



3. First-Order Nonlinear Partial Differential Equations

Preliminary remarks. For first-order partial differential equations with two independent variables, an exact solution

$$w = \Phi(x, y, C_1, C_2) \quad (1)$$

that depends on two arbitrary constants C_1 and C_2 is called a complete integral. The general integral (general solution) can be represented in parametric form by using the complete integral (1) and the two equations

$$\begin{aligned} C_2 &= f(C_1), \\ \frac{\partial \Phi}{\partial C_1} + \frac{\partial \Phi}{\partial C_2} f'(C_1) &= 0, \end{aligned} \quad (2)$$

where f is an arbitrary function and the prime stands for the derivative.

References

Kamke, E., *Differentialgleichungen: Lösungsmethoden und Lösungen, II, Partielle Differentialgleichungen Erster Ordnung für eine gesuchte Funktion*, Akad. Verlagsgesellschaft Geest & Portig, Leipzig, 1965.

Polyanin, A. D., Zaitsev, V. F., and Moussiaux, A., *Handbook of First Order Partial Differential Equations*, Taylor & Francis, London, 2002.

3.1. Equations Quadratic in One Derivative

1. $\frac{\partial w}{\partial x} + a \left(\frac{\partial w}{\partial y} \right)^2 = by.$
2. $\frac{\partial w}{\partial x} + a \left(\frac{\partial w}{\partial y} \right)^2 + by^2 = 0.$
3. $\frac{\partial w}{\partial x} + a \left(\frac{\partial w}{\partial y} \right)^2 = f(x) + g(y).$
4. $\frac{\partial w}{\partial x} + a \left(\frac{\partial w}{\partial y} \right)^2 = f(x)y + g(x).$
5. $\frac{\partial w}{\partial x} + a \left(\frac{\partial w}{\partial y} \right)^2 = f(x)w + g(x).$
6. $\frac{\partial w}{\partial x} - f(w) \left(\frac{\partial w}{\partial y} \right)^2 = 0.$
7. $f_1(x) \frac{\partial w}{\partial x} + f_2(y) \left(\frac{\partial w}{\partial y} \right)^2 = g_1(x) + g_2(y).$
8. $\frac{\partial w}{\partial x} + a \left(\frac{\partial w}{\partial y} \right)^2 + b \frac{\partial w}{\partial y} = f(x) + g(y).$
9. $\frac{\partial w}{\partial x} + a \left(\frac{\partial w}{\partial y} \right)^2 + b \frac{\partial w}{\partial y} = f(x)y + g(x).$
10. $\frac{\partial w}{\partial x} + a \left(\frac{\partial w}{\partial y} \right)^2 + b \frac{\partial w}{\partial y} = f(x)w + g(x).$

3.2. Equations Quadratic in Two Derivatives

1. $a\left(\frac{\partial w}{\partial x}\right)^2 + b\left(\frac{\partial w}{\partial y}\right)^2 = c.$
2. $\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = a - 2by.$
3. $\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \frac{a}{\sqrt{x^2 + y^2}} + b.$
4. $\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = f(x).$
5. $\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = f(x) + g(y).$
6. $\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = f(x^2 + y^2).$
7. $\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = f(w).$
8. $\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{x^2}\left(\frac{\partial w}{\partial y}\right)^2 = f(x).$
9. $\left(\frac{\partial w}{\partial x}\right)^2 + f(x)\left(\frac{\partial w}{\partial y}\right)^2 = g(x).$
10. $\left(\frac{\partial w}{\partial x}\right)^2 + f(y)\left(\frac{\partial w}{\partial y}\right)^2 = g(y).$
11. $\left(\frac{\partial w}{\partial x}\right)^2 + f(w)\left(\frac{\partial w}{\partial y}\right)^2 = g(w).$
12. $f_1(x)\left(\frac{\partial w}{\partial x}\right)^2 + f_2(y)\left(\frac{\partial w}{\partial y}\right)^2 = g_1(x) + g_2(y).$

3.3. Equations with Arbitrary Nonlinearities in Derivatives

1. $\frac{\partial w}{\partial x} + f\left(\frac{\partial w}{\partial y}\right) = 0.$
2. $\frac{\partial w}{\partial x} + f\left(\frac{\partial w}{\partial y}\right) = g(x).$
3. $\frac{\partial w}{\partial x} + f\left(\frac{\partial w}{\partial y}\right) = g(x)y + h(x).$

4. $\frac{\partial w}{\partial x} + f\left(\frac{\partial w}{\partial y}\right) = g(x)w + h(x).$
5. $\frac{\partial w}{\partial x} - F\left(x, \frac{\partial w}{\partial y}\right) = 0.$
6. $\frac{\partial w}{\partial x} + F\left(x, \frac{\partial w}{\partial y}\right) = aw.$
7. $\frac{\partial w}{\partial x} + F\left(x, \frac{\partial w}{\partial y}\right) = g(x)w.$
8. $F\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right) = 0.$
9. $w = x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + F\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right).$
10. $F_1\left(x, \frac{\partial w}{\partial x}\right) = F_2\left(y, \frac{\partial w}{\partial y}\right).$
11. $F_1\left(x, \frac{\partial w}{\partial x}\right) + F_2\left(y, \frac{\partial w}{\partial y}\right) + aw = 0.$
12. $F_1\left(x, \frac{1}{w}\frac{\partial w}{\partial x}\right) + w^k F_2\left(y, \frac{1}{w}\frac{\partial w}{\partial y}\right) = 0.$
13. $F_1\left(x, \frac{\partial w}{\partial x}\right) + e^{\lambda w} F_2\left(y, \frac{\partial w}{\partial y}\right) = 0.$
14. $F_1\left(x, \frac{1}{w}\frac{\partial w}{\partial x}\right) + F_2\left(y, \frac{1}{w}\frac{\partial w}{\partial y}\right) = k \ln w.$
15. $\frac{\partial w}{\partial x} + yF_1\left(x, \frac{\partial w}{\partial y}\right) + F_2\left(x, \frac{\partial w}{\partial y}\right) = 0.$
16. $F\left(\frac{\partial w}{\partial x} + ay, \frac{\partial w}{\partial y} + ax\right) = 0.$
17. $\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = F\left(x^2 + y^2, y\frac{\partial w}{\partial x} - x\frac{\partial w}{\partial y}\right).$
18. $F\left(x, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right) = 0.$
19. $F\left(ax + by, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right) = 0.$
20. $F\left(w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right) = 0.$
21. $F\left(ax + by + cw, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right) = 0.$

$$22. \quad F\left(x, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, w - y \frac{\partial w}{\partial y}\right) = 0.$$

$$23. \quad F\left(w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y}\right) = 0.$$

$$24. \quad F\left(ax + by, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, w - x \frac{\partial w}{\partial x} - y \frac{\partial w}{\partial y}\right) = 0.$$

$$25. \quad F\left(x, \frac{\partial w}{\partial x}, G\left(y, \frac{\partial w}{\partial y}\right)\right) = 0.$$

The EqWorld website presents extensive information on solutions to various classes of ordinary differential equations, partial differential equations, integral equations, functional equations, and other mathematical equations.

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<http://eqworld.ipmnet.ru/en/solutions/fpde/fpdetoc3.pdf>