计算机科学中的数学基础 Exercise7

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Warmup1

1 When we analyzed the Josephus problem in Chapter 1, we represented an arbitrary positive integer n in the form $n = 2^m + l$, where $0 \leqslant l < 2^m$. Give explicit formulas for l and m as functions of n, using floor and/or ceiling brackets.

由 $2^m \leq n \leq 2^m + l$,$0 \leq l < 2^m$ 知，

$$2^m \leq n < 2^{m+1} \tag{1}$$

对上式取对数，有:

$$m \leq logn < m + 1 \tag{2}$$
$$m = \lfloor logn \rfloor \tag{3}$$
$$m + 1 = \lceil logn \rceil \tag{4}$$

故，答案为:

$$m = \lfloor logn \rfloor \tag{5}$$
$$= \lfloor logn \rfloor - 1 \tag{6}$$
$$l = n - 2^m \tag{7}$$
$$= n - 2^{\lfloor logn \rfloor} \tag{8}$$
$$= n - 2^{\lceil logn \rceil - 1} \tag{9}$$

Warmup2

2 What is a formula for the nearest integer to a given real number x? In case of ties, when x is exactly halfway between two integers, give an expression that rounds (a) up—that is, to $\lceil x \rceil$; (b) down—that is, to $\lfloor x \rfloor$.

首先，很容易发现离 x 最近的整数相当于对 x 进行四舍五入，当然，*.5 的情况有稍微差异。不妨设离 x 最近的整数为 y，所以很容易写出:

$$(a) \quad y = \lfloor x + 0.5 \rfloor \tag{10}$$
$$(b) \quad y = \lceil x - 0.5 \rceil \tag{11}$$

Warmup3

3 Evaluate $\lfloor \lfloor m\alpha \rfloor n/\alpha \rfloor$, when m and n are positive integers and α is an irrational number greater than n.

不妨令:

$$\begin{aligned} \alpha &= k + \alpha \qquad (12) \\ k &= \lfloor k \rfloor \qquad (13) \end{aligned}$$

有，

$$\begin{aligned} \lfloor \frac{\lfloor m\alpha \rfloor n}{n} \rfloor &= \lfloor \frac{(m\alpha - m\alpha)n}{\alpha} \rfloor \qquad (14) \\ &= mn - \lfloor \frac{mn\alpha}{\alpha} \rfloor \qquad (15) \\ &= mn - 1 \qquad (16) \end{aligned}$$

Warmup4

4 The text describes problems at levels 1 through 5. What is a level 0 problem? (This, by the way, is *not* a level 0 problem.)

指不需要给出严谨的证明，可以根据经验、意识、感觉猜测出来的问题。

Warmup5

5 Find a necessary and sufficient condition that $\lfloor nx \rfloor = n\lfloor x \rfloor$, when n is a positive integer. (Your condition should involve $\{x\}$.)

由定义，有:

$$\begin{aligned} \lfloor nx \rfloor &= \lfloor n\lfloor x \rfloor + nx \rfloor \qquad (17) \\ &= n\lfloor x \rfloor + \lfloor nx \rfloor \qquad (18) \end{aligned}$$

根据题意，有:

$$\lfloor nx \rfloor = n\lfloor x \rfloor \qquad (19)$$

故有:

$$\lfloor n\lfloor x \rfloor \rfloor = 0 \qquad (20)$$

因为 n 为正整数，所以有:

$$\begin{aligned} 0 \le nx &< 1 \qquad (21) \\ 0 \le x &< \frac{1}{n} \qquad (22) \\ x &< \frac{1}{x} \qquad (23) \end{aligned}$$

Warmup6

6 Can something interesting be said about $\lfloor f(x) \rfloor$ when $f(x)$ is a continuous, monotonically *decreasing* function that takes integer values only when x is an integer?

结论：

$$\lfloor f(x) \rfloor = \lfloor f(\lceil x \rceil) \rfloor \tag{24}$$

由题意，f(x) 只在整数处取整数值。因此，不妨令 $\lfloor f(x) \rfloor =$ k, 即，$k \leq f(x) < k+1$, 再令 $f(x_0) = k$, 故由 f(x) 单调递减，对前述 x，有 $\lceil x \rceil = x_0$, 有，

$$\lfloor f(x) \rfloor = \lfloor f(\lceil x \rceil) \rfloor \tag{25}$$