

Systems of Ordinary Differential Equations > Nonlinear Systems of Three and More Equations

8.
$$x_{tt}'' = \frac{\partial F}{\partial x}$$
, $y_{tt}'' = \frac{\partial F}{\partial y}$, $z_{tt}'' = \frac{\partial F}{\partial z}$, where $F = F(r)$, $r = \sqrt{x^2 + y^2 + z^2}$.

Equations of motion of a point mass under gravitational force.

The equations can be rewritten in the vector form

$$\mathbf{r}_{tt}^{"} = \operatorname{grad} F$$
 or $\mathbf{r}_{tt}^{"} = \frac{F'(r)}{r} \mathbf{r}$,

where $\mathbf{r} = (x, y, z)$.

1°. First integrals:

$$(\mathbf{r}'_t)^2 = 2F(r) + C_1$$
 (law of conservation of energy),
 $[\mathbf{r} \times \mathbf{r}'_t] = \mathbf{C}$ (law of conservation of areas),
 $(\mathbf{r} \cdot \mathbf{C}) = 0$ (trajectories are plane curves).

2°. Solution:

$$\mathbf{r} = \mathbf{a} r \cos \varphi + \mathbf{b} r \sin \varphi$$
.

Here, the constant vectors **a** and **b** must satisfy the conditions

$$|\mathbf{a}| = |\mathbf{b}| = 1, \quad (\mathbf{a} \cdot \mathbf{b}) = 0,$$

and the functions r = r(t) and $\varphi = \varphi(t)$ are defined by the relations

$$t = \int \frac{r \, dr}{\sqrt{2r^2 F(r) + C_1 r^2 - C_3^2}} + C_2, \quad \varphi = C_3 \int \frac{dr}{r \sqrt{2r^2 F(r) + C_1 r^2 - C_3^2}}, \quad C_3 = |\mathbf{C}|,$$

where C_1 , C_2 , and C_3 are arbitrary constants.

Reference

Kamke, E., Differentialgleichungen: Lösungsmethoden und Lösungen, I, Gewöhnliche Differentialgleichungen, B. G. Teubner, Leipzig, 1977.

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