\delta y} = 0, \tag{2.22}$$

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which depends explicitly on the coefficients of the generator .

If we have a Lie algebra of vector fields of dimension 2 or more, then a stationary value of the difference functional (2.15) along the entire flow will be achieved on the intersection of the solutions of all equations of the type (2.22), i.e. on the quasiextremals (2.21).

On the other hand, equations (2.21) can be interpreted as a three-point difference scheme of the form (1.1),(1.2). For instance, given two points $(x, y)$ and $(x_-, y_-)$, we can calculate $(x_+, y_+)$. In the continuous limit both of these equations will provide the same second-order differential equation. Thus, one of the quasiextremal equations can be identified with eq. (1.1) and the difference between the two of them with the lattice equation (1.2).

It has been shown elsewhere [19,25,27], that if the functional (2.15) is invariant under some group $G$, then the quasiextremal equations (2.21) are also invariant with respect to $G$. As in the continuous case, the quasiextremal equations can be invariant with respect to a larger group than the corresponding Lagrangian.

A useful operator identity, valid for any Lagrangian $\mathcal{L}(x, x_+, y, y_+)$ and any vector field $X$ is ([19,25]):

$$\xi\frac{\partial \mathcal{L}}{\partial x} + \xi^+\frac{\partial \mathcal{L}}{\partial x_+} + \eta\frac{\partial \mathcal{L}}{\partial y} + \eta^+\frac{\partial \mathcal{L}}{\partial y_+} + \mathcal{L}\underset{+h}{D}(\xi) =$$

$$= \xi\left(\frac{\partial \mathcal{L}}{\partial x} + \frac{h_-}{h_+}\frac{\partial \mathcal{L}^-}{\partial x} - \underset{+h}{D}(\mathcal{L}^-)\right) + \eta\left(\frac{\partial \mathcal{L}}{\partial y} + \frac{h_-}{h_+}\frac{\partial \mathcal{L}^-}{\partial y}\right) + \qquad (2.23)$$

$$+ \underset{+h}{D}\left(h_-\eta\frac{\partial \mathcal{L}^-}{\partial y} + h_-\xi\frac{\partial \mathcal{L}^-}{\partial x} + \xi\mathcal{L}^-\right).$$

From eq. (2.23) we obtain the following discrete analog of Noether's theorem.

**Theorem 2.1** Let the Lagrangian density $\mathcal{L}$ be divergence invariant under a Lie group $G$ of local point transformations generated by vector fields $X$ of the form (1.3), i.e. let us have

$$\mathbf{pr}X(\mathcal{L}) + \mathcal{L}\underset{+h}{D}(\xi) = \underset{+h}{D}(V) \qquad (2.24)$$

for some function $V(x, y)$. Then each element $X$ of the Lie algebra corresponding to $G$ provides us with a first integral of the quasiextremal equations (2.21), namely

$$K = h_-\eta\frac{\partial \mathcal{L}^-}{\partial y} + h_-\xi\frac{\partial \mathcal{L}^-}{\partial x} + \xi\mathcal{L}^- - V. \qquad (2.25)$$

*Proof.* [19,25] On solutions of the quasiextremal equations (2.21) eq. (2.23) reduces to

$$\underset{+h}{D}\left(h_-\eta\frac{\partial \mathcal{L}^-}{\partial y} + h_-\xi\frac{\partial \mathcal{L}^-}{\partial x} + \xi\mathcal{L}^-\right) = \underset{+h}{D}(V) \qquad (2.26)$$

(we have used eq. (2.23)). The result (2.25) follows immediately. $\qquad \square$

The fundamental equation (2.25) is the discrete analog of eq. (2.9) for ODEs.

Let us compare the situation for second order ODEs and for three-point difference schemes. For a second order ODE a Lagrangian that is divergence invariant under a two-dimensional symmetry group provides two integrals of motion. From them we can eliminate the remaining first derivative and obtain the general solution, depending on two arbitrary constants (the two first integrals). Moreover, we do not really need a Lagrangian. Once we have a two dimensional symmetry group of the ODE, we can integrate in quadratures.

For three-point difference schemes we have two equations to solve, namely the system (1.1),(1.2). Equivalently, we have a set of points $(x_n, y_n)$, labeled by an integer $n$. Any 3 neighboring points are related by two equations that we can write e.g. as

$$y_{n+1} = F_1(x_n, y_n, x_{n-1}, y_{n-1}), \qquad x_{n+1} = \Omega_1(x_n, y_n, x_{n-1}, y_{n-1}). \qquad (2.27)$$

Alternatively, the system could be solved for $x_{n-1}, y_{n-1}$. We mention that we use notations like $x_{n-1} = x_-$, $x_n = x$, $x_{n+1} = x_+$, $y_{n-1} = y_-$, $y_n = y$, $y_{n+1} = y_+$ interchangeably.

Given some starting values $(x_0, y_0, x_{-1}, y_{-1})$, we can solve (2.27) for $(x_n, y_n)$ with $n \geq 1$, and $n \leq -2$. The solution will depend on four constants $K_i, i = 1, ..., 4$, and can be written as

$$y_n = y_n(x_n, K_1, K_2, K_3, K_4), \qquad (2.28)$$

$$x_n = x_n(K_1, K_2, K_3, K_4). \qquad (2.29)$$

The two quasiextremal equations (2.21) correspond to the system (2.27).

A one-parameter symmetry group of the Lagrangian $\mathcal{L}$ will provide us with a first integral (2.25), i.e. an equation of the form

$$f(x_n, y_n, x_{n+1}, y_{n+1}) = K_1. \qquad (2.30)$$

compatible with the system (2.21). We can solve (2.30) for e.g. $y_{n+1}$, substitute into (2.27) and thus simplify this system.

A two-dimensional symmetry group will provide two first integrals of the form (2.25). We can solve for $x_{n+1}$ and $y_{n+1}$. Then system (2.27) is reduced to a two-point difference scheme. Quite often it is possible to solve it by integration methods that allow one to integrate a two-point difference scheme explicitly.

A three-dimensional symmetry group provides three first integrals of the type (2.25). From them we can express $x_{n-1}, y_{n-1}$ and $y_n$ in terms of $x_n$. This provides us with the solution (2.28) and a two-point difference equation relating $x_{n+1}$ and $x_n$. If this equation can be solved, we have a complete solution of the problem. Finally, if we have four first integrals, then we get the general solution of the system by purely algebraic manipulations.

An alternative method can be proposed when the Lagrangian is invariant with respect to a two-dimensional Lie group. The discrete Lagrangian corresponding to a given continuous one is not unique and it is possible to introduce a family of Lagrangians:

$$\mathcal{L}_i = \mathcal{L}_i(x, x_+, y, y_+, \alpha_i, \beta_i), \quad i = 1, 2, 3, ... \tag{2.31}$$

depending on parameters $\alpha_i, \beta_i$, all satisfying

$$\lim_{(x_+, y_+) \to (x, y)} \mathcal{L}_i(x, x_+, y, y_+, \alpha_i, \beta_i) = \mathcal{L}(x, y, y')$$

for the same continuous Lagrangian $\mathcal{L}(x, y, y')$.

Let us take three different Lagrangians in the family (2.31), corresponding to constants $\alpha_1, \beta_1, \alpha_2, \beta_2$ and $\alpha_3, \beta_3$. Each of them will lead to two first integrals and two quasiextremals. In examples considered below we will show that it is possible to fine-tune the constants $\alpha_i, \beta_i$ in such a manner as to get a system of two invariant equations of the form (1.1),(1.2) and three first integrals, yielding a set of solutions to the two quasiextremal equations. It is these two equations that will constitute the invariant difference system.

In Section 2.2 we described a procedure for obtaining invariant Lagrangians for given second-order differential equation. For difference equations our starting point will be a discretization of the continuous Lagrangian. This is obviously not unique and we shall make use of the inherent arbitrariness. Once an invariant difference Lagrangian with a correct continuous limit is chosen we construct the invariant difference scheme in the manner described above.

In our previous article [1] we gave a classification of difference schemes and used all realizations of Lie algebras that provide such schemes. Any algebra containing a two-dimensional subalgebra realized by linearly connected vector fields such as

$$\left( \frac{\partial}{\partial x}, \quad y\frac{\partial}{\partial x} \right), \quad \left( \frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial x} \right),$$

leads to a linear differential equation and its discretization.

Below we shall consider only genuinely nonlinear difference schemes presented in Ref [1] that have nonlinear differential equations as their limit.

# 3 Equations corresponding to Lagrangians invariant under one and two-dimensional groups

## 3.1 One-dimensional symmetry group

We start with the simplest case of a symmetry group, namely a one–dimensional group. Its Lie algebra is generated by one vector field of the form (1.3). By an

appropriate change of variables we take this vector field into its rectified form. Thus we have

$$\mathbf{D_{1,1}}: \qquad X_1 = \frac{\partial}{\partial y}. \tag{3.1}$$

The most general second order ODE invariant under $X_1$ is

$$y'' = F(x, y'), \tag{3.2}$$

where $F$ is an arbitrary given function.

Eq. (3.2) is actually already reduced to a first order equation for $u = y'$. If $X_1$ is a variational symmetry of eq. (3.2) and we know the Lagrangian $L$ that it comes from, we can do better. An invariant Lagrangian density will by necessity have the form $L = L(x, y')$ (see eq. (2.6)). The Euler-Lagrange equation (2.4) reduces to

$$\frac{\partial^2 L}{\partial x \partial y'} + y'' \frac{\partial^2 L}{\partial y'^2} = 0 \tag{3.3}$$

Substituting for $y''$ from eq. (3.2), we obtain a linear partial differential equation for $L(x, y')$. This of course has an infinity of solutions. Let us assume that we know a solution $L(x, y)$ explicitly. Eq. (2.9), i.e. Noether's theorem, provides us with a first integral

$$\frac{\partial L}{\partial y'}(x, y') = K. \tag{3.4}$$

We can solve (in principle) eq. (3.4) for $y'$ as a function of $x$ (and $K$). The general solution is then obtained by a quadrature:

$$y' = \phi(x, K), \qquad y(x) = y_0 + \int_0^x \phi(x, K) dt. \tag{3.5}$$

In the discrete case the situation is similar. Let us assume that we know a Lagrangian $\mathcal{L}(x, x_+, y, y_+)$, invariant under the group of transformations of $y$, generated by $X_1$ of eq. (3.1). It will have the form

$$\mathcal{L} = \mathcal{L}(x, x_+, y_x), \qquad y_x = \frac{y_+ - y}{x_+ - x}. \tag{3.6}$$

The corresponding quasiextremal equations, to be identified with the system (1.1) and (1.2), are

$$\frac{\delta \mathcal{L}}{\delta y} = -\frac{\partial \mathcal{L}}{\partial y_x}(x, x_+, y_x) + \frac{\partial \mathcal{L}}{y_{\bar{x}}}(x_-, x, y_{\bar{x}}) = 0; \tag{3.7}$$

$$\frac{\delta \mathcal{L}}{\delta x} = h_+ \frac{\partial \mathcal{L}}{\partial x}(x, x_+, y_x) + y_x \frac{\partial \mathcal{L}}{\partial y_x}(x, x_+, y_x) - \mathcal{L}(x, x_+, y_x)$$

$$+ h_- \frac{\partial \mathcal{L}}{\partial x}(x_-, x, y_{\bar{x}}) - y_{\bar{x}} \frac{\partial \mathcal{L}}{\partial y_{\bar{x}}}(x_-, x, y_{\bar{x}}) + \mathcal{L}(x_-, x, y_{\bar{x}}) = 0. \tag{3.8}$$

The first integral (2.25) can be read off form eq. (3.7) and is

$$\frac{\partial \mathcal{L}}{\partial y_x}(x, x_+, y_x) = K. \tag{3.9}$$

We can solve eq. (3.9) for $y_x$ and by down shifting obtain $y_{\bar{x}}$:

$$y_x = \phi(x, x_+, K), \qquad y_{\bar{x}} = \phi(x_-, x, K). \tag{3.10}$$

Substituting into the quasiextremal equation (3.8), we obtain a relation between $x_+$, $x_-$ and $x$, i.e. a single three-point relation for the variable $x$. For $y$ we then obtain a two point equation

$$y_+ - y = (x_+ - x)\phi(x, x_+, K). \tag{3.11}$$

Equation (3.11) is really a discrete quadrature: a first order inhomogeneous linear equation for $y$.

**Example 3.1** Consider the Lagrangian

$$\mathcal{L} = x_n^a x_{n+1}^b \exp(y_x). \tag{3.12}$$

The quasiextremal equations are

$$-x_n^a x_{n+1}^b \exp(y_x) + x_{n-1}^a x_n^b \exp(y_{\bar{x}}) = 0; \tag{3.13}$$

$$h_+ a x_n^{a-1} x_{n+1}^b \exp(y_x) + y_x x_n^a x_{n+1}^b \exp(y_x) - x_n^a x_{n+1}^b \exp(y_x)$$
$$+ h_- b x_{n-1}^a x_n^{b-1} \exp(y_{\bar{x}}) - y_{\bar{x}} x_{n-1}^a x_n^b \exp(y_{\bar{x}}) + x_{n-1}^a x_n^b \exp(y_{\bar{x}}) = 0. \tag{3.14}$$

The first integral is

$$x_n^a x_{n+1}^b \exp(y_x) = K. \tag{3.15}$$

From (3.15) we have

$$y_x = \ln(K x_n^{-a} x_{n+1}^{-b}), \qquad y_{\bar{x}} = \ln(K x_{n-1}^{-a} x_n^{-b}). \tag{3.16}$$

Eq. (3.13) is satisfied identically. Eq. (3.14) reduces to a three point equation for $x$:

$$a x_{n+1} + (b-a) x_n - b x_{n-1} + x_n(-b \ln(x_{n+1}) + (b-a) \ln(x_n) + a \ln(x_{n-1})) = 0. \tag{3.17}$$

This lattice equation can be reduced to a two-point equation for a new variable $\lambda_n = x_{n+1}/x_n$:

$$a\lambda_n + (b-a) - \frac{b}{\lambda_{n-1}} = a \ln(\lambda_{n-1}) + b \ln(\lambda_n). \tag{3.18}$$

In particular, one can choose the solution $\lambda_n = \lambda_{n+1} = \lambda$, where $\lambda$ satisfies the equation

$$a\lambda + (b - a) - \frac{b}{\lambda} = (a + b)\ln(\lambda).$$

It provides us with the lattice $x_n = x_0\lambda^n$.

Substituting the lattice into (3.15), we obtain a two point equation for $y$:

$$y_{n+1} - y_n = (x_{n+1} - x_n)\ln(Kx_n^{-a}x_{n+1}^{-b}). \tag{3.19}$$

The fact that we could solve eq. (3.17) explicitly is specific for the considered example. The fact that we obtain a three-point equation involving only the independent variables is true in general.

## 3.2 Two-dimensional symmetry groups

$\mathbf{D_{2,1}}$ The Abelian Lie algebra with non-connected basis elements

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y} \tag{3.20}$$

corresponds to the invariant ODE

$$y'' = F(y'), \tag{3.21}$$

where $F$ is an arbitrary function.

The equation can be obtained from the Lagrangian

$$L = y + G(y'), \qquad F(y') = \frac{1}{G''(y')}. \tag{3.22}$$

The Lagrangian admits symmetries $X_1$ and $X_2$:

$$\mathrm{pr}X_1L + LD(\xi_1) = 0;$$

$$\mathrm{pr}X_2L + LD(\xi_2) = 1 = D(x).$$

With the help of Noether's theorem we obtain the following first integrals:

$$J_1 = y + G(y') - y'G'(y'), \qquad J_2 = G'(y') - x. \tag{3.23}$$

As we mentioned in the Section 2.2, it is sufficient to have two first integrals to write out the general solution of a second order ODE without quadratures. More explicitly, we can solve the second equation (3.23) for $y'$ in terms of $x$ and obtain

$$y' = H(J_2 + x), \quad H(J_2 + x) = [G']^{-1}(J_2 + x). \tag{3.24}$$

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Substituting into the first equation, we obtain

$$y(x) = J_1 - G[H(J_2 + x)] + (J_2 + x)H(J_2 + x). \tag{3.25}$$

Now we are in a position to show how one can find a variational discrete model and its conservation laws by means of Lagrange-type technique. Let us choose a difference Lagrangian in the form

$$\mathcal{L} = \frac{y + y_+}{2} + G(y_x), \tag{3.26}$$

then

$$\mathbf{pr}X_1\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_1) = 0;$$

$$\mathbf{pr}X_2\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_2) = 1 = \underset{+h}{D}(x).$$

The variations of $\mathcal{L}$ yield the following quasiextremal equations:

$$\frac{\delta\mathcal{L}}{\delta y} : \quad G'(y_x) - G'(y_{\bar{x}}) = \frac{h_+ + h_-}{2}; \tag{3.27}$$

$$\frac{\delta\mathcal{L}}{\delta x} : \quad -\frac{y + y_+}{2} - G(y_x) + y_x G'(y_x) + \frac{y + y_-}{2} + G(y_{\bar{x}}) - y_{\bar{x}} G'(y_{\bar{x}}) = 0. \tag{3.28}$$

Due to the invariance of the Lagrangian with respect to the operators $X_1$ and $X_2$, the difference analog of Noether's theorem yields two first integrals

$$I_1 = y + G(y_x) - y_x G'(y_x) + \frac{x_+ - x}{2} y_x, \tag{3.29}$$

$$I_2 = G'(y_x) - \frac{x + x_+}{2}. \tag{3.30}$$

As in the case of the algebra $\mathbf{D_{1,1}}$ we can solve for $y_x$ to obtain

$$y_x = \Phi_1(I_2, x + x_+). \tag{3.31}$$

Substituting into the equation for $I_1$ we obtain

$$y = \Phi_2(I_1, I_2, x, x_+). \tag{3.32}$$

Calculating $y_x$ from eq. (3.32) and setting it equal to (3.31), we obtain a three point recursion relation for $x$. Solving it (if we can), we turn eq. (3.32) into an explicit general solution of the difference scheme (3.27),(3.28).

**Example 3.2** Let us consider the case

$$\mathcal{L} = \frac{y + y_+}{2} + \exp(y_x). \tag{3.33}$$

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The two first integrals (3.29),(3.30) in this case are the following:

$$I_1 = y + \exp(y_x) - y_x \exp(y_x) + \frac{x_{n+1} - x_n}{2} y_x, \qquad (3.34)$$

$$I_2 = \exp(y_x) - \frac{x_{n+1} + x_n}{2}.$$

Equation (3.31) and (3.32) reduce to

$$y_x = \ln\left(I_2 + \frac{x_{n+1} + x_n}{2}\right), \qquad (3.35)$$

$$y = I_1 - I_2 - \frac{x_{n+1} + x_n}{2} + (I_2 + x_n)\ln\left(I_2 + \frac{x_{n+1} + x_n}{2}\right), \qquad (3.36)$$

The recursion relation for $x$ is

$$\frac{-x_{n+1} + x_{n-1}}{2} + (I_2 + x_n)[\ln(2I_2 + x_{n+1} + x_n) - \ln(2I_2 + x_n + x_{n-1})]. \qquad (3.37)$$

The last equation is difficult to solve. We have however reduced a system of two three-point equations to a single three-point one. We shall return to this case in Section 5 using an alternative method.

$\mathbf{D_{2,2}}$ The non–Abelian Lie algebra with non-connected elements

$$X_1 = \frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \qquad (3.38)$$

yields the invariant ODE

$$y'' = \frac{1}{x}F(y'). \qquad (3.39)$$

We define a function $G(y')$ by the equation

$$F(y') = \frac{G'(y')}{G''(y')}. \qquad (3.40)$$

Then the ODE (3.39) is the Euler–Lagrangian equation for the Lagrangian

$$L = \frac{1}{x}G(y'), \qquad (3.41)$$

which admits $X_1$ and $X_2$ as variational symmetries:

$$\mathrm{pr}X_1 L + LD(\xi_1) = 0;$$

$$\mathrm{pr}X_2 L + LD(\xi_2) = 0.$$

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Noether's theorem provides us with two first integrals:

$$J_1 = \frac{1}{x} G'(y'), \qquad J_2 = G(y') + \left(\frac{y}{x} - y'\right) G'(y').$$

Let us take the difference Lagrangian

$$\mathcal{L} = \frac{2}{x + x_+} G(y_x),$$

which satisfies

$$\mathbf{pr} X_1 \mathcal{L} + \mathcal{L} \underset{+h}{D}(\xi_1) = 0;$$

$$\mathbf{pr} X_2 \mathcal{L} + \mathcal{L} \underset{+h}{D}(\xi_2) = 0.$$

Then the variations of $\mathcal{L}$ yield the following quasiextremal equations:

$$\frac{\delta \mathcal{L}}{\delta y} : \quad \frac{2}{x + x_+} G'(y_x) - \frac{2}{x_- + x} G'(y_{\bar{x}}) = 0;$$

$$\frac{\delta \mathcal{L}}{\delta x} : \quad \begin{aligned} &-\frac{2h_+}{(x + x_+)^2} G(y_x) + \frac{2}{(x + x_+)} G'(y_x) y_x - \frac{2}{(x + x_+)} G(y_x) \\ &-\frac{2h_-}{(x_- + x)^2} G(y_{\bar{x}}) - \frac{2}{(x_- + x)} G'(y_{\bar{x}}) y_{\bar{x}} + \frac{2}{(x_- + x)} G(y_{\bar{x}}) = 0. \end{aligned} \qquad (3.42)$$

Since the Lagrangian is invariant with respect to the operators $X_1$ and $X_2$, we find the first integrals

$$I_1 = \frac{2G'(y_x)}{x + x_+}, \qquad I_2 = \frac{4xx_+}{(x + x_+)^2} G(y_x) + \frac{2G'(y_x)}{x + x_+} (y - xy_x) \qquad (3.43)$$

for the solutions of (3.42).

As in the case of the algebra $\mathbf{D_{2,1}}$, we can solve for $y_x$, using the integral $I_1$. We obtain:

$$y_x = \Phi_1(I_1, x + x_+). \qquad (3.44)$$

The second integral allows us to express $y$ as a function of $x$ and $x_+$

$$y = x\Phi_1 + \frac{I_2}{I_1} - \frac{4xx_+}{I_1(x + x_+)^2} G(\Phi_1). \qquad (3.45)$$

# 4 Equations corresponding to Lagrangians invariant under three-dimensional Lie groups

Among the "prototype equations" of our previous article [1], many have three-dimensional symmetry groups. In this section we shall consider two of these cases. Both of them come from Lagrangians that also have three-dimensional symmetry groups, i.e. all these symmetries are Lagrangian ones.

**D$_{3,1}$** Let us first consider a family of solvable Lie algebras depending on one constant $k$:

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial x} + ky\frac{\partial}{\partial y}, \qquad k \neq 0, \frac{1}{2}, 1, 2. \qquad (4.1)$$

The invariant equation has the form

$$y'' = y'^{\frac{k-2}{k-1}}. \qquad (4.2)$$

This equation can be obtained by the usual variational procedure from the Lagrangian

$$L = \frac{(k-1)^2}{k}(y')^{\frac{k}{k-1}} + y, \qquad (4.3)$$

which admits operators $X_1$ and $X_2$ for any parameter $k$ and $X_3$ for $k = -1$:

$$\mathrm{pr}X_1 L + LD(\xi_1) = 0;$$

$$\mathrm{pr}X_2 L + LD(\xi_2) = 1 = D(x);$$

$$\mathrm{pr}X_3 L + LD(\xi_3) = (k+1)L.$$

It is possible to show that there is no Lagrangian function $L(x, y, y')$ which gives eq. (4.2) with $k \neq -1$ as its Euler's equation and is divergence invariant for all three symmetries (4.1).

For arbitrary $k$ there are two first integrals

$$J_1 = \frac{(1-k)}{k}(y')^{\frac{k}{k-1}} + y = A^0, \qquad J_2 = (k-1)(y')^{\frac{1}{k-1}} - x = B^0.$$

Eliminating $y'$ we find the general solution:

$$y = \frac{1}{k}\left(\frac{1}{k-1}\right)^{(k-1)}(x + B^0)^k + A^0. \qquad (4.4)$$

In the case $k = -1$ we have the further first integral corresponding to the symmetry $X_3$:

$$J_3 = \frac{2}{\sqrt{y'}}(y - xy') + xy = C^0.$$

It is functionally dependent on $J_1$ and $J_2$ since a second order ODE can possess only two functionally independent first integrals. Let us mention that the first integral $J_3$ is basic:

$$J_1 = X_1(J_3), \qquad J_2 = -X_2(J_3),$$

since

$$[X_1, X_3] = X_1, \qquad [X_2, X_3] = kX_2.$$

In this case we have the following relation:

$$4 - J_1 J_2 - J_3 = 0. \tag{4.5}$$

Thus, the integral $J_3$ is not independent and is of no use in the present context.

Now let us turn to the discrete case and consider **k = −1** only. Other values of $k$ will be considered in Section 5, using a different approach. Let us choose the Lagrangian to be

$$\mathcal{L} = -4\sqrt{y_x} + \frac{y + y_+}{2} \tag{4.6}$$

as a discrete Lagrangian, which is invariant for $X_1$ and $X_3$ and divergence invariant for $X_2$:

$$
\begin{aligned}
&\mathrm{pr}X_1\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_1) = 0; \\
&\mathrm{pr}X_2\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_2) = 1 = \underset{+h}{D}(x); \\
&\mathrm{pr}X_3\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_3) = 0.
\end{aligned}
\tag{4.7}
$$

From the Lagrangian we obtain the quasiextremal equations:

$$
\frac{\delta\mathcal{L}}{\delta y} : \quad -\frac{4}{h_- + h_+}\left(\frac{1}{\sqrt{y_x}} - \frac{1}{\sqrt{y_{\bar{x}}}}\right) = 1;
$$

$$
\tag{4.8}
$$

$$
\frac{\delta\mathcal{L}}{\delta x} : \quad 4\left(\sqrt{y_x} - \sqrt{y_{\bar{x}}}\right) - \frac{y + y_+}{2} + \frac{y_- + y}{2} = 0.
$$

This system of equations is invariant with respect to all three operators (4.1). The application of the difference analog of the Noether theorem gives us three first integrals:

$$
I_1 = -2\sqrt{y_x} + \frac{y + y_+}{2} = A, \qquad I_2 = -\frac{2}{\sqrt{y_x}} - \frac{x + x_+}{2} = B,
$$

$$
\tag{4.9}
$$

$$
I_3 = \frac{2(x_+ y - y_+ x)}{h_+\sqrt{y_x}} + \frac{x_+ y + y_+ x}{2} = C.
$$

In contrast to the continuous case the three difference first integrals $I_1$, $I_2$ and $I_3$ are functionally independent and instead of eq. (4.5) we have the following relation:

$$
4 - I_1 I_2 - I_3 = \frac{1}{4}h_+^2 y_x = \frac{4\varepsilon^2}{(\varepsilon + 2)^2}. \tag{4.10}
$$

This coincides with eq. (4.5) in the continuous limit $\varepsilon \to 0$. We see that the expression $h_+^2 y_x$ is also a first integral of (4.8). This allows to introduce a convenient lattice, namely:

$$\frac{1}{4}h_-^2 y_{\bar{x}} = \frac{1}{4}h_+^2 y_x = \frac{4\varepsilon^2}{(\varepsilon + 2)^2}, \quad \varepsilon = const, \quad 0 < \varepsilon \ll 1. \tag{4.11}$$

Substituting $y_x$ from eq. (4.11) into $I_2$, we obtain a two-term recursion relation for $x$, namely

$$x_{n+1} - (1 + \varepsilon)x_n - \varepsilon B = 0, \tag{4.12}$$

or

$$-(1 + \varepsilon)x_{n+1} + x_n - \varepsilon B = 0, \tag{4.13}$$

depending on the sign choice for $\sqrt{y_x}$. These equations can be solved and we obtain a lattice satisfying

$$x_n = (x_0 + B)(1 + \varepsilon)^n - B, \quad x_0 > -B \tag{4.14}$$

for the first equation and a lattice satisfying

$$x_n = (x_0 + B)(1 + \varepsilon)^{-n} - B, \quad x_0 < -B \tag{4.15}$$

for the second equation. Using the expressions for $I_1$, we get the general solution for $y$ (the same for both lattices (4.14) and (4.15)) as

$$y_n = A - \frac{4}{x_n + B}\frac{1 + \varepsilon}{(1 + \frac{\varepsilon}{2})^2}. \tag{4.16}$$

This agrees with the continuous case up to order $\varepsilon$.

We have used the three integrals $I_1$, $I_2$ and $I_3$ to obtain the general solution of the difference scheme (4.8). Indeed, the solution (4.14),(4.16) for $x_n$, $y_n$ depends on 4 constants $(A, B, x_0, \varepsilon)$, as it should.

The difference scheme is not compatible with a regular lattice, but requires an exponential one, as in eq. (4.14). The only non-algebraic step in the integration was the solution of eq. (4.12): a linear two point equation with constant coefficients.

**D$_{3,2}$** The group given by the operators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} \tag{4.17}$$

corresponds to the invariant differential equation

$$y'' = y^{-3}. \tag{4.18}$$

This equation can be obtained from the Lagrangian function

$$L = y'^2 - \frac{1}{y^2}, \tag{4.19}$$

which admits all three operators:

$$\mathrm{pr}X_1 L + LD(\xi_1) = 0;$$

$$\mathrm{pr}X_2 L + LD(\xi_2) = 0; \tag{4.20}$$

$$\mathrm{pr}X_3 L + LD(\xi_3) = 2y'y = D(y^2).$$

Consequently, the symmetries yield the following first integrals

$$J_1 = y'^2 + \frac{1}{y^2} = A^0, \qquad J_2 = 2\frac{x}{y^2} - 2(y - y'x)y' = 2B^0,$$
$$\tag{4.21}$$
$$J_3 = \frac{x^2}{y^2} + (y - xy')^2 = C^0.$$

Using the integrals $A^0$ and $B^0$ we write the general solution $y(x)$ as

$$A^0 y^2 = (A^0 x - B^0)^2 + 1. \tag{4.22}$$

We see that the third integral, denoted $J_3$ is not needed, is not useful and indeed, is not independent. The integrals $J_1$, $J_2$ and $J_3$ are related as follows:

$$\left(\frac{J_2}{2}\right)^2 - J_1 J_3 + 1 = 0. \tag{4.23}$$

Now let us consider the discrete case. Let us consider the discrete Lagrangian function

$$\mathcal{L} = y_x^2 - \frac{1}{yy_+}, \tag{4.24}$$

which admits the same symmetries as the continuous one:

$$\mathbf{pr}X_1 \mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_1) = 0;$$

$$\mathbf{pr}X_2 \mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_2) = 0; \tag{4.25}$$

$$\mathbf{pr}X_3 \mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_3) = \underset{+h}{D}(y^2).$$

The Lagrangian generates the invariant quasiextremal equations:

$$\frac{\delta \mathcal{L}}{\delta y} : \quad 2(y_x - y_{\bar{x}}) = \frac{h_+}{y^2 y_+} + \frac{h_-}{y^2 y_-};$$

$$\frac{\delta \mathcal{L}}{\delta x} : \quad (y_x)^2 + \frac{1}{yy_+} - (y_{\bar{x}})^2 - \frac{1}{yy_-} = 0. \tag{4.26}$$

The quasiextremal equations have three functionally independent first integrals

$$I_1 = y_x{}^2 + \frac{1}{yy_+} = A, \qquad I_2 = \frac{x + x_+}{yy_+} + 2y_x(x_+ y_x - y_+) = 2B,$$

$$I_3 = \frac{xx_+}{yy_+} + (x_+ y_x - y_+)^2 = C. \tag{4.27}$$

In the discrete case the integrals $I_1$, $I_2$ and $I_3$ are independent. Eq. (4.23) no longer holds and instead we have

$$\left(\frac{I_2}{2}\right)^2 - I_1 I_3 + 1 = \frac{1}{4}\left(\frac{h_+}{yy_+}\right)^2. \tag{4.28}$$

In order to integrate the system (4.26) we will use three first integrals $A$, $B$ and the one in eq. (4.28), namely

$$\frac{h_+}{yy_+} = \varepsilon. \tag{4.29}$$

Eliminating $y_x$, $x_+$ and $y_+$, we obtain the solution

$$Ay^2 = (Ax - B)^2 + 1 - \frac{\varepsilon^2}{4}. \tag{4.30}$$

This agrees with the continuous limit (4.22) up to order $\varepsilon^2$.

Calculating $y_x$ from eq. (4.30) and substituting into the expression for $A$ in eq. (4.27) we obtain a two-point difference equation for $x$ and hence we obtain the lattice. In this case the equation has the form of a fractional linear mapping, i.e. it is a discrete version of the Riccati equation (with constant coefficients).

Explicitly we obtain

$$x_{n+1} = \frac{\alpha x_n + \beta}{\gamma x_n + \delta} \tag{4.31}$$

$$\alpha = 1 - \varepsilon B - \tfrac{1}{2}\varepsilon^2; \quad \beta = \frac{\varepsilon}{A}(1 + B^2 - \tfrac{1}{4}\varepsilon^2);$$

$$\gamma = -\varepsilon A; \qquad \delta = 1 + \varepsilon B - \tfrac{1}{2}\varepsilon^2. \tag{4.32}$$

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We see that the coefficients in the discrete Riccati equation (4.31) satisfy

$$\alpha\delta - \beta\gamma = 1, \qquad \alpha + \delta = 2 - \varepsilon^2. \tag{4.33}$$

Like the continuous Riccati equation, eq. (4.31) can be linearized. To do this we introduce a linear system

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} \tag{4.34}$$

If $u$ and $v$ satisfy eq. (4.34), then

$$x = \frac{u}{v} \tag{4.35}$$

will satisfy equation (4.31). Eq. (4.34) can be solved by standard methods. Indeed, both $u$ and $v$ must satisfy

$$u_{n+2} - (\alpha + \delta)u_{n+1} + (\alpha\delta - \beta\gamma)u_n = 0. \tag{4.36}$$

The characteristic equation

$$\lambda^2 - (\alpha + \delta)\lambda + (\alpha\delta - \beta\gamma) = 0 \tag{4.37}$$

is obtained by putting $u_n = \lambda^n$.

In view of eq. (4.33) the roots of eq. (4.37) are complex. The final result is that the solution of eq. (4.31) is

$$x_n = \frac{1}{A}\sqrt{1 - \frac{\varepsilon^2}{4}}\tan(\omega n + \rho) + \frac{B}{A}, \tag{4.38}$$

where $\rho$ is an integration constant and $\omega$ satisfies

$$\tan\omega = \frac{2\varepsilon}{2 - \varepsilon^2}\sqrt{1 - \frac{\varepsilon^2}{4}} \tag{4.39}$$

Eqs. (4.30) and (4.38) provide an explicit analytic solution of the system (4.26). It is the general solution and involves 4 constants: $A$, $B$, $\varepsilon$ and $\rho$. It follows from eq. (4.38) that the independent variable $x$ varies between $-$ and $+$ infinity as $\omega n + \rho$ varies between $\pm\frac{\pi}{2}$.

# 5 Integration of difference schemes with two variational symmetries

## 5.1 The method of perturbed Lagrangians

We have mentioned in Section 3.2 that a two-dimensional group of Lagrangian symmetries is always sufficient to reduce the original system of two three-point equations to a single three-point equation for the independent variable alone.

Here we shall show that in some cases we can do better. Using a different approach, we will actually obtain a complete solution of a difference scheme approximating a differential equation with a Lagrangian, divergence invariant under a two-dimensional symmetry group.

The case we shall consider is the equation (3.21) and hence the two-dimensional Abelian group $\mathbf{D_{2,1}}$ corresponding to the algebra (3.20). We shall make use of the fact that the Lagrangian is not unique. Indeed we will consider three different Lagrangians, all having the same continuous limit (3.22). Instead of writing the Lagrangian (3.26) in the discrete case, we shall use a family of Lagrangians, parametrized by two constants, $\alpha$ and $\beta$:

$$\mathcal{L} = \alpha G(y_x) + \beta y + (1-\beta)y_+, \quad \alpha \approx 1, \quad 0 \leq \beta \leq 1. \tag{5.1}$$

Each Lagrangian provides its own quasiextremal system

$$\alpha\left[-G'(y_x) + G'(y_{\bar{x}})\right] + \beta h_+ + (1-\beta)h_- = 0 \tag{5.2}$$

$$\alpha\left[y_x G'(y_x) - y_{\bar{x}} G'(y_{\bar{x}}) - G(y_x) + G(y_{\bar{x}})\right] - \beta(y - y_-) - (1-\beta)(y_+ - y) = 0 \tag{5.3}$$

We shall view one Lagrangian, with $\alpha_3 = 1$ and $\beta_3 = 0.5$ as the basic one, the other two as its perturbations.

Each Lagrangian in the family is divergence invariant under $X_1 = \partial_x$ and $X_2 = \partial_y$ and hence provides two first integrals of the corresponding quasiextremal equations (5.2) and (5.3):

$$\alpha\left[-y_x G'(y_x) + G(y_x)\right] + y + (1-\beta)h_+ y_x = A \tag{5.4}$$

$$\alpha G'(y_x) - x - \beta h_+ = B. \tag{5.5}$$

Let us now choose three different pairs $(\alpha_i, \beta_i)$. They provide six integrals (and six quasiextremal equations). We shall show that by appropriately fine tuning the constants $\alpha_i$ and $\beta_i$ and choosing some of the constants $A_i$ and $B_i$ we can manufacture a consistent difference system, representing both the equation and the lattice. Moreover, we can explicitly integrate the equations in a manner that approximates the exact solution obtained in the continuous limit.

Let us take one equation of the form (5.4) and two of the form (5.5). In these three equations we choose $\alpha_3 = 1$, $\beta_3 = 0.5$ and $B_2 = B_3 = B$. We then take the difference between the two equations involving $B$ to finally obtain the following system of three two-point equations:

$$\alpha_1\left[-y_x G'(y_x) + G(y_x)\right] + y + (1-\beta_1)h_+ y_x = A \tag{5.6}$$

$$G'(y_x) - x - \frac{1}{2}h_+ = B \tag{5.7}$$

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$$(1 - \alpha_2)G'(y_x) - (\frac{1}{2} - \beta_2)h_+ = 0. \tag{5.8}$$

From eq. (5.7) and (5.8) we have

$$G'(y_x) = \frac{x_+ + x + 2B}{2} \tag{5.9}$$

$$x_+ - (1 + \varepsilon)x - \varepsilon B = 0, \tag{5.10}$$

where we have put

$$\varepsilon = \frac{2(1 - \alpha_2)}{\alpha_2 - 2\beta_2}. \tag{5.11}$$

The continuous limit will correspond to $\varepsilon \to 0$.

Eq. (5.10) coincides with eq. (4.12) obtained using three Lagrangian symmetries in a special case. Here it appears in a much more general setting. The general solution of eq. (5.10)

$$x_n = (x_0 + B)(1 + \varepsilon)^n - B \tag{5.12}$$

depends on one integration constant $x_0$. This solution gives a lattice satisfying $h_- > 0$ and $h_+ > 0$ for $x_0 > -B$ if $\varepsilon > 0$ and for $x_0 < -B$ if $\varepsilon < 0$. For the other cases, namely $x_0 < -B$ if $\varepsilon > 0$ and for $x_0 > -B$ if $\varepsilon < 0$, formula (5.12) gives a lattice with a reverse order of points: $h_- < 0$ and $h_+ < 0$.

Using (5.12) and (5.9), we can express $y_x$ in terms of $x$. We have

$$G'(y_x) = \left(1 + \frac{\varepsilon}{2}\right)(B + x). \tag{5.13}$$

Denoting the inverse function of $G'(y_x)$ as $H$, we have

$$y_x = H\left[\left(1 + \frac{\varepsilon}{2}\right)(B + x)\right]. \tag{5.14}$$

Using (5.6) and (5.14), we can now write the general solution of the system (5.6), (5.7) and (5.8) as

$$y(x) = A - \alpha_1 G(H) + (x + B)H, \tag{5.15}$$

where we have put

$$\alpha_1\left(1 + \frac{\varepsilon}{2}\right) - (1 - \beta_1)\varepsilon = 1. \tag{5.16}$$

The value of $\alpha_1$, still figuring in the solution (5.15), must be so chosen as to obtain a consistent scheme. Indeed, $x_n$ and $y_n$ given in eq. (5.12) and (5.15) will satisfy the system (5.6), (5.7) and (5.8). We must however assure that $y_x$ of eq. (5.14) and $y_x = (y_{n+1} - y_n)/(x_{n+1} - x_n)$ coincide. A simple computation shows that this equality requires that $\alpha_1$ should satisfy

$$\alpha_1 = (1 + \varepsilon)^{n+1}(x_0 + B)\frac{H_{n+1} - H_n}{G(H_{n+1}) - G(H_n)}. \tag{5.17}$$

This equation is consistent only if the right hand side is a constant (independent on $n$). The constants $\alpha_i$ and $\beta_i$ can depend upon the constant $\varepsilon$ and for $\varepsilon \to 0$ we must have $\alpha_1, \alpha_2 \to 1$; $\beta_1, \beta_2 \to 0.5$.

From eq. (5.8) we have

$$\frac{h_+}{G'(y_x)} = \frac{2(1 - \alpha_2)}{1 - 2\beta_2} \tag{5.18}$$

This expression must vanish for $\varepsilon \to 0$. To achieve this while respecting eq. (5.11) we put

$$\alpha_2 = 1 + \varepsilon^2, \qquad \beta_2 = \frac{1}{2} + \varepsilon + \frac{\varepsilon^2}{2}. \tag{5.19}$$

Eq. (5.11) is satisfied exactly and we have

$$\frac{h_+}{G'(y_x)} = \frac{2\varepsilon}{\varepsilon + 2}. \tag{5.20}$$

We can view eq. (5.12) and (5.15) as the general solution of the following three point difference scheme:

$$G'(y_x) - G'(y_{\bar{x}}) - \frac{x_+ - x_-}{2} = 0,$$
$$\frac{h_+}{G'(y_x)} = \frac{h_-}{G'(y_{\bar{x}})}. \tag{5.21}$$

The system (5.21) is invariant under the group corresponding to $\mathbf{D_{2,1}}$. Strictly speaking, this is not a quasiextremal system, since it can not be derived from any single Lagrangian. The arbitrary constants $A$, $B$ and $\varepsilon$ come from three first integrals (5.6), (5.7) and (5.20) that are associated with three different Lagrangians.

We have not proven that eq. (5.17) is consistent for arbitrary functions $G(y_x)$. We shall however show below that in at least two interesting special cases the above integration scheme is consistent.

The results of this section can be summed up as a theorem.

**Theorem 5.1** The ODE (3.21) obtained from the Lagrangian (3.22) can be approximated by the difference system (5.21). If $\alpha_1$ of eq. (5.17) is constant, then the general solution of this system is given by

$$x_n = (x_0 + B)(1 + \varepsilon)^n - B,$$
$$y(x_n) = A - \alpha_1 G(H_n) + (x_n + B)H_n, \tag{5.22}$$

where $A$, $B$, $\varepsilon$ and $x_0$ are arbitrary constants. For $\varepsilon \to 0$, $y(x_n)$ agrees with the solution (3.25) of the ODE (3.21).

As applications of this theorem let us consider two different equations, each invariant under a three-dimensional group with $\mathbf{D_{2,1}}$ as an invariant subgroup. In both cases the Lagrangian is only divergence invariant under the subgroup $\mathbf{D_{2,1}}$.

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## 5.2 A polynomial nonlinearity.

$$\mathbf{D_{3,1}}: \qquad X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial x} + ky\frac{\partial}{\partial y}, \quad k \neq 0, \frac{1}{2}, \pm 1, 2 \quad (5.23)$$

This algebra for $k = -1$ was treated in Section 4, now we consider the generic case. We take

$$G(y_x) = \frac{(k-1)^2}{k} y_x^{\frac{k}{k-1}} \qquad (5.24)$$

and hence

$$G'(y_x) = (k-1)y_x^{\frac{1}{k-1}} = \left(1 + \frac{\varepsilon}{2}\right)(x + B). \qquad (5.25)$$

Eq. (5.14) reduces to

$$y_x = H_n(x) = \left(\frac{x+B}{k-1}\right)^{k-1} \left(1 + \frac{\varepsilon}{2}\right)^{k-1} \qquad (5.26)$$

and we have

$$G(H_n) = \frac{(k-1)^2}{k} \left(\frac{x+B}{k-1}\right)^k \left(1 + \frac{\varepsilon}{2}\right)^k. \qquad (5.27)$$

Substituting into (5.17), we find

$$\alpha_1 = \frac{k(1+\varepsilon)((1+\varepsilon)^{k-1} - 1)}{(k-1)(1+\frac{\varepsilon}{2})((1+\varepsilon)^k - 1)} \qquad (5.28)$$

so that we have $\alpha_1 = 1 + O(\varepsilon^2)$.

Thus, $\alpha_1$ is a constant, close to $\alpha_1 = 1$ for $\varepsilon \ll 1$. The solution $y_n$ of (5.22) specializes to

$$y_n = A + \frac{(x+B)^k}{(k-1)^{k-1}} \frac{\varepsilon \left(1 + \frac{\varepsilon}{2}\right)^{k-1}}{(1+\varepsilon)^k - 1}. \qquad (5.29)$$

This agrees with the solution (4.4) of the ODE (4.2) up to $O(\varepsilon^2)$.

It is interesting to note that for $k = -1$ $\alpha_1$ becomes independent on $\varepsilon$ and we obtain $\alpha_1 = 1$, $\beta_1 = 0.5$. The solution (5.29) provides us with the solution (4.16), which was obtained in Section 4 with the help of a different method.

## 5.3 An exponential nonlinearity

We consider another three-dimensional group and its Lie algebra, namely:

$$\mathbf{D_{3,3}}: \qquad X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y}. \qquad (5.30)$$

The corresponding invariant ODE is

$$y'' = \exp(-y') \qquad (5.31)$$

and can be obtained from the Lagrangian

$$L = \exp(y') + y. \tag{5.32}$$

We have

$$\begin{aligned}\mathrm{pr}X_1 L + LD(\xi_1) &= 0;\\ \mathrm{pr}X_2 L + LD(\xi_2) &= 1 = D(x);\end{aligned} \tag{5.33}$$

The corresponding first integrals of eq. (5.31) are

$$\exp(y')(1 - y') + y = A, \qquad \exp(y') - x = B. \tag{5.34}$$

Finally, the general solution of eq. (5.31) is

$$y = (x + B)(\ln(x + B) - 1) + A. \tag{5.35}$$

Now let us consider the discrete case, following the method of Section 5.1. We have

$$G(y_x) = \exp(y_x) \tag{5.36}$$

and hence

$$\begin{aligned}G'(y_x) &= \exp(y_x) = (x_n + B)\left(1 + \tfrac{\varepsilon}{2}\right)\\ H_n &= y_x = \ln(x_n + B) + \ln\left(1 + \tfrac{\varepsilon}{2}\right)\end{aligned} \tag{5.37}$$

Substituting into eq. (5.17), we find

$$\alpha_1 = \frac{(1 + \varepsilon)\ln(1 + \varepsilon)}{\varepsilon\left(1 + \tfrac{\varepsilon}{2}\right)} \tag{5.38}$$

so that $\alpha_1$ is indeed a constant and moreover we have $\alpha_1 = 1 + O(\varepsilon^2)$.

The solution $y(x)$ on the lattice given in eq. (5.22) is

$$y_n = A + (x_n + B)\ln(x_n + B) + (x_n + B)\left[\ln\left(1 + \frac{\varepsilon}{2}\right) - \frac{(1 + \varepsilon)\ln(1 + \varepsilon)}{\varepsilon}\right]. \tag{5.39}$$

This agrees with the solution (5.35) of the ODE (5.31) up to $O(\varepsilon^2)$.

# 6 Concluding remarks

We see that variational symmetries, and the first integrals they provide, play a crucial role in the study of exact solutions of invariant difference schemes. Much more so than in the theory of ordinary differential equations.

The procedure that we followed in this article can be reformulated as follows. We start from the continuous case where we know a Lagrangian density $L(x, y, y')$, invariant under a group $G_0$ of local point transformations, i.e. satisfying condition (2.6), or (2.7). We hence also know the corresponding Euler-Lagrange equation, invariant under the same group, or a larger group containing $G_0$ as a subgroup.

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Let us assume that we can approximate this Lagrangian by a "discrete Lagrangian density" $L(x, y, x_+, y_+)$ invariant under the same group $G_0$. Even in the absence of any symmetry group, the Lagrangian will provide us with the quasiextremal equations (2.21), i.e. with a discrete Euler-Lagrange system. This system can be identified with the difference system (1.1), (1.2).

If the Lagrangian is invariant under a one-dimensional symmetry group, we can reduce the quasiextremal system to a three-point relation for x alone, plus a "discrete quadrature" for y (see Section 3.1). If the symmetry group of the Lagrangian is two-dimensional, we can always reduce the quasiextremal system to one three-point equation for x alone, and write the solution $y_n(x)$ directly (see Section 3.2).

If the invariance group of the Lagrangian is (at least) three-dimensional then we can integrate the system explicitly (Section 4).

Finally, we have shown that if the symmetry group of the Lagrangian is two-dimensional, but the quasiextremal system has a third (non-Lagrangian) symmetry, we can also integrate explicitly.

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