

Systems of Ordinary Differential Equations > Linear Systems of Two Equations

12. 
$$x_{tt}^{"} = a(ty_t^{'} - y), \quad y_{tt}^{"} = b(tx_t^{'} - x).$$

The transformation

$$u = tx_t - x, \quad v = ty_t' - y \tag{1}$$

leads to the first-order system

$$u_t' = atv, \quad v_t' = btu.$$

The general solution of this system is given by

with 
$$ab > 0$$
: 
$$\begin{cases} u(t) = C_1 a \exp\left(\frac{1}{2}\sqrt{ab}\,t^2\right) + C_2 a \exp\left(-\frac{1}{2}\sqrt{ab}\,t^2\right), \\ v(t) = C_1 \sqrt{ab} \exp\left(\frac{1}{2}\sqrt{ab}\,t^2\right) - C_2 \sqrt{ab} \exp\left(-\frac{1}{2}\sqrt{ab}\,t^2\right), \end{cases}$$
with  $ab < 0$ : 
$$\begin{cases} u(t) = C_1 a \cos\left(\frac{1}{2}\sqrt{|ab|}\,t^2\right) + C_2 a \sin\left(\frac{1}{2}\sqrt{|ab|}\,t^2\right), \\ v(t) = -C_1 \sqrt{|ab|} \sin\left(\frac{1}{2}\sqrt{|ab|}\,t^2\right) + C_2 \sqrt{|ab|} \cos\left(\frac{1}{2}\sqrt{|ab|}\,t^2\right), \end{cases}$$
(2)

where  $C_1$  and  $C_2$  are arbitrary constants. On substituting (2) into (1) and integrating the resulting expressions, one arrives at the general solution of the original system:

$$x = C_3 t + t \int \frac{u(t)}{t^2} dt, \quad y = C_4 t + t \int \frac{v(t)}{t^2} dt,$$

where  $C_3$  and  $C_4$  are arbitrary constants.

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http://eqworld.ipmnet.ru/en/solutions/sysode/sode0112.pdf