计算机科学中的数学基础–exercise2

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Warmup:Q6

Some of the regions defined by n lines in the plane are infinite, while others are bounded. What's the maximum possible number of bounded regions?

Answer

This question is a little different from the example from the textbook,but we can also try to solve the problem by visualizing the recurrent process.

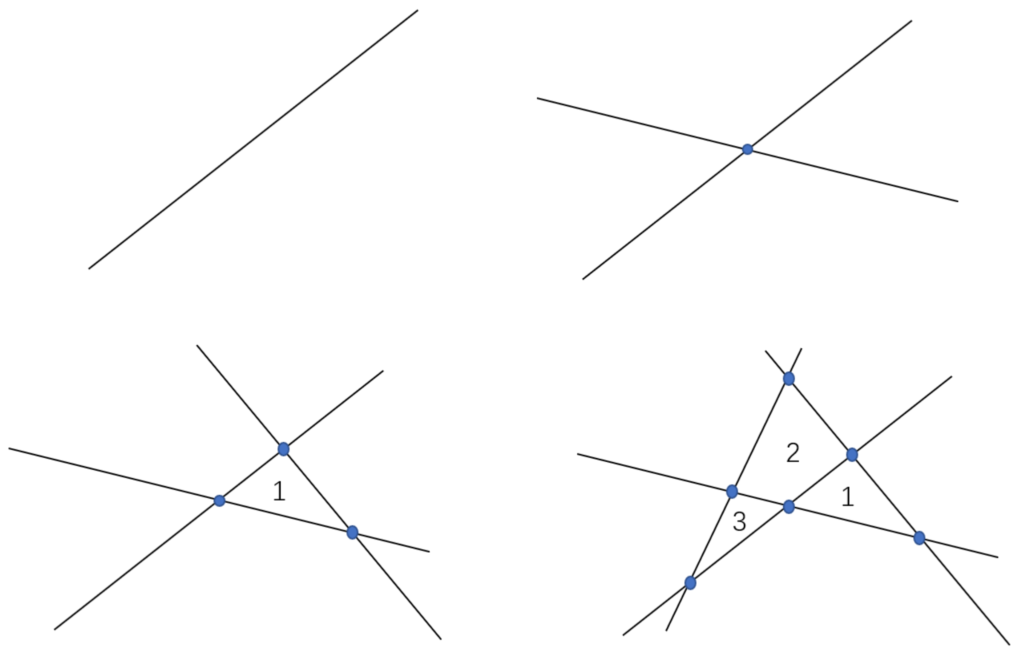

图 1: Q6-demo

So, we can kind of easily draw the conclusion that:the number of added bounded region equals the number of the lines which the new line intersect with minus one(in fact initially we can just).So,we get the following recursive formula:

$$T_n = 0 \quad (n = 1) \tag{1}$$

$$T_n = T_{n-1} + n - 2 \quad (n \geq 2) \tag{2}$$

We can get the answer by accumulating all the items together:

$$T_n = 0 \quad (n = 0) \tag{3}$$

$$T_n = \frac{(n-1)(n-2)}{2} \quad (n \geq 2) \tag{4}$$

Warmup:Q7

Let $H(n) = J(n+1) - J(n)$. Equation (1.8) tells us that $H(2n) = 2$, and $H(2n+1) = J(2n+2) - J(2n+1) = (2J(n+1)-1) - (2J(n)+1) = 2H(n)-2$, for all $n \geqslant 1$. Therefore it seems possible to prove that $H(n) = 2$ for all n, by induction on n. What's wrong here?

Answer

We can easily notice that H(0)=1$\neq$2,so the foudation of the inductive method collapses.

Homework:Q8

Solve the recurrence

$$Q_0 = \alpha; \qquad Q_1 = \beta;$$
$$Q_n = (1+Q_{n-1})/Q_{n-2}, \qquad \text{for } n > 1.$$

Assume that $Q_n \neq 0$ for all $n \geqslant 0$. *Hint:* $Q_4 = (1+\alpha)/\beta$.

Answer

Well,nothing had flashed across my mind when I was trying to solve the problem initially.What could I do was keeping calculating.Then,I got the answer.The series is periodic!

$$Q_0 = \alpha \tag{5}$$
$$Q_1 = \beta \tag{6}$$
$$Q_2 = \frac{1+\beta}{\alpha} \tag{7}$$
$$Q_3 = \frac{1+\alpha+\beta}{\alpha\beta} \tag{8}$$
$$Q_4 = \frac{1+\alpha}{\beta} \tag{9}$$
$$Q_5 = \alpha \tag{10}$$

Homework:Q9

Sometimes it's possible to use induction backwards, proving things from n to $n-1$ instead of vice versa! For example, consider the statement

$$P(n): \quad x_1 \ldots x_n \leqslant \left(\frac{x_1 + \cdots + x_n}{n}\right)^n, \quad \text{if } x_1, \ldots, x_n \geqslant 0.$$

This is true when $n = 2$, since $(x_1+x_2)^2 - 4x_1x_2 = (x_1-x_2)^2 \geqslant 0$.

a By setting $x_n = (x_1 + \cdots + x_{n-1})/(n-1)$, prove that $P(n)$ implies $P(n-1)$ whenever $n > 1$.

b Show that $P(n)$ and $P(2)$ imply $P(2n)$.

c Explain why this implies the truth of $P(n)$ for all n.

Answer

(a) If we want to prove P(n-1) using P(n),we can manage this by proving the following equation :

$$x_1 \cdots x_{n-1}(\frac{x_1 + \cdots x_{n-1}}{n-1}) \leq (\frac{x_1 + \cdots x_{n-1}}{n-1})^n \tag{11}$$

It's quite easy,as we can use the inequality:

$$(\frac{x_1 + \cdots x_{n-1}}{n-1})^{n-1} \geq (\frac{x_1 + \cdots x_n}{n})^n \tag{12}$$

(b) From P[n],we can get:

$$x_1 \cdots x_n x_{n+1} \cdots x_{2n} \leq ((\frac{x_1 + \cdots + x_n}{n})(\frac{x_{n+1} + \cdots + x_{2n}}{n}))^n \tag{13}$$

then using P(2),we can further get :

$$((\frac{x_1 + \cdots + x_n}{n})(\frac{x_{n+1} + \cdots + x_{2n}}{n}))^n \leq (\frac{x_1 + \cdots + x_{2n}}{2n})^2 \tag{14}$$

then the question got proved.

(c) Using the conclusion we got from (b),since we got P(2) now,then we can got P(4),using (a),we can get P(3).So,we can keep getting P(2^k) using (b),and get P($2^{k-1}+1$) to P(2^k-1).

Homework:Q10

Let Q_n be the minimum number of moves needed to transfer a tower of n disks from A to B if all moves must be *clockwise* — that is, from A to B, or from B to the other peg, or from the other peg to A. Also let R_n be the minimum number of moves needed to go from B back to A under this restriction. Prove that

$$Q_n = \begin{cases} 0, & \text{if } n = 0; \\ 2R_{n-1}+1, & \text{if } n > 0; \end{cases} \qquad R_n = \begin{cases} 0, & \text{if } n = 0; \\ Q_n + Q_{n-1} + 1, & \text{if } n > 0. \end{cases}$$

(You need not solve these recurrences; we'll see how to do that in Chapter 7.)

Answer

To solve the problem,we are supposed to be aware that the process of moving the tower from A to B euqals to from B to C and C to A as well.Also,the process of moving the tower from A to C,from B to A,from C to B are the same,which is R_n under the context.

So,the first set of equations got proved,because we should move the upper n-1 plates from A to C which uses R_n,then move the biggest one from A to B which uses one step,finally move the n- tower from C to B which is also at the cost of R_n.

As to the second set of formula,we should move the upper n-1 plates from A to C,use R_{n-1} steps,then the biggest one from A to B ,one step,then the n-1 tower from C to A,Q_{n-1} steps,then move the biggest one to C,at last move the n-1 tower from A to C within R_{n-1} steps.We got:

$$R_n = R_{n-1} + 1 + Q_{n-1} + R_{n-1} + 1 \tag{15}$$

With $Q_n = 2R_{n-1} + 1$ replacing Q_n,we got the right answer!.