计算机科学中的数学基础 – exercise W2

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2023 年 2 月 20 日

Homework9

Sometimes it's possible to use induction backwards, proving things from n to $n-1$ instead of vice versa! For example, consider the statement

$$P(n): \quad x_1 \ldots x_n \leqslant \left(\frac{x_1 + \cdots + x_n}{n}\right)^n, \quad \text{if } x_1, \ldots, x_n \geqslant 0.$$

This is true when $n = 2$, since $(x_1 + x_2)^2 - 4x_1x_2 = (x_1 - x_2)^2 \geqslant 0$.

a By setting $x_n = (x_1 + \cdots + x_{n-1})/(n-1)$, prove that $P(n)$ implies $P(n-1)$ whenever $n > 1$.

b Show that $P(n)$ and $P(2)$ imply $P(2n)$.

c Explain why this implies the truth of $P(n)$ for all n.

(a) If we want to prove P(n-1) using P(n),we can manage this by proving the following equation :

$$x_1 \cdots x_{n-1}\left(\frac{x_1 + \cdots x_{n-1}}{n-1}\right) \leq \left(\frac{x_1 + \cdots x_{n-1}}{n-1}\right)^n \tag{1}$$

It's quite easy,as we can use the inequality:

$$\left(\frac{x_1 + \cdots x_{n-1}}{n-1}\right)^{n-1} \geq \left(\frac{x_1 + \cdots x_n}{n}\right)^n \tag{2}$$

(b) From P[n],we can get:

$$x_1 \cdots x_n x_{n+1} \cdots x_{2n} \leq \left(\left(\frac{x_1 + \cdots + x_n}{n}\right)\left(\frac{x_{n+1} + \cdots + x_{2n}}{n}\right)\right)^n \tag{3}$$

then using P(2),we can further get :

$$\left(\left(\frac{x_1 + \cdots + x_n}{n}\right)\left(\frac{x_{n+1} + \cdots + x_{2n}}{n}\right)\right)^n \leq \left(\frac{x_1 + \cdots + x_{2n}}{2n}\right)^2 \tag{4}$$

then the question got proved.

(c) Using the conclusion we got from (b),since we got P(2) now,then we can got P(4),using (a),we can get P(3).So,we can keep getting $P(2^k)$ using (b),and get $P(2^{k-1}+1)$ to $P(2^k-1)$.

Homework11

A Double Tower of Hanoi contains $2n$ disks of n different sizes, two of each size. As usual, we're required to move only one disk at a time, without putting a larger one over a smaller one.

a How many moves does it take to transfer a double tower from one peg to another, if disks of equal size are indistinguishable from each other?

b What if we are required to reproduce the original top-to-bottom order of all the equal-size disks in the final arrangement? [*Hint:* This is difficult—it's really a "bonus problem."]

a 仿照经典汉诺塔问题的思路，由于最下面的两个盘子是等大小的，所以可以一起处理，即，每次处理两个相同大小的盘子。所以，过程就是：现将除去最下面两个盘子的其余盘子移到中间柱上，然后用 2 步移动最大的两个盘子到目标柱上。因此，可以得到递归式：

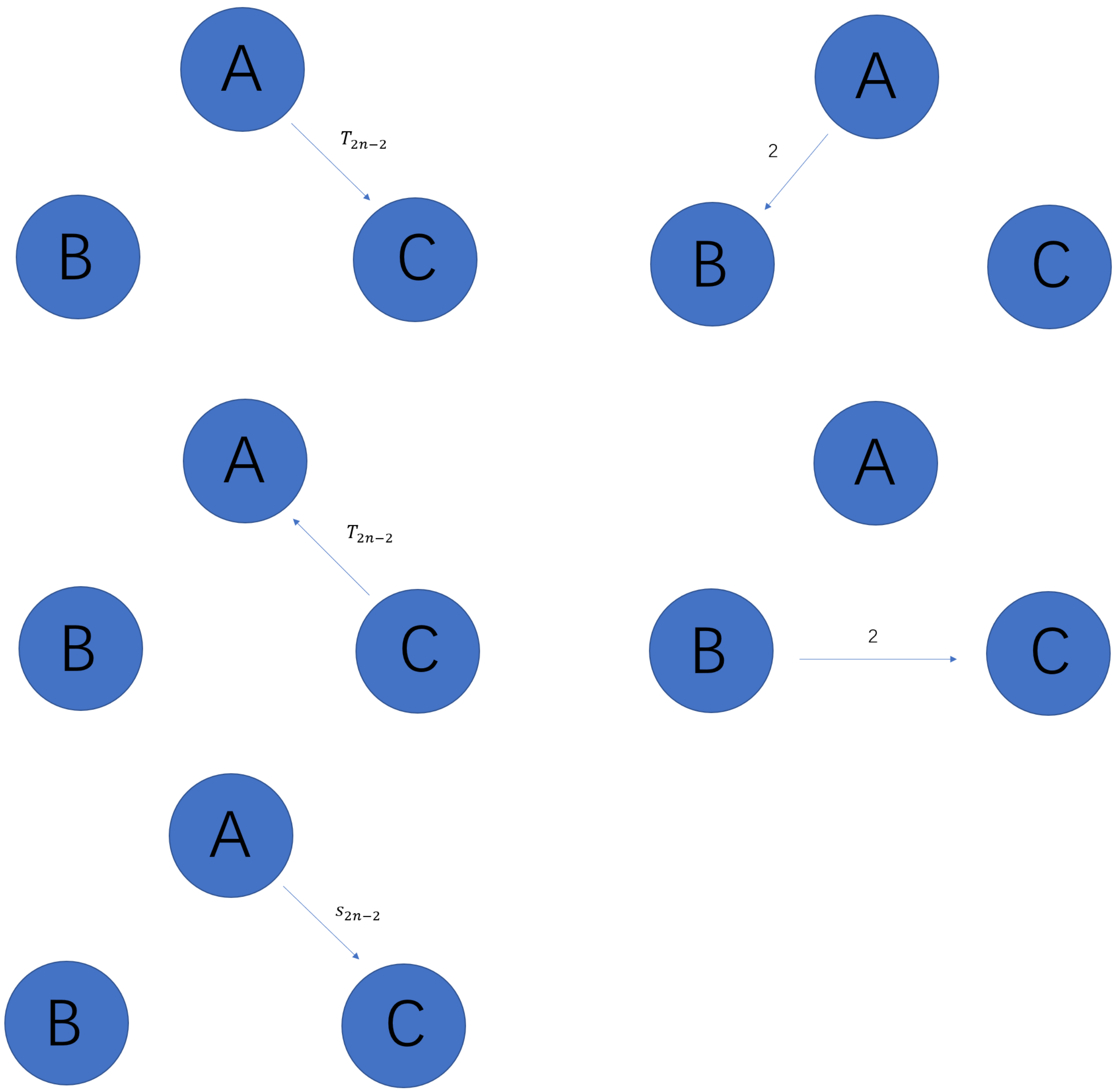
$$T_{2n} = 2 * T_{2n-2} + 2 \quad (n \geq 2) T_{2n} = 2 \quad (n = 1) \tag{5}$$

然后使用递归树，或者书中经典解法对递归式进行求解，可以得到：

$$T_{2n} = 2^{n+1} - 2 \tag{6}$$

b 易知，对于（a）中的过程，进行两次和（a）中过程一样的移动，得到的盘子和原先顺序一致。

因此，最少步骤的移动方式为：



由此，我们可以得到递归式：

$$S_{2n} = 2 * T_{2n-2} + 4 + S_{2n-2} \tag{7}$$

求解得，

$$S_{2n} = 2^{n+2} - 5 \tag{8}$$

Warmup1

What does the notation

$$\sum_{k=4}^{0} q_k$$

mean?

由本章第一节中对记号的定义，原式表达的含义是从 k=4 到 k=0 之间的 a_k 累加。而 $4 \leq k \leq 0$ 中包含 0 个符合条件的 k，故原式为 0。

Warmup2

Simplify the expression $x \cdot \big([x>0]-[x<0]\big)$.

题干中 [x > 0] 的含义为：

$$x = \begin{cases} 0 & \text{if } x <= 0, \\ 1 & \text{if } x > 0. \end{cases} \tag{9}$$

[x < 0] 类似。

因此，原式的含义为 $|x|$.

Warmup3

Demonstrate your understanding of $\sum$-notation by writing out the sums

$$\sum_{0 \leqslant k \leqslant 5} a_k \qquad \text{and} \qquad \sum_{0 \leqslant k^2 \leqslant 5} a_{k^2}$$

in full. (Watch out — the second sum is a bit tricky.)

由课本中关于求和符号的定义：我们将

$$\sum_{P(k)} a_k \tag{10}$$

记为所有项 a_k 之和的缩写，其中的 k 是满足给定性质 P(k) 的一个整数.

因此，由不等式 $0 \leq k \leq 5$ 解出 k=0,1,2,3,4,5, 因此

$$\sum_{0 \leq k \leq 5} a_k = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 \tag{11}$$

同理，由不等式 $0 \leq k^2 \leq 5$ 解出 k=0,-1,1,-2,2, 因此

$$\sum_{0 \leq k \leq 5} a_k = a_0 + a_1 + a_1 + a_2 + a_2 \tag{12}$$

Express the triple sum

$$\sum_{1\leqslant i<j<k\leqslant 4} a_{ijk}$$

as a three-fold summation (with three $\sum$'s),

a summing first on k, then j, then i;

b summing first on i, then j, then k.

Also write your triple sums out in full without the $\sum$-notation, using parentheses to show what is being added together first.

Warmup4

a 可以首先求出 i，j,k 的范围，分别是 $1 \le i \le 2, 2 \le j \le 3, 3 \le k \le 4$, 所以可以写出：

$$\sum_{1\le i\le 2}\sum_{i+1\le j\le 3}\sum_{j+1\le k\le 4} a_{ijk} = ((a_{123}+a_{124})+a_{134})+a_{234} \tag{13}$$

b 由（a）中 i,j,k 的范围，再结合求和的顺序，可以写出：

$$\sum_{3\le k\le 4}\sum_{2\le j\le k-1}\sum_{1\le i\le j-1} a_{ijk} = ((a_{134}+a_{234})+a_{124})+a_{123} \tag{14}$$

Warmup5

What's wrong with the following derivation?

$$\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{k=1}^{n}\frac{1}{a_k}\right) = \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{a_j}{a_k} = \sum_{k=1}^{n}\sum_{k=1}^{n}\frac{a_k}{a_k} = \sum_{k=1}^{n} n = n^2.$$

虽然 j 和 k 的取值范围相同，但是 j 和 k 是不同的指标，a_j 和 a_k 也未必是相同值。使用统一的指标 k 去替代两个独立的，不相同的指标显然是错误的。

尽管当 $a_k = a_j(1 \le k \le n, 1 \le j \le n)$，结果是正确的，但是推导过程依然是错误的。