1. Linear Difference and Functional Equations with One Independent Variable

1.1. Linear Difference and Functional Equations Containing Unknown Function with Two Different Arguments

First-order linear difference equations.

1. $y(x+1) - ay(x) = 0.$
   First-order constant-coefficient linear homogeneous difference equation.

2. $y(x+1) - ay(x) = f(x).$
   First-order constant-coefficient linear nonhomogeneous difference equation.

3. $y(x+1) - xy(x) = 0.$

4. $y(x+1) - a(x-b)(x-c)y(x) = 0.$

5. $y(x+1) - R(x)y(x) = 0, \qquad R(x) = a\dfrac{(x-\lambda_1)(x-\lambda_2)\ldots(x-\lambda_n)}{(x-\mu_1)(x-\mu_2)\ldots(x-\mu_m)}.$

6. $y(x+1) - f(x)y(x) = 0.$

7. $y(x+a) - by(x) = 0.$

8. $y(x+a) - by(x) = f(x).$

9. $y(x+a) - bxy(x) = 0, \qquad a, b > 0.$

10. $y(x+a) - f(x)y(x) = 0.$

Linear functional equations containing $y(x)$ and $y(ax)$.

11. $y(ax) - by(x) = 0, \qquad a, b > 0.$

12. $y(ax) - by(x) = f(x).$

Linear functional equations containing $y(x)$ and $y(a-x)$.

13. $y(x) - y(a-x) = 0.$

14. $y(x) + y(a-x) = 0.$

15. $y(x) + y(a-x) = b.$

16. $y(x) + y(a-x) = f(x).$

17. $y(x) - y(a-x) = f(x).$

18. $y(x) + g(x)y(a-x) = f(x).$

Linear functional equations containing $y(x)$ and $y(z)$, where $z = \varphi(x)$.

19. $y(x^a) - by(x) = 0, \qquad a, b > 0.$

20. $y(x) - y(a/x) = 0.$

21. $y(x) + y(a/x) = 0.$

22. $y(x) + y(a/x) = b.$

23. $y(x) + y(a/x) = f(x).$

24. $y(x) - y(a/x) = f(x).$

25. $y(x) + g(x)y(a/x) = f(x).$

26. $y(x) - y\left(\dfrac{a - x}{1 + bx}\right) = 0.$

27. $y(x) + y\left(\dfrac{a - x}{1 + bx}\right) = 0.$

28. $y(x) + y\left(\dfrac{a - x}{1 + bx}\right) = f(x).$

29. $y(x) - y\left(\dfrac{a - x}{1 + bx}\right) = f(x).$

30. $y(x) - cy\left(\dfrac{a - x}{1 + bx}\right) = f(x), \qquad c \neq \pm 1.$

31. $y(x) + g(x)y\left(\dfrac{a - x}{1 + bx}\right) = f(x).$

32. $y(x) + cy\left(\dfrac{ax - \beta}{x + b}\right) = f(x), \qquad \beta = a^2 + ab + b^2.$

33. $y(x) + cy\left(\dfrac{bx + \beta}{a - x}\right) = f(x), \qquad \beta = a^2 + ab + b^2.$

34. $y(x) + g(x)y\left(\dfrac{ax - \beta}{x + b}\right) = f(x), \qquad \beta = a^2 + ab + b^2.$

35. $y(x) + g(x)y\left(\dfrac{bx + \beta}{a - x}\right) = f(x), \qquad \beta = a^2 + ab + b^2.$

36. $y(x) - y\left(\sqrt{a^2 - x^2}\right) = 0, \qquad 0 \le x \le a.$

37. $y(x) + y\left(\sqrt{a^2 - x^2}\right) = 0, \qquad 0 \le x \le a.$

38. $y(x) + y\left(\sqrt{a^2 - x^2}\right) = b, \qquad 0 \le x \le a.$

39. $y(x) + y\left(\sqrt{a^2 - x^2}\right) = f(x), \qquad 0 \le x \le a.$

40. $y(x) - y\left(\sqrt{a^2 - x^2}\right) = f(x), \qquad 0 \le x \le a.$

41. $y(x) + g(x)y\left(\sqrt{a^2 - x^2}\right) = f(x), \qquad 0 \le x \le a.$

Linear functional equations containing $y(\sin x)$ and $y(\cos x)$.

42. $y(\sin x) - y(\cos x) = 0.$

43. $y(\sin x) + y(\cos x) = 0.$

44. $y(\sin x) + y(\cos x) = a.$

45. $y(\sin x) + y(\cos x) = f(x).$

46. $y(\sin x) - y(\cos x) = f(x).$

47. $y(\sin x) + g(x)y(\cos x) = f(x).$

Linear functional equations containing $y(x)$ and $y(\omega(x))$, where $\omega\big(\omega(x)\big) = x$.

48. $y(x) - y\big(\omega(x)\big) = 0,$ **where** $\omega\big(\omega(x)\big) = x.$

49. $y(x) + y\big(\omega(x)\big) = 0,$ **where** $\omega\big(\omega(x)\big) = x.$

50. $y(x) + y\big(\omega(x)\big) = b,$ **where** $\omega\big(\omega(x)\big) = x.$

51. $y(x) + y\big(\omega(x)\big) = f(x),$ **where** $\omega\big(\omega(x)\big) = x.$

52. $y(x) - y\big(\omega(x)\big) = f(x),$ **where** $\omega\big(\omega(x)\big) = x.$

53. $y(x) + g(x)y\big(\omega(x)\big) = f(x),$ **where** $\omega\big(\omega(x)\big) = x.$

1.2. Other Linear Difference and Functional Equations

Second-order linear difference equations, $y_n = y(n)$.

1. $y_{n+2} + ay_{n+1} + by_n = 0.$
    Second-order constant-coefficient linear homogeneous difference equation.

2. $y_{n+2} + ay_{n+1} + by_n = f_n.$
    Second-order constant-coefficient linear nonhomogeneous difference equation.

3. $y(x + 2) + ay(x + 1) + by(x) = 0.$
    Second-order constant-coefficient linear homogeneous difference equation.

4. $y(x + 2) + ay(x + 1) + by(x) = f(x).$
    Second-order constant-coefficient linear nonhomogeneous difference equation.

5. $y(x + 2) + a(x + 1)y(x + 1) + bx(x + 1)y(x) = 0.$

Other functional equations.

6. $Ay(ax) + By(bx) + y(x) = 0.$

7. $Ay(x^a) + By(x^b) + y(x) = 0.$

8. $y\big(y(x)\big) - x = 0.$

9. $y\big(y(x)\big) + ay(x) + bx = 0.$

10. $y\big(y(y(x))\big) - x = 0.$

11. $Ay(x) + By\Big(\dfrac{ax - \beta}{x + b}\Big) + Cy\Big(\dfrac{bx + \beta}{a - x}\Big) = f(x), \qquad \beta = a^2 + ab + b^2.$

12. $f_1(x)y(x) + f_2(x)y\Big(\dfrac{ax - \beta}{x + b}\Big) + f_3(x)y\Big(\dfrac{bx + \beta}{a - x}\Big) = g(x), \quad \beta = a^2 + ab + b^2.$

13. $y_{n+m} + a_{m-1}y_{n+m-1} + \ldots + a_1 y_{n+1} + a_0 y_n = 0.$
mth-order constant-coefficient linear nonhomogeneous difference equation.

14. $y_{n+m} + a_{m-1}y_{n+m-1} + \ldots + a_1 y_{n+1} + a_0 y_n = f_n.$
mth-order constant-coefficient linear nonhomogeneous difference equation.

15. $y(x + n) + a_{n-1}y(x + n - 1) + \ldots + a_1 y(x + 1) + a_0 y(x) = 0.$
nth-order constant-coefficient linear homogeneous difference equation.

16. $y(x + n) + a_{n-1}y(x + n - 1) + \ldots + a_1 y(x + 1) + a_0 y(x) = f(x).$
nth-order constant-coefficient linear nonhomogeneous difference equation.

17. $y(x + b_n) + a_{n-1}y(x + b_{n-1}) + \ldots + a_1 y(x + b_1) + a_0 y(x) = 0.$

18. $y\big(x^{a_n}\big) + b_{n-1}y\big(x^{a_{n-1}}\big) + \ldots + b_1 y\big(x^{a_1}\big) + b_0 y(x) = 0.$

19. $y(a_n x) + b_{n-1}y(a_{n-1}x) + \ldots + b_1 y(a_1 x) + b_0 y(x) = 0.$

20. $y^{[n]}(x) + a_{n-1}y^{[n-1]}(x) + \ldots + a_1 y(x) + a_0 x = 0, \qquad y^{[n]}(x) = y\big(y^{[n-1]}(x)\big).$