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12.
$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$
 $(a_n \neq 0)$.

Algebraic equation of general form of degree n; the coefficients a_k are real or complex numbers.

 1° . For brevity, denote the left-hand side of the equation, which is a polynomial of degree n, by

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \qquad (a_n \neq 0).$$
 (1)

A number $x=\xi$ is called a root of the equation in question, and also a root of the polynomial $P_n(x)$, if $P_n(\xi)=0$. A number $x=\xi$ is called a root of multiplicity m if $P_n(x)=(x-\xi)^mQ_{n-m}(x)$, where m is a positive integer $(1 \le m \le n)$ and $Q_{n-m}(x)$ is a polynomial of degree n-m such that $Q_{n-m}(\xi) \ne 0$.

- 2° . Main theorem of polynomial algebra. An algebraic equation of degree n has exactly n roots (real or complex), provided a root of multiplicity m is counted m times.
- 3°. If an algebraic equation has roots x_1, x_2, \ldots, x_s with respective multiplicities k_1, k_2, \ldots, k_s $(k_1 + k_2 + \cdots + k_s = n)$, then the left-hand side of the equation can be factorized so that

$$P_n(x) = a_n(x - x_1)^{k_1}(x - x_2)^{k_2} \dots (x - x_s)^{k_s}.$$

- 4°. An algebraic equation of odd degree with real coefficients has always at least one real root.
- 5°. Suppose an algebraic equation with real coefficients has a complex root $\xi = \alpha + i\beta$. Then this equation has also the complex conjugate root $\eta = \alpha i\beta$, with the multiplicities of ξ and η being the same.
- 6° . An algebraic equation of degree n with integer coefficients a_k cannot have rational roots other than irreducible fractions p/q with p being a divisor of a_0 and q a divisor of a_n . If $a_n = 1$, then all rational roots of the algebraic equation (if any) are integer divisors of the free term a_0 and can be found by simple search.
- 7° . Any equation with $n \le 4$ is solvable in terms of radicals, which means that its roots can be expressed in terms of the equation coefficients using the operations of addition, subtraction, multiplication, division, and root extraction. For n > 4, an algebraic equation is generally unsolvable in terms of radicals; this fact is known as the Ruffini–Abel theorem.
- 8° . Vieta's theorem. The following relations between the roots on an algebraic equation (taking into account their multiplicities) and its coefficients hold true:

$$x_1 + x_2 + \dots + x_n = \sum_{i=1}^n x_i = -a_{n-1}/a_n,$$

$$x_1 x_2 + x_1 x_3 + \dots + x_{n-1} x_n = \sum_{1 \le i < j}^n x_i x_j = a_{n-2}/a_n,$$

$$x_1 x_2 x_3 + x_1 x_2 x_4 + \dots + x_{n-2} x_{n-1} x_n = \sum_{1 \le i < j < k}^n x_i x_j x_k = -a_{n-3}/a_n,$$

$$x_1 x_2 x_3 \dots x_n = (-1)^n a_0/a_n.$$

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