计算机科学中的数学基础 Exercise9

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Warmup1

1 What is the smallest positive integer that has exactly k divisors, for $1 \leqslant k \leqslant 6$?

分别是 1,2,4,6,16,18.

Warmup2

2 Prove that $\gcd(m, n) \cdot \text{lcm}(m, n) = m \cdot n$, and use this identity to express $\text{lcm}(m, n)$ in terms of $\text{lcm}(n \bmod m, m)$, when $n \bmod m \neq 0$. *Hint:* Use (4.12), (4.14), and (4.15).

由 4.14，

$$k = gcd(m,n) \Leftrightarrow k_p = min(m_p, n_p) \quad ,for \quad any \quad p \tag{1}$$
$$k = lcm(m,n) \Leftrightarrow k_p = max(m_p, n_p) \quad ,for \quad any \quad p \tag{2}$$

故，

$$gcd(m,n) \times lcm(m,n) \Leftrightarrow k_p = min(m_p, n_p) + max(m_p, n_p) \quad ,for \quad any \quad p \tag{3}$$
$$\Leftrightarrow k_p = m_p + n_p \quad for \quad any \quad p \tag{4}$$
$$\tag{5}$$

因为 $m + n = max(m, n) + min(m, n)$，所以，

$$m \times n \Leftrightarrow k_p = m_p + n_p \quad for \quad any \quad p \tag{6}$$
$$gcd(m,n) \times lcm(m,n) = m \times n \tag{7}$$

Warmup3

3 Let $\pi(x)$ be the number of primes not exceeding x. Prove or disprove:

$$\pi(x) - \pi(x-1) = [x \text{ is prime}] .$$

显然对于整数，该式成立，因为 π(x) 与 π(x-1) 的差取决于 x 是否为素数.
对于 x 为实数的情况，显然，借助向下取整函数,π(x)=π($\lfloor x \rfloor$)，因此，实数的情形可以转化为整数的情况，

$$\pi(x) - \pi(x-1) = [\lfloor x \rfloor \quad is \quad prime] \tag{8}$$

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14 Prove or disprove:

a $\gcd(km, kn) = k\gcd(m, n)$;

b $\mathrm{lcm}(km, kn) = k\,\mathrm{lcm}(m, n)$.

结论：当 k 为非负整数时，该式成立.

首先，由式 4-14,4-15，有

$$gcd(m,n) = \prod_p p^{min(m_p,n_p)} \tag{9}$$

$$lcm(m,n) = \prod_p p^{max(m_p,n_p)} \tag{10}$$

而 k 也可以表示为 $\langle k_1, k_2, \cdots \rangle$, 即 $\prod_p p^{k_p}$。设 $km = m^{'}, kn = n^{'}$, 则有：

$$gcd(m^{'},n^{'}) = \prod_p p^{min(m^{'}_p,n^{'}_p)} \tag{11}$$

$$lcm(m^{'},n^{'}) = \prod_p p^{max(m^{'}_p,n^{'}_p)} \tag{12}$$

$$m^{'}_p = k_p + m_p \tag{13}$$

$$n^{'}_p = k_p + n_p \tag{14}$$

从而，

$$\begin{aligned} gcd(m^{'},n^{'}) &= \prod_p p^{min(m^{'}_p,n^{'}_p)} &(15)\\ &= \prod_p p^{min(m_p,n_p)+k_p} &(16)\\ &= \prod_p p^{min(m_p,n_p)} \times k &(17)\\ lcm(m^{'},n^{'}) &= \prod_p p^{max(m^{'}_p,n^{'}_p)} &(18)\\ &= \prod_p p^{max(m_p,n_p)+k_p} &(19)\\ &= \prod_p p^{max(m_p,n_p)} \times k &(20) \end{aligned}$$

即，

$$gcd(km,kn) = \prod_p p^{min(m_p,n_p)} \times k \tag{21}$$

$$lcm(km,kn) = \prod_p p^{max(m_p,n_p)} \times k \tag{22}$$

Basics15

15 Does every prime occur as a factor of some Euclid number e_n?

不是，由欧拉数的定义，显然可以看出，

$$e_1 = 2 \tag{23}$$
$$e_2 = 3 \tag{24}$$
$$e_3 = 7 \tag{25}$$
$$e_4 = 43 \tag{26}$$
$$\tag{27}$$

因此，我们可以假设欧拉数 e (mod 5) = 2 或 3.

若欧拉数 $e_{n-1}(mod5) = 2$，有 $e_n = e_{n-1} \times (e_{n-1} - 1) = (5k+2) \times (5k+1) + 1 = 25k^2 + 15k + 3$, 则 $e_n(mod5) = 3$, 若欧拉数 $e_{n-1}(mod5) = 3$，有 $e_n = e_{n-1} \times (e_{n-1} - 1) = (5k+3) \times (5k+2) + 1 = 25k^2 + 15k + 7$, 则 $e_n(mod5) = 2$, 故，可以从 2 开始使用数学归纳法。