2. Нелинейные системы двух дифференциальных уравнений в частных производных параболического типа

2.1. Системы уравнений массо- и теплообмена реагирующих сред и уравнений математической биологии вида

$$\frac{\partial u}{\partial t}=a\frac{\partial^2 u}{\partial x^2}+F(u,w),\quad \frac{\partial w}{\partial t}=b\frac{\partial^2 w}{\partial x^2}+G(u,w)$$

Предварительные замечания. Подобные системы дифференциальных уравнений широко используются в теории массотеплопереноса реагирующих сред, в теории химических реакторов, в теории горения, в математической биологии и биофизике.

В общем случае такие системы инвариантны относительно сдвигов по независимым переменным (и относительно замены x на $-x$) и допускает точные решения типа бегущей волны $u_m = u_m(z)$, где $z = kx + \lambda t$. Такие решения, а также вырожденные решения, когда одна из искомых величин равны нулю, здесь не рассматриваются.

Далее $f(\ldots), g(\ldots), h(\ldots)$ — произвольные функции соответствующего аргумента.

1. $\dfrac{\partial u}{\partial t}=a\dfrac{\partial^2 u}{\partial x^2}+u\exp\left(k\dfrac{w}{u}\right)f(u),\quad \dfrac{\partial w}{\partial t}=a\dfrac{\partial^2 w}{\partial x^2}+\exp\left(k\dfrac{w}{u}\right)[wf(u)+g(u)].$

2. $\dfrac{\partial u}{\partial t}=a\dfrac{\partial^2 u}{\partial x^2}+uf(bu-cw)+g(bu-cw),$

$\dfrac{\partial w}{\partial t}=a\dfrac{\partial^2 w}{\partial x^2}+wf(bu-cw)+h(bu-cw).$

3. $\dfrac{\partial u}{\partial t}=a\dfrac{\partial^2 u}{\partial x^2}+e^{\lambda u}f(\lambda u-\sigma w),\quad \dfrac{\partial w}{\partial t}=b\dfrac{\partial^2 w}{\partial x^2}+e^{\sigma w}g(\lambda u-\sigma w).$

4. $\dfrac{\partial u}{\partial t}=a\dfrac{\partial^2 u}{\partial x^2}+uf\left(\dfrac{u}{w}\right),\quad \dfrac{\partial w}{\partial t}=a\dfrac{\partial^2 w}{\partial x^2}+wg\left(\dfrac{u}{w}\right).$

5. $\dfrac{\partial u}{\partial t}=a\dfrac{\partial^2 u}{\partial x^2}+uf\left(\dfrac{u}{w}\right),\quad \dfrac{\partial w}{\partial t}=b\dfrac{\partial^2 w}{\partial x^2}+wg\left(\dfrac{u}{w}\right).$

6. $\dfrac{\partial u}{\partial t}=a\dfrac{\partial^2 u}{\partial x^2}+uf\left(\dfrac{u}{w}\right)+g\left(\dfrac{u}{w}\right),\quad \dfrac{\partial w}{\partial t}=a\dfrac{\partial^2 w}{\partial x^2}+wf\left(\dfrac{u}{w}\right)+h\left(\dfrac{u}{w}\right).$

7. $\dfrac{\partial u}{\partial t}=a\dfrac{\partial^2 u}{\partial x^2}+uf\left(\dfrac{u}{w}\right)+\dfrac{u}{w}h\left(\dfrac{u}{w}\right),\quad \dfrac{\partial w}{\partial t}=a\dfrac{\partial^2 w}{\partial x^2}+wg\left(\dfrac{u}{w}\right)+h\left(\dfrac{u}{w}\right).$

8. $\dfrac{\partial u}{\partial t}=a\dfrac{\partial^2 u}{\partial x^2}+u^3f\left(\dfrac{u}{w}\right),\quad \dfrac{\partial w}{\partial t}=a\dfrac{\partial^2 w}{\partial x^2}+u^3g\left(\dfrac{u}{w}\right).$

9. $\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}+au-u^3f\left(\dfrac{u}{w}\right),\quad \dfrac{\partial w}{\partial t}=\dfrac{\partial^2 w}{\partial x^2}+aw-u^3g\left(\dfrac{u}{w}\right).$

10. $\dfrac{\partial u}{\partial t}=a\dfrac{\partial^2 u}{\partial x^2}+u^nf\left(\dfrac{u}{w}\right),\quad \dfrac{\partial w}{\partial t}=b\dfrac{\partial^2 w}{\partial x^2}+w^ng\left(\dfrac{u}{w}\right).$

11. $\dfrac{\partial u}{\partial t}=a\dfrac{\partial^2 u}{\partial x^2}+uf\left(\dfrac{u}{w}\right)\ln u+ug\left(\dfrac{u}{w}\right),$

$\dfrac{\partial w}{\partial t}=a\dfrac{\partial^2 w}{\partial x^2}+wf\left(\dfrac{u}{w}\right)\ln w+wh\left(\dfrac{u}{w}\right).$

12. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + uf\Big(\dfrac{w}{u}\Big) - wg\Big(\dfrac{w}{u}\Big) + \dfrac{u}{\sqrt{u^2+w^2}}h\Big(\dfrac{w}{u}\Big),$
$\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + wf\Big(\dfrac{w}{u}\Big) + ug\Big(\dfrac{w}{u}\Big) + \dfrac{w}{\sqrt{u^2+w^2}}h\Big(\dfrac{w}{u}\Big).$

13. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + uf\Big(\dfrac{w}{u}\Big) + wg\Big(\dfrac{w}{u}\Big) + \dfrac{u}{\sqrt{u^2-w^2}}h\Big(\dfrac{w}{u}\Big),$
$\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + wf\Big(\dfrac{w}{u}\Big) + ug\Big(\dfrac{w}{u}\Big) + \dfrac{w}{\sqrt{u^2-w^2}}h\Big(\dfrac{w}{u}\Big).$

14. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + uf(u^n w^m), \quad \dfrac{\partial w}{\partial t} = b\dfrac{\partial^2 w}{\partial x^2} + wg(u^n w^m).$

15. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + u^{1+kn} f(u^n w^m), \quad \dfrac{\partial w}{\partial t} = b\dfrac{\partial^2 w}{\partial x^2} + w^{1-km} g(u^n w^m).$

16. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + cu\ln u + uf(u^n w^m), \quad \dfrac{\partial w}{\partial t} = b\dfrac{\partial^2 w}{\partial x^2} + cw\ln w + wg(u^n w^m).$

17. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + uf(u^2+w^2) - wg(u^2+w^2),$
$\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + wf(u^2+w^2) + ug(u^2+w^2).$

18. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + uf(u^2-w^2) + wg(u^2-w^2),$
$\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + wf(u^2-w^2) + ug(u^2-w^2).$

19. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + uf(u^2+w^2) - wg(u^2+w^2) - w\operatorname{arctg}\Big(\dfrac{w}{u}\Big)h(u^2+w^2),$
$\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + wf(u^2+w^2) + ug(u^2+w^2) + u\operatorname{arctg}\Big(\dfrac{w}{u}\Big)h(u^2+w^2).$

20. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + uf(u^2-w^2) + wg(u^2-w^2) + w\operatorname{arth}\Big(\dfrac{w}{u}\Big)h(u^2-w^2),$
$\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + wf(u^2-w^2) + ug(u^2-w^2) + u\operatorname{arth}\Big(\dfrac{w}{u}\Big)h(u^2-w^2).$

21. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + u^{k+1} f(\varphi), \quad \dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + u^{k+1}[f(\varphi)\ln u + g(\varphi)],$
$\varphi = u\exp\Big(-\dfrac{w}{u}\Big).$

22. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + uf(u^2+w^2) - wg\Big(\dfrac{w}{u}\Big),$
$\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + ug\Big(\dfrac{w}{u}\Big) + wf(u^2+w^2).$

23. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + uf(u^2-w^2) + wg\Big(\dfrac{w}{u}\Big),$
$\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + ug\Big(\dfrac{w}{u}\Big) + wf(u^2-w^2).$

24. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + uf\Big(u^2+w^2, \dfrac{w}{u}\Big) - wg\Big(\dfrac{w}{u}\Big),$
$\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + wf\Big(u^2+w^2, \dfrac{w}{u}\Big) + ug\Big(\dfrac{w}{u}\Big).$

25. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + uf\Big(u^2 - w^2, \dfrac{w}{u}\Big) + wg\Big(\dfrac{w}{u}\Big),$
$\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + wf\Big(u^2 - w^2, \dfrac{w}{u}\Big) + ug\Big(\dfrac{w}{u}\Big).$

26. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + F(u,w), \quad \dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bF(u,w).$

27. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + uf(bu - cw) + g(bu - cw) + c\Phi(u,w),$
$\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + wf(bu - cw) + h(bu - cw) + b\Phi(u,w).$

2.2. Системы уравнений массо- и теплообмена реагирующих сред и уравнений математической биологии вида

$$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + F(u,w), \quad \frac{\partial w}{\partial t} = \frac{b}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + G(u,w)$$

Предварительные замечания. Подобные системы дифференциальных уравнений широко используются в теории массотеплопереноса реагирующих сред, в теории химических реакторов, в теории горения, в математической биологии и биофизике. Значения $n = 1$ и $n = 2$ соответствуют задачам с осевой и центральной симметрией.

Далее $f(\ldots), g(\ldots), h(\ldots)$ — произвольные функции соответствующего аргумента.

1. $\dfrac{\partial u}{\partial t} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\Big(x^n\dfrac{\partial u}{\partial x}\Big) + uf(bu - cw) + g(bu - cw),$
$\dfrac{\partial w}{\partial t} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\Big(x^n\dfrac{\partial w}{\partial x}\Big) + wf(bu - cw) + h(bu - cw).$

2. $\dfrac{\partial u}{\partial t} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\Big(x^n\dfrac{\partial u}{\partial x}\Big) + e^{\lambda u}f(\lambda u - \sigma w),$
$\dfrac{\partial w}{\partial t} = \dfrac{b}{x^n}\dfrac{\partial}{\partial x}\Big(x^n\dfrac{\partial w}{\partial x}\Big) + e^{\sigma w}g(\lambda u - \sigma w).$

3. $\dfrac{\partial u}{\partial t} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\Big(x^n\dfrac{\partial u}{\partial x}\Big) + uf\Big(\dfrac{u}{w}\Big), \quad \dfrac{\partial w}{\partial t} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\Big(x^n\dfrac{\partial w}{\partial x}\Big) + wg\Big(\dfrac{u}{w}\Big).$

4. $\dfrac{\partial u}{\partial t} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\Big(x^n\dfrac{\partial u}{\partial x}\Big) + uf\Big(\dfrac{u}{w}\Big), \quad \dfrac{\partial w}{\partial t} = \dfrac{b}{x^n}\dfrac{\partial}{\partial x}\Big(x^n\dfrac{\partial w}{\partial x}\Big) + wg\Big(\dfrac{u}{w}\Big).$

5. $\dfrac{\partial u}{\partial t} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\Big(x^n\dfrac{\partial u}{\partial x}\Big) + uf\Big(\dfrac{u}{w}\Big) + \dfrac{u}{w}h\Big(\dfrac{u}{w}\Big),$
$\dfrac{\partial w}{\partial t} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\Big(x^n\dfrac{\partial u}{\partial w}\Big) + wg\Big(\dfrac{u}{w}\Big) + h\Big(\dfrac{u}{w}\Big).$

6. $\dfrac{\partial u}{\partial t} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\Big(x^n\dfrac{\partial u}{\partial x}\Big) + u^k f\Big(\dfrac{u}{w}\Big), \quad \dfrac{\partial w}{\partial t} = \dfrac{b}{x^n}\dfrac{\partial}{\partial x}\Big(x^n\dfrac{\partial w}{\partial x}\Big) + w^k g\Big(\dfrac{u}{w}\Big).$

7. $\dfrac{\partial u}{\partial t} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\Big(x^n\dfrac{\partial u}{\partial x}\Big) + uf\Big(\dfrac{u}{w}\Big)\ln u + ug\Big(\dfrac{u}{w}\Big),$
$\dfrac{\partial w}{\partial t} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\Big(x^n\dfrac{\partial w}{\partial x}\Big) + wf\Big(\dfrac{u}{w}\Big)\ln w + wh\Big(\dfrac{u}{w}\Big).$

8. $\dfrac{\partial u}{\partial t} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\Big(x^n\dfrac{\partial u}{\partial x}\Big) + uf(x, u^k w^m),$
$\dfrac{\partial w}{\partial t} = \dfrac{b}{x^n}\dfrac{\partial}{\partial x}\Big(x^n\dfrac{\partial w}{\partial x}\Big) + wg(x, u^k w^m).$

9. $$\frac{\partial u}{\partial t}=\frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right)+u^{1+kn}f(u^nw^m),$$
$$\frac{\partial w}{\partial t}=\frac{b}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right)+w^{1-km}g(u^nw^m).$$

10. $$\frac{\partial u}{\partial t}=\frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right)+cu\ln u+uf(x,u^kw^m),$$
$$\frac{\partial w}{\partial t}=\frac{b}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right)+cw\ln w+wg(x,u^kw^m).$$

11. $$\frac{\partial u}{\partial t}=\frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right)+uf(u^2+w^2)-wg(u^2+w^2),$$
$$\frac{\partial w}{\partial t}=\frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right)+wf(u^2+w^2)+ug(u^2+w^2).$$

12. $$\frac{\partial u}{\partial t}=\frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right)+uf(u^2-w^2)+wg(u^2-w^2),$$
$$\frac{\partial w}{\partial t}=\frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right)+wf(u^2-w^2)+ug(u^2-w^2).$$

13. $$\frac{\partial u}{\partial t}=\frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right)+uf(u^2+w^2)-wg\left(\frac{w}{u}\right),$$
$$\frac{\partial w}{\partial t}=\frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right)+wf(u^2+w^2)+ug\left(\frac{w}{u}\right).$$

14. $$\frac{\partial u}{\partial t}=\frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right)+uf(u^2-w^2)+wg\left(\frac{w}{u}\right),$$
$$\frac{\partial w}{\partial t}=\frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right)+wf(u^2-w^2)+ug\left(\frac{w}{u}\right).$$