PERTURBATION METHOD FOR LINEAR DIFFERENCE EQUATIONS WITH SMALL PARAMETERS

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We consider a boundary value problem for a linear difference equation with several widely different coefficients. We study the existence and uniqueness of its solution and we give successive asymptotic approximations for this solution, obtained by a simple iterative method. This method improves the *singular perturbation method*, it offers considerable reduction and simplicity in computation since it does not require to compute *boundary layer correction solutions*.

1. Introduction

In recent years, several methods have been developed for the study of boundary value problems for difference equations, see, for example, [2, 3, 8]. In this paper, we consider the $(m + n)$ th-order difference equation:

$$\varepsilon^m a_{m+n,k} y_{k+n+m} + \cdots + \varepsilon^2 a_{n+2,k} y_{k+n+2} + \varepsilon a_{n+1,k} y_{k+n+1}$$
$$+ a_{n,k} y_{k+n} + a_{n-1,k} y_{k+n-1} + \cdots + a_{1,k} y_{k+1} + a_{0,k} y_k = f_k, \tag{1.1}$$
$$k = 0, 1, \ldots, N - n - m,$$

where ε is a small parameter, $(a_{i,k})$, $0 \le i \le m + n$, (f_k), $0 \le k \le N - n - m$, are given discrete real functions, and N is a fixed integer. We associate to (1.1) the boundary conditions

$$y_k = \alpha_k, \quad k = 0, \ldots, n - 1, \qquad y_{N-k} = \beta_k, \quad k = 0, \ldots, m - 1, \tag{1.2}$$

where α_k, $0 \le k \le n - 1$, and β_k, $0 \le k \le m - 1$, are given constants.

We are concerned with the boundary value problem (P_ε) described by (1.1) and (1.2), it is a classical representation of multi-time-scale digital systems. Such systems are

prevalent in engineering and other great applications especially digital control theory [9–11, 14] and their perturbation analysis is gaining momentum [4–7, 13]. The presence of small parameters increases the order of the system and exhibits time-scale phenomena. The high dimensionality coupled with the time-scale behavior makes the system computationally *stiff* resulting in the use of extensive numerical routines.

Recently, the particular case $m = 1$, $n = 1$ of problem (P_ε) was studied in [15]. Through this paper, we studied the existence and uniqueness of the solution and we developed an iterative convergent method to get successive asymptotic approximations for this solution.

The boundary problem (P_ε) was also considered by Comstock and Hsiao [1] in the homogeneous case $f_k = 0$ for $m = 1$, $n = 1$. The time-invariant case of problem (P_ε) was considered by Naidu and Rao (see [12, Chapter 1]) and also by Krishnarayalu [7], where small parameters are multiple. These authors developed a singular perturbation method, a formal procedure, to give approximate solutions which consist of *outer solutions* and *boundary layer correction solutions*. Notice that in general this method cannot be extended for the general case of time-variant problems (see Section 3.2, Remark 3.3, for the arguments).

The aim of this paper is to extend for problem (P_ε) the perturbation method developed in [15]. We give sufficient conditions on the coefficients of (1.1), to ensure existence and uniqueness of the solution of problem (P_ε), and successive approximations of this solution, obtained by a simple procedure. A proof is given of uniform convergence of the iterative method. The most distinguished feature of this method, besides order reduction, is the decoupling of the original boundary value problem into initial value problems, which facilitates considerable treatment of the boundary value problem. The proposed method consists simply of writing the problem (P_ε) in a matrix form (see the proof of Theorem 2.1 for the details), and can be easily applied to initial value problems.

Our method is proposed to improve the singular perturbation method, it offers considerable reduction and simplicity in computation because it does not require to compute *boundary layer correction solutions*. The difference between both methods lies in the definition of boundary conditions of the *degenerate system*, obtained by suppressing the perturbation parameter in the initial system (1.1)-(1.2).

The remainder of the paper is organized as follows. In Section 2, we give the main results. We study the existence and uniqueness of the problem (P_ε), we present our procedure to get approximate solutions, and we give proof of uniform convergence of the proposed iterative method. Section 3 is mainly devoted to validate the effectiveness of our method, compared with the singular perturbation method. We consider a *right end perturbation* (small parameters are situated on the right), and we deduce the results from Section 2. The comparison with the other formal method requires the analysis of its results. We conclude with Section 4.

2. Main result

2.1. Formal asymptotic solution.   In this section, we develop a perturbation method to obtain asymptotic approximate solutions for the whole order. This iterative method facilitates a considerable reduction and simplicity in computation. Like in any perturbation

method, the solution $y_k(\varepsilon)$, $0 \le k \le N$, of problem (P_ε) is assumed as a power series in ε, we seek for a solution of the natural form

$$y_k(\varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j y_k^{(j)}, \quad 0 \le k \le N. \tag{2.1}$$

Substituting the formal expansion (2.1) into (1.1)-(1.2), and equating the coefficients at the same powers of ε, a set of equations is obtained. For the zeroth-order asymptotic approximation, the resulting equations are given by

$$y_k^{(0)} = \alpha_k, \quad k = 0, 1, \dots, n - 1,$$

$$a_{n,k} y_{k+n}^{(0)} + a_{n-1,k} y_{k+n-1}^{(0)} + \cdots + a_{0,k} y_k^{(0)} = f_k, \quad k = 0, 1, \dots, N - m - n, \tag{2.2}$$

$$y_{N-k}^{(0)} = \beta_k, \quad k = 0, 1, \dots, m - 1.$$

The system described by (2.2) corresponds to the *degenerate problem* of (P_ε), it is obtained by suppressing the perturbation parameter in (1.1)-(2.1). Notice that (2.2) is an initial value problem. It defines the sequence $(y_0^{(0)}, \dots, y_{N-m}^{(0)})$ if and only if $a_{n,k} \ne 0$ for $0 \le k \le N - n - m$. The final conditions in (2.2) define the sequence $(y_{N-m+1}^{(0)}, \dots, y_N^{(0)})$.

The terms $y_k^{(0)}$, $0 \le k \le N - n - m$, can be computed without any knowledge of the boundary conditions $y_{N-k} = \beta_k$, $0 \le k \le m - 1$. By analogy with the case of differential equations, we can say that there are m *boundary layers* located at the *right*, that is, at the final values.

For the jth-order asymptotic approximation, $j \ge 1$, we agree that

$$y_k^{(l)} \equiv 0, \quad l < 0, \ 0 \le k \le N, \tag{2.3}$$

to give a compact writing of the resulting equations which are given by

$$y_k^{(j)} = 0, \quad k = 0, 1, \dots, n - 1,$$

$$a_{n,k} y_{k+n}^{(j)} + a_{n-1,k} y_{k+n-1}^{(j)} + \cdots + a_{0,k} y_k^{(j)}$$

$$= -a_{n+1,k} y_{k+n+1}^{(j-1)} - a_{n+2,k} y_{k+n+2}^{(j-2)} \tag{2.4}$$

$$- \cdots - a_{n+m,k} y_{k+n+m}^{(j-m)}, \quad k = 0, \dots, N - n - m,$$

$$y_{N-k}^{(j)} = 0, \quad k = 0, 1, \dots, m - 1.$$

Notice that (2.4) is an initial value problem. It defines the sequence $(y_0^{(j)}, \dots, y_{N-m}^{(j)})$ if and only if $a_{n,k} \ne 0$ for $0 \le k \le N - n - m$. The final conditions in (2.4) define the sequence $(y_{N-m+1}^{(j)}, \dots, y_N^{(j)})$.

2.2. Existence and convergence. In this section, we present the main results of this paper. We give sufficient conditions that guarantee, for problem (P_ε), existence and uniqueness

of the solution, and we prove the convergence of the series (2.1). The following theorem includes these results.

THEOREM 2.1. *Assume that $a_{n,k} \neq 0$ for $0 \leq k \leq N - n - m$. Then there exists positive real number ε_0, if $|\varepsilon| < \varepsilon_0$, the solution $(y_k(\varepsilon))$ of (P_ε) exists and is unique and satisfies (2.1) uniformly for $0 \leq k \leq N$, where $y_k^{(0)}$ and $y_k^{(j)}$ are the solutions of (2.2) and (2.4), respectively. More precisely, for all $n \geq 0$ and all $0 \leq k \leq N$,*

$$\left| y_k(\varepsilon) - \sum_{j=0}^{n} \varepsilon^j y_k^{(j)} \right| \leq C \frac{\left(|\varepsilon|/\varepsilon_0 \right)^{n+1}}{1 - |\varepsilon|/\varepsilon_0}, \tag{2.5}$$

where C is a constant independent of n and ε.

Proof. For all $k = 0, 1, \ldots, N - n - m$, we use the transformations

$$u_1(k + n + 2) = \varepsilon y_{k+n+2},$$
$$u_2(k + n + 3) = \varepsilon u_1(k + n + 3),$$
$$\cdots \tag{2.6}$$
$$u_{m-1}(k + n + m) = \varepsilon u_{m-2}(k + n + m),$$

then problem (P_ε) becomes

$$y_k = \alpha_k, \quad k = 0, 1, \ldots, n - 1,$$
$$\varepsilon \left(a_{m+n,k} u_{m-1}(k + m + n) + \cdots + a_{n+2,k} u_1(k + n + 2) + a_{n+1,k} y_{k+n+1} \right)$$
$$+ a_{n,k} y_{k+n} + a_{n-1,k} y_{k+n-1} + \cdots + a_{0,k} y_k = f_k, \quad k = 0, 1, \ldots, N - n - m, \tag{2.7}$$
$$y_{N-k} = \beta_k, \quad k = 0, 1, \ldots, m - 1.$$

We can now write the system (2.7) in the matrix form

$$A_\varepsilon y = (A_0 + \varepsilon U) y = f, \tag{2.8}$$

where

$$y = (y_0, y_1, \ldots, y_N, u_1, \ldots, u_{m-1})^t,$$
$$u_j = (u_j(n + j + 1), \ldots, u_j(N - m + j + 1))^t, \quad j = 1, \ldots, m - 1, \tag{2.9}$$
$$f = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, f_0, f_1, \ldots, f_{N-n-m}, \beta_{m-1}, \beta_{m-2}, \ldots, \beta_0, 0, \ldots, 0),$$

and the matrix A_0 is given by

$$
\begin{pmatrix}
1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
a_{0,0} & \cdots & a_{n-1,0} & a_{n,0} & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & a_{0,N-n-m} & \cdots & a_{n,N-n-m} & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1
\end{pmatrix},
\tag{2.10}
$$

whereas matrixes A_ε and U can be deduced easily from (2.7), (2.8), (2.9), and (2.10). We indifferently denote by $\|\cdot\|$ the infinity norm in $\mathbb{R}^{N+1+(m-1)\times(N-n-m+1)}$ and the associated matrix norm. Since $a_{n,k} \neq 0$ for $0 \leq k \leq N - n - m$, matrix A_0 is nonsingular and we can define the positive number

$$
\varepsilon_0 = \frac{1}{\|UA_0^{-1}\|}.
\tag{2.11}
$$

If $|\varepsilon| < \varepsilon_0$, we deduce from (2.8) that

$$
A_0^{-1} \sum_{l=0}^{+\infty} \left(-\varepsilon UA_0^{-1} \right)^l = A_0^{-1}\left(I + \varepsilon UA_0^{-1} \right)^{-1} = A_\varepsilon^{-1},
\tag{2.12}
$$

that is, the inverse of matrix A_ε is well defined for $|\varepsilon| < \varepsilon_0$, and (2.8) has a unique solution $y(\varepsilon)$ given by

$$
y(\varepsilon) = A_\varepsilon^{-1} f.
\tag{2.13}
$$

We denote

$$
y^{(l)} = A_0^{-1}\left(-UA_0^{-1} \right)^l f, \quad l \geq 0,
\tag{2.14}
$$

$$
y^{(l)} = \left(y_0^{(l)}, y_1^{(l)}, \ldots, y_N^{(l)}, u_1^{(l)}, \ldots, u_{m-1}^{(l)} \right)^t \in \mathbb{R}^{N+1+(m-1)\times(N-n-m+1)}, \quad y_k^{(l)} \in \mathbb{R},
$$
$$
u_j^{(l)} = \left(u_j^{(l)}(n+j+1), \ldots, u_j^{(l)}(N-m+j+1) \right)^t, \quad j = 1, \ldots, m-1,
\tag{2.15}
$$

from (2.12), (2.13), and (2.14) we deduce that, for $|\varepsilon| < \varepsilon_0$, the solution $y(\varepsilon)$ of (2.8) can be represented in the convergent series:

$$
y(\varepsilon) = \sum_{l=0}^{+\infty} \varepsilon^l y^{(l)}.
\tag{2.16}
$$

We can easily verify that the $N + 1$ first components of $y^{(0)}$ (resp., $y^{(l)}$, $l \geq 1$), that is, $y_k^{(0)}$, $0 \leq k \leq N$, (resp., $y_k^{(l)}$, $0 \leq k \leq N$, $l \geq 1$) satisfy problem (2.2) (resp., problem (2.4)).

To prove estimate (2.5), we compute the remainder of series (2.12),

$$
\begin{aligned}
\left\| A_\varepsilon^{-1} - A_0^{-1} \sum_{l=0}^{n} \left(-\varepsilon U A_0^{-1} \right)^l \right\| &\leq \| A_0^{-1} \| \sum_{l=n+1}^{+\infty} \| \varepsilon U A_0^{-1} \|^l \\
&= \frac{\| A_0^{-1} \| \| \varepsilon U A_0^{-1} \|^{n+1}}{1 - \| \varepsilon U A_0^{-1} \|} \leq \| A_0^{-1} \| \frac{\left(|\varepsilon| / \varepsilon_0 \right)^{n+1}}{1 - |\varepsilon| / \varepsilon_0}.
\end{aligned}
\tag{2.17}
$$

We denote by C the real positive number

$$
C = \| A_0^{-1} \| \| f \|,
\tag{2.18}
$$

we deduce from (2.13), (2.17), and (2.18) that

$$
\left\| y(\varepsilon) - \sum_{j=0}^{n} \varepsilon^j y^{(j)} \right\| \leq C \frac{\left(|\varepsilon| / \varepsilon_0 \right)^{n+1}}{1 - |\varepsilon| / \varepsilon_0},
\tag{2.19}
$$

with the chosen norm, we obtain (2.5).   □

3. Comparison with singular perturbation method

In this section, we consider a *right end perturbation* [12]. We are concerned with the boundary value problem

$$
\begin{aligned}
y_k &= \alpha_k, \quad k = 0, \ldots, m - 1, \\
a_{m+n,k} y_{k+m+n} + \cdots &+ a_{m+1,k} y_{k+m+1} + a_{m,k} y_{k+m} \\
&+ \varepsilon a_{m-1,k} y_{k+m-1} + \cdots + \varepsilon^{m-1} a_{1,k} y_{k+1} + \varepsilon^m a_{0,k} y_k \\
&= f_k, \quad k = 0, 1, \ldots, N - n - m, \\
y_{N-k} &= \beta_k, \quad k = 0, \ldots, n - 1.
\end{aligned}
\tag{3.1}
$$

The stationary case of problem (3.1) was considered in [12]. The singular perturbation method was developed to get asymptotic expansions for the solution.

In order to validate the effectiveness of our method, we compare the results given by both methods. First, we give our results which are easy to deduce from Section 2, then we analyze the formal expansions obtained in [12].

3.1. Perturbation method. Using the transformation

$$
z_k = y_{N-k}, \quad 0 \leq k \leq N,
\tag{3.2}
$$

problem (3.1) is brought in the form (1.1)-(1.2) and we can already state the results.

Let $y_k^{(0)}$, $0 \le k \le N$, be the solution of the problem

$$y_k^{(0)} = \alpha_k, \quad k = 0, 1, \ldots, m - 1,$$

$$a_{m+n,k} y_{k+m+n}^{(0)} + \cdots + a_{m+1,k} y_{k+m+1}^{(0)} + a_{m,k} y_{k+m}^{(0)}$$

$$= f_k, \quad k = 0, 1, \ldots, N - m - n, \tag{3.3}$$

$$y_{N-k}^{(0)} = \beta_k, \quad k = 0, 1, \ldots, n - 1.$$

The sequence $(y_0^{(0)}, \ldots, y_{N-m}^{(0)})$ in (3.3) can be computed using the final values $y_m^{(0)}, \ldots, y_N^{(0)}$ if and only if $a_{m,k} \neq 0$ for $0 \le k \le N - n - m$. The values $y_0^{(0)}, \ldots, y_{m-1}^{(0)}$ are fixed. The m *boundary layers* are located at the *left*, at the initial values.

Once the coefficients $y_k^{(0)}$, $0 \le k \le N$, are fixed from (3.3), we can define the following problems recursively for $j \ge 1$.

Let $y_k^{(j)}$, $0 \le k \le N$, $j \ge 1$, be the solution of the problem

$$y_k^{(j)} = 0, \quad k = 0, 1, \ldots, m - 1,$$

$$a_{m+n,k} y_{k+m+n}^{(j)} + \cdots + a_{m+1,k} y_{k+m+1}^{(j)} + a_{m,k} y_k^{(j)}$$

$$= -a_{m-1,k} y_k^{(j-1)} - a_{m-2,k} y_k^{(j-2)} - \cdots \tag{3.4}$$

$$- a_{0,k} y_k^{(j-m)}, \quad k = 0, 1, \ldots, N - m - n,$$

$$y_{N-k}^{(j)} = 0, \quad k = 0, 1, \ldots, n - 1.$$

The sequence $(y_0^{(j)}, \ldots, y_{N-m}^{(j)})$ can be computed using the final values $y_{N-n+1}^{(j)}, \ldots, y_N^{(j)}$ in (3.4) if and only if $a_{m,k} \neq 0$ for $0 \le k \le N - n - m$. The values $y_0^{(j)}, \ldots, y_{m-1}^{(j)}$ are fixed.

COROLLARY 3.1. *Assume that $a_{m,k} \neq 0$ for $0 \le k \le N - n - m$. Then there exists positive real number ε_0, if $|\varepsilon| < \varepsilon_0$, the solution $(y_k(\varepsilon))$ of (3.1) exists and is unique, and satisfies*

$$y_k(\varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j y_k^{(j)}, \quad 0 \le k \le N, \tag{3.5}$$

uniformly for $0 \le k \le N$, where $y_k^{(0)}$ and $y_k^{(j)}$ are the solutions of (2.2) and (2.4), respectively. More precisely, for all $n \ge 0$ and all $0 \le k \le N$,

$$\left| y_k(\varepsilon) - \sum_{j=0}^{n} \varepsilon^j y_k^{(j)} \right| \le C \frac{\left(|\varepsilon|/\varepsilon_0 \right)^{n+1}}{1 - |\varepsilon|/\varepsilon_0}, \tag{3.6}$$

where C is a constant independent of n and ε.

Proof. A direct consequence of (3.2) and Theorem 2.1.   □

3.2. Singular perturbation method. The following problem:

$$y_k = \alpha_k, \quad k = 0,\ldots,m-1,$$

$$y_{k+m+n} + \cdots + a_m y_{k+m} + \varepsilon a_{m-1} y_{k+m-1} + \cdots + \varepsilon^m a_0 y_k$$
$$= f_k, \quad k = 0,1,\ldots,N-n-m,$$

$$y_{N-k} = \beta_k, \quad k = 0,\ldots,n-1,$$

(3.7)

was considered in [12], it corresponds to the stationary case of problem (3.1). By analogy with the case of ordinary differential equations, the authors developed a singular perturbation method. They wrote the solution $y_k(\varepsilon)$, $0 \le k \le N$, of (3.7) in the form

$$y_k(\varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j y_{t,k}^{(j)} + \varepsilon^k \sum_{j=0}^{\infty} \varepsilon^j w_{0,k}^{(j)} + \cdots + \varepsilon^{k-m+1} \sum_{j=0}^{\infty} \varepsilon^j w_{m-1,k}^{(j)}, \quad 0 \le k \le N, \quad (3.8)$$

where $\sum_{j=0}^{\infty} \varepsilon^j y_{t,k}^{(j)}$ is the *outer series* and $\sum_{j=0}^{\infty} \varepsilon^j w_{s,k}^{(j)}$, $s = 0,\ldots,m-1$, is the *correction series* introduced to recover the *m boundary layers* located in the initial conditions.

The coefficients $y_{t,k}^{(0)}$ are the solutions of the final value problem (3.9)-(3.10),

$$a_{m+n} y_{t,k+m+n}^{(0)} + a_{m+n-1} y_{t,k+m+n-1}^{(0)} + \cdots + a_m y_{t,k+m}^{(0)}$$
$$= f_k, \quad k = N-n-m, N-n-m-1,\ldots,$$

(3.9)

$$y_{t,N}^{(0)} = \beta_0, y_{t,N-1}^{(0)} = \beta_1,\ldots,y_{t,N-n+1}^{(0)} = \beta_{n-1}. \quad (3.10)$$

The coefficients $w_{s,k}^{(0)}$, $s = 0,1,\ldots,m-1$, are the solutions of the initial value problem (3.11)-(3.12),

$$a_m w_{s,k+m}^{(0)} + a_{m-1} w_{s,k+m-1}^{(0)} + \cdots + a_0 w_{s,k}^{(0)} = 0, \quad k = 0,1,\ldots, \quad (3.11)$$

$$w_{s,s}^{(0)} = \alpha_s - y_{t,s}^{(0)}, \quad w_{s,k}^{(0)} = 0 \quad \text{if } k \ne s, \ s = 0,1,\ldots,m-1. \quad (3.12)$$

To give a single writing for the problems which define the coefficients $y_{t,k}^{(j)}$, $w_{s,k}^{(j)}$, $j \ge 1$, $0 \le s \le m-1$, we agree that for all k, $0 \le s \le m-1$, $j < 0$,

$$y_{t,k}^{(j)} \equiv 0, \qquad w_{s,k}^{(j)} \equiv 0. \quad (3.13)$$

The coefficients $y_{t,k}^{(j)}$ are the solutions of the final value problem (3.14)-(3.15),

$$a_{m+n} y_{t,k+m+n}^{(j)} + a_{m+n-1} y_{t,k+m+n-1}^{(j)} + \cdots + a_m y_{t,k+m}^{(j)}$$
$$= -a_{m-1} y_{t,k+m-1}^{(j-1)} - a_{m-2} y_{t,k+m-2}^{(j-2)} - \cdots$$
$$- a_0 y_{t,k}^{(j-m)}, \quad k = N-n-m, N-n-m-1,\ldots,$$

(3.14)

$$y_{t,N}^{(j)} = y_{t,N-1}^{(j)} = \cdots = y_{t,N-n+1}^{(j)} = 0. \quad (3.15)$$

The coefficients $w_{s,k}^{(j)}$, $0 \le s \le m-1$, $j \ge 1$, are the solutions of the initial value problem (3.16)-(3.17),

$$a_m w_{s,k+m}^{(j)} + a_{m-1} w_{s,k+m-1}^{(j)} + \cdots + a_0 w_{s,k}^{(j)}$$

$$= -a_{m+1} w_{s,k+m+1}^{(j-1)} - a_{m+2} w_{s,k+m+2}^{(j-2)} - \cdots \tag{3.16}$$

$$- a_{m+n} w_{s,k+m+n}^{(j-n)}, \quad k = 0,1,\dots,$$

$$w_{s,s}^{(j)} = -y_{t,s}^{(j)}, \quad w_{s,k}^{(j)} = 0 \quad \text{if } k \ne s,\, 0 \le s \le m-1. \tag{3.17}$$

This formal procedure was not justified. The expansion (3.8) is not asymptotic when the order is equal to $N - n - m + 2$, see the following proposition.

PROPOSITION 3.2. *The series (3.19) is not an asymptotic expansion of the solution $y_k(\varepsilon)$ of order $N - n - m + 2$.*

Proof. Since $\tilde{y}_{N-n+1}^{(0)} = \beta_{n-1}$ and $y_{t,N-n+1}^{(j)} = 0$ for $j \ge 1$, from (3.8) we get

$$y_{N-n+1}(\varepsilon) = \beta_{n-1} + \varepsilon^{N-n-m+2} w_{m-1,N-n+1}^{(0)} + \cdots. \tag{3.18}$$

Since in general $w_{m-1,N-n+1}^{(0)} \ne 0$, (3.8) is not an asymptotic expansion of $y_{N-n+1}(\varepsilon) = \beta_{n-1}$ of order $N - n - m + 2$. □

Remark 3.3. To get the correction terms $w_{s,k}^{(j)}$, $0 \le s \le m-1$, $j \ge 0$, we must compute the values $y_{t,0}^{(0)}, y_{t,1}^{(0)}, \dots, y_{t,m-1}^{(0)}$ and $y_{t,0}^{(j)}, y_{t,1}^{(j)}, \dots, y_{t,m-1}^{(j)}$. This computation requires to solve the difference equations (3.9) and (3.14) for $k = -1, -2, \dots, -m$. In general, for the time-variant case, we do not have at our disposal the values $a_{m,-l}, a_{m+1,-l}, \dots, a_{m+n-1,-l}, f_{-l}, l = 1, 2, \dots, m$ which allow us to compute the coefficients $y_{t,s}^{(0)}$ and $y_{t,s}^{(j)}$, $j \ge 1$. Consequently, we cannot define the problems (3.11)-(3.12) and (3.16)-(3.17).

3.2.1. There is no need of correction series. In this section, we compare the expansion (3.8) and our expansion given in Section 3.1, for the time-invariant case. From (3.8) we see that

$$y_k(\varepsilon) = \sum_{j=0}^{+\infty} \varepsilon^j \tilde{y}_k^{(j)}, \quad 0 \le k \le N, \tag{3.19}$$

where

$$\tilde{y}_k^{(j)} = y_{t,k}^{(j)} + w_{0,k}^{(j-k)} + w_{1,k}^{(j-k+1)} + \cdots + w_{m-1,k}^{(j-k+m-1)}, \quad 0 \le k \le N,\, j \ge 0. \tag{3.20}$$

PROPOSITION 3.4. *The coefficients of series (3.19) satisfy*

$$\tilde{y}_s^{(0)} = \alpha_s, \quad 0 \le s \le m-1, \qquad \tilde{y}_k^{(0)} = \beta_{N-k}, \quad N-n+1 \le k \le N,$$

$$\tilde{y}_s^{(j)} = 0, \quad 0 \le s \le m-1, \qquad \tilde{y}_k^{(j)} = 0, \quad N-n+1 \le k \le N,\, 1 \le j \le N-n-m+1. \tag{3.21}$$

Proof. From (3.12) and (3.20), we deduce

$$\widetilde{y}_s^{(0)} = y_{t,s}^{(0)} + w_{0,s}^{(-s)} + w_{1,s}^{(-s+1)} + \cdots + w_{m-1,s}^{(-s+m-1)} = y_{t,s}^{(0)} + w_{s,s}^{(0)} = \alpha_s, \quad s = 0, 1, \ldots, m-1. \tag{3.22}$$

Equations (3.10) and (3.20) give

$$\widetilde{y}_k^{(0)} = y_{t,k}^{(0)} \equiv \beta_{N-k}, \quad k = N - n + 1, N - n + 2, \ldots, N. \tag{3.23}$$

In addition, (3.17) and (3.20) give

$$\widetilde{y}_s^{(j)} = y_{t,s}^{(j)} + w_{0,s}^{(j-s)} + w_{1,s}^{(j-s+1)} + \cdots + w_{m-1,s}^{(j-s+m-1)} = y_{t,s}^{(j)} + w_{s,s}^{(j)} \equiv 0, \quad s = 0, 1, \ldots, m-1. \tag{3.24}$$

Since $1 \leq j \leq N + 1 - n - m$, for all $N - n + 1 \leq k \leq N$, we have

$$j - k < j - k + 1 < \cdots < j - k + m - 1 < 0, \tag{3.25}$$

then, from (3.15) and (3.20), we deduce

$$\widetilde{y}_k^{(j)} = y_{t,k}^{(j)} \equiv 0, \quad N - n + 1 \leq k \leq N, \ 1 \leq j \leq N + 1 - n - m. \tag{3.26}$$

$\square$

PROPOSITION 3.5. *The coefficients of order 0 in (3.19) satisfy the equation*

$$\widetilde{y}_{k+m+n}^{(0)} + a_{m+n-1}\widetilde{y}_{k+m+n-1}^{(0)} \cdots + a_m \widetilde{y}_{k+m}^{(0)} = f_k, \quad k = 0, 1, \ldots, N - n - m. \tag{3.27}$$

The coefficients of order j, $1 \leq j \leq N - n - m + 1$, in (3.19) satisfy the equation

$$\widetilde{y}_{k+m+n}^{(j)} + a_{m+n-1}\widetilde{y}_{k+m+n-1}^{(j)} + \cdots + a_m \widetilde{y}_{k+m}^{(j)}$$
$$= -a_{m-1}\widetilde{y}_{k+m-1}^{(j-1)} - a_{m-2}\widetilde{y}_{k+m-2}^{(j-2)} - \cdots \tag{3.28}$$
$$- a_0 \widetilde{y}_k^{(j-m)}, \quad k = 0, 1, \ldots, N - n - m.$$

Proof. From (3.20), we get $\widetilde{y}_k^{(j)} = y_{t,k}^{(j)} + w_{0,k}^{(j-k)} + w_{1,k}^{(j-k+1)} + \cdots + w_{m-1,k}^{(j-k+m-1)}$, that is,

$$\widetilde{y}_{k+m+l}^{(0)} = y_{t,k+m+l}^{(0)}, \quad \forall l \geq 0, \ k \geq 0. \tag{3.29}$$

Then (3.9) is equivalent to the difference equation (3.27). We show now that the coefficients $\widetilde{y}_k^{(j)}$, $k = 0, \ldots, N$, satisfy the difference equation given in (3.28).

First, notice that with the convention (3.13), we get a unique writing for (3.11) and (3.16), thus we does not have to consider several cases. If we replace the variable j by $j + s - k - m$ in (3.16) that we multiply by -1, we obtain

$$w_{s,k+n+m}^{(j+s-k-n-m)} + a_{n+m-1}w_{s,k+n+m-1}^{(j+s-k-n-m+1)} + \cdots + a_m w_{s,k+m}^{(j+s-k-m)}$$
$$= -a_{m-1}w_{s,k+m-1}^{(j+s-k-m)} - \cdots - a_0 w_{s,k}^{(j+s-k-m)}, \quad k = 0, 1, \ldots. \tag{3.30}$$

Let us write (3.30) for various values of s, $s = 0, 1, \ldots, m-1$, the sum of these m equations and (3.14) gives the following difference equation:

$$
\begin{aligned}
a_{m+n} & \left(y_{t,k+m+n}^{(j)} + w_{0,k+m+n}^{(j-k-m-n)} + \cdots + w_{m-1,k+m+n}^{(j-k-n-1)} \right) \\
& + a_{m+n-1} \left(y_{t,k+m+n-1}^{(j)} + w_{0,k+m+n-1}^{(j-k-m-n+1)} + \cdots + w_{m-1,k+m+n-1}^{(j-k-n)} \right) \\
& + \cdots + a_m \left(y_{t,k+m}^{(j)} + w_{0,k+m}^{(j-k-m)} + \cdots + w_{m-1,k+m}^{(j-k-1)} \right) \\
= & -a_{m-1} \left(y_{t,k+m-1}^{(j-1)} + w_{0,k+m-1}^{(j-k-m)} + \cdots + w_{m-1,k+m-1}^{(j-k-1)} \right) \\
& - \cdots - a_0 \left(y_{t,k}^{(j-m)} + w_{0,k}^{(j-k-m)} + \cdots + w_{m-1,k}^{(j-k-1)} \right), \quad k = 0, 1, 2, \ldots.
\end{aligned}
\tag{3.31}
$$

From (3.20), we make sure that (3.31) is nothing other than the difference equation (3.28). □

Conclusion 3.6. Propositions 3.4 and 3.5 show that when $j \le N + 1 - n - m$, the coefficients of expansion (3.8) satisfy the same problems given by our method in Section 3.1. By the uniqueness of the solutions of problems (3.3) and (3.4), we conclude that we have the same expansions of order $N + 1 - n - m$.

4. Conclusion

In this paper, we extended the perturbation method developed in [15], for a linear time-variant classical digital control system with several widely different coefficients. We considered a class of boundary value problems and we achieved two objectives. First, we gave sufficient conditions to ensure the existence and uniqueness of the solution of the boundary problem. Second, we gave an iterative method to get successive asymptotic approximations. This method is simple, straightforward, and convergent. It improves the singular perturbation method, indeed it offers considerable reduction and simplicity in computation since it does not require to compute *boundary layer correction solutions*. This method can be easily extended for initial value problems.

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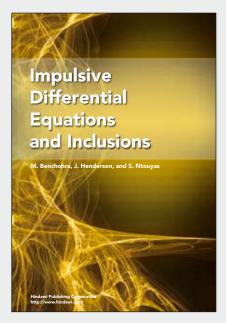
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Impulsive Differential Equations and Inclusions

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and Inclusions

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Functional differential equations and inclusions have assumed an increasing relevance, mainly due to the great variety of problems arising in sciences, engineering, and technology which can be described by such types of equations and inclusions. This book is devoted to this rapidly developing theory of differential equations and inclusions with special emphasis on such equations and inclusions subject to impulses. In addition, the book includes a wide account of various applications to physical problems.

The book consists of twelve chapters. Initial value problems and boundary value problems for both impulsive differential equations and differential inclusions, as well as for both impulsive functional differential equations and functional differential inclusions, or for both neutral functional differential equations and neutral functional differential inclusions, are studied. Moreover, results on impulsive semilinear ordinary differential inclusions and functional differential inclusions satisfying nonlocal boundary conditions are given extensive attention. Positive solutions and multiple positive solutions for impulsive ordinary differential equations and functional differential equations, as well as impulsive differential inclusions satisfying periodic boundary conditions, are also studied.

Some of these results are extended to functional differential equations and functional differential inclusions with variable times, to semilinear functional differential equations and functional differential inclusions for operators that are nondensely defined on a Banach space, to impulsive hyperbolic differential inclusions, and to impulsive dynamic equations on time scales.

Each chapter concludes with a section devoted to notes and bibliographical remarks. This book is addressed to a wide audience of specialists such as mathematicians, engineers, biologists, and physicists.